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ON CERTAIN INTEGRAL EQUATION WITH DELAY

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The purpose of this paper is to discuss the problem of a global existence of a solution (as well as a maximal and a minimal solution) for integral equation with delay

$$\begin{cases} z(t)=r(t), \quad t\in E_{t_0} \\ z(t)=r(t)+\displaystyle\int_{t_0}^{t} K(t,\, s,\, z(s),\, z[h_1(s)],\, \dots,\, z[h_m(s)]\,\mathrm{d}s, \quad t\geqq t_0 \end{cases} \tag{1}$$

Here $h_i(t)\in C[I, R]$, $I=(a, \infty)$ such that $h_i(t)\leqq t$ for each $t\in I$, $i=1, \dots, m$; $E_{t_0}=\bigcup_{i=1}^{m} E^i_{t_0}$ is the initial set, $t_0\in(a, \infty)$, where $E^i_{t_0}=\left(\inf_{t\in J} h_i(t),\, t_0\right]$, $i=1, \dots, m$ and we shall assume that for every $i=1, \dots, m$ there exists $t_i\in[t_0, \infty)=J$ such that $h_i(t_i)=t_0$. If $\inf_{t\in J} h_i(t)=\min_{t\in J} h_i(t)$ we shall put $E^i_{t_0}=\left[\inf_{t\in J} h_i(t),\, t_0\right]$.

Several properties of solutions of (1) are demonstrated and accompanying estimates of solutions are obtained.

Let us denote $z_T(t)$ the continuous solution of (1) which exists on the interval $[T_0, T)\subset J$.

Definition 1. If $\bar{T}<T(T, \bar{T}\in(t_0, \infty))$ and $z_T(t)=z_{\bar{T}}(t)$ for $t\in[t_0, T)$, then a solution $z_{\bar{T}}(t)$ is called an extension of the solution $z_T(t)$ to $[t_0, \bar{T})$ and the solution $z_T(t)$ is called a restriction of the solution $z_{\bar{T}}(t)$. Moreover, the solution $z_T(t)$ is called total (outside the $[t_0, T)$) if it is not a restriction of no one solution of (1).

Theorem 1. Let $K(t, s, v, u_1, \dots, u_m)\in C[L\times R^{n(m+1)}, R]$, where $L=\{(t, s)\colon a\leqq s\leqq t<\infty\}$, be the nondecreasing function in each of variables $v, u_1, \dots, u_m$ for all $(t, s)\in L$. Let $r(t)\in C[E_{t_0}\cup J, R]$. Let $c>0$, $b<\infty$ and let there exist a continuous solution $z_b(t)$ of (1). Then, if for every γ: $0<\gamma<c$ there exists a function $m_\gamma(t, s)\in C[L, R_+]$ such that $\sup_{\substack{|v|\leqq\gamma,\, |u_i|\leqq\gamma,\\ i=1, \dots, m}} |K(t, s, v, u_1, \dots, u_m)|\leqq m_\gamma(t, s)$

and $\sup_{t_0 \leqq t < b} |z(t)| < \infty$, there exists a number $d > 0$ such that the solution $z_b(t)$ can be extended to $[t_0, b+d)$.

Proof. Since $|K(t, s, z_b(s), z_b[h_1(s)], \ldots, z_b[h_m(s)])| \leqq m_\gamma(t, s)$ where $\gamma = \sup_{t \in [t_0, b)} \{[z_b(t)], |z_b[h_1(t)]|, \ldots, |z_b[h_m(t)]|\}$, $t_0 \leqq s \leqq b$, the function $K(t, s, z_b(s), z_b[h_1(s)], \ldots, z_b[h_m(s)])$ is an integrable function for $s \in [t_0, b]$ and for each $t \in [b, \infty)$ and the function

$$v(t) = \int_{t_0}^{b} K(t, s, z_b(s), z_b[h_1(s)], \ldots, z_b[h_m(s)])\, ds \tag{2}$$

is continuous for $b \leqq t < \infty$ as well. We will show that

$$\lim_{t \to b^-} z_b(t) = r(b) + v(b). \tag{3}$$

Indeed, for $t \in (t_0, b)$ we have $|z_b(t) - r(b) - v(b)| \leqq \int_{t_0}^{t} |K(t, s, z_b(s), z_b[h_1(s)], \ldots, z_b[h_m(s)]) - K(b, s, z(s), z[h_1(s)], \ldots, z[h_m(s)])|\, ds + \int_{t}^{b} |K(b, s, z_b(s), z_b[h_1(s)], \ldots, z_b[h_m(s)])|\, ds + |r(b) - r(t)|$ wherefrom with respect to the continuity of r we obtain (3).

Let $\gamma > 0$ be such that $|r(b) + v(b)| < \gamma$ (we may observe that from the above conditions it follows $|r(b) + v(b)| < \infty$). We denote

$$S = \{u \in C[I, R] : |u(t)| \leqq c,\ |u[h_i(t)]| \leqq c,\ b \leqq t \leqq b + d,\ i = 1, \ldots, m\}.$$

On the set S we define an operator Z in this way: for $u \in S$

$$Zu(t) = r^0(t),\quad t \in E_b$$

$$Zu(t) = r(t) + \int_b^t K(t, s, u(s), u[h_1(s)], \ldots, u[h_m(s)])\, ds + v(t),$$

where

$$r^0(t) = \begin{matrix} r(t), & t \in E_{t_0} \cap E_b \\ z(t), & t \in [t_0, b) \cap E_b \end{matrix}$$

and the set E_b (an initial set) is defined by $E_b = \bigcup_{i=1}^{m} E_b^i$, $E_b^i = \left[\inf_{t \in [b, b+d)} h_i(t), b\right]$, $i = 1, \ldots, m$ and we assume that for every $i = 1, \ldots, m$ there exists $t_i \in [b, b+d)$ such that $h_i(t_i) = b$.

Since for $u(t) \in S$ the inequalities $|Zu(t)| \leqq \int_b^t |K(t, s, u(s), u[h_1(s)], \ldots, u[h_m(s)])\, ds + |r(t) + v(t)| \leqq \int_b^t |K(t, s, c, \ldots, c)|\, ds + |r(t) + v(t)| \leqq$

$\int_b^t m_c(t, s)\, ds + |r(t)+v(t)-r(b)-v(b)| + |r(b)+v(b)|$ are true we see that for d sufficiently small $ZS \subset S$. Besides $Zu \in C[[b, b+d), R]$ and by [4], p. 123, Z is a completely continuous operator. Therefore, by Schauder's theorem there exists a solution $w(t)$ of the equation $u = Zu$. It can be easily shown that

$$x(t) = \begin{cases} r(t), & t \in E_{t_0} \\ z(t), & t \in [t_0, b) \\ w(t), & t \in [b, b+d) \end{cases}$$

is a continuous function on $[t_0, b+d)$ satisfying (1) and it is an extension of the solution $z_b(t)$ which completes the proof.

Corollary. If $[t_0, b)$, $b < \infty$ is the maximal interval of existence of the continuous solution $z(t)$ of (1) then $\lim\limits_{t \to b^-} \sup |z(t)| = \infty$.

Proof. This result follows immediately from Theorem 1.

Theorem 2. Let K, r possess the same properties as in Theorem 1. Then each continuous solution $z_T(t)$ $(T \in J)$ of (1) is either a total solution or it is a restriction of at least one total solution.

Proof. To each continuous solution $z_1(t)$, $t_0 \leqq t < t_1$ of (1) we appoint T_1 — supreme of those values of t_1 for which there exists an extension of $z_1(t)$. (We remark that there can be $T_1 = t_1$ as well as $T_1 = \infty$). First of all we consider the case $T_1 < \infty$. We form an extension $z_2(t)$ of the solution $z_1(t)$ up to $t_2 < T_1 - 1 \leqq t_1$ – if such a one exists. Obviously $T_2 \leqq T_1$ (the meaning of T_2 for $z_2(t)$ is the same as of T_1 above). Therefore $T_2 - t_2 < 1$. By proceeding analogically a sequence of solutions $\{z_n(t)\}$ of (1) can be constructed each of which is an extension of preceding one up to t_n such that

$$T_n - t_n < \frac{1}{n}. \tag{3}$$

The sequence $\{T_n\}$ is nonincreasing. Unification of all $z_n(t)$ is denoted $z(t)$. Then $z(t)$ is the solution of (1) which exists for $t_0 \leqq t < T_0 = \lim\limits_{n \to \infty} t_n$. We claim this solution is total. If not, there exists its extension up to $T_0 + d$ $(d > 0)$ and simultaneously each of $z_n(t)$ can be extended up to $T_0 + d$. This is a contradiction with (3) for $\frac{1}{n} < d$. The claim is proved.

Now let $T_1 = \infty$. If the solution $z_1(t)$ is a restriction of the solution $z_2(t)$ for which $T_2 < \infty$, then $z_2(t)$ — and therefore also $z_1(t)$ — are in accordance with the above proved restrictions of a total solution of (1).

Finally, let $T = \infty$ for all extensions $z(t)$ of the solution $z_1(t)$. Then a sequence of solutions $\{z_n(t)\}$ can be constructed such that each of which is an extension of the preceding one up to t_1+n. Unification of all $z_n(t)$ is again a solution of (1) which is an extension $z_1(t)$ into infinity, i.e. a total solution.

Theorem 3. Let K, r possess the same properties as in Theorem 1. If for a function $p(t)\in C[E_{t_0}\cup J, R]$ the inequalities

$$\begin{cases} p(t)<r(t), & t\in E_{t_0} \\ p(t)<r(t)+\int_{t_0}^{b} K(t, s, p(s), p[h_1(s)], \dots, p[h_m(s)])\,ds, & t\in J \end{cases} \tag{4}$$

hold and $z(t)$ is a (continuous) solution of (1) existing on J, then

$$p(t)<z(t) \tag{5}$$

is true for each $t\in J$.

Remark. For any given r, a solution of (1) need not be unique. Theorem 3 gives the estimate for every continuous solution $z(t)$ of (1).

Proof. From (1) and (4) we have $p(t_0)<z(t_0)$. Since the both functions p and z are continuous, the inequality (5) holds on some right neighbourhood of t_0. Now suppose that (5) does not hold for all $t\in J$. Then there exists $t_1>t_0$ such that $p(t)<z(t)$, $t_0\leqq t<t_1$, $p(t_1)=z(t_1)$. But from (1) and (4) it follows $z(t_1)-$ $-p(t_1)>\int_{t_0}^{t_1}[K(t_1, s, z(s), z[h_1(s)], \dots, z[h_m(s)])-K(t_1, s, p(s), p[h_1(s)], \dots,$ $p[h_m(s)])]\,ds\geqq 0$ since K is nondecreasing function. This is a contradiction and, thus (5) holds on whole J.

Remark. If the inequality (4) is replaced by reverse, then (5) changes its sign analogically as well.

Definition 2. A solution $\bar{z}(t)(\underline{z}(t))$ is called the maximal (the minimal) solution of (1) if for any other solution $z(t)$ of (1) the inequality $z(t)\leqq$ $\bar{z}(t)(z(t)\geqq\underline{z}(t))$ holds on the common interval of existence.

Remark. From the above definition it follows that every restriction of the maximal (the minimal) solution is also the maximal (the minimal) solution and if $z_1(t)$, $z_2(t)$ are maximal (minimal) solutions of (1) on $[t_0, t_0+b_1)$, $[t_0, t_0+b_2)$ respectively, where $b_1<b_2$ then $z_1(t)=z_2(t)$ for $t_0\leqq t<t_0+b_1$.

Theorem 4. Let K, r possess the same properties as in Theorem 1. Let functions $z_j(t)\in C[E_{t_0}\cup JR]$, $j=1, 2$ be such that $|z_j(t)|<c$, $|z_j[h_i(t)]|<c$, $j=1, 2$; $i=1, \dots, m$; $0<c<\infty$ and let the following inequalities

$$\begin{aligned} &z_1(t)>r(t), \quad t\in E_{t_0} \\ &z_1(t)>r(t)+\int_{t_0}^{t} K(t, s, z_1(s), z_1[h_1(s)], \dots, z_1[h_m(s)])\,ds, \quad t\in J \\ &z_2(t)<r(t), \quad t\in E_{t_0} \\ &z_2(t)<r(t)+\int_{t_0}^{t} K(t, s, z_2(s), z_2[h_1(s)], \dots, z_2[h_m(s)])\,ds, \quad t\in J \end{aligned} \tag{6}$$

hold. Then the following statements are true:

i) each continuous solution $z_b(t)$ of (1) ($b<\infty$) can be extended on whole J;
ii) both maximal and minimal solutions of (1) $\bar{z}(t)$, $\underline{z}(t)$ respectively, exist on J;
iii) the inequalities

$$z_2(t)<\underline{z}(t)\leqq z^0(t)\leqq \bar{z}(t)<z_1(t), \quad t\in J \tag{7}$$

hold, where $z^0(t)$ is a total solution of (1).

Proof. First of all we will show that the solution $z(t)$ of (1) can be extended on whole J. If we suppose the contrary, then by Theorem 2 there exists a number d such that $[t_0, d)$ is the maximal interval of its existence. By Theorem 3 for $t\in[t_0, d)$ it holds $z_2(t)<z(t)<z_1(t)$ wherefrom one obtains $|z(t)|<\max\limits_{j=1,2}|z_j(t)|$, $|z[h_i(t)]|<\max\limits_{j=1,2}|z_j[h_i(t)]|$, $i=1, \dots, m$ for $t\in[t_0, d)$ which yields the inequality $\sup\limits_{t\in[t_0, d)}|z(t)|=\sup\limits_{t\in[t_0, d)}\max\limits_{j=1,2}|z_j(t)|$. This is a contradiction with the Corollary of Theorem 1. Thus each continuous solution $z(t)$ of (1) can be extended on whole J and for this total solution $z^0(t)$ the inequalities $z_2(t)<z^0(t)<z_1(t)$ hold for all $t\in J$.

To prove the existence of a maximal and a minimal solutions of (1) it is sufficient to consider the only one case since the latter one is similar. To prove the existence of a maximal solution of (1) on J, let $\{u_n(t)\}$ be a sequence defined by

$$\begin{aligned}
&u_n(t)=r(t), \quad t\in E_{t_0}, \quad n=1, 2, \dots,\\
&u_1(t)=z_1(t), \quad t\in J\\
&u_{n+1}(t)=r(t)+\int_{t_0}^{t} K(t, s, u_n(s), u_n[h_1(s)], \dots, u_n[h_m(s)])\, ds+\\
&\qquad +\frac{1}{n+1}v(t), \quad t\in J, \quad n=1, 2, \dots,
\end{aligned} \tag{8}$$

where

$$v(t)=z_1(t)-\int_{t_0}^{t} K(t, s, z_1(s), z_1[h_1(s)], \dots, z_1[h_m(s)])\, ds-r(t), \quad t\in J.$$

Then for $t\in J$ $z_1(t)-u_2(t)=u_1(t)-u_2(t)=\frac{1}{2}v(t)>0$, wherefrom $u_2<c$ and also $u_2(t)-z_2(t)\geqq\int_{t_0}^{t}[K(t, s, z_1(s), z_1[h_1(s)], \dots, z_1[h_m(s)])-K(t, s, z_2(s), z_2[h_1(s)], \dots, z_2[h_m(s)])]\, ds+\frac{1}{2}v(t)>0$, from where $u_2>-c$ so that $|u_2(t)|<c$, $t\in J$.

Now let for some integer $n\geqq 2$

$$z_2(t) < u_n(t) < u_{n-1}(t) < z_1(t), \quad t \in J \tag{9}$$

is true so that $|u_n(t)| < c$, $t \in J$. Then the function $u_{n+1}(t)$ is defined on J and $u_n(t) - u_{n+1}(t) = \int_{t_0}^{t} [K(t, s, u_{n-1}(s), u_{n-1}[h_1(s)], \ldots, u_{n-1}[h_m(s)]) - K(t, s, u_n(s), u_n[h_1(s)], \ldots, u_n[h_m(s)])]\, ds + \frac{1}{n(n+1)} v(t) > 0$, $u_{n+1}(t) - z_2(t) \geqq \int_{t_0}^{t} [K(t, s, u_n(s), u_n[h_1(s)], \ldots, u_n[h_m(s)]) - K(t, s, z_2(s), z_2[h_1(s)], \ldots, z_2[h_m(s)])\, ds + \frac{1}{n+1} v(t) > 0$.
Thus the inequalities (9) are true for all integer n and we get from the above $|u_n(t)| < c$, $t \in J$, $n = 1, 2, \ldots$

Because of $u_n(t) - \int_{t_0}^{t} K(t, s, u_n(s), u_n[h_1(s)], \ldots, u_n[h_m(s)])\, ds - r(t) = u_n(t) - u_{n+1}(t) + \frac{1}{n+1} v(t) > 0$, from (9) with the aid of Theorem 3 one obtains

$$z_2(t) < z^0(t) < u_n(t) < u_{n-1}(t) < z_1(t), \quad t \in J. \tag{10}$$

Hence

$$u(t) = \lim_{n \to \infty} u_n(t) = z^0(t), \quad t \in J. \tag{11}$$

To finish the proof we will show that the function $u(t)$ satisfies (8) and $u(t) \in C[J, R]$. Since the function K is continuous, $K(t, s, u_n[h_1(s)], \ldots, u_n[h_m(s)])$ is uniformly bounded for any fixed $t \in J$ and any $u_n(t)$ satisfying (10) and the inequalities $K(t, s, z_2[h_1(s)], \ldots, z_2[h_m(s)]) \leqq K(t, s, u_n(s), u_n[h_1(s)], \ldots, u_n[h_m(s)]) \leqq K(t, s, z_1(s), \ldots, z_1[h_m(s)])$ are valid, we have $\lim_{n \to \infty} K(t, s, u_n(s), u_n[h_1(s)], \ldots, u_n[h_m(s)]) = K(t, s, u(s), u[h_1(s)], \ldots, u[h_m(s)])$.

Hence, applying Lebesgue's theorem the formula (8) leads to

$$u(t) = r(t), \quad t \in E_{t_0}$$

$$u(t) = r(t) + \int_{t_0}^{t} K(t, s, u(s), u[h_1(s)], \ldots, u[h_m(s)])\, ds, \quad t \in J.$$

Consequently, $u(t)$ is the solution of (1) for all $t \in J$.

Let $t_0 \leqq t_1 < d < \infty$. Then, appealing to conditions of the theorem it follows from (10) that $|u(t)| < c$, $|u[h_i(t)]| < c$, $i = 1, \ldots, m$. Since for $t_1, t_2 \in [t_0, d)$ $(t_1 > t_2)$ the inequality $|u(t_2) - u(t_1)| \leqq \int_{t_0}^{t_2} |K(t_2, s, u(s), u[h_1(s)], \ldots, u[h_m(s)]) - K(t_1, s, u(s), u[h_1(s)], \ldots, u[h_m(s)])|\, ds + \int_{t_2}^{t_1} |K(t_1, s, u(s), u[h_1(s)], \ldots, u[h_m(s)])|\, ds + |r(t_1) - r(t_2)|$ holds, the continuity of $u(t)$ by t_2 tends to t_1 from the left is evident. The continuity from the right can be proved in a similar way. Because of

$\lim\limits_{t \to t_0^+} u(t) = r(t_0)$ and the formula (11) is true, $u(t)$ is the maximal solution of (1) for $t \in J$ and the proof is complete.

Remark. It can be seen that the Theorems 1—4 remain correct without a change also in the case if we consider vector functions instead of the scalar ones. It is necessary only to introduce an appropriate partially ordered vector space and to choose a fixed, convenient vector norm.

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SÚHRN

O ISTEJ INTEGRÁLNEJ ROVNICI S ONESKORENÍM

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V práci sa skúma úloha globálnej existencie riešení istej integrálnej rovnice s oneskorením. Sú určené podmienky existencie riešenia (tiež maximálneho a minimálneho riešenia) na nekonečnom intervale, vyšetrené niektoré ich vlastnosti a stanovené odhady týchto riešení.

РЕЗЮМЕ

ОБ ОДНОМ ИНТЕГРАЛЬНОМ УРАВНЕНИИ С ЗАПАЗДЫВАНИЕМ

В. Роса, Братислава

В работе исследована задача общего существования решений какого-то интегрального уравнения с запаздывающим аргументом. Определены условия существования решений (также максимальных и минимальных решений) на бесконечном промежутке, и рассмотрены свойства и установлены оценки этих решений.