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A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM GENERATED BY A PARABOLIC PROBLEM

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1. Introduction. Let Ω be a bounded domain in R^m $(m \geqslant 1)$ with the sufficiently smooth boundary $\partial\Omega$. We shall consider an elliptic boundary value problem for the quasilinear system of $p \geqslant 1$ equations with p-unknown functions $u = (u_1, \ldots, u_p)$

$$L(x, D_x)u + \lambda u := \sum_{|k|=2b} A_k(x) D_x^k u + \lambda u = f(x, D_x^\gamma u) \qquad (1)$$

in Ω, where $f = (f_1, \ldots, f_p)$ and $0 \leqslant |\gamma| \leqslant 2b - 1$ ($b \geqslant 1$ is an integer) and λ is a complex parameter.

Our aim is to prove an existence theorem and to investigate a structure of the classic solution of the given problem. Furthermore we state the relation between this solution and the solution of the associated nonlinear parabolic system in infinite cylinder $Q_\infty = \langle 0, \infty) \times \Omega$ from [3]. These questions are studied with help of a priori estimates of the Green's matrix for linear problem. The coefficients A_k are smooth matrix functions and the vector function f is assumed to be continuous in the Hölder sense on a bounded or unbounded set. Hölder continuity on the bounded set allows to solve the equations of the type (1) with the strong nonlinearities for f.

2. Formulation of the problem. First of all we introduce some notions and notations which will be used throughout the whole paper.

By $x = (x_1, \ldots, x_m)$ we denote the point of a bounded domain Ω and H means the Cartesian product

$$\Omega \times \prod_{|\gamma|=0}^{2b-1} \prod_{j=1}^{p} \{-\infty < u_\gamma^j < \infty\} \text{ and } H(B) := \Omega \times \prod_{|\gamma|=0}^{2b-1} \prod_{j=1}^{p} \{-B \leqslant u_\gamma^j \leqslant B\}.$$

Here B is a positive real number and $s = \sum_{r=0}^{2b-1} t(r)$, where $t(r)$ means the number of multiindices $\gamma=(\gamma_1, \ldots, \gamma_m)$ with the length $|\gamma|:=\sum_{i=1}^{m}\gamma_i=r$ (γ_i is a non-negative integer for $i=1, \ldots, m$). Then the number of all elements of the vector $D_x^\gamma u=\{(D_x^\varkappa u_1, \ldots, D_x^\varkappa u_p)\}_{|\varkappa|=0}^{2b-1}$ from (1) or of the vector $u_\gamma=\{(u_\varkappa^1, \ldots, u_\varkappa^p)\}_{|\varkappa|=0}^{2b-1}$ is ps. By J, O, E_1 and E we shall denote the $(p\times 1)$-unit vector, the $(p\times 1)$-zero vector, the $(p\times p)$-matrix with all elements equal to 1 and the $(p\times p)$-unit matrix respectively.

The set of all Hölder continuous functions $u\colon D\subset R^m\to R^p$ with exponent $0<\varrho\leq 1$ will be denoted by $H_{\varrho,p}(x, D)$ and

$$\|u\|_{l,D}:=\max_{j=1,\ldots,p}\left\{\sum_{i=0}^{l}\sum_{|k|=i}\sup_{x\in D}|D_x^k u_j(x)|\right\}$$

for a non-negative integer l.

We shall investigate the system (1) for $x\in\Omega$ with the boundary conditions

$$B_q(x, D_x)u\Big|_{\partial\Omega}:=\sum_{|k|\leq r_q}(B_k^{(q)}(x), D_x^k u)\Big|_{\partial\Omega}=0 \tag{2}$$

for $r_q\leq 2b-1$ and $q=1, \ldots, bp$. Here $A_k(.)=(a_k^{hj}(.))_{h,j=1}^{p}$ is a matrix function for $|k|=2b$ and $B_k^{(q)}(.)=(b_k^{q1}(.), \ldots, b_k^{qp}(.))$ for $|k|\leq r_q$ and $q=1, \ldots, bp$ is a vector function on Ω and $f=(f_1, \ldots, f_p)\colon H\to R^p$. Our results are essentially connected with linear stationary parabolic problems for the equations

$$D_t u+L(x, D_x)u+\lambda u=g(x),\ (t, x)\in Q_\infty \tag{3}$$

and

$$D_t u+L(x, D_x)u=g(x),\ (t, x)\in Q_\infty \tag{4}$$

with the initial data

$$u(0, x)=O,\ x\in\Omega \tag{5}$$

and the boundary conditions

$$B_q(x, D_x)u\Big|_{\Gamma_\infty}=0 \tag{6}$$

for $q=1, \ldots, bp$ and $\Gamma_\infty=\langle 0, \infty)\times\partial\Omega$. The equation (1) and operator $B_k^{(q)}(x, D_x)$ from (2) are assumed to satisfy the following correctness conditions (see [2] and [4]):

(A) The operator $L(x, D_x)$ is such that the associated system (3) is uniformly parabolic in the sense of I. G. Petrovskij.

(B) The operator $B_q(x, D_x)$ and the system (3) are connected by the strong "uniform supplementary" condition.

(C) For the complex number λ the inequality $\operatorname{Re}\lambda > A$ holds.

$A>0$ is the constant from the following estimation of the Green's function H_1 for the problem (4), (5), (6) in the infinite cylinder Q_∞ (see [1] and [3]):

$$|D_t^{k_0} D_x^k H_1(t-\tau, x, \xi)| \leqslant \tag{7}$$
$$\leqslant C_1(t-\tau)^{-(m+2bk_0+|k|)/2b} \exp\{A(t-\tau) - c|x-\xi|^{2ba}/(t-\tau)^a\} E_1 \leqslant$$
$$\leqslant C(t-\tau)^{-\mu} |x-\xi|^{2b\mu-(m+2bk_0+|k|)} e^{A(t-\tau)} E_1,$$

where $a=1/(2b-1)$, $0\leqslant\tau<t<\infty$ and $x, y, \xi \in R^m$, $x\neq\xi$ and $2bk_0+|k|\leqslant 2b+l$ ($l\geqslant 0$ is an integer); $\mu\leqslant(m+2bk_0+|k|)/2b$ and C_1, C, c are positive constants.

($D_{l+\alpha}$) The coefficients A_k and $B_k^{(q)}$ and boundary $\partial\Omega$ satisfy the condition (C_l) from [1] (or the modified condition ($C_{l+\alpha}$) for the domain Ω from [3]), where $\alpha\in(0, 1)$ and $l\geqslant 0$ is an integer.

The problem (1), (2) will be solved in a special space of Hölder functions $C_x^{2b-1+\alpha}(\Omega)$ defined as follows: $u\in C_x^{2b-1+\alpha}(\Omega)$ iff

$$\|u\|_{2b-1+\alpha} := \max_{j=1,\dots,p}\left\{\sum_{i=0}^{2b-1}\sum_{|k|=i}\sup_{x\in\Omega}|D_x^k u_j(x)| + \right. \tag{8}$$
$$\left. + \sum_{|k|=2b-1}\sup_{\substack{x,y\in\Omega\\ x\neq y}} |D_x^k u_j(x) - D_x^k u_j(y)|\,|x-y|^{-\alpha}\right\} < \infty.$$

Remark 1. If $u\in C_x^{2b-1+\alpha}(\Omega)$, then the derivative $D_x^k u$ for $|k|=2b-1$ can be continuously and boundedly extended on the whole R^m. Further, using the mean value theorem and the relation

$$K_m\sum_{i=1}^m |x_i| \leqslant |x| \leqslant \sum_{i=1}^m |x_i| \tag{9}$$

for $x\in R^m$ and $K_m\in(0, (1/\sqrt{2})^{m-1})$ one obtains that $D_x^k u \in H_{1,p}(x, \Omega)$ *if* $|k| = 0, 1, \dots, 2b-2$. In our consideration we shall need the estimations for the Green's matrix of a linear elliptic problem from

Theorem 1. (See [1].) Let the assumptions (A), (B), (C), ($D_{l+\alpha}$) be satisfied. Then there is the Green's matrix function $G(x, \xi; \lambda)$ of (1), (2) (with $f=O$) such that ($a=1/2b$)

$$|D_x^k G(x, \xi; \lambda)| \leqslant E_1 C \exp\{-c_0\delta^a|x-\xi|\} \times \tag{10}$$
$$\times \begin{cases} 1 & \text{for } m+|k|<2b \\ 1+|\ln|x-\xi|| & \text{for } m+|k|=2b \\ |x-\xi|^{-m-|k|+2b} & \text{for } m+|k|>2b \end{cases}$$

for $x, y, \xi\in R^m$, $x\neq\xi$ and $|k|\leqslant 2b+l\cdot C$, c_0 are positive constants independent of x, y, ξ and λ and $\delta=\operatorname{Re}\lambda - A$.

The integral representation of solution of linear problems is given by

Theorem 2. (See [1].) If (A), (B), (C), $(D_{l+\alpha})$ hold and $\varphi \in H_{\alpha,p}(x, \Omega)$, then the function $u\colon \Omega \to R^p$ given by

$$u(x) = \int_\Omega G(x, \xi; \lambda)\varphi(\xi)\, d\xi$$

is a solution of the linear equation $L(x, D_x)u + \lambda u = \varphi(x)$ on Ω and satisfies data (2).

For brevity, in the following text L denotes an arbitrary positive constant.

3. The existence of a solution. To derive the fundamental existence theorem, we must prove some preliminary results.

Lemma 1. The space $(C_x^{2b-1+\alpha}(\Omega), \|.\|_{2b-1+\alpha})$ is complete.

Proof. Let $\{u_n\}_{n=1}^\infty$ be the Cauchy sequence of vector functions $u_n = (u_{n1}, \ldots, u_{np})\colon \Omega \to R^p$ such that $u_n \in C_x^{2b-1+\alpha}(\Omega)$ for $n = 1, 2, \ldots$. Then the sequence of derivatives $\{D_x^k u_{nj}\}_{n=1}^\infty$ uniformly converges on Ω for $j = 1, \ldots, p$ and $|k| = 0, 1, \ldots, 2b-1$. If we denote $v_j(x) := \lim_{n\to\infty} u_{nj}(x)$ so $\lim_{n\to\infty} D_x^k u_{nj}(x) = D_x^k v_j(x)$. Hence and by the inequalities

$$|D_x^k u_{nj}(x) - D_x^k u_{rj}(x) - D_x^k u_{nj}(y) + D_x^k u_{kj}(y)| \leq \varepsilon\, |x-y|^\alpha J,$$

$$|D_x^k u_{rj}(x) - D_x^k u_{rj}(y)| \leq L\, |x-y|^\alpha J,$$

$$|D_x^k u_{rj}(x)| \leq LJ$$

letting $r \to \infty$,we obtain $\|u_n - v\|_{2b-1+\alpha} < L\varepsilon$ for all $n > n_0\varepsilon$, where $n_0(\varepsilon)$ is a positive integer $(\varepsilon > 0)$ and $v = (v_1, \ldots, v_p) \in C_x^{2b-1+\alpha}(\Omega)$. This proves Lemma 1.

Lemma 2. Let the conditions (A), (B), (C), (D_α) $(l = 0)$ be fulfilled. Then for any $\beta \in (0, 1)$ and $x, y \in \Omega$ and $|k| = 0, 1, \ldots, 2b-1$ we have

$$I_{1,k}(x) := \left|\int_\Omega |D_x^k G(x, \xi; \lambda)|\, d\xi\right| \leq LE_1 \tag{11}$$

and

$$I_{2,k}(x, y) := \left|\int_\Omega |D_x^k G(x, \xi; \lambda) - D_x^k G(y, \xi; \lambda)|\, d\xi\right| \leq \tag{12}$$

$$\leq L|x-y|^\beta g(|x-y|)E_1,$$

where $g(z) = z^{1-\beta-\varrho[|k|/(2b-1)]}$ and $0 < \varrho < 1 - \beta (< 1)$ for $z > 0$. (The expression $[x]$ in the exponent denotes the integer for which $[x] \leq x < [x] + 1$.)

Proof. The estimation (11) for $m + |k| < 2b$ follows directly by (10). In the case $m + |k| = 2b$ let us choose $0 < r < m$. Since the function $h(x, \xi) = |x-\xi|^r |\ln |x-\xi||$ is bounded on $\Omega \times \Omega$, then we get from (10)

$$I_{1,k}(x) \leq E_1 \int_\Omega (1+|\ln|x-\xi||)\,\mathrm{d}\xi \leq \{\text{means } \Omega + L\int_\Omega |x-\xi|^{-r}\,\mathrm{d}\xi\} E_1$$

which proves the inequality (11).

For $m+|k|>2b$ (11) is true. Really, in the corresponding estimation from (10) the exponent satisfies the condition $0<m+|k|-2b<m$ for $|k|=0, 1, \ldots, 2b-1$.

For $0 \leq |k| \leq 2b-2$ the inequality (12) will be proved by the mean value theorem. There is $\tilde{x}_i=(y_1, \ldots, y_{i-1}, \xi_i, x_{i+1}, \ldots, x_m) \in R^m$ such that for x, $y=(y_1, \ldots, y_m)$ and ξ from Ω we have

$$|D_x^k G(x, \xi; \lambda) - D_x^k G(y, \xi; \lambda)| \leq \sum_{i=1}^m |x_i - y_i|\,|D_x^{k(i)} G(\tilde{x}_i, \xi; \lambda)|, \tag{13}$$

where the modulo of the multiindex $k(i)$ satisfies the condition $0 \leq |k(i)| = |k|+1 \leq 2b-1$ and ξ_i lies between x_i and y_i and $|x-y|>|\tilde{x}_i - x|$ for all $i=1, \ldots, m$. From (13) and (10) by the same way as in the proof of (11) one obtains

$$I_{2,k}(x, y) \leq L\sum_{i=1}^m |x_i - y_i| E_1 \leq (L/K_m)|x-y| E_1.$$

For $|k|=2b-1$ we proceed as follows:

Denote $\Omega_1=\{\xi \in \Omega: |\xi - x|>2|x-y|\}$ and $\Omega_2=\Omega-\Omega_1$. Then from (13) for $|k(i)|=2b (m+|k(i)|>2b)$ we have

$$I_{2,k}(x, y) \leq L\sum_{i=1}^m |x_i - y_i| \left|\int_{\Omega_1} |\tilde{x}_i - \xi|^{-m}\,\mathrm{d}\xi\right| E_1 + \tag{14}$$

$$+\left|\int_{\Omega_2} |D_x^k G(x, \xi; \lambda)|\,\mathrm{d}\xi\right| + \left|\int_{\Omega_2} |D_x^k G(y, \xi; \lambda)|\,\mathrm{d}\xi\right| := J_1 + J_2 + J_3.$$

If $\xi \in \Omega_1$ then we get $|x-y|<|\tilde{x}_i - \xi|$ and by $|\tilde{x}_i - x|<|x-y|$ we have $|x-\xi|<2|\tilde{x}_i - \xi|$. Hence

$$J_1 \leq 2^m L\sum_{i=1}^m |x_i - y_i| \left|\int_{\Omega_1} |x-\xi|^{-m+r}|x-\xi|^{-r}\,\mathrm{d}\xi\right| E_1 \leq$$

$$\leq 2^{m-r}(L/K_m)|x-y|^{1-r}\left|\int_{\Omega_1} |x-\xi|^{-m+r}\,\mathrm{d}\xi\right| E_1 \leq L_1|x-y|^{1-r}E_1,\ L_1>0.$$

If $m+|k|=2b (|x-\xi| \leq 2|x-y|)$ so

$$J_2 \leq L\left|\int_{\Omega_2} (1+|\ln|x-\xi||)|x-\xi|^{1-r}|x-\xi|^{r-1}\,\mathrm{d}\xi\right| E_1 \leq$$

$$\leq 2^{1-r}L|x-y|^{1-r}\left|\int_{\Omega_2} (1+|\ln|x-\xi||)|x-\xi|^{r-1}\,\mathrm{d}\xi\right| E_1$$

and for $m+|k|>2b$

$$J_2 \leqslant 2^{1-r} L\,|x-y|^{1-r} \left| \int_{\Omega_2} |x-\xi|^{-m+r}\, d\xi \right| E_1 .$$

In both cases we have

$$J_2 \leqslant L_2\,|x-y|^{1-r} E_1,\ L_2>0 .$$

Using $|\xi - y| \leqslant 3\,|x-y|$ for $\xi \in \Omega_2$ similarly as in the case of J_2 we estimate

$$J_3 \leqslant L_3\,|x-y|^{1-r} E_1,\ L_3>0 .$$

Putting the estimation for J_1, J_2, J_3 into (14) we get (12) which finishes the proof.

In the following consideration we shall need the operator

$$A(x)u = \int_{\Omega} G(x, \xi; \lambda) f[\xi, D_x^{\gamma} u(\xi)]\, d\xi . \tag{15}$$

Lemma 3. Let the condition (A), (B), (C), (D_α) be fulfilled and let $f\colon H \to R^p$ be continuous and bounded in the sense $\|f\|_{0,H}<M$, $M>0$. Then there is a real number $B(M)>0$ such that $A(x) C_x^{2b-1+\alpha}(\Omega) \subset S_B$, where the sphere $S_B = \{u \in C_x^{2b-1+\alpha}(\Omega)\colon \|u\|_{2b-1+\alpha} \leqslant B\}$.

Proof. Let $u \in C_x^{2b-1+\alpha}(\Omega)$. Then for $|k|=0, 1, \ldots, 2b-1$

$$|D_x^k A(x) u| \leqslant M I_{1,k}(x) J$$

and

$$|D_x^k A(x) u - D_x^k A(y) u| \leqslant M I_{2,k}(x, y) J .$$

Hence and by (11) and (12) for $\beta = \alpha$ we see that it is sufficient to take $B(M) \geqslant L[s + t(2b-1)]$.

Remark 2. If we assume the boundedness of f only on $H(B_0)$, $B_0>0$ instead of one on H so $A(x) S_{B_0} \subset S_B$. If moreover $B_0 \geqslant B$ then $A(x) S_{B_0} \subset S_{B_0}$.

New we are able to formulate the existence

Theorem 3. Let hypotheses (A), (B), (C), (D_α) be satisfied and let $f\colon H \to R^p$ be continuous and bounded vector function in the norm $\|.\|_{0,H}$ by $M>0$. Further, the Hölder condition

$$|f(x, u_\gamma) - f(y, v_\gamma)| \leqslant \{q_0 |x-y|^{\beta} + (q_\gamma, |u_\gamma - v_\gamma|^{\beta_\gamma})\} J \tag{16}$$

holds for $\beta, \beta_\gamma \in (0, 1\rangle$ and (x, u_γ), $(y, v_\gamma) \in H$, where $q_0>0$ and $q_\gamma = \{(q_\varkappa^1, \ldots, q_\varkappa^p)\}_{|\varkappa|=0}^{2b-1}$ is a vector of R^{ps} with non-negative components. Then the problem (1), (2) has at least one solution $u \in C_x^{2b-1+\alpha}(\Omega)$ such that $\|u\|_{2b-1+\alpha} \leqslant B_0$, where $B_0 \geqslant B(M)$ and $B(M)>0$ is the constant from Lemma 3.

Proof. Consider the non-empty, convex, bounded and closed sphere S_{B_0} in the

Banach space $C_x^{2b-1+\alpha}(\Omega)$ (see Lemma 3). Then for any $v \in S_{B_0}(D_x^k v \in H_{1,p}(x, \Omega)$ for $|k|=0, \dots, 2b-2$ and $D_x^k v \in H_{\alpha,p}(x, \Omega)$ for $|k|=2b-1)$ the function $f_v(x) = f[x, D_x^\gamma v(x)]$ satisfies the inequality

$$|f_v(x)-f_v(y)| = |f[x, D_x^\gamma v(x)] - f[y, D_x^\gamma v(y)]| \leqslant$$
$$\leqslant \{q_0|x-y|^\beta + \sum_{i=0}^{2b-2} \sum_{|\varkappa|=i} (\bar{q}_\varkappa, |x-\xi|^{\beta_\varkappa} J) + \sum_{|\varkappa|=2b-1} (\bar{q}_\varkappa, |x-\xi|^{\alpha\beta_\varkappa} J)\} J,$$

where $x, y \in \Omega$ and $\bar{q}_\varkappa = (\bar{q}_\varkappa^1, \dots, \bar{q}_\varkappa^p) \subset R^p$ and $q_\gamma = \{\bar{q}_\varkappa\}_{|\varkappa|=0}^{2b-1}$. Put $v = \min\limits_{0\leqslant|\gamma|\leqslant 2b-1} (\beta, \alpha\beta_\gamma) \leqslant \alpha$. Then $f_v \in H_{\gamma,p}(x, \Omega)$ and from (D_α) follows (D_v). Theorem 2 guarantees the mutual equivalence between the operator equation $A(x)u = u$ and the problem (1), (2) on S_{B_0}. Therefore the existence may be investigated by Leray—Schauder fixed point theorem. The inclusion $A(x)S_{B_0} \subset S_{B_0}$ is true (Lemma 3). It is sufficient to prove the continuity and compactness of $A(x)$ on S_{B_0}.

Let $u, u_n \in S_{B_0}$ for $n = 1, 2, \dots$ such that $\|u_n - u\|_{2b-1+\alpha} \to 0$ as $n \to \infty$. In virtue of (11), (12) and (16) there is n_0 such that for all $n > n_0$ and $|k| = 0, 1, \dots, 2b-1$

$$|D_x^k A(x)u_n - D_x^k A(x)u| \leqslant$$
$$\left\{\left|\int_\Omega |D_x^k G(x, \xi; \lambda)|(q_\gamma, |D_x^\gamma u_n(\xi) - D_x^\gamma u(\xi)|^{\beta_\gamma})d\xi\right|\right\}J \tag{17}$$
$$\leqslant L\varepsilon I_{1,k}(x)J$$

and for $|k| = 2b-1$

$$|D_x^k A(x)u_n - D_x^k A(x)u - D_x^k A(y)u_n + D_x^k A(y)u| \leqslant$$
$$\leqslant \left\{\left|\int_\Omega |D_x^k G(x, \xi; \lambda) - D_x^k G(y, \xi; \lambda)|(q_\gamma, |D_x^\gamma u_n(\xi) - D_x^\gamma u(\xi)|^{\beta_\gamma})\, d\xi\right|\right\}J \tag{18}$$
$$\leqslant L\varepsilon I_{2,k}(x, y)J$$

for any $\varepsilon > 0$. From (17) and (18) we have $\|A(x)u_n - A(x)u\|_{2b-1+\alpha} \to 0$ as $n \to \infty$ what proves the continuity of $A(x)$.

Now we derive the relative compactness of $A(x)S_{B_0}$.

Let $\{v_n\}_{n=1}^\infty$ be a subsequence of $A(x)S_{B_0}$, where $v_n(x) = (v_{n1}(x), \dots, v_{np}(x))$ for $n = 1, 2, \dots$. There exists $u_n \in S_{B_0}$ such that $v_n(x) = A(x)u_n$. Hence and by the inequality

$$|D_x^k v_n(x) - D_x^k v_n(y)| \leqslant MI_{2,k}(x, y)J \leqslant L|x-y|^\alpha J, \; x, y \in \Omega$$

the uniform boundedness and equicontinuity of sequence $\{D_x^k v_n(x)\}_{n=1}^\infty$ holds on Ω. Then there is a subsequence $\{v_{n_i}(x)\}_{i=1}^\infty = \{A(x)u_{n_i}(x)\}_{i=1}^\infty$ of the sequence

$\{v_n(x)\}_{n=1}^{\infty}$ and a vector function $v_0=(v_{01}, \dots, v_{0p})\colon \Omega\to R^p$ such that $\|D_x^k v_{n_l} - D_x^k v_0\|_{0,\Omega}\to 0$ as $l\to\infty$ for all $|k|=0, 1, \dots, 2b-1$.

Letting $l\to\infty$ in the following estimations

$$|D_x^k v_0(x)| \leq |D_x^k v_0(x) - D_x^k v_{n_l}(x)| + MI_{1,k}(x)J,$$

$$|D_x^k v_0(x) - D_x^k v_0(y)| \leq$$

$$\leq |D_x^k v_0(x) - D_x^k v_{n_l}(x)| + MI_{2,k}(x, y)J + |D_x^k v_{n_l}(y) - D_x^k v_0(y)|$$

we get $v_0\in C_x^{2b-1+\alpha}(\Omega)$. The closure S_0^* of $A(x)S_{B_0}$ in the norm $\|.\|_{2b-1,\Omega}$ is a subset of $C_x^{2b-1+\alpha}(\Omega)$. We must prove the same inclusion in the norm $\|.\|_{2b-1+\alpha}$.

From the estimation (12) for $v=(v_1, \dots, v_p)\in S_0^*$

$$\lim_{x\to y} \langle D_x^k v_j(x)\rangle_\alpha := \lim_{x\to y} |D_x^k v_j(x) - D_x^k v_j(y)|\ |x-y|^{-\alpha}=0$$

if $|k|=2b-1$ and $j=1, \dots, p$. Then we find $\delta>0$ such that for every $x\in\Omega$ and $|k|=2b-1$ for which $0<|x-y|<\delta$ the estimation

$$\langle D_x^k v_{n_l j}(x) - D_x^k v_{0j}(x)\rangle_\alpha < \varepsilon,\ \varepsilon>0 \tag{19}$$

is true for $j=1, \dots, p$. Since $\lim_{l\to\infty} \|v_n - v_0\|_{2b-1,\Omega}=0$ so for all $l>n(\varepsilon)>0$ and $|x-y|\geq\delta$

$$\langle D_x^k v_{n_l j}(x) - D_x^k v_{0j}(x)\rangle_\alpha \leq$$

$$\leq \delta^{-\alpha} \max_{j=1,\dots,p} \Big\{\sup_{x\in\Omega} |D_x^k v_{n_l j}(x) - D_x^k v_{0j}(x)| +$$

$$+\sup_{x\in\Omega} |D_x^k v_{n_l j}(y) - D_x^k v_{0j}(y)|\Big\} < \varepsilon,\ |k|=2b-1. \tag{20}$$

From (19) and (20) we get

$$\|v_{n_l} - v_0\|_{2b-1+\alpha} \leq \max_{j=1,\dots,p} \Big\{\|v_{n_l j} - v_{0j}\|_{2b-1,\Omega} +$$

$$+\sum_{|k|=2b-1} \max\Big[\sup_{\substack{x,y\in\Omega\\ 0<|x-y|<\delta}} \langle D_x^k v_{n_l j}(x) - D_x^k v_{0j}(x)\rangle_\alpha;$$

$$\sup_{\substack{x,y\in\Omega\\ |x-y|\geq\delta}} \langle D_x^k v_{n_l j}(x) - D_x^k v_{0j}(x)\rangle_\alpha\Big]\Big\} < \varepsilon[s+t(2b-1)]$$

for $l>n(\varepsilon)$, hence $\lim_{l\to\infty} \|v_{nl} - v_0\|_{2b-1+\alpha}=0$. This concludes the proof of Theorem 3.

This proof of Theorem 3 and Remark 2 allow to weaken the assumption on boundedness of f.

Corollary. Let (A), (B), (C), (D_α) be satisfied and let $f\colon H(B_0)\to R^p$ be continuous and bounded in the norm $\|.\|_{0, H(B_0)}$, where $B_0 \geq B$ (B is the constant from Lemma 3). If (16) holds on $H(B_0)$, then there exists a solution u of (1), (2) from $C_x^{2b-1+\alpha}(\Omega)$ such that $\|u\|_{2b-1+\alpha} \leq B_0$.

Remark 3. a) For $0 < \varrho \leq \alpha$ the solution of (1), (2) belongs to be space $C_x^{2b-1+\varrho}(\Omega)$ too.

b) The Corollary of Theorem 3 permits to consider the rapidly increasing functions f on $H(B_0)$ which are unbounded on H.

4. The structure of solution. In this section we deal with the relation between the solution of an elliptic and parabolic boundary value problem.

We easily see that if $H_1(t-\tau, x, \xi)$ is the Green's matrix of problem (4), (5), (6), then the Green's matrix of (3), (5), (6) is given by formula

$$H(t-\tau, x, \xi; \lambda) = e^{-\lambda(t-\tau)} H_1(t-\tau, x, \xi).$$

We can formulate

Theorem 4. If the hypotheses (A), (B), (C), $(D_{2b-1+\alpha})$ hold and f satisfies the conditions from Theorem 3 for $\beta_\gamma = 1$ and $|\gamma| = 0, 1, \ldots, 2b-1$ and $spCL_1L_2 \max\limits_{|\gamma|=0,1,\ldots,2b-1} q_\gamma < 1$, then the solution u of (1), (2) fulfils the equation $\lim\limits_{t\to\infty} \|u - v(t, .)\|_{2b-1, \Omega} = 0$, where v is a solution of the nonlinear stationary parabolic problem for equation

$$D_t v + L(x, D_x)v + \lambda v = f(x, D_x^\gamma v) \tag{21}$$

on Q_∞ with data (5), (6). Here L_1, L_2 are positive constants from estimations

$$\int_0^t e^{-\delta\varphi} \varphi^{-\mu}\, d\varphi < L_1 \quad \text{for} \quad t \in \langle 0, \infty),$$

$$\int_\Omega |x-\xi|^{2b\mu-(m+|k|)}\, d\xi < L_2 \quad \text{for} \quad x \in \Omega,$$

where $\mu \in \langle |k|/2b, 1)$ for $0 \leq |k| \leq 2b-1$.

Proof. By Theorem 3 we have a solution $u \in C_x^{2b-1+\alpha}(\Omega)$ of (1), (2). With respect to Theorem 3 from [3] the solution $v \in C_{x,t,f(A,x,\mu,\nu)}^{2b-1+\alpha,(2b-1+\alpha)/2b}(Q_\infty)$ of (21), (5), (6) exists too. The Green's matrix $G(x, \xi; \lambda)$ of the linear elliptic problem (1), (2) with $(f = O)$ can be expressed by the Green's matrix H_1 as follows (see [1])

$$G(x, \xi; \lambda) = \int_0^\infty e^{-\lambda\varphi} H_1(\varphi, x, \xi)\, d\varphi.$$

Then

$$u(x) = \int_\Omega \left[\int_0^\infty e^{-\lambda\varphi} H_1(\varphi, x, \xi)\, d\varphi\right] f[\xi, D_x^\gamma u(\xi)]\, d\xi$$

for $x \in \Omega$ and

$$v(t, x)=\int_{\Omega}\left\{\int_0^t e^{-\lambda\varphi}H_1(\varphi, x, \xi)f[\xi, D_x^\gamma v(\xi, \tau)]\,d\varphi\right\}d\xi$$

for $(t, x)\in Q_\infty$, whereat $\varphi=t-\tau$. Hence the difference

$$|D_x^k u(x)-D_x^k v(t, x)|\leq$$

$$\leq\|f\|_{0,H}\left|\int_{\Omega}\left[\int_t^{+\infty}e^{-\lambda\varphi}|D_x^k H_1(\varphi, x, \xi)|\,d\varphi\right]d\xi\right|J+$$

$$+\left|\int_{\Omega}\left\{\int_0^t e^{-\lambda\varphi}|D_x^k H_1(\varphi, x, \xi)|\left|f[\xi, D_x^\gamma u(\xi)]-\right.\right.\right.$$

$-f[\xi, D_x^\gamma v(\xi, \tau)]\Big|d\varphi\Big\}d\xi\Big|:=\|f\|_{0,H}J_1+J_2$ for $|k|=0, 1, \ldots, 2b-1$. Using the estimations from (7) and the Lipschitz condition on f we get

$$J_1\leq pC_1\left|\int_{\Omega}\left\{\int_t^{\infty}e^{-\delta\varphi}\varphi^{-(m+|k|)/2b}\times\right.\right.$$

$$\left.\left.\times\exp\left[-c|x-\xi|^{2ba}/(t-\tau)^a\,d\varphi\right\}d\xi\right|J\leq pL_1\operatorname{diam}\Omega\,e^{-\delta t}$$

for $t>1$ and

$$J_2\leq pC\max_{|\gamma|=0, 1, \ldots, 2b-1}q_\gamma\|u-v(t, .)\|_{2b-1,\Omega}\times$$

$$\times\int_0^t e^{-\delta\varphi}\varphi^{-\mu}\,d\varphi\left|\int_{\Omega}|x-\xi|^{2b\mu-(m+|k|)}\,d\xi\right|.$$

Because both last integrals are bounded functions in their variables for $|k|/2b\leq\mu$ $\mu<1$ then

$$(1-spCL_1L_2\max_{|\gamma|=0, 1, \ldots, 2b-1}q_\gamma)\|u-v(t, .)\|_{2b-1,\Omega}\leq spC_1\operatorname{diam}\Omega\|f\|_{0,H}e^{-\delta t}$$

for $t>1$. This finishes the proof of Theorem 4.

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SÚHRN

NELINEÁRNA ELIPTICKÁ OKRAJOVÁ ÚLOHA GENEROVANÁ PARABOLICKOU ÚLOHOU

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Práca sa zaoberá existenciou klasického riešenia u nelineárnej eliptickej okrajovej úlohy (1), (2) metódou apriórnych odhadov Greenových funkcií pre lineárne úlohy. Ďalej sa vyšetruje štruktúra riešenia a dokazuje sa rovnosť $\lim_{t\to\infty} \|u - v(t, .)\|_{2b-1+\alpha} = 0$, kde v je riešenie asociovanej parabolickej úlohy v nekonečnom valci.

РЕЗЮМЕ

НЕЛИНЕЙНАЯ ЭЛЛИПТИЧЕСКАЯ КРАЕВАЯ ЗАДАЧА ПОРОЖДЕННАЯ ПАРАБОЛИЧЕСКОЙ ЗАДАЧЕЙ

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В этой статье исследуется существование классического решения u нелинейной эллиптической краевой задачи (1), (2) методом априорных оценок функции Грина для линейных задач. Кроме того исследуется структура решения и доказывается равенсто

$$\lim_{t\to\infty} \|u - v(t, .)\|_{2b-1+\alpha} = 0,$$

где v значит решение отвечающей параболической задачи в бесконечном цилиндре.