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A LIMIT THEOREM FOR WEIGHTED SUMS OF RANDOM VARIABLES IN *F*-LATTICES

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In [1] Rohatgi proved the following theorem. Let f_n be independent real random variables with zero expectations such that $P\{|f_n|\geqslant a\}\leqslant P\{|f_1|\geqslant a\}$ for all $a>0$, $n\geqslant 1$. Let $\{a_{nk}\}$ be a double sequence of real numbers satisfying

$$\lim_n a_{nk}=0\ \forall k$$

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}|<\infty$$

If $\max\limits_k |a_{nk}|=0(n^{-\nu})$ for $\nu>0$, then $E|f_1|^{1+\frac{1}{\nu}}<\infty$ implies $\sum\limits_{k=1}^{\infty} a_{nk}f_k\to 0$ a.s.

This result belongs to the class of theorems which do not extend directly to Banach spaces. A counterexample can be found in [2]. My aim is to give an extension of Rohatgi's theorem to vector lattices. It turns out that an additional condition on weights $\{a_{nk}\}$ is needed. My terminology will follow [2] and [3].

Definition 1. Let (Z, S, P) be a probability space. A sequence $\{f_n\}$ of functions from Z to a vector lattice E converges to a function f almost uniformly if for every $\varepsilon>0$ there exists a set $A\in S$ such that $P\{A\}<\varepsilon$ and $\{f_n\}$ converges relatively uniformly to f uniformly on $Z-A$; i.e. there exists a sequence $\{a_n\}$ of real numbers converging to 0 and an element $r\in E$ such that $|f_n(z)-f(z)|\leqslant a_n r$ for each $z\in Z-A$.

Definition 2. A function $f\colon Z\to E$ is called a random variable if there exists a sequence $\{f_n\}$ of countably valued random variables such that $\{f_n\}$ converges to f almost uniformly.

From now on E means an Archimedean vector lattice, P a complete probability measure.

I have proved in [4] that each random variable with values in a Frechet lattice E is a measurable map from Z to E, i.e. a random element in the sense of [3].

Thanks to this, independent and symmetric random variables are defined in the usual manner, i.e. these definitions are straightforward extensions of the real case.

Theorem 1. Let E be a σ-complete F-lattice with the σ-property, $\{a_{nk}\}$ be a double array of real numbers satisfying

$$\lim_n a_{nk} = 0 \; \forall k$$

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty,$$

$\{f_n\}$ be a sequence of pairwise independent, symmetric random variables in E such that $P\{|f_n| \leqslant a\} \geqslant P\{|f_1| \leqslant a\}$ for all $a \in E$, $a > 0$ and all n and, moreover, $\sum_{n=1}^{\infty} nP\{|f_1| \leqslant na\}^C < \infty$ for some positive $a \in E$. (C stands for the set complement.) If $\lim_n \sum_{k=1}^{\infty} |a_{nk}| = 0$, then $\sum_{k=1}^{\infty} a_{nk} f_k \to 0$ relatively uniformly on a set of probability 1. (for definition of σ-property see [4].)

Proof. For each n let $\{f_n^k\}$ be a sequence of countably valued random variables converging almost uniformly to f_n. By definition 1 there exists a set Z_0 of probability 1 such that $f_n^k(z) \to f_n(z)$ relatively uniformly on Z_0 for all n with at most countably many different regulators of the convergence. Because of this, the inequality

$$|f_n| \leqslant |f_n - f_n^k| + |f_n^k|$$

which holds for each natural n and k, and the assumption that E has the σ-property, we obtain that all the values of f_n belong to a principal ideal of E (i.e. the ideal generated by a single element, say u, $u \in E$, $u > 0$, $a \leqslant u)I_u$. Denote the set of all values of f_n^k by $\{y_n\}_{n=1}^{\infty}$ and put $y_0 = u$. Consider the countable set $A = \left\{\sum_{i=0}^{n} a_i y_i;\ n = 0, 1, \dots\right\}$ of all linear combinations of y_i with the rational coefficients a_i. The set $B = \bigcap_{r \in Q} \bigcup_{a \in A} \{x \in I_u;\ |x - a| \leqslant ru\}$ where Q stands for the set of all rational numbers is a linear subspace of I_u. Denote this subspace by B.

It is well-known that I_u equipped with the o-unit norm is a B-space. So is B as a closed subset of I_u. Moreover, B is separable. Indeed, for each $x \in B$ and each $\varepsilon > 0$ there exists an element $a \in A$ such that $\|x - a\|_u < \varepsilon$; $\|\ \|_u$ means the norm induced by u. This space will be denoted by $(B, \|\ \|_u)$.

I shall prove that f_n are pairwise independent, symmetric random variables from Z_0 to B. Since B is separable, its Borel σ-algebra is generated by open balls. Denote this Borel σ-algebra by W_S and denote by W_T the σ-algebra generated by subsets of B open with respect to the original topology. It is sufficient to show that $W_S \subset W_T$. We have the following equality for an open ball

$$\{x \in B;\ \|x - x_i\|_u < \varepsilon\} = \bigcup_n \{x \in B;\ \|x - x_i\|_u \leqslant \varepsilon(1 - n^{-1})\} =$$

$$= \bigcup_n B \cap \{x \in I_u;\ \|x - x_i\|_u \leqslant \varepsilon(1 - n^{-1})\} = B \cap \bigcup_n \{x \in I_u;\ |x - x_i| \leqslant \varepsilon(1 - n^{-1})u\}.$$

It means that f_n are pairwise independent and symmetric random variables in $(B, \|\ \|_u)$.

By hypothesis we have

$P\{\|f_n\|_u \geqslant b\} \leqslant P\{\|f_1\|_u \geqslant b\}$ for all $b > 0$. Moreover, for $1 < v < 2$

$$E\|f_1\|_u^{1+\frac{1}{v}} \leqslant 1 + 3\sum_{n=1}^{\infty} nP\{\|f_1\|_u > n\} = 1 + 3\sum_{n=1}^{\infty} nP\{|f_1| \leqslant nu\}^C < \infty.$$

Now an application of [5], th. 2 yields that $\sum_{k=1}^{\infty} a_{nk} f_k \to 0$ in norm almost surely. Owing to the definition of the order-norm and because $P\{Z_0\} = 1$, we have that $\sum_{k=1}^{\infty} a_{nk} f_k \to 0$ relatively uniformly on a set of probability 1.

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SÚHRN

LIMITNÁ VETA O VÁŽENÝCH SÚČTOCH NÁHODNÝCH PREMENNĆH V F-ZVÄZOCH

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Autor rozširuje platnosť Rohatgiho limitnej vety o vážených súčtoch náhodných premenných na prípad, keď hodnotový priestor je F-zväz.

РЕЗЮМЕ

ОДНА ПРЕДЕЛЬНАЯ ТЕОРЕМА ДЛЯ СУММ СЛУЧАЙНЫХ ВЕЛИЧИН В РЕШЕТКАХ ФРЕШЕ

Р. Потоцки

Доказываемая в работе предельная теорема о сходимости по упорядочению сумм случайных величин является обобщением одного результата Рохатги.