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STRONG LAWS OF LARGE NUMBERS

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The purpose of this paper is to discuss strong laws of large numbers in Banach lattices. They are only a quarter century old as a subject of study. In fact, there were no laws of large numbers before the year 1953. Over the past 25 years several investigators have devoted their attention to this area and have produced a number of interesting results. In Banach lattices three types of laws of large numbers can be considered depending upon the modes of convergence; accordingly one speaks about laws of large numbers in the norm topology or the weak topology or else with respect to the ordering. While many results are known for the norm topology, the theory is much less developed for the weak topology and almost completely neglected is the case of the convergence with respect to the ordering. The latter is in the primary focus of this paper.

We will study mappings having values in a Banach lattice. To avoid misunderstanding, let us agree on this definition.

By a Banach lattice X we understand a normed lattice, i.e. a vector lattice with a monotonous norm, complete with respect to this norm. A Banach lattice is said to have o-continuous norm if each decreasing sequence with zero infimum converges to zero in norm.

A well known theorem asserts that every separable, countably o-complete Banach lattice has o-continuous norm (see [1], th. 5.14). In the rest of this paper we restrict our attention to separable, countably o-complete Banach lattices.

Definition 1. Let (Ω, S, P) be a probability space. A strongly measurable mapping f from Ω into X is said to be a random variable. The expectation of f (if it exists) is well defined by Bochner integral.

It is well known that f is integrable if and only if $\|f\|$ is integrable and it is if and only if $|f|$ is integrable.

Proposition 1. A mapping f: $\Omega \to X$ is a random variable iff there exists a sequence $\{f_n\}$ of elementary random variables such that $f_n(\omega)$ converges relatively uniformly to $f(\omega)$ for each ω, i.e. there is $v(\omega) \in X$ such that for every n $|f_n(\omega) - f(\omega)| \leq n^{-1} v(\omega)$.

Proof. See [8].

Definition 2. The variance of f is defined by $\sigma^2(f)=E\|f-Ef\|^2$ and the standard deviation of f, $\sigma(f)$ is the square root of the variance.

Definition 3. For $1\leqslant p\leqslant\infty$, $L^p(X)$ denotes the Lebesgue—Bochner function space of X-valued random variables for which $E\|f\|^p$ is finite.

It is a Banach lattice with the usual norm and ordering.

Definition 4. A sequence $\{f_n\}$ of random variables obeys the strong law of large numbers in the norm topology if

$$\left\|n^{-1}\sum_1^n f_i(\omega)\right\|\to 0 \quad \text{a.s.}$$

A sequence $\{f_n\}$ obeys the strong law of large numbers in the weak topology if, except for a single set A of probability 0,

$$T\left(n^{-1}\sum_1^n f_i(\omega)\right)\to 0$$

for each $T\in X'$ (the topological dual of X) and $\omega\notin A$.

A sequence $\{f_n\}$ obeys the strong laws of large numbers with respect to the ordering, if $n^{-1}\sum_1^n f_i(\omega)\to 0$ a.s. in the order.

One usually proves strong laws of large numbers by investigating the series of random variables. The key role is played by two lemmas.

Lemma 1. (Toeplitz). If $x_n\to x$ in the norm topology (the weak topology, in the order) then $n^{-1}\sum_1^n x_i\to x$ in the norm topology (the weak topology, in the order).

Proof. See [2].

Lemma 2 (Kronecker). In each Banach lattice the norm convergence (weak convergence, o-convergence) of the series $\sum_1^\infty n^{-1}x_n$ implies the norm (weak, order) convergence of $n^{-1}\sum_1^n x_i\to 0$.

Proof. See [2].

Theorem 1. ([3], th. III. 13). Let $\{f_n\}$ be a sequence of independent random variables with $Ef_n=0$. If $\sum_1^\infty n^{-2}\sigma^2(f_n)$ converges and $n^{-1}\sum_1^n \sigma(f_i)\to 0$, then $\{f_n\}$ obeys SLLN in the norm topology.

Theorem 2. ([4], th. 3.3). Let $\{f_n\}$ be a sequence of independent random variables with $Ef_n=0$. Let X' be separable. If $\sum_1^\infty n^{-2}\sigma^2(f_n)<\infty$ and $n^{-1}\sum_1^n \sigma(f_i)$ is a bounded sequence, then $\{f_n\}$ obeys SLLN in the weak topology.

Theorem 3. ([2], th. 1). Let $\{f_n\}$ be a sequence of random variables. If φ_n: $R^+ \to R^+$ are continuous non-decreasing, strictly positive on the real half-line and such that $t\varphi_n^{-1}(t)$ are non-decreasing, then the convergence of

$$\sum_1^\infty E\varphi_n \|f_n\|(\varphi_n \|f_n\| + \varphi_n(n))^{-1}$$

implies the validity of SLLN with respect to the ordering.

Corollary. Let $\{f_n\}$ be a sequence of random variables. Then the convergence of $\Sigma n^{-\beta} E \|f_n\|^\beta$, $\beta \leq 1$ implies the validity of SLLN in the order.

In comparison with the previous theorems, the latter one has the advantage of being valid without independence of summands and without the assumption $Ef_n = 0$.

In what follows we will show that SLLN holds in spaces that share some properties with Hilbert lattices and are, in general, a result of a geometric and a probabilistic condition.

Definition 5. X is B-convex if there is an integer $k \geq 2$ and an $\varepsilon > 0$ such that for each choice of $x_1, \ldots, x_k$ from the unit ball of X $\|\pm x_1 \pm \ldots \pm x_k\| \leq k(1-\varepsilon)$ for some choice of $+$ and $-$ signs.

Theorem 4. ([5], th. 1). X is B-convex iff every sequence $\{f_n\}$ of independent random variables, $Ef_n = 0$ and with $\|f_n(\omega)\| \leq 1$ obeys SLLN in the norm topology.

Definition 6. A semi-inner product on X is a real-valued function $[.,.]$ on $X \times X$ with the following properties

(i) $[x+y, z] = [x, z] + [y, z]$
$[\lambda x, y] = \lambda[x, y]$, $\forall$ real λ
(ii) $[x, x] > 0$ when $x \neq 0$
(iii) $|[x, y]|^2 \leq [x, x][y, y]$
(iv) $[x, x] = \|x\|^2$

In smooth spaces there exists a unique function with these properties (for the definition of a smooth space and related results see [6]).

Definition 7. Let f and g be random variables. They are orthogonal in $L^p(X)$, if $[g, f] = 0$ with respect to semi-inner product on $L^p(X)$. A sequence $\{f_n\}$ is mutually orthogonal in $L^p(X)$ if $[f_i, f_j] = 0$ for $i \neq j$.

Among smooth space the so-called G_α-spaces are important from the point of view of SLLN.

Definition 8. For $0 < \alpha \leq 1$, X is a G_α-space if there exists a mapping G: $X \to X'$ such that

(i) $\|G(x)\| = \|x\|^\alpha$
(ii) $G(x)x = \|x\|^{1+\alpha}$
(iii) $\|G(x) - G(y)\| \leq A \|x - y\|^\alpha$ for $x, y \in X$ and some $A > 0$.

As we have already mentioned, each G_α-space is smooth. Also true is that X being a G_α-space implies the same for $L^{1+\alpha}(X)$ (see [6]).

Theorem 5. ([7], th. 2) Let $\{f_n\}$ be a sequence of independent random variables, $Ef_n=0$ in a G_α-space, $0<\alpha\leqslant 1$. If $\sum_1^\infty n^{-(1+\alpha)}E\|f_n\|^{1+\alpha}$ converges, then $\{f_n\}$ obeys SLLN in the norm.

All l^p and separable L^p, $p\geqslant 2$ satisfy Kolmogorov's SLLN (i.e. th. 5). It does not hold in l^p and L^p, $1<p<2$.

In contrast with this theorem, the next result is developed for orthogonal rather than independent sequences of random variables.

Theorem 6. Let $\{f_n\}$ be a sequence of random variables in a G_α-space X, $0<\alpha\leqslant 1$, with $\{|f_n|\}$ mutually orthogonal in $L^{1+\alpha}(X)$. If $\sum_1^\infty n^{-(1+\alpha)}E\|f_n\|^{1+\alpha}$, then $\{f_n\}$ obeys SLLN with respect to the ordering.

Proof. In the proof the following inequality plays the key role:

$E\left\|\sum_1^n f_i(\omega)\right\|^{1+\alpha}\leqslant A\sum_1^n E\|f_i(\omega)\|^{1+\alpha}$, where A is the G_α-constant. This inequality is estiblished in [8], th. 1. Using this, we have that the series $\Sigma n^{-1}\|f_n\|$ converges almost surely. The rest of the proof follows from the order-continuity of the norm and Kronecker's lemma.

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SÚHRN

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