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ON PROJECTIVE LIMITS OF SMALL SYSTEMS

JURAJ RIEČAN, Bratislava

The paper deals with the very well known Marczewski and Ryll Nardzewski generalization (see [6]) of the famous Kolmogorov consistency theorem. Instead of a limit measure generated by a projective system of measures we shall construct a so-called small system. In the classical case a small system generated by a measure P can be defined as a sequence $\{N_n\}_{n=1}^{\infty}$ of sets of measurable sets with $N_n = \left\{E;\, P(E)<\frac{1}{n}\right\}$. So as a special case of our projective limit theorem we get the Kolmogorov theorem (of course, in the mentioned general form). Moreover, our theorem can be applied also to the theory of subadditive measures (Section 4).

1. Compact approximation

Definition 1. Let X be a set, $\mathscr{A}\subset 2^X$ be an algebra. A sequence $\{N_n\}_{n=1}^{\infty}$ of subsystems of the system $\mathscr{A}$ will be called a small content, if the following properties are satisfied:

1. $\emptyset\in N_n$ for all $n\in Z^+$.
2. *To any* $m\in Z^+$ there exists a sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers such that to any $m\in Z^+$ and arbitrary sets $E_i\in N_{k_i}$ it is $\bigcup_{i=1}^{m} E_i\in N_n$ (subadditivity).
3. If $E\in N_n$ and $F\subset E$, $F\in\mathscr{A}$, then $F\in N_n$ (monotonity).

Definition 2. Let X be a set, $\mathscr{A}\subset 2^X$ be an algebra. A small content $\{N_n\}_{n=1}^{\infty}$ of subsystems of the system $\mathscr{A}$ will be called:

a) a σ-subadditive small content, if to any $n\in Z^+$ there exists a sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers such that to arbitrary sets $E_i\in N_{k_i}$ is holds $\bigcup_{i=1}^{\infty} E_i\in N_n$ (σ-subadditivity)

b) a continuous small content, if to arbitrary sets $E_i \in \mathcal{A}$, $E_{i+1} \subset E_i$, $i = 1, 2, \ldots$ such that $\bigcap_{i=1}^{\infty} E_i = \emptyset$ and to any $n \in Z^+$ there exists $m \in Z^+$ such that $E_m \in N_n$ (continuity)

Definition 3. A σ-subadditive and continuous small content is called a small system.

Definition 4. Let X be a set, $\mathcal{A} \subset 2^X$ be an algebra and $\{N_n\}_{n=1}^{\infty}$ be a small content of subsystems of the system $\mathcal{A}$. We say that a system $\mathcal{S} \subset \mathcal{A}$ approximates a system $\mathcal{B} \subset \mathcal{A}$, if to any set $B \in \mathcal{B}$ the following property is satisfied:

$$\forall n \in Z^+ \; \exists S_n \in \mathcal{S} \; \exists A_n \in \mathcal{A}\colon S_n \subset B \subset S_n \cup A_n,\ A_n \in N_n .$$

Lemma 5. Let X be a set, $\mathcal{A} \subset 2^X$ be an algebra and $\{N_n\}_{n=1}^{\infty}$ be a small content of subsystems of the system $\mathcal{A}$. Let a system $\mathcal{S} \subset \mathcal{A}$ approximate a system $\mathcal{B} \subset \mathcal{A}$. Then the following properties are satisfied:

a) $\mathcal{S}^{\cup}$ approximates $\mathcal{B}^{\cup}$

b) $\mathcal{S}^{\cap}$ approximates $\mathcal{B}^{\cap}$

Proof. a) Let n be an arbitrary positive integer. If $B \in \mathcal{B}^{\cup}$, then there exist $B_i \in \mathcal{B}$, $B = \bigcup_{i=1}^{m} B_i (m \in Z^+)$. Since $\mathcal{S}$ approximates $\mathcal{B}$, there exist $S_i \in \mathcal{S}$ and $A_i \in \mathcal{A}$ such that

$$S_i \subset B_i \subset S_i \cup A_i,\ A_i \in N_{k_i} \quad (i = 1, \ldots, m).$$

We obtain

$$\bigcup_{i=1}^{m} S_i \subset \bigcup_{i=1}^{m} B_i \subset \bigcup_{i=1}^{m} S_i \cup \bigcup_{i=1}^{m} A_i,\ \bigcup_{i=1}^{m} A_i \in N_n .$$

b) Let n be an arbitrary positive integer. Similarly as in the part a) we obtain

$$S_i \subset B_i \subset S_i \cup A_i,\ A_i \in N_{k_i}$$

and from this we get

$$\bigcap_{i=1}^{m} S_i \subset \bigcap_{i=1}^{m} B_i \subset \bigcap_{i=1}^{m} S_i \cup \bigcup_{i=1}^{m} A_i,\ \bigcup_{i=1}^{m} A_i \in N_n .$$

Q.E.D.

Lemma 6. Let $\mathcal{A} \subset 2^X$ be an algebra of subsets of a set X. Let $\{N_n\}_{n=1}^{\infty}$ be a small content of subsystems of the system $\mathcal{A}$. Let I be an index set. Let $\mathcal{S}_\alpha \subset \mathcal{A}$ approximate $\mathcal{B}_\alpha \subset \mathcal{A}$ for all $\alpha \in I$. Then $\bigcup_{\alpha \in I} \mathcal{S}_\alpha$ approximates $\bigcup_{\alpha \in I} \mathcal{B}_\alpha$.

Proof. The assertion of Lemma 6 is evident, since any set $B_0 \in \bigcup_{\alpha \in I} \mathcal{B}_\alpha$ can be approximated by the system $\mathcal{S}_{\alpha_0}$ such that $B_0 \in \mathcal{B}_{\alpha_0}$. Q.E.D.

Lemma 7. Let $\mathcal{A} \subset 2^X$ be an algebra of subsets of a set X and $\{N_n\}_{n=1}^{\infty}$ be a small content of subsystems of the system $\mathcal{A}$. Let I be an index set. Let $\mathcal{S}_\alpha \subset \mathcal{A}$ approximate an algebra $\mathcal{B}_\alpha \subset \mathcal{A}$ for all $\alpha \in I$. Then $\left(\bigcup_{\alpha \in I} \mathcal{S}_\alpha\right)^{\cap\cup}$ approximates the algebra $A\left(\bigcup_{\alpha \in I} \mathcal{B}_\alpha\right)$.

Proof. We shall use the following lemma (see [6], Corollary (0.2)): Let $\{\mathcal{A}_\alpha\}_{\alpha \in I}$ be a family of subalgebras of the algebra $\mathcal{A}$. Then $\mathcal{A}\left(\bigcup_{\alpha \in I} \mathcal{A}_\alpha\right) = \bigcup_{I_0 \in J_0} \left(\bigcup_{\alpha \in I_0} \mathcal{A}_\alpha\right)^{\cap\cup}$, where $J_0 = \{I_0 \subset I;\ I_0 \text{ is a finite set}\}$.

By Lemma 5 and Lemma 6 we get: The system $\left(\bigcup_{\alpha \in I} \mathcal{S}_\alpha\right)^{\cap\cup}$ approximates the system $\left(\bigcup_{\alpha \in I} \mathcal{B}_\alpha\right)^{\cap\cup}$ and by the preceding lemma $\left(\bigcup_{\alpha \in I} \mathcal{B}_\alpha\right)^{\cap\cup} \supset \mathcal{A}\left(\bigcup_{\alpha \in I} \mathcal{B}_\alpha\right)$. Hence $\left(\bigcup_{\alpha \in I} \mathcal{S}_\alpha\right)^{\cap\cup}$ approximates the algebra $\mathcal{A}\left(\bigcup_{\alpha \in I} \mathcal{B}_\alpha\right)$. Q.E.D.

Definition 8. Let X be a set, $\mathcal{C} \subset 2^X$. The system $\mathcal{C}$ is called a compact system, if to any sequence $\{C_i\}_{i=1}^{\infty}$ of sets from $\mathcal{C}$ such that $\bigcap_{i=1}^{n} C_i \neq \emptyset$ for all $n \in Z^+$ also $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$.

Definition 9. Let $\mathcal{A} \subset 2^X$ be an algebra, X be a set and $\{N_n\}_{n=1}^{\infty}$ be a small content of subsystems of the system $\mathcal{A}$. We say that a system $\mathcal{B} \subset \mathcal{A}$ is compactly approximable, if there exists a compact system $\mathcal{C} \subset \mathcal{A}$ such that $\mathcal{C}$ approximates $\mathcal{B}$.

Lemma 10. Let $\mathcal{A}$ be an algebra of subsets of X and $\{N_n\}_{n=1}^{\infty}$ be a small content of subsystems of the system $\mathcal{A}$. Let $\{\mathcal{C}_\alpha\}_\alpha$ be an algebraically independent system. Let I be an index set and for every $\alpha \in I$, $\mathcal{B}_\alpha$ be an algebra compactly approximable by a compact system $\mathcal{C}_\alpha$. Then the algebra $\mathcal{A}\left(\bigcup_{\alpha \in I} \mathcal{B}_\alpha\right)$ is compactly approximable by the system $\left(\bigcup_{\alpha \in I} \mathcal{C}_\alpha\right)^{\cap\cup}$.

Proof. By Lemma 7 $\left(\bigcup_{\alpha \in I} \mathcal{C}_\alpha\right)^{\cap\cup}$ approximates $\mathcal{A}\left(\bigcup_{\alpha \in I} \mathcal{B}_\alpha\right)^{\cap\cup}$. By [6] Lemma (1.3), Lemma (1.4) and Theorem (1.6), $\left(\bigcup_{\alpha \in I} \mathcal{C}_\alpha\right)^{\cap\cup}$ is a compact system. Q.E.D.

Lemma 11 *(Alexandrov's theorem).* Let $\mathcal{A}$ be an algebra of subsets of á set X and $\{N_n\}_{n=1}^{\infty}$ be a small content of subsystems of the algebra $\mathcal{A}$. Let $\mathcal{A}$ be a compactly approximable system. Then $\{N_n\}_{n=1}^{\infty}$ is a continuous small content.

Proof (see [2], [4]). Let $\{N_n\}_{n=1}^{\infty}$ be a small content and n be a positive integer. Let $\{k_i\}_{i=1}^{\infty}$ be a sequence of positive integers such that to any $n \notin Z^+$ and to arbitrary sets $E_i \in N_{k_i}$ $(i=1, 2, \ldots)$ it is $\bigcup_{i=1}^{m} E_i \in N_{n_0}$.

Let us take $\{E_i\}_{i=1}^{\infty}$, $E_i \in \mathcal{A}$, $E_{i+1} \subset E_i$, $\bigcap_{i=1}^{\infty} E_i = \emptyset$. Since $\mathcal{A}$ is a compactly approximable system, there exists a compact system $\mathscr{C}$ such that to any k_i and to any $E_i \in \mathcal{A}$ there exists $C_i \in \mathscr{C}$, $C_i \subset E_i \subset C_i \cup (E_i - C_i)$ and $(E_i - C_i) \in N_{k_i}$. Since $\bigcap_{i=1}^{\infty} C_i \subset \bigcap_{i=1}^{\infty} E_i = \emptyset$, we have $\bigcap_{i=1}^{\infty} C_i = \emptyset$. Since $C_i \in \mathscr{C}$, there exists a positive integer m such that $\bigcap_{i=1}^{m} C_i = \emptyset$.

We obtain $E_m = \bigcap_{i=1}^{m} E_i = \bigcap_{i=1}^{m} E_i - \bigcap_{i=1}^{m} C_i \subset \bigcup_{i=1}^{m} (E_i - C_i)$. Hence $E_m \in N_{n_0}$ because $E_i - C_i \in N_{k_i}$ and $\{N_n\}_{n=1}^{\infty}$ is a subadditive small content. Q.E.D.

Theorem 12. Let $\{\mathcal{A}_\alpha\}_{\alpha \in I}$ be a family of algebraically σ-independent algebras. Let $\{N_n\}_{n=1}^{\infty}$ be a small content of subsystems of the algebra $\mathfrak{A}\left(\bigcup_{\alpha \in I} \mathcal{A}_\alpha\right)$. Let $\mathcal{A}_\alpha$ be a compactly approximable algebra for all $\alpha \in I$. Then $\mathfrak{A}\left(\bigcup_{\alpha \in I} \mathcal{A}_\alpha\right)$ is a compactly approximable system and $\{N_n\}_{n=1}^{\infty}$ is a continuous small content.

Proof. By Lemma 10 the algebra $\mathfrak{A}\left(\bigcup_{\alpha \in I} \mathcal{A}_\alpha\right)$ is compactly approximable. By Alexandrov's theorem (Lemma 11) $\{N_n\}_{n=1}^{\infty}$ is a continuous small content.

Remark. Let $\{N_n\}_{n=1}^{\infty}$ be a σ-subadditive small content. Let $\{N_n\}_{n=1}^{\infty}$ satisfy the assumptions of Lemma 11. Then $\{N_n\}_{n=1}^{\infty}$ is a small system (see Definition 2 and Definition 3).

2. Projective limits

Definition 13. Let M be a set and $<$ be a relation on the set M. $(M, <)$ is called a directed set, if the following properties are satisfied:

1. $<$ is reflexive and transitive
2. To any a, $b \in M$ there exists $c \in M$ such that $a < c$ and $b < c$.

Definition 14. a) Let $(M, <)$ be a directed set. A system $S = \{X_\alpha\, ;\, \alpha \in M\}$ will be called a projective system in the category of compact topological spaces, if the following properties are satisfied:

1. X_α is a compact topological space for all $\alpha \in M$.
2. To any $\alpha < \beta \in M$ there exists a continuous function $\pi_{\beta\alpha}\colon X_\beta \to X_\alpha$ such that $\pi_{\alpha\alpha}$ is an identical function for every $\alpha \in M$.

b) Let I be an index set and M be a set of all finite subsets of I. ($(M, \subset)$ is a directed set.) Let X_i be a locally compact, σ-compact topological space for all $i \in I$. A system $\mathscr{S} = \{X_\alpha\, ;\, \alpha \in M\}$ will be called a projective system in the category of locally compact σ-compact topological spaces, if the following properties are satisfied:

1. $X_\alpha = \prod_{i \in \alpha} X_i$

2. To any $\alpha \subset \beta \in M$ such that $\alpha = \{x_{i_1}, \ldots, x_{i_n}\}$ $\beta = \{x_{j_1}, \ldots, x_{j_m}\}$, $\pi_{\beta\alpha}: X_\beta \to X_\alpha$ is the projection, i.e. $\pi_{\beta\alpha}(x_{j_1}, \ldots, x_{j_m}) = (x_{i_1}, \ldots, x_{i_n})$.

c) Let the assumptions a) be satisfied. Then the projective limit of the system $\mathcal{S}$ in the category of compact topological spaces is X_∞ such that

1. X_∞ is a compact topological space.

2. There exist continuous functions $\pi_\alpha: X_\infty \to X_\alpha$ for all $\alpha \in M$ such that $\pi_\alpha = \pi_{\beta\alpha} \circ \pi_\beta$ for all $\alpha < \beta \in M$.

d) Let the assumptions b) be satisfied. X_∞ will be called a projective limit in the category of locally compact σ-compact topological spaces, if the following properties are satisfied:

1. $X_\infty = \prod_{i \in I} X_i$

2. There exist continuous functions $\pi_\alpha: X_\infty \to X_\alpha$ for all $\alpha \in M$ such that to any $x \in X_\infty$, $\pi_\alpha(x) = (x_{i_1}, \ldots, x_{i_n})$ for all $\alpha = \{i_1, \ldots, i_n\}$.

3. Generalized Kolmogorov theorem for small systems

Definition 15. $\mathcal{B}(\alpha)$ is the smallest σ-ring containing the system of all compact sets in the space X_α. $\mathcal{B}(\alpha)$ is called the system of all Borel subsets of X_α.

Remark. Since X_α is a compact space or a σ-compact space, $\mathcal{B}(\alpha)$ is a α-algebra.

Definition 16. A set $A \subset X_\infty$ is called a measurable cylinder, if there exist $\alpha \in M$ and $E \in \mathcal{B}(\alpha)$ such that $A = \pi_\alpha^{-1}(E)$. By $\mathcal{V}$ we shall denote the set of measurable cylinders.

Definition 17. Let $\{N_n(\alpha)\}_{n=1}^\infty$ be a small system of subsystems of $\mathcal{B}(\alpha)$ for all $\alpha \in M$. $\{\{N_n(\alpha)\}_{n=1}^\infty; \alpha \in M\}$ is called a consistent system of small systems, if to any $\alpha_1 < \alpha_2 \in M$ and to an arbitrary set $E \in X_{\alpha_1}$ the following property is satisfied:

$$\pi_{\alpha_2\alpha_1}^{-1}(E) \in N_n(\alpha_2) \Leftrightarrow E \in N_n(\alpha_1)$$

Definition 18. Let X_∞ be a projective limit of a projective system $\mathcal{S} = \{X_\alpha; \alpha \in M\}$. Let $\mathcal{M} = \{\{N_n(\alpha)\}_{n=1}^\infty; \alpha \in M\}$ be a consistent system of small systems such that $N_n(\alpha) \subset \mathcal{B}(\alpha)$ for all $\alpha \in M$ and all positive integers n.

We define a small content $\{N_n\}_{n=1}^\infty$ induced by the system $\mathcal{M}$ by the following formula:

$$N_n = \{E \in \mathcal{V}; \exists \alpha \in M, \exists A \in N_n(\alpha): E = \pi_\alpha^{-1}(A)\} \quad n = 1, 2, \ldots$$

Lemma 19. Let $\mathscr{S}=\{X_\alpha;\ \alpha\in M\}$ be a projective system. Then the inverse image $\pi_\alpha^{-1}\mathscr{K}=\{\pi^{-1}(K),\ K\in\mathscr{K}\}$ is a compact system. ($\mathscr{K}$ is an arbitrary compact system.)

Proof. Let $\mathscr{K}$ be a compact system. Let $\{\pi_\alpha^{-1}(K_i)\}_{i=1}^\infty$ be a sequence such that $K_i\in\mathscr{K}$ for every i. Let $\bigcap_{i=1}^n \pi_\alpha^{-1}(K_i)\neq\emptyset$ for eny positive integer n. Then evidently $\pi_\alpha^{-1}\left(\bigcap_{i=1}^n K_i\right)\neq\emptyset$ *hence* $\bigcap_{i=1}^n K_i\neq\emptyset$ for any n. Therefore $\bigcap_{i=1}^\infty K_i\neq\emptyset$ and hence $\bigcap_{i=1}^\infty \pi^{-1}(K_i)=\pi^{-1}\left(\bigcap_{i=1}^\infty K_i\right)\neq\emptyset$. Q.E.D.

Lemma 20 ([3], theorem 3). $\mathscr{B}(\alpha)$ is a compactly approximable system.

Lemma 21. $\pi_\alpha^{-1}\mathscr{B}(\alpha)$ is a compactly approximable system and $\pi_\alpha^{-1}\mathscr{B}(\alpha)$ is an algebra.

Proof. The first assertion follows from Lemma 19, Lemma 20 and Definition 18. The second assertion is evident.

Theorem 22 (the generalized Kolmogorov theorem for small systems). Let $(M,<)$ be a directed set. Let $\mathscr{S}=\{X_\alpha;\ \alpha\in M\}$ be a projective limit of this system, $\mathscr{M}=\{\{N_n(\alpha)\}_{n=1}^\infty;\ \alpha\in M\}$ be a consistent system of small systems (see Definition 17) and $\{N_n\}_{n=1}^\infty$ be the small content induced by $\mathscr{M}$. Then $\{N_n\}_{n=1}^\infty$ is a continuous small content.

Proof. $\{N_n\}_{n=1}^\infty$ is a small content of subsystems of the system of all measurable cylinders, hence it is a small content of subsystems of the algebra $\mathscr{A}(\mathscr{V})$. By Lemma 20 and Lemma 21 $\{\pi_\alpha^{-1}\mathscr{B}(\alpha);\ \alpha\in M\}$ is a system of compactly approximable algebras. Evidently $\{\pi_\alpha^{-1}\mathscr{B}(\alpha);\ \alpha\in M\}$ is a system of alg. σ-independent algebras. By Lemma 10 $\mathfrak{A}(\mathscr{V})$ is compactly approximable and hence by the Alexandrov theorem (Lemma 11) $\{N_n\}_{n=1}^\infty$ is a continuous small content. Q.E.D.

Remark. Let $\{N_n\}_{n=1}^\infty$ be a σ-subadditive small content and the assumptions of Theorem 22 be satisfied. Then $\{N_n\}_{n=1}^\infty$ is a small system.

4. Consequences for submeasures

Definition 23. A non-negative σ-subadditive set function semicontinuous from above in the empty set is called a submeasure.

Definition 24. Let X be a set, $\mathscr{A}\subset 2^X$ be an algebra and let P be a subadditive non-negative set function defined on A such that $P(\emptyset)=0$. A system $\mathscr{B}\subset\mathscr{A}$ is compactly approximable, if there exists a compact system $\mathscr{C}$ such that

$$\forall\varepsilon>0\ \forall B\in\mathscr{B}\ \exists C\in\mathscr{C}:\ C\subset B,\ P(B-C)<\varepsilon.$$

Lemma 25. Let $\mathscr{A}\subset 2^X$ be an algebra, X be a set and P be a subadditive non-negative set function defined on $\mathscr{A}$ and such that $P(\emptyset)=0$. Let $N_n=$

$\left\{E \in \mathcal{A};\ P(E) < \frac{1}{n}\right\}$, $n = 1, 2, \dots$ Then $\{N_n\}_{n=1}^{\infty}$ is a small content.

Proof. 1. $P(\emptyset) < \frac{1}{n}$ for any positive integer n, hence $\emptyset \in N_n$ for any n.

2. Let $n \in Z^+$, $k_i = \frac{1}{2^i n}$, $i = 1, 2, \dots$ Hence to arbitrary sets $E_i \in N_{k_i}$ and to any positive integer m

$$P\left(\bigcup_{i=1}^{m} E_i\right) \leq \sum_{i=1}^{m} \frac{1}{2^i n} < \sum_{i=1}^{\infty} \frac{1}{2^i n} = \frac{1}{n},$$

hence $\bigcup_{i=1}^{m} E_i \in N_n$.

3. Let $F \subset E \in N_n$. Then $P(F) \leq P(E) < \frac{1}{n}$, hence $F \in N_n$.

Q.E.D.

Lemma 26. Let $\mathcal{A} \subset 2^X$ be an algebra and P be a submeasure on the algebra $\mathcal{A}$. Let $N_n = \left\{E \in \mathcal{A};\ P(E) < \frac{1}{n}\right\}$ $n = 1, 2, \dots$ Then $\{N_n\}_{n=1}^{\infty}$ is a small system.

Proof. Let n be a positive integer. Put $k_i = 2^i n$. If $E_i \in N_{k_i}$ then similarly as in the preceding lemma it can be shown that $\bigcup_{i=1}^{\infty} E_i \in N_n$. Let $\{E_i\}_{i=1}^{\infty}$ is now such a sequence that $E_i \supset E_{i+1}$, $E_i \in \mathcal{A}$ $(i = 1, 2, \dots)$ and $\bigcap_{i=1}^{\infty} E_i = \emptyset$, then $\lim_{i \to \infty} P(E_i) = P(\emptyset) = 0$. Hence to any positive integer n there exists a positive integer i_0 such that to any $i > i_0$ it is $P(E_i) < \frac{1}{n}$. Therefore to any $n \in Z^+$ there exists $m \in Z^+$ such that $E_m \in N_n$.

Q.E.D.

Lemma 27. Let $\mathcal{A} \subset 2^X$ be an algebra and P be a subadditive, non-negative function such that $P(\emptyset) = 0$. Let $\{N_n\}_{n=1}^{\infty}$ be the small content of subsystems of the system $\mathcal{A}$ defined by

$$N_n = \left\{E \in \mathcal{A};\ P(E) < \frac{1}{n}\right\}$$

for any positive integer n. Then $\mathcal{A}$ is compactly approximable by Definition 9 if and only if it is compactly approximable by Definition 24.

Proof. $\Rightarrow$

Let $B \in \mathcal{A}$ and $\varepsilon > 0$. Let $\mathscr{C} \subset \mathcal{A}$ be a compact system approximating $\mathcal{A}$ by Definition 9. Let n be a positive integer such that $\frac{1}{n} \leq \varepsilon$. Hence there exists $C \in \mathscr{C}$ such that

$$C \subset B \subset C \cup (B - C) \text{ and } P(B - C) < \frac{1}{n} \leqslant \varepsilon.$$

$\Leftarrow$

Let $B \subset \mathscr{A}$ and n be a positive integer. Let $\mathscr{C} \subset \mathscr{A}$ be a compact system approximating $\mathscr{A}$ by Definition 24. Let ε be a positive real number such that $\varepsilon \leqslant \frac{1}{n}$. Then there exists $C \in \mathscr{C}$ such that $C \subset B \subset C \cup (B - C)$ and $P(B - C) < \varepsilon \leqslant \frac{1}{n}$, hence $(B - C) \in N_n$. Q.E.D.

Lemma 28. Subadditive non-negative set function, semicontinuous fom above in the empty set is a submeasure.

Proof is evident.

Lemma 29. Let $\mathscr{A} \subset 2^X$ be an algebra, P be a subadditive, non-negative set function defined on $\mathscr{A}$ such that $P(\emptyset) = 0$ and $\{N_n\}_{n=1}^{\infty}$ be a continuous small content such that $N_n = \left\{E \in \mathscr{A};\ P(E) < \frac{1}{n}\right\}$ for any positive integer n. Then P is a set function, semicontinuous from above in the empty set.

Proof. Let $\{E_i\}_{i=1}^{\infty}$ be a sequence such that $E_i \in \mathscr{A}$, $E_{i+1} \subset E_i$ $(i = 1, 2, \ldots)$ and $\bigcap_{i=1}^{\infty} E_i = \emptyset$. Let ε be a positive number and n be a positive integer such that $\varepsilon \geqslant \frac{1}{n}$. Then there exists a positive integer m_0 such that $P(E_{m_0}) < \frac{1}{n} \leqslant \varepsilon$, hence $\lim_{m \to \infty} P(E_m) = 0$. Q.E.D.

Theorem 30. Let I be an index set and $\{\mathscr{A}_\alpha\}_{\alpha \in I}$ be a family of σ-independent algebras. Let P be a subadditive, non-negative set function such that $P(\emptyset) = 0$ defined on the algebra $\mathscr{A}\left(\bigcup_{\alpha \in I} \mathscr{A}_\alpha\right)$ generated by $\bigcup_{\alpha \in I} \mathscr{A}_\alpha$. Let $\mathscr{A}_\alpha$ be compactly approximable for any $\alpha \in I$. Then $\mathscr{A}\left(\bigcup_{\alpha \in I} \mathscr{A}_\alpha\right)$ is a compactly approximable system and P is a submeasure.

Proof. Let $\{N_n\}_{n=1}^{\infty}$ be a sequence of subsystems of the algebra $\mathscr{A}\left(\bigcup_{\alpha \in I} \mathscr{A}_\alpha\right)$ such that $N_n = \left\{E \in \mathscr{A}\left(\bigcup_{\alpha \in I} \mathscr{A}_\alpha\right);\ P(E) < \frac{1}{n}\right\}$. By Lemma 25 $\{N_n\}_{n=1}^{\infty}$ is a small content. By Lemma 27 $\mathscr{A}_\alpha$ are compactly approximable systems (by Definition 9). By Lemma 12 $\mathscr{A}\left(\bigcup_{\alpha \in I} \mathscr{A}_\alpha\right)$ is a compactly approximable system (by Definition 9). By Lemma 27 $\mathscr{A}\left(\bigcup_{\alpha \in I} \mathscr{A}_\alpha\right\}$ is a compactly approximable system (by Definition 24) and by Lemma 29 P is semicontinuous from above in the empty set. Hence by Lemma 28 P is a submeasure. Q.E.D.

Definition 31. Let $(M, <)$ be a directed set and $\mathcal{S}=\{X_\alpha\,;\,\alpha\in M\}$ be a projective system (see Definition 14). Let P_α be a submeasure defined on $\mathcal{B}(\alpha)$ for any $\alpha\in M$. Then $\{P_\alpha\,;\,\alpha\in M\}$ is a consistent system, if

$\forall\alpha,\beta\in M,\ \alpha<\beta,\ \forall E\in\mathcal{B}(\alpha)\colon P_\beta(\pi_{\beta\alpha}^{-1}(E))=P_\alpha(E)$

Assumptions:

(i) Let $(M, <)$ be a directed set and $\mathcal{S}=\{X_\alpha\,;\,\alpha\in M\}$ be a projective system with the projective limit X_∞.

(ii) Let P_α be a submeasure defined on $\mathcal{B}(\alpha)$ and $\{P_\alpha\,;\,\alpha\in M\}$ be a consistent system of submeasures.

(iii) Let $\{N_n(\alpha)\}_{n=1}^\infty$ be a small system of subsystems of the system $\mathcal{B}(\alpha)$ for any $\alpha\in M$ and $N_n(\alpha)=\left\{E\in\mathcal{B}(\alpha)\,;\,P_\alpha(E)<\frac{1}{n}\right\}$ for any $\alpha\in M$ and for any positive integer n.

Lemma 32. Let the assumptions (i), (ii) and (iii) be satisfied. Then $\{\{N_n(\alpha)\}_{n=1}^\infty\,;\,\alpha\in M\}$ is a consistent system of small systems.

Proof. Let $\alpha,\beta\in M$, $\alpha<\beta$. Evidently $E\in N_n(\alpha)$ if and only if $\frac{1}{n}>P_\alpha(E)=P_\beta(\pi_{\beta\alpha}^{-1}(E))$ and it is if and only if $\pi_{\beta\alpha}^{-1}(E)\in N_n(\beta)$.

Lemma 33. Let the assumptions (i) and (ii) be satisfied. Then there exists exactly one set function P on $\mathcal{V}$ ($\mathcal{V}$ is the system of all measurable cylinders — see definition 16) with the following property: $\forall\beta\in M\ \forall B\in\mathcal{B}(\beta)\colon P(\pi_\beta^{-1}(B))=P_\beta(B)$.

Proof. 1. *Existence.* Let $A=\pi_\beta^{-1}(B)=\pi_\gamma^{-1}(C)$ and $\alpha\in M$ be such that $\beta<\alpha$, $\gamma<\alpha$. We shall prove that $P_\beta(\pi_\beta(A))=P_\gamma(\pi_\gamma(A))$. Since $\{P_\alpha\,;\,\alpha\in M\}$ is a consistent system of submeasures, we obtain

$$P_\beta(\pi_\beta(A))=P_\alpha(\pi_{\alpha\beta}^{-1}(\pi_\beta(A)))=P_\alpha(\pi_\alpha(\pi_\beta^{-1}(B)))=$$
$$=P_\alpha(\pi_\alpha(\pi_\gamma^{-1}(C)))=P_\alpha(\pi_{\alpha\gamma}^{-1}(C))=P_\alpha(\pi_{\alpha\gamma}^{-1}(\pi_\gamma(A))=P_\gamma(\pi_\gamma(A)).$$

2. *Uniqueness* can be obtained by the definition of P. Q.E.D.

Definition 34. Let the assumptions (i) and (ii) be satisfied. Then we define a set function P on the system $\mathcal{V}$ such that

$$\forall\beta\in M\ \forall A=\pi_\beta^{-1}(B)\colon P(\pi_\beta^{-1}(B))=P_\beta(B)$$

Lemma 35. The function P from Definition 34 is subadditive, non-negative and such that $P(\emptyset)=0$.

Proof. 1. Subadditivity. *Let* $A, B\subset\mathcal{V}$. Then there exist $\alpha,\beta\in M$, $C\in\mathcal{B}(\alpha)$, $D\in\mathcal{B}(\beta)$ such that $A=\pi_\alpha^{-1}(C)$, $B=\pi_\beta^{-1}(D)$. Let $\gamma\in M$ be such that $\alpha<\gamma$, $\beta<\gamma$.

Then

$$P_\gamma(\pi_{\gamma\alpha}^{-1}(C)) = P_\alpha(C),\ P_\gamma(\pi_{\gamma\beta}^{-1}(D)) = P_\beta(D)).$$

Hence $P(A)+P(B) = P_\alpha(C)+P_\beta(D) = P_\gamma(\pi_{\gamma\alpha}^{-1}(C))+P_\gamma(\pi_{\gamma\beta}^{-1}(D)) \geq P_\gamma((\pi_{\gamma\alpha}^{-1}(C)) \cup (\pi_{\gamma\beta}^{-1}(D))) = P_\gamma(\pi_\gamma(\pi_\alpha^{-1}(C)) \cup \pi_\gamma(\pi_\beta^{-1}(D))) = P_\gamma(\pi_\gamma(A \cup B)) = = P(A \cup B)$.

2. The non-negativity of P is evident by Definition 34.

3. Evidently $P(\emptyset) = P_\alpha(\pi_\alpha^{-1}(\emptyset)) = P_\alpha(\emptyset) = 0$. Q.E.D.

Lemma 36. Let the assumption (i), (ii) and (iii) be satisfied. Let $\{N_n\}_{n=1}^\infty$ be the small content induced by the system $\{\{N_n(\alpha)\}_{n=1}^\infty;\ \alpha \in M\}$ and P be the function defined by Definition 34. Then $N_n = \left\{E \in \mathscr{V};\ P(E) < \frac{1}{n}\right\}$ for any positive integer n.

Proof. $\subset$

Let $A \in N_n$, n being a positive integer. Then there exists $\alpha \in M$, $B \in \mathscr{B}(\alpha)$ such that $P_\alpha(B) = P(A) < \frac{1}{n}$, hence $B \in N_n(\alpha)$. Since $P_\alpha(B) < \frac{1}{n}$, we have $P(A) < \frac{1}{n}$.

$\supset$

Let $P(A) < \frac{1}{n}$, n being a positive integer. Then there exist $\alpha \in M$, $B \in \mathscr{B}(\alpha)$ such that $P_\alpha(B) < \frac{1}{n}$. Since $B \in N_n(\alpha)$, we obtain $A \in N_n$.

Theorem 37. Let the assumptions (i) and (ii) be satisfied. Then the set function P defined in Definition 34 is a submeasure.

Proof. Put $N_n(\alpha) = \left\{E \in \mathscr{B}(\alpha);\ P_\alpha(E) < \frac{1}{n}\right\}$ for any $\alpha \in M$ and any positive integer n. By Lemma 26 and Lemma 32 $\mathscr{M}$ is a consistent system of small systems. By Theorem 22 there exists a small content of subsystems of $\mathscr{V}\{N_n\}_{n=1}^\infty$ induced by $\mathscr{M}$. (See Definition 18) and $\{N_n\}_{n=1}^\infty$ is a continuous small content.

By Lemma 33 there exists a set function P such that $\forall \beta \in M$ $\forall A = \pi_\beta^{-1}(B) \in \mathscr{V}$: $P(\pi_\beta^{-1}(B)) = P_\beta(B)$. By Lemma 36 we obtain $N_n = \left\{E \in \mathscr{V};\ P(E) < \frac{1}{n}\right\}$, $n = 1, 2, \ldots$ By Lemma 35 P is subadditive, non-negative and such that $P(\emptyset) = 0$. By Lemma 29 P is semicontinuous from above in the empty set. By Lemma 29 P is semicontinuous from above in the empty set. By Lemma 29 P is a submeasure. Q.E.D.

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Juraj Riečan
Katedra teórie pravdepodobnosti
a matematickej štatistiky
Matematický pavilón
Mlynská dolina
842 15 Bratislava

SÚHRN

O PROJEKTÍVNYCH LIMITÁCH MALÝCH SYSTÉMOV

J. Riečan, Bratislava

Práca sa zaoberá Marczewského a Rylla—Nardzewského zovšeobecnením Kolmogorovovej vety o rozšírení miery. Namiesto limitnej miery generovanej projektívnym systémom mier zostrojuje sa tzv. malý systém. V klasickom prípade každá miera P generuje malý systém, t. j. postupnosť $\{N_n\}_{n=1}^{\infty}$ systémov merateľných množín, kde $N_n = \left\{E\,;\, P(E) < \frac{1}{n}\right\}$. Preto špeciálnym prípadom vety o projektívnej limite malých systémov je aj Kolmogorovova veta. Navyše, vetu dokazovanú v článku možno použiť aj v teórii subaditívnych mier.

РЕЗЮМЕ

ОБ ОБРАТНЫХ ПРЕДЕЛАХ МАЛЫХ СИСТЕМ

Ю. Риечан, Братислава

Работа занилмается обобщением Марчевского и Рыля—Нарджевского теоремы Колмогорова о продолжении меры. Вместо пределной меры порожденной обратной системой мер построена так называемая малая система. В классическом случае всякая мера P порождает малую систему, т. е. последовательность $\{N_n\}_{n=1}^{\infty}$ систем измеримых множеств, где

$$N_n = \left\{E\,;\, P(E) < \frac{1}{n}\right\}.$$

Ввиду того, частным случаем теоремы об обратном пределе малых систем является и теорема Колмогорова. Более того, теоремы доказываемую в статье возможно применить и в теории полуаддитивных мер.