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ON REGULAR VECTOR MEASURES

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Introduction

The purpose of the paper is to generalize certain results of the measure theory, above all [4]. The measure we are working with has values in a vector lattice. Since we have a vector measure, we cannot apply an ε-technique. Therefore we use an assertion from [12], which is presented in this paper as Lemma 1.

Let G be a vector lattice. If $x, y \in G$ we shall denote by $x \vee y (x \wedge y)$ the supremum (infimum) of elements x, y. We say that G is boundedly σ-complete if each sequence (x_n) in G, which is bounded from above (below), has the supremum (infimum) in G. We shall denote the supremum (infimum) of a sequence (x_n) by $\bigvee_n x_n$, $\sup x_n \left(\bigwedge_n x_n, \inf x_n \right)$. If (x_n) is a non-decreasing sequence converging to x, we shall write $\lim x_n = x$ or $x_n \nearrow x$. If (x_n) is a non-increasing convergent sequence, we shall write also $\lim x_n = x$ or $x_n \searrow x$. We define the sum $\sum_{n=1}^{\infty} x_n$ of a sequence (x_n) in the following way:

$$\sum_{n=1}^{\infty} x_n = \bigvee_n \sum_{i=1}^{n} x_i .$$

We shall denote by ∞ the supremum of any unbounded set of elements in G^+ ($G^+ = \{x \in G;\, x \geqq 0\}$) and $G \cup \{\infty\}$ by $\bar{G}$.

Let X be an arbitrary non-empty set, $\mathcal{S}$ be a σ-ring of subsets of X. A map $m\colon \mathcal{S} \to \bar{G}$ is called a vector measure if

1. $m(\emptyset) = 0$;
2. $m(B) \geqq 0$ for each $B \in \mathcal{S}$;
3. m is σ-additive.

A regular measure

Let L be any infinite set, L^N the set of all functions $f\colon N \to L$. Let G be a boundedly σ-complete vector lattice and $\bar{G}$ be $G \cup \{\infty\}$.

Definition 1. We say that a family $\{a_{i,\lambda};\ i \in N,\ \lambda \in L\}$ has the property (P) if

(i) $a_{i,\lambda} \geqq 0$ for each $i \in N$, $\lambda \in L$;

(ii) for each $f \in L^N$ there is $\bigvee_i a_{i,f(i)}$ (we denote this supremum by a_f);

(iii) for each $i \in N$ holds $\bigwedge_{\lambda \in L} a_{i,\lambda} = 0$.

Let $m\colon \mathcal{S} \to \bar{G}$ be a measure, where $\mathcal{S}$ is a σ-ring of subsets of an arbitrary non-empty set X. Let $\mathcal{X}$, $\mathcal{T}$ be non-empty subsystems of $\mathcal{S}$.

Definition 2. We say that $B \in \mathcal{S}$ with $m(B) < \infty$ is inner regular (with respect to m) if there is a family $\{a_{i,\lambda};\ i \in N,\ \lambda \in L\}$ with the property (P) such that for each $f \in L^N$ there is $C_f \in \mathcal{X}$ such that $C_f \subset B$ and $m(B - C_f) \leqq a_f$; it is outer regular (with respect to m) if there is a family $\{b_{i,\lambda};\ i \in N,\ \lambda \in L\}$ with the property (P) such that for each $f \in L^N$ there is $U_f \in \mathcal{T}$ such that $B \subset U_f$ and $m(U_f - B) \leqq b_f$.

Definition 3. We say that $B \in \mathcal{S}$ with $m(B) = \infty$ is inner regular if there is a non-decreasing sequence (B_n) of inner regular sets of a finite measure in $\mathcal{S}$ such that $B = \bigcup_{n=1}^{\infty} B_n$; it is outer regular if there is $U \in \mathcal{T}$ such that $U \supset B$ (then trivially holds $m(B) = \bigwedge\{m(U);\ U \supset B,\ U \in \mathcal{T}\}$).

Definition 4. We say that $B \in \mathcal{S}$ is regular (with respect to m) if B is both inner and outer regular simultaneously.

The following assertion is proved in [12].

Lemma 1. Let $\{a_{n,i,\lambda};\ i \in N,\ \lambda \in L\}$ be a sequence of families with the property (P), i.e. for each $n \in N$ holds

(i) $a_{n,i,\lambda} \geqq 0$ for each $i \in N$, $\lambda \in L$;

(ii) for each $f \in L^N$ there is $\bigvee_i a_{n,i,f(n+i)}$ (this supremum we denote by a_f^n);

(iii) for each $i \in N$ holds $\bigwedge_{\lambda \in L} a_{n,i,\lambda} = 0$.

Then for every $a \in G^+$ there is a family $\{a_{i,\lambda};\ i \in N,\ \lambda \in L\}$ with the property (P) such that for each $f \in L^N$ holds

$$a \wedge \sum_{n=1}^{\infty} a_f^n \leqq a_f .$$

Let $\mathcal{X}$, $\mathcal{T}$ and $\mathcal{S}$ be families of subsets of an arbitrary nonempty set X with the following properties:

P1 $\emptyset \in \mathcal{X} \cap \mathcal{T}$;

P2 if $U_i \in \mathcal{T}$ $(i=1, 2, \dots)$, then $\bigcup_{n=1}^{\infty} U_n \in \mathcal{T}$;

P3 if $C, D \in \mathcal{X}$, then $C \cup D \in \mathcal{X}$;

P4 if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$;

P5 for each $C \in \mathcal{X}$ there is $U \in \mathcal{T}$ and $D \in \mathcal{X}$ such that $C \subset U \subset D$;

P6 for each $C \in \mathcal{X}$ and $U \in \mathcal{T}$ is $U - C \in \mathcal{T}$, $C - U \in \mathcal{X}$;

P7 $\mathcal{T} \subset \mathcal{S}$, where $\mathcal{S}$ is a σ-ring generated by $\mathcal{X}$.

Definition 5. A set $B \subset X$ is said to be bounded if there exists $C \in \mathcal{X}$ such that $B \subset C$.

Lemma 2. a) For any sequence of sets (C_n) from $\mathcal{X}$, the intersection $\bigcap_{n=1}^{\infty} C_n \in \mathcal{X}$.

b) If $B \in \mathcal{S}$, then there exists a non-decreasing sequence (B_n) of bounded sets in $\mathcal{S}$ such that $B = \bigcup_{n=1}^{\infty} B_n$

c) If $B \in \mathcal{S}$, then there is $U \in \mathcal{T}$ such that $B \subset U$.

Proof. ([4], 38—40).

Let m be a finite measure in $\mathcal{X}$ in the following text.

Definition 6. Let m be a $\bar{G}$-valued measure on $\mathcal{S}$. m is said to be regular at ∞ if for each $B \in \mathcal{S}$ with $m(B) < \infty$ there exists $U \in \mathcal{T}$ such that $B \subset U$ and $m(U) < \infty$.

Theorem 1. Let m be a $\bar{G}$-valued measure which is regular at ∞. Then the union of a sequence of outer regular sets in $\mathcal{S}$ is outer regular.

Proof. Let (B_n) be a sequence of outer regular sets in $\mathcal{S}$, $B = \bigcup_{n=1}^{\infty} B_n$. If $m(B) = \infty$, evidently B is outer regular. Let $m(B) < \infty$. From the assumption it follows that there is $U \in \mathcal{T}$ such that $B \subset U$ and $m(U) < \infty$. Since B_n is outer regular, there is a family $\{a_{n,i,\lambda};\ i \in N, \lambda \in L\}$ with the property (P) such that for each $f \in L^N$ there exists $U_n \in \mathcal{T}$ such that $B_n \subset U_n \subset U$ and $m(U_n - B_n) \leqq a_f^n$. Put $V = \bigcup_{n=1}^{\infty} U_n$, then $V \in \mathcal{T}$, $B \subset V \subset U$. By Lemma 1 there is a family $\{a_{i,\lambda};\ i \in N, \lambda \in L\}$ with the property (P) such that for each $f \in L^N$ holds

$$m(U) \wedge \sum_{n=1}^{\infty} a_f^n \leqq a_f .$$

Then

$$m(V - B) \leqq m(U) \wedge \sum_{n=1}^{\infty} m(U_n - B_n) \leqq a_f .$$

Theorem 2. The intersection of a non-increasing sequence of outer regular sets of a finite measure in $\mathcal{S}$ is outer regular.

Proof. Let (B_n) be a non-increasing sequence of outer regular sets of a finite measure. Then $\lim m(B_n) = m(B)$. Since B_n is outer regular, so there exists a family $\{a_{n,i,\lambda};\ i \in N,\ \lambda \in L\}$ with the property (P) such that for each $f \in L^N$ there is $U_n \supset B_n$, $U_n \in \mathscr{T}$ and such that $m(U_n - B_n) \leqq a_f^n$. Put $V_n = \bigcap_{i=1}^{n} U_i$. Then $V_n \in \mathscr{T}$, $V_n \supset B_n$, $m(V_n - B_n) \leqq a_f^n$ and $m(V_n) \leqq m(U_1)$. Since $m(U_1) = m(B_1) + m(U_1 - B_1) \leqq m(B_1) + a_f^1 < \infty$, so by Lemma 1 there is a family $\{c_{i,\lambda};\ i \in N,\ \lambda \in L\}$ with the property (P) such that

$$m(U_1) \wedge \sum_{n=1}^{\infty} a_f^n \leqq c_f .$$

Therefore,

$$m(V_n) \leqq m(B_n) + c_f .$$

Put $b_n = m(B_n) - m(B)$ and let $\{\lambda_i;\ i = 1, 2, \ldots\}$ be a simple sequence of elements in L (i.e. holds, if $n \neq m$ then $\lambda_n \neq \lambda_m$). Put $b_{n,\lambda_1} = b_1, \ldots, b_{n,\lambda_k} = b_k, \ldots$ and for $\lambda \in L - \{\lambda_i;\ i = 1, 2, \ldots\}$ $b_{n,\lambda} = b_n$. Then the family $\{b_{i,\lambda};\ i \in N,\ \lambda \in L\}$ has the property (P) and hence there is a family $\{a_{i,\lambda};\ i \in N,\ \lambda \in L\}$ with the property (P) such that $c_f + b_f \leqq a_f$.

Since $m(B) = m(B_n) - b_n$ and for each $f \in L^N$ there is n such that $b_f \geqq b_n$, so $m(B) \geqq m(V_n) - a_f$.

Theorem 3. The intersection of a sequence of inner regular sets of a finite measure in $\mathscr{S}$ is inner regular.

The proof ot this Theorem is similar as in Theorem 1.

Theorem 4. The union a non-decreasing sequence of inner regular sets of a finite measure in $\mathscr{S}$ is inner regular.

The proof of this Theorem is similar as in Theorem 2.

Theorem 5. The difference of two regular sets of a finite measure in $\mathscr{S}$ is regular.

Proof. Let A, B be regular sets of a finite measure. Then there are families $\{a_{i,\lambda};\ i \in N,\ \lambda \in L\}$, $\{b_{i,\lambda};\ i \in N,\ \lambda \in L\}$ such that for each $f \in L^N$ there are $C, D \in \mathscr{X}$, $U, V \in \mathscr{T}$ such that

$$U \supset A \supset D,\ m(U - A) \leqq a_f,\ m(A - D) \leqq a_f,$$

$$V \supset B \supset C,\ m(V - B) \leqq b_f,\ m(B - C) \leqq b_f .$$

By P6 $U - C \in \mathscr{T}$, $D - V \in \mathscr{X}$ and $U - C \supset A - B \supset D - V$. Let $\{c_{i,\lambda};\ i \in N,\ \lambda \in L\}$ is a family with the property (P) and such that

$$a_f + b_f \leqq c_f .$$

Then

$$m((U - C) - (A - B)) \leqq m((U - A) \cup (B - C)) \leqq c_f$$

and

$$m((A-B)-(D-V))\leqq c_f.$$

Theorem 6. The union of two regular sets of a finite measure is regular.

Proof. This assertion is trivial. It follows from Lemmal, Theorem 1 and relation

$$m(A\cup B-C\cup D)\leqq m((A-C)\cup(B-D)),$$

where $C\subset A$, $D\subset B$.

Theorem 7. A necessary and sufficient condition that every set in $\mathscr{Z}$ be outer regular is that every bounded set in $\mathscr{T}$ be inner regular.

Proof. Let every set in $\mathscr{Z}$ be outer regular, $U\in\mathscr{T}$ be a bounded, i.e. $U\subset C$, where $C\in Z$. Then by P6 is $C-U\in\mathscr{Z}$ and, therefore, there is a family $\{a_{i,\lambda};\, i\in N, \lambda\in L\}$ with the property (P) such that for each $f\in L^N$ there is $V\in\mathscr{T}$ such that

$$V\supset C-U,\ m(V-(C-U))\leqq a_f.$$

Since $U=C-(C-U)\supset C-V\in\mathscr{Z}$, so

$$m(U-(C-V))\leqq m(V-(C-U))\leqq a_f,$$

hence U is inner regular.

Let now every bounded set in $\mathscr{T}$ be inner regular, $C\in\mathscr{Z}$. Let $U\in\mathscr{T}$ be bounded and such that $U\supset C$. Then $U-C$ is bounded and $U-C\in\mathscr{T}$. Evidently, for each $f\in L^N$ there is $D\in\mathscr{Z}$ such that

$$D\subset U-C,\ m((U-C)-D)\leqq a_f.$$

Since $C=U-(U-C)\subset U-D\in\mathscr{T}$, so

$$m((U-D)-C)=m((U-C)-D)\leqq a_f.$$

Hence C is outer regular.

Theorem 8. If m is regular at ∞, then the outer regularity of every set in $\mathscr{Z}$ or the inner regularity of every bounded set in $\mathscr{T}$ is necessary and sufficient condition for the regularity of m.

Proof. The necessary condition is evident. We prove the sufficient condition.

Let every set in $\mathscr{Z}$ be outer regular. With respect to Theorem 1 and 4, Lemma 2 is sufficient, when we prove that each bounded set in $\mathscr{S}$ is regular.

Let $E\in\mathscr{S}$ be an arbitrary bounded set, $E\subset C$, where $C\in\mathscr{Z}$, let

$$\mathscr{Y}=\{A\in\mathscr{S};\ A\subset C,\ A \text{ is regular}\}.$$

Evidently, $C\cap\mathscr{Z}\subset\mathscr{Y}$, where $C\cap\mathscr{Z}$ is a system of all sets A for which there exists $B\in\mathscr{Z}$ such that $A=B\cap C$. Further, $\mathscr{Y}$ is a σ-ring (with respect to Theorems 1—6 and ([2], Theorem A, p. 27)). We denote by $\mathscr{W}$ the least σ-ring generated by

$C\cap\mathscr{X}$. Then $\mathscr{W}\subset\mathscr{Y}$ and with respect to ([2], Theorem E, p. 25) we obtain that $\mathscr{W}=\mathscr{S}\cap C$, hence $\mathscr{S}\cap C\subset\mathscr{Y}$.

Therefore, if $E\in\mathscr{S}$, $E\subset C$, thus $E\in\mathscr{S}\cap C$, i.e. E is regular.

Corollary 1. Let m be regular at ∞. If for every bounded set $U\in\mathscr{T}$ there is a sequence of sets (C_n) in $\mathscr{X}$ such that $U=\bigcup_{n=1}^{\infty} C_n$, then m is regular.

Corollary 2. Let m be regular at ∞. If for $\mathscr{X}$ and $\mathscr{T}$ holds $\mathscr{X}\subset\mathscr{T}$ or $\mathscr{T}\subset\mathscr{X}$, then m is regular.

Theorem 9. Let m be regular at ∞ and $\mathscr{W}=\left\{A\in\mathscr{T};\ A=\bigcup_{n=1}^{\infty} C_n,\ C_n\in\mathscr{X}\right\}$. Then m is regular measure on a σ-ring $\mathscr{S}(\mathscr{W})$ generated by $\mathscr{W}$.

Proof. The proof of these assertions is simple.

Example 1. Let X be a locally compact Hausdorff space, $\mathscr{X}$ be a family of all compact subsets of X, $\mathscr{T}$ be a family of all open sets belonging to $\mathscr{S}$, where $\mathscr{S}$ is a σ-ring generated by $\mathscr{X}$. $\mathscr{S}$ is said to be a family of Borel sets. $\mathscr{X}$, $\mathscr{T}$ and $\mathscr{S}$ have the properties P1—P7 ([4], ex. 1, p. 45).

Example 2. Let X be a locally compact Hausdorff space, $\mathscr{X}$ be a family of all compact G_δ*'s* subsets of X, $\mathscr{T}$ be a family of all open sets belonging to $\mathscr{S}$, where $\mathscr{S}$ is a σ-ring generated by $\mathscr{X}$. $\mathscr{S}$ is said to be a family of Baire sets and a measure on $\mathscr{S}$ finite on $\mathscr{X}$ is said to be Baire measure. $\mathscr{X}$, $\mathscr{T}$ and $\mathscr{S}$ have properties P1—P7.

Example 3. Let X be an arbitrary topological space, $\mathscr{X}$ be a family of all closed sets, $\mathscr{T}$ be a family of all open sets, $\mathscr{S}$ be a σ-ring generated by $\mathscr{X}$. Then $\mathscr{T}$, $\mathscr{X}$, $\mathscr{S}$ have properties P1—P7.

Consequently, we can apply Theorem 8 in the examples 1—3.

Theorem 10. Let X be a metric space, $\mathscr{X}$ be a family of all closed sets, m be a measure defined on a σ-ring $\mathscr{S}$ generated by $\mathscr{X}$. If m is regular at ∞, then m is regular.

Proof. Let $U\in\mathscr{T}$, then $U=\bigcup_{n=1}^{\infty} C_n$, $C_n\in\mathscr{X}$. By Corollary 1 is m regular.

Theorem 11. Every Baire measure is regular if and only if it is regular at ∞.

Proof. If m is regular, then m is regular at ∞. It is evident.

Let m be a Baire measure regular at ∞. If $C\in\mathscr{X}$, then $C=\bigcap_{n=1}^{\infty} G_n$, $G_n\in\mathscr{T}$, $G_n\supset G_{n+1}$ and G_n is a finite measure for each n. Consequently $\lim m(G_n)=m(C)$. Put $b_n=m(G_n)-m(C)$, then $b_n\searrow 0$. We define a family $\{b_{i,\lambda};\ i\in N,\ \lambda\in L\}$ in the following way: let $\{\lambda_n;\ n=1,2,\dots\}$ be a simple sequence of elements in L, then $b_{i,\lambda_1}=b_1,\ \dots,\ b_{i,\lambda_n}=b_n,\ \dots$ and for $\lambda\in L-\{\lambda_n;\ n=1,2,\dots\}$ $b_{i,\lambda}=b_i$ for each i. The family $\{b_{i,\lambda};\ i\in N,\ \lambda\in L\}$ has the property (P) and for each $f\in L^N$ there is n such that $b_f\geqq b_n$. Since $m(C)=m(G_n)-b_n$, then $m(C)\geqq m(G_n)-b_f$ for suitable n.

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РЕЗЮМЕ

О РЕГУЛЯРНОЙ МЕРЕ

Э. Храховина, Братислава

В этой стати определенна регулярная векторовая мера и доказаны необходимые и достаточные условия для регулярности мери. Это какое-нибудь разьирение [4].

SUMMARY

ON REGULAR VECTOR MEASURES

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The article is introduced a notion of a regular vector measure and are proved the necessary and sufficient conditions for the regularity of measure. It is generalized of [4].