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THE CONSTRUCTION OF AN INVARIANT MEASURE

JOZEF KALAS, Bratislava

In the first part of the paper a measure μ is constructed on the σ-ring generated by some class of the closed subsets of a topological space X. The measure μ is invariant under a group of autohomeomorphisms of X.

In the second part two special cases are discussed. Let $(X, \mathcal{T})$ be a topological space, $\mathcal{F}$ be a class of some closed subsets of the space X, G be a group of authomeomorphisms of the space X, $\mathcal{P}$ be a class of open coverings of the space X. Let $\mathcal{F}$, $\mathcal{P}$ and G fulfill the following conditions

(i) Let F be any set in $\mathcal{F}$ and $\mathcal{U}$ be any open covering in $\mathcal{P}$. Then there exist $E_i \in \mathcal{U}$, $f_i \in G$, $i=1, 2, \ldots, n$ such that $\bigcup_{i=1}^{n} f_i^{-1}(E_i) \supset F$.

(ii) For every disjoint sets F_1, $F_2 \in \mathcal{F}$ there exists $\mathcal{W} \in \mathcal{P}$ fulfilling the following condition: for every $f \in G$, $E \in \mathcal{W}$ whenever $f(E) \cap F_1 \neq \emptyset$ then $f(E) \cap F_2 = \emptyset$

(iii) for every $\mathcal{U}$, $\mathcal{V} \in \mathcal{P}$ there exists $\mathcal{W} \in \mathcal{P}$ such that $\mathcal{W} \geqslant \mathcal{U}$, $\mathcal{W} \geqslant \mathcal{V}$ (for every $B \in \mathcal{W}$ there exists $C \in \mathcal{U}$ or $\mathcal{V}$ respectively, such that $B \subset C$).

Now let F be any set in $\mathcal{F}$. Put

$$\mathcal{K}_F = \{ U \subset X: U \text{ is an open set and there are } f_i \in G \ (i=1, \ldots, n) \text{ such that } \bigcup_{i=1}^{n} f_i^{-1}(U) \supset F \}.$$

Evidently $X \in \mathcal{K}_F$ for every $F \in \mathcal{F}$. Denote by $\mathcal{K} = \bigcap \{\mathcal{K}_F : F \in \mathcal{F}\}$

(iv) Suppose that there exists a set $A \in \mathcal{K}$ such that $\bar{A} \in \mathcal{F}$.

Remark. Further we shall denote by A only this set.

Theorem 1. There exists a non-trivial set function λ on the class $\mathcal{F}$ satisfying the following properties:

(a) λ is finite and non negative,

(b) λ is monotone,

(c) if $E, F \in \mathscr{F}$ and also $E \cup F \in \mathscr{F}$, then

$$\lambda(E \cup F) \leq \lambda(E) + \lambda(F) \text{ (i.e. } \lambda \text{ is subadditive)}$$

(d) if, moreover, $E \cap F = \emptyset$, then

$$\lambda(E \cup F) = \lambda(E) + \lambda(F) \text{ (i.e. } \lambda \text{ is additive)}$$

(a) if for $E \in \mathscr{F}$, $f \in G$ also $f(E) \in \mathscr{F}$, then

$$\lambda(f(E)) = \lambda(E) \text{ (i.e. } \lambda \text{ is invariant under } G).$$

Proof. For any set $F \in \mathscr{F}$ and for any $\mathscr{U} \in \mathscr{P}$ denote by $F\colon \mathscr{U} = \min\{n$: there exist $E_i \in \mathscr{U}$, $f_i \in G$ $(i = 1, 2, \ldots, n)$ such that

$$\left.\bigcup_{i=1}^{n} f_i^{-1}(E_i) \supset F\right\} \tag{1}$$

By the condition (i) it follows that such minimum exists.

Denote further by

$$F\colon A = \min\left\{n\colon \text{there exist } f_i \in G\ (i = 1, \ldots, n) \text{ such that } \bigcup_{i=1}^{n} f_i^{-1}(A) \supset F\right\} \tag{2}$$

Finally, denote by

$$A\colon \mathscr{U} = \min\left\{n\colon \text{there exist } E_i \in \mathscr{U},\ f_i \in G\ (i = 1, \ldots, n) \text{ such that } \bigcup_{i=1}^{n} f_i^{-1}(E_i) \supset A\right\} \tag{3}$$

Define now for every $\mathscr{U} \in \mathscr{P}$ the set function $\lambda_{\mathscr{U}}$ on the class $\mathscr{F}$ by

$$\lambda_{\mathscr{U}}(F) = \frac{F\colon \mathscr{U}}{A\colon \mathscr{U}} \text{ for every } F \in \mathscr{F} \tag{4}$$

We show now that it holds $\lambda_{\mathscr{U}}(F) \leq F\colon A$ for any $\mathscr{U} \in \mathscr{P}$ and $F \in \mathscr{F}$.

Let $F\colon A = s$. Then by the definition of $F\colon A$ it follows that there are $g_j \in G$ $(j = 1, \ldots, s)$ such that $\bigcup_{j=1}^{s} g_j^{-1}(A) \supset F$.

Let $A\colon \mathscr{U} = n$. Analogously, there are $E_i \in \mathscr{U}$, $f_i \in G$ $(i = 1, \ldots, n)$ such that

$$\bigcup_{i=1}^{n} f_i^{-1}(E_i) \supset A.$$

Since

$$\bigcup_{j=1}^{s}\bigcup_{i=1}^{n} (f_i \circ g_j)^{-1}(E_i) = \bigcup_{j=1}^{s}\bigcup_{i=1}^{n} g_j^{-1}(f_i^{-1}(E_i)) =$$

$$=\bigcup_{j=1}^{s} g_j^{-1}\left(\bigcup_{i=1}^{n} f_i^{-1}(E_i)\right) \supset \bigcup_{j=1}^{s} g_j^{-1}(A) \supset F$$

it holds with respect to the definition of $F\colon \mathscr{U}$ that $F\colon \mathscr{U} \leqslant s \cdot n$ and hence

$$\lambda_{\mathscr{U}}(F) \leqslant F\colon A\,. \tag{5}$$

Consider now the following system of the compact topological spaces $\{\langle 0, F\colon A\rangle, \mathscr{T}_F);\ F \in \mathscr{F}\}$, where $\mathscr{T}_F$ is the relative topology of the usual topology on the real line.

Denote by Ω a topological product of these compact topological spaces. It follows by the Tychonov theorem that Ω is a compact topological space. By (5) it follows that $\lambda_{\mathscr{U}} \in \Omega$ (evidently, $\lambda_{\mathscr{U}}(F) \geqslant 0$ for every $F \in \mathscr{F}$) for every $\mathscr{U} \in \mathscr{P}$. Denote by $\mathscr{L}$ the following class of subsets of the space Ω

$$\mathscr{L} = \{\delta_{\mathscr{U}}\colon \delta_{\mathscr{U}} = \{\lambda_{\mathscr{V}}\colon \mathscr{V} \geqslant \mathscr{U},\ \mathscr{V} \in \mathscr{P}\},\ \mathscr{U} \in \mathscr{P}\}$$

By the condition (iii) it follows that for any $\delta_{\mathscr{U}_i}$ $(i=1, \dots, n)$ there exists $\mathscr{W} \in \mathscr{P}$ such that $\delta_{\mathscr{W}} \subset \bigcap_{i=1}^{n} \delta_{\mathscr{U}_i}$ Since $\delta_{\mathscr{W}}$ always contains $\lambda_{\mathscr{W}}$ and it is therefore non empty, the class $\mathscr{L}$ has the finite intersection property. The compactness of Ω implies that there is a point λ in the intersection of the closures of all $\delta_{\mathscr{U}}$, i.e. $\lambda \in \bigcap\{\bar{\delta}_{\mathscr{U}}\colon \mathscr{U} \in \mathscr{P}\}$ (Further we shall denote $\mathscr{L}_0 = \bigcap\{\bar{\delta}_{\mathscr{U}}\colon \mathscr{U} \in \mathscr{P}\}$.)

We shall prove that λ is the required set function, i.e. it satisfies the properties (a)—(d).

(a) it is obvious that λ is a finite and non negative set function

(b) let E and F be any sets in $\mathscr{F}$ such that $E \subset F$. By (1) it follows that $E\colon \mathscr{U} \leqslant F\colon \mathscr{U}$ for any $\mathscr{U} \in \mathscr{P}$. Then also

$$\lambda_{\mathscr{U}}(E) \leqslant \lambda_{\mathscr{U}}(F) \quad \text{for any} \quad \mathscr{U} \in \mathscr{P} \tag{6}$$

Denote by Ω_1 the following subset of Ω

$$\Omega_1 = \{t \in \Omega\colon t(E) \leqslant t(F), \ \text{for given } E, F \in \mathscr{F}\}$$

Evidently, Ω_1 is a closed subset of Ω and by (6) it follows that $\lambda_{\mathscr{U}} \in \Omega_1$ for every $\mathscr{U} \in \mathscr{P}$. This means that $\delta_{\mathscr{U}} \subset \Omega_1$ for every $\mathscr{U} \in \mathscr{P}$. Since Ω_1 is closed, then also $\mathscr{L}_0 \subset \Omega_1$ and hence $\lambda(E) \leqslant \lambda(F)$. This means that λ is a monotone set function.

(c) Let E, F be any sets in $\mathscr{F}$ such that $E \cup F \in \mathscr{F}$. Evidently, it holds for any $\mathscr{U} \in \mathscr{F}$ that

$$E \cup F\colon \mathscr{U} \leqslant E\colon \mathscr{U} + F\colon \mathscr{U} \quad \text{and hence also}$$

$$\lambda_{\mathscr{U}}(E \cup F) \leqslant \lambda_{\mathscr{U}}(E) + \lambda_{\mathscr{U}}(F)$$

Denote again by $\Omega_2 = \{t \in \Omega\colon t(E \cup F) \leqslant t(E) + t(F))\}$

It follows from the same reasons as in the proof of the property (b) that $\lambda \in \Omega_2$, i.e.

$$\lambda(E \cup F) \leq \lambda(E) + \lambda(F) \text{ for any } E, F \in \mathscr{F} \text{ such that } E \cup F \in \mathscr{F}.$$

(d) Moreover, let E and F be disjoint. By the condition (ii) it follows that there exists $\mathscr{U}^0 \in \mathscr{P}$ such that for every $f \in G$ and $C \in \mathscr{U}^0$ whenever $f(C) \cap E \neq \emptyset$, then $f(C) \cap F = \emptyset$. (7)

Let now $\mathscr{V} \in \mathscr{P}$ and $\mathscr{V} \geqslant \mathscr{U}^0$. If we put $E \cup F: \mathscr{V} = n$ then it follows (by the definition of $E \cup F: \mathscr{V}$) that there exist $E_i^0 \in \mathscr{V}$, $f_i \in G$ $(i = 1, \ldots, n)$ such that

$$\dot{\bigcup_{i=1}^{n}} f_i^{-1}(E_i) \supset E \cup F \tag{8}$$

Let $T_1 = \bigcup\{f_i^{-1}(E_i^0) \cap E \neq \emptyset\}$ and $T_2 = \bigcup\{f_i^{-1}(E_i^0) \cap F \neq \emptyset\}$. Then by (7) and (8) we have $T_1 \supset E$, $T_2 \supset F$ and $n = \text{card } T_1 + \text{card } T_2$. This implies the following inequality $n = E \cup F: \mathscr{V} \geqslant E: \mathscr{V} + F: \mathscr{V}$ and hence also $\lambda_{\mathscr{V}}(E \cup F) \geqslant \lambda_{\mathscr{V}}(E) + \lambda_{\mathscr{V}}(F)$. Then it holds (the opposite inequality has been proved abready)

$$\lambda_{\mathscr{V}}(E \cup F) = \lambda_{\mathscr{V}}(E) + \lambda_{\mathscr{V}}(F) \text{ for any } \mathscr{V} \geqslant \mathscr{U}^0 \tag{9}$$

Denote again by

$$\Omega_3 = \{t \in \Omega: t(E \cup F) = t(E) + t(F)\}$$

The subset Ω_3 is evidently closed. Then it follows by (9): $\bar{\delta}_{\mathscr{U}_0} \subset \Omega_3$ and hence also $\mathscr{L}_0 \subset \Omega_3$, i.e.

$$\lambda(E \cup F) = \lambda(E) + \lambda(F) \text{ for any } E, F \in \mathscr{F} \text{ such that}$$

$$E \cup F \in \mathscr{F} \text{ and } E \cap F = \emptyset.$$

(e) Let $F \in \mathscr{F}$, $f \in G$ and let also $f(E) \in \mathscr{F}$. It follows from the equalities

$$f^{-1}\left(\dot{\bigcup_{i=1}^{n}} f_i^{-1}(E_i)\right) = \dot{\bigcup_{i=1}^{n}} (f_i \circ f)^{-1}(E_i)$$

and

$$f\left(\dot{\bigcup_{i=1}^{n}} f_i^{-1}(E_i)\right) = \dot{\bigcup_{i=1}^{n}} (f_i \circ f^{-1})^{-1}(E_i) \text{ that } E: \mathscr{U} = f(E): \mathscr{U}$$

for any $\mathscr{U} \in \mathscr{P}$ and hence also

$$\lambda_{\mathscr{U}}(E) = \lambda_{\mathscr{U}}(f(E)) \text{ for any } \mathscr{U} \in \mathscr{P} \tag{10}$$

Denote again by $\Omega_4 = \{t \in \Omega: t(E) = t(f(E))\}$.

The subset Ω_4 is closed. By (10) it holds $\lambda_{\mathscr{U}} \in \Omega_4$ for any $\mathscr{U} \in \mathscr{P}$. Then $\bar{\delta}_{\mathscr{U}} \subset \Omega_4$ and hence also $\mathscr{L}_0 \subset \Omega_4$. Since $\lambda \in \mathscr{L}_0$, we get

$$\lambda(E) = \lambda(f(E)).$$

We shall suppose that the class $\mathscr{F}$ still fulfils the following conditions:

(K_1) Let $E \in \mathscr{F}$, $E \subset \bigcup_{i=1}^{\infty} U_i$, U_i be open sets $(i = 1, 2, \ldots)$.

Then there exists $n \in N$ such that $E \subset \bigcup_{i=1}^{n} U_i$.

(K_2) If $F \in \mathscr{F}$ and U and V are open sets such that $F \subset U \cup V$, then there exist F_1, $F_2 \in \mathscr{F}$ such that $F_1 \subset U$, $F_2 \subset V$, and $F = F_1 \cup F_2$.

(K_3) For every $E \in \mathscr{F}$ there exist sets U and F such that U is an open set, $F \in \mathscr{F}$, and $E \subset U \subset F$.

(K_4) If $E \in \mathscr{F}$, $f \in G$, then also $f(E) \in \mathscr{F}$

We shall now construct (going out from the set function λ) on the σ-ring $\mathscr{S}(\mathscr{F})$ generated by $\mathscr{F}$ a non trivial measure, invariant with respect to the group G. Define first on the system of all open sets a set function λ_* by

$$\lambda_*(U) = \sup \{\lambda(F) : F \subset U, F \in \mathscr{F}\} \tag{11}$$

Remark. By (K_2) it follows that $\emptyset \in \mathscr{F}$.

Theorem. The set function λ_* vanishes at $\emptyset$, it is monotone, and countably subadditive.

Proof. The first two properties of λ_* are evident. We prove now the third property. We show a subadditivity of the function λ_* at first. If U and V are open sets and if $E \in \mathscr{F}$ is such that $E \subset U \cup V$, then by (K_2) there exist $E_1 \in \mathscr{F}$ and $E_2 \in \mathscr{F}$ such that $E_1 \subset U$, $E_2 \subset V$, and $E = E_1 \cup E_2$. Since

$$\lambda(E) \leqslant \lambda(E_1) + \lambda(E_2) \leqslant \lambda_*(U) + \lambda_*(V), \text{ it follows that}$$

$$\lambda_*(U \cup V) = \sup \{\lambda(E) : E \subset U \cup V, E \in \mathscr{F}\} \leqslant \lambda_*(U) + \lambda(V),$$

i.e. that λ_* is subadditive. It follows immediately, by the mathematical induction, that λ_* is finitely subadditive. If $\{U_i\}_{i=1}^{\infty}$ is a sequence of open sets and if $E \in \mathscr{F}$, such that $E \subset \bigcup_{i=1}^{\infty} U_i$, then by ($K_1$) there is a positive integer n such that $E \subset \bigcup_{i=1}^{n} U_i$. It follows that

$$\lambda(E) \leqslant \lambda_*\left(\bigcup_{i=1}^{n} U_i\right) \leqslant \sum_{i=1}^{n} \lambda_*(U_i) \leqslant \sum_{i=1}^{\infty} \lambda_*(U_i)$$

and therefore

$$\lambda_*\left(\bigcup_{i=1}^{\infty} U_i\right) = \sup \left\{\lambda(E) : E \subset \bigcup_{i=1}^{\infty} U_i, E \in \mathscr{F}\right\} \leqslant \sum_{i=1}^{\infty} \lambda_*(U_i)$$

i.e. λ_* is countably subadditive.

Let now E be any subset of X. Put

$$m^*(E) = \inf \{\lambda_*(U) : E \subset U, \ U \text{ is a open set.}\}$$

Theorem. The set function m^* is an outer measure defined on the system of all subsets of X.

Proof. The set function m^* is non negative, monotone and vanishes at $\emptyset$ evidently. We prove that m^* is also countably subadditive (i.e. it has all properties of an outer measure).

If $\{E_i\}_{i=1}^{\infty}$ is a sequence of subsets of X, then, for every $\varepsilon > 0$ and for every $i = 1, 2, \ldots$ there exists an open set U_i such that $E_i \subset U_i$ and $\lambda_*(U_i) \leqslant m^*(E_i) + \frac{\varepsilon}{2^i}$.

It follows that

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leqq \lambda_*\left(\bigcup_{i=1}^{\infty} U_i\right) \leqslant \sum_{i=1}^{\infty} \lambda_*(U_i) \leqslant \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon$$

the arbitrariness of $\varepsilon > 0$ implies the countable subadditivity of m^*.

Remark. It follows immediately by the definition of m^*, that $m^*(U) = \lambda_*(U)$ for any open set U.

Denote by $\mathscr{B}$ the system of all m^*-measurable sets. We use the terminology according to [1]. In [1] it is proved that $\mathscr{B}$ is a σ-ring and the set function μ^* defined on $\mathscr{B}$ by $\mu^*(E) = m^*(E)$ for any $E \in \mathscr{B}$ is a measure on the $\mathscr{B}$. By the same method as in [1] (p. 234) it is possible to prove: any set F in $\mathscr{F}$ is m^*-measurabe. Then it follows that if $\mathscr{S}(\mathscr{F})$ is the σ-ring generated by $\mathscr{F}$, it holds $\mathscr{S}(\mathscr{F}) \subset \mathscr{B}$. Hence the set function μ defined on $\mathscr{S}(\mathscr{F})$ by $\mu(E) = m^*(E)$ for every $E \in \mathscr{S}(F)$ is a measure on $\mathscr{S}(\mathscr{F})$.

Theorem. The measure μ fulfils the following conditions:

(a) μ is a non trivial measure,

(b) μ is a finite measure on $\mathscr{F}$

(c) μ is an invariant measure under G.

Proof.

(a) First we show for any F in $\mathscr{F}$ that $\mu(F) \geqslant \lambda(F)$. Let U is any open set such that $F \subset U$. Since $\lambda_*(U) \geqslant \lambda(F)$ it follows that

$$\mu(F) = m^*(F) = \inf \{\lambda_*(U) : F \subset U\} \geqq \lambda(F).$$

Since $A \subset \bar{A}$ it follows that

$$\lambda_{\mathscr{U}}(\bar{A}) = \frac{\bar{A} : \mathscr{U}}{A : \mathscr{U}} \geqslant 1 \ (10) \text{ for any } \mathscr{U} \text{ in } \mathscr{P}.$$

The set $\Omega_5 = \{t \in \Omega : t(\bar{A}) \geqslant 1\}$ is closed and by (10) $\lambda_{\mathscr{U}} \in \Omega_5$ for every $\mathscr{U}$ in $\mathscr{P}$. It follows that $\mathscr{L}_0 \subset \Omega_5$ and hence also $\lambda(\bar{A}) \geqslant 1$. Since $\mu(\bar{A}) \geqslant \lambda(\bar{A})$, then also $\mu(\bar{A}) \geqslant 1$.

(b) Let F is any set in $\mathscr{F}$. By (K_3) there exist an open set U and a set E in $\mathscr{F}$ such that $F \subset U \subset E$. Let C be any set in $\mathscr{F}$ such that $C \subset U$. Since $\lambda(C) \leqslant \lambda(E)$, it follows that

$$\lambda_*(U) = \sup \{\lambda(C) \colon C \subset U,\ C \in \mathscr{F}\} \leqslant \lambda(E).$$

It follows from the following inequalities that μ is finite:

$$\mu(F) = m^*(F) \leqslant m^*(U) = \lambda_*(U) \leqslant \lambda(E) < \infty .$$

(c) First we prove for any open set u and for any f in G that

$$\lambda_*(U) = \lambda_*(f(U)) \colon$$

$$\lambda_*(U) = \sup \{\lambda(E) \colon E \subset U,\ E \in \mathscr{F}\} = \sup \{\lambda(f(E)) \colon$$

$$E \subset U,\ E \in \mathscr{F}\} = \sup \{\lambda(F) \colon F \subset f(U),\ F \in \mathscr{F}\} = \lambda_*(f(U)).$$

Analogously $m^*(E) = \inf \{\lambda_*(U) \colon E \subset U,\ U$ is an open set$\} = \inf \{\lambda_*(f(U)) \colon E \subset U,\ U$ is an open set$\} = \inf \{\lambda_*(V) \colon f(E) \subset V,\ V$ is an open set$\} = m^*(f(E))$ and hence if $E \in \mathscr{S}(\mathscr{F})$ then $\mu(f(E)) = m^*(f(E)) = m^*(E) = \mu(E)$ i.e. that μ is invariant under G.

II

We show now some special cases of our general theory.

Example 1. Let X be a locally compact topological group (the group operation is denoted by $\cdot$), $\mathscr{F}$ be a class of all compact sets in X and

$$G = \{f_a \colon f_a(x) = a \cdot x,\ a \in X\}$$

Denote by $\mathscr{U}_e$ the class of all neighbourhoods of the identity element e of the topological group X.

Put

$$\mathscr{P} = \{\{a \cdot U \colon a \in X\},\ U \in \mathscr{U}_e\}.$$

We show that the conditions (i)—(iv) given in section I. are fulfilled:

(i) Let F be any compact set and

$$\mathscr{U} = \{a \cdot U \colon a \in X\} \text{ be any element in } \mathscr{P}.$$

Since $\bigcup\{a \cdot U \colon a \in X\} \supset F$ by the compactness of F and by the properties of the topological group, there is a positive integer n such that $\bigcup_{i=1}^{n} a_i U \supset F$, i.e.

$$\bigcup_{i=1}^{n} f_e^{-1}(a_i U) \supset F.$$

(ii) The validity of this condition follows from the properties of the topological group.

Let F_1, F_2 be any disjoint compact sets of X. There exists U in $\mathcal{U}_e$ such that $(F_1 \cdot U) \cap (F_2 \cdot U) = \emptyset$. Let V be a set in $\mathcal{U}_e$ such that $V^{-1} \cdot V \subset U$.

Denote by $\mathcal{W} = \{a \cdot V: a \in X\}$. We show for any b in X: whenever $f_b(a \cdot V) \cap F_1 = (b \cdot a \cdot V) \cap F_1 \neq \emptyset$ then $f_b(a \cdot V) \cap F_2 = (b \cdot a \cdot V) \cap F_2 = \emptyset$. We prove this implication by the contradiction.

Let there be $c \in (b \cdot a \cdot V) \cap F_1$ and $z \in (b \cdot a \cdot V) \cap F_2$. Then $c = b \cdot a \cdot v_1$, $z = b \cdot a \cdot v_2$, v_1, $v_2 \in V$ and this implies that $z = c \cdot v_1^{-1} \cdot v_2$ i.e. $z \in c \cdot V^{-1} \cdot V \subset \subset F_1 U$.

This is a contradiction, since $z = z \cdot e \in F_2 U$.

(iii) Let $\mathcal{U} = \{a \cdot U: a \in X\}$ and $\mathcal{V} = \{a \in V: a \in X\}$, where U and V are in $\mathcal{U}_e$.

Put $\mathcal{W} = \{a \cdot (U \cap V): a \in X\}$. Then evidently $U \cap V$ is in $\mathcal{U}_e$ and $\mathcal{W} \geqslant \mathcal{V}$, $\mathcal{W} \geqslant \mathcal{U}$.

(iv) Let U_e^0 be such neighbourhood of the identity element e that $\bar{U}_e^0$ is a compact set and let F be any compact set. Since $\bigcup\{aU_e^0: a \in F\} \supset F$, by the compactness of F there is a positive integer n such that

$$\bigcup_{i=1}^{n} a_i U_e^0 \supset F \text{ i.e. } \bigcup_{i=1}^{n} \overset{-1}{f}_{a_i^{-1}}(U_e) \supset F.$$

We proved the existence of a set A considered in the section I.

The conditions (K_1)—(K_4) are evidently fulfilled. We get now the following known theorem from the results of the section I.

Theorem. Let X be a locally compact topological group. There exists a non trivial Borel measure μ such that $\mu(E) = \mu(a \cdot E)$ for any $a \in X$ and any Borel set E.

Remark: The measure μ is called a Haar measure.

Example 2. Let (X, ϱ) be a compact metric space, $\mathcal{F}$ be a class of all compact sets and G be a group of autohomeomorphisms of the space X such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\varrho(f(x_1), f(x_2)) < \varepsilon$ for every x_1, $x_2 \in X$ such that

$$\varrho(x_1, x_2) < \delta \text{ and for every } f \in G. \tag{1}$$

Put $\mathcal{P} = \{\{O(a, \varepsilon), a \in X\}, \varepsilon > 0\}$, where $O(a, \varepsilon) = \{x \in X: \varrho(a, x) < \varepsilon\}$

We show again that the conditions (i)—(iv) are fulfilled.

(ii) Let F be any compact set

$\mathcal{U} = \{O(a, \varepsilon), a \in X\}$ be any element in $\mathcal{P}$ and f be any autohomeomorphism in G. Since

$\bigcup\{f^{-1}(O(a, \varepsilon)); a \in X\} \supset F$, by the compactness of F there exist $a_1, \ldots, a_n$ such that

$$\bigcup_{i=1}^{n} f^{-1}(O(a_i, \varepsilon)) \supset F.$$

(ii) Let F_1, F_2 be any disjoint compact subsets of X and f be any autohomeomorphism in G. Denote by $d = \inf\{\varrho(x, x_2) : x_1 \in F_1, x_2 \in F_2\}$. The distance d is positive since X is a compact metric space. By the condition (1) it follows that there exists $\delta > 0$ such that $\varrho(f(x_1), f(x_2)) < d$ for every $f \in G$ whenever $\varrho(x_1, x_2) < \delta$.

Put $\mathcal{U} = \left\{O\left(a, \frac{\delta}{2}\right), \ a \in X\right\}$. We show by a contradiction that whenever $f\left(O\left(a, \frac{\delta}{2}\right)\right) \cap F_1 \neq \emptyset$ then $f\left(O\left(a, \frac{\delta}{2}\right)\right) \cap F_2 = \emptyset$:

Let $z_1 \in f\left(O\left(a, \frac{\delta}{2}\right)\right) \cap F_1$ and $z_2 \in f\left(O\left(a, \frac{\delta}{2}\right)\right) \cap F_2$. Then $z_1 = f(x_1)$, $x_1 \in O\left(a, \frac{\delta}{2}\right)$, $z_2 = f(x_2)$, $x_2 \in O\left(a, \frac{\delta}{2}\right)$. Since $\varrho(x_1, x_2) < \delta$, we get that $\varrho(f(x_1), f(x_2)) = \varrho(z_1, z_2) < d$. This is a contradiction since $z_1 \in F_1$, $z_2 \in F_2$.

(iii) Let $\mathcal{U} = \{O(a, \varepsilon_1) : a \in X\}$, $\mathcal{V} = \{O(a, \varepsilon_2) : a \in X\}$. Denote by $\varepsilon = \min\left(\frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2}\right)$ and put $\mathcal{W} = \{O(a, \varepsilon) : a \in X\}$. Then $\mathcal{W} \geqslant \mathcal{U}$ and $\mathcal{W} \geqq \mathcal{V}$.

(iv) Since $X \in \mathcal{K}$ (the definition of $\mathcal{K}$ is given in the section I.) we can put $A = X$.

Evidently the conditions (K_1)—(K_4) are satisfied too. Then we get the following theorem from the results of the section I.

Theorem. Let X be a compact metric space and G a group of autohomeomorphisms of X fulfilling the condition (1). There exists a probability Borel measure P invariant under G, i.e. $P(f(E)) = P(E)$ for any Borel set E and $f \in G$.

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Autor's address:

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Jozef Kalas,
Katedra teórie pravdepodobnosti
a matematickej štatistiky MFF UK,
Mlynská dolina — Matematický pavilón,
842 15 Bratislava

SÚHRN

KONŠTRUKCIA INVARIANTNEJ MIERY

J. Kalas, Bratislava

Prvá časť práce sa zaoberá konštrukciou miery na σ-okruhu generovanom istou triedou uzavretých množín nejakého topologického priestoru X. Konštruovaná miera je invariantná vzhľadom na nejakú grupu autohomeomorfizmov priestoru X.

V druhej časti práce sú uvedené dva špeciálne prípady uvažovanej konštrukcii.

РЕЗЮМЕ

ПОСТРОЕНИЕ ИНВАРИАНТНОЙ МЕРЫ

Й. Калас, Братислава

Первая часть работы содержит построение меры, определенной на σ-кольце порожденном некоторым классом замкнутых множеств какого-нибудь топологического пространства X. Построенная мера является инвариантной относительно даной группы гомеоморфных отображений пространства X.

Вторая часть работы содержит два специальных случая этого построения.