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# ON MEASURABILITY OF SUPERPOSITIONS

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An important question, when dealing with differential equations, is to establish measurability of the superposition $f(x, g(x))$ where $g: X \to Y$, $f: X \times Y \to Z$. In case $X$, $Y$, $Z$ are equipped with $\sigma$-algebras, the superposition measurability has been studied by I. V. Šragin, see e.g. [2], [3]. The aim of the present paper is to give sufficient conditions for the superposition measurability suitable also for the case when the measurable sets considered do not form a $\sigma$-algebra but a $q$-algebra only, i.e. a family closed under disjoint countable unions and complementation.

We are going to introduce a modified notion of measurability that will play the key role in formulating the main result.

**Definition 1.** Let $X$, $Y$ be nonempty sets, $\mathscr{X}$ any family of subsets of $X$ and $\mathscr{V}$ any family of subsets of $X \times Y$. We say that $g: X \to Y$ is $(\mathscr{X}, \mathscr{V})$-projection measurable iff for each $V \in \mathscr{V}$ the set $\{x: (x, g(x)) \in V\}$ is in $\mathscr{X}$.

In common situations the projection measurability is closely linked with the usual measurability of $g$ defined by $g^{-1}(B) \in \mathscr{X}$ for all $B \in \mathscr{Y}$ where $\mathscr{Y}$ is the family of "measurable" sets in $Y$.

**Proposition 1.** Let $\mathscr{X} \subset 2^X$, $\mathscr{Y} \subset 2^Y$, $\mathscr{V} \subset 2^{X \times Y}$, $g: X \to Y$. If for every $B \in \mathscr{Y}$ there exists $\tilde{B} \in \mathscr{X}$ such that $g^{-1}(B) \subset \tilde{B}$ and $\tilde{B} \times B \in \mathscr{V}$, then $(\mathscr{X}, \mathscr{V})$-projection measurability of $g$ implies the usual $(\mathscr{X}, \mathscr{Y})$-measurability of $g$.

**Proof.** Let $B \in \mathscr{Y}$, then $g^{-1}(B) = \tilde{B} \cap g^{-1}(B) = \{x: (x, g(x)) \in \tilde{B} \times B\} \in \mathscr{X}$.

**Example 1.** This example will show that the hypothesis $g^{-1}(B) \subset \tilde{B} \in \mathscr{X}$ is essential in Proposition 1. Put $X = \{1, 2, 3\}$, $Y = \{1, 2\}$, $\mathscr{X} = \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$, $\mathscr{Y} = \{\emptyset, \{1, 2\}\}$, $\mathscr{V} = \{\emptyset, \{1, 2\} \times \{1, 2\}, \{1, 3\} \times \{1, 2\}, \{2, 3\} \times \{1, 2\}\}$ ($\mathscr{V}$ is the $q$-ring generated by the measurable rectangles). Then $g(x) = 1$ is $(\mathscr{X}, \mathscr{V})$-projection measurable but evidently not $(\mathscr{X}, \mathscr{Y})$-measurable.

**Example 2.** Now we show that, in general, $(\mathscr{X}, \mathscr{Y})$-measurability of $g$ does not imply its $(\mathscr{X}, \mathscr{V})$-projection measurability even if $\mathscr{V} = \{A \times B: A \in \mathscr{X}, B \in \mathscr{Y}\}$. Put $X = Y = \{1, 2, 3, 4\}$, $\mathscr{X} = \mathscr{Y} = \{A \subset X: \text{card } A \text{ is even}\}$, $g(x) = x$. Then $g$ is evidently $(\mathscr{X}, \mathscr{Y})$-measurable but for $E=\{1,2\}\times\{1,3\}\in\mathscr{V}$ we have $\{x\colon (x, g(x))\in E\} = \{1\}\notin\mathscr{X}$.

**Proposition 2.** In any of the following situations $(\mathscr{X}, \mathscr{Y})$-measurability of $g$ implies its $(\mathscr{X}, \mathscr{V})$-projection measurability.

a) $\mathscr{X}$ is a $\sigma$-ring in $X$, $\mathscr{Y}$ is a $\sigma$-ring in $Y$, and $\mathscr{V}$ is the $\sigma$-ring generated by the measurable rectangles $A\times B$ $(A\in\mathscr{X},\ B\in\mathscr{Y})$

b) $\mathscr{X}$ is a $q$-algebra in $X$, $\mathscr{Y}$ is a $q$-algebra in $Y$, and $\mathscr{V}$ consists of $A\times Y$ and $X\times B$ with $A\in\mathscr{X}$, $B\in\mathscr{Y}$.

**Proof.** a) Denote $\mathscr{L}=\{C\subset X\times Y\colon \ \{x\colon \ (x, g(x))\in C\}\in\mathscr{X}\}$. The $(\mathscr{X}, \mathscr{Y})$-measurability of $g$ implies that for any rectangle $A\times B$ with $A\in\mathscr{X}$, $B\in\mathscr{Y}$ we have $\{x\colon (x, g(x)) \in A\times B\} = A\cap g^{-1}(B)\in\mathscr{X}$, hence $\mathscr{L}$ contains all such rectangles. Evidently $\mathscr{L}$ is a $\sigma$-ring, and so $\mathscr{L}\supset\mathscr{V}$, which means that $g$ is $(\mathscr{X}, \mathscr{V})$-projection measurable.

b) Clearly, $\{x\colon (x, g(x))\in A\times Y\}=A\in\mathscr{X}$, and $\{x\colon (x, g(x)) \in X\times B\} = g^{-1}(B)\in\mathscr{X}$ due to $(\mathscr{X}, \mathscr{Y})$-measurability of $g$.

**Theorem 1.** If $\mathscr{X}\subset 2^X$, $\mathscr{Y}\subset 2^Y$, $\mathscr{V}\subset 2^{X\times Y}$, $\mathscr{Z}\subset 2^Z$, $f\colon X\times Y\to Z$ is $(\mathscr{V}, \mathscr{Z})$-measurable and $g\colon X\to Y$ is $(\mathscr{X}, \mathscr{V})$-projection measurable, then $f(\cdot, g(\cdot))$ is $(\mathscr{X}, \mathscr{Z})$-measurable.

**Proof.** For $C\in\mathscr{Z}$, we have $\{x\colon f(x, g(x))\in C\} = \{x\colon (x, g(x)) \in f^{-1}(C)\}\in\mathscr{X}$ by the hypotheses.

**Corollary.** Let $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ be $q$-algebras on $X$, $Y$, $Z$, respectively, let $f\colon X\times Y\to Z$ and $g\colon X\to Y$. Any of the following conditions is sufficient for $(\mathscr{X}, \mathscr{Z})$-measurability of the superposition $f(\cdot, g(\cdot))$.

a) $f$ is $(\mathscr{V}, \mathscr{Z})$-measurable, where $\mathscr{V}=\{A\times Y, X\times B\colon A\in\mathscr{X}, B\in\mathscr{Y}\}$, and $g$ is $(\mathscr{X}, \mathscr{Y})$-measurable

b) $f$ is $(\mathscr{V}, \mathscr{Z})$-measurable with $\mathscr{V}$ being the $q$-algebra generated by the rectangles $A\times B$ $(A\in\mathscr{X},\ B\in\mathscr{Y})$, and $g$ is $(\mathscr{X}, \mathscr{V})$-projection measurable.

Other conditions implying superposition measurability can be obtained by combining our results with those of T. Neubrunn [1]. Recall that a subfamily $\mathscr{A}$ of a $q$-algebra is strongly compatible iff for any $A, B\in\mathscr{A}$ the intersection $A\cap B$ is in the smallest $q$-algebra containing $\mathscr{A}$. Now Theorem 1 of [1] together with Part b of the last Corollary yields

**Theorem 2.** Let $\mathscr{X}$ be a $q$-algebra of subsets of $X$, let $Y$ be a second-countable topological space and $\mathscr{Y}$ a $\sigma$-algebra of subsets of $Y$ containing all open sets. Let $\mathscr{B}$ denote the Borel $\sigma$-algebra on the real line $R$. Let $f\colon X\times Y\to R$ be such that $f(x, \cdot)$ is continuous for every $x$ and there exists a dense set $D\subset Y$ such that $\{\{x\colon f(x, y)\in B\}\colon y\in D,\ B\in\mathscr{B}\}$ is a strongly compatible subfamily of $X$. Let $\mathscr{V}$ denote the $q$-algebra generated by rectangles $A\times B$ $(A\in\mathscr{X},\ B\in\mathscr{Y})$. If $g\colon X\to Y$ is $(\mathscr{X}, \mathscr{V})$-projection measurable, then $f(\cdot, g(\cdot))$ is $(\mathscr{X}, \mathscr{B})$-measurable.

The last theorem can be modified in the manner of Theorem 2 of [1], using the notion of P-system.

## REFERENCES

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## SÚHRN

### O MERATEĽNOSTI SUPERPOZÍCIÍ

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V článku sa podávajú postačujúce podmienky merateľnosti superpozície $f(\cdot, g(\cdot))$, kde $f: X \times Y \to Z$, $g: X \to Y$, pričom triedy merateľných množín v priestoroch $X$, $Y$, $Z$ nemusia byť $\sigma$-algebry, ale všeobecnejšie systémy.


## РЕЗЮМЕ

### ОБ ИЗМЕРИМОСТИ СУПЕРПОЗИЦИЙ

Й. Дравецки, Братислава


В статье дадутся достаточные условия измеримости суперпозиции $f(\cdot, g(\cdot))$, где $f: X \times Y \to Z$, $g: X \to Y$ и классы измеримых множеств в пространствацх $X$, $Y$, $Z$ более общие системы чем счётно аддитивные кольца.