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ON WEAK FORMS OF CONTINUITY AND PRODUCT SPACES

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The paper deals with questions of some types of generalized continuity of mappings and multivalued mappings, and their appropriate weak variants of continuity. It is supplementary to some results of the paper [5] and supplements, and also enlarges some results of [3], [4], [8]. In the case when the domain of a mapping is a product topological space, we introduce a new class of mapping, called somewhat continuous mappings with respect to one of variables which contains the class of quasicontinuous mappings with respect to one of variables.

The results are concerned mainly with the case in which the range is a quasiregular topological space.

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In what follows X, Y and Z denote topological spaces. Let A be a subset of a topological space X. We shall denote the closure of A and the interior of A in X by $\bar{A}^x$ and $\operatorname{int}_x A$ respectively. If there is no misunderstanding, x will be omitted. Besides of mappings $f: X \to Y$ we consider also multivalued mappings $F: X \to P(Y)$, where $P(Y)$ is the power set of Y. For simplicity they are denoted $F: X \to Y$ and we suppose that $F(x) \neq \emptyset$ for each $x \in X$. If F is a multivalued mapping defined on the product space $X \times Y$, we shall call an x-section for given $x \in X$ the multivalued mapping F_x defined on Y as $F_x(y) = F(x, y)$. The y-section F_y for a given $y \in Y$ is defined analogically. In the case of a single valued mapping we denote x-sections and y-sections f_x and f_y respectively. The continuity of a multivalued mapping $F: X \to Y$ is defined by the means of its upper and lower continuity (see [2]). We give the definitions of various types of generalized continuity only for multifunctions. The corresponding notions of generalized continuity for single valued functions may be obtained in a natural way.

Definition 1 [2]. A multivalued mapping $F: X \to Y$ is said to be upper (lower) semi-continuous at a point $x \in X$ if for any open set V, $F(x) \subset V$ ($F(x) \cap V \neq \emptyset$) there exists a neighborhood U of x such that for any $y \in U$ $F(y) \subset V$ ($F(y) \cap V \neq \emptyset$). It is said to be upper (lower) semi-continuous, if it is upper (lower) semi-continuous at any $x \in X$.

Definition 2 [7]. A multivalued mapping $F: X \to Y$ is said to be upper (lower) semi-quasicontinuous at a point $x \in X$ if for any neigborhood U of x and any open set V, $F(x) \subset V$ ($F(x) \cap V \neq \emptyset$) there exists a nonempty open set $G \subset U$ such that for any $y \in G$ $F(y) \subset V$ ($F(y) \cap V \neq \emptyset$). It is said to be upper (lower) semi-quasicontinuous, if it is upper (lower) semi-quasicontinuous at any $x \in X$.

Note that we can immediately obtain the corresponding weak notions substituting in Def. 1 and Def. 2 the relations $F(y) \subset V$ or $F(y) \cap V \neq \emptyset$ by $F(y) \subset \bar{V}$ or $F(y) \cap \bar{V} \neq \emptyset$ respectively.

The notion of the upper (lower) inverse image $F^+(A)$ ($F^-(A)$) is frequently used. It is defined for $A \subset Y$ as

$$F^+(A) = \{x: x \in X, F(x) \subset A\}, \quad F^-(A) = \{x: x \in X, F(x) \cap A \neq \emptyset\}.$$

By means of F^+ and F^- one can define upper (lower) somewhat continuity.

Definition 3 [7]. A multivalued mapping $F: X \to Y$ is said to be upper (lower) somewhat continuous if for any open set V such that $F^+(V) \neq \emptyset$ ($F^-(V) \neq \emptyset$) we have int $F^+(V) \neq \emptyset$ (int $F^-(V) \neq \emptyset$).

Note that we can immediately obtain the corresponding weak notions substituting in Def. 3 the relations int $F^+(V) \neq \emptyset$ or int $F^-(V) \neq \emptyset$ by int $F^+(\bar{V}) \neq \emptyset$ or int $F^-(\bar{V}) \neq \emptyset$ respectively.

Remark 1. If the multivalued mapping is closed valued and if its range is a normal topological space then any of such weak upper continuity notions coincides with the original one. If the range of any multivalued mapping is a regular topological space then any of such weak lower continuity notions coincides with the original one.

The following proposition shows that the mapping $f: X \to Y$ in Proposition 1 of [5] can be replaced by multivalued mapping $F: X \to Y$.

Proposition 1. A multivalued mapping $F: X \to Y$ is weakly upper (lower) semi-quasicontinuous on X if and only if there exists a base $\mathscr{B}$ of open sets in X such that for any $B \in \mathscr{B}$ the restriction F/B is weakly upper (lower) somewhat continuous.

Proof. For the proof see [4, lemma].

Definition 4 [3]. A space X is said to be quasiregular if for any nonempty open set $G \subset X$ there exists a nonempty open set $D \subset G$ such that $\bar{D} \subset G$.

Definition 5 [6]. A family $\mathscr{S}$ of subsets of X is upper (lower) dense in X if and only if for any nonempty open set $G \subset X$ there exists a set $S \in \mathscr{S}$ such that $S \subset G$ ($S \cap G \neq \emptyset$).

The following proposition is a slight improvement of Proposition 2 in [5].

Proposition 2. Let Y be a quasiregular space and $F\colon X \to Y$ a multivalued mapping such that the family $F(X)$ of subsets of Y is upper (lower) dense in Y. Then F is upper (lower) somewhat continuous if and only if it is weakly upper (lower) somewhat continuous.

Proof. Necessity is evident. We shall prove the sufficiency. Let V be an open set of Y suchthat $F^+(V) \neq \emptyset$. By the quasiregularity of Y there exists an open set $D \subset V$ such that $\bar{D} \subset V$. Since the family $F(X)$ is upper dense in Y there exists $S \in F(X)$ such that $S \subset D$. Hence $F^+(D) \neq \emptyset$, from upper weak somewhat continuity we have $\emptyset \neq \operatorname{Int} F^+(\bar{D}) \subset \operatorname{Int} F^+(V)$. Therefore F is upper somewhat continuous. In the case that F is lower somewhat continuous the proof is analogous.

Example 2 in [5] shows, that if the range of the mapping f is quasiregular space, then weak quasicontinuity does not imply quasicontinuity neither in the case when f is onto Y. The following example shows that a weak quasicontinuity and a weak continuity do not imply quasicontinuity and continuity either respectively in the case when f is one-to-one, onto Y.

Example 1. Let $X = (0, 1)$, where the topology of X consists of the sets of the form $G \cup N$, where G is an open set in the usual topology for $(0, 1)$ and $N \subset \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$. Put $Y = (0, 1)$ with the topology generated by the sets $\left\{1 - \frac{1}{n}\right\}$, $Y \setminus \left\{1 - \frac{1}{n}\right\}$ for $n = 2, 3, 4, \ldots$ and by the set Q of the rationals.

Let G be any open set of Y, then G contains some point in the shape $1 - \frac{1}{n}$. The sets $\left\{1 - \frac{1}{n}\right\}$ for $n = 2, 3, 4, \ldots$ are open and closed simultaneously. Therefore Y is a quasiregular space. Let $f\colon X \to Y$ be defined as $f(x) = 1 - x$ for any $x \in X$. Then f is weakly continuous at any point $x \in X$. Indeed, if $x = \frac{1}{n}$ for $n = 2, 2, 4, \ldots$, it is clear that f is weakly continuous at the point x. Indeed, if $x = \frac{1}{n}$ for $n = 2, 3, 4, \ldots$, it is clear that f is weakly continuous at the point x. Suppose that $x \neq \frac{1}{n}$ for $n = 2, 3, 4, \ldots$, then for each open set V such that $f(x) \in V$ we have $Q \setminus \bigcup_{n=2}^{\infty} \left\{1 - \frac{1}{n}\right\} \subset V$. For $U \subset X$ it is enough to take an arbitrary open neighborhood of x which does not contain any point $\frac{1}{n}$ for $n = 2,3,4,\ldots$ and we get $f(U) \subset Q \setminus \bigcup_{n=2}^{\infty} \left\{1 - \frac{1}{n}\right\} \subset \bar{V}$. Hence f is a weakly continuous mapping at any $x \in X$. From the fact that for any open set U which does not contain any point $\frac{1}{n}$ for $n=2, 3, 4, \ldots$ there exists no nonempty open set $G \subset U$ such that $f(G) \subset Q$ it follows that f is not quasicontinuous at the point $x \in X$ if $x=\frac{1}{n}$ for $n=2, 3, \ldots$.

In the paper [5] (Theorem 2) the following characterization of a normal space is given. A space Y is normal if and only if any of the following conditions is true for any space X.

(i) If F: $X \to Y$ is a closed valued multifunction, the F is upper somewhat continuous if and only if it is weakly upper somewhat continuous.

(ii) If F: $X \to Y$ is a closed valued multifunction, then F is upper semi-quasicontinuous if and only if it is wealky upper semi-quasicontinuous.

The following theorem shows that the normal spaces can be characterized using the equivalence of the upper semicontinuity and the weak upper semi-continuity in the case of multivalued mappings.

Theorem 1. A space Y is normal if and only if the following condition is true for any space X. If F: $X \to Y$ is a closed valued multifunction, then F is upper semi-quasicontinuous if and only if it is weakly upper semi-continuous.

Proof. Necessity. Let $x \in X$ be any point, V an open set of Y such that $F(x) \subset V$. Since $F(x)$ is a closed set, by the normality there exists an open set G such that
$F(x) \subset G \subset \bar{G} \subset V$. By the weak upper semi-continuity there exists a neighborhood U of x such that for any $y \in U$ $F(y) \subset \bar{G} \subset V$. This implies that F is upper semi-continuous at x.

Sufficiency. It is enough to show that if Y is not a normal space, then there exists a space X and a closed valued multifunction F: $X \to Y$ such that F is weakly upper semi-continuous but not upper semi-continuous.

Let Y be not a normal space. Then there exists a closed set $K \subset Y$ and an open set $G \supset K$ such that for every open set $H \supset K$ we have $\bar{H}^Y \cap (Y \setminus G) \neq \emptyset$. Put $X=Y$ and define the topology on X as follows. A set $A \subset X$ is open if and only if $A \subset X \setminus K$ or there exists the set $P \subset X$ such that $K \subset \mathrm{Int}_Y P$ and $\bar{P}^Y \subset A$. Define the multifunction F: $X \to Y$ as $F(x)=\overline{\{x\}}^Y \cup K$ for any $x \in X$. Then F is a closed valued multifunction. Let $x \in X$ be any point, V an open set in Y containing $F(x)$. Then there exists a neighborhood U in X of x such that for each $y \in U$, $F(y) \subset \bar{V}^Y$. Take $U=\bar{V}^Y$. Thus F is weakly upper semi-continuous at any point $x \in X$. Suppose that F is upper semi-continuous at some point $x \in K$. Then for any open set V containing $F(x)$ there exists a neighborhood U of x such that for each $y \in U$ we have $F(y) \subset V$. Let us take $V=G$. Since $x \in U$ there exists a set M such that $K \subset \mathrm{Int}_Y M$ and $\bar{M}^Y \subset U$. Put $H=\mathrm{Int}_Y M$ then H is an open set of Y such that $K \subset H$ and $\bar{H}^Y \cap (Y \setminus G) \subset F(U) \cap (Y \setminus G) \subset G \cap (Y \setminus G)=\emptyset$. This is a contradition. We have shown that F is not upper semi-continuous because it is not upper semi-continuous at any $x \in K$.

In the case of a single valued mapping the equivalence of the mentioned types of continuities to their weak forms depends on the regularity of the space of values. The following characterization of regular spaces can be given.

Theorem 2. A space Y is regular if and only if any of the following conditions is true for any space X.

(i) If $f\colon X \to Y$ is a bijective mapping, then f is continuous if and only if it is weakly continuous.

(ii) If $f\colon X \to Y$ is an arbitrary mapping, then f is quasicontinuous if and only if it is weakly quasicontinuous.

(iii) If $f\colon X \to Y$ is an arbitrary mapping, then f is somewhat continuous if and only if it is weakly somewhat continuous.

Proof. The necessity of all the conditions is proved analogically as in the Theorem 1. The sufficiency of the condition (i) and conditions (ii), (iii) respectively is proved analogically as in the Theorem 1 and as in [5] (Theorem 2) respectively.

Kempisty's classical theorem on the quasicontinuity of separately quasicontinuous functions in [1] is generalized in various ways (see [3], [4], [7]). The following example shows that the condition in [4] (Theorem 2) that the space of values be regular cannot be replaced by the condition that the space of values be quasiregular.

Example 2. Let $X = Y = \langle -1, 1\rangle$ with the usual topology. Put $Z = \langle 0, 1\rangle$ where the open base $\mathcal{B}$ of topology on Z is defined as follows. A set $B \in \mathcal{B}$ if and only if it belongs to one of the six families of sets:

$$\mathcal{A}_1 = \left\{(a, b) \cup (1-b, 1-a) \ : \quad 0 < a < b < \frac{1}{2}\right\}$$

$$\mathcal{A}_2 = \{\langle 0, a) \cup (1-a, 1\rangle \ : \quad 0 < a < 1\}$$

$$\mathcal{A}_3 = \left\{\left(\frac{1}{2} - a, \frac{1}{2} + a\right) \ : \quad 0 < a \leq \frac{1}{2}\right\}$$

$$\mathcal{A}_4 = \left\{A \setminus T\colon A \in \mathcal{A}_1 \quad \text{and } T = \left\{\frac{2}{3}, \frac{3}{4}, \dots, \frac{n-1}{n}, \dots\right\}\right\}$$

$$\mathcal{A}_5 = \left\{A \setminus T\colon A \in \mathcal{A}_2 \quad \text{and } T = \left\{\frac{2}{3}, \frac{3}{4}, \dots, \frac{n-1}{n}, \dots\right\}\right\}$$

$$\mathcal{A}_6 = \left\{A \setminus T\colon A \in \mathcal{A}_3 \quad \text{and } T = \left\{\frac{2}{3}, \frac{3}{4}, \dots, \frac{n-1}{n}, \dots\right\}\right\}$$

To prove the quasiregularity of the space Z it is enough to prove that for any set $B \in \mathcal{B}$ there exists some set $D \in \mathcal{B}$ such that $\bar{D} \subset B$. Let $B \in \mathcal{B}$, then there exists a point $x \in B$ such that $\frac{1}{2} < x < 1$ and $x \notin T$. Since $x \notin T$ and $x \neq 1$, there exists $\varepsilon > 0$

such that $\langle x-\varepsilon, x+\varepsilon\rangle \subset B$ and in addition $\langle x-\varepsilon, x+\varepsilon\rangle \cap T=\emptyset$. Put $D=(1-x-\varepsilon,\ 1-x+\varepsilon)\cup(x-\varepsilon, x+\varepsilon)$. Suppose that $y\notin\langle 1-x-\varepsilon, 1-x+\varepsilon\rangle \cup \langle x-\varepsilon, x+\varepsilon\rangle$ then there exists $\delta>0$ such that

$$[(y-\delta, y+\delta)\cup(1-y-\delta, 1-y+\delta)]\cap[\langle 1-x-\varepsilon, 1-x+\varepsilon\rangle \cup\langle x-\varepsilon, x+\varepsilon\rangle]=\emptyset$$

hence $y\notin\bar{D}$. This implies that $\bar{D}\subset[\langle 1-x-\varepsilon,\ 1-x+\varepsilon\rangle \cup \langle x-\varepsilon, x+\varepsilon\rangle]\subset B$. Denote $A_n=\left\langle -\frac{1}{n}, \frac{1}{n}\right\rangle \times\left\langle -\frac{1}{n}, \frac{1}{n}\right\rangle$ for $n=1, 2, 3, \ldots$ and H is a dense set in $X\times Y$ such that $(0, 0)\in H$ and in addition $\overline{\overline{[\{x\}\times Y]\cap H}}=1$ and $\overline{\overline{[X\times\{y\}]\cap H}}=1$ for each $x\in X$ and $y\in Y$. In other words, each horizontal and vertical stright line has just one point common with the set H. *Let* $f\colon X\times Y\to Z$ be defined as $f(x, y)=\frac{1}{n}$ if $(x, y)\in(A_{n-1}\setminus A_n)\setminus H$, $f(x, y)=0$ if $(x, y)=(0, 0)$ and $f(x, y)=1-\frac{1}{n}$ if $(x, y)\in(A_{n-1}\setminus A_n)\cap H$ for $n=2, 3, 4, \ldots$ Let $y\in Y$ be any point. If $y=0$, it is easy to see that f_0 is quasicontinuous. Suppose that $y\neq 0$. Then there exists a point $(h, y)\in[X\times\{y\}]\cap H$. For the proof that f_y which is defined as $f_y(x)=\frac{1}{n+1}$ if $(x, y)\in A_n\setminus A_{n+1}$ but $x\neq h$ and $f_y(x)=1-\frac{1}{n+1}$ if $(x, y)\in A_n\setminus A_{n+1}$ and $x=h$ for $n=1, 2, 3, \ldots$ is quasicontinuous, it is enough to realize that for each open set V such that $1-\frac{1}{n+1}\in V$ we have $\frac{1}{n+1}\in V$. Then it is easy to see that f is quasicontinuous. Since f_x for each $x\in X$ is defined, analogically we have that f_x is also quasicontinuous for each $x\in X$. It is easy to verify that f is not somewhat continuous.

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Now we shall define the notion of a somewhat continuous mapping with respect to y, which is similar to the notion of a quasicontinuous mapping with respect to y, introduced by Kempisty in [1].

Definition 6. Let X, Y and Z be topological spaces. A mapping $f\colon X\times Y\to Z$ is said to be somewhat continuous with respect to y at a point $(p, q)\in X\times Y$ if for any open set V such that $f(p, q)\in V$ we have $(X\times\{q\})\cap\operatorname{Int} f^{-1}(V)\neq\emptyset$. It is said to be somewhat continuous with respect to y if it is somewhat continuous with respect to y at any $(p, q)\in X\times Y$.

In an analogical way there is defined the notion of the somewhat continuity with respect to x. Obviously any somewhat continuous mapping with respect to one of the variables is somewhat continuous.

The following result is known (see [8]): Let X be a Baire space, Y be the first countable and Z a regular space. If $f\colon X\times Y\to Z$ is a mapping such that f_x is continuous for all $x\in X$ and f_y is quasicontinuous for all $y\in Y$, then f is quasicontinuous with respect to y.

The following example shows that if f_y is somewhat continuous for all $y\in Y$ and f_x is continuous for all $x\in X$, then f need not be somewhat continuous with respect to y.

Example 3. Let $X=Y=(-\infty,\infty)$, let the family $\mathcal{B}=\{(-\infty,0),\{0\},(a,b)\colon 0\leq a<b\}$ be a base for topology on X and the family $\mathcal{S}=\{\langle -a,a\rangle,(-\infty,-a),(a,\infty)\colon a>0\}$ be a subbase for topology on Y. It is easy to see that X is a Baire space and Y is the first countable. Put $Z=\{0,1\}$ with the discrete topology. It is clear that Z is a regular space. Let $f\colon X\times Y\to Z$ be defined as $f(x,y)=0$ if $x=0$ if $x<0$ and $|y|>\frac{-1}{x}$ or if $x>0$ and $|y|\leq\frac{1}{x}$ and let $f(x,y)=1$ if $x<0$ and $|y|\leq\frac{-1}{x}$ or if $x>0$ and $|y|>\frac{1}{x}$.

Let $y\in Y$ be any point. If $y\neq 0$, then $f_y(x)=0$ if $x\in A$ where $A=\left(-\infty,-\frac{1}{|y|}\right)\cup\left\langle 0,\frac{1}{|y|}\right\rangle$, and $f_y(x)=1$ if $x\in B$ where $B=\left\langle -\frac{1}{|y|},0\right)\cup\left(\frac{1}{|y|},\infty\right)$. From the fact that $\operatorname{Int} A\neq\emptyset$ and $\operatorname{Int} B\neq\emptyset$ it results that f_y is somewhat continuous. If $y=0$, then $f_0(x)=0$ if $x\in\langle 0,\infty)$, and $f_0(x)=1$ if $x\in(-\infty,0)$. Since $\langle 0,\infty)$ and $(-\infty,0)$ are open sets, hence $f_0(x)$ is also somewhat continuous. Let $x\in X$ be any point. If $x<0$, then $f_x(y)=0$ if $x\in\left(-\infty,\frac{1}{x}\right)\cup\left(-\frac{1}{x},\infty\right)$, and $f_x(y)=1$ if $x\in\left\langle\frac{1}{x},-\frac{1}{x}\right\rangle$. It is easy to verify that the inverse image of each open set in Z is an open set in Y. If $x>0$, the situation is similar. If $x=0$, then $f_0(y)=1$ for all $y\in Y$. Hence $f_x(y)$ is continuous for all $x\in X$. We shall prove that f is not somewhat continuous with respect to y at any point $(x,0)$ where $x<0$. Suppose the opposite of this statement, then $[X\times\{0\}]\cap\operatorname{Int} f^{-1}(\{1\})\neq\emptyset$. Then there exists $(x_0,0)$ such that $(x_0,0)\in[X\times\{0\}]\cap\operatorname{Int} f^{-1}(\{1\})$ hence $(x_0,0)$ is the interior point of $f^{-1}(\{1\})$ and so there exists an open set $U\subset Y$ such that $0\in U$ and $(-\infty,0)\times U\subset\operatorname{Int} f^{-1}(\{1\})$. This is a contradiction to the definition of the mapping f.

The next example shows that a mapping $f\colon X\times Y\to Z$ such that f_x and f_y are quasicontinuous for each $x\in X$ and $y\in Y$ need not be somewhat continuous with respect to y and to x respectively.

Example 4. Let $X=Y=Z=\langle -1,1\rangle$ with usual topology. Let $f\colon X\times Y\to Z$ be defined as $f(x,y)=0$ if $x<0$ and $y>0$ or f $x>0$ and $y<0$ and $f(x,y)=1$ if $x\leq 0$ and $y\leq 0$ or f $x\geq 0$ and $y\geq 0$. It is easy to verify that f_x and f_y are quasicontinuous for each $x\in X$ and $y\in Y$ but the mapping f is not somewhat continuous with respect to y and to x respectively at the point $(0,0)$.

It is clear that any mapping which is quasicontinuous with respect to one of the variables is somewhat continuous with respect to this variable. The converse of this statement is not true. The following example shows that a mapping somewhat continuous with respect to x and y such that f_x and f_y are quasicontinuous need not be quasicontinuous with respect to y or with respect to x.

Example 5. Let X, Y and Z have the same meaning as in Example 4. Let $f: X\times Y\to Z$ be defined as $f(x, y)=0$ if $x<-\frac{1}{2}$, if $-\frac{1}{2}\leq x<0$ and $y\in\left(-1, -\frac{1}{2}\right)\cup\left(0, \frac{1}{2}\right)$ or if $0\leq x<\frac{1}{2}$ and $y<0$ and let $f(x, y)=1$ if $x\geq\frac{1}{2}$, if $0\leq x\leq\frac{1}{2}$ and $y>0$ or if $-\frac{1}{2}\leq x<0$ and $y\in\left\langle -\frac{1}{2}, 0\right\rangle\cup\left\langle\frac{1}{2}, 1\right\rangle$. It is easy to verify that f is somewhat continuous with respect to y and x and in addition f_x and f_y are quasicontinuous for each $x\in X$ and $y\in Y$ but it is not quasicontinuous with respect to y and to x respectively at the point $(0, 0)$.

The following example shows that a mapping somewhat continuous with respect to y and x such that f_x is continuous for all $x\in X$ need not be quasicontinuous with respect to y.

Example 6. Let $X=Z=\langle -1, 1\rangle$ with the usual topology. Put $Y=\langle -1, 1\rangle$ where the family $\mathscr{B}=\{\langle -1, 0\rangle, (a, b), (b, 1\rangle : 0\leq a<b<1\}$ is a base for topology on Y. Let $f: x\times Y\to Z$ be defined as $f(x, y)=0$ if $x\in\left(-\frac{1}{2}, \frac{1}{2}\right)\setminus\{0\}$ and $y\leq 0$ and let $f(x, y)=1$ if $x\in\left\langle -1, -\frac{1}{2}\right\rangle\cup\left\langle\frac{1}{2}, 1\right\rangle\cup\{0\}$ or if $x\in\left(-\frac{1}{2}, \frac{1}{2}\right)\setminus\{0\}$ and $y>0$. It is easy to verify that f is somewhat continuous with respect to y and x such that f_x is continuous for all $x\in X$ but it is not quasicontinuous with respect to y and to x respectively at the point $(0, 0)$.

The relation between somewhat continuity with respect to one of the variables and quasicontinuity with respect to this variable may be described in an analogical way as it is described in the case of somewhat continuity and quasicontinuity in [4]. This is shown by the following proposition.

Proposition 3. Let X, Y and Z be topological spaces. Then a mapping $f: X\times Y\to Z$ is quasicontinuous with respect to y if and only if there exists a base $\mathscr{B}$ of the space X such that for any element $B\in\mathscr{B}$ the restriction $f/B\times Y$ is somewhat continuous with respect to y.

Proof. Necessity. Let $B\in\mathscr{B}$ and (x, y) be any point of $B\times Y$. Since f is quasicontinuous with respect to y, there exist nonempty open sets $G\subset B$ and $U\subset Y$, $y\in U$ such that $f(G\times U)\subset V$ hence $B\times\{y\}\cap\text{Int}\,(f/B\times Y)^{-1}(V)\neq\emptyset$.

Sufficiency. Let $(x, y)\in X\times Y$ be any point, H be any open set of $X\times Y$ containing (x, y) and V be any open set of Z containing $f(x, y)$. Let $B\in\mathscr{B}$ be such that $x\in B$ and $B\times\{y\}\subset H$. Consider the restriction $f/B\times Y$. According to the assumption we have $B\times\{y\}\cap\text{Int}\,(f/B\times Y)^{-1}(V)\neq\emptyset$.

Let $(x_0, y) \in B \times \{y\} \cap \text{Int}\,(f/B \times Y)^{-1}(V)$, hence (x_0, y) is an interior point of $H \cap \text{Int}\,(f/B \times Y)^{-1}(V)$ then there exist open sets $G \subset B$, $x_0 \in G$ and $U \subset Y$, $y \in U$ such that $f(G \times U) \subset V$. The quasicontinuity with respect to y is proved.

In the case of a multivalued mapping we define the notions of the upper and lower quasicontinuity with respect to one of the variables as follows.

Definition 7. Let X, Y and Z be topological spaces. A multivalued mapping $F: X \times Y \to Z$ is said to be upper (lower) semi-quasicontinuous with respect to y at a point $(p, q) \in X \times Y$ if for any open set V such that $F(p, q) \subset V$ $(F(p, q) \cap V \neq \emptyset)$ and for any neighborhood $G \times U$ of (p, q) there exists a neighborhood U' of q contained in U and nonempty open set $G' \subset G$ such that $F(x, y) \subset V$ $(F(x, y) \cap V \neq \emptyset)$ for all $(x, y) \in G' \times U'$. It is said to be upper (lower) semi-quasicontinuous with respect to y if it is upper (lower) semi-quasicontinuous with respect to y at any $(p, q) \in X \times Y$.

Analogically we define the notion of upper (lower) semi-quasicontinuity with respect to x.

Obviously any upper (lower) semi-quasicontinuous multivalued mapping with respect to one of the variables is upper (lower) semi-quasicontinuous.

The following theorems generalize some results of [3] (Theorem 4) and as a special case we obtain Theorem 1 from [8].

Theorem 3. Let X be a Baire space, Y the first countable and Z a normal space. If $F: X \times Y \to Z$ is a closed valued multifunction such that F_x is upper semi-continuous for all $x \in X$ and F_y is upper and lower semi-quasicontinuous for all $y \in Y$ then F is upper semi-quasicontinuous with respect to y.

Proof. Suppose that F is not upper semi-quasicontinuous with respect to y at $(p, q) \in X \times Y$, hence there exists an open set $V \subset Z$, $F(p, q) \subset V$ and a neighborhood $G \times U$ of (p, q) such that for each neighborhood U' of q, $U' \subset U$ and for each nonempty open set $G' \subset G$ there exists a point $(x_0, y_0) \in G' \times U'$ such that $F(x_0, y_0)$ is not the subset of V. The set $F(p, q)$ is closed. By the norality of Z there exists an open set V_1 such that $F(p, q) \subset V_1$ and $\bar{V}_1 \subset V$. Since F_q is upper semi-quasicontinuous at p, there exists a nonempty open set $H \subset G$ such that $F_q(x) \subset V_1$ for each $x \in H$.

Let $\{U_n\}_{n=1}^{\infty}$ be the elements of a local base at q such that $U_n \subset U$ for each $n = 1, 2, 3, \ldots$. Put $A_n = \{x \in H: F(x, y) \subset V_1 \text{ for each } y \in U_n\}$. Let $x \in H$ be any point. Since F_x is upper semi-continuous, there exists an open set $K \subset Y$ such that $q \in K$ and for each $y \in K$ we have $F(x, y) \subset V_1$. But there exists n such that $U_n \subset K$. This implies that $x \in A_n$. Thus $H = \bigcup_{n=1}^{\infty} A_n$. Let $W \subset H$ be any open set. Take $W \times U_n$ for given n. According to the assumption we have that there exists $(x_0, y_0) \in W \times U_n$ such that $F(x_0, y_0) \not\subset V$, hence we have $\emptyset \neq F(x_0, y_0) \cap (Z \setminus V) \subset F(x_0, y_0) \cap (Z \setminus \bar{V}_1)$. Since F_{y_0} is lower semi-quasicontinuous, there exists a nonempty open set $W_1 \subset W$ such that $F_{y_0}(x) \cap (Z \setminus \bar{V}_1) \neq \emptyset$ for each $x \in W_1$. This implies that $F(x, y_0) \not\subset V_1$ for each $x \in W_1$. Since $y_0 \in U_n$ we have $x \notin A_n$. Thus $W_1 \cap A_n = \emptyset$. This means that A_n is nowhere dense and the set $H = \bigcup_{n=1}^{\infty} A_n$ is of the first category. This is a contradiction.

The following example shows that the condition that F_y is lower semi-quasicontinuous for all $y \in Y$ is essential.

Example 7. Let $X = Y = Z = \langle -1, 1 \rangle$ with usual topology. Let H have the same meaning as in Example 2. Put F: $X \times Y \to Z$ as follows:

$$F(x, y) = 0 \qquad \text{if } (x, y) \in (X \times Y) \setminus H$$
$$F(x, y) = \langle -1, 1 \rangle \quad \text{if } (x, y) \in H$$

It is easy to verify that F_x and F_y are upper semi-continuous for each $x \in X$ and $y \in Y$ but F is not upper semi-quasicontinuous. Thus it is not upper semi--quasicontinuous with respect to y.

Theorem 4. Let X be a Baire space, Y the first countable and Z a regular space. If F: $X \times Y \to Z$ is a multivalued mapping such that F_x is lower semi-continuous for all $x \in X$ and F_y is lower and upper semi-quasicontinuous for all $y \in Y$, then F is lower semi-quasicontinuous with respect to y.

Proof. Suppose that F is not lower semi-quasicontinuous with respect to y at $(p, q) \in X \times Y$, hence there exists an open set V, $F(p, q) \cap V \neq \emptyset$ and a neighborhood $G \times U$ of (p, q) such that for each neighborhood U' of q, $U' \subset U$ and for each nonempty open set $G' \subset G$ there exists a point $(x_0, y_0) \in G' \times U'$ such that $F(x_0, y_0) \cap V = \emptyset$. Since $F(p, q) \cap V \neq \emptyset$ there exists $z \in Z$ such that $z \in F(p, q) \cap V$, by the regularity of Z there exists an open set V_1 such that $z \in V_1$ and $\bar{V}_1 \subset V$. Since F_q is lower semi-quasicontinuous at p, there exists an open set $H \subset G$ such that $F_q(x) \cap V_1 \neq \emptyset$ for each $x \in H$. Let $\{U_n\}_{n=1}^{\infty}$ be the elements of a local base at q such that $U_n \subset U$ for each $n = 1, 2, 3, \ldots$. Put $A_n = \{x \in H\colon F(x, y) \cap V_1 \neq \emptyset$ for each $y \in U_n\}$. Let $x \in H$ be any point. Since F_x is lower semi-continuous, there exists an open set $K \subset Y$ such that $q \in K$ and for each $y \in K$ we have $F(x, y) \cap V_1 \neq \emptyset$. But there exists n such that $U_n \subset K$. This implies that $x \in A_n$ and thus $H = \bigcup_{n=1}^{\infty} A_n$. Let $W \subset H$ be any open set. Take $W \times U_n$ for given n. According to the assumption we have that there exists $(x_0, y_0) \in W \times U_n$ such that $F(x_0, y_0) \cap V = \emptyset$. Hence

$$F(x_0, y_0) \subset Z \setminus V \subset Z \setminus \bar{V}_1.$$

Since F_{y_0} is upper semi-quasicontinuous there exists a nonempty set $W_1 \subset W$ such that $F_{y_0}(x) \subset Z \setminus \bar{V}_1$ for each $x \in W_1$. Since $y_0 \in U_n$ we have $x \notin A_n$. Thus $W_1 \cap A_n = \emptyset$. This means that A_n is nowhere dense and the set $H = \bigcup_{n=1}^{\infty} A_n$ is of the first category. This is a contradiction.

The following example shows that the condition that F_y is upper semi-quasicontinuous for all $y \in Y$ is essential.

Example 8. Let X, Y, Z and H have the same meaning as in Example 7. Put F: $X \times Y \rightarrow Z$ as follows: $F(x, y) = 0$ if $(x, y) \in H$ and $F(x, y) = \langle -1, 1 \rangle$ if $(x, y) \in (X \times Y) \setminus H$. It is easy to verify that F_x and F_y are lower semi-continuous for each $x \in X$ and $y \in Y$ but F is not lower semi-quasiconstinuous. Thus it is not lower semi-quasicontinuous with respect to y.

Corollary 1. Let X, Y be first countable Baire spaces and Z a normal space. If F: $X \times Y \rightarrow Z$ is a closed valued multifunction such that F_x and F_y are upper and lower semi-continuous for each $x \in X$ and $y \in Y$, then F is upper and lower semi-quasicontinuous with respect to y and x.

Remark 2. The converse of the Corollary 1 is not true as it is shown in [3] (Example 2).

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SÚHRN

SLABÉ FORMY SPOJITOSTI A SÚČINY PRIESTOROV

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Práca sa zaoberá otázkami niektorých typov zovšeobecnenej spojitosti funkcií a multifunkcií a k nim prislúchajúcich slabých variantov spojitosti. Pomocou týchto pojmov sa okrem iného, udáva charakterizácia normálnych a regulárnych topologických priestorov. V prípade, že definičný obor je súčin priestorov je zavedená nová trieda funkcií tzv. skoro spojité funkcie vzhľadom na jednu premennú.