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A UNIFIED APPROACH TO THE TRANSFINITE CONVERGENCE AND GENERALIZED CONTINUITY

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The problems, as to whether a given set of functions is closed with respect to the transfinite convergence, were studied in various papers (see [6, 8, 9, 11, 12, 15]). The sets of functions which are considered are mostly sets of functions which are continuous in some generalized sense. Most of the known results concern functions on metric spaces.

The aim of the present paper is to give such results for general topological spaces. On the other hand we want to unify some methods which are used in the theory of the transfinite convergence of functions.

Basic notions and some results

Throughout all the paper a transfinite sequence is a sequence $\{a_\xi\}_{\xi<\Omega}$, where Ω is the first uncountable ordinal number. If the members a_ξ belong to a topological space, then we define the transfinite convergence as follows.

(I) A transfinite sequence $\{a_\xi\}_{\xi<\Omega}$ of elements of a topological space X is said to be convergent to an element $a \in X$ (we write $\lim\limits_{\xi<\Omega} a_\xi = a$), if to any neighbourhood U of a there exists $\xi_0 < \Omega$ such that $a_\xi \in U$ for any $\xi \geqq \xi_0$.

The following lemma was proved by Sierpiński ([13]) for transfinite sequences of real numbers. In the form stated here it may be found e. g. in [6].

Lemma 1. Let X be first-countable T_1-space and $\{a_\xi\}_{\xi<\Omega}$, $a_\xi \in X$ a transfinite sequence of elements convering to $a \in X$. Then there exists $\xi_0 < \Omega$ such that $a_\xi = a$ for $\xi \geqq \xi_0$.

The preceding lemma motivates a new definition of the transfinite convergence, which may be used in any abstract set. The idea of such definition is due to J. S. Lipiński [8].

(II) Let X be a set and $\{a_\xi\}_{\xi<\Omega}$ a transfinite sequence of its elements. Then $\{a_\xi\}_{\xi<\Omega}$ is said to be convergent to an element $a \in X$, if there exists $\xi_0 < \Omega$ such, that $a_\xi = a$ for $\xi \geqq \xi_0$.

The following proposition follows immediately from Lemma 1.

Proposition 1. Let X be first-countable T_1-space. Then the convergences in both the senses (I), (II) coincide.

We give two simple examples, showing that the assumptions on X in Proposition 1 are essential.

Example 1. Let $X = \{a, b\}$ and $\tau = \{\emptyset, X\}$ a topology on X. Putting $a_\xi = a$ for $\xi < \Omega$ we see that $\{a_\xi\}_{\xi<\Omega}$ converges in the sense (I) both to a and b, while in the sense (II) the convergence is uniquely determined. The space X in this case is not a T_1-space.

Example 2. Consider $X = \{\xi: \xi \leqq \Omega\}$ with the order topology (see [4], p. 87). Put $a_\xi = \xi$ for $\xi < \Omega$. The sequence $\{a_\xi\}_{\xi<\Omega}$ converges in the sense (I) to Ω, while it does not converge in the sense (II).

Begining from now, if a topology on X is given, the convergence always means convergence in the sense (I). Of course if the topology will not be given on X the convergence is taken in the sense (II). In all topological cases where the convergence of $\{a_\xi\}_{\xi<\Omega}$ will be considered, the spaces will be first-countable and T_1, so there will be no misunderstandings.

Let X, Y be two sets, $\{f_\xi\}_{\xi<\Omega}$ a sequence of functions $f_\xi: X \to Y$. The (pointwise) convergence of $\{f_\xi\}_{\xi<\Omega}$ to a function $f: X \to Y$ means that $\lim_{\xi<\Omega} f_\xi(x) = f(x)$ for every $x \in X$ (shortly $\lim_{\xi<\Omega} f_\xi = f$).

From the definition of the pointwise convergence and from a property of the set of all ordinal numbers $\xi < \Omega$ the following known theorem follows (For the case of real functions see [13]).

Lemma 2. Let $f_\xi: X \to Y$, $f: X \to Y$ be such that $\lim_{\xi<\Omega} f_\xi = f$. Let $M \subset X$ be a countable set in X. Then there exists $\xi_0 < \Omega$ such that $f_{\xi_0}(x) = f(x)$ for $\xi \geqq \xi_0$ and every $x \in M$.

Continuity

To unify various results on transfinite convergence and continuity and to obtain some new results, we chose the following approach.

Suppose Y to be a set. Let to any $y \in Y$ a collection $\mathcal{S}_y$ of subsets of Y be given such that the following is true:

$$S \in \mathscr{S}_y \Rightarrow y \in S$$

Given such a collection we say that Y is an $\mathscr{S}$-space.

Let X be a topological space and Y an $\mathscr{S}$-space and $f\colon X \to Y$ a mapping. Then f is said to be $\mathscr{S}$-continuous at $x \in X$ if for any $S \in \mathscr{S}_{f(x)}$ there exists a neighbourhood U of x such that $f(U) \subset S$. If f is $\mathscr{S}$-continuous at any $x \in X$, we say that it is $\mathscr{S}$-continuous.

Remark 1. Obviously if both X and Y are topological space and if for any $y \in Y$ the collection $\mathscr{S}_y$ is the collection of all (open) neighbourhoods of y, then $\mathscr{S}$-continuity coincides with the usual notion of continuity.

Theorem 1. Let X be a first-countable topological space, Y an $\mathscr{S}$-space. Let $f_\xi\colon X \to Y(\xi < \Omega)$ be $\mathscr{S}$-continuous and let $\{f_\xi\}_{\xi<\Omega}$ converge to f. Then f is $\mathscr{S}$-continuous.

Proof. Suppose f not to be $\mathscr{S}$-continuous at x_0. Then there exists $S \in \mathscr{S}_{f(x_0)}$ such that for any neighbourhood U of x_0 there is $x \in U$ such that $f(x) \notin S$. Since X is first countable, there exists a countable basis $\{U_n\}_{n=1}^{\infty}$ of neighbourhoods of x_0 and we can choose $x_n \in U_n$ such that $f(x_n) \notin S$ for $n = 1, 2, \ldots$ By Lemma 2 we can choose $\xi_0 < \Omega$ such that

$$f_\xi(x_n) = f(x_n) \quad \text{for} \quad n = 0, 1, 2, \ldots, \ \xi \geqq \xi_0. \tag{1}$$

By $\mathscr{S}$-continuity of f_{ξ_0} at x_0 there exists a neighbourhood U of x_0 such that $f_{\xi_0}(U) \subset S$. Hence for some n $f_{\xi_0}(U_n) \subset f_{\xi_0}(U) \subset S$. Thus $f_{\xi_0}(x_n) \in S$. The last and (1) give $f(x_n) \in S$. It is a contradiction.

Corollary 1. Let X be a first-countable topological space, Y first-countable T_1-space. Let $\{f_\xi\}_{\xi<\Omega}$ be a transfinite sequence of continuous functions defined on X with values in Y and converging to f. Then f is continuous.

Proof. It follows directly from Theorem 1 (see also Remark 1).

Remark 2. Corollary 1 covers some known results on transfinite convergence of continuous functions (see [12, 13, 15]).

In [12] an example was given, showing that there exists a topological space, and a sequence $\{f_\xi\}_{\xi<\Omega}$ of functions $f_\xi(\xi < \Omega)$, $f_\xi\colon X \to R$, which are continuous and the limit function is not continuous. The example was given by means of the continuum hypothesis. The following example shows the same fact without the continuum hypothesis.

Example 3. Let $X = \{\xi\colon \xi \leqq \Omega\}$ with the order topology (see [4] p. 87). Let $\{\eta_\xi\}_{\xi<\Omega}$, $\eta_\xi < \Omega$ be a transfinite sequence of non-limit ordinal numbers such that $\{\eta_\xi\}_{\xi<\Omega}$ is cofinal with Ω. For $\xi < \Omega$ define $f_\xi\colon X \to R$ as

$$f_\xi(x) = \begin{cases} 1, & \text{if } x \leqq \eta_\xi \\ 0, & \text{if } x > \eta_\xi \end{cases}$$

Then $\lim\limits_{\xi<\Omega} f_\xi(x)=1$, if $x \in X$, $x \neq \Omega$, $\lim\limits_{f_\xi<\Omega} \xi(\Omega)=0$. The limit function is not continuous, because it is not continuous at Ω, while f_ξ $(\xi<\Omega)$ are continuous.

Remark 3. Note that the limit function in Example 3 is not even weakly continuous, while $f_\xi(\xi<\Omega)$ are obviously weakly continuous.

There are also another types of continuities for which the preserving under transfinite convergence may be proved by means of Theorem 1.

If X, Y are topological spaces then a function $f\colon X \to Y$ is said to be continuous in the sense of Singal (see [13]) at a point $x \in X$, if for any neighbourhood V of $f(x)$ there exists a neighbourhood U of x such that $f(U) \subset (\bar{V})^0$ (E^0 denotes the interior of a set E). The continuity in the sense of Singal (on X) is defined in an obvious way.

One can see that the last notion of continuity may be obtained as $\mathscr{S}$-continuity where for any $y \in Y$ we take $\mathscr{S}_y$ as the set of all $(\bar{V})^0$ where V is a neighbourhood of y. So from Theorem 1 we obtain the following.

Corollary 2. Let X be first-countable, Y first countable and T_1. Let $f_\xi\colon X \to Y(\xi<\Omega)$ be continuous in the sense of Singal. Let $\lim\limits_{\xi<\Omega} f_\xi = f$. Then f is continuous in the sense of Singal.

Quasicontinuity

The classical notion of quasicontinuity of a function $f\colon X \to Y$, where X, Y are topological spaces (see [4], [7]) may be formulated also if Y is an $\mathscr{S}$-space.

Suppose that X is a topological space, Y an $\mathscr{S}$-space. A function $f\colon X \to Y$ is said to be $\mathscr{S}$-quasicontinuous at $x_0 \in X$, if for any $S \in \mathscr{S}_{f(x_0)}$ and any neighbourhood U of x_0 there exists a nonempty open set $G \subset U$ such that $f(G) \subset \mathscr{S}_{f(x_0)}$. It is said to be $\mathscr{S}$-quasicontinuous, if it is $\mathscr{S}$-quasicontinuous at any $x \in X$.

Remark 4. If both X, Y are topological spaces and if we take for any $y \in Y$ the collection $\mathscr{S}_y$ to be the collection of all neighbourhoods of y, then the corresponding $\mathscr{S}$-quasicontinuity coincides with the usual quasicontinuity.

Remark 5. If X, Y are topological spaces and if for $y \in Y$ we take $\mathscr{S}_y$ as the set of all closures $\bar{V}$ of all neighbourhoods V of y, we obtain weak quasicontinuity.

A topological space will said to be strongly separable if any of its subspace is separable (in the relative topology). It is said to be strongly locally separable, if for any $x \in X$ there exists a neighbourhood U of x, which is a strongly separable subspace of X.

Theorem 2. Let X be strongly locally separable space, Y an $\mathscr{S}$-space. Let $\{f_\xi\}_{\xi<\Omega}$ be a transfinite sequence of $\mathscr{S}$-quasicontinuous functions $f_\xi\colon X \to Y$ converging to f. Then f is $\mathscr{S}$-quasicontinuous.

Proof. Suppose f not be $\mathscr{S}$-quasicontinuous at $x_0 \in X$. Then there exists

$S \in \mathscr{S}_{f(x_0)}$ and a neighbourhood U of x_0 such that for any nonempty open $G \subset U$ there exists a point $x \in G$ for which $f(x) \notin S$. Since X is strongly locally separable, we may suppose U to be strongly separable. The set

$$M = \{x: x \in U, f(x) \notin S\} \tag{2}$$

is dense in U. Hence M contains a countable dense set D, which is dense in U. Since $\{f_\xi\}_{\xi<\Omega}$ is convergent on $D \cup \{x_0\}$, we have from Lemma 2

$$f_\xi(x) = f(x) \text{ for any } x \in D \cup \{x_0\} \text{ and any } \xi \geqq \xi_0,$$

where $\xi_0 < \Omega$ is suitably chosen.

The $\mathscr{S}$-quasisontinuity of f_{ξ_0} at x_0 implies that a nonempty open set $W \subset U$ exists such that $f_{\xi_0}(W) \subset S$. Since $W \cap D \neq \emptyset$, there exists $x \in W \cap D$ with $f_{\xi_0}(x) \in S$. Hence, by (3), $f(x) \in S$. Since $x \in M$, it is a contradiction (see (2)).

If X satisfies the first-countability axiom we can omit in Theorem 2 the assumption of strong local separability of X. We obtain

Theorem 3. Let X be locally separable, first countable topological space. Then, under the same assumptions on Y and $f_\xi(\xi < \Omega)$ as in Theorem 2, the limit function f is $\mathscr{S}$-quasicontinuous.

Proof. Suppose f not to be $\mathscr{S}$-quasicontinuous at x_0. Then a set $S \in \mathscr{S}_{f(x_0)}$ and a neighbourhood U of x_0 exists such that for any nonempty open $G \subset U$ there is $x \in G$ with $f(x) \notin S$. U may be supposed separable. Let M be defined by (2) as in the proof of Theorem 2. Let D be any countable dense set in U. For $s \in D$ let $\{B_n^s\}$ $(n = 1, 2, \dots)$ be a countable basis of neighbourhoods of s. We may suppose $B_n^s \subset U$. Hence there is $x_n^s \in B_n^s$ such that $x_n^s \in M$. The set

$$T = \{x_n^s: s \in D \; n = 1, 2, \dots\}$$

is a countable dense set in U. Using Lemma 2 we obtain $\xi_0 < \Omega$ such that

$$f_\xi(x) = f(x) \text{ for any } x \in T \cup \{x_0\} \text{ and any } \xi \geqq \xi_0$$

Now we use the quasicontinuity of f_{ξ_0} at x_0 and the proof ends in the same way as the proof of Theorem 2.

Corollary 3. Let X be either strongly locall separable or locally separable and first countable topological space. Let Y be first countable T_1-space, $f_\xi: X \to Y(\xi < \Omega)$ quasicontinuous functions such that $\lim_{\xi<\Omega} f_\xi = f$. Then f is quasicontinuous.

Proof. It follows from Theorem 1 and 2 (see also Remark 5).

Note that Corollary 1 generalizes the result on quasicontinuity of the limit of a transfinite sequence of quasicontinuous functions defined on a locally separable metric space (see [10]).

As another corollary we way obtain a new result concerning transfinite convergence of weakly quasicontinuous functions.

Corollary 4. Let X, Y satisfy the assumptions of Corollary 1. Let f_ξ: $X \to Y$, $\xi < \Omega$, by weakly quasicontinuous and converging to f. Then f is weakly quasicontinuous.

Note that the assumptions of strong local separability and local separability and fist countability appearing in Theorem 2 and Theorem 3 respectively, are independent each of other. One can see it from the following examples.

Example 4. Let X be an uncountable set with a topology consisting of all the sets with finite complements and of the empty set (cofinite fopology). It is easy to see that X is strongly separable, but not first countable.

Example 5. Let X be the set of all ordered pairs of real numbers. Define the topology by means of the basis of neighbourhoods in each point $(a, b) \in X$. As a basis at each $(a, b) \in X$ we take the intervals $\langle a, a+\varepsilon) \times \langle b, b+\eta)$, where $\varepsilon > 0$, $\eta > 0$. The space is evidently first countable. But if we take any $(a, b) \in X$ and any neighbourhood U of (a, b), then there exists a subspace of U, which is not separable. The idea of the proof of this fact is developped in [3] p. 88. In fact, take any neighbourhood of the type $\langle a, a+\varepsilon) \times \langle b, b+\varepsilon)$, $\varepsilon > 0$, of a point (a, b). For the simplicity put $a = 0$, $b = 0$. Take now the set M of all (x, y) for which $(x, y) \in \langle 0, \varepsilon) \times \langle 0, \varepsilon)$ and $y = -x + \frac{\varepsilon}{2}$. Then the only dense set in the space M is the set M itself. Thus X is not strongly separable space.

Somewhat continuity

The notion of somewhat continuity for a function f: $X \to Y$, where X, Y are topological spaces, was introduced in [3]. f is called somewhat continuous, if for any open $G \subset Y$ for which $f^{-1}(G) \neq \emptyset$, we have $(f^{-1}(G))^0 \neq \emptyset$. An equivalent definition of somewhat continuity may be given using the following necessary and sufficient condition of somewhat continuity.

Proposition 2 (see [3]). A function f: $X \to Y$ is somewhat continuous if and only if for any set D dense in X the set $f(D)$ is dense in $f(X)$.

To formulate a definition of somewhat continuity for functions with values in $\mathcal{S}$-spaces we introduce the notion of S-density.

Let, A, $B \subset Y$. The set A is said to be $\mathcal{S}$-dense in B if for any $y \in B$ and any $S \in \mathcal{S}_y$ containing y there is $S \cap A \neq \emptyset$.

Remark 6. Obviously the notion of $\mathcal{S}$-density coincides with that of density, if Y is a topological space and $\mathcal{S}_y$ $(y \in Y)$ is defined by means of its topology.

Let X be a topological space and Y a $\mathcal{S}$-space. A function f: $X \to Y$ is said to be $\mathcal{S}$-somewhat continuous if for any set D dense in X the set $f(D)$ is $\mathcal{S}$-dense in $f(X)$.

Theorem 4. Let X be a strongly separable topological space, Y an $\mathcal{S}$-space.

Let $\{f_\xi\}_{\xi<\Omega}$ be a transfinite sequence of $\mathscr{S}$-somewhat continuous functions, f_ξ: $X \to Y$, converging to a function f. Then f is $\mathscr{S}$-somewhat continuous.

Proof. Let M be a set dense in X. We have to prove that $f(M)$ is $\mathscr{S}$-sense in $f(X)$. There exists a countable dense set in M and hence in X. It is sufficient to prove that the image of this countable set is dense in X. So we may suppose that M is countable. According to the convergence of $\{f_\xi\}_{\xi<\Omega}$, there exists ξ_0 such that $f_\xi(x)=f(x)$ for $\xi \geqq \xi_0$ and every $x \in M$. Let $y_0 \in f(X)$. There exists $x_0 \in X$ such that $f(x_0)=y_0$. For sufficiently large ξ we have $f_\xi(x_0)=f(x_0)$. With no loss of generality we may suppose that ξ_0 was chosen such that $f_\xi(x_0)=f(x_0)$ for $\xi \geqq \xi_0$. The set $f_{\xi_0}(M)$ is $\mathscr{S}$-dense in $f_{\xi_0}(X)$ due to the somewhat continuity of f_{ξ_0}. Hence for any S containg y_0, there exists $y \in f_{\xi_0}(M)=f(M)$. Thus $f(M)$ is $\mathscr{S}$-dense in $f(X)$.

Corollary 5. Let X be a strongly separable topological space and Y a first countable T_1-space. Then any convergent transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of somewhat continuous functions f_ξ: $X \to Y$ converges to a somewhat continuous function.

Remark 7. The above Corollary is obviously valid for metric spaces. The version for metric spaces is given in [10]. In [11] there was also given an example that for locally separable metric spaces such theorem is not valid. So Theorem 4 can not be extended to strongly locally separable topological spaces.

The notion of the weak somewhat continuity can be formulated in the following way.

A function f: $X \to Y$, where X, Y are topological spaces, is said to be weakly somewhat continuous, if for any open set $G \subset Y$ for which $f^{-1}(G) \neq \emptyset$ we have $(f^{-1}(\bar{G}))^0 \neq \emptyset$.

Let Y be a topological space. A set $A \subset Y$ will be said to be weakly dense in a set $B \subset Y$ if for any point $y \in B$ and any neighbourhood V of y we have $\bar{V} \cap A \neq \emptyset$.

The following proposition will be useful.

Proposition 3. Let X, Y be topological spaces. A function f: $X \to Y$ is weakly somewhat continuous if and only if for any set $D \subset X$, dense in X, the set $f(D)$ is weakly dense in $f(X)$.

Proof. Let f be weakly somewhat continuous. Let $y_0 \in f(X)$ and V any neighbourhood of y_0. Then $f^{-1}(V) \neq \emptyset$ and by the weak somewhat continuity $(f^{-1}(\bar{V}))^0 \neq \emptyset$. By density of D, there exists a point $x \in D$, $x \in (f^{-1}(\bar{V}))^0$. Hence $f(x) \in \bar{V}$. So $f(D)$ is weakly dense in $f(X)$.

Conversely let $f(D)$ be weakly dense in $f(X)$ for any set D, dense in X. Suppose f not to be weakly somewhat continuous. Then there exists V open in Y such that $f^{-1}(V) \neq \emptyset$, $(f^{-1}(\bar{V}))^0 = \emptyset$. Put $D = X - f^{-1}(\bar{V})$. Then D is dense in X. But

$$f(D) = f(X - f^{-1}(\bar{V})) = f(X) - \bar{V}$$

is not weakly dense in $f(X)$, because $\bar{V} \cap f(D) = \emptyset$.

The weak somewhat continuity may be interpretted as $\mathscr{S}$-somewhat continuity if we take for any $y \in Y$ the set $\mathcal{S}_y$ as the set of all $\bar{V}$ where V is a neighbourhood of y.

So we may now obtain a result for transfinite of weakly somewhat continuous functions.

Theorem 5. Let X be a strongly separable topological space, Y a first countable T_1-space. Let $\{f_\xi\}_{\xi<\Omega}$ be a transfinite sequence of weakly somewhat continuous functions, $f_\xi \colon X \to Y$, converging to f. Then f is weakly somewhat continuous.

Proof. It follows from Theorem 4 and Proposition 3.

Cliquishness

The cliquish functions are defined in the literature for functions defined on a topological space X with values in a metric space Y with the metric ϱ. (see e.g. [9]).

Such a function is said to be cliquish at $x_0 \in X$ if for any $\varepsilon > 0$ and any neighbourhood U of x_0 there exists a nonempty open set $G \subset U$ such that for any $x_1, x_2 \in G$ $\varrho(f(x_1), f(x_2)) < \varepsilon$. It is said to be cliquish if it is cliquish at any $x \in X$.

It is possible to define the notion of the cliquish function for a function with values in an uniform topological space. But we can define it in an abstract way similarly as we have introduced the previous continuity notions.

Let Y be a set. Consider a collection $\mathcal{S}$ of subsets of $Y \times Y$ such that the diagonal, i. e. the set $\Delta = \{(y, y) \colon y \in Y\}$ is a subset of any $S \in \mathcal{S}$. Given such a collection $\mathcal{S}$ on $Y \times Y$ we say that Y is an uniform $\mathcal{S}$-space with the $\mathcal{S}$-uniformity $\mathcal{S}$.

Let X be a topological space, Y an uniform $\mathcal{S}$-space. A function $f \colon X \to Y$ is said to be $\mathcal{S}$-cliquish at $x_0 \in X$, if for any $S \in \mathcal{S}$ and any neighbourhood U of x_0 there exists a nonempty open set $G \subset U$ such that for any $x, y \in G$ we have $f(x), f(y)) \in S$.

Remark 8. If Y is a uniform topological space with the uniformity $\mathcal{V}$ (see [2], p. 203), then taking $\mathcal{S} = \mathcal{V}$ we obtain uniform $\mathcal{S}$-space.

In case of metric spaces, taking the natural uniformity, we obtain from the corresponding $\mathcal{S}$-cliquisnes the usual cliquishness.

Theorem 6. Let X be a first-countable locally separable topological space. Let Y be an $\mathcal{S}$-uniform space. Let a transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of $\mathcal{S}$-cliquish functions defined on X taking values in Y converge to f. Then f is $\mathcal{S}$-cliquish.

Proof. Suppose f not to be $\mathcal{S}$-cliquish at $x_0 \in X$. Then there exists $S \in \mathcal{S}$ and an open set U containing x_0 such that for any nonempty open set $G \subset U$ there is a pair (y, z), $y \in G$, $z \in G$ with $(f(y), f(z)) \notin S$. Let M be the set of all pairs (y, z), $y \in U$, $z \in U$ for which $(f(y), f(z)) \notin S$. With no loss of generality we may suppose that U is separable. Let $D \subset U$ be a countable dense set in U. For any $s \in D$ let $\{B_n^s\}$, $B_n^s \subset U$ be a countable basis of neighbourhoods of the point s. Take $(y_n^s, z_n^s) \in M$, where $y_n^s, z_n^s \in B_n^s$.

Put

$$T = \bigcup_{s \in D} \bigcup_{n=1}^{\infty} \{y_n^s\} \cup \{z_n^s\}$$

The set T is countable and since $\{f_\xi\}_{\xi<\Omega}$ is convergent to f, there exists $\xi_0 < \Omega$ such that for $\xi \geqq \xi_0$ $f_\xi(t) = f(t)$ for any $t \in T$. Since f_{ξ_0} is $\mathcal{S}$-cliquish at x_0, there exists a set $G \subset U$ with $(f_{\xi_0}(y), f_{\xi_0}(z)) \in S$ for any $y, z \in G$. But for suitably chosen n we have $\{B_n^s\} \subset G$, hence $y_n^s, z_n^s \in G$. Since $f_{\xi_0}(y_n^s) = f_{\xi_0}(y_n^s)$ and $f_{\xi_0}(z_n^s) = f(z_n^s)$ we have $(f(y_n^s), f(z_n^s) \in S$. It is a contradiction because $(y_n^s, z_n^s) \in M$.

Corollary 6. Let X be a locally separable, first countable topological space. Let Y be a uniform first countable topological space. Then any convergent transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of cliquish functions defined on X with values in Y converges to a cliquish function.

The preceding Corollary covers a result known for transfinite sequence of cliquish functions defined on a locally separable metric space with values in a metric space (see [9]).

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SÚHRN

JEDNOTNÝ POHĽAD NA TRANSINITNÚ KONVERGENCIU
A ZOVŠEOBECNENÚ SPOJITOSŤ

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Z jednotného hľadiska sa skúma zachovávanie rôznych typov spojitosti pri konvergencii transfinitných postupností. Nové výsledky sa dosahujú najmä pre funkcie definované na lokálne separabilných topologických priestoroch, ktoré nie sú nevyhnutne metrickými.

РЕЗЮМЕ

ОБЩАЯ ТОЧКА ЗРЕНИЯ НА ТРАНСФИНИТНУЮ СХОДИМОСТЬ
И ОБОБЩЕН7Ю НЕПРЕРЫВНОСТЬ

А. Нойбруннова, Братислава

С общей точки зрения исследуется сохранение различных типов непрерывности при сходимости трансфинитных последовательностей. Новые результаты доказаны именно для функций определенных на локально сепарабельных топологических, необязательно метрических, пространствах.