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THE STRUCTURE OF THE RANGE OF PERIODIC FUNCTIONS

MÁRIA BERŽELOVÁ, Bratislava

It is proved in the paper [4], that the set of the limit points of the sequence $\{\sin n\}_{n=1}^{\infty}$ coincides with the interval $\langle -1, 1\rangle$. One might ask whether it would be possible to generalize this result. The authors of [4] have given a certain kind of generalization in one of the exercises in article [4], but one can prove that this generalization is not correct, if we consider the notion of a piecewise continuous function by using the definition in [1] (see p. 469).

We will give the following counter-example.

Example 1. Let w be an irrational number, $w>0$. Let us choose the function f in the following way:

$$f(x)=0 \quad \text{if} \quad x\in \left\langle 0, \frac{w}{2}\right)\cup\left(\frac{w}{2}, w\right\rangle \quad \text{and} \quad f\left(\frac{w}{2}\right)=1$$

Let us extend f periodically on the set of all real numbers. The function f is evidently piecewise continuous on interval $\langle 0, w\rangle$ and it is equal to 1 only for the points of the form $\frac{w}{2}\pm kw$, where k is an integer. But $\frac{w}{2}\pm kw\neq n$ $(n=1, 2, \ldots)$ for every integer k. Therefore $\{f(1), f(2), \ldots\} = \{0\}$, and it is not dense in the set $\{0, 1\}$.

But if we consider a piecewise continuous function in the following way, then it will be all right, and it could easily be proved by the Theorem 1. The function $f\colon \langle a, b\rangle\to \mathrm{R}$ is called piecewise continuous, if there exist disjoint non-degenerated intervals $I_1, \ldots, I_s$, such that $\langle a, b\rangle=\bigcup_{k=1}^{s} I_k$ and for every $k=1, \ldots, s$ $f\,|\,I_k$ is a continuous function on I_k.

We shall give a generalization of the main result of [4] by using the notion of a somewhat continuous function (see [3]).

Definition 1. Let X, Y be topological spaces. The function $f\colon X\to Y$ is said to be somewhat continuous, if for every open set $V \subset Y$ such that $f^{-1}(V) \neq \emptyset$ we have $\operatorname{int} f^{-1}(V) \neq \emptyset$.

The next theorem was proved in [4].

Theorem A. Let α be an irrational number. Then the limit points of the sequence $\{\{n\alpha\}\}_{n=1}^{\infty}$ cover the interval $\langle 0, 1\rangle$. ($\{a\} = a - [a]$ for every real number a.)

This Theorem follows also from the fact that the sequence $\{n\alpha\}_{n=1}^{\infty}$, where α is an irrational number, is uniformly distributed mod 1 (see [2], p. 8).

Definition 2. A function $f\colon X \to \boldsymbol{R}$, $X \subset \boldsymbol{R}$ is said to be periodic with the period $w > 0$, if the domain X of f contains with x all the numbers $x + kw$ (k is an arbitrary integer), and $f(x + kw) = f(x)$.

Theorem 1, Let $f\colon X \to \boldsymbol{R}$, $X \subset \boldsymbol{R}$ be a somewhat continuous periodic function with an irrational period $w > 0$. Let $\boldsymbol{R} - X$ be a nowhere dense set. Then the range of $f|(N \cap X)$ is dense in $f(X)$.

Proof. We can assume that $w > 1$. Let $t \in f(X)$. Then there exists $c \in X$ such that $t = f(c)$. We can assume $0 \leqq c < w$. Let $\varepsilon > 0$. We shall show that the interval $(t - \varepsilon, t + \varepsilon)$ contains a point from $f|(N \cap X)$. By choosing the point c we have $f^{-1}((t - \varepsilon, t + \varepsilon)) \neq \emptyset$ and because f is a somewhat continuous function we have int $f^{-1}((t - \varepsilon, t + \varepsilon)) \neq \emptyset$. Therefore there exists an interval $I \subset \langle 0, w\rangle$ such that

$$f(I \cap X) \subset (t - \varepsilon, t + \varepsilon) \tag{1}$$

Because $\boldsymbol{R} - X$ is a nowhere dense set, there exists an interval $J \subset I$ such that

$$J \cap (\boldsymbol{R} - X) = \emptyset \tag{2}$$

From the last equation we get $J \subset X$ and so $J \cap X = J$. Let $J = (a, b)$, where $0 \leqq a < b < w$, so that $\left(\frac{a}{w}, \frac{b}{w}\right) \subset \langle 0, 1)$. Because w is an irrational number, therefore according to Theorem A, there exists $m \in N$ such that

$$\left\{\frac{m}{w}\right\} = \frac{m}{w} - \left[\frac{m}{w}\right] \in \left(\frac{a}{w}, \frac{b}{w}\right)$$

It follows from this that

$$x = w\left\{\frac{m}{w}\right\} \in (a, b) = J.$$

By using (1) we get

$$|f(x) - f(c)| < \varepsilon \tag{3}$$

But we have

$$x = w\left(\frac{m}{w} - \left[\frac{m}{w}\right]\right) = m - w\left[\frac{m}{w}\right]$$

Because of periodicity of function f we get $f(x)=f(m)$, and now using relation (3) we get $|f(m)-f(c)|<\varepsilon$. This ends the proof.

Corollary. Let f: $\boldsymbol{R}\to\boldsymbol{R}$ be a periodic function with an irrational period $w>0$. Let $I_1, \ldots, I_s$ be disjoint, non-degenerated intervals such that $\langle 0, w\rangle=\bigcup_{k=1}^{s} I_k$ and each of the functions $f\,|\,I_k$ $(k=1, 2, \ldots, s)$ is continuous on I_k. Then the set $f(N)$ is dense in $f(\boldsymbol{R})$.

The Theorem 1 gives a relatively strong generalization of the main result of [4]. The next example will show that a somewhat continuous function can be discontinuous at every point.

Example 2. (see [3]) Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of all ordered pairs of rational numbers, $\alpha_n=(r_n, s_n)$, where $r_n\neq s_n$ for every $n=1, 2, \ldots$. Let us define the function f: $\langle 0, 1\rangle\to\boldsymbol{R}$ in the following way:

$$g(0)=0$$

$$g(x)=r_n, \quad \text{if} \quad x\in\left(\frac{1}{n+1}, \frac{1}{n}\right\rangle, \quad x \text{ is a rational number}$$

$$g(x)=s_n, \quad \text{if} \quad x\in\left(\frac{1}{n+1}, \frac{1}{n}\right\rangle, \quad x \text{ is an irrational number}$$

Obviously function g is discontinuous at every point of the interval $\langle 0, 1\rangle$. But g is evidently a somewhat continuous function.

By a small modification of the example 2 we can get an example of a somewhat continuous, everywhere discontinuous periodic function.

Example 3. Let us define the function g: $\boldsymbol{R}\to\boldsymbol{R}$ in the following way. Let $w>0$ be an irrational number. Let $\{(s_n, r_n)\}_{n=1}^{\infty}$ be a sequence of all ordered pairs of rational numbers, such that $s_n\neq r_n$ for every $n=1, 2, \ldots$. Let

$$g(x)=r_n, \quad \text{if} \quad x\in\left(\frac{w}{n+1}, \frac{w}{n}\right\rangle, \quad x \text{ is a rational number}$$

$$g(x)=s_n, \quad \text{if} \quad x\in\left(\frac{w}{n+1}, \frac{w}{n}\right\rangle, \quad x \text{ is an irrational number}$$

Function g is defined on the interval $(0, w\rangle$, g is discontinuous at every point of this interval. Let us extend the function g periodically on the whole line. We shall get a somewhat continuous periodic function, with period $w>0$, which is discontinuous at every point.

We shall give now the following example to show the application of the proved Theorem 1.

Example 4. Each of the functions sin x, cos x, tg x, cotg x fulfils the assumptions of the Theorem 1, therefore e.g. the set $\{\cos 1, \cos 2, \ldots, \cos n, \ldots\}$ is dense in $\langle -1, 1\rangle$; $\{\mathrm{tg}\, 1, \mathrm{tg}\, 2, \ldots, \mathrm{tg}\, n, \ldots\}$ is dense in $(-\infty, \infty)$.

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Author's address:

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Mária Perželová
ÚTK SAV, Dúbravská cesta 9
842 35 Bratislava

SÚHRN

ŠTRUKTÚRA OBORU HODNÔT PERIODICKYCH FUNKCIÍ

M. Perželová, Bratislava

Práca sa zaoberá štruktúrou oboru hodnôt periodických funkcií. Jej hlavným výsledkom je dokázaná Veta 1. Táto Veta tvrdí, že postupnosť $\{f(n)\}_{n=1}^{\infty}$ je hustá v obore hodnôt funkcie f, kde f je ľubovoľná trochu spojitá funkcia s iracionálnou periódou. Tvrdenie Vety 1 je relatívne silné, pretože trochu spojitá funkcia môže byť nespojitá v každom bode.

РЕЗЮМЕ

СТРУКТУРА ВЕЛИЧИНЫ ЗНАЧЕНИЙ ПЕРИОДИЧЕСКИХ ФУНКЦИЙ

М. Пержелова, Братислава

В данной работе исследуется структура величины значений периодических функций. Главным результатом является доказанная Теорема 1. С помощью этой теоремы удалось показать, что для любой несколько непрерывной функции f с иррациональным периодом последовательность $\{f(n)\}_{n=1}^{\infty}$ плотная в множестве значений функции f. Это доказательство содержит довольно сильное утверждение, потому что несколько непрерывная функция может быть разрывна в каждой точке.