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Jahr: 1984

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ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XLIV—XLV—1984

ON SEQUENTIAL CHARACTERIZATIONS OF QUASICONTINUOUS MULTIFUNCTIONS

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The upper and the lower continuity of multifunctions admits a sequential characterization (see e.g. [1], [5]). Such a characterization seems to be usefull in some applications. (See [2]). If quasicontinuity of multifunctions is discussed, then it is known (see [3]) that it cannot be in general characterized as a continuity on a quasiopen set. In spite of it, a sequential characterization of the lower quasicontinuity of a multifunction F by means of quasiopen sets is possible in some cases when the characterization by means of its continuity on a quasiopen set fails.

Throughout the paper, if nothing else is said, X, Y denote the first countable Hausdorff topological spaces. Multifunction is a mapping defined on X with the values in the collection of nonempty subsets of Y. We write for shortness F: $X \to Y$ to denote a multifunction F defined on X with the values in the potence set of Y. A multifunction F: $X \to Y$ is said to be lower quasicontinuous at $x_0 \in X$ if for any open set V for which $F(x_0) \cap V \neq \emptyset$ and for any open U containing x_0 there exists a nonempty open $G \subset U$ such that $F(x \cap V \neq \emptyset$ for any $x \in G$. F is said to be upper quasicontinuous at x_0, if for any V open containing $F(x_0)$ and any U open containing x_0, there exists a nonempty open set $G \subset U$ such that $F(x) \subset V$ for any $x \in G$.

The notion lower semi-quasicontinuous and upper semi-quasicontinuous would be more appropriate. It would correspond to upper semi-continuous and lower semi-continuous respectively. We omit the word "semi" for shortness.

If f: $X \to Y$ is a single valued function, then any of the above definitions of upper (lower) quasicontinuity gives the notion of the quasicontinuity of f at x_0. Of course, f should be interpretted as a multifunction assigning to any $x \in X$ the value $\{f(x)\}$.

1. Lower quasicontinuity

Lemma 1. Let X, Y be first countable topological spaces. Let $x_0 \in X$ be a point which is not isolated. Suppose that $\{P_k\}_{k=1}^{\infty}$ is a sequence of properties such that to any neighbourhod U of x_0 a sequence $\{M_k\}_{k=1}^{\infty}$ of nonempty open sets exists such that P_k is satisfied on M_k. Then a descending base $\{W_k\}_{k=1}^{\infty}$ of neighbourhoods of x_0 and a sequence $\{G_k\}_{k=1}^{\infty}$ of mutually disjoint open sets exist such that $G_k \subset W_k$, $G_k \cap W_{k+1} = \emptyset$, P_k is satisfied on G_k for $k = 1, 2, \ldots$ Evidently then the set $A = \bigcup_{k=1}^{\infty} G_k \cup \{x_0\}$ is quasiopen.

Proof. Let $\{U_n\}_{n=1}^{\infty}$ be a descending base of neighbourhoods of x_0. Put $n_1 = 1$ and take U_{n_1}. There exists a nonempty open $M_1 \subset U_{n_1}$ such that P_1 is satisfied on M_1. Using the fact that X is Hausdorff and $M_1 \neq \{x_0\}$, we can find $n_2 > n_1$ with $M_1 - \bar{U}_{n_2} \neq \emptyset$. Put $G_1 = M_1 - \bar{U}_{n_2}$, $W_1 = U_{n_1}$, $W_2 = U_{n_2}$. Evidently $G_1 \subset W_1$, $G_1 \cap W_2 = \emptyset$ and P_1 is satisfied on G_1. Now let us proceed by induction. Suppose that for $k \geqq 1$ the sets $W_1 \supset W_2 \supset \ldots W_k \supset W_{k+1}$ are constructed such that $W_i = U_{n_i}$, where $n_i < n_{i+1}$ for $i = 1, 2, \ldots, k$. Further a sequence $G_1, G_2, \ldots, G_k$ is such that $G_i \cap G_j = \emptyset$ for $i \neq j$, G_i are nonempty open $\left(\bigcup_{i=1}^{k} G_i\right) \cap U_{n_{k+1}} = \emptyset$, and P_i is satisfied on G_i for $i = 1, 2, \ldots, k$. Then $M_{k+1} \subset U_{n_{k+1}}$ exists such that P_{k+1} is satisfied on M_{k+1}, and M_{k+1} is nonempty, and open. Again, since X is Hausdorff and $M_{k+1} \neq \{x_0\}$ there exists $n_{k+2} > n_{k+1}$ such that $M_{k+1} - \bar{U}_{n_{k+2}} \neq \emptyset$.

Put $G_{k+1} = M_{k+1} - \bar{U}_{n_{k+2}}$. The sets $G_1, \ldots, G_{k+1}$ are pairwise disjoint. It is sufficient now to put $W_{k+2} = U_{n_{k+2}}$.

Theorem 1. A multifunction F: $X \to Y$ is lower quasicontinuous at $x_0 \in X$ if and only if for any $y \in F(x_0)$ a quasiopen set A containing x_0 exists such that for any sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in A$, $x_n \to x_0$ there exists $\{y_n\}_{n=1}^{\infty}$, $y_n \in F(x_n)$, $y_n \to y$.

Proof. Suppose F to be lower quasicontinuous at x_0. If x_0 is an isolated point then we can take $A = \{x_0\}$. So let x_0 be not isolated. Let $\{V_n\}_{n=1}^{\infty}$ be a descending sequence of open neighbourhoods which is a base at y. Let n be positive integer. Denote P_n the property which asserts that to V_n and any neighbourhod U of x_0 there exists a nonempty open set $M \subset U$ such that $F(x) \cap V_n \neq \emptyset$ for any $x \in M$. By the lower quasicontinuity of F at x_0 such a set exists. By lemma 1 there exists a sequence $\{G_k\}_{k=1}^{\infty}$ of mutually disjoint open sets such that $A = \bigcup_{k=1}^{\infty} G_k \cup \{x_0\}$ is quasiopen. Moreover, $G_k \subset W_k$ for $k = 1, 2, \ldots$, where $\{W_k\}_{k=1}^{\infty}$ is a descending base of neighnourhoods at x_0, $G_k \cap W_{k+1} = \emptyset$ and $F(x) \cap V_k \neq \emptyset$ for any $x \in G_k$.

Now let $x_n \in A$, $x_n \to x_0$. With no loss of generality we may suppose $x_n \neq x_0$ for $n = 1, 2, \ldots$. Since $x_n \in A$, $x_n \neq x_0$, there exists $i(n)$ such that $x_n \in G_{i(n)}$, hence $F(x_n) \cap V_{i(n)} \neq \emptyset$. Take $y_n \in F(x_n) \cap V_{i(n)}$. Let V be any neighbourhood of y. Choose k such that $V_k \subset V$. Since $x_n \to x_0$, there exists N such that $x_n \in W_k$ for $n \geqq N$. In this case we have $i(n) \geqq k$, by the construction of the sequence $\{G_k\}_{k=1}^{\infty}$. So $V_{i(n)} \subset V_k \subset V$. Thus for $n \geqq N$, $y_n \in F(x_n) \cap V_{i(n)} \subset V$. The convergence $y_n \to y$ is proved.

Now let us prove the sufficient part. Suppose that for any $y \in F(x_0)$ the quasiopen set A with the mentioned properties exists. Suppose F not to be lower quasicontinuous at x_0. So we have an $y \in V$ and an open set U containing x_0 such that $V \cap F(x) = \emptyset$ for any x belonging to a dense set $D \subset U$. Evidently $\{x_0\}$ is not open. Take the quasiopen set A. Then $A \cap U$, as a nonempty intersection of an open and quasiopen set, is quasiopen and the interior $(A \cap U)^0$ is nonempty. Moreover, $x_0 \in \overline{(A \cap U)^0}$. From the last and from the first countability of X and density of D there exists a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in D$, $x_n \in A$, $x_n \to x_0$. Hence $F(x_n) \cap V = \emptyset$ for $n = 1, 2, \ldots$ and there is not a sequence $\{y_n\}_{n=1}^{\infty}$ with $y_n \in F(x_n)$, $y_n \to y$. It is a contradiction.

Remark. The set A in the above Theorem depends obviously on the point $y \in F(x_0)$. To show that in general it cannot be chosen independently on $y \in F(x_0)$ we can use an example given in another connections in [3].

Example. Let F: $\langle 0, 1) \to R$ be defined as

$$F(0) = \{1, 2\}$$

$$F(x) = \begin{cases} \{1\} & \text{if } x \in \left\langle \dfrac{1}{2n}, \dfrac{1}{2n-1} \right), \\ \{2\} & \text{if } x \in \left\langle \dfrac{1}{2n+1}, \dfrac{1}{2n} \right) \end{cases}$$

$n = 1, 2, \ldots$

It is easy to show (see [3]) that F is lower quasicontinuous. Let A be a quasiopen set with the property that for any $\{x_n\}_{n=1}^{\infty}$, $x_n \in A$, there exists $\{y_n\}_{n=1}^{\infty}$, $y_n \in F(x_n)$, $y_n \to 1$, Then A necessarily contains a sequence $\{z_n\}_{n=1}^{\infty}$, $z_n \neq 0$, $n = 1, 2, \ldots$ $z_n \in \bigcup_{n=1}^{\infty} \left\langle \frac{1}{2n}, \frac{1}{2n-1} \right)$.

Then for any $y_n \in F(z_n)$ we have $y_n = 1$, and so A may not serve for the point $2 \in F(0)$.

The above example was used in [3] to show that lower quasicontinuity may not be characterized by means of the lower continuity of its restriction on a quasiopen set. In this connection we can give a necessary and sufficient condition for such a characterization.

Theorem 2. Let F be a lower quasicontinuous multifunction at a point x_0. In order that a quasiopen set A exists such that $F|A$ is lower continuous at x_0 the following condition (C) is necessary and sufficient.

(C) There exists a quasiopen set A containing x_0 (not depending on $y \in F(x_0)$) such that for any $y \in F(x_0)$ and any sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in A$, $x_n \to x_0$, there exists $y_n \in F(x_n)$, $y_n \to y$.

Proof. Let A be quasiopen, $x_0 \in A$ and $F|A$ is lower continuous. We can consider the relative topology on A, and the sequential characterization of the lower continuous function $F|A$ (see [4]) is possible, i.e. for any $y \in F|A(x_0)$ $= F(x_0))$ and any sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in A$, $x_n \to x_0$, there exists $y_n \in F|A(x_n)$ $(= F(x_n))$ such that $y_n \to y$.

Now let (C) be satisfied and suppose that $F|A$ is not lower continuous at x_0. Then an open set V, which is a neighbourhood of some $y \in F(x_0)$, exists such that for any neighbourhood U of x_0 there is a point $x \in U \cap A$ for which $F(x) \cap V = \emptyset$. So taking a countable base $\{U_n\}_{n=1}^{\infty}$ of the point x_0, we obtain a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in U_n \cap A$ with $F(x_n) \cap V = \emptyset$. It is a contradiction to (C).

There is a possibility to characterize the lower quasicontinuity of F at x_0 by means of selectors. Recall that a selector for F is a single valued function f such that $f(x) \in F(x)$ for any $x \in X$.

Theorem 3. A multifunction F is lower quasicontinuous at $x_0 \in X$ if and only if for any $y \in F(x_0)$ there exists a selector f which is quasicontinuous at x_0 and $f(x_0) = y$.

Proof. Let F be lower quasicontinuous at x_0. If x_0 is an isolated point then any f such that $f(x_0) = y$, $f(x) \in F(x)$ if $x \neq x_0$ is a quasicontinuous selector at x_0. So let x_0 be not isolated. Let $y \in F(x_0)$ and $\{V_k\}_{k=1}^{\infty}$ be a descending base of neighbourhoods at the point y. Let $\{G_k\}_{k=1}^{\infty}$, $\{W_k\}_{k=1}^{\infty}$ have the same meaning as in Theorem 1. Thus G_k $(k = 1, 2, \dots)$ are nonempty open mutually disjoint, $G_k \subset W_k$, $G_k \cap W_{k+1} = \emptyset$, where $\{W_k\}$ is a descending base of neighbourhoods at x_0, $A = \bigcup_{n=1}^{\infty} G_k \cup \{x_0\}$ a quasiopen set and $F(x) \cap V_k \neq \emptyset$ if $x \in G_k$. Define f as follows:

$$\begin{aligned} f(x_0) &= y \\ f(x) &\in F(x) \cap V_k \quad \text{if} \quad x \in G_k, \\ f(x) &\in F(x) \quad \text{if} \quad x \notin A \end{aligned}$$

The function f is evidently a selector for F. It is quasicontinuous at x_0. In fact let V be any neighbourhood of y and U any neighbourhood of x_0. Let k be such that $V_k \subset V$. There exists $i > k$ such that $W_i \subset U$, hence $G_i \subset U$. If $x \in G_i$ then $f(x) \in V_i \subset V_k \subset V$. The quasicontinuity of f at x_0 is proved.

Sufficiency. Let $y \in F(x_0)$, V any neighbourhood of y and U any neighbourhood of x_0. Let f be a quasicontinuous selector at x_0 with $f(x_0) = y$. Then there is a nonempty open $G \subset U$ such that $f(x) \in V$ if $x \in G$. Since $f(x) \in F(x)$, we have $F(x) \cap V \supset \{f(x)\} \neq \emptyset$. Thus F is lower quasicontinuous at x_0.

A question may arise, whether an analogical characterization as given in Theorem 1 may be obtained if we omit the first countability axiom substituing sequences by nets. The following example (used also in [3] in another connection) shows that the answer is negative.

Example 2. Let $X = \{x : x \leqq \omega_1\}$ where ω_1 is the first uncountable ordinal.

Consider the order topology on X. Let $Y=(-\infty, \infty)$ with the usual topology. It is well known that any $x \in X$ may be expressed in an unique manner as $x=\xi+n$, where ξ is a limit ordinal number and n a nonnegative integer. Define now a single-valued function $f\colon X \to R$ such that $f(\omega_1)=0$,

$$f(x)=\begin{cases} \dfrac{1}{n} & \text{if} \quad x=\xi+n, \quad n \neq 0, \\ 1 & \text{if} \quad x<\omega_1, \quad\quad n=0. \end{cases}$$

Then f is quasicontinuous (hence both upper and lower quasicontinuous, because f is single-valued) at ω_1. It follows from the fact that in any neighbourhood of ω_1 there is an isolated point of the form $\xi+n$ with sufficiently large n. On the other hand, there is not a quasiopen set A containing ω_1 such that for arbitrary net $\{x_d\}$, $x_d \in A$, converging to ω_1 the net $\{f(x_d)\}$ converges to 0. In fact, if A is quasiopen $\omega_1 \in A$, then we may choose a transfinite sequence $\{x_\xi\}(\xi<\omega_1)x_\xi \neq \omega_1$, such that it converges to ω_1 However the net $\{f(x_\xi)\}(\xi<\omega_1)$ does not convege to 0. In fact, if $\{f(x_\xi)\}(\xi<\omega_1)$ converges to 0, then $f(x_\xi)=0$ begining from certain $\eta<\omega_1$, because it is a transfinite sequence of real numbers (see [4]). However it is a contradiction to the definition of f.

2. Remarks on the upper quasicontinuity

The upper quasicontinuity may be under fairly general assumptions characterized by means of the upper continuity on a quasiopen set. Thus in these cases the sequential characterization may be obtained from the sequential characterization of upper continuous multifunctions. Thus the problem of a sequential characterization of an upper quasicontinous multifunction will follow immediately from two konwn results. The first one (Lemma 2) concerns the characterization of the upper quasicontinuous multifunction by means of the upper continuity of its restriction. The second (Lemma 3) is a sequential characterization of the upper continuity.

Lemma 2. ([5] Theorem 2) Let X be a first countable Hausdorff space and Y a second countable Hausdorff space. Let $F\colon X \to Y$ be such that $F(x_0)$ is compact. Then F is upper quasicontinuous at x_0 if and only if there exists a quasiopen set A containing x_0 such that $F|A$ is upper continuous at x_0.

Remark. In fact, Theorem 2 in [5] is formulated in slightly different way but the proof of Lemma 2 is the same.

Lemma 3 ([5] Theorem 4). Let X be first countable, Y second countable Hausdorff spaces. Let $F\colon X \to Y$ and $F(x_0)$ compact. Then F is upper continuous at x_0 if and only if the following condition (D) is satisfied.

(D) If $x_n \to x_0$ and $y_n \in F(x_n)$, then there exists $\{y_{n_k}\}_{k=1}^{\infty}$ which is a subsequence of $\{y_n\}_{n=1}^{\infty}$ such that $y_{n_k} \to y \in F(x_0)$.

The above lemma coincides with Theorem 4 of [4]. An unessential change in the proof of Theorem 4 of [5] should be done.

Theorem 4. Let X be first countable and Y second countable Hausdorff spaces. Let $x_0 \in X$, $F(x_0)$ compact. Then F is upper quasicontinuous at x_0 if and only if a quasiopen set containing x_0 exists such that the following is true. If $x_n \to x_0$, $x_n \in A$, $y_n \in F(x_n)$ $(n = 1, 2, \ldots)$, then there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y \in F(x_0)$.

Proof. From Lemma 2 it follows that F is upper quasicontinuous if and only if $F \mid A$ is upper continuous on a quasiopen set A containing x_0. However $F \mid A$ satisfies the assumptions of Lemma 3 when the relative topology on A is considered. Hence $F \mid A$ is upper continuous if and only if for any $x_n \in A$, $x_n \to x_0$ and $y_n \in F(x_n)$ there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y \in F(x_0)$. The proof is finished.

Of course another sequential characterizations of the upper quasicontinuity may be obtained in the case when the upper quasicontinuity is characterized by means of the upper continuity of its restriction $F \mid A$ on a quasiopen set. It is sufficient to use the appropriate sequential characterizations of the upper continuity. As to the latter it was discussed in [1] and [5].

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Received: 16. 8. 1982

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Даются характеризации квазинепрерывности сверху и снизу при помощи последовательностей на некоторых топологических пространствах.