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TWO PROPERTIES OF THE SEQENCE $n\alpha(\text{mod } 1)$

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1. The first property is connected with an example of an infinite set A of positive integers (ordered by increasing) which does not contain any infinite arithmetic progression. If differences between consecutive terms of A are unbounded, then A has this property automaticaly. If these differences are bounded, then this property again is possible. A familiar example is the set $A = Z^+ - B$, where B is a set having at least one element common with each infinite arithmetic progression of positive integers and the differences between consecutive terms of B are greater than one. An entirely different example is the set

$$A = \{n \in Z^+;\ \{n\alpha\} \in I\} \tag{1}$$

where α is irrational, $\{n\alpha\}$ is a fractional part of $n\alpha$, and I is a subinterval of $[0, 1]$ the length $|I|$ of which satisfies

$$0 < |I| < 1$$

This set has the following properties

Theorem 1.

(i) The set A contains no infinite aritmetic progression.

(ii) For the set A there exist positive integers k_1, k_2 such that

$$n + k_1 \in A \quad \text{or} \quad n + k_2 \in A$$

for all $n \in A$.

(iii) The set A contains an arbitrarily long arithmetic progression.

Proof. The assertion (i) is an immediate consequence of the fact that a polynomial sequence $P(n)$, $n = 1, 2, \ldots$ is uniformly distributed mod 1 for every polynomial $P(x)$ with at least one irrational (nonabsolute) coefficient (see [1, Theorem 3.2, p. 27]). In our case $P(n) = (an + b)\alpha$.

To prove (ii), let us suppose $n \in A$, i.e. $\{n\alpha\} \in I$. The point $\{n\alpha\}$ divides the interval I into two subintervals, left I_1 and right I_2. Let us translate the interval I_1 to the point 1 and the interval I_2 to the point 0 and let us denote these intervals as I_1' and I_2', i.e.

$$I_1' = I_1 + 1 - \{n\alpha\}$$
$$I_2' = I_2 - \{n\alpha\}$$

Clearly, it is true

$$\{(n+k)\alpha\} \in I_1 \Leftrightarrow \{k\alpha\} \in I_1'$$
$$\{(n+k)\alpha\} \in I_2 \Leftrightarrow \{k\alpha\} \in I_2'$$

for all $k \in Z^+$. Since the sequence

$$\{n\alpha\}; \quad n = 1, 2, \ldots \tag{2}$$

of fractional part is dense in [0, 1], there exist k_1, $k_2 \in Z^+$ such that

$$|1 - \{k_1\alpha\}| < \frac{|I|}{2}$$

$$|0 - \{k_2\alpha\}| < \frac{|I|}{2}$$

Moreover, at least one from the intervals I_1', I_2' has the length $\geqq$ $|I|/2$ and therefore either $\{k_1\alpha\} \in I_1'$ or $\{k_2\alpha\} \in I_2'$. Thus, the proof of (ii) is finished.

The assertion (iii) is an consequence of the Erdős—Szemerédi's theorem [2] and of the fact that the set A from (1) has the asymptotic density equal to $|I| > 0$. Using proof of (ii) we can prove (iii) directly in a more general form:

Theorem 2. Let $D+1$ be a length of an arithmetic sequence and let A be a set defined by (1). Then there exist positive integers K_1, K_2 such that either

$$\{n, n+K_1, n+2K_1, \ldots, n+DK_1\} \subset A$$

or

$$\{n, n+K_2, n+2K_2, \ldots, n+DK_2\} \subset A$$

for all $n \in A$. It is sufficient to choose K_1, K_2 such that

$$|1 - \{K_1\alpha\}| < \frac{|I|}{2D}$$

$$|0 - \{K_2\alpha\}| < \frac{|I|}{2D}$$

Proof. For every positive integer $i \leqq D$ it is true

$$0 < i - i\{K_1\alpha\} < \frac{i|I|}{2D} < 1$$

From it

$$0 < i - i\{K_1\alpha\} = i - iK_1\alpha + i[K_1\alpha] = 1 - \{iK_1\alpha\} < |I|/2$$

Similarly,

$$i\{K_2\alpha\} = \{iK_2\alpha\} < \frac{|I|}{2}$$

2. The sequence (2) has also the following interesting property:

Let $\{I_i\}$ be a sequence of pairwise disjoint subintervals of [0, 1] which covers (2) such that the series $\Sigma|I_i|$ of the lengths of $\{I_i\}$ satisfies

$$\Sigma|I_i| < 1$$

Let n_i denote the number of terms of

$$\{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\} \tag{3}$$

which belong to the interval I_i. Using uniform distribution of (2) we have

$$\frac{n_i}{n} \to |I_i|$$

as $n \to \infty$. On the other hand,

$$1 = \frac{n}{n} = \frac{n_1}{n} + \frac{n_2}{n} + \ldots \tag{4}$$

Thus the left side of (4) is equal to 1 and the right side of (4) is by terms convergent to $\Sigma|I_i| < 1$ (i.e. the asymptotic density is not σ-additive). Where is a "loss"? If the irrational α has bounded partial quotients in the continued fraction expansion we have a following answer:

Let N denote the number of intervals from $\{I_i\}$ which contain any terms from (3). Then

$$\frac{N}{n} \to 1 - \Sigma|I_i| \tag{5}$$

as $n \to \infty$. A sequence which has the property (5) we call "uniformly quick", see [3, Definition 9].

Definition. An infinite sequence $\{y_i\}$ from a finite interval I is said to be uniformly quick in I, if for every sequence $\{I_i\}$ of pairwise disjoint subintervals of I for which $\{y_i\} \subset \cup I_i$ we have

$$\text{card}\,\frac{\{I_j \in \{I_i\}\,;\ I_j \cap \{y_i\}_{i \leq n} \neq \emptyset\}}{n} \to \frac{|I - \bigcup I_i|}{|I|}$$

as $n \to \infty$. We shall prove

Theorem 3. The sequence (2) is uniformly quick if and only if irrational α has bounded partial quotients.

Proof. Order the finite sequence (3) to an increasing sequence

$$0<\{\alpha i_1\}<\{\alpha i_2\}<\ldots<\{\alpha i_n\}<1 \tag{6}$$

Evidently

$$N=n-N_1$$

where N_1 denotes the number of pairs of consecutive points $\{\alpha i_j\}$, $\{\alpha i_{j+1}\}$ from (6) which belong to the same interval from $\{I_i\}$. Similarly

$$N\geqq n'-N_1'$$

where n' denotes the number of terms from (3) which are contained in

$$[0,\ 1]-\bigcup_{i\leqq k} I_i$$

and also N_1' is just so reduced N_1, i.e. N_1' denotes the number of pairs of consecutive points from (6) which belong to the same interval from $\{I_i\}_{i>k}$.

Using uniform distribution of (2) we have

$$n'\geqq(1-\varepsilon)\Big(1-\sum_{i\leqq k}|I_i|\Big)n \tag{7}$$

for all sufficiently large n. Put

$$d=\min_{1\leqq j<n}\ \{\alpha i_{j+1}\}-\{\alpha i_j\}$$

Clearly,

$$N_1'\leqq\frac{\sum_{i>k}|I_i|}{d} \tag{8}$$

Next, let

$$\alpha=[a_0;\ a_1,\ a_2,\ \ldots,\ a_n,\ \ldots]$$

be a continued fraction expansion of α and $a_n\leqq c$ for all n. Since, see [4, p. 36],

$$\left|\alpha-\frac{p}{q}\right|>\frac{1}{q^2(c+2)(c+1)^2}=\frac{1}{c'q^2}$$

for every integers p, $q>0$, then

$$d\geqq\frac{1}{c'n} \tag{9}$$

Summary, by (7), (8), (9)

$$\frac{N}{n}\geqq\Big(1-\sum_{i\leqq k}|I_i|\Big)(1-\varepsilon)-\sum_{i>k}|I_i|c'$$

Thus

$$\frac{N}{n} \geqq (1-\Sigma|I_i|)(1-\varepsilon)$$

for every $\varepsilon > 0$ and for all sufficiently large n.

On the other hand,

$$\frac{n-\sum_{j\leqq k} n_j}{n} \geqq \frac{N-k}{n}$$

Thus

$$1-\Sigma|I_i| \geqq \frac{N}{n} - \varepsilon$$

for every $\varepsilon > 0$ and for all sufficiently large n. From it follows (5).

Now, let α be an irrational number which has unbounded partial quotient. Then by [5, p. 435] the sequence (2) is not eutaxic, by [3, Corollary 1] also it is not quick and therefore also not uniformly quick. From it follows that for this α there exists a sequence $\{I_i\}$ of pairwise disjoint subintervals of [0, 1], which covers (2),

$$\Sigma|I_i| < 1$$

and

$$\liminf_{n\to\infty} \frac{N}{n} = 0$$

The proof of Theorem 3 is finished.

By Definition every uniformly quick sequence is also uniformly distributed and also can be characterized by integration of continued functions.

Theorem 4. A sequence $\{y_i\}$ is uniformly quick in I if and only if for every sequence $\{I_i\}$ of pairwise disjoint subintervals of I which covered $\{y_i\}$ and for every bounded continued function f on I it is true

$$\frac{1}{n}\Big(\sum_{n_j>0} \frac{1}{n_j} \sum_{\substack{y_i\in I_j\\ i\leqq n}} f(y_i)\Big) \to \frac{1}{|I|}\int_{I-\bigcup I_i} f(y)\,\mathrm{d}y \tag{10}$$

as $n\to\infty$. Here n_j denotes the number of terms from $\{y_i\}_{i\leqq n}$ which belong to I_j.

Proof. It is sufficient to prove (10) for every step function. By Definition it is true for f = constant. Let f/J = constant, where J is a subinterval of I, $|J|>0$. The part of $\{y_i\}$ which belongs to J is also uniformly quick in J and therefore the sequence from (10) which is reduced on J convergents to

$$\frac{1}{|J|}\int_{J-\bigcup I_i} f(y)\,\mathrm{d}y$$

Since

$$\frac{\operatorname{card}\{y_i \in J;\, i \leqq n\}}{n} \to \frac{|J|}{|I|}$$

the proof is finished.

Similarly we obtain

$$\frac{1}{n}\Big(\sum_{i \leqq n} f(y_i) - \sum_{n_j > 0} \frac{1}{n_j} \sum_{\substack{y_i \in I_j \\ i \leqq n}} f(y_i)\Big) \to \frac{1}{|I|} \int_{\bigcup I_i} f(y)\, dy \tag{11}$$

as $n \to \infty$.

The limits (10), (11) can be perhaps applied in numerical integrations.

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SÚHRN

DVE VLASTNOSTI POSTUPNOSTI $n\alpha$(mod 1)

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V práci je okrem iného ukázané, že ak α je iracionálne číslo s ohraničenými koeficientami v reťazovom rozvoji a $\{I_i\}$ je postupnosť po dvoch disjunktných intervalov z $[0, 1]$, ktoré pokrývajú postupnosť necelých častí $\{n\alpha\}$ $(n = 1, 2, \ldots)$, potom integrál z ľubovolnej spojitej funkcie cez množinu $[0, 1] - \bigcup I_i$ môžeme vypočítať podľa (10) ako limitu z aritmetického priemeru istých aritmetických priemerov. Ohraničenosť koeficientov v reťazovom rozvoji α je i nutná podmienka.

РЕЗЮМЕ

О ДВУХ СВОЙСТВАХ ПОСЛЕДОВАТЕЛЬНОСТИ $n\alpha(\mathrm{mod}\ 1)$

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В работе между прочим показано, что если α – ирациональное число, непрерывная дробь которого имеет ограниченные коэффиценты и $\{I_i\}$ – последовательность непересекающихся промежутков из [0, 1] покрывающих последовательность дробных долей $\{n\alpha\}$ $(n=1, 2, \ldots)$, то интеграл любой непрерывной функции относительно множества $[0, 1]-\bigcup I_i$ можно вычислить по формуле (10) как предел среднего арифметического некоторых средних арифметических. Ограниченность коеффэффициентов непрерывной дроби числа α является необходимым условием.