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SOME NEW CRITERIONS FOR SEQUENCES WHICH SATISFY DUFFIN-SCHAEFFER CONJECTURE, II.

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All our criterions in papers [1, 2] for sequences $\{q_i\}$ of positive integers which satisfy Duffin—Schaeffer conjecture (abbreviated D.S.C., see introduction [2]) are applications [1, Theorem 2] and they are based on the fact that a small subsum of the sum

$$\sum\left(\frac{x}{q_i}-\frac{y}{q_j}\right)$$

has few summands. Here x/q_i, y/q_j are rational numbers in the canonical form from the interval [0, 1], $m<i\neq j\leqq n$, $x/q_i-y/q_j>0$.

Summarizing, our primary problem is as follows:

Let $\{t_i\}$ be a finite (not necessarily one-to-one) sequence of nonnegative reals and let Σ be a subsum of Σt_i with k terms, i.e.

$$\sum = t_{i(1)}+t_{i(2)}+\ldots+t_{i(k)} \qquad (1)$$

$$i(1)\neq i(2)\neq\ldots\neq i(k)$$

Our aim is to find a connection between the magnitude of the sum Σ and the number k of its terms in a form

$$k\leqq F(\Sigma)+k_o \qquad (2)$$

where $F(u)$ is a nondecreasing function, $F(u)\to 0$ as $u\to 0$ and k_o is a constant. In the papers [1, 2] we have solved this problem by two following methods:

Method 1. Let $\{t_i\}$ be a nondecreasing sequence. We need to find a nondecreasing sequence $\{g(i)\}$ such that

$$g(i)\leqq t_i \qquad (3)$$

for every i and such that the sum

$$\sum_{1\leqq i\leqq k} g(i)$$

has a good calculable lower estimation dependent on k (some initial segment of $\{g(i)\}$ can be negative). For example, we can use the integral

$$G(k)=\int_0^k g(u)\,\mathrm{d}u \leqq \sum_{1\leqq i\leqq k} g(i) \tag{4}$$

Evidently, by (3), (4) also

$$G(k)\leqq\Sigma \tag{5}$$

for every sum Σ of a form (1). For $k\geqq k_0$ we find a function $G_o(k)$ such that $G_o(k)\leqq G(k)$ and G_o has a good calculable inverse function G_o^{-1}, thus

$$k\leqq G_o^{-1}(\Sigma)+k_o \tag{6}$$

i.e. we found an estimation (2).

In applications, a nondecreasing order of the sequence $\{t_i\}$ need not be known (e. g. if $\{t_i\}$ is a sequence of lengths of rational numbers). In this cases we exchange (3) with equivalent inequality

$$g\Big(\sum_{t_i\leqq t} 1\Big)\leqq t \tag{7}$$

for every $t>0$, or (if g is increasing)

$$\sum_{t_i\leqq t} 1\leqq g^{-1}(t)$$

The preceding Method 1 we have used in [1, Proposition 4], where we assumed that the sequence $\{t_i\}$ satisfies

$$g(i)=\frac{i-c_2}{c_1}\leqq t_i$$

for every i which is equivalent to

$$\sum_{t_i\leqq t} 1\leqq c_1t+c_2$$

for every $t>0$. From it

$$k\leqq\sqrt{6c_1\Sigma}+3c_2$$

We can generalize this result by the following

Example 1. Let $\{t_i\}$ be nondecreasing sequence which satisfies

$$g(i)=\frac{i^s-c_2}{c_1}\leqq t_i$$

for every i (where s, c_1, c_2 are positive constants). It is equivalent to

$$\left(\sum_{t_i \leqq t} 1\right)^s \leqq c_1 t + c_2 \tag{8}$$

By (5) for every sum $\sum$ from (1) we have

$$\int_0^k \frac{u^s - c_2}{c_1}\, du = \frac{\dfrac{k^{s+1}}{s+1} - c_2 k}{c_1} \leqq \Sigma$$

From there it follows, if

$$k \geqq (2c_2(s+1))^{1/s}$$

then

$$k \leqq (2(s+1)c_1\Sigma)^{1/(s+1)}$$

Summarizing

$$k \leqq (2(s+1)c_1\Sigma)^{1/(s+1)} + (2c_2(s+1))^{1/s} \tag{9}$$

Method 2. This method is based on an estimation of a sum

$$\sum_{t_i \leqq t} t_i$$

Let for nondecreasing $\{t_i\}$ exist a nondecreasing function $g(u)$, $g(u)>0$ as $u>0$, such that

$$g\left(\sum_{i \leqq j} t_i\right) \leqq t_j \tag{10}$$

for every j, what is equivalent to

$$g\left(\sum_{t_i \leqq t} t_i\right) \leqq t \tag{11}$$

for all $t>0$. From (10) follows

$$1 \leqq \frac{t_j}{g\left(\sum_{i \leqq j} t_i\right)}$$

$$k \leqq \frac{t_1}{g(t_1)} + \frac{t_2}{g(t_1+t_2)} + \ldots + \frac{t_k}{g\left(\sum_{i \leqq k} t_i\right)} \tag{12}$$

The sum of the right side of (12) is a lower integral sum of the nonincreasing function $1/g(u)$ on the interval $\left[0, \sum_{i \leqq k} t_i\right]$ which is divided into following intervals

$$[0, t_1], [t_1, t_1+t_2], [t_1+t_2, t_1+t_2+t_3], \ldots$$

Thus this sum is not greater than the integral of $1/g(u)$ on the interval $\left[0, \sum_{i\leqq k} t_i\right]$, i.e. by (12)

$$k \leqq G\Big(\sum_{i\leqq k} t_i\Big) \leqq G(\Sigma) \tag{13}$$

for every sum Σ from (1), where

$$G(\Sigma) = \int_0^{\Sigma} \frac{du}{g(u)} \tag{14}$$

If the integral (14) is divergent in 0, then we take t_1 as a lower bound for integration and we have

$$k-1 \leqq G\Big(\sum_{i\leqq k} t_i\Big) - G(t_1) \tag{15}$$

Usually, in applications it is $\Sigma t_i \leqq c_o$, $G(c_o) \leqq 0$, $-G(t_1) \to +\infty$ as $t_1 \to 0$. Hence

$$k-1 \leqq -G(t_1) \tag{16}$$

I.e. we obtain also an upper estimation for the total number of terms in the finite sequence $\{t_i\}$ which is dependent on $t_1 = \min\{t_i\}$ and which is better than a trivial estimation c_o/t_1. Thus, we obtain (2) where the right side is not dependent on Σ and $k_o = -G(t_1) + 1$.

Example 2. Let a sequence $\{t_i\}$ satisfy the estimation

$$\sum_{t_i \leqq t} t_i \leqq ct^{1+\varepsilon} \ (\varepsilon > 0) \tag{17}$$

for every $t > 0$. In this case

$$g(u) = \left(\frac{u}{c}\right)^{\frac{1}{1+\varepsilon}}$$

and using (13), (14), we have

$$k \leqq c^{\frac{1}{1+\varepsilon}} \frac{1+\varepsilon}{\varepsilon} (\Sigma)^{\frac{\varepsilon}{1+\varepsilon}} \tag{18}$$

for every sum Σ of the form (1) (for $\varepsilon = 1$ we obtain [1, (23)]).

Example 3. Let a sequence $\{t_i\}$ satisfy the estimation

$$\sum_{t_i \leqq t} t_i \leqq ct \tag{19}$$

for every $t > 0$. In this case $g(u) = u/c$ and the integral (14) is divergent. Therefore by (16) we obtain

$$k-1 \leqq -c \log t_1 \tag{20}$$

if we assume that $\Sigma t_i \leqq 1$. Thus we obtain an estimation for the number of terms of $\{t_i\}$ which is better than $1/t_1$.

Example 4. Let

$$\left(\sum_{t_i \leqq t} t_i\right)^{1+\delta} \leqq ct \ (\delta > 0) \tag{21}$$

Then according to (16) we have

$$k-1 \leqq \frac{c}{\delta} \frac{1}{t_1^{\delta}} \tag{22}$$

From preceding Examples 1—4 and by means of [1, Theorem 2] we can obtain new criterions for a quick (and also for a uniformly quick) sequence $\{y_i\}$ (see [1, Definitions 3,9]) by the following:

Let us order the finite section $\{y_i\}_{m<i\leqq n}$ to a nondecreasing sequence

$$y_{i(m+1)} \leqq y_{i(m+2)} \leqq \ldots \leqq y_{i(n)}$$

and put

$$\{t'_i\} = \{y_{i(j+1)} - y_{i(j)};\ m<j<n\}$$

Let us choose $\{t_i\}$ such that

$$0 \leqq t_i \leqq t'_i \tag{23}$$

for every i (if we shall use Method 2 it must be $t_i > 0$). Let us put constants in the assumptions (8), (17), (19), (21) such that their consequences (9), (18), (20), (22) have the form

$$k \leqq (n-m)c_m F(\Sigma) + (n-m)c'_m \tag{24}$$

Concretely:

$$\begin{gathered} \text{in (8) } c_1 = (n-m)^{s+1}c_m,\ c_2 = (n-m)^s c'_m, \text{ in (17)} c = (n-m)^{1+\varepsilon}c_m, \\ \text{in (19) } c = c'_m(n-m)/\log(n-m), \text{ in (21) } c = c'_m(n-m)^{1-K\delta} \end{gathered} \tag{25}$$

We assume here in two last cases that

$$t_1 \geqq \frac{1}{(n-m)^K} \tag{26}$$

for a fixed positive K.

Since in all our cases we obtain in (24) a function $F(u)$ nondecreasing and $F(u) \to 0$ as $u \to 0$, we can use [1, Theorem 2] and from it

Theorem 1. Let $\{y_i\}$ be a positively distributed (uniformly distributed) sequence for which there exist sequences $\{t_i\}$ which satisfy (23) and which satisfy at least one inequality from (8), (17), (19), (21) (with constants from (25)) for infinitely many m (such that $c'_m \to 0$ as $m \to \infty$ and c_m is arbitrarily large), for all sufficiently large n and for every $t>0$. Then $\{y_i\}$ is a quick (uniformly quick) sequence (see [1, Definitions 1, 3, 9]).

Next, let $\{q_i\}$ be a one-to-one sequence of positive integers and $\{f(q_i)\}$ be a sequence of positive reals and let $\{y_i\}$ be the one-to-one sequence of all rational numbers from the interval [0, 1] denominators of which (in a canonical form) are contained in the sequence $\{q_i\}$, and ordering of which is induced by the place of occurrence of q_i in $\{q_i\}$. Using Theorem 1 for this $\{y_i\}$ and [1, Part 5] we can obtain new criterions for sequences $\{q_i\}$ which satisfy D.S.C. In this case let us denote

$$\{t'_i\} = \left\{\frac{x}{q_i} - \frac{y}{q_j} > 0;\ m < i \neq j \leqq n,\ 0 < x < q_i,\ 0 < y < q_j,\ (x, q_i) = (y, q_j) = 1\right\} \tag{27}$$

$$0 < t_i \leqq t'_i \tag{28}$$

for all i. Note that we can use in (27) neighbouring rational numbers x/q_i, y/q_j only. Now, in (25) we replace $n-m$ by

$$A = \sum_{m<i\leqq n} \varphi(q_i) \quad (\varphi \text{ — Euler's function}) \tag{29}$$

Thus, the inequalities (8), (17), (19), (21) have a form

$$\left(\sum_{t_i \leqq t} 1\right)^s \leqq c_m t A^{s+1} + c'_m A^s \tag{30}$$

$$\sum_{t_i \leqq t} t_i \leqq c_m t^{1+\varepsilon} A^{1+\varepsilon} \tag{31}$$

$$\sum_{t_i \leqq t} t_i \leqq c'_m t \frac{A}{\log A} \tag{32}$$

$$\left(\sum_{t_i \leqq t} t_i\right)^{1+\delta} \leqq c'_m t A^{1-(2+\varepsilon)\delta} \tag{33}$$

In two last cases we used the relation

$$t'_1 = \min\{t'_i\} \geqq \frac{1}{A^{2+\varepsilon}} \quad (\varepsilon > 0 \text{ is arbitrarily small}) \tag{34}$$

for all $\{q_i\}$, and we assume that also $t_1 = \min\{t_i\}$ satisfies (34). From preceding results we obtain the following theorem

Theorem 2. Let $\{q_i\}$ be a one-to-one sequence of positive integers for which there exist sequences $\{t_i\}$ which satisfy (28) and which also satisfy at least one inequality from (30)—(33) for infinitely many m (such that $c'_m \to 0$ as $m \to \infty$ and c_m

is arbitrarily large), for all sufficiently large n and for every $t>0$. Then $\{q_i\}$ satisfies D.S.C. with every nonincreasing $\{f(q_i)\}$.*

Note 1. Let us suppose that in the sequence $\{t'_i\}$ from (27) we have only the lengths $x/q_i - y/q_j > 0$ for which x/q_i, y/q_j are neighbouring. We can diminish these lengths to $1/q_k^2$, where $q_k = \max\{q_i, q_j\}$. The number of neighbouring x/q_i, y/q_j for which $\max\{q_i, q_j\} = q_k$ is not greater than $2\varphi(q_k)$. Thus in the sequence $\{t_i\}$ we can place $2\varphi(q_k)$ times the number $1/q_k^2$ for every k for which $m<k\leqq n$. In this case it is sufficient to prove any estimation from (30)—(33) only for every $t=1/q_k^2$, $m<k\leqq n$. Next without loss of generality, we can assume that the finite section $\{q_i\}_{m<i\leqq n}$ is ordered by increasing and from it

$$\sum_{t_i \leqq 1/q_k^2} 1 = 2 \sum_{k\leqq i\leqq n} \varphi(q_i)$$

$$\sum_{t_i \leqq 1/q_k^2} t_i = 2 \sum_{k\leqq i\leqq n} \varphi(q_i)/q_i^2$$

for every k, $m<k\leqq n$. By preceding and by Theorem 2 we can derive some new criterions for sequences $\{q_i\}$ which satisfy D.S.C., e.g. using (30) we have

Theorem 3. Let us have

$$\Big(\sum_{k\leqq i\leqq n} \varphi(q_i)\Big)^s \leqq c_m(1/q_k^2)\Big(\sum_{m<i\leqq n} \varphi(q_i)\Big)^{s+1} + c'_m\Big(\sum_{m<i\leqq n} \varphi(q_i)\Big)^s \tag{35}$$

for infinitely many m (such that $c'_m \to 0$ as $m\to\infty$ and c_m is arbitrarily large), for all sufficiently large n and for every k, $m<k\leqq n$ (s is a fixed positive constant). Then $\{q_i\}$ satisfies D.S.C. with every nonincreasing $\{f(q_i)\}$.

Note 2. If we divide the sequence $\{t_i\}$ into finite fixed parts such that every part satisfies an inequality of the form (2) then also $\{t_i\}$ satisfies (2) (with other F, k_o).

Next, the sequence $\{t'_i\}$ from (27) we can divide into two parts $\{t'_i\}^1$, $\{t'_i\}^2$, where into $\{t'_i\}^1$ we put these lengths $x/q_i - y/q_j$ for which

$$\frac{x}{q_i} - \frac{y}{q_j} \geqq \frac{\log q_{ij}}{q_{ij}d_{ij}} \tag{36}$$

whereas $\{t'_i\}^2$ contains all the remaining ones. Here

$$d_{ij} = (q_i, q_j),\ q_{ij} = q_i q_j / d_{ij}^2 \tag{37}$$

By [2, Theorem 4] the number of $x/q_i - y/q_j > 0$ which are not satisfying (36) (for fixed i, j) is not greater than

$$B = c_o \frac{\log q_{ij}}{q_{ij}d_{ij}} \varphi(q_i)\varphi(q_j).$$

* Since in (31) $c'_m = 0$, it is sufficient to assume that this estimation holds for some m.

and we can diminish each of them to $1/q_{ij}d_{ij}$. Let us put in $\{t_i\}^2$ these $1/q_{ij}d_{ij}$ by $[B]$-times for every $i, j, m<i\neq j\leqq n$ and let us give $\{t_i\}^1=\{t'_i\}^1$. Let $\{t_i\}$ be the sum of $\{t_i\}^1$ and $\{t_i\}^2$. Then $\{t_i\}$ satisfies (28) and $\{t_i\}^1$ satisfies (see [2, Theorem 4]) (30) with $s=1$, $c'_m=0$, $c_m=c_o$. For $\{t_i\}^2$; the sum $\sum_{t_i\leqq t} 1$ has an estimation ΣB, where the summation through over all i, j for which $m<i\neq j\leqq n$, $1/q_{ij}d_{ij}\leqq t$. Similarly the sum $\sum_{t_i\leqq t} t_i$ has an upper estimation

$$c_o \sum_{\substack{1/q_{ij}d_{ij}\leqq t\\ m<i\neq j\leqq n}} \frac{\log q_{ij}}{(q_{ij}d_{ij})^2}\varphi(q_i)\varphi(q_j) \tag{38}$$

By Cauchy inequality (38) is not greater than $A\sqrt{C}$, where

$$C=c_o^2 \sum_{\substack{1/q_{ij}d_{ij}\leqq t\\ m<i\neq j\leqq n}} \frac{\log^2 q_{ij}}{q_{ij}^3 d_{ij}^2}\,\frac{\varphi(q_i)}{q_i}\,\frac{\varphi(q_j)}{q_j} \tag{39}$$

(also see (29)). If we replace in (31)—(33) the sum $\sum_{t_i\leqq t} t_i$ by $A\sqrt{C}$ and using Theorem 2 we obtain some new criterions, e.g. if we use (31), then we have

Theorem 4. A sequence $\{q_i\}$ satisfies D.S.C. with every nonincreasing $\{f(q_i)\}$ if

$$C\leqq c_m t^{2+\varepsilon} A^{\varepsilon} \tag{40}$$

for some m, for all sufficiently large n and for every $t>0$.

From it for the k-th powers we have

Theorem 5. A sequence $\{q_i^k\}$ satisfies D.S.C. for every nonincreasing $\{f(q_i^k)\}$ if

$$\sum_{m<i\neq j\leqq n} d_{ij}^{k\varepsilon}\leqq c_m\Big(\sum_{m<i\leqq n}\varphi(q_i^k)\Big)^{\varepsilon} \tag{41}$$

for some m and for every sufficiently large n. Here $d_{ij}=(q_i, q_j)$, $k>1$ is an integer, c_m is arbitrarily large, ε is a positive constant for which *

$$0<\varepsilon<1-\frac{1}{k} \tag{42}$$

Proof. From C in (39) we can omit $\varphi(q_i)\varphi(q_j)/q_iq_j$. For $d_{ij}=$constant the number of solutions of $q_{ij}=q$ (see (37)) is not greater than $2^{\upsilon(q)}$ ($\upsilon(q)$ denotes the number of different prime factors of q). Therefore,

$$C\leqq c_o^2 \sum_{m<i\neq j\leqq n} \frac{1}{d_{ij}^2} \sum_{1/d_{ij}t\leqq q} \frac{\log^2 q}{q^3}\, 2^{\upsilon(q)} \tag{43}$$

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* In (41) the summation through over all pairwise different d_{ij}

The second sum is not greater than the integral from $c/q^{3-\delta}$ (δ is sufficiently small, c is sufficiently large) in the interval $[1/d_{ij}t, +\infty)$. If we replace $\{q_i\}$ by the sequence $\{q_i^k\}$ of the k-th powers, then we must replace also d_{ij}, q by d_{ij}^k, q^k and the second sum in (43) is not greater than the integral of $c/q^{3k-\delta}$ on the interval $[1/d_{ij}t^{1/k}, +\infty)$. Thus

$$C \leqq ct^{3-(\delta+1)/k} \sum d_{ij}^{k(1-(\delta+1)/k)} \tag{44}$$

Next we substitute (44) to (40) and we choose δ such that $\delta > 0$, $3k - \delta > 1$ and

$$3 - \frac{\delta+1}{k} = 2 + \varepsilon$$

i.e. such that the exponents of t in (44) and in (40) are equal. From it and from Theorem 4 there follows Theorem 5.

We note than we can prove Theorem 5 also using (30) in Theorem 2.

Using [2, Theorem 2] we can prove a new criterion which is based also on a property of d_{ij}.

Theorem 6. Suppose that

$$d_{ij} \leqq (q_i q_j)^{(1/2)-\varepsilon} \tag{45}$$

for every $i \neq j$ and

$$\sum \frac{\log^2 q_i}{q_i^{2\varepsilon}} < +\infty \tag{46}$$

Then $\{q_i\}$ satisfies D.S.C. with every $\{f(q_i)\}$.

Proof. If the series

$$\sum \frac{\log^2 q_{ij}}{q_{ij}} = \sum \frac{d_{ij}^2 \log^2 \left(\frac{q_i q_j}{d_{ij}^2}\right)}{q_i q_j} = \sum \frac{\log^2 \left(\frac{q_i q_j}{d_{ij}^2}\right)^{d_{ij}}}{q_i q_j} \tag{47}$$

is convergent, then $\{q_i\}$ satisfies the assumption of [2, Theorem 2]. Moreover, the function

$$\left(\frac{q_i q_j}{u^2}\right)^u$$

is increasing in $(0, \sqrt{q_i q_j}/e]$ and decreasing in $[\sqrt{q_i q_j}/e, +\infty)$. From that and by (45) it follows that the series from (47) is majorized by

$$\sum 4\varepsilon^2 \frac{\log^2 q_i q_j}{(q_i q_j)^{2\varepsilon}}$$

and if it holds (46), then this series is convergent.

Example 5. Let $\{q_i\} = \{u_i v_i\}$ such that

(i) $\{u_i\}$ are relative primes

(ii) $v_i \leqq u_i^{1/2\varepsilon - 1}$ for all i

If $\{q_i\}$ satisfies (46) then also satisfies D.S.C with every $\{f(q_i)\}$. Really, by (ii)

$$(v_i v_j)^{\varepsilon} \leqq (u_i u_j)^{1/2-\varepsilon}$$

and from it by (i)

$$d_{ij} \leqq (v_i v_j)^{1/2} \leqq (u_i v_i u_j v_j)^{1/2-\varepsilon}$$

Thus $\{q_i\}$ satisfies the assumptions of Theorem 6.

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SÚHRN

NIEKTORÉ NOVÉ KRITÉRIÁ PRE POSTUPNOSTI, KTORÉ SPĹŇAJÚ DUFFIN—SCHAEFFEROVU HYPOTÉZU, II

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V práci je okrem iného ukázané, že ak postupnosť k-tých mocnín $\{q_i^k\}$ spĺňa (41), postupnosť kladných reálnych čísel $\{f(q_i^k)\}$ je nerastúca a

$$\Sigma f(q_i^k)\varphi(q_i^k) = +\infty$$

potom skoro pre všetky α má nerovnosť

$$\left|\alpha - \frac{x}{q_i^k}\right| < f(q_i^k)$$

pre nekonečne veľa i celočíselné riešenie x také, že x, q_i sú nesúdeliteľné.

РЕЗЮМЕ

НЕКОТОРЫЕ НОВЫЕ ПРИЗНАКИ ДЛЯ ПОСЛЕДОВАТЕЛЬНОСТЕЙ УДОВЛЕТВОРЯЮЩИХ ГИПОТЕЗЕ ДАФФИН—ШАФФЕРА, II.

О. Штраух, Братислава

В работе между прочим показано, что если последовательность k-тых степеней $\{q_i^k\}$ удовлетворяет условию (41), последовательность положительных действительных чисел $\{f(q_i^k)\}$ — неубывающая и

$$\Sigma f(q_i^k)\varphi(q_i^k) = +\infty,$$

то для почти всех α неравенство

$$\left|\alpha - \frac{x}{q_i^k}\right| < f(q_i^k)$$

имеет целочисленное решение x для бесконечно многих i, такое, что x, q_i — взаимно простые.