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ON A METRIC RESULT IN THE THEORY OF CONTINUED FRACTIONS

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In the metric theory of continued fractions the well-known theorem of Lévy on the distribution of numbers 1, 2, ... in continued fractions of real numbers has a cetral position (cf. [1], p. 110; [3]). In the paper [4] it is shown that the set of all real numbers for which the theorem of Lévy holds is a set of the first Baire category. In this paper we shall prove an analogous result related to the following well-known theorem from the metric theory of continued fractions:

For almost all irrational numbers $x \in (0, 1)$, $x = [0; a_1(x), a_2(x), \dots]$ (the continued fraction of x) we have

$$\lim_{n\to\infty} \sqrt[n]{(a_1(x) \cdot a_2(x) \dots a_n(x))} = \prod_{r=1}^{\infty} \left(1 + \frac{1}{r(r+2)}\right)^{\frac{\log r}{\log 2}} \tag{1}$$

(cf. [1], p. 111).

In what follows denote by $\{b_n\}'_n$ the set of all limit points of the sequence $\{b_n\}_{n=1}^{\infty}$.

Theorem. Denote by H the set of all irrational numbers $x \in (0, 1)$ for which we have

$$\{\sqrt[n]{(a_1(x) \cdot a_2(x) \dots a_n(x))}\}'_n = \langle 1, +\infty\rangle$$

Then H is a residual set in $(0, 1)$.

Corollary. The set M of all irrational numbers $x \in (0, 1)$ for which (1) holds, is a set of the first Baire category.

Proof of Theorem. Denote by X the metric space of all irrational numbers $x \in (0, 1)$ with the Euclidean metric.

Let $z \in (1, +\infty)$ and let k, m be positive integers. Denote by $H(z, k, m)$ the set of all $x \in X$ for which

$$|\sqrt[m]{(a_1(x)\cdot a_2(x)\dots a_m(x))}-z|<\frac{1}{k}.$$

Put

$$H(z)=\bigcap_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}H(z,k,m) \tag{2}$$

Then clearly $H(z)$ is the set of all $x\in X$ for which

$$z\in\{\sqrt[n]{(a_1(x)\cdot a_2(x)\dots a_n(x))}\}_n'$$

We shall show that $H(z)$ is a residual set in $(0, 1)$.

In agreement with [1] the symbol $E\begin{pmatrix}1, & 2, & \dots, & n\\ k_1, & k_2, & \dots, & k_m\end{pmatrix}$ denotes the interval of all such numbers $x\in(0, 1)$, $x=[0; a_1(x), a_2(x), \dots]$ for which $a_j(x)=k_j$ $(j=1, 2, \dots, m)$ (cf. [1], p. 73). Then the set

$$E\begin{pmatrix}1, & 2, & \dots, & m\\ k_1, & k_2, & \dots, & k_m\end{pmatrix}\cap X \tag{3}$$

is an open set in X. It is evident from the definition of the set $H(z, k, m)$ that it is a union of some sets of the form (3). Hence $H(z, k, m)$ is an open set in X and we see from (2) that $H(z)$ is a G_δ-set in X.

We shall prove that $H(z)$ is a dense set in X.

It suffices to prove that if p is an arbitrarily chosen positive integer and $k_1, k_2, \dots, k_p$ are arbitrary positive integers, then there exists such an $y=[0; a_1(y), a_2(y), \dots]\in X$ that

$$a_j(y)=k_j \quad (j=1, 2, \dots, p) \tag{4}$$

$$z\in\{\sqrt[n]{(a_1(y)\cdot a_2(y)\dots a_n(y))}\}_n' \tag{5}$$

Let $p, k_1, k_2, \dots, k_p$ have the previous meaning. Put $t=[z]+1>z$. Since

$$\lim_{n\to\infty}\sqrt[n]{a}=1 \tag{6}$$

for each $a>0$, there exists such a positive integer v_1 that

$$\sqrt[p+v_1]{(k_1\cdot k_2\dots k_p\cdot 1\cdot 1\dots 1)}<z$$

(the number of 1's being equal to v_1).

On account of the definition of the number t there exists such a positive integer v_2 that

$$\sqrt[p+v_1+v_2]{(k_1 \cdot k_2 \dots k_p \cdot 1 \cdot 1 \dots 1 \cdot t \cdot t \dots t)} \geqq z$$

(the number of 1's being v_1, the number of t's being v_2).

Let v_2 be the least positive integer such that the foregoing inequality holds. Then according to the choice of v_2 we have

$$\sqrt[p+v_1+v_2-1]{(k_1 \cdot k_2 \dots k_p \cdot 1 \cdot 1 \dots 1 \cdot t \cdot t \dots t)} < z$$

(the number of 1's is v_1 and the number of t's is v_2-1).

Further, according to (6) there exists such a positive integer v_3 that

$$\sqrt[p+v_1+v_2+v_3]{(k_1 \cdot k_2 \dots k_p \cdot 1 \cdot 1 \dots 1 \cdot t \cdot t \dots t \cdot 1 \cdot 1 \dots 1)} < z$$

(the number of 1's in the second group of 1's is equal to v_3), a.s.o.

Put $s_0=0$, $s_n=v_1+v_2+\dots+v_n$ $(n=1, 2, \dots)$, $a_j(y)=k_j$ $(j=1, 2, \dots, p)$, $a_j(y)=t$ $(p+s_{2k+1}<j \leqq p+s_{2k+2})$, $a_j(y)=1$ $(p+s_{2k}<j \leqq p+s_{2k+1})$ $(k=0, 1, 2, \dots)$. Then $y=[0; a_1(y), a_2(y), \dots] \in X$ and from the construction of the numbers v_j $(j=1, 2, \dots)$ we get

$$\sqrt[p+s_{2k}]{(a_1(y) \cdot a_2(y) \dots a_{p+s_{2k}}(y))} \geqq z \qquad (k=1, 2, \dots), \tag{7}$$

$$\sqrt[p+s_{2k}-1]{(a_1(y) \cdot a_2(y) \dots a_{p+s_{2k}-1}(y))} < z \qquad (k=1, 2, \dots). \tag{8}$$

By a simple estimation we get from (8)

$$\sqrt[p+s_{2k}]{a_1(y) \cdot a_2(y) \dots a_{p+s_{2k}-1}(y))} < z \tag{8'}$$

From (7) and (8′) we obtain

$$\frac{1}{\sqrt[p+s_{2k}]{t}} < \frac{z}{\sqrt[p+s_{2k}]{(a_1(y)a_2(y) \dots a_{p+s_{2k}}(y))}} \leqq 1 \qquad (k=1, 2, \dots).$$

From this for $k \to \infty$ we get

$$z = \lim_{k \to \infty} \sqrt[p+s_{2k}]{(a_1(y) \cdot a_2(y) \dots a_{p+s_{2k}}(y))}$$

Hence (5) is true. The validity of (4) follows from the construction of y.

Thus $H(z)$ is a dense G_δ-set in X. Therefore (cf. [2], p. 49) the set $H(z)$ is residual in X and so it is residual in (0, 1), too.

Now, let $r_1, r_2, \dots, r_n, \dots$ be the enumeration of all rational numbers of the interval $(1, +\infty)$. It is easy to check that $H = \bigcap_{n=1}^{\infty} H(r_n)$ and according to the previous part of the proof each of the sets $H(r_n)$ $(n = 1, 2, \ldots)$ is residual in $(0, 1)$. Therefore H is a residual set in $(0, 1)$. This ends the proof.

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SÚHRN

O ISTOM METRICKOM VÝSLEDKU V TEÓRII REŤAZOVÝCH ZLOMKOV

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Je známe, že pre skoro všetky

$$x = [0; a_1(x), a_2(x), \ldots] \in (0, 1)$$

platí

$$\lim_{n \to \infty} \sqrt[n]{a_1(x) \ldots a_n(x)} = \prod_{r=1}^{\infty} \left(1 + \frac{1}{r(r+2)}\right)^{\frac{\log r}{\log 2}} \tag{1}$$

V práci je dokázané, že množina všetkých tých $x \in (0, 1)$, pre ktoré platí (1), je množina prvej Baireovej kategórie.

РЕЗЮМЕ

ОБ ОДНОМ МЕТРИЧЕСКОМ РЕЗУЛЬТАТЕ В ТЕОРИИ ЦЕПНЫХ ДРОБЕЙ

Т. Шалат, Братислава

Известно, что для почти всех

$$x = [0\,;\, a_1(x),\, a_2(x),\, \ldots] \in (0,\, 1)$$

имеет место

$$\lim_{n \to \infty} \sqrt[n]{a_1(x) \ldots a_n(x)} = \prod_{r=1}^{\infty} \left(1 + \frac{1}{r(r+2)}\right)^{\frac{\log r}{\log 2}}.$$

В работе доказывается, что множество всех $x \in (0,\, 1)$, для которых имеет место (1) — множество первой категории Бэра.