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ON EQUIVALENCE RELATIONS

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Dedicated to Prof. O. Borůvka on the occasion of his eightieth birthday

In this paper, the product of two binary relations R, S is denoted by RS, the converse relation of R by R^-: We use the symbol E_M for the identity relation in a set M. If R is an arbitrary binary relation in M, we denote the transitive hull of R by $t(R)$; so

$$t(R) = R \cup R^2 \cup R^3 \cup \ldots$$

Let R, S be equivalence relations in a set A. It is well known that RS is an equivalence relation if and only if $RS = SR$ (see e.g. [3], [4], [2]). One can easily verify that the condition $RS = SR$ is equivalent to the equality

$$RS = (R \cup S)^2$$

We will next consider analogous questions for $n \geqq 2$ equivalence relations.

Theorem 1. Let $R_1, \ldots, R_n$ be equivalence relations in A ($n \geqq 2$). Then the following statements are equivalent:

a) $R_1 \ldots R_n$ is an equivalence relation,

b) $R_1 \ldots R_n = (R_1 \cup \ldots \cup R_n)^n$,

c) $R_1 \ldots R_n = t(R_1 \cup \ldots \cup R_n)$.

Proof. Denote by I_n the set of all finite sequences $(i_1, \ldots, i_n)$ of the numbers $1, \ldots, n$. If $R_1 \ldots R_n$ is an equivalence relation then $(R_1 \ldots R_n)^k = R_1 \ldots R_n$ for every $k \geqq 2$. Specially $(R_1 \ldots R_n)^n = R_1 \ldots R_n$, and since the relations $R_1, \ldots, R_n$ are reflexive, if follows that

$$R_{i_1} \ldots R_{i_n} \subset R_1 \ldots R_n$$

for every $(i_1, \ldots, i_n) \in I_n$. Therefore by the well known distributivity properties

$$(R_1 \cup \ldots \cup R_n)^n = R_1 \ldots R_n$$

Suppose now that b) holds. Then

$$R_1 \dots R_n = \bigcup_{(i_1, \dots, i_n) \in I_n} R_{i_1} \dots R_{i_n}$$

and

$$(R_1 \cup \dots \cup R_n)^{n+1} = (R_1 \cup \dots \cup R_n)^n (R_1 \cup \dots \cup R_n) =$$
$$= R_1 \dots R_n (R_1 \cup \dots \cup R_n)$$

For every $k \in \{1, \dots, n\}$ we have

$$R_1 R_2 \dots R_n R_k \subset R_1 (R_1 \cup \dots \cup R_n)^n = R_1 \dots R_n$$

Thus

$$(R_1 \cup \dots \cup R_n)^{n+1} = (R_1 \cup \dots \cup R_n)^n$$

and consequently

$$(R_1 \cup \dots \cup R_n)^k = R_1 \dots R_n$$

for each $k \geqq n$. Finally

$$t(R_1 \cup \dots \cup R_n) = R_1 \dots R_n$$

The implication c) $\Rightarrow$ a) is obvious.

Corollary. Let R, S be equivalence relations in A and n a positive integer. Then

a) $(RS)^n$ is an equivalence relation if and only if

$$(RS)^n = (SR)^n$$

b) $(RS)^n R$ is an equivalence relation if and only if

$$(RS)^n R \supset S(RS)^n$$

Proof. Put in Theorem 1

$$R_1 = R_3 = \dots = R_{2n-1} = R$$
$$R_2 = R_4 = \dots = R_{2n} = S$$

If $(RS)^n$ is an equivalence relation then by Theorem 1

$$(RS)^n = (R \cup S)^{2n} = (RS)^n \cup (SR)^n$$

Thus

$$(SR)^n \subset (RS)^n$$

and since

$$(RS)^n = ((SR)^n)^{-} \subset ((RS)^n)^{-} = (SR)^n$$

we obtain the equality

$$(RS)^n = (SR)^n$$

The converse follows from the equality

$$(R \cup S)^{2n} = (RS)^n \cup (SR)^n$$

The proof of b) is similar. Note that inclusion in b) can not be replaced in general by equality (see [5], p. 172).

Let now R be an equivalence relation in A and f a mapping of A onto a set B, i.e. $f^-f=E_B$. Define the binary relation S in B as follows: y_1Sy_2 means that there exist elements x_1, $x_2\in A$ such that x_1Rx_2, $f(x_1)=y_1$, $f(x_2)=y_2$. In other words

$$S=f^-Rf$$

We ask under which conditions S is an equivalence relation in B. The answer is given in the following theorem (F denotes the equivalence relation ff^-).

Theorem 2. The following conditions are equivalent:

a) S is an equivalence relation,

b) FRF is an equivalence relation,

c) $FRF=t(F\cup R)$.

Proof. If S is an equivalence relation, then

$$FRF\supset E_A$$
$$(FRF)^-=FRF$$
$$(FRF)^2=ff^-Rff^-ff^-Rff^-=ff^-Rff^-Rff^-=fSf^-=ff^-Rff^-=FRF$$

So FRF is an equivalence relation.

By Theorem 1, we have b) $\Leftrightarrow$ c).

Suppose that FRF is an equivalence relation. Since

$$f^-FRFf=S$$

we get

$$E_B\subset S$$
$$S^-=S$$
$$S^2=f^-FRFff^-FRFf=f^-(FRF)^2f=f^-FRFf=S$$

Therefore S is an equivalence relation and Theorem 2 is proved.

In the following theorem we characterize a special situation of this kind (cf. [1]).

Theorem 3. In the notations of Theorem 2, the following statements are equivalent:

a) if X_1, X_2 are different classes of R, then either $f(X_1)=f(X_2)$ or $f(X_1)\cap f(X_2)=\emptyset$,

b) $FRF=FR$,

c) $FR=RF$.

Proof. Suppose that a) holds and x_1FRFx_2. Denote by X_1, resp. X_2 the class of x_1, resp. x_2 in R. By hypothesis, there are such x_1', $x_2'\in M$ that $x_1'Rx_2'$, $f(x_1)=$ $=f(x_1')=y_1$, $f(x_2)=f(x_2')=y_2$. Let X_3 be the class of the elements x_1', x_2' in R. Then

$$y_1\in f(X_1)\cap f(X_3)$$
$$y_2\in f(X_2)\cap f(X_3)$$

and consequently

$$f(X_1)=f(X_2)=f(X_3)$$

Thus there exists such $x_2'' \in X_2$ that $x_1 f y_1 f^- x_2''$, i.e. $x_1 FR x_2$. We have proved that $FRF \subset FR$; with the evident inclusion $FR \subset FRF$ we get the equality b).

Suppose now that b) holds. Then

$$(FR)^2=(FRF)R=FRR=FR$$
$$(FR)^-=(FRF)^-=FRF=FR$$
$$FR \supset E_A$$

i.e. FR is an equivalence relation.

Suppose finally that the condition c) is satisfied and a) does not hold. Then there are such classes X_1, X_2 of R that $f(X_1)\neq f(X_2)$ and $f(X_1)\cap f(X_2)\neq\emptyset$. Let e.g. $y_1 \in f(X_1)\cap f(X_2)$, $y_2 \in f(X_2)-f(X_1)$, $y_1=f(x_1)$, $y_2=f(x_2)$. Then $x_1 f y_1 f^- R x_2$, but there is no $x \in X_1$ such that $x_1 R x f f^- x_2$, since otherwise we should have $f(x)=$ $=f(x_2)=y_2 \in f(X_1)$.

Remark. The equalities

$$R_1R_2R_1=R_1R_2$$
$$R_1R_2R_1=R_2R_1$$
$$R_1R_2=R_2R_1$$

are equivalent for arbitrary equivalence relations R_1, R_2 in M.

Theorem 4. The following conditions are equivalent:

a) the images of different classes of R in f are disjoint,
b) $FRF=R$,
c) $F\subset R$.

Proof. If a) holds and $x_1 FRF x_2$ then there are x_1', x_2' with $f(x_1')=f(x_1)$, $f(x_2')=f(x_2)$, $x_1' R x_2'$. Let X_1, X_2 be the classes of x_1 and x_2 respectively, and X the class containing x_1', x_2'. Then $X_1=X_2=X$ by a) and thus $x_1 R x_2$. So a) implies b).

If $FRF=R$ then $F\subset FRF=R$, i.e. b) implies c).

Suppose now that $F\subset R$ and X_1, X_2 are classes of R with $f(X_1)\cap f(X_2)\neq\emptyset$. Then there exist such $x_1 \in X_1$, $x_2 \in X_2$ that $f(x_1)=f(x_2)$ and by hypothesis it follows that $x_1 R x_2$. Consequently $X_1=X_2$ and the theorem is proved.

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SÚHRN

O EKVIVALENCIÁCH

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Vo vete 1 sú nájdené nutné a dostatočné podmienky, aby súčin konečného počtu ekvivalencií bol ekvivalenciou. Vo vetách 2, 3 a 4 sa študujú zobrazenia rozkladov na množine.

РЕЗЮМЕ

ОБ ОТНОШЕНИЯХ ЭКВИВАЛЕНТНОСТИ

Л. Космак, Братислава

В теореме 1 указаны необходимые и достаточные условия для того, чтобы произведение $n \geqq 2$ отношений эквивалентности было отношением эквивалентности. В теоремах 2, 3 и 4 исследуются отображения разбиений на множестве.