UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XXXIX — 1980

INTRINSIC TOPOLOGIES ON ORDERED SETS

MILAN KOLIBIAR, Bratislava

Dedicated to Professor O. Borůvka on the occasion of his 80th birthday

Several authors stated conditions under which the interval topology of an ordered set (eventually of some specific class) is Hausdorff. The present note contains some generalizations of these results.[1]) Also connections between the interval topology and a kind of "between topology" on a lattice [5] are considered.

In the whole paper $\mathscr{P}=(P;\leqq)$ will denote an ordered (i.e. partially ordered) set, $\mathscr{L}=(L;\wedge,\vee)$ a lattice. Given $a, b\in P$, denote $(a,b)=\{x\in P\colon a<x<b\}$, $[a,b]=\{x\in P\colon a\leqq x\leqq b\}$, $(a]=\{x\in P\colon x\leqq a\}$ and symmetrically $[a)$. $a\|b$ will denote that a and b are incomparable. A subset $A\subset P$ will be said to have the finite maximal property if to each element $a\in A$ there is a maximal element m of A with $m\geqq a$ and the set of all maximal elements of A is finite. The finite minimal property is defined dually.

1. Hausdorff interval topology

The interval topology on $\mathscr{P}$ is that which takes the family of all intervals $(a]$ and $[a)$ $(a\in P)$ as a subbase for closed sets. Denote $N(a)=\{x\in P\colon x\|a\}$ and $N(a,b)=P-(((a]\cap(b])\cup([a)\cap[b)))$ (set-theoretic difference). A subset $A\subset P$ will be called a separating set of P whenever each element of P is comparable with some element of A.

The following theorems give conditions on $\mathscr{P}$ to have Hausdorff interval topology.

Theorem A [4]. A sufficient (but not necessary) condition for an ordered set $\mathscr{P}$ to have Hausdorff interval topology is that for each $a\in P$, $N(a)$ have a finite separating set.

[1]) Some results of paragraph 1 of the present note were published without proofs earlier [3].

Theorem B [6]. A necessary (but not sufficient) condition for a lattice to have Hausdorff interval topology is that for each $a, b \in L$, $a<b$, the interval (a, b) have a finite separating set.

Theorem C [1]. A necessary and sufficient condition for the interval topology of a lattice $\mathscr{L}$ to be Hausdorff is that for each $a, b \in L$, $a<b$, $N(a, b)$ have a finite separating set.

Theorem D [2,6]. The interval topology of a Boolean lattice $\mathscr{L}$ is Hausdorff if and only if to each element $a \in L$, $a \neq 0$, there is an atom $p \leqq a$.

In the present note we shall show that theorem C may be enlarged (with a suitable modification) to arbitrary ordered sets. Also we modify the condition of Theorem A to be sufficient as well as necessary. Further we generalize theorem D to a class of complemented modular lattices (1.10.1). Theorems A, B, C are corollaries of our theorem 1.2.

1.1. The following known assertion will be useful

Let $\mathscr{S}$ be a subbase of closed sets of a topological space X. X is Hausdorff if and only if to any $a, b \in X$, $a \neq b$, there is a finite subset $\mathscr{F}$ of $\mathscr{S}$ with the property

($*$) $\mathscr{F}$ covers X (i.e. $\cup\mathscr{F}=X$) and no element of $\mathscr{F}$ contains both elements a and b.

1.2. Theorem. The following conditions on $\mathscr{P}$ are equivalent.

(i) The interval topology of $\mathscr{P}$ is Hausdorff.

(ii) For each $a, b \in P$, $a \neq b$, $N(a, b)$ has a finite separating set.

(iii) If for an $a \in P$, $N(a)$ has no finite separating set then for each $b \in P$, $b \neq a$, $N(a, b)$ has a finite separating set.

(iv) Given $a, b \in P$, $a \neq b$, then:

if $a \| b$ then one of the sets $N(a)$, $N(b)$ and $N(a, b)$ has a finite separating set;

if $a<b$ then either there is $c \in (a, b)$ such that $N(c)$ has a finite separating set or $N(a, b)$ has a finite separating set.

Proof. (i)$\Rightarrow$(ii). Let $a, b \in P$, $a \neq b$. According to 1.1 there is a set of intervals of the form $[u)$ or $(u]$, where u runs over a finite set F, satisfying ($*$). Obviously $F \cap N(a, b)$ is a separating set of $N(a, b)$.

The implication (ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(iv). If $a \| b$ the assertion is trivial. Let $a<b$. Let there exist an element $c \in (a, b)$. If $N(c)$ does not have a finite separating set then both $N(a, c)$ and $N(c, b)$ do have. Since $N(a, b)=N(a, c) \cup N(c, b) \cup\{c\}$, $N(a, b)$ has a finite separating set. If b covers a then $N(a, b)$ has a finite separating set because of $N(a, b)=N(a) \cup N(b)$ and the supposition (iii).

(iv)$\Rightarrow$(i). Let $a, b \in P$, $a \neq b$. If $a \| b$ and $N(a, b)$ or one of the sets $N(a)$ and $N(b)$, say $N(a)$, has a finite separating set F, then the set of intervals $(u]$, $[u)$,

$u \in F \cup \{a\}$, has the property (*) in 1.1. Let $a < b$. If $a < c < b$ and $N(c)$ has a finite separating set F then the set of intervals $(u]$, $[u)$, $u \in F \cup \{c\}$, has the property (*). If $N(a, b)$ has a finite separating set G then the set of intervals $(a]$, $[b)$, $(u]$, $[u)$, $(u \in G)$ has the property (*). This completes the proof.

1.3. Theorems A, B, C are corollaries of theorem 1.2. Indeed, theorem A follows from the implication (iii)$\Rightarrow$(i). Theorem B follows from the implication (i)$\Rightarrow$(ii): Let F be *a* finite separating set of $N(a, b)$ $(a < b)$. Then $\{u \vee a : u \in F$ and $u \leqq x$ for some $x \in (a, b)\} \cup \{u \wedge b : u \in F$ and $x \leqq u$ for some $x \in (a, b)\}$ is a (finite) separating set of (a, b). Theorem C follows from the equivalence (i)$\Leftrightarrow$(ii) because of $N(a, b) = N(a \wedge b, a \vee b)$.

1.4. Lemma. Let an interval (a, b) $(a < b)$ of a relatively complemented lattice have a finite separating set. Then the interval $[a, b]$ contains a prime interval (i.e. elements u, v such that v covers u).

Proof. Suppose $[a, b]$ does not contain a prime interval. It suffices to show:

(a) Any subset $\{a_1, a_2, \ldots, a_n\}$ of the interval $(a, b) = A$ is not a separating set of A.

The proof of (a) proceeds by induction on n. If $n = 1$ and a_1' is a relative complement of a_1 in the interval $[a, b]$ then a_1' belongs to A and is incomparable with a_1. Suppose the assertion is true for n and let $a_1, \ldots, a_{n+1}$ be elements of A. According to the hypothesis there is an element $c \in A$ which is incomparable with a_i, $i = 1, \ldots, n$. First we shall prove:

(b) To each $u \in A$, $u < c$, there is an element t such that $u \leqq t < c$ and $t \| a_i$, $i = 1, \ldots, n$.

To prove this we find elements $t_1, \ldots, t_n \in A$ such that t_i is incomparable with $a_1, \ldots, a_i$ and $u \leqq t_i < c$ $(i = 1, \ldots, n)$. If $u \| a_1$ we set $t_1 = u$. Otherwise $u < a_1$ and we choose $t_1 \in (a_1 \wedge c, c)$. Supposing we have elements t_i, $i \leqq k < n$, choose t_{k+1} as follows. If $t_k \| a_{k+1}$ we set $t_{k+1} = t_k$. Otherwise $t_k < a_{k+1}$ hence $u \leqq c \wedge a_{k+1}$. Let $c \wedge a_j$ be a maximal element of the set $[u) \cap \{c \wedge a_i : i = 1, \ldots, k+1\}$. Choose $t_{k+1} \in (c \wedge a_j, c)$. The element $t = t_n$ satisfies (b). This completes the proof of (b).

Now we continue the proof of (a). If $c \| a_{n+1}$ the assertion is proved. Let $c \geqq a_{n+1}$ (the case $c \leqq a_{n+1}$ is dual). If $c > a_{n+1}$ let u be *a* relative complement of a_{n+1} in the interval $[a, c]$. According to (b) there is an element $t \in [u, c)$, incomparable with $a_1, \ldots, a_n$. Obwiously $t \| a_{n+1}$ too. If $c = a_{n+1}$, there is an element $c_1 \in (c, b)$. According to the assertion dual to (b) there is an element t, $c < t \leqq c_1$, incomparable with $a_1, \ldots, a_n$ and if suffices to take t instead of c to get the previous case. This completes the proof of the lemma.

1.5. Combining 1.4 with theorem B we get

Theorem. Let the interval topology of a relatively complemented lattice $\mathscr{L}$ be Hausdorff. Then each interval $[a, b]$ $(a < b)$ in $\mathscr{L}$ contains a prime interval.

1.6. Lemma. Let $\mathscr{L}$ be *a* lattice with 0 in which each element of any interval $[0, c]$ has in $[0, c]$ a minimal relative complement. If each interval $[0, b]$ $(0 < b)$ in

$\mathscr{L}$ contains a prime interval, then to any element $a \neq 0$ of $\mathscr{L}$ there is an atom $p \leqq a$.

Proof. Given $a > 0$, there are elements $u, v \in [0, a]$ such that v covers u. If $u > 0$ then a minimal relative complement of u in $[0, v]$ is a desired atom.

1.7. Applying 1.5 and 1.6 we get

1.7.1. Theorem. Let $\mathscr{L}$ be a relatively complemented lattice with 0 in which each element of any interval $[0, c]$ has in $[0, c]$ a minimal relative complement. If the interval topology of $\mathscr{L}$ is Hausdorff then to any element $a \neq 0$ of $\mathscr{L}$ there is an atom $p \leqq a$.

1.7.2. Corollary. If the interval topology of a relatively complemented modular lattice with 0 is Hausdorff then to each element $a \neq 0$ of $\mathscr{L}$ there is an atom $p \leqq a$.

1.8. The proof of the following assertions is straightforward.

1.8.1. Let $\mathscr{L}$ be a relatively complemented lattice with 0 such that to each element a of $\mathscr{L}$ there is an atom $p \leqq a$. Then to any two distinct elements a, b of $\mathscr{L}$ there is an atom satisfying exactly one of the relations $p \leqq a$, $p \leqq b$.

1.8.2. Let p be an atom of a relatively complemented lattice $\mathscr{L}$ with 0 and I, and let $a \in L$, $a \not\geqq p$. Then a complement b of p exists with $a \leqq b$.

1.9. Theorem. Let $\mathscr{L}$ be a relatively complemented lattice with 0 and I such that to any element $a \neq 0$ of $\mathscr{L}$ there is an atom $p \leqq a$, and the set of all complements of each atom has the finite maximal property. Then the interval topology of $\mathscr{L}$ is Hausdorff.

Proof. Let a, b be distinct elements of $\mathscr{L}$. According to 1.8.1 there is an atom p such that, e.g., $p \leqq a$, $p \not\leqq b$. Let $q_1, \ldots, q_n$ be all maximal complements of p. According to 1.8.2 the intervals $[p, I]$, $[0, q_i]$ $(i = 1, \ldots, n)$ cover L and $a \in [p, I]$, $b \notin [p, I]$ and $a \in [0, q_i]$ for no i. Now it suffices to apply 1.1.

1.10. In the case of modular lattices analogous theorems may be formulated more simply.

1.10.1. Theorem. Let $\mathscr{L}$ be a complemented modular lattice in which each atom has only finitely many complements. The following conditions are equivalent.

(i) The interval topology of $\mathscr{L}$ is Hausdorff.

(ii) Each interval $[a, b]$ $(a < b)$ in $\mathscr{L}$ contains a prime interval.

(iii) Each interval $[0, a]$ $(0 < a)$ in $\mathscr{L}$ contains a prime interval.

(iv) To each element $a \neq 0$ of $\mathscr{L}$ there is an atom $p \leqq a$.

Proof. (i)$\Rightarrow$(ii) follows from 1.5. (ii)$\Rightarrow$(iii) is trivial. (iii)$\Rightarrow$(iv): Let $a > 0$ and let $[u, v]$ be a prime interval contained in $[0, a]$. A relative complement of u in the interval $[0, v]$ is the desired atom. The implication (iv)$\Rightarrow$(i) follows from 1.9.

1.10.2. Remarks.

1.10.2.1. In a relatively complemented lattice $\mathscr{L}$ with 0 the implication (ii)$\Rightarrow$(iv) (or (iii)$\Rightarrow$(iv)) holds whenever $\mathscr{L}$ satisfies the condition

(+) If $x \vee y$ covers x then y covers $x \wedge y$.

The implication (iv)$\Rightarrow$(iii) holds whenever $\mathscr{L}$ satisfies the dual of (+). In a relatively complemented modular lattice with 0 the conditions (ii), (iii) and (iv) are equivalent.

1.10.2.2. The part (i)$\Leftrightarrow$(iv) of theorem 1.10.1 gives a generalization of theorem D.

2. Interval topology and between topology

2.0. Consider a ternary relation in a lattice: axb if and only if $(a\wedge x)\vee(x\wedge b)= =x=(a\vee x)\wedge(x\vee b)$ (x is between a and b) [7]. Y. Matsushima [5] considered the topology $\mathcal{B}^*$ on a lattice $\mathcal{L}$ whose subbase of closed sets is the family of all sets $B^*(a,b)=\{x\in L\colon abx\}$ $(a, b\in L)$, and studied inter-relations among the topology $\mathcal{B}^*$ and the interval topology $\mathcal{I}$. In this paragraph some results in [5] are supplemented.

2.1. In any lattice $\mathcal{I}\subset\mathcal{B}^*$.[1])

Proof. It suffices to show that $[a)$ is a $\mathcal{B}^*$-closed set. This is true because of $[a)=\bigcap\{B^*(u,a)\colon u\leqq a\}$ $(=A)$: If $x\in A$ then $x\in B^*(a\wedge x, a)$ hence $(a\wedge x)ax$; on the other hand $a(a\wedge x)x$ so that $a\wedge x=a$, i.e. $x\in[a)$. The converse inclusion is trivial.

2.2. Theorem. Let $\mathcal{L}$ satisfy the following condition (i) and its dual:

(i) For each two elements $a, b\in L$, $a<b$, the set $M(a,b)=\{x\in L\colon a\vee x=b\}$ has the finite minimal property.

Then $\mathcal{B}^*=\mathcal{I}$.

Proof. In view of 2.1 it suffices to show $\mathcal{B}^*\subset\mathcal{I}$. This follows from the relations $B^*(a,b) = B^*(a\vee b, b)\cap B^*(a\wedge b, b)$ and, for $a<b$, $B^*(a,b) = \bigcup([m)\colon :m\in M)$ where M is the set of all minimal elements of the set $M(a,b)$ and an analogous relation for $a>b$. (Note that $B^*(a,a)=L$.) The proof of these relations is straightforward (see [5, Th. 1.3].)

Remark. In the theorem the condition (i) cannot be replaced by the condition (i′) that the set of all minimal elements of $M(a,b)$ is finite as the following example shows. Let L consist of the infinite chain $A\colon a_0>a_1>a_2>\dots$ and of two elements u, v such that $u<v<a_0$, $u<a_n$ and $v\,\|\,a_n$ for each $n>0$. The condition (i′) and its dual are satisfied but the set $B^*(v, a_0)=A$ fails to be closed in $\mathcal{I}$.

2.3. Theorem. Let $\mathcal{L}$ be a lattice with 0 and I. Suppose that for each $a\in L$,

(i) the set of all relative complements in $[0,a]$ of each element of $[0,a]$ is non-empty and has the finite minimal property,

(ii) the set of all relative complements in $[a,I]$ of each element of $[a,I]$ is non-empty and has the finite maximal property.

Then $\mathcal{I}=\mathcal{B}^*$.

[1]) In [5] this assertion is proved under supposition that the lattice has 0 and I.

Proof. Let a, $b \in L$, $a < b$, and let K be the set of all minimal relative complements of a in $[0, b]$. According to 2.2 (and the duality) it suffices to show that a) $K \subset M(a, b)$, and b) to each $x \in M(a, b)$ there is $t \in K$ with $t \leqq x$. The relation a) is obvious. As for b), let $a \vee x = b$. A relative complement t of $a \wedge x$ in $[0, x]$ has the desired property.

2.4. Corollary [5, Th. 1.2]. If in a complemented modular lattice the number of complements of each element is finite then $\mathcal{I} = \mathcal{B}^*$. In particular $\mathcal{I} = \mathcal{B}^*$ in any Boolean lattice [5, Th. 1.1].

Proof. It suffices to realize that 1) any two complements of an element are incomparable, and 2) each relative complement t of an element $x \in [0, a]$ in the interval $[0, a]$ is of the form $t = a \wedge y$ where y is a complement of x in $\mathcal{L}$ (and to use the duality).

REFERENCES

[1] Baer, R. M.: A characterization theorem for lattices with Hausdorff interval topology. J. Math. Soc. Japan 7 (1955), 177—187.

[2] Katětov, M.: Remarks on Boolean algebras. Colloquium Math. 2 (1951), 229—235.

[3] Kolibiar, M.: Bemerkungen über Intervalltopologie in halbgeordneten Mengen. General topology and its relations to modern analysis and algebra, Proceedings of the Symposium held in Prague in September, 1961.

[4] Matsushima, Y.: Hausdorff interval topology on a partially ordered set. Proc. Amer. Math. Soc. 11 (1960), 233—235.

[5] Matsushima, Y.: Between topology for lattices. J. Math. Soc. Japan 16 (1964), 335—341.

[6] Northam, E. S.: The interval topology of a lattice. Proc. Amer. Math. Soc. 4 (1953), 824—827.

[7] Pitcher, E.—Smiley, M. F.: Transitivities of betweenness. Trans. Amer. Math. Soc. 52 (1942), 95—114.

Received March 20, 1979

Author's address:
Katedra algebry a teórie čísel MFFUK, Mlynská dolina,
816 31 Bratislava

SÚHRN

VNÚTORNÉ TOPOLÓGIE NA USPORIADANÝCH MNOŽINÁCH

M. Kolibiar, Bratislava

V práci sú uvedené podmienky pre (čiastočne) usporiadané množiny a niektoré triedy zväzov, aby ich intervalová topológia bola Hausdorffova. Zovšeobecňujú sa pritom niektoré výsledky prác [1], [2], [4], [6]. V druhej časti práce sa skúma vzájomný vzťah intervalovej topológie a topológie odvodenej z relácie „medzi" [5] vo zväzoch.

РЕЗЮМЕ

ВНУТРЕННИЕ ТОПОЛОГИИ НА УПОРЯДОЧЕННЫХ МНОЖЕСТВАХ

М. Колибиар, Братислава

Изучаются условия для частично упорядоченных множеств и решеток некоторых классов, чтобы их интервальная топология была Хаусдорфовой. При этом обобщаются некоторые результаты работ [1], [2], [4], [6]. Вторая часть работы посвящена отношениям между интервальной топологией и топологией связанной с отношением «между» [5] в решетках.

UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XXXIX — 1980

ON THE A-CONTINUITY OF REAL FUNCTIONS

JOZEF ANTONI—TIBOR ŠALÁT, Bratislava

Dedicated to Professor O. Borůvka on the occasion of his 80th birthday

In [1] the following problem is formulated: Let $x_n \in \mathrm{R}$ $(n = 1, 2, \dots)$. We shall write $x_n \underset{C}{\longrightarrow} x_0$ if

$$\frac{x_1 + x_2 + \dots + x_n}{n} \to x_0,$$

so $x_n \underset{C}{\longrightarrow} x_0$ means $(C, 1) - \lim x_n = x_0$, see [4], page 27, and $(C, 1)$ is the first Cesàro mean defined by the matrix

$$\mathbf{C} = \begin{pmatrix} 1, & 0, & 0, & 0, & \dots \\ \frac{1}{2}, & \frac{1}{2}, & 0, & 0, & \dots \\ \frac{1}{3}, & \frac{1}{3}, & \frac{1}{3}, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

We can say, that a function f: $\mathrm{R} \to \mathrm{R}$ is C-continuous at the point x_0 if $f(x_n) \underset{C}{\longrightarrow} f(x_0)$ whenever $x_n \underset{C}{\longrightarrow} x_0$. In the solution of the mentioned problem is showed, that the function f: $\mathrm{R} \to \mathrm{R}$ is linear if f is C-continuous at least at one point of R.

In conection with this result the problem arises to investigate the following four types of "continuity" of functions f: $\mathrm{R} \to \mathrm{R}$ at the point x_0:

1. $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$ (the obvious continuity)
2. $x_n \underset{C}{\longrightarrow} x_0 \Rightarrow f(x_n) \underset{C}{\longrightarrow} f(x_0)$ (the C-continuity)
3. $x_n \to x_0 \Rightarrow f(x_n) \underset{C}{\longrightarrow} f(x_0)$
4. $x_n \underset{C}{\longrightarrow} x_0 \Rightarrow f(x_n) \to f(x_0)$

The cases 1 and 2 are explained. Let us investigate the case 4.

Theorem 1. Let f: R→R have the following property: there exists such a point $x_0 \in$ R that the following implication

(1) $$x_n \xrightarrow[C]{} x_0 \Rightarrow f(x_n) \to f(x_0)$$

is valid.

Then f is a constant function.

Proof. The assertion (1) implies the following

$$x_n \xrightarrow[C]{} x_0 \Rightarrow f(x_n) \xrightarrow[C]{} f(x_0)$$

The last implication and the quoted result of [1] gives that f is a linear function. Put $a = x_0 - 1$, $b = x_0 + 1$. We can construct a sequence $\{x_n\}_{n=1}^{\infty} = a, b, a, b, \ldots$. It is easy to see that $x_n \xrightarrow[C]{} x_0$. It follows from (1) that $\{f(x_n)\}_{n=1}^{\infty} = f(a), f(b), f(a), f(b), \ldots$ converges. The last statement yields

(2) $$f(b) = f(a)$$

Since f is a linear function it follows from (2) that f is a constant function.

The case 3. seems as some generalization of the notion of continuity of functions. It is not so as it seems. In the paper [2] there is namely the following result proved:

Let **A** be a regular matrix (see [4], p. 7—8) and f: R→R a function. Let $A - \lim f(x_n)$ exist in R whenever $\{x_n\}_{n=1}^{\infty}$ converges. Then f is a continuous function on R.

In connection with the result from [1] the following question arises: Is it possible to generalize the mentioned result by replacing the Cesàro matrix by a nother regular matrix? We can formulate this question more precisely. Let **A** be a regular matrix. We shall say that the function f: R→R is A-continuous at the point $x_0 \in$ R if the following statement is true

$$A - \lim x_n = x_0 \Rightarrow A - \lim f(x_n) = f(x_0)$$

Immediately the question arises: Does the A-continuity of the function f imply that f is a linear function? Simple examples show that in general the answer is negative. For example if **A** is a matrix equivalent to convergence (see [4], p. 11, 31) so every continuous function is also A-continuous. In the next theorem we shall give a sufficient condition for linearity of A-continuous functions.

We shall say that a regular matrix **A** has the property $L(a)$ if there exists such a sequence $\{\eta_n\}_{n=1}^{\infty}$, $\eta_n = 0$ or 1 $(n = 1, 2, \ldots)$ for which $A - \lim \eta_n = a$.

Using the same consideration as in the proof of the quoted result from [2] the following auxiliary result can be proved.

Lemma 1. Let **A** be a regular matrix and f: R→R be an A-continuous function at a point $x_0 \in$ R. Then f is continuous at the point x_0.

Theorem 2. Let $\mathbf{A}=(a_{mn})$ be a regular matrix with property $L(a)$ for a number a, $a \neq 0, 1$ and let f: R→R be an A-continuous function at each point of R. Then f is a linear function.

Proof. Let $x, y \in$ R. Let A-lim $\eta_n = a$, $a \neq 0, 1$ and $\eta_n = 0$ or 1 $(n=1, 2, \dots)$. Let us put $t_n = \eta_n y + (1-\eta_n)x$ $(n=1, 2, \dots)$. Then we have

$$A-\lim t_n = \lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} t_n = y \lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn}\eta_n + \tag{3}$$

$$+ x \lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn}(1-\eta_n) = ay + (1-a)x$$

It is easy to see that $f(t_n) = \eta_n f(y) + (1-\eta_n)f(x)$ $(n=1, 2, \dots)$. In the same way as (3) we have

$$A-\lim f(t_n) = af(y) + (1-a)f(x) \tag{4}$$

Since the function f is A-continuous at the point $ay+(1-a)x$, by (3) and (4) we obtain

$$f(ay+(1-a)x) = af(y) + (1-a)f(x) \tag{5}$$

(for arbitrary $x, y \in$ R).

According to Lemma 1 and assumptions of Theorem 2 f is a continuous function on R. Further the function f fulfills the following functional equation

$$g((1-a)x+ay) = (1-a)g(x) + ag(y)$$

($x, y \in$ R and $a \neq 0$, $1-a \neq 0$) (see (5)).

On account of the well-known results on the functional equations of this type (see [3] p. 68—70) we can conclude that f is the linear function and the proof is complete.

Remark. Obviously the Cesàro matrix has property $L\left(\frac{1}{2}\right)$ because the sequence 0, 1, 0, 1, ... is $(C, 1)$-summable to $\frac{1}{2}$.

There exist regular matrices with property $L(a)$, $a \neq 0, 1$, which are not equivalent to Cesàro matrix. Such a matrix is the following

$$\mathbf{A}_0 = \begin{bmatrix} \frac{1}{2}, & \frac{1}{2}, & 0, & 0, & \dots & & & & \\ 0, & 0, & \frac{1}{2}, & \frac{1}{2}, & 0, & 0, & \dots & & \\ 0, & 0, & 0, & 0, & \frac{1}{2}, & \frac{1}{2}, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

The matrix $\mathbf{A}_0$ has the property $L\left(\frac{1}{2}\right)$. We can take $\eta_{2k}=1$, $\eta_{2k-1}=0$ $(k=1,2,\dots)$ and we obtain that $A-\lim \eta_n=\frac{1}{2}$. If we take $\varepsilon_{4k}=1$ $(k=1,2,\dots)$ and $\varepsilon_{4k+r}=0$ $(r=1,2,3$ and $k=1,2,\dots)$ we obtain the sequence which is $(C,1)$-summable to $\frac{1}{4}$ and which is not A_0-summable.

We already have mentioned that if the function $f\colon \mathrm{R}\to\mathrm{R}$ is C-continuous at a point $x_0\in\mathrm{R}$, then f is a linear function (cf. [1]). If we compare this assertion with Theorem 2 the question arises whether the assumption of the A-continuity of f at each point of R can be replaced by the assumption that f is A-continuous at one point $x_0\in\mathrm{R}$. The following example gives a negative answer to this question.

Example. The matrix

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2}, & \frac{1}{2}, & 0, & 0, & \dots & & & \\ 0, & \frac{1}{2}, & \frac{1}{2}, & 0, & 0, & \dots & & \\ 0, & 0, & \frac{1}{2}, & \frac{1}{2}, & 0, & 0, & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

is evidently regular. Let us put $f(x)=-1$ for $x\leqq -1$, $f(x)=x$ for $-1<x<1$, $f(x)=1$ for $x\geqq 1$. Then f is a continuous, but not a linear function.

We shall show that the function f is A-continuous at 0 and it is not A-continuous at any point $x\in\mathrm{R}$, $x\neq 0$.

Let $A-\lim x_n=0$. We prove that $A-\lim f(x_n)=0$ $(=f(0))$. We put for brevity: $y_n=\frac{1}{2}(x_n+x_{n+1})$, $t_n=\frac{1}{2}(f(x_n)+f(x_{n+1}))$ $(n=1,2,\dots)$. Since $y_n\to 0$ $(n\to\infty)$, there exists such a n_0 that for each $n\geqq n_0$ we have

$$(6) \qquad |x_n+x_{n+1}|<1$$

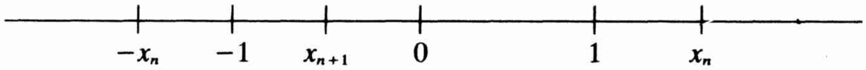
For fixed $n \geqq n_0$ we have the following four possibilities:

1. $|x_n| \geqq 1$, $|x_{n+1}| \geqq 1$;
2. $|x_n| \geqq 1$, $|x_{n+1}| < 1$;
3. $|x_n| < 1$, $|x_{n+1}| \geqq 1$;
4. $|x_n| < 1$, $|x_{n+1}| < 1$.

1. According to (6) we have either $x_n \geqq 1$, $x_{n+1} \leqq -1$ or $x_n \leqq -1$, $x_{n+1} \geqq 1$. In any case we get $t_n = 0$.

2. Let e.g. $x_n \geqq 1$, $|x_{n+1}| < 1$. Then according to (6) we have $-1 < x_{n+1} < 0$ and so we get $t_n = \frac{1}{2}(1 + x_{n+1})$. Therefore

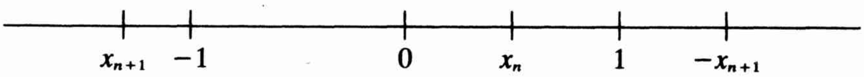
$$|t_n - y_n| = \frac{1}{2}|1 - x_n| \leqq \frac{1}{2}|x_n + x_{n+1}| = |y_n|$$



If $x_n \leqq -1$ (and $|x_{n+1}| < 1$) then in an analogous way we get $|t_n - y_n| \leqq |y_n|$.

3. Let e.g. $x_{n+1} \leqq -1$. Then according to (6) we have $0 < x_n < 1$ and so we get $t_n = \frac{1}{2}(x_n - 1)$. Hence

$$|t_n - y_n| = \frac{1}{2}|1 + x_{n+1}| \leqq |y_n|$$



If $x_{n+1} \geqq 1$ (and $|x_n| < 1$) we obtain similarly the estimation $|t_n - y_n| \leqq |y_n|$.

4. We have $t_n = y_n$.

Hence in all cases we have (for $n \geqq n_0$) $t_n = 0$ or $|t_n - y_n| \leqq |y_n|$. Since $y_n \to 0$ $(n \to 0)$, we get $t_n \to 0$ $(n \to \infty)$.

We shall show that f is A-continuous at no point $x \in \mathrm{R}$, $x \neq 0$. Let $x \in \mathrm{R}$, $x \neq 0$. Choose a positive integer k such that $(k-2)|x| > 1$. Let us put $x_{2n-1} = kx$, $x_{2n} = -(k-2)x$ $(n = 1, 2, \ldots)$. Then evidently we have $A - \lim x_n = x$ and simultaneously $t_n = 0$ $(n = 1, 2, \ldots)$. Hence $A - \lim f(x_n) = 0 \neq f(x)$.

REFERENCES

[1] Problem 4216 [1946, 470] Amer. Math. Monthly. Proposed H. Robins. Solution by R. C. Buck, Amer. Math. Monthly 55 (1948), 36.

[2] Posner, E. C.: Summability-preserving functions. Proc. Amer. Math. Soc. 12 (1961), 73—76.

[3] Aczél, J.: Vorlesungen über Funkcionalgleichungen und ihre Anwendungen. VEB Deutscher Verlag der Wissenschaften, Berlin 1961.

[4] Petersen, G. M.: Regular matrix transformations. McGraw-Hill, New York—Toronto—Sydney, 1966.

Received March 20, 1979

Author's addresser:
Jozef Antoni
MÚ SAV, Obrancov mieru 41,
886 25 Bratislava

Tibor Šalát
Katedra algebry a teórie čísel MFFUK, Mlynská dolina,
816 31 Bratislava

SÚHRN

O A-SPOJITOSTI REÁLNYCH FUNKCIÍ

J. Antoni—T. Šalát, Bratislava

Nech $\mathbf{A}$ je regulárna matica. Funkcia f: $R \to R$ sa nazýva A-spojitá v bode $x_0 \in R$, ak z $A - \lim x_n = x_0$ vyplýva $A - \lim f(x_n) = f(x_0)$. V práci sú dokázané postačujúce podmienky k tomu, aby z A-spojitosti funkcie vyplývala lineárnosť funkcie.

РЕЗЮМЕ

О A-НЕПРЕРЫВНОСТИ ВЕСЩЕСТВЕННЫХ ФУНКЦИЙ

Й. Антони–Т. Шалат

Пусть $\mathbf{A}$ регуларная матрица. Функция f: $R \to R$ называется A-непрерывной в точке $x_0 \in R$ эсли из $A - \lim x_n = x_0$ вытекает $A - \lim f(x_n) = f(x_0)$. В работе даны достаточные условия для того чтобы из A-непрерывности функций вытекала линейность функций.