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BEHAVIOR OF NONOSCILLATORY SOLUTIONS OF SOME NONLINEAR DIFFERENTIAL EQUATIONS

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Dedicated to Professor O. Borůvka on the occasion of his 80th birthday

We consider the differential equation

$$L_n y + h(t, y, y', \dots, y^{(n-1)}) = 0 \tag{E}$$

where $n > 1$,

$$L_n y = a_n(t)(a_{n-1}(t)(\dots(a_1(t)(a_0(t)y)')'\dots)')')$$

$a_i(t)$, $i = 0, 1, \dots, n$, are positive and continuous functions on $[0, \infty)$ and $h(t, y_0, y_1, \dots, y_{n-1})$ is real valued and continuous on $\Omega = [0, \infty) \times E_n$, E_n being Euclidean n dimensional vector space. Under a solution $y(t)$ of (E) we will understand a solution existing on some ray $[T_y, \infty)$ and satisfying the condition

$$\sup\{|y(t)| : t_0 \leqq t < \infty\} > 0 \quad \text{for any} \quad t_0 \in [T_y, \infty)$$

A solution $y(t)$ will be called nonoscillatory if there is $t_1 \geqq T_y$ such that $y(t) \neq 0$ for $t \geqq t_1$. Sometimes we will require the following conditions to be satisfied:

$$yh(t, y, y_1, \dots, y_{n-1}) > 0 \quad \text{or} \quad < 0 \quad \text{for} \quad y \neq 0 \tag{1}$$

$$\int^\infty \frac{dt}{a_i(t)} = \infty, \quad i = 1, 2, \dots, n-1 \tag{2}$$

The expressions $L_0 y = a_0(t)y$, $L_i y = a_i(t)(L_{i-1}y)'$, $i = 1, 2, \dots, n$, will be called quasi-derivatives of y. We restrict our considerations only to nonoscillatory solutions of (E); we'll examine their properties and the properties of their quasi-derivatives. The equations of the type (E) were examined by various authors, see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]. There were mainly examined the conditions which guarantee that nonoscillatory solutions tend to a finite limit (for example to zero) as $t \to \infty$. We will also examine the asymptotic behavior of quasi-derivatives.

We start with some lemmas that elucidate the behavior of quasiderivatives of some function $y(t)$ in relation with the properties of $a_i(t)$, $i=1, 2, \ldots, n-1$.

Lemma 1. Let $y(t) \neq 0$ be such that $L_n y$ exists for $t \geqq t_0 \geqq 0$ and let $y(t) L_n y(t) \geqq 0$ or $\leqq 0$ ($=$can hold only at isolated points) hold for $t \geqq t_0$. Then $\lim_{t \to \infty} L_i y(t)$, $i=0, 1, \ldots, n-1$, exist, are finite or infinite and there is $T \geqq t_0$ such that $L_i y(t) \neq 0$ for $t \geqq T$, $i=0, 1, \ldots, n-1$. If moreover (2) is satisfied, then for $t \geqq T$

$$\lim_{t \to \infty} L_i y(t) \,.\, \operatorname{sgn} y(t) \geqq 0, \quad i=0, 1, \ldots, n-1 \tag{3}$$

Proof. Let be $L_n y(t) \geqq 0$ ($\leqq 0$) ($=$holding at isolated points eventually) for $t \geqq t_0 \geqq 0$. Then $(L_{n-1} y(t))' = a_n^{-1}(t)\, L_n y(t) \geqq 0$ ($\leqq 0$) for $t \geqq t_0$. Thus $L_{n-1} y(t)$ is strictly monotone, therefore $\lim_{t \to \infty} L_{n-1} y(t)$ exists (finite or infinite) and there is a $t_1 \geqq t_0$ such that $L_{n-1} y(t) \neq 0$ for $t \geqq t_1$. Successively we get the strict monotonicity of $L_i y(t)$, $i=n-2, n-3, \ldots, 1, 0$ and the existence of their limits for $t \to \infty$ and the existence of such a T that $L_i y(t) \neq 0$ for $t \geqq T$, $i=0, 1, \ldots, n-1$.

Let be $y(t) > 0$ for $t > t_0$ and let (3) be not true for some $i \in (1, 2, \ldots, n-1)$, i.e. let $\lim_{t \to \infty} L_i y(t) = -c < 0$. Then there is such a $b_i \geqq t_0$ that $L_i y(t) < -c/2 < 0$ for $t \geqq b_i$. Then

$$(L_{i-1} y(t))' = \frac{1}{a_i(t)} L_i y(t) < -\frac{c}{2} \frac{1}{a_i(t)}$$

for $t \geqq b_i$. Integrating on (b_i, ∞) we get

$$\lim_{t \to \infty} L_{i-1} y(t) \leqq L_{i-1} y(b_i) - \frac{c}{2} \int_{b_i}^{\infty} \frac{\mathrm{d}t}{a_i(t)} = -\infty$$

Repeating the same consideration for $i-1, i-2, \ldots, 1$ we get that $\lim L_0 y(t) = -\infty$ as $t \to \infty$ which contradicts the assumption that $y(t) > 0$. This proves the validity of (3). For $y(t) < 0$ a similar reasoning gives us the validity of (3).

Remark 1. It follows from the proof of this lemma that if $y(t) \,.\, L_n y(t) \geqq 0$ ($\leqq 0$), where $=$holds only at isolated points and $L_n y(t) \geqq 0$ ($\leqq 0$), $y(t)$ must be nonoscillatory.

Lemma 2. Let $y(t) \neq 0$ be such that $L_n y(t)$ exists on $[t_0, \infty)$ and let $y(t) \,.\, L_n y(t) \leqq 0$ be true in this interval whereby $=$holds at isolated points eventually. Let (2) be valid. Then there exists a $T \geqq 0$ and a $k \in (0, 1, \ldots, n-1)$ such that: if n is even, then

a) $(-1)^{i+1} y(t)\, L_i y(t) > 0$, $i=k+1, k+2, \ldots, n-1$, for $t > T$,

b) $\lim L_i y(t) = 0$ as $t \to \infty$, $i=k+1, k+2, \ldots, n-1$,

c) $\lim L_k y(t)$ exists and is finite as $t \to \infty$,

d) $\lim L_i y(t) = \infty \cdot \operatorname{sgn} y(t)$ as $t \to \infty$, $i = 0, 1, \ldots, k-1$;
if n is odd, then b), c), d) is still true and instead of a) it holds
a′) $(-1)^i y(t)\, L_i y(t) > 0$, $i = k+1, k+2, \ldots, n-1$, for $t > T$

Proof. Let be $y(t) > 0$ for $t \geqq t_0 \geqq 0$. Then following Lemma 1 there is a $T \geqq t_0$ such that $L_i y(t) \neq 0$ for $t \geqq T$ and $\lim L_i y(t) \neq 0$ as $t \to \infty$, $i = 0, 1, \ldots, n-1$. It is impossible that all $\lim L_i y(t) = \infty$ as $t \to \infty$ because $L_n y(t) \leqq 0$ and therefore $L_{n-1} y(t)$ is positive and decreases. Thus $k \leqq n-1$. If for some $i \in (1, 2, \ldots, n-1)$ $\lim L_i y(t) > 0$ as $t \to \infty$ then an easy calculus in which we respect (2) gives that $\lim L_j y(t) = \infty$ as $t \to \infty$ for $j = 0, 1, \ldots, i-1$. On the other hand the similar reasoning gives that if for some $i \in (0, 1, \ldots, n-2)$ $\lim L_i y(t) = 0$ as $t \to \infty$, then $\lim L_j y(t) = 0$ as $t \to \infty$ for $j = i+1, i+2, \ldots, n-1$. Therefore there is a $k \in (0, 1, \ldots, n-1)$ such that $\lim L_i y(t) = \infty$ as $t \to \infty$ for $i = 0, 1, \ldots, k-1$, $\lim L_j y(t) = 0$ as $t \to \infty$ for $j = k+1, k+2, \ldots, n-1$ and $\lim L_k y(t)$ is finite as $t \to \infty$. Thus the properties b), c), d) are true. In the case that $y(t) < 0$ for $t \geqq t_0 \geqq 0$ a similar reasoning gives the validity of b), c), d).

Now, let n be even. Suppose that $y(t) > 0$ for $t > t_0 \geqq 0$. Then $L_n y(t) \leqq 0$ for $t > t_0$ and therefore $L_{n-1} y(t)$ decreases and must be positive to avoid the contradiction with the fact that $y(t) > 0$. Thus $0 < L_{n-1} y(t) = (-1)^{n-1+1} L_{n-1} y(t)$. From this fact and from the fact that $\lim L_i y(t) = 0$ as $t \to \infty$ for $i = k+1, k+2, \ldots, n-1$ it follows that $L_i y(t) \,.\, L_{i+1} y(t) < 0$, $i = k+1, k+2, \ldots, n-2$. Therefore $(-1)^{i+1} L_i y(t) > 0$ for $t > T$, $i = k+1, k+2, \ldots, n-2$.

If $y(t) < 0$ for $t > t_0$, then $L_n y(t) \geqq 0$ for $t > t_0$ and $L_{n-1} y(t)$ increases. It must be negative to avoid the contradiction with the fact that $y(t) < 0$. In this case we have $(-1)^{n-1+1} y(t) \,.\, L_{n-1} y(t) > 0$ for $t > t_0$. From this and from the fact that $\lim L_i y(t) = 0$ as $t \to \infty$ for $i = k+1, k+2, \ldots, n-1$ it follows that $L_i y(t) \,.\, L_{i+1} y(t) < 0$, $i = k+1, k+2, \ldots, n-2$. Finally we get $(-1)^{i+1} y(t).\, L_i y(t) > 0$ for $t > t_0$, $i = k+1, k+2, \ldots, n-1$. The validity of a) is proved.

Let n be odd. Let be $y(t) > 0$ for $t \geqq t_0$. Then $L_n y(t) \leqq 0$ for $t \geqq t_0$ and therefore $L_{n-1} y(t)$ decreases there. It must be positive to avoid the contradiction with the fact that $y(t) > 0$ for $t \geqq t_0$. Thus we have $0 < L_{n-1} y(t) = (-1)^{n-1} L_{n-1} y(t)$. The same reasoning as above gives that $L_i y(t) \,.\, L_{i+1} y(t) < 0$ for $i = k+1, k+2, \ldots, n-2$ and $t > t_0$ and $(-1)^i L_i y(t) > 0$ for $t > t_0$ and $i = k+1, k+2, \ldots, n-1$. Thus validity of a′) is true. If $y(t) < 0$ for $t > t_0$ a similar reasoning as above in the case of n even proves the validity of a′).

Remark 2. If (2) is not satisfied, Lemma 2 can not be true. For example, let $y(t) = t$, $a_0(t) = 1$, $a_1(t) = t^{-1}$, $a_2(t) = 1$, $a_3(t) = \frac{1}{2} t^4$, $a_4(t) = t^{-1}$. Then $L_0 y = t$, $L_1 y = t^{-1}$, $L_2 y = -t^{-2}$, $L_3 y = t$, $L_4 y = t^{-1}$, and if $a_5(t) = 1$, then $L_5 y = -t^{-2}$.

Definition 1. Let be $0 \leqq c < t$. Denote

$$P_0(t,c)=1,\quad P_i(t,c)=\int_c^t \frac{ds_1}{a_1(s_1)}\int_c^{s_1}\frac{ds_2}{a_2(s_2)}\cdots\int_c^{s_{i-1}}\frac{ds_i}{a_i(s_i)} \tag{4}$$

$$i=1,2,\ldots,n-1$$

$$Q_n(t,c)=1,\quad Q_j(t,c)=\int_c^t \frac{ds_{n-1}}{a_{n-1}(s_{n-1})}\int_c^{s_{n-1}}\frac{ds_{n-2}}{a_{n-2}(s_{n-2})}\cdots\int_c^{s_{j+1}}\frac{ds_j}{a_j(s_j)}$$

$$j=1,2,\ldots,n-1$$

Lemma 3. Let (2) be valid. Then

$$\lim P_i(t,c)=\infty,\quad \lim Q_i(t,c)=\infty \quad \text{as} \quad t\to\infty \tag{5}$$

$$\text{for} \quad i=1,2,\ldots,n-1$$

$$\lim\frac{P_j(t,c)}{P_i(t,c)}=\infty \quad \text{for} \quad 0\leqq i<j\leqq n-1, \quad \text{as} \quad t\to\infty$$

$$\lim\frac{Q_i(t,c)}{Q_j(t,c)}=\infty \quad \text{for} \quad 0<i<j\leqq n \quad \text{as} \quad t\to\infty \tag{6}$$

for $t_0>c$ there are constants $\alpha_{ij}>0$ and $\beta_{ij}>0$ such that for $t\geqq t_0$ and $0<i<j\leqq$ $\leqq n-1$

$$\alpha_{ij}P_i(t,c)<P_j(t,c),\quad \beta_{ij}Q_j(t,c)<Q_i(t,c) \tag{7}$$

Proof. It follows from (2) that $\lim P_1(t,c)=\infty$ as $t\to\infty$. Suppose that $\lim P_i(t,c)=\infty$ as $t\to\infty$ for some $i\in(1,2,\ldots,n-2)$. Then for $P_{i+1}(t,c)$ we get

$$P_{i+1}(t,c)=\int_c^t\frac{ds_1}{a_1(s_1)}\int_c^{s_1}\frac{ds_2}{a_2(s_2)}\cdots\int_c^{s_{i-1}}\frac{ds_i}{a_i(s_i)}\cdot$$

$$\cdot\left[\int_c^b\frac{ds_{i+1}}{a_{i+1}(s_{i+1})}+\int_b^{s_i}\frac{ds_{i+1}}{a_{i+1}(s_{i+1})}\right]\geqq\int_c^b\frac{ds_{i+1}}{a_{i+1}(s_{i+1})}\cdot P_i(t,c)$$

for $t\geqq b$. Thus $\lim P_i(t,c)=\infty$ as $t\to\infty$ implies $\lim P_{i+1}(t,c)=\infty$ as $t\to\infty$. Thus the first part of (5) is proved. The second part of (5) can be proved also in the same manner. Then we get (6) from (5) by use of l'Hospital rule. Let now $t_0>c$ and $0\leqq i<j\leqq n-1$. Then from continuity and positivity of $\dfrac{P_j(t,c)}{P_i(t,c)}$ for $t\geqq t_0$ and from (5) it follows that $\dfrac{P_j(t,c)}{P_i(t,c)}$ has on $[t_0,\infty)$ a positive absolute minimum which we denote by m. Then for $\alpha_{ij}=\dfrac{m}{2}$ we get the first part of (7). A similar reasoning gives also the second part of (7).

Remark 3. It is easy to see from the proof of Lemma 3 that for validity of (5) it is sufficient that $\lim P_1(t, c)=\infty$ as $t\to\infty$ and $\lim Q_{n-1}(t, c)=\infty$ as $t\to\infty$, respectively.

Lemma 4. Let the assumptions of Lemma 2 be satisfied. Let k be the same as in Lemma 2. Then there is a $T_1>0$ such that for $t>T_1$

$$\operatorname{sgn} y(t)=\operatorname{sgn} L_k y(t). \tag{8}$$

If $n+k$ is even, then $|L_k y(t)|$ increases and there exist two constants $0<c_1<c_2$ such that for $t>T_1$

$$0<c_1<|L_k y(t)|<c_2 \tag{9}$$

and

$$0<c_1<\left|\lim_{t\to\infty}\frac{a_0(t)y(t)}{P_k(t, c)}\right|<c_2, \quad \lim_{t\to\infty}\frac{a_0(t)y(t)}{P_{k+1}(t, c)}=0 \tag{10}$$

If $n+k$ is odd, then $|L_k y(t)|$ decreases and there is a constant $c>0$ such that for $t>T_1$

$$0<|L_k y(t)|<c \tag{11}$$

and

$$0\leqq\left|\lim_{t\to\infty}\frac{a_0(t)y(t)}{P_k(t, c)}\right|<c, \quad \lim_{t\to\infty}\frac{a_0(t)y(t)}{P_{k+1}(t, c)}=0 \tag{12}$$

Proof. Let $n+k$ be even. Let n be even and k even. Then from Lemma 2, a) we get $(-1)^{k+2}y(t)L_{k+1}y(t) = y(t)L_{k+1}y(t)>0$ for $t>t_k$. If $y(t)>0$ for $t>t_k$, then $L_{k+1}y(t)>0$ for $t>t_k$ and $L_k y(t)$ increases and must be positive for $t>T\geqq t_k$. If no, we have $L_k y(t)<0$ for $t>t_k$, which means that $L_{k-1}y(t)$ decreases and therefore $\lim L_{k-1}y(t)<\infty$ which is in contradiction with the property d) of Lemma 2. Thus we have that there exists $T_1>0$ and positive constants c_1, c_2 such that (9) is valid for $t>T_1$ as also (8).

If $y(t)<0$ for $t>t_k$, then $L_{k+1}y(t)<0$ and $L_k y(t)$ decreases and must be negative on (T_k, ∞) for some $T_k\geqq t_k$ for to avoid the contradiction with the fact that $\lim_{t\to\infty} L_{k-1}y(t)=-\infty$. It is clear that (8) and (9) is valid also in this case. In the case that n is odd and k is odd too, using the property a') of Lemma 2 and by a similar reasoning as above we get also the validity of (8) and (9). In both cases the validity of (10) follows from (9) and the l'Hospital rule.

Let now $n+k$ is odd. Examine first the case: n even and k odd. Then from the property a) of Lemma 2 we have $(-1)^{k+2}y(t)L_{k+1}y(t)>0$ for $t>t_k$. Thus $y(t)L_{k+1}y(t)<0$ for $t>t_k$. Let $y(t)>0$ for $t>t_k$. Then $L_{k+1}y(t)<0$ and therefore $L_k y(t)$ decreases for $t>t_k$ and must be positive in order to avoid the contradiction with the fact that $\lim_{t\to\infty} L_{k-1}y(t)=\infty$ (property d) of Lemma 2). Thus (8) and (11) is true for $t>t_k$. If $y(t)<0$ for $t>t_k$, then from the property a) of Lemma 2 $L_{k+1}y(t)>0$ for $t>t_k$. Thus $L_k y(t)$ increases and it must be negative for $t>t_k$ in order to avoid the contradiction with the fact that $\lim_{t\to\infty} L_{k-1}y(t)=-\infty$ (property d) of Lemma 2). It is clear that also in this case (8) and (11) are true for all t large enough.

If n is odd and k even, the proof follows the same line using a′) instead of a) of Lemma 2. The validity of (12) follows from (11) and the use of l'Hospital rule.

Lemma 5. Let $y(t)\neq 0$ be such that $L_n y(t)$ exists and let $y(t)\, L_n y(t)\geqq 0$ ($=$holding only at isolated points eventually). Let (2) be valid. Then:

Either there exists a $T\geqq 0$ and a $k\in(0, 1, \ldots, n-1)$ such that the following is true:

If n is even, then

α) $(-1)^i y(t)\, L_i y(t)>0$, $i=k+1, k+2, \ldots, n-1$, for $t>T$;

β) $\lim_{t\to\infty} L_i y(t)=0$, $i=k+1, k+2, \ldots, n-1$;

γ) $\lim_{t\to\infty} L_k y(t)$ exists and is finite;

δ) $\lim_{t\to\infty} L_i y(t)=\infty\cdot\operatorname{sgn} y(t)$, $i=0, 1, \ldots, k-1$.

If n is odd, then β), γ), δ) are true and instead of α), α′) $(-1)^{i+1} y(t)\, L_i y(t)>0$ is valid for $t>T$, $i=k+1, k+2, \ldots, n-1$.

Or

$$\varepsilon)\quad \lim_{t\to\infty} L_i y(t)=\infty.\operatorname{sgn} y(t), \quad i=0, 1, \ldots, n-1.$$

The proof can be made in the same way as it was done in the proof of Lemma 2.

Lemma 6. Let the assumptions of Lemma 5 be satisfied. Then there is a $T>0$ such that the following is true:

1. If $0\leqq k\leqq n-1$, then

$$\operatorname{sgn} y(t)=\operatorname{sgn} L_k y(t) \quad \text{for} \quad t>T \tag{13}$$

and if moreover $n+k$ is odd, then $|L_k y(t)|$ increases and there exist positive constants c_1, c_2 such that

$$0<c_1<|L_k y(t)|<c_2 \quad \text{for} \quad t>T \tag{14}$$

and

$$0<c_1<\left|\lim_{t\to\infty}\frac{a_0(t)y(t)}{P_k(t, c)}\right|<c_2, \quad \lim_{t\to\infty}\frac{a_0(t)y(t)}{P_{k+1}(t, c)}=0 \tag{15}$$

but if $n+k$ is even, then $|L_k y(t)|$ decreases and there is a positive constant c_3 that

$$0<|L_k y(t)|<c_3 \quad \text{for} \quad t>T \tag{16}$$

$$0 \leqq \left|\lim_{t\to\infty} \frac{a_0(t)y(t)}{P_k(t, c)}\right| < c_3, \quad \lim_{t\to\infty} \frac{a_0(t)y(t)}{P_{k+1}(t, c)} = 0 \tag{17}$$

2. If the case ε) from Lemma 5 takes place, then

$$\left|\lim_{t\to\infty} \frac{a_0(t)y(t)}{P_{n-1}(t, c)}\right| = \infty \tag{17'}$$

The proof of this lemma is similar to that of Lemma 4.

Definition 2. We say that the function $y(t)$ has the property (V_k), $k \in (0, 1, \ldots, \ldots, n-1)$, with respect to L_n if it has the properties a) resp. a′), b), c), d) from Lemma 2 or α) resp. α′), β), γ), δ) from Lemma 5. We say that $y(t)$ has the property (V_n) if it has the property ε) from Lemma 5. We will denote by V_k the set of all solutions of (E) with the property (V_k).

Theorem 1. Let (1) and (2) be satisfied. Then each nonoscillatory solution of (E) belongs precisely to one set V_k, $k = 0, 1, \ldots, n$ and Lemma 4 and Lemma 6, respectively, is valid.

Proof. If (1) and (2) are satisfied and if $y(t)$ is a nonoscillatory solution of (E), then the assumptions of Lemma 2 and 5, respectively are satisfied and therefore $y(t)$ has one and only one of the properties (V_k), $k = 0, 1, \ldots, n$. The validity of Lemma 4 and 6, respectively is clear.

The problem we want to discuss is: how to insure that for every $y(t) \in V_k$, $k = 0, 1, \ldots, n-1$, which is a solution of (E), $\lim L_k y(t) = 0$ as $t \to \infty$. To this aim we will need to know more precisely the formula expressing $L_k y(t)$ of a solution $y(t)$ of (E) which belongs to V_k. Integrating the equation (E) successively and respecting the properties b) and β), respectively, we get

$$L_i y(t) = (-1)^{n-i+1} \int_t^\infty \mathrm{d}s \, \frac{h(s, \bar{y}(s))}{a_n(s)} \, Q_{i+1}(s, t) \tag{18}$$

$$i = n-1, n-2, \ldots, k+1$$

where $\bar{y}(s) = (y(s), y'(s), \ldots, y^{(n-1)}(s))$ and $Q_{i+1}(s, t)$ is given by the Definition 1. Denoting $\lim L_k y(t) = L$ as $t \to \infty$ we get

$$L_k y(t) = L + (-1)^{n-k+1} \int_t^\infty \mathrm{d}s \, \frac{h(s, \bar{y}(s))}{a_n(s)} \, Q_{k+1}(s, t) \tag{19}$$

Denote by D_{n-1} the set of all functions f defined and continuous on $J = [t_0, \infty)$ such that all quasi-derivatives $L_i f$, $i = 0, 1, \ldots, n-1$, with respect to the functions $a_m(t)$, $m = 0, 1, \ldots, n-1$, exist and are continuous on J. Denote by $D_{n-1,k}$ the set of all $f \in D_{n-1}$ which have bounded quasiderivatives $L_k f, L_{k+1} f, \ldots, L_{n-1} f$. By the norm

$$\|f\| = \max_{k \leqq i \leqq n-1} \left\{\sup_J |L_i f(t)|\right\} + \sum_{j=0}^{k-1} |L_j f(t_0)| \tag{20}$$

$D_{n-1,k}$ becomes Banach space. The convergence with respect to this norm $\|\cdot\|$ implies the uniform convergence of the quasiderivatives of order i, $i=0, 1, \dots, \dots, n-1$, on every compact subinterval of J and moreover the uniform convergence on J of the quasi-derivatives of order j, $j=k, k+1, \dots, n-1$.

Theorem 2. Let (2) be valid. Let $F(t, u)$ be a function which is continuous, nonnegative on $J\mathrm{x}(\infty, \infty)$ and nondecreasing in u. Let be more

$$|h(t, y)| \leqq F(t, y) \quad \text{for all} \quad (t, y) \in J\mathrm{x}(-\infty, \infty) \tag{21}$$

Let $r>0$ be a real number, $0 \leqq k \leqq n-1$ an integer and

$$g_k(t, t_0, r) = r \frac{1}{a_0(t)} \sum_{i=0}^{k} P_i(t, t_0) \tag{22}$$

$$\int_{t_0}^{\infty} Q_{k+1}(t, t_1) \frac{1}{a_n(t)} F(t, g_k(t, t_0, r))\, dt < \infty, \quad 0 < t_1 < t_0 < t \tag{23}$$

for all $r>0$ and

$$\lim_{r\to\infty} \frac{1}{r} \int_{t_0}^{\infty} Q_{k+1}(t, t_1) \frac{1}{a_n(t)} F(t, g_k(t, t_0, r))\, dt = 0 \tag{24}$$

Let $c_0, c_1, \dots, c_k$ be given real numbers. Then the equation

$$L_n y + h(t, y) = 0 \tag{E$_1$}$$

has at least one solution $y(t)$ defined on J and satisfying the conditions

$$\begin{aligned} &L_i y(t_0) = c_i, \quad i = 0, 1, \dots, k-1 \\ &\lim_{t\to\infty} L_k y(t) = c_k, \\ &\lim_{t\to\infty} L_i y(t) = 0, \quad i = k+1, \dots, n-1 \end{aligned} \tag{25}$$

Proof. Let $y(t)$ be a solution of (E$_1$) satisfying (25). Then an easy calculus gives that $y(t)$ is a solution of the integral equation

$$y(t) = \frac{1}{a_0(t)} \left\{ \sum_{i=0}^{k} c_i P_i(t, t_0) + (-1)^{n-k+1} \cdot \right.$$

$$\left. \cdot \int_{t_0}^{t} \frac{ds_1}{a_1(s_1)} \int_{t_0}^{s_1} \frac{ds_2}{a_2(s_2)} \cdots \int_{t_0}^{s_{k-1}} \frac{ds_k}{a_k(s_k)} \int_{s_k}^{\infty} \frac{h(s, y(s))}{a_n(s)} \cdot Q_{k+1}(s, s_k)\, ds \right.$$

for $t \geqq t_0$ and every solution of this integral equation is a solution of (E$_1$) satisfying the conditions (25). For the quasiderivatives of $y(t)$ we get

$$L_i y(t) = (-1)^{n-i+1} \int_{t}^{\infty} \frac{h(s, y(s))}{a_n(s)} Q_{i+1}(s, t)\, ds, \quad t \geqq t_0$$

$$i = k+1, k+2, \dots, n-1 \tag{27}$$

$$L_k y(t) = c_k + (-1)^{n-k+1} \int_t^\infty \frac{h(s, y(s))}{a_n(s)} Q_{k+1}(s, t)\, ds, \quad r \geqq t_0$$

Denote by $G_r = \{f \in D_{n-1,k}: \|f\| \leqq r\}$. Then $|L_i f| \leqq r$, $i = k, k+1, \ldots, n-1$ and $|f| \leqq g_k(t, t_0, r)$. We define on G_r the operator T as follows: if $f \in G_r$, then

$$v(t) = Tf(t) = \frac{1}{a_0(t)} \left\{ \sum_{i=0}^{k} c_i P_i(t, t_0) + \right.$$

$$+ (-1)^{n-k+1} \cdot \int_{t_0}^{t} \frac{ds_1}{a_1(s_1)} \int_{t_0}^{s_1} \frac{ds_2}{a_2(s_2)} \cdots \int_{t_0}^{s_{k-1}} \frac{ds_k}{a_k(s_k)} \int_{s_k}^{\infty} \frac{h(s, f(s))}{a_n(s)} \cdot Q_{k+1}(s, s_k)\, ds, \quad t \geqq t_0. \tag{28}$$

Respecting (21) and (23) it is easy to verify that this operator is well defined on G_r. Moreover, we have

$$L_i v(t_0) = c_i, \quad i = 0, 1, \ldots, k-1 \quad \lim_{t \to \infty} L_k v(t) = c_k$$

$$L_k v(t) = c_k + (-1)^{n-k+1} \int_t^\infty \frac{h(s, f(s))}{a_n(s)} Q_{k+1}(s, t)\, ds, \quad t \geqq t_0$$

$$L_i v(t) = (-1)^{n-i+1} \int_t^\infty \frac{h(s, f(s))}{a_n(s)} Q_{i+1}(s, t)\, ds, \quad t \geqq t_0 \tag{29}$$

$$\lim_{t \to \infty} L_i v(t) = 0, \quad i = k+1, k+2, \ldots, n-1$$

Thus $Tf(t) = v(t) \in D_{n-1,k}$. Moreover, respecting (21) and the properties of F and Q_i we get

$$|L_k v(t)| \leqq |c_k| + \int_{t_0}^\infty \frac{F(s, g_k(s, t_0, r))}{a_n(s)} Q_{k+1}(s, t_1)\, ds$$

$$|L_i v(t)| \leqq \int_{t_0}^\infty \frac{F(s, g_k(s, t_0, r))}{a_n(s)} Q_{i+1}(s, t_1)\, ds, \quad i = k+1, \ldots, n-1 \tag{30}$$

From Lemma 3 we get that there are constants $\beta_{k+1,i}$, $i = k+2, \ldots, n-1$, such that

$$\beta_{k+1,i} Q_i(t, t_1) < Q_{k+1}(t, t_1) \quad \text{for} \quad t > t_0 (> t_1)$$

Then it is easy to see that from (23)

$$\int_{t_0}^\infty \frac{F(t, g_k(t, t_0, r))}{a_n(t)} Q_i(t, t_1)\, dt < \infty \tag{31}$$

$$i = k+2, \ldots, n$$

and from (24)

$$\lim_{r \to \infty} \frac{1}{r} \int_{t_0}^\infty \frac{F(t, g_k(t, t_0, r))}{a_n(t)} Q_i(t, t_1)\, dt = 0$$

$$i = k+2, \ldots, n,$$

follow. Hence we can conclude that there is an $r_0 > 0$ such that

$$\int_{t_0}^{\infty} \frac{F(t, g_k(t, t_0, r))}{a_n(t)} Q_i(t, t_1)\, dt \leqq \frac{r}{2}$$

for all $r \geqq r_0$ and $i = k+1, \ldots, n-1$. Then we have

$$|L_k v(t)| \leqq |c_k| + \frac{r}{2}, \quad |L_i v(t)| \leqq \frac{r}{2}$$

for $r \geqq r_0$ and $i = k+1, k+2, \ldots, n-1$, and

$$\|Tf\| = \|v(t)\| \leqq \sum_{i=0}^{k} |c_i| + \frac{r}{2}, \quad \text{for} \quad r \geqq r_0$$

Thus, there exists an $R \geqq r_0$ such that

$$\sum_{i=0}^{k} |c_i| + \frac{R}{2} \leqq R$$

That means that T maps the ball G_R into G_R.

Now, we are going to prove the continuity of T on G_R. Let be $f_m, f \in G_R$, $\|f_m - f\| \to 0$ as $m \to \infty$. Then

$$\|Tf_m - Tf\| = \max_{k \leqq i \leqq n-1} \left\{ \sup_J \left| \int_t^{\infty} \frac{Q_{i+1}(s, t)}{a_n(s)} [h(s, f_m(s)) - h(s, f(s))]\, ds \right| \right\} \to 0$$

as $m \to \infty$ using the continuity of h, (21), (23), (31) and the Lebesgue dominated convergence theorem.

Denote $TG_R = H$ and $H^{(j)}$, $j = 0, 1, \ldots, n-1$, the set of the quasi-derivatives of order j of all functions from H. Evidently the functions of $H^{(j)}$, $j = k, k+1, \ldots, \ldots, n-1$, are uniformly bounded by R. They are also equicontinuous on J. In fact, for $v \in H$ we have

$$L_{n-1} v(t) = \int_t^{\infty} \frac{h(s, f(s))}{a_n(s)}\, ds,$$

where $f \in G_R$ is a suitable function and

$$|L_{n-1} v(t) - L_{n-1} v(t')| \leqq \left| \int_t^{t'} \frac{h(s, f(s))}{a_n(s)}\, ds \right| \leqq \left| \int_t^{t'} \frac{F(s, g_k(s, t_0, R))}{a_n(s)}\, ds \right|$$

From this and from (31) the equicontinuity on J of all functions from $H^{(n-1)}$ follows. Then from the uniform boundedness of $H^{(j)}$, $j = k, k+1, \ldots, n-1$, the equicontinuity of $H^{(j)}$, $j = k-1, k, \ldots, n-2$, on J follows. Taking in account (21), the monotonicity of $F(t, y)$ in y, the fact that for $f \in G_R$ $|f| \leqq g_k(t, t_0, R)$ and the formulae for $L_i v$, $i=0, 1, \ldots, k-1$, we get that $H^{(j)}$, $j=0, 1, \ldots, k-1$, are uniformly bounded and equicontinuous on every compact subinterval of J.

Denote by M the convex and closed envelope of H. Then M is convex and closed and $M^{(j)}$, $j=0, 1, \ldots, n-1$, are uniformly bounded and equicontinuous on every compact subinterval of J (See [3].) M being the convex and closed envelope of H and $H=TG_R \subset G_R$, we have that $M \subset G_R$. Then $TM \subset TG_R = H \subset M \subset G_R$. Now, it is easy to see that TM is compact. Application of Schauder's theorem gives the existence of at least one fixed point of T in M, which is a solution of (26) and also of (E_1) with the properties (25).

Remark 4. We needed the assumption (24) only to guarantee the existence of such a ball G_R that $TG_R \subset G_R$. This assumption is not satisfied for example if $F(t, y)$ is linear in y. But we took t_0 fixed in our considerations. If we don't do it we can eliminate the assumption (24) in the following manner: We choose a $r > \sum_{i=0}^{k} |c_i|$ and a t_0 such that

$$\int_{t_0}^{\infty} \frac{F(t, g_k(t, t_0, r))}{a_n(t)} Q_i(t, t_0)\, dt \leqq r - \sum_{j=0}^{k} |c_j|$$

$i=k+1, \ldots, n$. Then certainly $TG_r \subset G_r$ and in the same way as Theorem 2 the next theorem can be proved.

Theorem 3. Let all assumptions of Theorem 2 with $r > \sum_{i=0}^{k} |c_i|$ be satisfied except the assumption (24) and let the assumption (23) be substituted by

$$\int_{t}^{\infty} \frac{F(s, g_k(s, t, r))}{a_n(s)} Q_{k+1}(s, t)\, ds < \infty \tag{32}$$

Then there is a $t_0 > 0$ such that the equation (E_1) has at least one solution $y(t)$ defined on $[t_0, \infty)$ and satisfying the conditions (25).

From Theorem 2 and 3, respectively, immediately follows

Theorem 4. Let all assumptions of Theorem 2 and Theorem 3, respectively, be satisfied. Let k be an integer, $0 \leqq k \leqq n-1$, and let c_k be a real number. Then there are k solutions of the equation (E_1) which are linearly independent and the k-th quasi-derivatives of which have the limit for $t \to \infty$ equal to c_k.

Remark 5. We have supposed that h is a function of two variables t, y only. We can extend our considerations to the case that $h=h(t, y, y_1, \ldots, y_{n-1})$. Then it is necessary to change the definition of the set D_{n-1}: it will be the set of all functions f defined on J which have continuous derivative of the order $n-1$ and all quasi-derivatives $L_i f$, $i=0, 1, \ldots, n-1$, with respect to the functions $a_i(t)$, $i= =0, 1, \ldots, n-1$. The last requirement is satisfied for example if we suppose that $a_i(t) \in C^{n-i}(J)$, $i=0, 1, \ldots, n-1$. Then $D_{n-1,k}$ denotes the Banach space of all functions $f \in D_{n-1}$ which have the bounded quasi-derivatives $L_i f$, $i = k, k+1, \dots, n-1$. The norm is given by (20).

Theorem 5. The conclusions of Theorems 2, 3 and 4, respectively still remain valid if we substitute the function $h(t, y)$ by $h(t, y, y_1, \dots, y_{n-1})$ and instead of (21) we have

$$|h(t, y, y_1, \dots, y_{n-1})| \leqq F(t, y) \tag{33}$$

The proof of this theorem can be made in the same way as the proof of Theorem 2, but it is necessary to respect Remark 5 and to make the needed changes.

Remark 6. If $F(t, y) = \beta_0(t) + \beta(t)|y|$, then condition (23) will be satisfied if

$$\int_{t_0}^{\infty} \frac{\beta_0(t)}{a_n(t)} Q_{k+1}(t, t_0)\, dt < \infty \tag{34}$$

and

$$\int_{t_0}^{\infty} \frac{\beta(t)}{a_n(t) a_0(t)} P_k(t, t_0) Q_{k+1}(t, t_0)\, dt < \infty \tag{35}$$

In the case that $h(t, y)$ satisfies the Lipschitz condition

$$|h(t, y_1) - h(t, y_2)| \leqq \beta(t)|y_1 - y_2|$$

then $F(t, y) = |h(t, 0)| + \beta(t)|y|$ and the condition (35) and

$$\int_{t_0}^{\infty} \frac{|h(t, 0)|}{a_n(t)} Q_{k+1}(t, t_0)\, dt < \infty$$

guarantee the fulfillement of (23).

After this we can start to solve the problem: to state the conditions which guarantee that if $y(t) \in V_k$ is a solution of (E), then $\lim L_k y(t) = 0$ as $t \to \infty$.

Theorem 6. Let (1) and (2) be satisfied. Let there exist functions $G(t, y) \geqq 0$, $F(t, y) \geqq 0$ defined on J, continuous and nondecreasing in y such that

$$G(t, y) \leqq |h(t, y, y_1, \dots, y_{n-1})| \leqq F(t, y) \tag{36}$$

for $(t, y, y_1, \dots, y_{n-1}) \in J \times E^n$, where E^n is the n-dimensional Euclidean vector space. Let be $k \in (0, 1, \dots, n-1)$. Then the condition

$$\int_{t}^{\infty} \frac{1}{a_n(s)} Q_{k+1}(s, t) F(s, g_k(s, t, r))\, ds = \infty \quad \text{for all} \quad r > 0 \tag{37}$$

is a necessary condition and

$$\int_{t}^{\infty} \frac{1}{a_n(s)} Q_{k+1}(s, t) G(s, g_k(s, t, r))\, ds = \infty \quad \text{for all} \quad r > 0 \tag{38}$$

a sufficient condition that for every solution $y(t) \in (E)$ such that $y(t) \in V_k$ holds $\lim L_k y(t) = 0$ as $t \to \infty$.

Proof. Suppose that the second part of (36) is true and that (37) is not true, i.e. that the integral in (37) converges for an $r > 0$. Then it is easy to verify that all assumptions of Theorem 5 are fulfilled. Therefore, if we choose $(c_0, c_1, \ldots, c_k)$, $c_k \neq 0$, arbitrarily, then there exists a $t_0 > 0$ such that the equation (E) has at least one nonoscillatory solution $y(t)$ which belongs to V_k and $\lim L_k y(t) = c_k \neq 0$ as $t \to \infty$. Thus the necessity of (37) is proved.

Let now be $y(t) \in V_k$ a nonoscillatory solution of (E) and let $\lim L_k y(t) = L \neq 0$ as $t \to \infty$. Then to $\frac{L}{2}$ there exists a $c > 0$ large enough such that $|L_k y(t)| > \frac{L}{2} > 0$ for $t \geqq c$ and $L_i y(t)$, $i = 0, 1, \ldots, k$, have the same sign for $t \geqq c$. Then the successive integration gives

$$|y(t)| \geqq \frac{1}{a_0(t)} \left[\sum_{i=0}^{k-1} |L_i y(c)| P_i(t, c) + \frac{L}{2} P_k(t, c) \right] \geqq g_k(t, c, r)$$

where $r = \min \left\{ |L_i y(c)|,\ i = 0, 1, \ldots, k-1, \frac{L}{2} \right\}$. Suppose that the left part of (36) holds. Then using the formula (27) where we set L instead of c_k we get

$$|L - L_k y(c)| \geqq \int_c^{\infty} \frac{1}{a_n(s)} Q_{k+1}(s, c) G(s, g_k(s, c, r)) \, ds$$

The left side is finite but the right side is infinite. This contradiction proves that $\lim_{t \to \infty} L_k y(t)$ can not be different from zero.

Corollary 1. Let (1) and (2) be satisfied. Let there exist functions $\alpha(t) \geqq 0$, $\beta(t) \geqq 0$ defined on J and such that

$$\alpha(t)|y| \leqq |h(t, y, y_1, \ldots, y_{n-1})| \leqq \beta(t)|y| \tag{36'}$$

for $(t, y, y_1, \ldots, y_{n-1}) \in J \times E^n$, where E^n is the n-dimensional Euclidean vector space. Let be $k \in (0, 1, \ldots, n-1)$. Then

$$\int_t^{\infty} \frac{\beta(s)}{a_0(s) a_n(s)} P_k(s, t) Q_{k+1}(s, t) \, ds = \infty \tag{37'}$$

is a necessary condition and

$$\int_t^{\infty} \frac{(s)}{a_0(s) a_n(s)} P_k(s, t) Q_{k+1}(s, t) \, ds = \infty \tag{38'}$$

is a sufficient condition that for every nonoscillatory solution $y(t)$ of (E) such that $y(t) \in V_k$, $\lim L_k y(t) = 0$ as $t \to \infty$ holds.

This corollary follows immediatly from Theorem 6. We have only to respect the monotonicity of F and G.

From Theorem 6 follows also

Theorem 7. Let $h(t, y) = p(t)y$, $p(t) \geqq 0$ $(\leqq 0)$ continuous on J, where the sign = in the last inequality can hold only at isolated points. Let (2) be valid and let $k \in (0, 1, \ldots, n-1)$. Then for every solution $u(t)$ of the equation

$$L_n y(t) + p(t)y = 0 \tag{E$_2$}$$

belonging to V_k $\lim L_k u(t) = 0$ as $t \to \infty$ holds if and only if

$$\int_t^\infty \frac{|p(s)|}{a_0(s)a_n(s)} P_k(s, t) Q_{k+1}(s, t) \, \mathrm{d}s = \infty \tag{39}$$

Corollary 2. Let $p(t)$ be as in Theorem 7. Then for all nonoscillatory solution $y(t)$ of

$$y^{(n)} + p(t)y = 0 \tag{E$_3$}$$

belonging to V_k $\lim y^{(k)} = 0$ as $t \to \infty$, $k = 0, 1, \ldots, n-1$, hold if and only if

$$\int_t^\infty |p(s)| s^{n-1} \, \mathrm{d}s = \infty \tag{40}$$

Proof. Because $a_i(t) = 1$, $i = 0, 1, \ldots, n$, we have in this case that $P_k(s, t) = (s-t)^k$, $Q_{k+1}(s, t) = (s-t)^{n-k-1}$. Therefore condition (39) has the form

$$\int_t^\infty |p(s)|(s-t)^{n-1} \, \mathrm{d}s = \infty$$

But this condition is satisfied if and only if (40) is true as it follows from an easy calculus.

Theorem 8. Let (2) be valid and let be

$$yh(t, y, y_1, \ldots, y_{n-1}) \geqq 0 \quad (\leqq 0) \quad \text{for} \quad y \neq 0$$

where = holds in isolated points at most. Suppose that the left part of (36) holds and (38) is true for some $k \in (0, 1, \ldots, n-1)$. Then the set V_k of solutions of (E) is empty if $n + k$ is even (odd).

Proof. From Theorem 6 it follows that if $y(t) \in V_k$ is a solution of (E), then $\lim L_k y(t) = 0$ as $t \to \infty$. But from Lemma 4 (from Lemma 6) it follows that $|L_k y(t)|$ increases if $n + k$ is even (odd).

Theorem 9. Let $p(t) \geqq 0$ $(\leqq 0)$ be continuous on J, where = can hold only in isolated points. Let (2) and (39) be valid. Then the set V_k of solutions of (E_2) is empty if $n + k$ is even (odd).

A similar reasoning as in the proof of Theorem 8 gives the proof of this theorem.

Corollary 3. Let $p(t) \geqq 0$ $(\leqq 0)$ be continuous on J, where $=$ can hold in isolated points only. Let (40) is valid. Then the sets V_k of solutions of (E_3) are empty for all $k \in (0, 1, \ldots, n-1)$ for which $n+k$ is even (odd).

This result is the same as by W. J. Kim [4]. Theorems 2, 3, 4 are generalizations of earlier results of M. Švec [1], [2], [3] and Theorem 6, 7 and Corollary 2 of the results M. Švec [1] and W. J. Kim [4], Theorems 8, 9 and Corollary 3 of W. J. Kim [4].

Remark 6. It can be seen from the proofs of theorems that the conditions concerning the continuity of $F(t, y)$ in Theorem 2 and of $G(t, y)$ and $F(t, y)$ in Theorem 6 as also of $\alpha(t)$ and $\beta(t)$ in Corollary 1 can be relaxed.

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SÚHRN

SPRÁVANIE SA NEOSCILATORICKÝCH RIEŠENÍ NIEKTORÝCH NELINEÁRNYCH DIFERENCIÁLNYCH ROVNÍC

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V práci sa vyšetruje správanie neoscilatorických riešení a ich kváziderivácií diferenciálnych rovníc typu $L_n y + h(t, y, y', \dots, y^{(n-1)}) = 0$, kde L_n je diskonjugovaný lineárny operátor na intervale $J = [0, \infty)$ a $yh(t, y, y', \dots, y^{(n-1)})$ má konštantné znamienko. Je prevedený disjunktný rozklad množiny neoscilatorických riešení na triedy V_k, $k = 0, 1, \dots, n$ a stanovujú sa postačujúce a nutné podmienky, aby pre riešenie $y(t) \in V_k$ platilo $\lim L_k y(t) = 0$ pre $t \to \infty$. ($L_k y(t)$ znamená k-tu kvázideriváciu $y(t)$.) Ďalej sú stanovené postačujúce a nutné podmienky, aby niektoré triedy V_k boli prázdne.

РЕЗЮМЕ

ПОВЕДЕНИЕ НЕОСЦИЛАЦИОННЫХ РЕШЕНИЙ НЕКОТОРЫХ НЕЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

М. Швец, Братислава

В настоящей статье исследуется поведение неосцилляционных решений и их квазипроизводных дифференциальных уравнений вида $L_n y + h(t, y, y', \dots, y^{(n-1)}) = 0$, где L_n – неосцилляционный (disconjugate) линейный оператор на $J = [0, \infty)$ и $yh(t, y, y', \dots, y^{(n-1)})$ не меняет знака. Установлено разбиение множества неосцилляционных решений на попарно не пересекающиеся классы V_k, $k = 0, 1, \dots, n$ и даны достаточные и необходимые условия для того, чтобы для решения $y(t) \in V_k$ имело место $\lim L_k y(t) = 0$ при $t \to \infty$, ($L_k y(t)$ обозначает k-ю квазипроизводную $y(t)$). Кроме того, приведены достаточные и необходимые условия для того, чтобы некоторые классы V_k были пустыми.