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BANDS OF SOLUTIONS OF A DIFFERENTIAL EQUATION OF THE THIRD ORDER WITH CONTINUOUS COEFFICIENTS AND THEIR APPLICATION

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To Professor O. Borůvka on the occasion of his 80th birthday

The paper studies so-called bands of solutions [1] of the differential equation of the third order

(a) $$y''' + p(x)y'' + q(x)y' + r(x)y = 0$$

where $p = p(x)$, $q = q(x)$, $r = r(x)$ are continuous functions of $x \in (\alpha, \infty)$, $-\infty \leqq \alpha$, and their applications in solving certain boundary value problems with parameter.

The inspiration for introducing the notion of bands of solutions in the theory of differential equation of the third order was the theory of linear differential equation of the second order, worked out by O. Borůvka, today elaborated in monograph [2] and other papers.

I

The adjoint differential equation to equation (a) is

(b) $$[(z' - pz)' + qz]' - rz = 0$$

Let w be a solution of the differential equation (b) on (α, ∞). In addition have the differential equation of the second order

(c) $$wy'' + (pw - w')y' + [qw + (w' - pw)']y = 0$$

The operator on the left side of the differential equation (c) follows from the relation between solutions of adjoint differential equations (a), (b), [3].

Suppose in following, that the differential equation of the second order

(b_1) $$(u' - pu)' + qu = 0$$

is disconjugated in (α, ∞), i.e. each of its solutions has at most one null-point in (α, ∞).

The differential equation (b) can be written in the form of linear differential system of the first order

$$\begin{aligned} z' &= -pz + u \\ u' &= -qz + v \\ v' &= rz \end{aligned}$$

from where it follows, that there exists for a number $x_0 \in (\alpha, \infty)$ and for every three numbers z_0, z_0', z_0'' unique solution of the differential equation (b), defined on (α, ∞), with the properties $z(x_0) = z_0$, $z'(x_0) = z_0'$, $(z' - pz)'(x_0) = z_0''$.

Denote

$$(b_2) \qquad [(v' - pv)' + qv]' = 0$$

It is clear, if u is a solution of the differential equation (b_1) with the property $u(x_0) = 0$, $u'(x_0) > 0$, then $u(x) > 0$ for $x > x_0$, $\alpha < x_0 < \infty$.

Let u_1, u_2 be a fundamental system of solutions of the differential equation (b_1), then their Wronskian is

$$W^0(x) = W^0(u_1, u_2) = \begin{vmatrix} u_1, & u_2 \\ u_1' - pu_1, & u_2' - pu_2 \end{vmatrix} = \begin{vmatrix} u_1, & u_2 \\ u_1', & u_2' \end{vmatrix}$$

The function

$$W_2^0(x, t) = \begin{vmatrix} u_1(x), & u_2(x) \\ u_1(t), & u_2(t) \end{vmatrix}$$

is a solution of the differential equation (b_1) for fixed $t \in (\alpha, \infty)$. $W_2^0(t, t) = 0$, $W_{2x}^{0\prime}(t, t) = -W^0(t) \neq 0$. Hence $W_2^0(x, t) \neq 0$ for $x > t$.

Lemma 1. Let u_1, u_2 be a fundamental system of solutions of the differential equation (b_1) and let $W^0(u_1, u_2) > 0$ for $x \in (\alpha, \infty)$. Then the function

$$v(x) = -k \int_{x_0}^{x} \frac{W_2^0(x, t)}{W^0(t)} \, dt, \quad \alpha < x_0 < \infty, \quad x \in (\alpha, \infty)$$

is a solution of the differential equation (b_2) with the property $v(x_0) = 0$, $v'(x_0) = 0$, $(v' - pv)'(x_0) = k \neq 0$ and moreover $v(x) \neq 0$ for $x > x_0$.

Proof. We easily see by the method of variation of parameters, that $v(x)$ is a solution of the differential equation of the second order

$$(u' - pu)' + qu = k$$

and therefore of the differential equation (b_2), too.

Clearly

$$v(x_0) = 0, \quad v'(x) = -k \int_{x_0}^{x} \frac{W_{2x}^{0\prime}(x, t)}{W^0(t)} \, dt$$

so $v'(x_0)=0$. We have also

$$v'(x)-p(x)v(x)=-k\int_{x_0}^{x}\frac{1}{W^0(t)}\begin{vmatrix}u_1'(x)-p(x)u_1(x), & u_2'(x)-p(x)u_2(x)\\ u_1(t), & u_2(t)\end{vmatrix}\,\mathrm{d}t$$

and

$$[v'(x)-p(x)v(x)]'=k-k\int_{x_0}^{x}\frac{1}{W^0(t)}\begin{vmatrix}[u_1'(x)-p(x)u_1(x)]', & [u_2'(x)-p(x)u_2(x)]'\\ u_1(t), & u_2(t)\end{vmatrix}\,\mathrm{d}t$$

and so is $[v'(x)-p(x)v(x)]'(x_0)=k$. If we suppose e.g. $k>0$, then $v(x)>0$ for $x>x_0$.

Lemma 2. Let $r(x)\geqq 0$ for $x\in(\alpha,\infty)$ and let $w=w(x)$ be the solution of the differential equation (b) with the property $w(x_0)=w'(x_0)=0$, $(w'-pw)'(x_0)=k>>0$. Then $w(x)>0$ for $x>x_0>\alpha$.

Proof. The differential equation (b) can be written in the form

$$[(z'-pz)'+qz]'=rz$$

It follows from the method of variation of parameters [1], that the solution w can be written in the form

$$w(x)=v(x)+\int_{x_0}^{x}\frac{W_3(x,t)}{W(t)}r(t)\,\mathrm{d}t, \tag{1}$$

where $v(x)$ is suitable solution of the differential equation (b_2) and $W(t)$ is the Wronskian of the fundamental system of solutions v_1, v_2, v_3 of the differential equation (b_2) and

$$W_3(x,t)=\begin{vmatrix}v_1(x), & v_2(x), & v_3(x)\\ v_1(t), & v_2(t), & v_3(t)\\ v_1'(t), & v_2'(t), & v_3'(t)\end{vmatrix}$$

$W_3(x,t)$ is for fixed t a solution of the differential equation (b_2) with double null-point in the point t. If $W(t)>0$, then from the lemma 1 follows $W_3(x,t)\geqq 0$ for $x\geqq t$. If $v(x)$ satisfies the conditions of the lemma 1, that is $v(x_0)=v'(x_0)=0$, $(v'-pv)'(x_0)=k>0$, then from the relation (1) follows the assertion of the lemma 2.

Remark 1. If $w(x)$ has the properties as in the lemma 2 and we substitute in the equation (c) $z=w(x)$, we get in the interval (x_0,∞) the differential equation of the second order ($w(x)\neq 0$ for $x>x_0$), which is regular on the interval (x_0,∞), i.e. for the solutions of the differential equation (c) holds the theorem concerning the separation of null-points.

Definition. The set of solutions of the differential equation (a) with the property $y(x_0)=0$ is called the band of solutions of the first type at the point x_0, the band at the point x_0 for short.

Remark 2. The band at the point x_0 satisfies a differential equation of the form

(c) and is regular in the interval (x_0, ∞), i.e. null-points of every two independent solutions of the band separate each other (if they exist).

Remark 3. The condition of disconjugacy of the differential equation (b_1) is equivalent to the condition of disconjugacy of the differential equation

$$(a_1) \qquad y''+py'+qy=0$$

in the interval (α, ∞).

The assertion follows from the relation between solutions of the adjoint differential equations (a_1) and (b_1). If namely $\bar{y}_1, \bar{y}_2$ is a fundamental system of solutions of the differential equation (a_1) with $\bar{y}_1(x_0)\bar{y}_2'(x_0) - \bar{y}_1'(x_0)\bar{y}_2(x_0)=1$, then

$$u_1=\bar{y}_1(x)\exp\left(\int_{x_0}^{x} p(t)\,\mathrm{d}t\right), \quad u_2=\bar{y}_2(x)\exp\left(\int_{x_0}^{x} p(t)\,\mathrm{d}t\right)$$

is the fundamental system of solutions of the differential equation (b_1).

Corollary 1. Let $r(x)\geqq 0$ for $x\in(\alpha,\infty)$. Let $y_1(x)$ be the solution of the differential equation (a) with the property $y_1(x_0)=y'(x_0)=0$, $y_1''(x_0)\neq 0$, $\alpha<x_0<$ $>\infty$. Then $y_1(x)\neq 0$ for $x<x_0$.

Proof. The assertion follows from the relations between the solutions of adjoint differential equations of the third order [2].

Let namely y_1, y_2, y_3 be a fundamental system of solutions of the differential equation (a) with the Wronskian $W(y_1, y_2, y_3)(x_0)=1$.

The fundamental system of solutions of the differential equation (b) is of the form

$$z_1=\begin{vmatrix} y_1, & y_2 \\ y_1', & y_2' \end{vmatrix} \exp\left(\int_{x_0}^{x} p\,\mathrm{d}t\right), \quad z_2=\begin{vmatrix} y_1, & y_3 \\ y_1', & y_3' \end{vmatrix} \exp\left(\int_{x_0}^{x} p\,\mathrm{d}t\right),$$

$$z_3=\begin{vmatrix} y_2, & y_3 \\ y_2', & y_3' \end{vmatrix} \exp\left(\int_{x_0}^{x} p\,\mathrm{d}t\right)$$

If $y_1(x_0)=y_1'(x_0)=0$, $y_1''(x_0)\neq 0$, $y_2(x_0)=y_2''(x_0)=0$, $y_2'(x_0)\neq 0$, then z_1 has the property $z_1'(x_0)=z_1(x_0)=0$, $(z_1'-pz_1)'(x_0) = -y_2'(x_0)y_1''(x_0)\neq 0$.

Now let $x_1<x_0$ and suppose, that $y_1(x_1)=0$. Then we could write y_1 in the form $y_1(x) = c_1\bar{y}_1(x) + c_2\bar{y}_2(x)$, where $\bar{y}_1, \bar{y}_2$ are the solution of (a) with the properties $\bar{y}_1(x_1) = \bar{y}_1'(x_1)=0$, $\bar{y}_1''(x_1)\neq 0$, $\bar{y}_2(x_1) = \bar{y}_2''(x_1)=0$, $\bar{y}_2'(x_1)\neq 0$, i.e. y_1 would belong to the band at the point x_1.

Hence there would exist such constants c_1, c_2 (not both zeros), that

$$c_1\bar{y}_1(x_0)+c_2\bar{y}_2(x_0)=0$$
$$c_1\bar{y}_1'(x_0)+c_2\bar{y}_2'(x_0)=0$$

would hold. But that would be in controversy with the assertion of the lemma 2 and with the regularity of the band at the point x_1.

Lemma 3. Let $r(x) \geqq 0$ for $x \in (\alpha, \infty)$ and let $x_1 > x_0 \in (\alpha, \infty)$ is the first null-point of the solution y_1 (with double null-point at the point x_0) to the right of x_0. Then each solution (independent of y_1) of the band at the point x_0 has exactly one null-point between x_0 and x_1.

Proof. Let $y = c_1 y_1 + c_2 y_2$ be the band at the point x_0 of the differential equation (a), where $y_1(x_0) = y_1'(x_0) = 0$, $y_1''(x_0) \neq 0$, $y_2(x_0) = y_2''(x_0) = 0$, $y_2'(x_0) \neq 0$, $\alpha < x_0 < \infty$ and let $x_1 > x_0$ be the first null-point of the solution y_1. Let further be $c_2 \neq 0$. Then we have

$$\left(\frac{y_1}{y}\right)' = \frac{y_1' y - y_1 y'}{y^2}$$

Suppose $y(x) \neq 0$ for $x \in (x_0, x_1)$. After the integration of the last equality from x_0 to x_1 we get

$$0 = \lim_{x \to x_0} \left[-c_2 \int_{x_0}^{x} \frac{z_1}{y^2} \exp\left(-\int_{x_0}^{t} p \, ds \right) dt \right] \neq 0$$

and we arrived to a contradiction. Hence the conclusion of the lemma is true.

Lemma 4. Let $r(x) \geqq 0$ for $x \in (\alpha, \infty)$. Let y_1 be the solution of the differential equation (a) with double null-point at a point $x_0 \in (\alpha, \infty)$ and let $y_1(x_1) = 0$, $x_1 > x_0$, where x_1 is the first null-point of the solution y_1 to the right of x_0. Let $x_0 < \bar{x} < x_1$ and let $\bar{y}_1$ be the solution of the differential equation (a) with double null-point at $\bar{x}$ and let $\bar{x}_1 > \bar{x}$ be its next nearest null-point. Then $\bar{x}_1 > x_1$ holds.

Proof. Suppose the contrary, i.e. $x_0 < \bar{x} < \bar{x}_1 \leqq x_1$. Then there exist a solution y_{x_0}, that belongs to the band at the point x_0 and that has a null-point at the point $\bar{x}$, hence that belongs to the band at the point $\bar{x}$. However from the properties of bands (lemma 3) it follows, that the solution y_{x_0} has another null-point between $\bar{x}$ and $\bar{x}_1$, i.e. between x_0 and x_1 it has two null-points and that is impossible. Thus the lemma is proved.

II

Consider a differential equation

$$\text{(A)} \qquad y''' + p(x)y'' + q(x)y' + \lambda r(x)y = 0$$

where $p = p(x)$, $q = q(x)$, $r = r(x)$ are continuous functions $x \in (\alpha, \infty)$, $r(x) > 0$ for $x \in (\alpha, \infty)$ and λ is a real parameter. Besides let the differential equation (a_1) be disconjugated in (α, ∞).

Lemma 5. Let the conditions mentioned above be fulfilled and let $\langle a, b \rangle \subset (\alpha, \infty)$, $\alpha < a < b < \infty$. Let $y_1(x, \lambda)$ be a solution of the differential equation (A) with the property $y_1(a, \lambda) = y_1'(a, \lambda) = 0$, $y_1''(a, \lambda) = k \neq 0$. Then there exist a $\bar{\lambda} > 0$, such that $y_1(x, \lambda)$ has at least one null-point in (a, b) for each $\lambda > \bar{\lambda}$.

Proof. Compare the differential equation (A) with

$$(\bar{A}) \qquad v''' + p(x)v'' + q(x)v' + \lambda k v = 0$$

where $k = \min r(x)$ for $x \in \langle a, b \rangle$. The differential equation (A) can be written in the form

$$y''' + p(x)y'' + q(x)y' + \lambda k y = \lambda[k - r(x)]y$$

From the method of variation of constants it follows for y_1, that

$$(2) \qquad \begin{aligned} y_1(x, \lambda) = \bar{v}_1(x, \lambda) - \lambda \int_a^x [r(t) - k] \\ W(x, t) \exp\left(\int_a^t p(s)\, ds\right) y_1(t, \lambda)\, dt \end{aligned}$$

where $W(x, t)$ is of the form $W_3(x, t)$ of the lemma 2, but, v_1, v_2, v_3 form the fundamental system of solutions of the differential equation $(\bar{A})$ and their Wronskian is equal to one at a. It is easy to see, that $W(x, t)$ is for fixed t the solution of $(\bar{A})$ with a double null-point at t. If y_1 and $\bar{v}_1$ are the solutions of (A) and $(\bar{A})$ with the condition

$$y_1(a, \lambda) = \bar{v}_1(a, \lambda) = y_1'(a, \lambda) = \bar{v}_1'(a, \lambda) = 0,$$

$y_1''(a, \lambda) = \bar{v}_1''(a, \lambda) > 0$, and if $x_1 > a$ is the first null-point of $\bar{v}_1$ on the right of a, then there is $W(x, t) \geqq 0$ for $a \leqq t \leqq x \leqq x_1$, which follows from the lemma 4 and then from the relation (2) we obtain the assertion, that the first nullpoint of y_1 on the right of a is less or equal to x_1. From this result and from the asymptotic formulas for the solutions of the differential equation $(\bar{A})$ [4, kap. II] follows the assertion of the lemma 5.

Theorem 1 (Oscillation theorem). Let the suppositions of the lemma 5 be fulfilled. Let $y(x, \lambda)$ be an arbitrary nontrivial solution of the differential equation (A) with the property $y(a, \lambda) = 0$. Then to any naturel number $v > 0$ there exists such a $\lambda_v > 0$, that for $\lambda > \lambda_v$ $y(x, \lambda)$ has at least v null-points in (a, b).

Proof. It is sufficient to prove the theorem for $y_1(x, \lambda)$. Then there follows the assertion for each y with null-point at the point a from the properties of bands of solutions.

Divide the interval $\langle a, b \rangle$ into $v + 1$ equal subintervals with dividing points

$$x_0 = a < x_1 < \ldots < x_v < b = x_{v+1}$$

For $\lambda = \bar{\lambda} > 0$ let each solution $\bar{y}_i(x)$ of the differential equation (A) with the properties $\bar{y}_i(x_i) = \bar{y}_i'(x_i) = 0$, $\bar{y}_i''(x_i) \neq 0$, $i = 0, 1, 2, \ldots$, have another null-point in the interval (x_i, x_{i+1}). The existence of such $\bar{\lambda}$ follows from the lemma 5. From the lemma 4 and from the properties of bands of solutions there follows, that it is sufficient to take $\lambda_v = \bar{\lambda}$ and this proves the theorem.

Lemma 6. Let the assumptions of the lemma 5 be fulfilled. Let $y(x, \lambda)$ be a nontrivial solution of the differential equation (A) with the properties $y(x_0, \lambda) = 0$, $\alpha < x_0 < \infty$. Then the null-points of the solution $y(x, \lambda)$, to the right of x_0, are continuous functions of the parameter λ, $\lambda > 0$ (if they exist).

The proof of the lemma 6 is exactly similar to the proof of the lemma 7 in [1].

Theorem 2 (Three point boundary value problem). Let the assumptions of the lemma 5 hold. Let $\alpha < a < b < c < \infty$. Let $\alpha(\lambda)$, $\alpha_1(\lambda)$, $\beta(\lambda)$, $\beta_1(\lambda)$ be continuous functions of parameter and let be

$$|\alpha(\lambda)| + |\alpha_1(\lambda)| \neq 0, \quad |\beta(\lambda)| + |\beta_1(\lambda)| \neq 0$$

and besides let be either $\beta(\lambda) \equiv 0$ or $\beta(\lambda) \neq 0$ for all $\lambda > 0$. Then there exist a natural number v and a sequence of values of λ (eigenvalues)

$$\lambda_v, \lambda_{v+1}, \ldots, \lambda_{v+p}, \ldots, \quad p = 0, 1, 2, \ldots$$

and a sequence of functions

$$y_v, y_{v+1}, \ldots, y_{v+p}, \ldots \quad \text{(eigenfunctions)}$$

such that $y_{v+p} = y(x, \lambda_{v+p})$ is a solution of the differential equation (A), satisfying boundary conditions

$$y(a, \lambda_{v+p}) = 0$$
$$\alpha_1(\lambda_{v+p}) y(b, \lambda_{v+p}) - \alpha(\lambda_{v+p}) y'(b, \lambda_{v+p}) = 0$$
$$\beta_1(\lambda_{v+p}) y(c, \lambda_{v+p}) - \beta(\lambda_{v+p}) y'(c, \lambda_{v+p}) = 0$$

and $y(x, \lambda_{v+p})$ has exactly $v + p$ null-points in (a, c).

The proof is similar to the proof of the theorem 24 in the paper [1].

REFERENCES

[1] Greguš, M.: Über die lineare homogene Differentialgleichungen dritter Ordnung. Wiss. Zeitschr. Univ. Halle, Math.-Nat. XII/3, 1963, 265—286.

[2] Borůvka, O.: Lineare Differentialtransformationen 2. Ordnung. VEB Deutscher Verlag der Wissenschaften, Berlin 1967.

[3] Sansone, G.: Equazioni differenziali nel campo reale, parte prima. Secondo edizione. Bologna 1948.

[4] Neumark, M. A.: Lineare Differentialoperatoren, Akademie-Verlag, Berlin, 1960, preklad z ruštiny.

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SÚHRN

ZVÄZKY RIEŠENÍ DIFERENCIÁLNEJ ROVNICE TRETIEHO RÁDU SO SPOJITÝMI KOEFICIENTAMI A ICH APLIKÁCIA

M. Greguš, Bratislava

V práci sa študujú tzv. zväzky riešení diferenciálnej rovnice tretieho rádu

(a) $$y''' + p(x)y'' + q(x)y' + r(x)y = 0$$

kde $p(x)$, $q(x)$, $r(x)$ sú spojité funkcie $x \in (a, \infty)$, $-\infty \leqq a$, a ich aplikácie na riešenie určitých okrajových úloh s parametrom.

РЕЗЮМЕ

СВЯЗИ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ТРЕТЬЕГО ПОРЯДКА С НЕПРЕРЫВНЫМИ КОЭФФИЦИЕНТАМИ И ИХ ПРИМЕНЕНИЕ

М. Грегуш, Братислава

В работе изучены т.н. связи решений дифференциального уравнения третьего порядка

(a) $$y''' + p(x)y'' + q(x)y' + r(x)y = 0$$

где $p(x)$, $q(x)$, $r(x)$ непрерывные функции от $x \in (a, \infty)$, $-\infty \leqq a$, и их применения для решения некоторых краевых задач с параметром.