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Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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REGULARIZATION OF THE POWER OF A GRAPH

MILAN SEKANINA—VLADIMÍR VETCHÝ, Brno

Dedicated to Academician Prof. O. Borůvka on his 80th birthday

A simple graph G with the vertex-set V and with the edge-set E will be denoted as $G=(V, E)$ (so undirected graphs without loops and multiple edges are dealt with). If multiple edges are allowed (but not loops), one speaks about a multigraph and then the edge- set will be denoted by $\mathscr{E}$, so one gets a multigraph $G=(V, \mathscr{E})$ (the simple graph is a special case of the multigraph). If a, b are two vertices in a multigraph $G=(V, \mathscr{E})$, $\mathscr{E}_G(a)$ denotes the set of all edges incident to a, $\mathscr{E}_G(a, b) = \mathscr{E}_G(a) \cap \mathscr{E}_G(b)$, $V_G(a)$ is the set of all verices connected with a. In our paper, we deal with finite multigraphs, i.e. the sets V and $\mathscr{E}$ are finite.

Definition 1 (Berge [1]). Let $G=(V, E)$ be a simple graph. A multigraph $G_0=(V, \mathscr{E})$ is a regularization of G of degree k, if for every a, b in V there holds

1. $\langle a, b \rangle \in E \equiv \mathscr{E}_{G_0}(a, b) \neq \emptyset$.
2. $\operatorname{card} \mathscr{E}_{G_0}(a) = k$ for each $a \in V$.

If there exists a regularization of G, G is called regularizable.

Proposition 1 (Berge [1]). Let $G=(V, E)$ be simple, no bipartite graph. G is regularizable, iff $\operatorname{card} X < \operatorname{card} \cup \{V_G(x)\colon x \in X\}$ for every independent subset $X \subset V$, $X \neq \emptyset$.

In the proof of 3.1. in [1] the following assertion was used:

Proposition 2. Let $G=(V, E)$ be a simple graph and let for every adge $\langle a, b \rangle \in E$ there exist a regular factor in G containing $\langle a, b \rangle$. Then G is regularizable.

Proof. We assign a regular factor $F_{\langle a, b \rangle}$ containing $\langle a, b \rangle$ to each $\langle a, b \rangle \in E$. Take such copies $(V, E_{\langle a, b \rangle})$ of all $F_{\langle a, b \rangle}$ for which $\langle a, b \rangle \neq \langle c, d \rangle \Rightarrow$ $\Rightarrow E_{\langle a, b \rangle} \cap E_{\langle c, d \rangle} = \emptyset$. Then $\left(V, \bigcup\limits_{\langle a, b \rangle \in E} E_{\langle a, b \rangle}\right)$ is a regularization of the graph G.

Definition 2. A graph (V, E) is strongly hamiltonian, iff for each $e \in E$ there exists a hamiltonian circuit containing e.

Proposition 2 implies

Proposition 3. Every strongly hamiltonian graph is regularizable.

Definition 3. Let $n \geqq 2$ be an integer and $G=(V, E)$ be a connected simple graph. Let $G^n=(V, E^n)$ where $\langle a, b\rangle \in E^n$ iff $a, b \in V, a \neq b$ and the distance of a from b in G is at most n.

The graph G^n is called the n-th power of G.

Proposition 4 ([2]). If $n \geqq 3$, G^n is strongly hamiltonian.

As corollary of Proposition 3 and Proposition 4 we get

Proposition 5. If $n \geqq 3$ and G is a connected simple graph, G^n is regularizable.

Now, we shall deal with these simple connected graphs G for which G^2 is regularizable. The Main Theorem states that this is not the case exactly for the i-graphs, which are defined in the following lines.

Definition 4. A simple graph $G=(V, E)$ is called an i-graph, if

1. card V is even.
2. There is $\frac{\text{card } V}{2}$ end-vertices in G.
3. Every not-end-vertex has exactly one end-vertex as a neighbor.

Proposition 6. For a connected i-graph $G=(V, E) G^2$ is not regularizable.

Proof. Let V_1 be the set of all end-vertices in G, $V_2=V-V_1$. We get $V_2=\cup\{V_{G^2}(x): x \in V_1\}$. It is card $V_1=$ card $V_2 \cdot G^2$ is not bipartite as it contains triangles. By Proposition 1 G^2 is not regularizable.

Proposition 7. In a connected simple graph $G=(V, E)$ there exists a spanning tree which is not an i-graph iff G is neither an i-graph nor a circuit of length 4.

Proof. If G is an i-graph or a circuit of length 4, then clearly all spanning trees of G are i-graphs.

Now, suppose G is neither an i-graph nor a circuit of length 4. Let $G^*= =(V, E^*)$ be a spanning tree of G. Suppose, G^* is an i-graph. It is $E^* \subset E$, $E^* \neq E$. There exists $\langle a, b\rangle \in E-E^*$ such that at least one of the vertices a, b is an end-vertex in (V, E^*) (otherwise G would be an i-graph).

a) Let a, b be end-vertices in (V, E^*) and $a=a_1, a_2, \ldots, a_m=b$ be a circuit in $(V, E^* \cup\{\langle a, b\rangle\})$. It is $m \geqq 3$. If a_2 is of degree 2 in (V, E^*), then $G'= =(V, E^* \cup\{\langle a, b\rangle\}-\{\langle b, a_{m-1}\rangle\})$ is a spanning tree in G, which is no i-graph. Namely, $a_3 \neq b$ as $G^*$is an i-graph and a_3 is no end-vertex in G', as G is not a circuit of length 4.

Let $\langle a_2, c\rangle \in E^*$, $a \neq c \neq a_3$. Then a_2 is not an end-vertex in $(V, E^*- -\{\langle a_{m-2}, a_{m-1}\rangle\})$. The graph $(V, E^* \cup\{\langle a, b\rangle\}-\{\langle a_{m-2}, a_{m-1}\rangle\})$ is a spanning tree of G and no i-graph, as a has no end-vertex as a neighbor in it.

b) Let a be no end-vertex in G, b an end-vertex in G. Let $a_1, \ldots, a_m$ have the upper meaning. As a has at least two end-vertices in $G'=(V, E^* \cup\{\langle a, b\rangle\}- -\{\langle a_{m-1}, a_m\rangle\})$ as its neighbors, G' is no i-graph and it is a spanning tree of G.

Definition 5. Let $G=(V, E)$ be a simple graph, $a, b \in V a \neq b$. The vertices a and b form the conjugate pair (a, b) of type 1, if they are end-vertices and have the distance 2 in G. The ordered pair (a, b) is a conjugate pair of G of type 2, if a is an end-vertex, b is a neighbor of a in G and is of degree 2 in G.

Generally we shall speak about a conjugate pair, if further specification is not needed.

Lemma 8. Let $G=(V, E)$ be a tree, $\operatorname{card} V \geqq 3$. Then

1. In G there exists a conjugate pair.

2. If G is an i-graph, then there exist two conjugate pairs of type 2 (a, b), (c, d) such that $\{a, b\} \cap \{c, d\} = \emptyset$.

Proof. Let $a_1, a_2, \ldots, a_k$ be one of the longest paths in G. If the degree of a_2 is 2, then (a_1, a_2) is a conjugate pair of type 2, if c is a neighbor of a_2, $a_1 \neq c \neq a_3$, (a_1, c) is a conjugate pair of type 1.

If G is an i-graph, then $k \geqq 4$ and (a_1, a_2) and (a_k, a_{k-1}) are conjugate pairs of type 2 with $\{a_1, a_2\} \cap \{a_k, a_{k-1}\} = \emptyset$.

Definition 6. Let $G=(V, E)$ be a simple graph, (a, b) its conjugate pair, $G_1=(V_1, E_1)$ a subgraph of G induced on $V_1 = V - \{a, b\}$. We write $G = G_1 \vee (a, b)$ and we say that G is constructed by a regular construction from G_1.

Proposition 9. Let $G_1 = (V_1, E_1)$ be a connected simple graph, $\operatorname{card} V_1 \geqq 3$. Let G_1^2 be regularizable and $G=(V, E) = G_1 \vee (a, b)$. Then G^2 is regularizable.

Proof. As G^2 is not bipartite, we can use Proposition 1. Let X be an non-empty indepent set in G^2. If $\{a, b\} \cap X = \emptyset$, X is independent in G_1^2 and $X^* = \cup\{V_{G_1^2}(x) \colon x \in X\} \subset \cup\{V_{G^2}(x) \colon x \in X\} = X^{**}$ and so $\operatorname{card} X < \operatorname{card} X^* \leqq \operatorname{card} X^{**}$.

$\{a, b\} \cap X$ can be at most one-element set as $\langle a, b\rangle \in E^2$. E.g. let $a \in X$, $X_1 = X - \{a\}$. X_1 is independent in G_1^2 (may be empty). If $X=\{a\}$, then $\operatorname{card} V_{G^2}(a) \geqq 2$. If $X_1 \neq \emptyset$ then (with the upper notation using X_1 instead of X) $\operatorname{card} X_1 < \operatorname{card} X_1^*$ and b non $\in X_1^*$ (b is not a vertex of G_1^2). Therefore card $\cup\{V_{G^2}(x) \colon x \in X\} > \operatorname{card} X$.

Proposition 10. Let $G=(V, E)$ be a tree, no i-graph and $\operatorname{card} V \geqq 7$. There exists $G_1 = (V_1, E_1)$ such that G_1 is no i-graph and one gets G from G_1 by a regular construction.

Proof. By Lemma 8 there exists a conjugate pair (a, b) in G such that $G = G_1 \vee (a, b)$, where $G_1 = (V_1, E_1)$, $V_1 = V - \{a, b\}$. Suppose G_1 is an i-graph.

a) Let (a, b) be of type 2 and c be the neighbor of b in G different from a. Then c is an end-vertex in G_1 (otherwise G would be an i-graph). Let (a_1, b_1) be a conjugate pair of type 2 in G_1 where $a_1 \neq c$. The induced subgraph of G $G_2 = (V_2, E_2)$, $V_2 = V - a_1$, b_1 is no i-graph, as c does not neighbor an end-vertex in G_2. It is $G = G_2 \vee (a_1, b_1)$.

b) Let (a, b) be of type 1 and c be a neighbor of a (i.e. of b, too). Let (a_1, b_1) be a conjugate pair in G_1, for which $a_1 \neq c \neq b_1$. The induced subgraph of G $G_2 = (V_2, E_2)$ $V_2 = V - \{a_1, b_1\}$ is no i-graph as c has at least two end-verteces as its neighbor. Again $G = G_2 \vee (a_1, b_1)$.

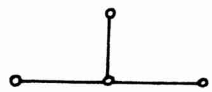
Definition 7. If $G_1, \ldots, G_k$ is a sequence of graphs such that G_i can be constructed by a regular construction from G_{i-1}, this sequence is called regular.

Proposition 11. If a tree $G=(V, E)$ is no i-graph, card $V \geqq 7$, then there exists a regular sequence $G_1, \ldots, G_k$ where G_i are no i-graphs, $G_k = G$ and G_1 has at most 6 vertices.

Proof follows immediately from Proposition 10.

Proposition 12. Let a tree $G=(V, E)$ be no i-graph, card $V \leqq 6$. Then G^2 is regularizable.

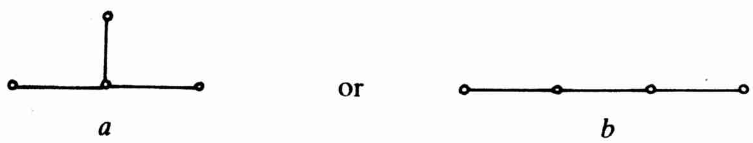
Proof. The assertion is clear for card $V \leqq 3$. If card $V=4$ then G is isomorphic to



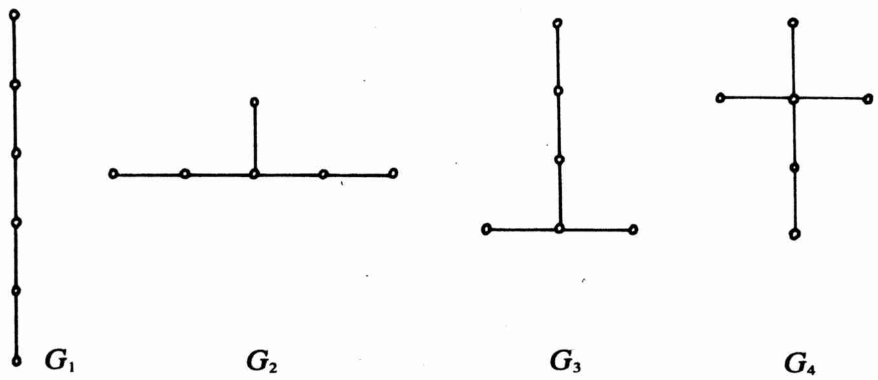
and G^2 is regular.

Let now card $V=5$. Let $G=G_1 \vee (a, b)$. Then G_1 is isomorphic to

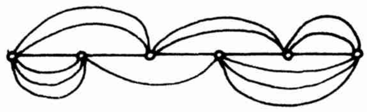
and G_1^2 is regularizable. By Proposition 9, G^2 is regularizable. Let card $V=6$ and $G=G_1 \vee (a, b)$. Then G_1 is isomorphic to



For a we conclude again by Proposition 9. For b G can be isomorphic to one of the following graphs G_1, G_2, G_3, G_4.



G_1^2 is regularizable as the multigraph,



is regular.

G_2 is an i-graph. This case does not occur.

G_3, G_4 can be constructed by a regular construction from graphs which have the regularizable second power.

Main Theorem. Let G be a connected simple graph. Then G^2 is regularizable iff G is no i-graph.

Proof. Necessity was proved in Proposition 6.

Let $G=(V,E)$ be a tree and no i-graph. Let card $V \leqq 6$. Then G^2 is regularizable by Proposition 12. Let now card $V \geqq 7$. Suppose our assertion to be valid for all G with at most card $V-1$ vertices. By Proposition 11, there exists a regular sequence $G_1, \ldots, G_k$ such that all G_i' s are no i-graphs, $G_k = G$ and G_1 has at most 6 vertices. Therefore $k \geqq 2$. By Definition 6 of a regular construction and Definition 7 of a regular sequence the graphs $G_1, \ldots, G_k$ are trees. By the induction supposition $(G_{k-1})^2$ is regularizable. By Proposition 9 $G_k^2 (= G^2)$ is regularizable.

Let $G=(V,E)$ be no tree. The assertion is clear, if G is a circuit. Suppose G is no circuit. By Proposition 7 there is a spanning tree $G^* = (V, E^*)$ of G, which is no i-graph. As $(G^*)^2$ contains triangles, no subgraph of G^2 containing $(G^*)^2$ is bipartite. As G^* is a tree, $(G^*)^2$ is regularizable. G^2 can be obtained from $(G^*)^2$ by succesive adding of edges. By 3.2 [1] an adding of an edge to a regularizable graph, which is not bipartite, gives a regularizable graph. Therefore G^2 is regularizable.

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Author's address:
Katedra algebry a geometrie přírodovědecké fakulty university J. E. Purkyně
Janáčkovo nám. 2a, Brno, ČSSR

SHRNUTÍ

REGULARIZOVATELNOST MOCNINY GRAFU

M. Sekanina—V. Vetchý, Brno

Obyčejný graf G, tj. graf bez smyček a paralelních hran, se nazývá regularizovatelným, když přidáním vhodných hran, paralelních k hranám v G, lze zkonstruovat konečný pravidelný multigraf. Graf nazýváme i-grafem, když právě polovina jeho uzlů je koncových a každý nekoncový uzel je spojen hranou právě s jedním koncovým. Je-li G konečný souvislý obyčejný graf, potom n-tá mocnina G^n grafu G je regularizovatelná pro $n \geqq 3$. Druhá mocnina G^2 je regularizovatelná, právě když G není i-graf.

РЕЗЮМЕ

РЕГУЛЯРИЗУЕМОСТЬ СТЕПЕНИ ГРАФА

М. Секанина–В. Ветхы, Брно

Изучаются обыкновенные графы, т. е. неориентируемые графы без петель и параллельных ребер. Обыкновенный граф G по дефиниции регуляризуемый тогда когда добавлением подходящих ребер параллельных к ребрам в G можно получить конечный правильный мультиграф. Граф G называется i-графом если точно половина узлов является концовыми и если каждый неконцовый узел соединен ребрем точно с одним концовым узлом. Пусть G конечный, связный, обыкновенный граф. Потом n-тая степень G^n графа G для $n \geqq 3$ является регуляризуемой. Вторая степень регуляризуемая тогда когда G не является i-графом.