UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XXXIX — 1980

A GENERALIZATION OF FLOQUET THEORY

FRANTIŠEK NEUMAN, Brno

Dedicated to Academician O. Borůvka on his 80th birthday anniversary

I. Introduction

The classical Floquet Theory is applicable on linear differential systems or equations of the n-th order, both with periodic coefficients. As a background for a generalization we take the fact that the change $x \mapsto x + d$ of the independent variable (d being a period) either does not change at all such a system or equation (supposed to be defined on the whole real line $\mathbf{R}$), or we obtain a continuation of the same system or equation.

Due to P. Stäckel [12] and E. J. Wilczynski [15] we know the most general transformations of linear differential equations and systems. We may study the systems and equations that can be transformed into themselves (or can be transformed into their continuation) not only by a change of the independent variable (and a very special one as in Floquet Theory) but by a transformation in its most general form.

A generalization of Floquet Theory for the 2nd order differential equations of the form $y'' = q(x)y$ given in [6] by M. Laitoch and studied also by S. Staněk [13, 14] was based on the theory of global transformations of the second order differential equations (especially on the theory of dispersions) developed by O. Borůvka [1], see also [2]. Invariant differential equations on homogeneous manifolds were studied in [3], changes of the independent variable that leave linear differential equations of the n-th order invariant were investigated in [10]. Problems concerning of possible generalizations of Floquet Theory were considered by D. R. Snow [11]. Transformations of linear differential systems were deeply studied by N. P. Jerugin, e.g. in [4, 5].

Our considerations take their origin from Borůvka's theory of transformations of the second order linear differential equations extended in [7] and [8] for equations of arbitrary orders. We also substantially use the form of solutions of vector functional equations described in [9].

II. Transformations of a linear differential system into itself

Let $C^n(I)$ denote the set of all real (if not explicitly stated else) scalar, or vector, or matrix functions defined on an open interval $I \subset \boldsymbol{R}$ and having here continuous derivatives of the n-th order; C^0 means continuity.

Consider a system of n linear homogeneous differential equations of the 1st order, $n \geqq 2$:

$$(1) \qquad y' = A(x)y, \quad A \in C^0(I), \quad ' = \mathrm{d}/\mathrm{d}x$$

with a fundamental n by n matrix solution Y, $Y \in C^1(I)$.

The most general pointwise transformation of any system of the form (1) into a system of the same form (see [15]) is

$$(2) \qquad \tau Y(x) = Z(t) := F(t) \cdot Y(\varphi(t))$$

with $\varphi(J) \subset I$, $\varphi \in C^1(J)$, $\mathrm{d}\varphi(t)/\mathrm{d}t \neq 0$ on J,

$$F \in C^1(J), \quad \det F(t) \neq 0 \quad \text{on } J$$

F is called the multiplier and φ the parametrization of the transformation $\tau = \langle F, \varphi \rangle$. If, moreover, $\varphi(J) = I$ we speak about the global transformation.

Consider another system of n linear homogeneous equations of the 1st order

$$(3) \qquad \dot{z} = B(t)z, \quad B \in C^0(J), \quad \dot{} = \mathrm{d}/\mathrm{d}t$$

with its fundamental n by n matrix solution Z.

Theorem 1. Let two differential systems (1) and (3) be given. There always exists a transformation (2) that transforms (1) into (3).

We may even prescribe a parametrization φ of the transformation, if only $\varphi \in C^1(J)$ and $\mathrm{d}\varphi(t)/\mathrm{d}t \neq 0$ on J. If φ is a bijection of J onto I, then the transformation is global.

Proof. For a given $\varphi \in C^1(J)$, $\mathrm{d}\varphi(t)/\mathrm{d}t \neq 0$ on J, define

$$F(t) := Z(t) \cdot Y^{-1}(\varphi(t)), \quad t \in J$$

We have $F \in C^1(J)$ and $Z(t) = F(t) \cdot Y(\varphi(t))$ on J. Hence $\tau = \langle F, \varphi \rangle$ transforms (1) into (3), and if φ is a bijection of J onto I, then the τ is also global. QED.

We get immediately

Corollary. Each system (1) can be globally transformed into itself. Moreover, we may prescribe arbitrarily the parametrization φ of the transformation if only φ is a bijection of I onto I, $\varphi \in C^1(I)$ and $\mathrm{d}\varphi(t)/\mathrm{d}t \neq 0$ on I.

III. Transformations of the n-th order linear differential equation into itself

Consider a linear homogeneous differential equation of the n-th order, $n \geqq 2$:

$$(P) \qquad y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_0(x)y = 0, \quad p_i \in C^0(I)$$

for $i = 1, \ldots, n-1$. Denote by the column vector $\boldsymbol{y}$ its n-tuple of linear independent solutions $(y_1, \ldots, y_n)^T$. Let (Q) be a linear differential equation of the same order defined on J, and $\boldsymbol{z}$ its n-tuple of independent solutions.

The most general pointwise transformation of (P) into (Q) is of the form

$$(4) \qquad T\boldsymbol{y}(x) = \boldsymbol{z}(t) := C \cdot f(t) \cdot \boldsymbol{y}(\varphi(t))$$

where C is a regular constant n by n matrix

$$\varphi(J) \subset I, \quad \varphi \in C^n(J), \quad \mathrm{d}\varphi(t)/\mathrm{d}t \neq 0 \text{ on } J$$
$$f \in C^n(J), \quad f: J \to \boldsymbol{R}, \quad f(t) \neq 0 \text{ on } J, \quad \text{see also [7]}$$

We shall write also $T = \langle Cf, \varphi \rangle$. If $\varphi(J) = I$ we speak about a global transformation.

All global transformations of all linear homogeneous differential equations of arbitrary orders n $(n \geqq 2)$ with respect to the composition form Ehresmann groupoid. Each component of the groupoid is Brandt groupoid. Objects of each of the Brandt groupoid (considered as a category) are globally equivalent differential equations, see [8].

For (P) let $B(P)$ denote all equations globally equivalent with (P). Let $G(P)$ denote the set of all global transformations of (P) into itself, it is a group. If $(Q) \in B(P)$ and T globally transforms (P) into (Q), then $T^{-1}G(P)T$ is a group of all global transformations of (Q) into itself, i.e.

$$G(Q) = T^{-1}G(P)T$$

see [8]. Hence the groups of global transformations of two globally equivalent linear differential equations into themselves are conjugate.

First we shall investigate when (P) restricted on an interval $j \subset I$ can be transformed by means of $T = \langle Cf, \varphi \rangle$ into its continuation on $\varphi(j) \subset I$, $j \cap \varphi(j) \neq \emptyset$, $\mathrm{d}\varphi(t)/\mathrm{d}t > 0$ on j. With respect to (4), we have the following vector functional equation

$$(5) \qquad \boldsymbol{y}(x) = C \cdot f(x) \cdot \boldsymbol{y}(\varphi(x)), \quad x \in j$$

If $W\boldsymbol{y}(x) = (\boldsymbol{y}(x), \boldsymbol{y}'(x), \ldots, \boldsymbol{y}^{(n-1)}(x))$ denotes Wronski matrix of $\boldsymbol{y}(x)$, then $\det W\boldsymbol{y}(x)$ is the Wronski determinant of $\boldsymbol{y}(x)$. For $\varphi(x) = x$ on j, the Wronski determinant of $\boldsymbol{y}(x)$ should be the same as that of $C \cdot f(x)\boldsymbol{y}(x)$ that is equal to $\det C \cdot (f(x))^n \cdot W\boldsymbol{y}(x)$. Hence $f(x)$ is a nonzero constant and $C \cdot f(x)$ is a regular constant matrix. Forming Wronski matrices of both sides of (5) we get that $C\cdot f(x)$ is the unit matrix E, and the transformation T in this case is trivial, i.e. $T=\langle E, id_i\rangle$. Hence, if T is not trivial, $\varphi(x)\neq x$ for some $x\in j$.

For an interval $i\subset \boldsymbol{R}$ and a constant $d\in \boldsymbol{R}$, define $i+d=\{x+d\,;\,x\in i\}$.

Hence our problem consists in solving the functional equation (5). We can apply results of [9], to get

Theorem 2. Let (P) restricted on j can be globally transformed by means of a nontrivial transformation $\langle Cf, \varphi\rangle$, $\varphi'>0$, onto (P) restricted on $\varphi(j)$, $j\cup\varphi(j)$ being an interval.

Then each $x\in j$, such that $\varphi(x)=x$, is an isolated point. On each interval $(a, b)\subset j$ where $\varphi(x)\neq x$ the equation (P) is globally equivalent to a linear homogeneous differential equation with real π-periodic coefficients on an interval (c, d). The equation (P) has an n-tuple $\boldsymbol{y}$ of independent solutions in the form

$$\boldsymbol{y}(x)=r(x)\cdot e^{D\cdot\varrho(x)}\cdot\boldsymbol{k}(\varrho(x)) \tag{6}$$

where $r\in C^n(j\cup\varphi(j))$, $r(x)>0$ on $j\cup\varphi(j)$,

D is a complex constant n by n matrix such that $e^D=C$,

$\varrho\in C^n(j\cup\varphi(j))$, $d\varrho/dx>0$ on $j\cup\varphi(j)$,

$\varrho(j)=i$, $\varrho(\varphi(j))=i+\varepsilon\pi$, $\varepsilon=\operatorname{sign}(\varphi(x)-x)$,

$\boldsymbol{k}\in C^n(i\cup(i+\varepsilon\pi)$, $\boldsymbol{k}$ is a generally complex periodic $(\boldsymbol{k}(t+\varepsilon\pi)=$ $=\boldsymbol{k}(t))$ for $f>0$,

or half periodic $(\boldsymbol{k}(t+\varepsilon\pi)=-\boldsymbol{k}(t))$ for $f<0$.

Moreover $c=-\infty$ if and only if $\varphi(a+)=a$,

$d=\infty$ if and only if $\varphi(b-)=b$.

Proof. The assertion about isolated points follows from Lemma 1 of [9]. From Theorem 2 of [9] we have the global equivalency between (P) on (a, b) where $\varphi(x)-x\neq 0$ and a differential equation with periodic coefficients. Theorem 1 of the same paper gives the form (6) of $\boldsymbol{y}$ with $d\varrho/dx\neq 0$. If we take $-\varrho$ instead of ϱ in the case of $d\varrho/dx<0$ (that is just when $\varphi(x)-x<0$), we get the above formulations. QED.

Now we investigate the case when (P) can be globally (on its whole interval of definition I) transformed into itself.

Theorem 3. Let a linear differential equation (P) of the n-th order, $n\geqq 2$, defined on I, can be globally transformed by a nontrivial transformation $\langle Cf, \varphi\rangle$, $\varphi'>0$, into itself. Each point x, such that $\varphi(x)=x$, is an isolated point. The set $I-\{x\in I\,;\,\varphi(x)=x\}$ is a union of finite number of countable many disjoint open intervals. On each of the intervals the equation (P) is globally equivalent to a linear differential equation with π-periodic coefficients defined on $(-\infty, \infty)$.

Proof. In our case, $\varphi(I)=I$. If (a, b) is one of the open intervals forming the decomposition of $I-\{x\in I;\ \varphi(x)=x\}$, it is always $\varphi(a)=a$ and $\varphi(b)=b$ ($\varphi(-\infty)=-\infty$ and $\varphi(\infty)=\infty$ are not excluded). According to Theorem 2, $c=-\infty$ and $d=\infty$. QED.

IV. A generalized Floquet Theory

Guided by the fact that in Floquet Theory a linear differential system or equation of the n-th order with π-periodic coefficients is not changed by the change $x\mapsto x+\pi$ of the independent variable, we have considered linear differential systems and equations that are transformed into themselves by transformations of the most possible general form.

Accotding to Corollary each system of n linear differential equations of the 1st order can be transformed into itself even by any prescribed parametrization φ ($\varphi\in C^1$, $\varphi'\neq 0$). Hence we do not obtain any restriction on the form of its solutions.

Of course, another situation occurs, when some conditions on the transformation are required, see e.g. N. P. Jerugin [4] and [5].

A different situation is also for linear differential equations of the n-th order, $n\geqq 2$. If such an equation can be globally transformed by a nontrivial transformation $\langle Cf, \varphi\rangle$, $\varphi'>0$, into itself, then on each maximal interval where $\varphi(x)\neq x$, the equation can be globally transformed into π-periodic linear differential equation on $(-\infty, \infty)$ and the form of its solutions is given by (6).

For the second order linear differential equations of the form $y''+q(x)y=0$ our approach gives exactly the generalization introduced by M. Laitoch in [6].

It is also possible to restrict our generalization of Floquet Theory only on the case of those linear differential equations of the n-th order that are on their whole interval of definition I globally equivalent to a linear differential equation with periodic coefficients. In the case the form (6) of solutions of the equations would be valid on the whole I.

REFERENCES

[1] Borůvka, O.: Lineare Differentialtransformationen 2. Ordnung. VEB, Berlin, 1967. In English: Linear Differential Transformations of the Second Order. The English Univ. Press, London 1971.

[2] Borůvka, O.: Теория глобальных свойств обыкновенных линейных дифференциальных уравнений второго порядка. Дифференциальные Уравнения, Минск **12** (1976), 1347—1383; translated in English: Theory of global properties of second order linear ordinary equations, Differential equations **12** (1976), 949—975 (1977).

[3] Helgason, S.: Invariant differential equations on homogeneous manifolds. Bull. AMS **83** (1977), 751—774.

[4] Jerugin, N. P.: Приводимые системы. Труды матем. института им. В. А. Стеклова **13** (1946).

[5] Jerugin, N. P.: Линейные системы обыкновенных дифференциальных уравнений. ИАН БССР Минск 1963.

[6] Laitoch, M.: Расширение метода Флоке для определения вида функциональной системы решений дифференциального уравнения второго порядка $y'' = Q(x)y$. Czech. Math. J. **5, (80)**, (1955), 164—174.

[7] Neuman, F.: Geometrical approach to linear differential equations of the n-th order. Rend. Mat. **5** (1972), 579—602.

[8] Neuman, F.: Categorial approach to global transformations of the n-th order linear differential equations. Časopis Pěst. Mat. **102** (1977), 350—355.

[9] Neuman, F.: On solutions of the vector functional equation $\mathbf{y}(\xi(x)) = f(x) \cdot A \cdot \mathbf{y}(x)$. Aequationes Math. **16** (1977), 245—257.

[10] Posluczny, J.—Rubel, L. A.: The motion of an ordinary differential equation. To appear.

[11] Snow, D. R.: A generalization of Floquet's Theorem by using a functional equation. Abstracts of short communications of International Congress of Mathematicians, Helsinki 1978, 173—174.

[12] Stäckel, P.: Über Transformationen von Differentialgleichungen J. Reine Angew. Math. **111** (1893), 290—302.

[13] Staněk, S.: Phase and dispersion theory of the differential equation $y'' = q(t)y$ in connection with the generalized Floquet Theory. Archivum Math. (Brno) **14** (1978), 109—122.

[14] Staněk, S.: On the structure of second-order linear differential equations with given characteristic multipliers in the generalized Floquet Theory. Archivum Math. (Brno) **14** (1978), 235—242.

[15] Wilczynski, E. J.: Projective differential geometry of curves and ruled surfaces. Teubner, Leipzig, 1906.

Received December 12, 1978

Author's address:
Mathematical Institute of the Czechoslovak Academy of Sciences branch in Brno
Janáčkovo nám. 2a
662 95 Brno, Czechoslovakia

SÚHRN

ZOBECNĚNÍ FLOQUETOVY TEORIE

F. Neuman, Brno

Za základ pro klasickou Floquetovu teorii pokládáme to, že lineární periodické diferenciální systémy či rovnice se nemění při transformaci nezávislé proměnné $x \mapsto x +$ perioda. V práci je podáno zobecnění tohoto principu v nejširším smyslu, neboť jsou popsány lineární diferenciální systémy 1. řádu a rovnice n-tého řádu připouštějící nejobecnější transformace na sebe, a je odvozen tvar jejich řešení.

РЕЗЮМЕ

ОБОБЩЕНИЕ ТЕОРИИ ФЛОКЕ

Ф. Нейман, Брно

Основным для классической теории Флоке мы считаем то, что линейные периодические дифференциальные системы или уравнения не изменятся посредством преобразования независимой переменной $x \mapsto x +$ период. В работе изложено обобщение этого принципа в самом широком смысле, так-как описаны линейные дифференциальные системы 1-го порядка и уравнения n-го порядка позволяющие самые общие преобразования в себе, и дана форма их решений.