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PRODUCTS OF RADICAL CLASSES OF LATTICE ORDERED GROUPS

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Dedicated to Professor O. Borůvka on the occassion of his 80th birthday

1. Introduction. The notion of radical class of lattice ordered groups has been introduced in [3]. Martinez [4] investigated torsion classes of lattice ordered groups. Each torsion class of lattice ordered groups is a radical class.

For any lattice ordered group G and any radical class A we denote by $\varrho(A)(G)$ the join of all convex l-subgroups of G belonging to the class A. Then $\varrho(A)(G)$ is an l-ideal of G; it is said to be the radical of G corresponding to the radical class A.

Let A, B be radical classes of lattice ordered groups. The class of all lattice ordered groups G having the property that $G/\varrho(A)(G)$ belongs to B will be denoted by AB.

For torsion classes the following assertions are known to be valid [4]:

(α_1) If A, B are torsion classes of lattice ordered groups, then AB is also a torsion class.

(α_2) If A_i, B_i $(i=1, 2)$ are torsion classes of lattice ordered groups such that $A_i \subseteq B_i$ $(i=1, 2)$, then $A_1A_2 \subseteq B_1B_2$.

(For further results on products of torsion classes cf. [4] and [2].)

In this paper it will be shown that if A, B are radical classes of lattice ordered groups, then AB is also a radical class. There will be described radical classes generated by linearly ordered groups and by means of this description the validity of the assertion (α_2) for radical classes will be investigated. It will be shown that if A is a radical class generated by linearly ordered groups, then A cannot be represented as a product BC of radical classes B, C with $B \neq \{\{0\}\} \neq C$. For any radical classes A, B, the lattice of radical classes generated by the set $\{A, B, AB\}$ will be characterized. It will be proved that if A, B are radical classes generated by linearly ordered groups and if $G \in AB$, then no convex l-subgroup of G can be expressed as a direct product of an infinite number of nonzero lattice ordered groups.

2. Preliminaries. Let us recall some basis definitions.

A nonempty class A of lattice ordered groups is said to be a radical class if it has the following properties:

(a) A is closed with respect to isomorphisms.

(b) If $H \in A$ and H_1 is a convex l-subgroup of H, then $H_1 \in A$.

(c) If G is a lattice ordered group and $\{H_i\}_{i \in I}$ is a family of convex l-subgroups of G such that each H_i belongs to A, then $\bigvee_{i \in I} H_i$ also belongs to A.

A radical class that is closed with respect to homomorphisms is called a torsion class. Each variety of lattice ordered groups is a torsion class (Holland [1]).

For each nonempty class L of lattice ordered groups we denote by $R(L)$ the intersection of all radical classes A having the property that $L \subseteq A$. Then $R(L)$ is obviously a radical class; $R(L)$ will be called the radical class generated by L.

If H_i $(i \in I)$ are lattice ordered groups, then we denote by $\Pi^r_{i \in I} H_i$ and $\Pi^c_{i \in I} H_i$ their restricted direct product and (complete) direct product, respectively. Let K_j $(j \in J)$ be convex l-subgroups of a lattice ordered group G. Consider the following conditions:

(i) For each $j \in J$ and each element $0 \leqq g \in G$ the set $[0, g] \cap K_j$ possesses the greatest element (this element will be denoted by g_j or $g(K_j)$).

(ii) $g = \bigvee_{j \in J} g_j$ holds for each $0 \leqq g \in G$.

(iii) $K_{j_1} \cap K_{j_2} = \{0\}$ whenever j_1 and j_2 are distinct elements of J.

(iv_1) If $0 \leqq h^j \in K_j$ for each $j \in J$, then there exists $g \in G$ such that $g_j = h^j$ for each $j \in J$.

(iv_2) For each $0 < g \in G$, the set of those g_j $(j \in J)$ that are distinct from 0 is finite.

If (i), (ii), (iii) and (iv_1) are valid, then G is said to be the (internal) direct product of its l-subgroups K_j $(j \in J)$ and we denote this fact by writing $G = \Pi_{j \in J} K_j$. We write $G = \Pi'_{j \in J} K_j$ if the conditions (i), (ii), (iii) and (iv_2) hold; in this case G is said to be a restricted (internal) direct product of its l-subgroups K_j $(j \in J)$. It is easy to verify that there exists an isomorphism of $\Pi_{j \in J} K_j$ onto $\Pi^c_{j \in J} K_j$ and an isomorphism of $\Pi'_{j \in J} K_j$ onto $\Pi^r_{j \in J} K_j$.

Let K_1 be a lattice ordered group and let $K_2, K_3, \ldots, K_n$ be linearly ordered groups. We denote by $K_1 \circ K_2 \circ \ldots \circ K_n$ (the lexicographic product of $K_1, \ldots, K_n$) the set of all n-tuples $(a_1, \ldots, a_n)$ such that $a_i \in K_i$ $(i = 1, \ldots, n)$ with a binary operation $+$ defined coordinatenwise and by a partial order $\leqq$ defined as follows: for a, $b \in K_1 \circ \ldots \circ K_n$, $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ with $a \neq b$ let $i_0 \in \{1, \ldots, n\}$ be such that $a_i = b_i$ for each $i \in \{1, \ldots, n\}$, $i > i_0$, and $a_{i_0} \neq b_{i_0}$; we put $a < b$ iff $a_{i_0} < b_{i_0}$. Then $K = K_1 \circ \ldots \circ K_n$ is a lattice ordered group; K is linearly ordered if and only if K_1 is linearly ordered.

Let G be a lattice ordered group and let $X \subseteq G$. We put

$$X^\delta = \{y \in G : |y| \wedge |x| = 0 \text{ for each } x \in X\}.$$

X^{δ} is said to be a polar of G; each polar of G is a closed convex l-subgroup of G (cf. Šik [5]).

The symbols N_0, R_0 and R_1 will denote the additive group of all integers, all reals or all rational numbers, respectively, with the natural linear order.

3. Radical classes generated by linearly ordered groups. The following assertion is contained in [3]:

3.1. Lemma. Let A be a radical class of lattice ordered groups. Let H_1 be a convex l-subgroup of a lattice ordered group H. Then $\varrho(A)(H_1) = H_1 \cap \varrho(A)(H)$.

3.2. Proposition. Let A, B be radical classes of lattice ordered groups. Then $C = AB$ is a radical class as well.

Proof. The validity of the condition (a) for C is obvious. Let $H \in C$ and let H_1 be a convex l-subgroup of H. For proving (b) we have to verify that $H_1/\varrho(A)(H_1)$ belongs to B. From 3.1 it follows that $H_1/\varrho(A)(H_1)$ is isomorphic with $(H_1 \vee \varrho(A)(H))/\varrho(A)(H)$. The lattice ordered group $(H_1 \vee \varrho(A)(H))/\varrho(A)(H)$ is a convex l-subgroup of $H/\varrho(A)(H)$. Since $H \in C$, we have $H/\varrho(A)(H) \in B$ and hence (because B fulfils (b)) we infer that $(H_1 \vee \varrho(A)(H))/\varrho(A)(H)$ belongs to B. Thus $H_1/\varrho(A)(H_1) \in B$ and therefore $H_1 \in C$. Hence C fulfils the condition (b).

Let H_i $(i \in I)$ be convex l-subgroups of a lattice ordered group G, $\bigvee_{i \in I} H_i = H$. Suppose that each H_i belongs to C. We have to verify that $H/\varrho(A)(H) \in B$. Let $i \in I$. Then $H_i/\varrho(A)(H_i) \in B$ and according to 3.1,

$$\varrho(A)(H_i) = H_i \cap \varrho(A)(H) \tag{1}$$

holds. Denote $\bar{H}_i = (H_i \vee \varrho(A)(H))/\varrho(A)(H)$, $\bar{H} = H/\varrho(A)(H)$. From (1) it follows that $H_i/\varrho(A)(H_i)$ is isomorphic with $\bar{H}_i$, hence $\bar{H}_i \in B$ for each $i \in I$. Thus $\bigvee_{i \in I} \bar{H}_i \in B$. Let H_0 be the set of all $g \in H$ such that $g + \varrho(A)(H) \in \bigvee \bar{H}_i$. From the fact that $\bigvee \bar{H}_i$ is a convex l-subgroup of $\bar{H}$ it follows that H_0 is a convex l-subgroup of H. Clearly $H_i \subseteq H_0$ for each $i \in I$ and thus $H_0 = H$. Therefore $\bigvee \bar{H}_i = \bar{H}$, which implies $\bar{H} \in B$, completing the proof.

3.3. Lemma. Let H_1 and H_2 be convex linearly ordered l-subgroups of a lattice ordered group G. Suppose that $H_1 \cap H_2 \neq \{0\}$. Then H_1 and H_2 are comparable (i.e., either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$).

Proof. From $H_1 \cap H_2 \neq \{0\}$ it follows that there exists $0 < g \in H_1 \cap H_2$. Assume that neither $H_1 \subseteq H_2$ nor $H_2 \subseteq H_1$ holds. Then there are elements $0 < h_1 \in H_1$, $0 < h_2 \in H_2$ such that $h_1 \notin H_2$ and $h_2 \notin H_1$. Put $u = h_1 - h_1 \wedge h_2$, $v = h_2 - h_1 \wedge h_2$. Thus $0 < u \in H_1$, $0 < v \in H_2$ and $u \wedge v = 0$. Now both u and v must be comparable with g, but each of the relations $g \geqq u$ and $g \leqq u$ leads to a contradiction. Therefore H_1 and H_2 are comparable.

3.4. Theorem. Let L be a nonempty class of linearly ordered groups. Suppose that L is closed with respect to isomorphisms. Let $\{0\} \neq G$ be a lattice ordered group. Then the following conditions are equivalent:

(i) $G \in R(L)$.

(ii) There are linearly ordered convex l-subgroups $\{0\} \neq A_i$ $(i \in I)$, A_{ij} $(j \in J(i),\ i \in I)$ of G such that each A_{ij} is a convex l-subgroup of a lattice ordered group belonging to L, $A_i = \bigcup_{j \in J(i)} A_{ij}$ is valid for each $i \in I$, and $G = \Pi'_{i \in I} A_i$.

Proof. Suppose that (ii) is valid. From (b) and (c) it follows that all A_i belong to $R(L)$. Clearly $\bigvee_{i \in I} A_i = G$ and hence according to (c), G is an element of $R(L)$. Thus (ii) $\Rightarrow$ (i).

Let T be the class of all lattice ordered groups G fulfilling the condition (ii). Then $L \subseteq T \subseteq R(L)$. Hence for proving that (i) implies (ii) it suffices to verify that T is a radical class. Obviously T fulfils (a). Let G be as in (ii) and let K be a convex l-subgroup of G. Then $K = \Pi'_{i \in I}(K \cap A_i)$ and $K \cap A_i = \bigcup_{j \in J(i)}(K \cap A_{ij})$ holds for each $i \in I$, whence $K \in T$. Thus T fulfils the condition (b).

Let G be any lattice ordered group and let $\{H_m\}$ $(m \in M)$ be a system of convex l-subgroups of G such that each H_m belongs to T and $H_m \neq \{0\}$. Thus each H_m fulfils (ii); let A_{mi} $(i \in I_m)$ and A_{mij} $(j \in J_m(i),\ i \in I_m)$ be the corresponding linearly ordered groups with properties analogous to those of A_i and A_{ij} in the condition (ii). Without loss of generality we can assume that all linearly ordered groups A_{mij} are nonzero. Let $m_1 \in M$ and $i_1 \in I_m$ be fixed. We denote by $(A_{m_1 i_1})^{\sim}$ the set of all linearly ordered groups A_{mi} such that $A_{mi} \cap A_{m_1 i_1} \neq \{0\}$. Then $X \cap Y \neq \{0\}$ for each $X, Y \in (A_{m_1 i_1})^{\sim}$ and hence according to 3.3, X and Y are comparable. Thus the set-theoretical union $A^*_{m_1 i_1}$ of all linearly ordered groups belonging to $(A_{m_1 i_1})^{\sim}$ is a convex linearly ordered l-subgroup of G.

We denote by $S = \{X_p\}_{p \in P}$ the set of all linearly ordered groups A^*_{mi} $(m \in M,$ $i \in I_m)$. Further we denote by H_0 the set of all elements of G^+ that can be expressed as a join of a finite number of elements of the set $\bigcup_{p \in P} X_p$. Then H_0 is a convex sublattice of the lattice $(G^+; \leqq)$ and, at the same time, H_0 is a subsemigroup of the semigroup $(G^+; +)$. From this it follows that the set $H = \{g_1 - g_2 : g_1, g_2 \in H_0\}$ is a convex l-subgroup of G.

Put $\bigvee_{m \in M} H_m = H'$. From the definition of H we obtain immediately that $H \subseteq H'$ holds. Let $m \in M$ and $0 < h \in H_m$. Then h is a join of a finite number of elements belonging to $\bigcup_{i \in I_m} A_{mi}$. Since $A_{mi} \subseteq A^*_{mi}$, we get $h \in H_0$. Therefore $H_m \subseteq H$ for each $m \in M$. From this and from the fact that H is a convex l-subgroup of G it follows $H' \subseteq H$. Hence $H' = H$.

For verifying that the condition (c) holds for the class T we have to show that H fulfils the condition (ii).

Consider the linearly ordered l-subgroups X_p $(p \in P)$ and A_{mij} $(m \in M,\ i \in I_m,$ $j \in J_m(i))$. According to the construction of X_p, each X_p is a union of some A_{mij}. Hence it suffices to verify that $H = \Pi'_{p \in P} X_p$ holds.

Let p_1 and p_2 be distinct elements of P. From the construction of X_{p_1} and X_{p_2} it follows immediately that $X_{p_1} \cap X_{p_2} = \{0\}$ is valid. Let $p \in P$ and $0 < g \in H$. Then there are distinct indices $p_1, p_2, \ldots, p_m \in P$ and elements $0 < x_1 \in X_{p_1}, \ldots, 0 < x_m \in X_{p_m}$ such that $g = x_1 \vee \ldots \vee x_m$. If $p \notin \{p_1, \ldots, p_m\}$, then $[0, g] \cap X_p = \{0\}$. Let $y \in [0, g] \cap X_{p_1}$. We have $y = y \wedge g = y \wedge (x_1 \vee \ldots \vee x_m) = y \wedge x_1$, whence x_1 is the greatest element of the set $[0, g] \cap X_{p_1}$; the situation for $p_2, \ldots, p_m$ is analogous. Thus (under the denotation as in the definition of the internal restricted direct product) we have $g_p = 0$ whenever $p \notin \{p_1, \ldots, p_m\}$, and $g_{p_1} = x_1, \ldots, g_{p_m} = x_m$, completing the proof.

3.5. Lemma. Let A be a radical class generated by linearly ordered groups and let A' be a radical class of lattice ordered groups such that A' is a subclass of A. Let L' be the class of all linearly ordered groups belonging to A'. Then $A' = R(L')$.

This follows immediately from 3.4.

3.6. Lemma. Let A be a radical class generated by linearly ordered groups. Let G_1 be a lattice ordered group that is not linearly ordered and let $G_2 \neq \{0\}$ be a linearly ordered group. Then $G_1 \circ G_2$ does not belong to A.

Proof. The lattice ordered group $G = G_1 \circ G_2$ is not linearly ordered and there exists $0 < g \in G$ such that $g > g_1$ holds for each element $g_1 \in G$ incomparable with 0. From 3.4 it follows that if $G \in A$, then no element g having the mentioned property can exist in G; hence $G \notin A$.

The radical class containing only the one-element lattice ordered group $\{0\}$ will be called trivial. Let L_0 be the class of all linearly ordered groups distinct from $\{0\}$.

3.7. Lemma. Let A be a radical class generated by linearly ordered groups. Then A cannot be represented as a product BC where B is a nontrivial radical class and $C \cap L_0 \neq \emptyset$.

Proof. Assume that there are radical classes B, C with $A = BC$, where B is a nontrivial radical class and $C \cap L_0 \neq \emptyset$. Hence there exist lattice ordered groups $\{0\} \neq G_1 \in B$, $G_2 \in C \cap L_0$. Put $G = (G_1 \times G_1) \circ G_2$, $H = \varrho(B)(G)$. The elements of G are triples (g_1, g_1', g_2) with $g_1, g_1' \in G_1$, $g_2 \in G_2$. Let H_1 be the set of all $(g_1, g_1', g_2) \in G$ with $g_2 = 0$. Then H_1 is a convex l-subgroup of G isomorphic with $G_1 \times G_1$, whence $H_1 \subset H$. Let G_2' be the set of all $g_2 \in G_2$ having the property that there are $g_1, g_1' \in G_1$ with $(g_1, g_1', g_2) \in H$. Then G_2' is a convex l-subgroup of G_2 (thus G_2' is linearly ordered) and $H = (G_1 \times G_1) \circ G_2'$. Hence $(G_1 \times G_1) \circ G_2' \in B$. From $A = BC$ it follows $B \subseteq A$, thus according to 3.5 B is a radical class generated by linearly ordered groups. From this and from 3.6 we obtain $G_2' = \{0\}$, hence $H = H_1$. Therefore $G/\varrho(B)(G)$ is isomorphic with G_2 and $G \in BC = A$. But this contradicts 3.6, since A is generated by linearly ordered groups.

3.8. Theorem. Let A be a radical class generated by linearly ordered groups. Then A cannot be represented as a product BC of nontrivial radical classes B and C.

Proof. Assume that there exist nontrivial radical classes B, C such that $A=BC$. Let A be generated by a class L of linearly ordered groups. Let $\{0\}\neq A_i\in L$. Then $A_i\in BC$, whence $A_i/\varrho(B)(A_i)\in C$. Since $A_i/\varrho(B)(A_i)$ is linearly ordered and since in view of 3.7 we have $C\cap L_0=\emptyset$, we infer that $A_i/\varrho(B)(A_i)=\{0\}$. Therefore $\varrho(B)(A_i)=A_i$, whence $L\subseteq B$. From this it follows $A\subseteq B$. On the other hand, from $A=BC$ we get $B\subseteq A$; thus $B=A$ and $A=AC$. There exists $\{0\}\neq G\in C$. Denote $G_1=\varrho(A)(G)$. If $G_1\neq\{0\}$, then according to 3.4 there exists a convex l-subgroup $\{0\}\neq G_2$ of G_1 such that G_2 is linearly ordered; in such a case we would have $G_2\in C$ (since G_2 is a convex l-subgroup of G), whence $C\cap L_0\neq\emptyset$, which is a contradiction. Thus $\varrho(A)(G)=\{0\}$. Hence $G/\varrho(A)(G)$ is isomorphic with G and therefore G belongs to AC. Because of $AC=A$ we have $G\in A$, whence $\varrho(A)(G)=G$, a contradiction.

4. The condition (α_2). In this paragraph it will be shown that the condition (α_2) does not hold, in general, for radical classes of lattice ordered groups; i.e., there are radical classes A_i, B_i with $A_i\subseteq B_i$ ($i=1,2$) such that A_1A_2 fails to be a subclass of B_1B_2.

4.1. Lemma. Let A be a radical class of lattice ordered groups. Let H_i ($i\in I$) be convex l-subgroups of a lattice ordered group H, $\bigvee_{i\in I}H_i=H$. Then $\varrho(A)(H)=\bigvee_{i\in I}\varrho(A)(H_i)$.

Proof. Denote $\varrho(A)(H)=K$. According to 3.1 we have

$$K=K\cap H=K\cap(\bigvee_{i\in I}H_i)=\bigvee_{i\in I}(K\cap H_i)=\bigvee_{i\in I}\varrho(A)(H_i)$$

Let us denote

$$G_1=N_0\circ R_0,\quad G_2=N_0\circ R_0\circ N_0$$

Let A and B be the radical class of lattice ordered groups that is generated by the set $\{G_1\}$ or $\{G_2\}$, respectively. If $K\neq\{0\}$ is a convex l-subgroup of G_1, then either $K=G_1$ or K is isomorphic with N_0. Similarly, if $K\neq\{0\}$ is a convex l-subgroup of G_2 and $K\neq G_2$, then K is isomorphic either with N_0 or with G_1. From this and from 3.4 we obtain:

4.2. Lemma. Let $G\neq\{0\}$ be a lattice ordered group. Then

(i) G belongs to A if and only if G can be expressed as a restricted direct product $G=\Pi'_{i\in I}A_i$ such that each A_i is isomorphic to some of the linearly ordered groups N_0, G_1;

(ii) G belongs to B if and only if G can be expressed in the form $G=\Pi'_{i\in I}A_i$ such that each A_i is isomorphic to some of the linearly ordered groups N_0, G_1, G_2.

Clearly A is a proper subclass of B. We shall show that A^2 fails to be a subclass of B^2; moreover, the class of nonisomorphic types of lattice ordered groups that belong to A^2 and do not belong to B^2 is a proper class.

Let I be a nonempty set. There exists a lattice ordered group $G(I)$ that can be expressed as $G(I)=\Pi'_{i\in I}G_i$, where each G_i is isomorphic with $H=N_0\circ R_0\circ N_0\circ R_0$.

For any nonempty sets I, I' the lattice ordered groups $G(I)$ and $G(I')$ are isomorphic if and only if card $I = $ card I'.

We have $\varrho(A)(H) = N_0 \circ R_0 \circ \{0\} \circ \{0\}$, whence $H/\varrho(A)(H)$ is isomorphic with $N_0 \circ R_0 = G_1$, and thus $H \in A^2$. Clearly $G(I) = \bigvee_{i \in I} G_i$. From this and from 4.1 we obtain $\varrho(A)(G(I)) = \bigvee_{i \in I} \varrho(A)(G_i) \in A$ (because $\varrho(A)(G_i)$ is isomorphic with $\varrho(A)(H)$). Hence $G(I) \in A^2$.

Similarly we have $\varrho(B)(H) = N_0 \circ R_0 \circ N_0 \circ \{0\}$ and thus $H/\varrho(B)(H)$ is isomorphic with R_0. Hence according to 4.2, $H/\varrho(B)(H)$ does not belong to B and therefore H does not belong to B^2. By applying 4.1 we infer that $G(I)$ does not belong to B^2. So we have verified that the condition (α_2) does not hold for products of radical classes.

We can also consider the question whether for each triple of radical classes A, B, C of lattice ordered groups the implications

$$(2) \qquad A \subseteq B \Rightarrow CA \subseteq CB,$$

$$(3) \qquad A \subseteq B \Rightarrow AC \subseteq BC$$

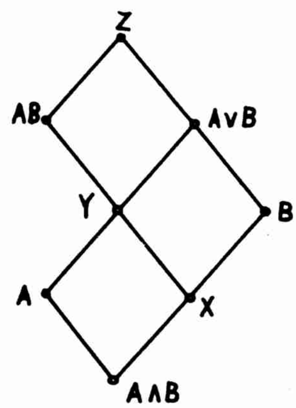
are valid.

The implication (2) follows immediately from the definition of the product of radical classes. If A and B have the same meaning as above, then we have $A \subset B$ and

$$A^2 \nsubseteq BA, \quad AB \nsubseteq B^2$$

(these relations can be proved by analogous reasoning as we did by proving $A^2 \nsubseteq B^2$); hence (3) does not hold in general.

Let A, B be radical classes of lattice ordered groups. Then $A \cap B$ is a radical class and clearly $A \cap B$ is the greatest radical class that is a subclass of both A and



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B. Thus we can denote $A \cap B = A \wedge B$. If C is a radical class such that $C \supseteq A$, $C \supseteq B$, and $C \subseteq C_1$ whenever C_1 is a radical class with $C_1 \supseteq A$, $C_1 \supseteq B$, then we shall write $C = A \vee B$.

4.3. Lemma. (Cf. [3].) For each pair of radical classes A, B of lattice ordered groups there exists a radical class C such that $C = A \vee B$. Moreover, for any triple of radical classes A_1, A_2, B we have $(A_1 \vee A_2) \wedge B = (A_1 \wedge B) \vee (A_2 \wedge B)$.

4.4. Proposition. Let A, B be radical classes of lattice ordered groups. Then the lattice L^1 of radical classes generated by the set $\{A, B, AB\}$ is a homomorphic image of the lattice L^0 on *Fig. 1*. There exist radical classes A, B such that the corresponding lattice L^1 is isomorphic to L^0.

Proof. From the distributivity of L^1 and from $A \subseteq AB$ it follows that L^1 is a homomorphic image of L^0. We denote by A and B the radical class generated by the set $\{N_0 \circ R_0\}$ or $\{N_0 \circ N_0, N_0 \circ R_1\}$ respectively. Put $(AB) \vee (A \vee B) = Z$, $(AB) \wedge (A \vee B) = Y$, $(AB) \wedge B = X$.

From 3.4 it follows that $N_0 \circ R_0 \notin B$, whence $A \wedge B \subset A$, and thus $X \subset Y$, $B \subset A \vee B$. Further we infer from 3.4 that $A \vee B$ is the class of all lattice ordered groups G that can be expressed as $G = \Pi'_{i \in I} G_i$, where each G_i is isomorphic to some of the linearly ordered groups N_0, $N_0 \circ N_0$, $N_0 \circ R_0$, $N_0 \circ R_1$. The lattice ordered group $N_0 \circ R_0 \circ N_0 \circ R_1$ belongs to AB and fails to belong to $A \vee B$, whence $Y \subset AB$ and thus $A \vee B \subset Z$. Clearly $N_0 \circ R_1 \in B$ and $(N_0 \circ R_1)/\varrho(A)(N_0 \circ R_1)$ is isomorphic with R_1, thus $N_0 \circ R_1$ does not belong to AB. Therefore $X \subset B$ and hence $Y \subset A \vee B$, $AB \subset Z$. Next we have $N_0 \circ N_0 \in B$, $N_0 \circ N_0 \in AB$ and $N_0 \circ N_0 \notin A$, whence $A \wedge B \subset X$, $A \subset Y$, completing the proof.

5. Direct decompositions. In this section the following result will be proved:

5.1. Theorem. Let A, B be radical classes generated by linearly ordered groups and let $G \in AB$. Let D be a convex l-subgroup of G. Then D cannot be expressed as a direct product of an infinite number of lattice ordered groups distinct from $\{0\}$.

For proving 5.1 we need some lemmas.

5.2. Lemma. Let A, B be radical classes of lattice ordered groups and let $G \in AB$. Then $(\varrho(A)(G))^\delta \in B$.

Proof. Denote $G_1 = \varrho(A)(G)$, $G_2 = (\varrho(A)(G))^\delta$. Then $G_1 \cap G_2 = \{0\}$ and $\varrho(A)(G_1 \vee G_2) = G_1$. Since AB is a radical class, $G_1 \vee G_2$ belongs to AB, hence $G_1 \vee G_2/G_1$ belongs to B. The lattice ordered groups $G_1 \vee G_2/G_1$ and $G_2/\{0\} = G_2$ being isomorphic, we obtain $G_2 \in B$.

5.3. Lemma. Let G be a lattice ordered group, $0 < a_i \in G$ $(i \in I)$, $0 < d_s \in G$ $(s \in S)$, $g \in G$, $g = \bigvee_{i \in I} a_i$, $g = \bigvee_{s \in S} d_s$. Suppose that $[0, a_i]$ is a chain for each $i \in I$. Further suppose that $a_i \wedge a_{i_1} = 0$ whenever i and i_1 are distinct elements of I, and that $d_s \wedge d_{s_1} = 0$ whenever s and s_1 are distinct elements of S. Then there exists a mapping φ of I onto S such that $d_s = \bigvee_{i \in \varphi^{-1}(s)} a_i$ holds for each $s \in S$.

Proof. Let $i \in I$. From

$$(4) \qquad a_i = a_i \wedge g = \bigvee_{s \in S}(a_i \wedge d_s)$$

it follows that there exists $s \in S$ with $a_i \wedge d_s > 0$. Let $s_1 \in S$, $s_1 \neq s$. If $a_i \wedge d_{s_1} > 0$, then $0 = d_s \wedge d_{s_1} \geqq (d_s \wedge a_i) \wedge (d_{s_1} \wedge a_i) > 0$ (because $d_s \wedge a_i$, $d_{s_1} \wedge a_i \in [0, a_i]$), which is a contradiction. Thus $a_i \wedge d_{s_1} = 0$ for each $s_1 \in S$, $s_1 \neq s$. Put $\varphi(i) = s$. Hence according to (4), $a_i = a_i \wedge d_{\varphi(i)}$.

Let s be any element of S. We have

$$d_s = d_s \wedge g = \bigvee_{i \in I}(d_s \wedge a_i)$$

hence there is $i \in I$ with $d_s \wedge a_i > 0$; thus $s = \varphi(i)$ and φ is a mapping of I onto S. Since $a_j \wedge d_s = 0$ whenever $j \notin \varphi^{-1}(s)$ and $a_j \wedge d_s = a_j$ if $j \in \varphi^{-1}(s)$, we obtain $d_s = \bigvee_{i \in \varphi^{-1}(s)} a_i$.

In what follows we assume that L and L' are nonempty classes of linearly ordered groups and that A and B are radical classes generated by L or L', respectively.

5.4. Lemma. Let G be a lattice ordered group and let D be a convex l-subgroup of $\varrho(A)(G)$. Then D cannot be expressed as a direct product of an infinite number of lattice ordered groups distinct from $\{0\}$.

Proof. Assume that D can be expressed in the form $D = \Pi_{s \in S_1} D_s$ with $D_s \neq \{0\}$ for each $s \in S_1$, card $S_1 \geqq \aleph_0$. Choose $0 < d^s \in D_s$ for each $s \in S_1$. There exists $0 < g \in D$ with $g(D_s) = d^s$ for each $s \in S_1$. Then we have $g = \bigvee_{s \in S_1} d^s$. Let A_i $(i \in I)$ be as in 3.4 with the distinction that we take $\varrho(A)(G)$ instead of G. There is a finite subset I_1 of I such that $g = \bigvee_{i \in I_1} g(A_i)$ and $0 < g(A_i)$ for each $i \in I_1$. Hence according to 5.3 we have card $S_1 \leqq$ card I_1, which is a contradiction.

If we take B instead of A, then from 5.2 and 5.4 we obtain

5.5. Lemma. Let $G \in AB$ and let D be a convex l-subgroup of $(\varrho(A)(G))^\delta$. Then D cannot be expressed as a direct product of an infinite number of lattice ordered groups distinct from $\{0\}$.

Let G be a lattice ordered group. Let $X \subseteq G$. The closed convex l-subgroup of G generated by X will be denoted by $c(X)$. If H is a convex l-subgroup of G, then $(c(H))^+$ is the set of all elements $g \in G$ having the property that there exists $0 \leqq b_s \in H$ $(s \in S)$ such that $g = \bigvee_{s \in S} b_s$. From this and from 3.4 it follows:

5.6. Lemma. Let G be a lattice ordered group. Let A_i $(i \in I)$ be as in 3.4 with the distinction that we take $\varrho(A)(G)$ instead of G. Let $0 < g \in G$. Then the following conditions are equivalent:

(i) $g \in c(\varrho(A)(G))$.

(ii) there are elements a_i $(i \in I)$ such that $0 \leqq a_i \in A_i$ for each $i \in I$ and $\bigvee_{i \in I} a_i = g$.

Proof. Clearly (ii) implies (i). Suppose that (i) holds. Then there are elements $0 < b_s \in \varrho(A)(G)$ $(s \in S)$ such that $g = \bigvee_{s \in S} b_s$. There exists a subset I_1 of I having the property that each b_s can be expressed as a join of some elements belonging to $\bigcup_{i\in I_1}A_i$. Hence there are elements $0<c_t$ $(t\in T)$ with $c_t\in\bigcup_{i\in I_1}A_i$ such that $g=\bigvee_{t\in T}c_t$. We can assume that $\{c_t\}_{t\in T}\cap A_i\neq\emptyset$ for each $i\in I_1$ and that, whenever $i\in I_1$ and $0<a_i\in A_i$ with $a_i\leqq g$, then $a_i\in\{c_t\}_{t\in T}$.

Let $i\in I_1$. Suppose that $g\geqq a_i$ for each $a_i\in A_i$. Choose $0<a_0\in A_i$. Then

$$g<a_0+g=a_0+\bigvee_{t\in T}c_t=\bigvee_{t\in T}(a_0+c_t)$$

If $c_t\in A_i$, then $a_0+c_t\in A_i$, hence there is $t_1\in T$ with $a_0+c_t=c_{t_1}$. If $c_t\notin A_i$, then $a_0\wedge c_t=0$, thus $a_0+c_t = a_0\vee c_t = c_{t_2}\vee c_t$, where $a_0=c_{t_2}$. From this it follows that $\bigvee_{t\in T}(a_0+c_t)\leqq\bigvee_{t\in T}c_t$, hence $a_0+g\leqq g$, which is a contradiction. Therefore for each $i\in I_1$ there exists $0<a_{0i}\in A_i$ such that $g\ngeqq a_{0i}$. Denote $a^i=g\wedge a_{0i}$. It is easy to verify that a^i is the greatest element of the set $\{a_i\in A_i\colon 0<a_i\leqq g\}$. Hence $g=\bigvee_{i\in I_1}a^i$. If we put $a^i=0$ for each $i\in I\setminus I_1$, then $g=\bigvee_{i\in I}a^i$.

Denote $\varrho(A)(G)=G_1$, $G_1^\delta=G_2$.

5.7. Lemma. Let $G\in AB$ and let D be a convex l-subgroup of $c(G_1)$. Then D cannot be expressed as a direct product of an infinite number of lattice ordered groups distinct from $\{0\}$.

Proof. Assume that $D=\Pi_{s\in S}D_s$ where S is infinite and $D_s\neq\{0\}$ for each $s\in S$. According to 5.4, D cannot be a convex l-subgroup of G_1. Hence there exists $0<g_1\in D$ such that g_1 does not belong to G_1. Further there exists $0<g_2\in D$ such that the set $\{s\in S\colon g_2(D_s)>0\}$ is infinite. Put $g=g_1\vee g_2$. Then $0<g\in D$, $g\notin G_1$ and the set $S(g)=\{s\in S\colon g(D_s)>0\}$ is infinite. From $G\in AB$ it follows $c(G_1)\in AB$. Since $\varrho(A)(c(G_1))=G_1$, we have $c(G_1)/G_1\in B$. Therefore according to 3.4 there exist linearly ordered groups B_j $(j\in J)$ (having analogous properties to those of A_i with the distinction that G is replaced by $c(G_1)/G_1$ and L is replaced by L') such that $c(G_1)/G_1=\Pi'_{j\in J}B_j$. In $c(G_1)/G_1$ we have $0<g+G_1$, hence there exists a finite subset $\emptyset\neq J_1$ of J such that $g+G_1=\bigvee_{j\in J_1}b_j$ where $0<b_j\in B_j$ for each $j\in J_1$. From this it follows that there are elements $0<b_j^-\in b_j$ $(j\in J_1)$ with $g=\bigvee_{j\in J_1}b_j^-$. Let $j\in J_1$. We have $b_j^-\in D$. Put $S_j=\{s\in S\colon b_j^-(D_s)>0\}$. Obviously $S(g)=\bigcup_{j\in J_1}S_j$; hence there exists $j\in J_1$ such that S_j is infinite; let such an index j be fixed. There exist $s_1, s_2, s_3, \ldots\in S_j$ such that $s_n\neq s_m$ whenever $n\neq m$. Further there exists $d\in D$ such that $d(D_s)=3b_j^-(D_s)$ for each $s\in\{s_2, s_4, s_6, \ldots\}$, and $d(D_s)=b_j^-(D_s)$ for each $s\in S\setminus\{s_2, s_4, s_6, \ldots\}$. Then $b_j^-<d<4b_j^-$, hence in $c(G_1)/$ $/G_1$ we have $d+G_1>0$ and $d+G_1\in B_j$. Since B_j is linearly ordered we infer that $d+G_1$ is comparable with $2b_j=2b_j^-+G_1$.

Denote $G_1=\bar{0}$ (zero element in $c(G_1)/G_1$), $x=d+G_1-(2b_j^-+G_1)=$ $=(d-2b_j^-)+G_1$. From the comparability of elements $d+G_1$, $2b_j^-+G_1$ it follows that either $x\vee\bar{0}=\bar{0}$ or $x\wedge\bar{0}=\bar{0}$. Hence either $(d-2b_j^-)\vee 0\in G_1$ or $(d-$ $-2b_j^-)\wedge 0\in G_1$. Put $(d-2b_j^-)\vee 0=y$. We have $y(D_s)=b_j^-(D_s)$ for each $s\in\{2, 4, 6, \ldots\}$ and $y(s)=0$ otherwise. Clearly $0<y\in c(G_1)$. According to 5.6 there is a subset $\emptyset\neq I_1\subseteq I$ and there are elements $0<a_i\in A_i$ $(i\in I_1)$ such that

$y = \bigvee_{i \in I_1} a_i$. Then from 5.3 and from the fact that the set $\{s \in S: y(s) \neq 0\}$ is infinite we obtain that y cannot belong to G_1. Similarly we can verify that $(d - 2b_i^-) \wedge 0$ does not belong to G_1 and so we arrived at a contradiction.

Proof of 5.1:

Assume that $D = \Pi_{s \in S} D_s$ where S is infinite and $D_s \neq \{0\}$ for each $s \in S$. As above, put $G_1 = \varrho(A)(G)$, $G_2 = (G_1)^\delta$. We denote $D^{(1)} = D \cap c(G_1)$, $D^{(2)} = D \cap G_2$, $D_s^{(1)} = D_s \cap c(G_1)$, $D_s^{(2)} = D_s \cap G_2$ for each $s \in S$. Then $D^{(1)}$ is a convex l-subgroup of $c(G_1)$ and $D^{(2)}$ is a convex l-subgroup of G_2. Since $c(G_1)$ and G_2 are closed convex l-subgroups of G, we have

$$D^{(1)} = \Pi_{s \in S} D_s^{(1)}, \quad D^{(2)} = \Pi_{s \in S} D_s^{(2)} \tag{5}$$

Put $S_1 = \{s \in S: D_s^{(1)} \neq \{0\}\}$, $S_2 = \{s \in S: D_s^{(2)} \neq \{0\}\}$. For each $s \in S$ we have either $D_s^{(2)} = D_s \neq \{0\}$ or $D_s^{(1)} \neq \{0\}$. Thus we have either $D^{(1)} = \Pi_{s \in S_1} D_s^{(1)}$ with card $S_1 \geqq \aleph_0$, or $D^{(2)} = \Pi_{s \in S_2} D_s^{(2)}$ with card $S_2 \geqq \aleph_0$; both these cases being impossible in view of 5.5 and 5.7, we have a contradiction.

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SÚHRN

SÚČINY RADIKÁLOVÝCH TRIED ZVÄZOVO USPORIADANÝCH GRÚP

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Pojem radikálovej triedy zväzovo usporiadaných grúp bol zavedený v autorovej práci [3]. V predloženej práci sa ukazuje, že súčin takýchto radikálových tried je opäť radikálová trieda a skúmajú sa niektoré vlastnosti tohto súčinu. Je charakterizovaná radikálová trieda generovaná ľubovoľnou triedou lineárne usporiadaných grúp.

РЕЗЮМЕ

ПРОИЗВЕДЕНИЯ РАДИКАЛЬНЫХ КЛАССОВ СТРУКТУРНО УПОРЯДОЧЕННЫХ ГРУП

Я. Якубик, Кошице

Понятие радикального класса структурно упорядоченных групп введено автором в работе [3]. В настоящей статье показывается, что произведение таких радикальных классов является радикальным классом и изучаются некоторые свойства этого произведения. Дана характеристика радикального класса, порожденного любым классом линейно упорядоченных групп.