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DIAMETER k-CRITICAL GRAPHS

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Under a graph we mean throughout the paper an undirected finite graph without loops and multiple lines.

This paper presents an attempt to generalize the notion of so-called e-critical graph, investigated in papers [2] - [5] . A graph G is said to be e-critical if the deleting of an arbitrary line from G increases its diameter. A graph is said to be k-critical, (k is a natural number) if the deleting of an arbitrary subset M of its lines increases its diameter if and only if $|M| \geq k$. We will investigate the question about the existence of such graphs. It is conjectured that there are no such graphs. This conjecture is settled in affirmative for approximately one half of all possible cases. Our terminology as well as denotation is besed on [1] except for the given here.

A graph $G = (V, E)$ has the point set $V = V(G)$ and the set of lines $E = E(G)$. Let $x \in V$. By $N(x)$ we denote the neighbourhood of x (i.e. $N(v) = \{u \mid d(u, v) = 1\}$). The eccentricity of x is denoted by $ec(x)$, ($ec(x) = \max_{y \in V(G)} d(x, y)$). We define the the distance of x and a set M as follows: $d(x, M) = \min_{y \in M} d(x, y)$

Let $N \subset E$. The graph with the point set $V(G-N) = V(G)$ and the set of lines $E(G-N) = E(G) - N$ is denoted by $G-N$. If N consists of only one line e, we will write $G-e$.

Let P be a v_0-v_n walk in G, with the points $v_0, v_1, \ldots$ $\ldots v_n$ ($v_i \in V(G)$, for $i = 0, 1, \ldots, n$). The length of P is denoted by $\lambda(P)$. We denote the v_n-v_0 walk with the points $v_n, v_{n-1}, \ldots, v_0$ by P^{-1}. Further, for any integers k, m with $1 \leq k < m \leq n$ we denote by $P(v_k$-$v_m)$ the v_k-v_m subwalk of P with the points $v_k, v_{k+1}, \ldots, v_m$. Let $N \subset E(G)$. The symbol $(P \cap N)_e$ denotes the set $\{e \mid e \in E(G) \wedge e \in N \wedge e \in P\}$. Let k be a natural number and for $i = 1, 2, \ldots, k$ P_i be a $v_{n_{i-1}}$-v_{n_i} walk in G with the points $v_{n_{i-1}}, v_{n_{i-1}+1}, \ldots, v_{n_i}$. The concatenation of P_i (in a given ordering) we define as the v_{n_0}-v_{n_k} walk with the points $v_{n_0}, v_{n_0+1}, \ldots, v_{n_{k-1}}, v_{n_{k-1}+1}, \ldots, v_{n_k}$. Let $P_1, P_2, \ldots, P_m$ be the v_0-v_n walks in a graph G. The symbol $(\bigcap_{i=1}^{m} P_i)_e$ denotes the set $\{e \mid e \in E(G) \wedge e \in P_i$, for $i = 1, 2, \ldots, m\}$.

To emphasize for a given invariant to be refered to a graph G we will use the subscript G, for example $d_G(x, y)$, $ec_G(x)$, etc.

Definition 1. Let k be a natural number. A graph $G = (V, E)$ is said to be diameter k-critical (in the next k-critical) with the diameter d, if the following holds:

1. $d(G) = d < +\infty$
2. $d(G-M) = d(G)$ if and only if $|M| < k$, for any subset M of E.

According to this definition an e-critical graph is a 1-critical graph. Throughout the paper we assume $k \geq 2$.

Assertion 1. There is no k-critical graph ($k \geq 2$), with the diameter one.

Assertion 2. Let $G = (V, E)$ be a k-critical graph with $k \geq 2$. Then $\deg_G(x) \geq k$, for every $x \in V$.

P r o o f. Since $d(G-M) = +\infty$ for $M = \{xt \mid t \in N_G(x)\}$, there must be $|M| \geqslant k$.

T h e o r e m 1. Let $G = (V, E)$ be a k-critical graph $(k \geqslant 2)$ with the diameter d. Then for any line e from E the graph G-e is (k-1)-critical with the diameter d.

P r o o f. Let us denote $G-e = G_e = (V_e, E_e)$. Because $k \geqslant 2$ then $d(G-e) = d(G) = d$ for any line $e \in E$. Let G_e be not (k-1)-critical. There are the next possibilities:

1. There is a subset M of E_e, with $|M| < k-1$ such that $d(G_e-M) > d(G_e)$. Then for $M_1 = M \cup \{e\}$ we have $d(G-M_1) > d(G)$ which is not possible because $|M_1| < k$.

2. There is a subset M of E_e with $|M| = k-1$ such that $d(G_e-M) = d(G_e)$. Because $e \notin M$, there also must be $d(G-M_1) = d(G)$ for $M_1 = M \cup \{e\}$. But this is impossible because $|M_1| = k$. This completes the proof.

C o r o l l a r y 1. Let $G = (V, E)$ be a k-critical graph with $k \geqslant 2$. Then for every subset M of E with $1 \leq |M| < k$, the graph G-M is $(k-|M|)$-critical with the same diameter as G.

C o r o l l a r y 2. If there is no k-critical graph $(k \geqslant 2)$ with the diameter d, then also there is no (k+1)-critical graph with the diameter d.

L e m m a 1. Let $G = (V, E)$ be a k-critical graph $(k \geqslant 2)$ with the diameter d. Let $M \subset E$, $|M| = k$, $M = \{e_1, e_2, \ldots, e_k\}$. Then for every two points $x, y \in V$ with $d_{G-M}(x, y) > d$ there are k x-y paths $\{Q_i\}_{i=1}^{k}$ such that $\lambda(Q_i) \leqslant d$ and $(Q_i \cap M)_e = \{e_i\}$, $(i = 1, 2, \ldots, k)$.

P r o o f. Let $M_i = M - \{e_i\}$. For every x-y path P with $\lambda(P) \leqslant d$ in $G-M_i$ must be $e_i \in P$. Because for any $j = 1, 2, \ldots, k$ $j \neq i$, the line e_j cannot belong to P there must be $(P \cap M)_e = \{e_i\}$. This completes the proof.

D e f i n i t i o n 2. Let $G = (V, E)$ be a graph, $x \in V$. The system $\{M_i\}_{i=0}^{m}$ of subsets of V is said to be a distance decomposition of V from the point x, if

1. $m = ec_G(x)$
2. $M_i = \{z \in V \mid d_G(z, x) = i\}$ for $i = 0, 1, \ldots, m$.

L e m m a 2. Let $G = (V, E)$ be a k-critical graph $(k \geq 2)$ with the diameter d $(d \geq 2)$. Let $x \in V$, $ec_G(x) = d$. Let $\{M_i\}$ be the distance decomposition of V from the point x. Then there is no line yz with $y, z \in M_d$ in G.

P r o o f. For $|M_d| = 1$, lemma trivially holds. Let $y, z \in \in M_d$ and $yz \in E$. From the Assertion 2 we have $deg_G(x) \geq k$. Let us denote $L = \{xx_1, xx_2, \ldots, xx_{k-1}, yz\}$, where $x_i \in N_G(x)$, and $x_i \neq x_j$ for $i \neq j$. Because G is a k-critical graph, there must be in $G-L$ such two points $t, q \in V$, for which $d_{G-L}(t, q) > 1$. Let $t \in M_m$, $q \in M_n$, for $0 \leq m, n \leq d$. According to Lemma 1 there are t-q paths Q_1, Q_2 with $\lambda(Q_i) \leq d$ for $i = 1, 2$ in G such that $(Q_1 \cap L)_e = \{xx_1\}$, $(Q_2 \cap L)_e = \{yz\}$.

There are two possibilities:

1. $m + n > d$. Because the length of any t-q path containing the line xx_1 is greater than or equal to $m + n$, $\lambda(Q_1) \geq m + n > d$. This is a contradiction.
2. $m + n \leq d$. The length of any t-q path containing the line yz is greater than or equal to $2d - m - n + 1 \geq d + 1$. So $\lambda(Q_2) > d$ which is a contradiction.

The line yz cannot belong to E. Hence the lemma holds.

T h e o r e m 2. There is no k-critical graph with the diameter two, for $k \geq 2$.

P r o o f. Let $G = (V, E)$ be 2-critical graph with the diameter two. Let $t \in V$, $ec_G(t) = 2$, and $\{M_i\}$ be the distance

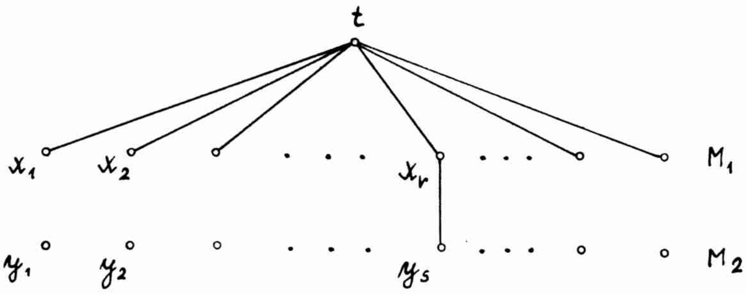

Fig. 1

decomposition of V, from t, $M_1 = \{x_1, x_2, \dots, x_n\}$, $M_2 = \{y_1, y_2, \dots, y_m\}$, (Fig. 1).

Obviously $n \geqslant 2$, $m > 0$. We show that $x_i y_j \in E$ for every $i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$. Let there be such numbers r, s ($1 \leqslant r \leqslant n$, $1 \leqslant s \leqslant m$) that $x_r y_s \notin E$. Then $d_G(y_s, x_r) = d_G(y_s, t) = 2$. For the distance decomposition $\{M_i'\}$ of V from the point y_s we get a contradiction with the Lemma 2, because $tx_r \in E$ and $t, x_r \in M_2'$. So, $x_r y_s \in E$. Let $R = \{x_i x_j \in E \mid i = 1, 2, \dots, n; j = 1, 2, \dots, m\}$. If $|R| \geqslant 2$ then $d(G-R) = d(G) = 2$ which is a contradiction. If $|R| = 0$, then G is the complete bipartite graph $K_{n,m+1}$ which is a contradiction, too. So there must be $|R| = 1$. Let $xx_1 \in E$. If $n > 2$ then for $j \geqslant 3$ we have a contradiction $d_{G-tx_j}(t, x_j) > 2$. Hence $n = 2$ and $d(G-\{tx_1, x_1y_1\}) = 2$. But this is impossible, because G is 2-critical with the diameter two.

The assertion of the theorem follows immediately by using the Corollary 2.

<u>L e m m a 3</u>. Let $G = (V, E)$ be a k-critical graph ($k \geqslant 2$) with the diameter $d \geqslant 2$. Let $x, q \in V$, $d_G(x, q) = d$. Let $y \in N_G(x)$,

$p \in N_G(q)$ and $L = \{xy, pq\}$. Let $\{M_i\}$ be the distance decomposition of V from x and $W \subset E$, $|W| = k$, $L \subset W$. Then $i + j = d$ for every two points $m \in M_i$, $n \in M_j$, $(0 \leq i, j \leq d)$ with $d_{G-W}(m, n) > d$.

P r o o f. (Fig. 2)

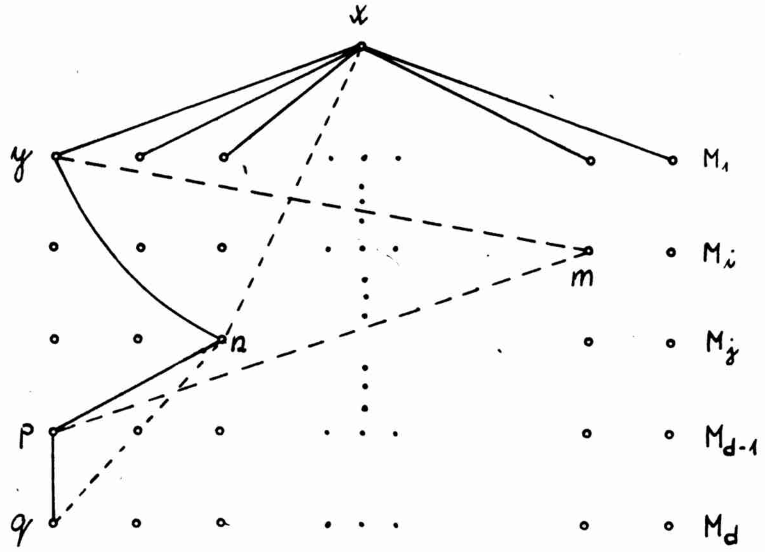

Fig. 2

By the Lemma 1 there must exist in G two m-n paths P_1, P_2 of the length less than or equal d, such that $(P_1 \cap W)_e = \{xy\}$, $(P_2 \cap W)_e = \{pq\}$. Because $xy \in P_1$, $pq \in P_2$, there must be $d \geq \lambda(P_1) \geq i + j$ and $d \geq \lambda(P_2) \geq 2d - (i + j)$. Hence, $i+j = d$. This completes the proof.

C o r o l l a r y 3. Let the assumptions of the Lemma 3 hold. Moreover, let any line from $W-L$ be incident with x or q. Then $d_G(m, n) = d$, for every two points $m, n \in V$ with $d_{G-W}(m, n) > d$.

Lemma 4. Let $G = (V, E)$ be a 2-critical graph with the diameter $d \geq 3$. Let $x, q \in V$, $d_G(x, q) = d$. Let $\{M_i\}$ be the distance decomposition of V from x. Then for every $p \in N_G(q)$ there is a point $t \in M_1$ such that $d_G(t, p) = d - 2$ and $d_{G-L_t}(x, q) = d$ for $L_t = \{xt, pq\}$.

Proof. (Fig. 3)

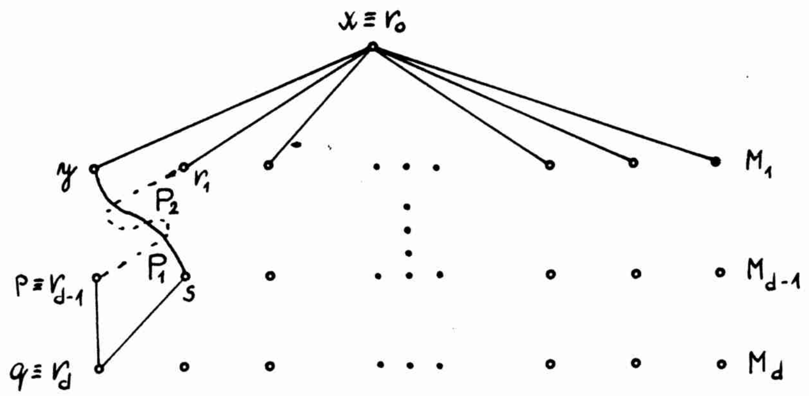

Fig. 3

Let $p \in N_G(q)$. By the Lemma 2 we have $p \in M_{d-1}$. There exists a point $y \in M_1$ such that $d_G(y, p) = d - 2$. If $d_{G-L_y}(x, q) = d$, then y is the required point. Let $d_{G-L_y}(x, q) > d$. By the Lemma 1 there are two x-p paths P_1, P_2 of the length d, such that $(P_1 \cap L_y)_e = \{xy\}$ and $(P_2 \cap L_y)_e = \{pq\}$.

Denote the points of P_2 as follows: $r_o = x, r_1, r_2, \ldots, \ldots, r_{d-1}, r_d$. Then $r_i \in M_i$ for $i = 0, 1, \ldots, d$ and $r_1 \neq y$, $r_{d-1} \equiv p$. To prove the theorem it is sufficient to put $t = r_1$. This completes the proof.

Lemma 5. Let $G = (V, E)$ be a graph with the diameter three. Let $x, q \in V$ and $d_G(x, q) = 3$. Let $\{M_i\}$ be the distance decomposition of V from x. If for every $p \in N_G(q)$ there is a point $r \in M_2$ such that $pr \in E$, then G is not 2-critical.

Proof. (Fig. 4)

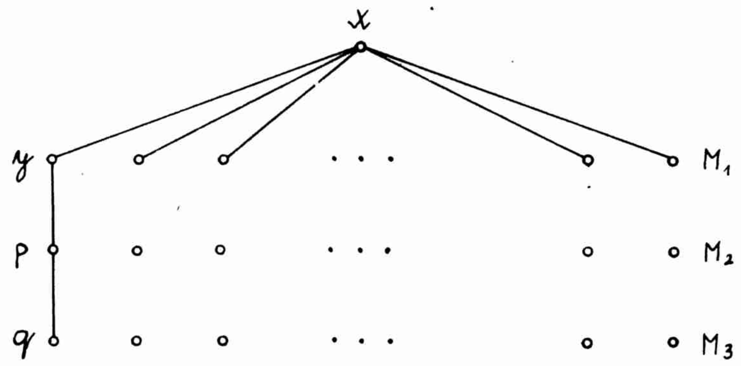

Fig. 4

Let $G = (V, E)$ be a 2-critical graph, $d(G) = 3$ and the assumptions of the lemma hold. Let $p \in N_G(q)$. By the Lemma 2 we have $p \in M_2$. According to Lemma 4 there is a point $y \in M_1$ such that $yp \in E$ and $d_{G-L_y}(x, q) = 3$ for $L_y = \{xy, pq\}$. Since G is 2-critical graph there are two points $m, n \in V$, $m \in M_i$, $n \in M_j$, $1 \leq i, j \leq 3$, for which $d_{G-L_y}(m, n) > 3$. In accordance with the Lemma 3 we have $d_G(m, n) = 3$ and $i + j = 3$. By the Lemma 1 there are two m-n paths Q_1, Q_2 of the length three such that $(Q_1 \cap L_y)_e = \{xy\}$, $(Q_2 \cap L_y)_e = \{pq\}$.

There are the next possibilities:

1. $m \equiv x$, $n \in M_3$, $n \neq q$. Since $d_G(x, \{p, q\}) = 2$, we have $\lambda(Q_2) > 3$, a contradiction.

2. $m \in M_1$, $n \in M_2$, $m \neq y$, $n \neq p$. Because $d_G(m, \{x, y\}) = 1$ the lines ny and mp belong to E. Then in $G - L_y$ there is a path $mpyn$, which is a contradiction.

3. $m \equiv y$, $n \in M_2$, $n \neq p$. Since $d_G(m, q) = 2$, we have $qn \in E$. By the assumption, there is a point $r \in M_2$ such that $rn \in E$. If $d_G(m, r) \leq 2$, then also $d_{G-L_y}(m, n) \leq 3$, which is a contradiction. Let $d_G(m, r) = 3$. Then for the distance decomposition $\{M_i'\}$ of V from the point y we have a contradiction with the Lemma 2, since r, n must belong to M_3'.

4. $m \neq y$, $m \in M_1$, $n \equiv p$. By the assumption, there is a point $r \in M_2$, for which $pr \in E$. Analogously to the case 3 we have $d_G(m, r) = 3$. Because $d_G(m, p) = 3$ we get a contradiction with the Lemma2 for the distance decomposition of V from the point m.

Thus we have proved $d(G-L_y) = d(G)$, a contradiction. This completes the proof.

T h e o r e m 3. There is no k-critical graph with the diameter three, for $k \geq 2$.

P r o o f. (Fig. 4) We prove the theorem for $k = 2$. The assertion of the theorem then directly follows from the Corollary 2.

Let $G = (V, E)$ be a 2-critical graph with the diameter three. Let $x, q \in V$, $d_G(x, q) = 3$. Let $\{M_i\}$ be the distance decomposition of V from x. By the Lemma 5 there is a point $p \in N_G(q)$ such that for every line $pf \in E$ $(f \in V)$, the point $f \notin M_2$. From the Lemma2 we have $p \in M_2$. Let $u \in M_1$. Denote $L_u = \{xu, pq\}$.

Let $S_p = \{t \in M_1 \mid d_G(p, t) = 1 \wedge d_{G-L_t}(x, q) = 3\}$. By the Lemma 4 $S_p \neq \emptyset$. Let $y \in S_p$. There are two points $m, n \in V$ such that $d_{G-L_y}(m, n) > 3$. If $m \in M_i$, $n \in M_j$ $(1 \leq i, j \leq 3)$, then by the Corollary 3 we have $i + j = 3$ and $d_G(m, n) = 3$. In accordance to Lemma 1 there are two m-n paths Q_1, Q_2 such that $\lambda(Q_1) = \lambda(Q_2) = 3$ and $(Q_1 \cap L_y)_e = \{xy\}$, $(Q_2 \cap L_y)_e = \{pq\}$.

We will distinguish the next cases:

1. $x \equiv m$, $n \in M_3$, $n \neq q$. Then $\lambda(Q_2) > 3$, a contradiction.

2. $m \in M_1$, $m \neq y$, $n \in M_2$, $n \neq p$. Then $mp, ny \in E$ and $d_{G-L_y}(m, n) = 3$, a contradiction.

3. $m \neq y$, $m \in M_1$, $n \equiv p$. If $d_{G-yp}(p, x) = 2$, then also $d_{G-L_y}(p, m) = 3$, a contradiction. So $d_{G-yp}(p, x) = 3$. In $G-yp$ there must exist a path pr_1r_2x, where $r_1, r_2 \in V$, $r_2 \in M_1$. By the assumption $r_1 \notin M_2$. If $r_1 \in M_1$ then $d_{G-yp}(p, x) = 2$, a contradiction. As follows from the definition of $\{M_i\}$, $r_1 \notin M_3$. So we have a contradiction.

Thus we have proved: either G is not a 2-critical graph, or for every $y \in S_p$ from $d_{G-L_y}(m, n) > 3$ it follows $m \equiv y$, $n \in M_2$ and $n \neq p$. We exclude this last case.

4. $m \equiv y$, $n \in M_2$, $n \neq p$. We will distinguish the following cases:

A. There are two points $v \in S_p$, $f \in M_1$ such that $vf \in E$. Put $v = y$. If $d_G(f, n) \leq 2$, then also $d_{G-L_y}(m, n) \leq 3$ which is a contradiction. If $d_G(f, n) = 3$, we get a contradiction with the Lemma 2, for the distance decomposition of V from n, because $yf \in E$.

B. For every point $v \in S_p$ from the $yz \in E$, $(z \in V)$ it follows $z \notin M_1$. There exists a point $f \in M_1$, $f \neq y$ such that $fn \in E$. Simultaneously $qn \in E$ (in the opposite case we have $\lambda(Q_2) > 3$, a contradiction). If $d_{G-pq}(p, n) \leq 2$, then also $d_{G-L_y}(p, n) \leq 2$. Hence $d_{G-L_y}(y, n) \leq 3$, a contradiction. Let $d_{G-pq}(p, n) = 3$. Then there exists a p-n path pr_1r_2n in $G - pq$, where $r_1r_2 \in V$. From the properties of p we have $r_1 \notin M_2$. Let $r_2 \in M_2$. Since $d_G(y, r_2) = 3$, we have a contradiction with the Lemma 2 for the distance decomposition of V from the point y $(r_2n \in E)$. So, $r_2 \notin M_2$. But then from the Lemma2 we have $r_1 \notin M_3$. Analogously $r_2 \notin M_3$. But then both r_1, r_2 must belong to M_1. Obviously

$r_1 \notin \{y, f\}$. Since $pr_1 \in E$, and $d_{G-L_{r_1}}(x, q) = 3$ (there is a path $xfnq$ in $G-L_{r_1}$), we have $r_1 \in S_p$. But this is a contradiction with the assumption, because $r_1 \in M_1$ and $r_1 r_2 \in E$.

So, there is no 2-critical graph with the diameter three. This completes the proof.

<u>L e m m a 6</u>. Let $G = (V, E)$ be a 3-critical graph with the diameter 4. Let $x, q \in V$, $d_G(x, q) = 4$. Let $\{M_i\}$ be the distance decomposition of V from the point x. For every two points $y, z \in M_1$, $y = z$, there is a point $p \in N_G(q)$ such that from $d_{G-L}(m, n) > 4$ $(m, n \in V)$ it follows $m, n \in M_2$, where $L = \{xy, xz, pq\}$.

<u>P r o o f</u>. (Fig. 5)

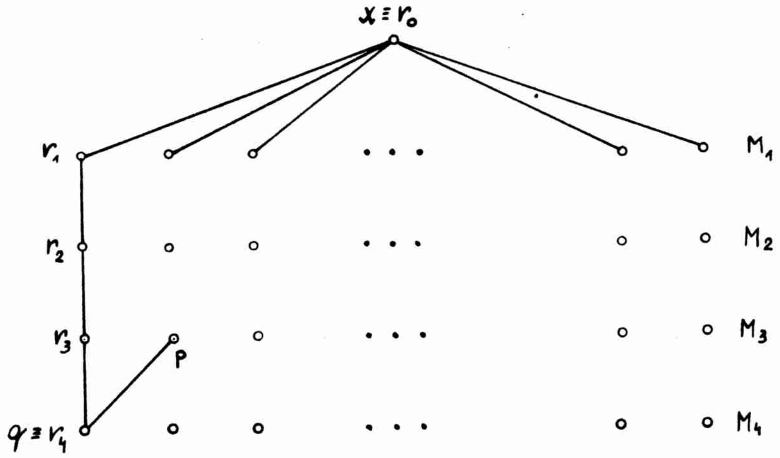

Fig. 5

Let $y, z \in M_1$, $y \neq z$. There is a point $r_1 \neq y, z$, $r_1 \in M_1$. $d_G(q, r_1) = 3$ (if $d_G(q, r_1) = 4$, then we have a contradiction with the Lemma 2 for the distance decomposition of V from q). So, there is an x-q path $P : r_0 \equiv x, r_1, r_2, r_3, r_4 \equiv q$, where $r_i \in M_i$ for $i = 0, 1, \ldots, 4$. Let $p \in N_G(q)$, $p \neq r_3$. Denote

$L = \{xy,\ xz,\ pq\}$. Let $m,\ n \in V$, $d_{G-L}(m,\ n) > 4$. If $m \in M_i$, $n \in M_j$, $0 \leq i,\ j \leq 4$, then by the Lemma 3 and Corollary 3 we have $i + j =$ $= d_G(m,\ n) = 4$. In accordance with Lemma 1 there are m-n paths $Q_1,\ Q_2,\ Q_3$ such that $\lambda(Q_1) = \lambda(Q_2) = \lambda(Q_3) = 4$ and $(Q_1 \cap L)_e = \{xy\}$, $(Q_2 \cap L)_e = \{xz\}$, $(Q_3 \cap L)_e = \{pq\}$.
We will distinguish the next cases:

1. $m \equiv x$, $n \in M_4$. If $n \equiv q$, then for the existence of P in $G-L$ we have $d_{G-L}(m,\ n) = 4$, a contradiction. If $n \neq q$, then $\lambda(Q_3) >$ > 4, a contradiction.

2. $m \in M_1$, $n \in M_3$. If $m \equiv y$, then $\lambda(Q_2) > 4$, a contradiction. If $m \equiv z$ then $\lambda(Q_1) > 4$, a contradiction. If $m \neq y,\ z$, then every m-n path which contains the line xy or xz must contain the line mx, too. But then $d_{G-\{xm,pq\}}(m,\ n) > 4$, a contradiction.

So, we have proved the lemma.

T h e o r e m 4. There is no k-critical graph with the diameter four, for $k \geq 3$.

P r o o f. (Fig. 6)

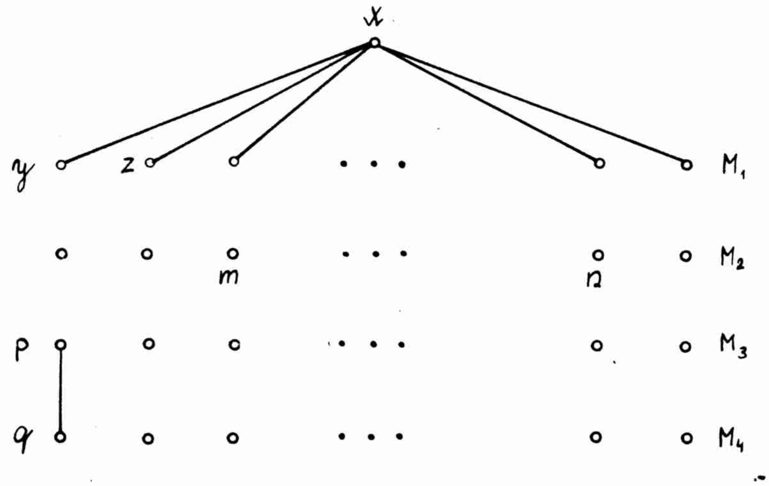

Fig. 6

We prove the theorem for $k = 3$. The assertion of the theorem then directly follows from the Corollary 2.

Let $G = (V, E)$ be a 3-critical graph with the diameter 4. Let $x, q \in V$, $d_G(x, q) = 4$. Let $\{M_i\}$ be the distance decomposition of V from the point x.

We will distinguish next two cases:

1. For every two points $y, z \in M_1$, $y \neq z$ there is a point $t \in M_2$ such that $ty, tz \in E$. Let p, L have the same meaning as in Lemma6. Then from $d_{G-L}(m, n) > 4$ it follows $m, n \in M_2$. In accordance with the definition of $\{M_i\}$ there are two points $f_1, f_2 \in \in M_1$, such that $f_1m, f_2n \in E$. Obviously $f_1 \neq f_2$. By the assumption there is a point $r \in M_2$, such that $rf_1, rf_2 \in E$. But then there is an m-n walk mf_1rf_2n in G-L. So, we have $d_{G-L}(m, n) \leq 4$ a contradiction.

2. There are two points $y, z \in M_1$ such that for every point $t \in M_2$, $\{ty, tz\} \not\subset E$. By the Lemma 6 there is a point $p \in N_G(q)$ such that from $d_{G-L}(m, n) > 4$ it follows $m, n \in M_2$, where $L = \{xy, xz, pq\}$. In accordance with Lemma 1 there are two m-n paths Q_1, Q_2 such that $\lambda(Q_1) = \lambda(Q_2) = 4$, and $(Q_1 \cap L)_e = \{xy\}$, $(Q_2 \cap L)_e = \{xz\}$.

Denote $S = \{rs \in E \mid r \in \{y, z\}, s \in \{m, n\}\}$. Obviously $|S| \leq 2$. If $|S| = 0$, then by the definition of $\{M_i\}$ there are two points $f_1, f_2 \in M_1$, $f_1, f_2 \notin \{y, z\}$, $f_1 \neq f_2$ such that $mf_1, nf_2 \in E$. Then we have $d_{G-L}(m, n) \leq 4$, a contradiction. Let $|S| = 1$. If $my \in E$, (or $ny \in E$), then $\lambda(Q_2) > 4$, a contradiction. If $mz \in E$ (or $nz \notin E$), then $\lambda(Q_1) > 4$, a contradiction. So, $|S| = 2$. Then $my, nz \in E$ (or $mz, ny \in E$, which is a symmetric case). If $d_G(m, M_1 - \{y, z\}) = d_G(n, M_1 - \{y, z\}) = 1$, then we have $d_{G-L}(m,n) \leq 4$, a contradiction. If $d_G(m, M_1 - \{y, z\}) \geq 2$, then $\lambda(Q_2) > 4$, L e m m a 7. Let $G = (V, E)$ be a k-critical graph with the diameter $d \geq 4$, and $k \geq 2$. Let $x, q \in V$, $d_G(x, q) = d$. Then there exists a 2d-angle in G which is an induced subgraph of G and contains the points x, q.

P r o o f. (Fig. 7)

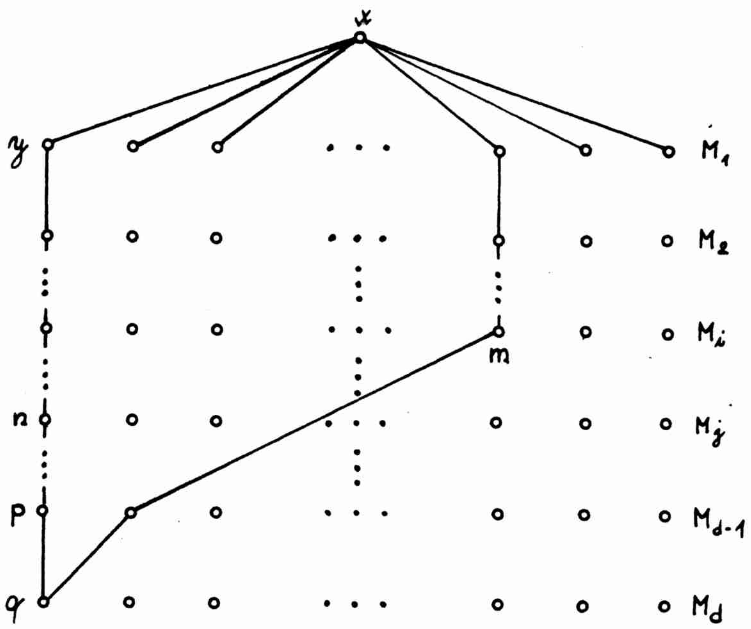

Fig. 7

Let $\{M_i\}$ be the distance decomposition of V from the point x. Since $\deg_G(x) \geq k$ and $\deg_G(q) \geq k$, there exists a subset $M \subset E$, $|M| = k$, such that the following holds:

1. Every line from M is incident with x or q.
2. There are $y \in M_1$, $p \in N_G(q)$ such that $\{xy, pq\} \subset M$.
3. $d_{G-M}(x, q) = d$.

Because G is a k-critical graph, there are two points $m, n \in V$, with $d_{G-M}(m, n) > d$. If $m \in M_i$, $n \in M_j$, $1 \leq i, j \leq d$, then by the Lemma 3 and Corollary 3 we have $d_G(m, n) = i + j = d$. Since $m, n \notin M_d$, then $i \geq 1$, $j \leq d-1$. We assume $i \leq j$ (the case $j \leq i$ can be proved analogically). In accordance with Lemma 1 there are m-n paths P, Q of the length $\lambda(P) = \lambda(Q) = d$, such that $(P \cap M)_e = \{xy\}$, $(Q \cap M)_e = \{pq\}$. Obviously $\lambda(P(m-x)) = i$, $\lambda(P(x-n)) = j$, $\lambda(Q(m-q)) = d - i$, $\lambda(Q(q-n)) = d - j$. We will show for P, Q to be point disjoint.

Let P, Q have the common point t, $t \in M_h$, $0 \leq h \leq d$. If $h < i$ then $t \notin Q$, a contradiction. If $h > j$, then $t \notin P$, a contradiction. If $h = i$, then $t \equiv m$, a contradiction.
If $h \equiv j$ then $t \equiv n$, a contradiction. So, we have $i < h < j$. Since $t \in P(x-n)$, $t \neq x, n$, the next inequality holds:

$$\lambda(P(t-n)) \leq \lambda(P(x-n)) - h = j - h$$

Analogically $t \in Q(m-q)$, $t \neq m, q$, and we have

$$\lambda(Q(m-t)) \leq \lambda(Q(m-q)) - d + h = h - i$$

By the concatenation of Q(m-t) and P(t-n) we get an m-n walk Y of the length $\lambda(Y) \leq j - i < d$. Since $q \notin Y$, $x \notin Y$ we have $d_{G-M}(m, n) < d$, a contradiction. So, P, Q are point-disjoint paths. By the concatenation of P and Q^{-1} we get the required 2d-angle C. We will show that C is an induced subgraph of G.

Let $v \in M_h$, $w \in M_r$, $0 \leq h, r \leq d$ such that $wv \in E$, $w, v \in C$, and $wv \notin C$. If $v \in P(m-x)$, $w \in P(x-n)$ or $v \in Q(m-q)$, $w \in Q(q-n)$, then we have $d_G(m, n) < d$, a contradiction. If $v \in P(x-n)$, $w \in Q(q-n)$ or $v \in P(m-x)$, $w \in Q(m-q)$, then $d_G(x, q) < d$, a contradiction. We will distinguish the next cases:

1. $v \in Q(q-n)$, $w \in P(m-x)$. By the concatenation of P(m-w), wv, and Q(v-n) we get an m-n walk Y of the length $\lambda(Y) = i-r+1+h-j$. Since $h - r \leq 1$ and $q \notin Y$, $x \notin Y$, we have $\lambda(Y) \leq 2+i-j \leq 2$, and also $d_{G-M}(m, n) \leq 2$, a contradiction.

2. $v \in P(x-n)$, $w \in Q(m-q)$. By the concatenation of $Q(m-w)$, wv, and $P(v-n)$ we get an $m-n$ walk Y of the length $\lambda(Y) = r - i + 1 + j - h$. Since $r - h \leq 1$, $i \geq 1$, we have $\lambda(Y) \leq d$. $w \neq q$, because in the opposite case $w \in Q(q-n)$, a contradiction. But then $q \notin Q(m-w)$. Obviously $q \notin P(x-n)$. So, $q \notin Y$. Analogously $x \notin Y$. Then $d_{G-M} \leq d$, a contradiction. This completes the proof.

L e m m a 8. Let k, $d \geq 2$, be natural numbers. Let $G = (V, E)$ be a k-critical graph with the diameter d. Let $x, y \in V$. Let $\mathcal{G}_G^{k,d}(x, y) = \{M \mid M \subset E \text{ and } |M| = k \wedge d_{G-M}(x, y) > d\}$. Then $|\mathcal{G}_G^{k,d}(x, y)| \leq d^k$.

P r o o f. Let $\mathcal{P}_{xy}(G) = \{P_1, P_2, \ldots, P_n\}$ (n is a natural number), be the set of all x-y paths in G of the length $\lambda(P_i) \leq d$ for $i = 1, 2, \ldots, n$.

We will prove the lemma by induction on k.

Let $k = 1$. If $\mathcal{G}_G^{1,d}(x, y) = \emptyset$, then the assertion of lemma trivially holds. Let $\mathcal{G}_G^{1,d}(x, y) \neq \emptyset$. Then $\mathcal{G}_G^{1,d}(x, y) = (\bigcap_{i=1}^{n} P_i)_e$ and we have $|\mathcal{G}_G^{1,d}(x, y)| \leq d$.

Let $k \geq 2$. If $\mathcal{G}_G^{k,d}(x, y) = \emptyset$, the assertion of lemma holds.

Let $\mathcal{G}_G^{k,d}(x, y) \neq \emptyset$. Let $R \in \mathcal{P}_{xy}(G)$ (since $d(G) = d$, $\mathcal{P}_{xy}(G)$ is nonempty set). For every $M \in \mathcal{G}_G^{k,d}(x, y)$ there is a line $e \in E$, such that $e \in (M \cap R)_e$ (because in the opposite case $d_{G-M}(x, y) \leq d$, a contradiction).

Let $\mathcal{M} = \{e \in E \mid e \in R$, and there is such $M \in \mathcal{G}_G^{k,d}(x, y)$ that $e \in M\}$.

Obviously $|\mathcal{M}| \leq d$. For every line $e \in \mathcal{M}$ we define:

$\mathcal{F}_e = \{M \in \mathcal{G}_G^{k,d}(x, y) \mid e \in M\}$. Then $\mathcal{F}_e \subset \mathcal{G}_G^{k,d}(x, y)$

Since for every set $M \in \mathcal{F}_e$ we have $M - \{e\} \in \mathcal{G}^{k-1,d}_{G-e}(x, y)$ and $|\mathcal{G}^{k-1,d}_{G-e}(x, y)| \leq d^{k-1}$ (by the induction assumption), we get $|\mathcal{F}_e| \leq d^{k-1}$. Then

$$|\mathcal{G}^{k,d}_G(x, y)| \leq \sum_{e \in \mathcal{M}} |\mathcal{F}_e| \leq \lambda(R).d^{k-1} \leq d.d^{k-1} = d^k$$

This completes the proof.

Theorem 5. Let $k, d \geq 2$ be natural numbers. Let $G = (V, E)$ be a k-critical graph with the diameter d. Then

$$\binom{m}{k} \leq \binom{n}{2} d^k - md^k + md^{k-1}$$

where $m = |E|$, $n = |V|$.

Proof. Let $\mathcal{Q}_G(k, d) = \bigcup_{x,y \in V} \mathcal{G}^{k,d}_G(x, y)$. Since (by the Lemma 8) $|\mathcal{G}^{k,d}_G(x, y)| \leq d^k$, we have $|\mathcal{Q}_G(k, d)| \leq \binom{n}{2} d^k$. But simultaneously $\mathcal{Q}_G(k, d) = \{M \subset E \mid |M| = k\}$. So we have $\binom{m}{k} \leq \binom{n}{2} d^k$. Since $G - xy$ is a (k-1)-critical graph with the diameter d, the inequality $|\mathcal{G}^{k-1,d}_{G-xy}(x, y)| \leq d^{k-1}$ holds for every line $xy \in E$. Then also $|\mathcal{G}^{k,d}_G(x, y)| \leq d^{k-1}$ for every $xy \in E$. So, $\binom{m}{k} \leq \binom{n}{2} d^k - md^k + md^{k-1}$. This completes the proof.

Lemma 9. Let k, d be natural numbers, $k \geq d \geq 10$. Then there is no k-critical graph with the diameter d.

Proof. We will prove the lemma for the case $k = d$. The assertion of lemma then immediately follows from the Corollary 2. Let $G = (V, E)$ be a k-critical graph with the diameter $k \geq 10$. Denote $|V| = n$, $|E| = m$. Then from the Teorem 5 we have

$$\binom{m}{k} \leq \binom{n}{2} k^k \tag{1}$$

Since (by the Assertion 2), $m \geq \frac{kn}{2}$, the next inequality is true:

$$nk\left(\frac{nk}{2} - 1\right) \ldots \left(\frac{nk}{2} - k + 1\right) \leqslant n(n-1)k!\, k^k \qquad (2)$$

Then also

$$nk\left(\frac{nk}{2} - k + 1\right)^{k-1} \leqslant k!\, n(n-1)k^k$$

For $k > 2$ we have $\frac{nk}{2} > n > k-1$. Hence

$$nk\left(\frac{nk}{2} - n\right)^{k-1} \leq k!\, n(n-1)k^k$$

and also

$$\frac{n^{k-1}}{n-1} \leqslant k!\, \left(\frac{2k}{k-2}\right)^{k-1}$$

Since $n^{k-2} = \frac{n^{k-1}}{n} < \frac{n^{k-1}}{n-1}$, we have

$$n^{k-2} < k!\, \left(\frac{2k}{k-2}\right)^{k-1}$$

Because (by the Lemma 7), $n > 2k$, the next inequality holds:

$$(2k)^{k-2} \leq k!\, \left(\frac{2k}{k-2}\right)^{k-1}$$

and also

$$1 < \frac{k!\; 2k}{(k-2)^{k-1}} \qquad (3)$$

For $k = 10$ the inequality (3) does not hold. We will show that the right side of the inequality (3), is not increasing in k. For $k \geqslant 10$, the next inequality is true:

$$2\left(1 + \frac{1}{k-2}\right)(k-1) \geqslant \left(1 + \frac{1}{k}\right)(k+1) \qquad (4)$$

because it is equivalent with the inequality $k^3 - 4k^2 + 5k + 2 \geqslant 0$, and for $k \geqslant 10$ we have $k^3 - 4k^2 + 5k + 2 > 0$. Simultaneously, for $k \geqslant 10$ we obtain

$$(1 + \frac{1}{k-2})^{k-2} \geqslant (1 + \frac{1}{8})^8 \geqslant 2 \qquad (5)$$

since the left side of the inequality (5) is increasing function in k. Then also

$$(1 + \frac{1}{k-2})^{k-1} (k - 1) \geqslant 2(1 + \frac{1}{k-2})(k - 1) \qquad (6)$$

From (6) and (4) then we have

$$(1 + \frac{1}{k-2})^{k-1} (k - 1) \geqslant (1 + \frac{1}{k})(k + 1)$$

and also

$$\frac{k}{(k-2)^{k-1}} \geqslant \frac{(k+1)^2}{(k-1)^k}$$

Hence

$$\frac{2k\ k!}{(k-2)^{k-1}} \geqslant \frac{2(k+1)(k+1)!}{(k-1)^k}$$

So we have a contradiction, because the inequality (3) does not hold for $k \geqslant 10$. We have proved that for $k = d \geqslant 10$, there is no k-critical graph with the diameter k. This completes the proof of lemma.

<u>Theorem 6</u>. Let k, d be natural numbers, $k \geqslant d \geqslant 2$. Then there is no k-critical graph with the diameter d.

<u>Proof</u>. The assertion of theorem for $d = 2, 3, 4$ follows from Theorems 2, 3, 4.

For $d \geqslant 10$ the assertion follows from Lemma 9. From the inequality (2) (Lemma 9) it follows:

1. if $k = 5$ then $n \leqslant 13$
2. if $k = 6$ then $n \leqslant 13$

3. if $k = 7$ then $n \leqslant 13$
4. if $k = 8$ then $n < 16$
5. if $k = 9$ then $n < 18$.

Let $5 \leqslant k = d \leqslant 9$. Let $G = (V, E)$ be a k-critical graph with the diameter d. By the Lemma 7 there is a 2k-angle in G, which is an induced subgraph of G. Denote the points of this 2k-angle by $x_1, x_2, \ldots, x_{2k} \equiv x_1$.

Since 2k-angle is 1-critical graph there must be $|V| = n > 2k$. This excludes the cases 3, 4, 5.

By the Assertion 2, $\deg_G(x_i) \geqslant k$ for $i = 1, 2, \ldots, 2k$. This excludes the cases 1, 2.

So, there is no k-critical graph with the diameter d, for $5 \leqslant k = d \leqslant 9$. By the Corollary 2 then there is no k-critical graph with the diameter d for $k \geqslant d$ and $5 \leqslant d \leqslant 9$.

One can see from the table in the Fig. 8 that the following cases remain open:

1. $d \geqslant 5$ and $2 \leqslant k \leqslant d$
2. $k = 2$ and $d = 4$.

We conjecture that there is no k-critical graph with the diameter d, for $k \geqslant 2$ and $d \geqslant 2$.

The symbol "E" indicates that there exists a k-critical graph with the diameter d. The symbol "-" indicates the opposite case.

k \ d	1	2	3	4	5	6	...
1	E	E	E	E	E	E	...
2	–	–	–				...
3	–	–	–	–			...
4	–	–	–	–			...
5	–	–	–	–	–		...
6	–	–	–	–	–	–	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Fig. 8

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S ú h r n

k-KRITICKÉ GRAFY S DANÝM PRIEMEROM

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Autor sa v práci zaoberá kritickými grafmi, vzhľadom na priemer grafu. Vyšetruje otázky existencie k-kritických grafov s daným priemerom, t.j. grafov (neorientovaných, bez slučiek a násobných hrán), v ktorých vynechním ľubovoľnej podmnožiny hrán grafu dostaneme graf s tým istým priemerom ako mal pôvodný graf práve vtedy, keď $|M| < k$. Je vyslovená hypotéza, že pre $k \geq 2$ také grafy neexistujú. Platnosť hypotézy je v práci dokázaná približne pre polovicu všetkých možných prípadov. Otázku existencie k-kritických grafov v ostatných prípadoch autor uvádza ako otvorený problém.

Р е з ю м е

ДИАМЕТРАЛЬНО к-КРИТИЧЕСКИЕ ГРАФЫ

Петер Кыш, Братислава

Автор в работе занимается критическими графами по диаметру. Он занимается вопросами существования так называемых к-критических графов. к-критическим называется граф /неориентированный, без петель и кратных ребер/, в котором после удаления любого подмножества M множества ребер, получим граф с диаметром равным диаметру основного графа тогда и только тогда, когда $|M| < к$. Приблизительно для одной половины всех возможных случаев доказано, что такие графы несуществуют. Открытым является вопрос существования графов для остальных случаев.

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ОБ ОДНОЙ КВАЗИОРТОГОНАЛЬНОЙ СИСТЕМЕ ВЕКТОРОВ В ТЕОРИИ ДЗЕТА-ФУНКЦИИ РИМАНА

ЯН МОЗЕР, Братислава

1. Пусть ([4] , стр. 94)

$$Z(t)=e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right) \tag{1}$$

где ([4] , стр. 383)

$$\vartheta(t)=-\frac{1}{2}t\ln\pi+\operatorname{Im}\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)=$$

$$=\frac{1}{2}t\ln\frac{t}{2\pi}-\frac{1}{2}t-\frac{1}{8}\pi+O\left(\frac{1}{t}\right). \tag{2}$$

Пусть, дальше, $\{t_\nu\}$ обозначает последовательность значений определенных соотношением

$$\vartheta(t_\nu)=\pi\nu, \tag{3}$$

где ν — целое положительное.

В работе [2] , исходя из формулы Римана-Зигеля ([3] , стр. 60, [4] , стр. 94)

$$Z(t)=2\sum_{m\leq\sqrt{\frac{t}{2\pi}}}\frac{1}{\sqrt{m}}\cos(\vartheta-t\ln m)+O(t^{-1/4}), \tag{4}$$

мы изучали семейство сумм

$$E(\tau;T,H)=\sum_{T\leq t_\nu\leq T+\overline{H}}Z(t_\nu)Z(t_\nu+\tau),\ \tau\neq 0, \tag{5}$$

где

$$\overline{H}=\sqrt{T}\,\psi(T), \tag{6}$$

[$\psi(T)$ - сколь угодно медленно возрастающая к $+\infty$ функция].

Продолжая изучение сумм (5), мы обнаружили одну закономерность, касающуюся совокупности значений функции $Z(t)$, относительно отрезка определенной арифметической последовательности.

А именно, имеет место

Теорема. Соотношение

$$\overset{*}{Z}(g_\nu) \sim \frac{2}{\pi} \sum_{p=1}^{L} \frac{(-1)^{p+1}}{2p-1} \left\{ \overset{*}{Z}(g_\nu + \bar{\tau}_p) + \overset{*}{Z}(g_\nu - \bar{\tau}_p) \right\}, \tag{7}$$

$$L \sim \{\psi(T)\}^{\varepsilon}, \quad 0 < \varepsilon < 1,$$

где

$$g_\nu = t_\nu + \frac{\pi}{\ln \frac{T}{2\pi}}, \quad \bar{\tau}_p = \frac{(2p-1)\pi}{\ln \frac{T}{2\pi}}, \quad \overset{*}{Z}(t) = \frac{Z(t)}{\sqrt{\ln \frac{T}{2\pi}}}, \tag{8}$$

имеет место для количества значений $g_\nu \in \langle T, T+U \rangle$ порядка $\sim \frac{1}{2\pi} \bar{H} \ln T$.

Содержание этой теоремы выразим еще с точки зрения интерполяции (в случае равноотстоящих узлов). Именно, если считать значения

$$\{\overset{*}{Z}(g_\nu \pm \bar{\tau}_p)\}_{p=1}^{L}$$

известными, то (вообще говоря) значение $\overset{*}{Z}(g_\nu)$ хорошо приближается формулой (7). При этом заметим (см. (8)), что совокупность

$$\{g_\nu \pm \bar{\tau}_p\}_{p=1}^{L}$$

представляет отрезок определенной арифметической последовательности.

2. В 6 части этой работы мы покажем, что имеет место

Лемма. Если $\tau', \tau'' = O\left(\frac{\psi^{\varepsilon}}{\ln T}\right)$, $0 < \varepsilon$, то, $(\tau' \neq \tau'')$,

$$\sum_{T \leq t_\nu \leq T + \bar{H}} Z(t_\nu + \tau') Z(t_\nu + \tau'') =$$

$$= \frac{1}{2\pi} \frac{\sin[(\tau'' - \tau') \ln P_0]}{(\tau'' - \tau') \ln P_0} \bar{H} \ln^2 \frac{T}{2\pi} + O(\sqrt{T} \ln^2 T), \tag{9}$$

$$\sum_{T \leqq t_\nu \leqq T + \bar{H}} Z^2(t_\nu + \tau') = \frac{\bar{H}}{2\pi} \ln^2 \frac{T}{2\pi} + O(\sqrt{T} \ln^2 T), \tag{10}$$

где $P_0 = \sqrt{\frac{T}{2\pi}}$.

Напомним ([1] , (23)), что

$$Q = Q(T, \bar{H}) = \sum_{T \leqq t_\nu \leqq T + \bar{H}} 1 = \frac{1}{2\pi} \bar{H} \ln \frac{T}{2\pi} + O\left(\frac{\bar{H}^2}{T}\right),$$

т.е. в силу (6),

$$Q \sim \frac{1}{2\pi} \bar{H} \ln \frac{T}{2\pi}, \tag{11}$$

и, положим

$$\tau_p = \frac{2\pi}{\ln \frac{T}{2\pi}} p, \quad p = -L+1, \dots, -1, 0, 1, \dots, L. \tag{12}$$

Теперь, в силу (9), (12), $(p \neq p')$,

$$\sum_{T \leqq t_\nu \leqq T + \bar{H}} Z(t_\nu + \tau_p) Z(t_\nu + \tau_{p'}) = O(\sqrt{T} \ln^2 T), \tag{13}$$

и, в силу (10),

$$\sum_{T \leqq t_\nu \leqq T + \bar{H}} Z^2(t_\nu + \tau_p) = \frac{\bar{H}}{2\pi} \ln^2 \frac{T}{2\pi} + O(\sqrt{T} \ln^2 T). \tag{14}$$

Следовательно, в силу (5), (11), (13), (14), (см. (8)),

$$\frac{1}{Q} \sum_{T \leqq t_\nu \leqq T + \bar{H}} \overset{*}{Z}(t_\nu + \tau_p) \overset{*}{Z}(t_\nu + \tau_{p'}) =$$

$$= \begin{cases} 1 + O\left(\frac{1}{\psi}\right), & p = p', \\ O\left(\frac{1}{\psi}\right), & p \neq p'. \end{cases} \tag{15}$$

Совокупность значений

$$\overset{*}{Z}(t_\nu),\quad T\leq t_\nu\leq T+\overline{H},$$

назовем вектором.

<u>Примечание 1.</u> В силу (15) естественно назвать систему векторов

$$\overset{*}{Z}(t_\nu+\tau_p),\quad T\leq t_\nu\leq T+H,\qquad p=-L+1,\dots,-1,0,1,\dots,L, \tag{16}$$

квазиортогональной.

План изложения таков: в частях 3 - 5 , предполагая лемму доказанной, мы завершим доказательство теоремы, в 6 части мы докажем лемму.

3. В этой части мы получим "коэффициенты Фурье" вектора

$$\overset{*}{Z}(t_\nu+\tau^0),\ \tau^0=\frac{\pi}{\ln\frac{T}{2\pi}},\quad T\leq t_\nu\leq T+\overline{H}. \tag{17}$$

Прежде всего заметим, что (см. (12))

$$\tau_p-\tau^0=\frac{(2p-1)\pi}{\ln\frac{T}{2\pi}}=\left(\pi p-\frac{\pi}{2}\right)\frac{1}{\ln P_0}, \tag{18}$$

и,

$$\frac{\sin[(\tau_p-\tau^0)\ln P_0]}{(\tau_p-\tau^0)\ln P_0}=\frac{\pi}{2}\,\frac{(-1)^{p+1}}{2p-1}. \tag{19}$$

Следовательно, в силу (11), (19)

$$A_p=\frac{1}{Q}\sum_{T\leq t_\nu\leq T+\overline{H}}\overset{*}{Z}(t_\nu+\tau^0)\overset{*}{Z}(t_\nu+\tau_p)=$$

$$=\frac{2}{\pi}\,\frac{(-1)^{p+1}}{2p-1}+O\left(\frac{1}{\Psi}\right). \tag{20}$$

<u>Примечание 2.</u> Величины A_p естественно назвать коэффициентами Фурье вектора $\overset{*}{Z}(t_\nu+\tau^0)$, $T\leq t_\nu\leq T+\overline{H}$, относительно системы векторов (16). Числа

$$\overline{A}_p = \frac{2}{\pi}\frac{(-1)^{p+1}}{2p-1} \tag{21}$$

назовем асимптотическими коэффициентами Фурье.

Заметим, что из (21) получается соотношение

$$\overline{A}_p = \overline{A}_{-p+1}. \tag{22}$$

Наконец (используя (22)) положим

$$P_{2L}[\overset{*}{Z}(t_\nu+\tau^0)] = \sum_{p=-L+1}^{L} \overline{A}_p \overset{*}{Z}(t_\nu+\tau_p) =$$

$$= \sum_{p=1}^{L} \overline{A}_p \overset{*}{Z}(t_\nu+\tau_p) + \sum_{p=1}^{L} \overline{A}_{1-p} \overset{*}{Z}(t_\nu+\tau_{1-p}) =$$

$$= \sum_{p=1}^{L} \overline{A}_p \{\overset{*}{Z}(t_\nu+\tau_p) + \overset{*}{Z}(t_\nu+\tau_{1-p})\} =$$

$$= \frac{2}{\pi}\sum_{p=1}^{L} \frac{(-1)^{p+1}}{2p-1}\{\overset{*}{Z}(t_\nu+\tau_p) + \overset{*}{Z}(t_\nu+\tau_{1-p})\}. \tag{23}$$

4. В этой части мы вычислим среднеквадратичное уклонение

$$\delta^2 = \frac{1}{Q}\sum_{T \leqq t_\nu \leqq T+\overline{H}} \{\overset{*}{Z}(t_\nu+\tau^0) - P_{2L}\}^2. \tag{24}$$

Прежде всего

$$\delta^2 = \frac{1}{Q}\sum_{(t_\nu)} \overset{*}{Z}{}^2(t_\nu+\tau^0) - \frac{2}{Q}\sum_{(t_\nu)} \overset{*}{Z}(t_\nu+\tau^0) P_{2L} +$$

$$+ \frac{1}{Q}\sum_{(t_\nu)} (P_{2L})^2 = W_1 + W_2 + W_3. \tag{25}$$

В силу (10), (11),

$$W_1 = 1 + O\left(\frac{1}{\psi}\right), \tag{26}$$

дальше, в силу (20), (23),

$$W_2 = -\frac{2}{Q} \sum_{(t_\nu)} \overset{*}{Z}(t_\nu + \tau^0) \sum_{p=-L+1}^{L} \bar{A}_p \overset{*}{Z}(t_\nu + \tau_p) =$$

$$= -2 \sum_{p=-L+1}^{L} \bar{A}_p \cdot \frac{1}{Q} \sum_{(t_\nu)} \overset{*}{Z}(t_\nu + \tau^0) \overset{*}{Z}(t_\nu + \tau_p) =$$

$$= -2 \sum_{p=-L+1}^{L} \bar{A}_p A_p, \tag{27}$$

и, в силу (15), (23)

$$W_3 = \sum_{p,q=-L+1}^{L} \bar{A}_p \cdot \bar{A}_q \cdot \frac{1}{Q} \sum_{(t_\nu)} \overset{*}{Z}(t_\nu + \tau_p) \overset{*}{Z}(t_\nu + \tau_q) =$$

$$= \sum_{p=-L+1}^{L} \bar{A}_p^2 \left(1 + O\left\{\frac{1}{\psi}\right\}\right) + O\left(\frac{1}{\psi} \sum_{\substack{p,q=-L+1 \\ p \neq q}}^{L} |\bar{A}_p \bar{A}_q|\right). \tag{28}$$

Так как (известный ряд Эйлера)

$$\sum_{p=1}^{\infty} \frac{1}{(2p-1)^2} = \frac{\pi^2}{8}, \tag{29}$$

то, в силу (21), (22), (напомним, что $L \sim \psi^{\varepsilon}$)

$$-2 \sum_{p=-L+1}^{L} \bar{A}_p A_p = -2 \sum_{p=-L+1}^{L} \bar{A}_p^2 + O\left(\frac{1}{\psi} \sum_{p=-L+1}^{L} 1\right) =$$

$$= -2 \sum_{p=-L+1}^{L} \bar{A}_p^2 + O\left(\frac{1}{\psi^{1-\varepsilon}}\right), \tag{30}$$

$$\frac{1}{\psi} \sum_{p=-L+1}^{L} \bar{A}_p^2 = O\left(\frac{1}{\psi} \sum_{p=1}^{\infty} \bar{A}_p^2\right) = O\left(\frac{1}{\psi}\right), \tag{31}$$

$$\sum_{p,q=-L+1}^{L} |\bar{A}_p \bar{A}_q| = O\left(\sum_{p=1}^{\infty} \bar{A}_p^2\right) = O(1), \tag{32}$$

и, наконец

$$\sum_{p=-L+1}^{L} A_p^2 = \frac{4}{\pi^2} \sum_{p=-L+1}^{L} \frac{1}{(2m-1)^2} = \frac{8}{\pi^2} \sum_{p=1}^{L} \frac{1}{(2m-1)^2} =$$

$$= \frac{8}{\pi^2}\left[\frac{\pi^2}{8} - \sum_{p=L+1}^{\infty} \frac{1}{(2m-1)^2}\right] = 1 + O\left(\frac{1}{L}\right) = 1 + O\left(\frac{1}{\psi^{\varepsilon}}\right). \tag{33}$$

В силу приведенного выше,

$$W_2 = -2 + O\left(\frac{1}{\psi^{\varepsilon}}\right), \quad W_3 = 1 + O\left(\frac{1}{\psi^{\varepsilon}}\right). \tag{34}$$

Теперь, из (25), в силу (26), (34) мы получаем

$$\delta^2 < \frac{A}{\psi^{\varepsilon}}. \tag{35}$$

5. Пусть R обозначает количество значений $\bar{t}_\nu \in \langle T, T+H \rangle$, для которых

$$\left|\overset{*}{Z}(\bar{t}_\nu + \tau^0) - P_{2L}[\overset{*}{Z}(\bar{t}_\nu + \tau^0)]\right| \geq \frac{1}{\psi^{\varepsilon/4}}. \tag{36}$$

Имеет место

$$R = o(\bar{H} \ln T). \tag{37}$$

Действительно. Пусть

$$R > A\bar{H}\ln T. \tag{38}$$

Тогда, в силу (24), (35), (36), (38), (11),

$$\frac{A}{\psi^{\varepsilon}} > \frac{1}{Q}\sum_{(\bar{t}_\nu)}\{\overset{*}{Z}(\bar{t}_\nu+\tau^0) - P_{2L}[\overset{*}{Z}(\bar{t}_\nu+\tau^0)]\}^2 >$$

$$> \frac{A\bar{H}\ln T}{Q}\frac{1}{\psi^{\varepsilon/2}} > \frac{A}{\psi^{\varepsilon/2}}, \tag{39}$$

что противоречиво. Следовательно, в силу (23), (36), (37), для количества значений $t_\nu \in \langle T, T+\bar{H}\rangle$ порядка $\sim\frac{1}{2\pi}\bar{H}\ln T$, имеет место

$$\overset{*}{Z}(t_\nu+\tau^0) \sim \frac{2}{\pi}\sum_{p=1}^{L}\frac{(-1)^{p+1}}{2p-1}\{\overset{*}{Z}(t_\nu+\tau_p) +$$

$$+\overset{*}{Z}(t_\nu+\tau_{1-p})\}. \tag{40}$$

Наконец, полагая

$$t_\nu+\tau^0 = g_\nu,\ \tau_p - \tau_0 = \frac{(2p-1)\pi}{\ln\frac{T}{2\pi}} = \bar{\tau}_p,$$

$$\tau_{1-p} - \tau^0 = -\frac{(2p-1)\pi}{\ln\frac{T}{2\pi}} = -\bar{\tau}_p, \tag{41}$$

получается (7).

6. Теперь мы покажем, что имеет место лемма, приведенная нами в части 2 этой работы. Для этого достаточно привести здесь соотношения, родственные приведенным в работе [2] .

Прежде всего, вместо соотношения [2] , (51) мы имеем

$$Z(t_\nu+\tau') = 2(-1)^\nu\sum_{m<P_0}\frac{1}{\sqrt{m}}\cos\left(t_\nu\ln m - \tau'\ln\frac{P_0}{m}\right)+$$

$$+O\{T^{-1/4}(\ln T)^{q-1}\psi^{\varepsilon}\},\ H\leq\sqrt{T}\,(\ln T)^{q},\ P_0=\sqrt{\frac{T}{2\pi}}, \tag{42}$$

[сомножитель ψ^{ε} в O - члене происходит от условия леммы $\tau'=O\left(\frac{\psi^{\varepsilon}}{\ln T}\right)$, которое теперь используется вместо соотношения $\tau'=O\left(\frac{1}{\ln T}\right)$, ср. [2], (11)], и, аналогичное соотношение для $Z(t_\nu+\tau'')$. Дальше, вместо [2], (54), мы имеем

$$\sum_{T\leq t_\nu\leq T+H} Z(t_\nu+\tau')Z(t_\nu+\tau'')=$$

$$=2\sum_{(m,n)}\sum\frac{1}{\sqrt{mn}}\sum_{(t_\nu)}\cos\left\{t_\nu\ln(mn)-\tau'\ln\frac{P_0}{n}-\right.$$

$$\left.-\tau''\ln\frac{P_0}{m}\right\}+$$

$$+2\sum_{(m,n)}\sum\frac{1}{\sqrt{mn}}\sum_{(t_\nu)}\cos\left\{t_\nu\ln\frac{n}{m}-\tau'\ln\frac{P_0}{n}+\right.$$

$$\left.+\tau''\ln\frac{P_0}{m}\right\}+O\{\bar{H}\,T^{-1/12}(\ln T)^{q+1}\psi^{\varepsilon}\}=$$

$$=S_1+S_2+O\{T^{5/12}(\ln T)^{2q+1}\psi^{\varepsilon}\}. \tag{43}$$

Теперь, вместо [2], (55),

$$S_1(m=n=1)=2\sum_{(t_\nu)}\cos[(\tau''+\tau')\ln P_0]=$$

$$=O(H\ln T)=O\{\sqrt{T}(\ln T)^{q+1}\}, \tag{44}$$

вместо [2], (56),

$$S_2(m=n)=2\sum_{(m)}\frac{1}{m}\sum_{(t_\nu)}\cos\left[(\tau''-\tau')\ln\frac{P_0}{m}\right]=$$

$$= 2 \sum_{(t_\nu)} 1 \cdot \sum_{(m)} \frac{1}{m} \cos\left[(\tau'' - \tau') \ln \frac{P_0}{m}\right], \tag{45}$$

и, следовательно, (ср. [2] , (60))

$$S_2(m = m) = \frac{1}{2\pi} \frac{\sin[(\tau'' - \tau') \ln P_0]}{(\tau'' - \tau') \ln P_0} H \ln^2 \frac{T}{2\pi} +$$

$$+ O\{\sqrt{T} (\ln T)^{\ell+1}\} \tag{46}$$

Применяя метод ван дер Корпута-Титчмарша ([5], [2], (62)) получается

$$S_1(m = m > 1),\ S_1(m \neq m),\ S_2(m \neq m) = O(\sqrt{T} \ln^2 T) \tag{47}$$

Собирая перечисленные соотношения при $\overline{H} = \sqrt{T}\ \psi(T)$, (т.е. $\ell = 1$), получается утверждение (9).

Наконец, переходя в соотношении (9) к пределу $\tau'' \to \tau'$, (что, в силу (43) - (47) допустимо), мы получаем (10).

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Súhrn

O JEDNEJ KVAZIORTOGONÁLNEJ SÚSTAVE VEKTOROV V TEÓRII RIEMANNOVEJ DZETA-FUNKCIE

Ján Moser, Bratislava

Nech $Z(t)=e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right)$, $\vartheta(t)=\frac{t}{2}\ln\frac{t}{2\pi}-\frac{t}{2}-\frac{\pi}{8}+O\left(\frac{1}{t}\right)$ a $\{t_\nu\}$ označuje postupnosť definovanú vzťahom $\vartheta(t_\nu)=\pi\nu$, /ν - prirodzené číslo/. V tejto práci, vychádzajúc z Riemann-Siegelovho vzorca

$$Z(t)=2\sum_{m\leq\sqrt{\frac{t}{2\pi}}}\frac{1}{\sqrt{m}}\cos(\vartheta-t\ln m)+O(t^{-1/4}),$$

zavádzame istú kvaziortogonálnu sústavu vektorov v teórii $\zeta(s)$.

Summary

ON CERTAIN QUASIORTOGONAL SYSTEM OF VECTORS IN THE THEORY OF THE RIEMANN ZETA-FUNCTION

Ján Moser, Bratislava

Let $Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$, $\vartheta(t) = \frac{t}{2}\ln\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O(\frac{1}{t})$

and $\{t_\nu\}$ denotes the seqence defined with the relation $\vartheta(t_\nu) = \pi\nu$,/ ν is an positive integer/. In this paper with respect to Riemann-Siegel formula

$$Z(t) = 2\sum_{m\leqq\sqrt{\frac{t}{2\pi}}} \frac{1}{\sqrt{m}}\cos(\vartheta - t\ln m) + O(t^{-1/4}),$$

we define certain quasiorthogonal system of vectors in the theory $\zeta(s)$.