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# ON MEASURABILITY OF BINARY RELATIONS

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In this paper two generalization of Theorem 6.2 of the paper [1] are given. We prove measurability of a relation defined from a function of two variables which is measurable in one variable and quasicontinuous or regular with respect to a P-system in the other one.

## DEFINITIONS AND BASIC PROPERTIES

Throughout this paper we consider a binary relation F as a subset of T x X (notation $F : T \longrightarrow X$), F being called a relation from T to X. For a relation F and any set M

$$F[M] = \{x \mid \exists t \in M : (t,x) \in F\}.$$

Instead of $F[\{t\}]$ we shall write $F(t)$. If $F^{-1}$ is the inverse relation to F then

$$F^{-1}[M] = \{t \in T \mid F(t) \cap M \neq \emptyset\}.$$

Let $2^M$ be the family of all subsets of M. The following definition generalizes measurability of functions.

D e f i n i t i o n 1. Consider a relation $F : T \longrightarrow X$, $\mathcal{M} \subset 2^T$, $\mathcal{N} \subset 2^X$. F is said to be $\mathcal{M}-\mathcal{N}$-measurable relation iff

$$\forall E \in \mathcal{N} : F^{-1}[E] \in \mathcal{M}$$

If the family $\mathcal{M}$ (or $\mathcal{N}$ ) is fixed we say simply that F is

$\mathcal{N}$-measureble (or $\mathcal{M}$-measurable, respectively).

In the following we shall call attention to the special case when a structure of a measurable space is given by means of a $\sigma$-algebra $\mathcal{A} \subset 2^T$ covering T and topological structure is defined on X.

Definition 2. Let $\mathcal{A}$ be a $\sigma$-algebra on T and X a topological space. The relation $F : T \longrightarrow X$ is said to be measurable (weakly measurable, B-measurable) iff $F^{-1}[M] \in \mathcal{A}$ for every closed (open, Borel) subset M of X.

Remark 1. Since for any sets A, B the inclusions

$$F^{-1}[A \cap B] \subset F^{-1}[A] \cap F^{-1}[B], \quad F^{-1}[A \setminus B] \supset F^{-1}[A] \setminus F^{-1}[B]$$

are in general strict, three types of measurability introduced above need not coicide. Some connections are shown by the next theorem.

Theorem 1. Let $F : T \longrightarrow X$ be a relation. Then B-measurability of F implies both measurability and weak measurability. If in X every open set is of the type $\mathcal{F}_\sigma$, then measurability implies weak measurability.

Proof. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{B}$ be the family of all closed, open, Borel subsets of the topological space X, respectively. The first part of the theorem follows from $\mathcal{F}, \mathcal{G} \subset \mathcal{B}$. When $O \in \mathcal{G}$ is a $\mathcal{F}_\sigma$-set in X and F is measurable, then

$$O = \bigcup_{n=1}^{\infty} A_n, \quad A_n \in \mathcal{F}$$

and therefore

$$F^{-1}[O] = \bigcup_{n=1}^{\infty} F^{-1}[A_n]$$

The example given below indicates that in general other implications are not true.

E x a m p l e. Let $\mathcal{A} \neq 2^T$ be a $\sigma$-algebra on $T$ and $(X, \mathcal{T})$ a topological space. If $\{a\} \subset X$ is closed and not open we can define a relation $F : T \to X$ as follows. Choose a non-measurable set $N \in 2^T$ and put

$$F^{-1}(a) = N$$

For any open set $O \subset X$ we define a relation $R(O) = T \times O$ in the case $a \notin O$. If $a \in O$, then we put $R(O) = T \times \{p\}$, where $p \in O \setminus \{a\}$. If we define

$$F = N \times \{a\} \cup \bigcup \{R(O) : O \in \mathcal{T}\},$$

then $F$ is weakly measurable since the inverse image of any open set is the set $T \in \mathcal{A}$. As $\{a\}$ is closed and $N \notin \mathcal{A}$, $F$ fails to be measurable.

If we demand, in the preceding example, $\{a\}$ to be open and not closed and $O$ to be closed, then we obtain the relation which is measurable but is not weakly measurable. Summarizing this example and Theorem 1 we have that weak measurability does not imply B-measurability and measurability fails to imply B-measurability either. A more interesting example is given in [1] p. 58 which shows that measurability fails to imply B-measurability even if $X$ is a topological $T_1$-space.

R e m a r k 2. If $F$ is a function, these three types of measurability are equivalent since the inclusions in Remark 1 become equalities.

SPECIAL TYPES OF RELATIONS

In this section we are going to consider some special relations defined from a function of two variables. Let $f : T \times X \longrightarrow Y$ be a function whose domain is the whole set $T \times X$. For each $t \in T$, we define the t-section of $f$ as a function $f_t : X \to Y$, $x \mapsto f_t(x) = f(t, x)$ and, for each $x \in X$, the x-section is a function ${}^x f : T \to Y$, $t \mapsto {}^x f(t) = f(t, x)$. In case that $Y$ is a topological space we may fix an open set $O$ in $Y$ and define the relation $F : T \to X$ such that

$$F(t) = \{x \in X : f(t, x) \in O\}.$$

The last type of relation is discussed in the following theorem proved in [1] p. 65.

<u>Theorem 2</u>. If $X$ is a separable metric space, ${}^x f$ is measurable for each $x \in X$, $f_t$ is continuous for each $t \in T$, then $F$ is a measurable relation from $T$ to $X$.

This theorem can be generalized assuming only $f_t$ to be quasicontinuous. We recall the definition of quasicontinuity ([3]), since the notion is less known than that of continuity.

<u>Definition 3</u>. A function $f$ which maps a topological space $X$ into a topological space $Y$ is said to be quasicontinuous at a point $x_0 \in X$ iff for every open set $V \ni f(x_0)$ and every open set $U \ni x_0$ there exists a nonempty set $G \subset U$ with

$$f[G] \subset V.$$

$f$ is said to be quasicontinuous iff it is quasicontinuous at every point of the domain of $f$.

<u>Theorem 3</u>. Let $X$ be a separable topological space, $Y$ be a topological space in which an open set $O$ is fixed and

let $T$ be a measurable space. If $f : T \times X \to Y$ is such a function that ${}^x f$ is measurable $(x \in X)$ and $f_t$ is quasicontinuous $(t \in T)$ then the relation

$$F(t) = f_t^{-1}[0]$$

is weakly measurable.

P r o o f. Choosing $U \subset X$ open, $U$ is separable due to the separability of $X$ and there exists a countable dense subset $H$ in $U$. We have to prove the measurability of the set

$$F^{-1}[U] = \{t \in T \mid F(t) \cap U \neq \emptyset\}.$$

We shall prove the equivalence $F(t) \cap U \neq \emptyset \Leftrightarrow F(t) \cap H \neq \emptyset$. Since $F(t) \cap U \neq \emptyset$, there exists $x \in U$ with $f(t, x) \in 0$ and hence there is $x \in U$ with $f_t(x) \in 0$. Making use of quasicontinuity of $f_t$ there exists an open set $G \subset U$, $G \neq \emptyset$ such that $f_t[G] \subset 0$. As $H$ is dense in $U$, $G \cap H \neq \emptyset$ which implies the existence of a $z \in G \cap H$. For the point $z$ we have $f_t(z) \in 0$, $z \in H$ i. e. $z \in F(t) \cap H$. Now as the implication $F(t) \cap H \neq \emptyset \Rightarrow F(t) \cap U \neq \emptyset$ is obvious, we have

$$F^{-1}[U] = \{t \in T \mid F(t) \cap H \neq \emptyset\} = \{t \in T \mid \exists z \in H : z \in F(t)\} =$$
$$= \bigcup_{z \in H} \{t \in T \mid f_t(z) \in 0\} = \bigcup_{z \in H} {}^z f^{-1}[0].$$

The last set is measurable as a countable union of measurable sets.

To prove another generalization of Theorem 2 we introduce the notion of a P-system. (See [2] pp. 149-150.)

D e f i n i t i o n 4. Let $(X, \mathcal{M})$ be a measurable space and let for each positive integer $k \in N$ an at most countable covering $\mathcal{P}^k$ of the set $X$ be given, which consists only of non-empty measurable sets. Then $\mathcal{P} = \{P \mid P \in \mathcal{P}^k,\ k \in N\}$ is called a P-system on $X$.

We shall always consider the P-system as a two-indexed system

$$\mathcal{P} = \{P_n^k \in \mathcal{M} \setminus \{\emptyset\} \mid k \in N,\ n \in N_k\}$$

where $N_k$ is a subset of the set $N$ of all positive integers.

D e f i n i t i o n 5. If on a measurable space $(X, \mathcal{M})$ a P-system $\mathcal{P}$ is given and $(Y, \mathcal{T})$ is a topological space, we shall say that a function $f : X \to Y$ is regular at a point $x_0 \in X$ with respect to $\mathcal{P}$ iff for any neighbourhood $G$ of $f(x_0)$ there is $k_0 \in N$ such that for $k > k_0$ and every $n \in N_k$, $x_0 \in P_n^k$ implies $f[P_n^k] \subset G$. A function is said to be regular iff it is regular at any point of its domain.

D e f i n i t i o n 6. A P-system $\mathcal{P}$ on a measurable space $(X, \mathcal{M})$ is said to be regular relative to a topology $\mathcal{T}$ on $X$ iff the identity mapping

$$i : X \to X$$

is regular with respect to $\mathcal{P}$.

If on a measurable space a P-system is given then the coverings of which this P-system consists, need not be partitions of the underlying set. However, the following lemma is true.

L e m m a. If $\mathcal{P} = \bigcup\{\mathcal{P}^k \mid k \in N\}$ is a P-system on a measurable space $(X, \mathcal{M})$ which is regular relative to a topology $\mathcal{Y}$ on $X$ and $f : X \to Y$ is a function regular with respect to the P-system $\mathcal{P}$ then there exists a P-system $\tilde{\mathcal{P}} = \bigcup\{\tilde{\mathcal{P}}^k \mid k \in N\}$ regular relative to the topology $\mathcal{Y}$ and such that $\tilde{\mathcal{P}}^k$ are measurable partitions of $X$, whereby $f$ remains regular with respect to $\tilde{\mathcal{P}}$.

P r o o f. Let be $\mathcal{P}^k = \{P_n^k \in \mathcal{M} \mid n \in N_k\}$. Form new sets

$$\tilde{P}_n^k = P_n^k \setminus \bigcup_{i<n} P_i^k \qquad (n \in N_k)$$

As $\mathcal{M}$ is a $\sigma$-ring, for every $(k, n) \in N \times N_k$ we have $\tilde{P}^k_n \in \mathcal{M}$ and these sets are pairwise disjoint and cover again the set X. Dropping out the empty set we obtain a measurable partition of X,

$$\tilde{\mathcal{P}}^k = \{\tilde{P}^k_n \in \mathcal{M} \setminus \{\emptyset\} \mid n \in \tilde{N}_k\}$$

where $\tilde{N}_k \subset N_k$. The only thing which has to be proved is regularity of f with respect to the P-system $\tilde{\mathcal{P}} = \bigcup\{\tilde{\mathcal{P}}^k \mid k \in N\}$. As a special case we obtain the regularity of $\mathcal{P}$ relative to $\mathcal{S}$. Since for every $(k, n) \in N \times \tilde{N}_k$ we have $\tilde{P}^k_n \subset P^k_n$, it follows

$$f[\tilde{P}^k_n] \subset f[P^k_n] .$$

It is clear now that f remains regular also with respect to $\tilde{\mathcal{P}}$. We make use of this lemma when proving the following theorem.

T h e o r e m 4. Suppose $(T, \mathcal{A})$, $(X, \mathcal{M})$ are measurable spaces, and $\mathcal{P}$ is a P-system on X regular relative to the topology $\mathcal{S}$ on X. Let $(Y, \mathcal{T})$ be a topological space and $f : T \times X \to Y$ such a function that every ${}^x f$ is measurable and every $f_t$ is regular with respect to $\mathcal{P}$. When we fix an open set $O \subset Y$ then

$$F : T \to X, \quad t \mapsto F(t) = \{x \in X : f(t, x) \in O\}$$

is an $\mathcal{A}$- $2^X$-measurable relation.

P r o o f. If $M \subset X$ then

$$F^{-1}[M] = \{t \mid F(t) \cap M \neq \emptyset\} = \{t \mid \exists x \in M : f(t, x) \in O\} =$$
$$= \{t \mid \exists x \in M : t \in {}^x f^{-1}[O]\} = \bigcup\{{}^x f^{-1}[O] \mid x \in M\}.$$

We prove that the last union can be written as a countable union of measurable sets in T. Supported by Lemma we can suppose $\mathcal{P} = \{P^k_n \in \mathcal{M} \setminus \{\emptyset\} \mid (k, n) \in N \times N_k\}$ to be such a P-system on X that for every $k \in N$ $\mathcal{P}^k = P^k_n \mid n \in N_k$ is a measurable partition of $X$ and $f_t$ is regular with respect to $\mathcal{P}$. For every pair $(k, n) \in N \times N_k$ form the set

$$M^k_n = M \cap P^k_n .$$

The system $\mathcal{M}^k = \{M^k_n \neq \emptyset \mid n \in N'_k\}$, where $N'_k \subset N_k$ is a partition of $M$ ( $N'_k = \emptyset$ iff $M = \emptyset$). Choosing a point $x^k_n$ in each set $M^k_n$ we obtain a countable set of representatives

$$R = \{x^k_n \in M^k_n \mid (k, n) \in N \times N'_k\}.$$

We prove the equation

$$\bigcup_{x \in M} {}^x f^{-1} [O] = \bigcup_{x \in R} {}^x f^{-1} [O]$$

The inclusion " $\supset$ " is true because $M \supset R$. Since

$$t \in \bigcup_{x \in M} {}^x f^{-1} [O] \Leftrightarrow \exists x \in M : f_t(x) \in O$$

$O$ is an open neighbourhood of $f_t(x)$ and due to the regularity of $f_t$ with respect to $\mathcal{P}$ it is true that

$$\exists k_o \in N \; \forall k > k_o : x \in P^k_n \Rightarrow f_t[P^k_n] \subset O .$$

However, $\forall k \in N \; \exists n \in N'_k : x \in M^k_n \subset P^k_n$ and therefore the representative $x^k_n \in M^k_n$ and we have

$$f_t(x^k_n) \in f_t[M^k_n] \subset f_t[P^k_n] \subset O .$$

This implies $t \in {}^{x^k_n} f^{-1}[O]$ , which proves the required equation. Considering measurability of each ${}^x f$ we obtain the claimed measurability of $F^{-1}[M]$.

R e m a r k 3. There exists a class of objects which fulfil the assumption of Theorem 4. In the paper [2] (Lemma 4.2) it is proved that in every topological space $(X, \mathcal{G})$ with a countable base of $\mathcal{G}$ there is a P-system regular relative to the topology $\mathcal{G}$ if it is measurable i. e. $\mathcal{G} \subset \mathcal{M}$. In such a case every continuous function is regular with respect to the P-system.

R e m a r k 4. It follows from Remark 3 and from the fact that every continuous function is quasicontinuous that Theorem 2 is a special case of Theorem 4.

R e f e r e n c e s


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## Súhrn

### O MERATEĽNOSTI BINÁRNYCH RELÁCIÍ

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V práci sú dokázané dve vety o merateľných reláciách. Nech F je binárna relácia z merateľného priestoru T do topologického priestoru X definovaná pomocou funkcie dvoch premenných f(t, x), ktorá zobrazuje T x X do topologického priestoru Y. Je dokázaná slabá merateľnosť F, keď f je merateľná v t a kvázispojitá v x. V prípade, že f je merateľná v t a regulárna vzhľadom k P-systému v premennej x, je dokázaná merateľnosť relácie F.



## Резюме

### ОБ ИЗМЕРИМОСТИ БИНАРНЫХ ОТНОШЕНИЙ

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В этой статье приведены две теоремы об измеримых отношениях. Пусть F отношение из измеримого пространства T в топологическое пространство X, которое определено при помощи функции двух переменных f(t, x) отображающей T x X в топологическое пространство Y. Доказывается слабая измеримость F когда f измерима в t и квазинепрерывна в x. Когда f измерима в t и регулярна по отношению к P-семейству в переменной x, тогда доказывается измеримость F.