UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XXXVIII - 1981

TENSOR PRODUCTS IN THE CATEGORY OF GENERALIZED TOPOLOGICAL SPACES

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0. Introduction

In [4] a structure of closed category for any initially structured category in the sense of [8] has been constructed. Moreover, the corresponding tensor product has been shown to be in a certain sense the smallest possible. The known closed structures, e.g. the known closed structure in the category $\mathcal{T}$ of topological spaces, in the category of partially ordered sets, the closed structures studied in [10], [4] and other, are either the special cases of the closed structure defined in [4] or the cartesian closed structures. In this paper, closed structures in the category $\mathcal{C}$ of all topological spaces in Čech's sense from the year 1937 (see [2]) and continuous maps will be studied. It will be shown that this category admits (up to a natural isomorphism) precisely two closed structures without being cartesian closed. This shows that the above mentioned two types of closed structures do not include all closed structures in initially structured categories. Moreover, it will be proved that in the full subcategory $\mathcal{F}$ of $\mathcal{C}$ consisting of all F-spaces there exists (up to an isomorphism) exactly one closed structure. Note that $\mathcal{F}$ is the reflective hull of the category $\mathcal{T}$ (which is a full subcategory of $\mathcal{C}$) and the coreflective hull of $\mathcal{T}$ in $\mathcal{C}$ is the category of closure spaces studied in [4]. The concept closed category is used in the sense of [7, p. 180] and it coincides with the concept symmetric monoidal closed category used in [5]. Recall that a tensor product is a symmetric monoidal structure extendable to a closed structure.

Throughout this paper all subcategories will be supposed to be full and isomorphism-closed.

1. Tensor products in the category $\mathcal{C}$

A C-space, i.e. a topological space in Čech's sense from the year 1937, is a pair (P, u) where P is a set and u: $2^P \longrightarrow 2^P$ is a map satisfying

(i) $u\emptyset = \emptyset$

(ii) $M \subset uM$ for each $M \in 2^P$

(iii) $L \subset M$ implies $uL \subset uM$ for all $L, M \in 2^P$

and called a C-topology.

If (P, u), (Q, v) are C-spaces, then a map f: $(P, u) \longrightarrow (Q, v)$ is said to be continuous provided that $f[uM] \subset vf[M]$ for each $M \in 2^P$.

The class of all C-spaces together with the class of all continuous maps between C-spaces form a category which will be denoted by $\mathcal{C}$. We shall often write only P instead of (P, u) and then $\overline{M}$ instead of uM for $M \subset P$.

Recall (see [8]) that $(\mathcal{A}, U)$ is said to be an initially structured category provided that $\mathcal{A}$ is a category and U: $\mathcal{A} \longrightarrow$ Set is a functor such that the following hold:

(1) If $(f_i: X \longrightarrow UA_i)_{i \in I}$ is a source in Set, then there exists an (epi, monosource)-factorization $(X \xrightarrow{e} UA \xrightarrow{Ua_i} UA_i)_{i \in I}$

in Set such that $(a_i)_{i \in I}$ is a U-initial source, i.e. for every source $(b_i: B \longrightarrow A_i)_{i \in I}$ and every map $g: UB \longrightarrow UA$ with $Ua_i \circ g = Ub_i$ for each $i \in I$ there exists $k: B \longrightarrow A$ for which $Uk = g$ and $a_i \circ k = b_i$ for all $i \in I$.

(2) U has small fibres, i.e. for every set X there is at most a set of pairwise non-isomorphic $\mathcal{A}$-objects A with $UA = X$.

(3) There exists (up to an isomorphism) precisely one $\mathcal{A}$-object Z such that $UZ = \{*\}$ (the singleton in Set).

Initially structured categories are closed under the formation of non-trivial coreflective and epi_U-reflective subcategories (k is an epi_U-morphism iff Uk is an epimorphism).

It is obvious that the category $\mathcal{C}$ together with the forgetful functor $U: \mathcal{C} \longrightarrow \mathrm{Set}$; $(P, u) \longmapsto P$, $f \longmapsto f$ is an initially structured category (we identify Uf and f).

Let X, Y, Z be sets and $f: X \times Y \longrightarrow Z$ a map. Then for each $a \in X$ f_a denotes the map $Y \longrightarrow Z$ defined by $f_a(y) = f(a, y)$ and for each $b \in Y$ f^b denotes the map $X \longrightarrow Z$ given by $f^b(x) = f(x, b)$. If $M \subset X \times Y$, then for $x \in X$ $xM = \{y \in Y: (x, y) \in M\}$ and for $y \in Y$ $My = \{x \in X: (x, y) \in M\}$.

1.1. Definition. Let (P, u), (Q, v) be C-spaces, $M \subset P \times Q$. Define $(u \otimes v)M = (\bigcup_{x \in P}(\{x\} \times v(xM))) \cup \cup (\bigcup_{y \in Q}(u(My) \times \{y\}))$. Then $(P \times Q, u \otimes v)$ is a C-space and it will be denoted by $(P, u) \otimes (Q, v)$.

1.2. Remark. $\times$ usually denotes the cartesian product in Set. All other categorical products will be denoted by Π.

1.3. Proposition. Let P, Q, S be C-spaces. Then the map $f: P \otimes Q \longrightarrow S$ is continuous if and only if for each $a \in P$ and $b \in Q$ f_a and f^b are continuous maps.

Proof. Immediate from the definition of $P \otimes Q$.

The map $\otimes : \mathrm{ob}\,\mathcal{C} \times \mathrm{ob}\,\mathcal{C} \longrightarrow \mathrm{ob}\,\mathcal{C}$ defined in 1.1. is a special case of that in [4]. By [4; Proposition 1.6.] it can be extended to a functor by putting $f \otimes g = f \times g$ for all $\mathcal{C}$-morphisms f, g and by [4; Theorem 1.9.] $\otimes$ is a tensor product. Following [4] the corresponding internal hom functor is defined as follows: Let (Q, v), (S, w) be C-spaces and $(S, w)^Q$ a (object part of) $\mathcal{C}$-power. Denote by $[(Q, v), (S, w)]$ the subspace of $(S, w)^Q$ consisting of all continuous maps $(Q, v) \longrightarrow (S, w)$. If $f: P' \longrightarrow P$, $g: Q \longrightarrow Q'$ are $\mathcal{C}$-morphisms, then the map $[f, g]: [P, Q] \longrightarrow [P', Q']$ is given by $[f, g](t) = g \circ t \circ f$. The assignments $(P, Q) \longmapsto [P, Q]$, $(f, g) \longmapsto [f, g]$ form the required internal hom functor. Thus we have:

1.4. Theorem. $(\mathcal{C}, \otimes, [-, -])$ is a closed category.

1.5. Definition. Let (P, u), (Q, v) be C-spaces. Put $(u \boxtimes v)M = \bigcup_{A \times B \subset M} (uA \times vB)$ for each $M \subset P \times Q$. Then $(P \times Q, u \boxtimes v)$ is a $\mathcal{C}$-space and it will be denoted by $(P, u) \boxtimes (Q, v)$.

1.6. Lemma. Let $f: P \longrightarrow P'$, $g: Q \longrightarrow Q'$ be $\mathcal{C}$-morphisms. Then $f \times g: P \boxtimes Q \longrightarrow P' \boxtimes Q'$ is a $\mathcal{C}$-morphism.

Proof. Let $M \subset P \boxtimes Q$ and $(a, b) \in \overline{M}$. Then there exist $A \subset P$, $B \subset Q$ such that $A \times B \subset M$ and $a \in \overline{A}$, $b \in \overline{B}$. Evidently, $f[A] \times g[B] = (f \times g)[A \times B] \subset (f \times g)[M]$. Since $f(a) \in \overline{f[A]}$ and $g(b) \in \overline{g[B]}$ we obtain that $(f \times g)(a, b) =$ $= (f(a), g(b)) \in \overline{f[A]} \times \overline{g[B]} \subset \overline{(f \times g)[M]}$.

1.7. Corollary. $\boxtimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$; $(P, Q) \longmapsto P \boxtimes Q$, $(f, g) \longmapsto f \times g$ is a functor.

1.8. Proposition. $(\mathcal{C}, \boxtimes)$ is a symmetric monoidal category.

P r o o f . Evidently, $\{*\}$ is a unit object of $\boxtimes$, $l_P: \{*\} \boxtimes P \longrightarrow P$; $(*, y) \longmapsto y$, $r_P: P \boxtimes \{*\} \longrightarrow P$; $(x, *) \longmapsto x$ and $c_{PQ}: P \boxtimes Q \longrightarrow Q \boxtimes P$; $(x, y) \longmapsto (y, x)$ are natural isomorphisms. Let P, Q, S be C-spaces and $\alpha_{PQS}: (P \boxtimes Q) \boxtimes S \longrightarrow$ $\longrightarrow P \boxtimes (Q \boxtimes S)$ be the map defined by $((x, y), z) \longmapsto (x, (y, z))$. Let $M \subset (P \boxtimes Q) \boxtimes S$. Then $((a, b), c) \in \overline{M}$ iff there exist $A \subset P$, $B \subset Q$, $C \subset S$ with $a \in \overline{A}$, $b \in \overline{B}$, $c \in \overline{C}$ and $(A \times B) \times C \subset M$. Clearly, $A \times (B \times C) \subset \alpha_{PQS}[M]$ and this is equivalent with $(a, (b, c)) = \alpha_{PQS}((a, b), c) \in \overline{\alpha_{PQS}[M]}$. Hence α_{PQS} is an isomorphism which is evidently natural. The coherence axioms are obviously fulfilled.

Recall (see [3]) that if $((P_a, u_a))_{a \in A}$ is a set-indexed family of C-spaces, $((j_a)_{a \in A}, P)$ is a Set-coproduct of $(P_a)_{a \in A}$ and u is the C-topology on P defined by $uM =$ $= \bigcup_{a \in A} j_a[u_a j_a^{-1}[M]]$, then $((P, u), (j_a)_{a \in A})$ is a $\mathcal{C}$-coproduct of $((P_a, u_a))_{a \in A}$. It is easy to see that it holds:

1.9. L e m m a . For each C-space Q the functor $- \boxtimes Q: \mathcal{C} \longrightarrow \mathcal{C}$ preserves $\mathcal{C}$-coproducts.

1.10. L e m m a . For each C-space Q the functor $- \boxtimes Q: \mathcal{C} \longrightarrow \mathcal{C}$ preserves $\mathcal{C}$-coequalizers.

P r o o f . Recall (see [3]) that (e, E) is a coequalizer of f, g in $\mathcal{C}$ iff (Ue, UE) is a coequalizer of Uf, Ug in Set ($U: \mathcal{C} \longrightarrow$ Set is the forgetful functor) and e is an extremal $\mathcal{C}$-epimorphism. A $\mathcal{C}$-morphism $h: (P, u) \longrightarrow (Q, v)$ is an extremal $\mathcal{C}$-epimorphism iff h is a surjection and for each $M \subset Q$ $vM = h[uh^{-1}[M]]$. Let $(e, E) = \mathrm{Coeq}(f, g)$ in $\mathcal{C}$. Evidently, $(Ue \times U1_Q, UE \times UQ)$ is a coequalizer of $Uf \times U1_Q$, $Ug \times U1_Q$ in Set. Thus it suffices to prove that the functor $- \boxtimes Q$ preserves extremal $\mathcal{C}$-epimorphisms. Let $e: P \longrightarrow K$ be an extremal $\mathcal{C}$-epimorphism, $M \subset K \boxtimes Q$ and $(a, b) \in \bar{M}$. Then there exist $A \subset K$, $B \subset Q$ such that $A \times B \subset M$ and $a \in \bar{A}$, $b \in \bar{B}$. Since e is an extremal $\mathcal{C}$-epimorphism there exists $c \in \overline{e^{-1}[A]}$ with $e(c) = a$. But $\overline{e^{-1}[A] \times B} \subset \overline{(e \times 1_Q)^{-1}[M]}$ and this implies that $(c, b) \in \overline{(e \times 1_Q)^{-1}[M]}$. Evidently, $(e \times 1_Q)(c, b) = (a, b)$ so that $e \times 1_Q$ is an extremal $\mathcal{C}$-epimorphism.

1.11. Proposition. For each C-space Q the functor $- \boxtimes Q: \mathcal{C} \longrightarrow \mathcal{C}$ preserves colimits.

Proof. Since $\mathcal{C}$ is cocomplete, well-powered and co-well-powered it follows from 1.9. and 1.10. (see [6]).

1.12. Theorem [6]. Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be a functor, $\mathcal{A}$ be well-powered, complete and have a coseparator. Then the functor F has a left adjoint functor if and only if F preserves limits.

1.13. Theorem. There exists an internal hom functor $H: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$ such that $(\mathcal{C}, \boxtimes, H)$ is a closed category.

Proof. The category $\mathcal{C}$ has a separator (every non-empty C-space) and is co-well-powered and cocomplete. Hence by 1.11. and the dualization of 1.12., for each C-space Q the functor $- \boxtimes Q: \mathcal{C} \longrightarrow \mathcal{C}$ has a right adjoint functor.

1.14. Remark. Denote by G^2 the Sierpinski doubleton. It is easy to check that $G^2 \boxtimes G^2$ is not isomorphic with $G^2 \otimes G^2$ and therefore $\boxtimes$ is not isomorphic with $\otimes$. Recall (see [3]) that if $((P_a, u_a))_{a \in A}$ is a set-indexed family of C-spaces, $(P, (p_a)_{a \in A})$ is a cartesian product of $(P_a)_{a \in A}$ and u is C-topology defined by $uM = \bigcap_{a \in A} p_a^{-1}[u_a p_a[M]]$ for each $M \subset P$, then $((P, u), (p_a)_{a \in A})$ is a $\mathcal{C}$-product of $((P_a, u_a))_{a \in A}$. Now denote by D_2 the discrete doubleton. It is easy to see that $D_2 \sqcap D_2$

is not discrete so that $\sqcap$ is not a tensor product. Thus $\mathcal{C}$ is not cartesian closed.

To show that any tensor product in $\mathcal{C}$ is isomorphic either with $\otimes$ or with $\boxtimes$ the next theorem will be very useful. Recall (see [7]) that a pair $(\mathcal{K}, V)$ is a concrete category provided that $\mathcal{K}$ is a category and $V: \mathcal{K} \longrightarrow \mathrm{Set}$ is a faithful functor.

1.15. Theorem [9]. Let $(\mathcal{K}, V)$ be a concrete category with the following properties:

(1) If $c: VA \longrightarrow VB$ is a constant map, then there exists a $\mathcal{K}$-morphism $k: A \longrightarrow B$ with $Vk = c$.

(2) For every bijection $f: VA \longrightarrow X$ in Set there exists a $\mathcal{K}$-isomorphism $h: A \longrightarrow B$ with $Vh = f$.

(3) There exists a $\mathcal{K}$-object A with card $VA \geq 2$.

Let there exist a closed structure $(\Box, H)$ on $\mathcal{K}$. Then there exists a closed structure $(\bigcirc, G)$ on $\mathcal{K}$ isomorphic with $(\Box, H)$ with the following properties:

(a) Card $VI = 1$ where I is the unit of $\bigcirc$.

(b) $VA \times VB \subset V(A \bigcirc B)$ for all $\mathcal{K}$-objects A, B.

(c) For each $f, g: A \bigcirc B \longrightarrow C$, $Vf|_{VA \times VB} = Vg|_{VA \times VB}$ implies $f = g$ ($|$ denotes a restriction of a map).

(d) $V(f \bigcirc g)|_{VA \times VB} = Vf \times Vg$ for each $f: A \longrightarrow A'$, $g: B \longrightarrow B'$.

(e) $VG(B, C) = \mathcal{K}(B, C)$ for any $\mathcal{K}$-objects B, C.

(f) $V(\theta f)(a) = (b \longmapsto Vf(a, b))$ where $\theta: \mathcal{K}(A \bigcirc B, C) \longrightarrow \mathcal{K}(A, G(B, C))$ is the corresponding adjunction.

(g) $V(\theta^{-1} g)(a, b) = (Vg(a))(b)$.

If, moreover, $(\mathcal{K}, V)$ fulfils

(4) $X \subset VA$ implies that there exists $j: B \longrightarrow A$ with $VB = X$ and $Vj(x) = x$ for all $x \in X$ and

(5) for every $\mathcal{K}$-epimorphism e Ve is a surjection,

then

(i) $VA \times VB = V(A \bigcirc B)$ for all $\mathcal{K}$-objects A, B.

Since $\mathcal{C}$ and $\mathcal{F}$ fulfil (1) - (5) of the last theorem we can adopt the following restriction without loss of generality.

1.16. Convention. Throughout the remainder of this paper all closed structures on $\mathcal{C}$, $\mathcal{F}$ will be assumed to fulfil (a) - (i) of 1.15.. It is easy to check that if $\mathcal{A} \in \{\mathcal{C}, \mathcal{F}\}$, $(\mathcal{A}, \Box, H)$ is a closed category satisfying this convention, a is the associativity and c the symmetry of $\Box$, then for any P, Q, S $\in$ ob$\mathcal{A}$ a_{PQS} is given by $((x, y), z) \longmapsto (x, (y, z))$ and c_{PQ} by $(x, y) \longmapsto (y, x)$.

1.17. Proposition. If $(\mathcal{C}, \Box, H)$ is a closed category, then for any C-spaces (P, u), (Q, v) $id_{P \times Q}$: $(P, u) \otimes (Q, v) \longrightarrow (P, u) \Box (Q, v)$ is a $\mathcal{C}$-morphism.

Proof. Immediate from [4; Proposition 1.12.].

Let (P, u), (Q, v) be C-spaces. We shall write $(P, u) \leqq (Q, v)$ iff $P = Q$ and for each $M \subset P$ $uM \subset vM$. Let ξ be an arbitrary cardinal. Choose a fixed point in ξ and denote it by 0_ξ. Put $A_\xi = \xi - \{0_\xi\}$. Now define the C-topology on ξ by $\overline{M} = M$ for each $M \in 2^\xi - \{A_\xi\}$ and $\overline{A}_\xi = \xi$. Denote the just defined C-space by G^ξ. If $\xi = n$ is a finite cardinal, then $0_n = 0$ $(n = \{0, 1, \ldots, n - 1\})$. Note that for any cardinal ξ G^ξ is also an F-space (i.e. the closure operation is idempotent). In [3] it has been proved that for any cardinal $\alpha \geqq 2$ the coreflective hull of the class $\mathcal{G}_\alpha = \{G^\xi : \xi \geqq \alpha\}$ in $\mathcal{C}$ coincides with $\mathcal{C}$ and then, evidently, the coreflective hull of $\mathcal{G}_\alpha$ in $\mathcal{F}$ coincides with $\mathcal{F}$ and for $\xi \geqq \zeta$ every map $f: G^\xi \longrightarrow G^\zeta$ such that $f(0_\xi) = 0_\zeta$ and $f[A_\xi] = A_\zeta$ is an extremal $\mathcal{C}$-epimorphism and also an extremal $\mathcal{F}$-epimorphism.

1.18. Proposition. Let $\mathcal{A} \in \{\mathcal{C}, \mathcal{F}\}$, $(\mathcal{A}, \Box, H)$ be a closed category and $\alpha \geqq 2$ a cardinal. Then the tensor product $\Box$ is uniquely determined by its values at (G^{ξ}, G^{ζ}) for all $\xi, \zeta \geqq \alpha$.

Proof. Let P, Q be $\mathcal{A}$-objects. Since the coreflective hull of $\mathcal{S}_\alpha$ in $\mathcal{A}$ coincides with $\mathcal{A}$ there exist extremal $\mathcal{A}$-epimorphisms $f: \bigsqcup_{a \in A} G^{\xi a} \longrightarrow P$, $g: \bigsqcup_{b \in B} G^{\xi b} \longrightarrow Q$ for suitable $G^{\xi a}$, $G^{\xi b}$ belonging to $\mathcal{S}_\alpha$ and sets A, B. The $\mathcal{A}$-morphism $f \Box g = f \times g: (\bigsqcup_{a \in A} G^{\xi a}) \Box (\bigsqcup_{b \in B} G^{\xi b}) = \bigsqcup_{(a, b) \in A \times B} (G^{\xi a} \Box G^{\xi b}) \longrightarrow P \Box Q$ is an extremal $\mathcal{A}$-epimorphism ($\Box$ preserves extremal $\mathcal{A}$-epimorphisms which coincide with regular ones in $\mathcal{A}$) so that the $\mathcal{A}$-topology of the space $P \Box Q$ is uniquely determined.

1.19. Proposition. Let $(\mathcal{A}, \Box, H)$ be a closed category, $\mathcal{A} \in \{\mathcal{C}, \mathcal{F}\}$. Then for any cardinals $\xi, \zeta \geqq 3$, $G^{\xi} \Box G^{\zeta} \cong G^{\xi} \circ G^{\zeta}$ where the $\mathcal{C}$-topology v of $G^{\xi} \circ G^{\zeta}$ is defined as follows:

(i) If $(x, y) \in vM - M$, then $x = 0_{\xi}$ or $y = 0_{\zeta}$.

(ii) $(0_{\xi}, y_0) \in vM - M$ for $y_0 \neq 0_{\zeta}$ iff $My_0 = \{x \in G : (x, y_0) \in M\} = A_{\xi}$.

(iii) $(x_0, 0_{\zeta}) \in vM - M$ for $x_0 \neq 0_{\xi}$ iff $x_0M = A_{\zeta}$.

(iv) $(0_{\xi}, 0_{\zeta}) \in vM - M$ iff for each $x \in A_{\xi}$ $xM \cup 0_{\zeta}M = A_{\zeta}$ or $0_{\zeta} \in xM$ and for each $y \in A_{\zeta}$ $My \cup M0_{\zeta} = A_{\xi}$ or $0_{\xi} \in My$.

Proof. For each $x \in A_{\xi}$ define $f_x: G^{\xi} \longrightarrow A_{\xi}$ (the subspace A_{ξ} of G^{ξ} is a discrete space) putting $f_x(y) = y$ for $y \neq 0_{\xi}$ and $f_x(0_{\xi}) = x$. Evidently, f_x is an extremal $\mathcal{A}$-epimorphism. If X is a discrete space, then $X = \bigsqcup_{x \in X} \{x\}$. Hence for an arbitrary $\mathcal{A}$-space Y $X \,\square\, Y = (\bigsqcup_{x \in X} \{x\}) \,\square\, Y = \bigsqcup_{x \in X} (\{x\} \,\square\, Y)$. Clearly, if Y is also a discrete space, then $X \,\square\, Y$ is discrete so that $A_\xi \,\square\, A_\xi$ is a discrete space. Denote the $\mathcal{A}$-topology of $G^\xi \,\square\, G^\xi$ by u.

(i) Let $M \subset G^\xi \,\square\, G^\xi$ and $(x, y) \in uM - M$. Let $x \neq 0_\xi$ and $y \neq 0_\xi$ and $x' \in G^\xi$, $y' \in G^\xi$ such that $0_\xi \neq x' \neq x$ and $0_\xi \neq y' \neq y$. Then $(f_{x'} \,\square\, f_{y'})(x, y) = (x, y) \notin (f_{x'} \,\square\, f_{y'})[M] =$
$= \overline{(f_{x'} \,\square\, f_{y'})[M]}$ - a contradiction. Hence $x = 0_\xi$ or $y = 0_\xi$.

(ii) Let $(0_\xi, y_0) \in uM - M$, $y_0 \neq 0_\xi$ and $y_1 \in A_\xi - \{y_0\}$. Then for each $x \in A_\xi$ we have the $\mathcal{A}$-morphism
$f_x \,\square\, f_{y_1} : G^\xi \,\square\, G^\xi \longrightarrow A_\xi \,\square\, A_\xi$ with $(f_x \,\square\, f_{y_1})(0_\xi, y_0) =$
$= (x, y_0) \in (f_x \,\square\, f_{y_1})[M] = \overline{(f_x \,\square\, f_{y_1})[M]}$. Since $(f_x \,\square\, f_{y_1})^{-1}(x, y_0) = \{(0_\xi, y_0), (x, y_0)\}$ and $(0_\xi, y_0) \notin M$ it follows that $(x, y_0) \in M$. Hence $My_0 = A_\xi$.

(iii) Analogously as (ii).

(iv) Let $(0_\xi, 0_\xi) \in uM - M$, $y_0 \in A_\xi$. Consider $1 \,\square\, f_{y_0}\colon G^\xi \,\square\, G^\xi \longrightarrow$
$\longrightarrow G^\xi \,\square\, A_\xi = \bigsqcup_{y \in A} (G^\xi \,\square\, \{y\})$ $(1 = 1_{G^\xi})$. $(1 \,\square\, f_{y_0})(0_\xi, 0_\xi) =$
$(0_\xi, y_0) \in \overline{(1 \,\square\, f_{y_0})[M]} = (\bigcup_{y \in A_\xi - \{y_0\}} (My \times \{y\})) \,\cup$
$\cup ((My_0 \cup M0_\xi) \times \{y_0\})$. Thus for each $y_0 \in A_\xi$,

$(0_\xi, y_0) \in (My_0 \cup M0_\xi) \times \{y_0\}$, i.e. for each $y_0 \in A_\xi$ $My_0 \cup M0_\xi =$
$= A_\xi$ or $0_\xi \in My_0$. Similarly it can be shown that for each $x \in A_\xi$ $xM \cup 0_\xi M = A_\xi$ or $0_\xi \in xM$.

1.20. R e m a r k . Let $\xi, \zeta \geq 2$ be cardinals. Denote by u the C-topology of $G^\xi \otimes G^\xi$ and by v the C-topology of $G^\xi \circ G^\xi$. If $(x, y) \in G^\xi \times G^\xi - \{(0_\xi, 0_\xi)\}$ and $M \subset G^\xi \times G^\xi$, then $(x, y) \in uM$ iff $(x, y) \in vM$. Hence, if $(\mathcal{C}, \square, H)$ is a closed category and w is the C-topology of $G^\xi \,\square\, G^\xi$, then for

$(j_A)_{A\in\mathcal{S}}$ uniquely determines the $\mathcal{F}$-morphism $p\colon \bigsqcup_{A\in\mathcal{S}} A \longrightarrow G^3 \square G^3$ which is evidently an extremal $\mathcal{F}$-epimorphism. But then $p \square 1$, $1 \square p$ are also extremal $\mathcal{F}$-epimorphisms which uniquely determine the F-topologies of $(G^3 \square G^3) \square G^3$ and $G^3 \square (G^3 \square G^3)$ respectively. Consider the map $a\colon (G^3 \square G^3) \square G^3 \longrightarrow G^3 \square (G^3 \square G^3)$; $((x, y), z) \longmapsto (x, (y, z))$. Since $(\mathcal{F}, \square, H)$ is a closed category (by 1.16.) a has to be an $\mathcal{F}$-isomorphism. Denote by w_1 the C-topology on $(G^3 \times G^3) \times G^3$ for which $p \square 1$ is an extremal $\mathcal{C}$-epimorphism and by w_2 the C-topology on $G^3 \times (G^3 \times G^3)$ for which $1 \square p$ is an extremal $\mathcal{C}$-epimorphism. Then the F-topologies of $(G^3 \square G^3) \square G^3$ and $G^3 \square (G^3 \square G^3)$ are the F-modifications of w_1 and w_2 respectively. Now consider the set $X =$
$= \{((0, 0), 1), ((2, 2), 1), (2, 2), 0), ((0, 1), 2), ((1, 0), 2), ((1, 1), 2)\}$. Then $((0, 0), 0) \in w_1 X$. Indeed, $h\colon G^5_{H_1} \longrightarrow G^3$;
$(0, 0) \longmapsto 0$, $(2, 2) \longmapsto 1$, $(0, 1) \longmapsto 2$, $(1, 0) \longmapsto 2$, $(1, 1) \longmapsto 2$
is an extremal $\mathcal{F}$-epimorphism. Evidently, $(h \square 1)^{-1}[H_1] = X$.
If $((0, 0), 0) \notin \overline{X}$ in $G^5_{H_1} \square G^3$, then it is easy to verify that $\overline{X} = X$ in $G^5_{H_1} \bigcirc G^3$ and therefore $\overline{X} = X$ in $G^5_{H_1} \square G^3$. Since $h \square 1$ is an extremal $\mathcal{F}$-epimorphism and $(h \square 1)^{-1}[H_1]$ is closed it follows that H_1 is closed in $G^3 \square G^3$ - a contradiction. Hence $((0, 0), 0) \in \overline{X}$ in $G^5_{H_1} \square G^3$ and therefore $((0, 0), 0) \in w_1 X$.
Since the F-modification of w_1 is coarser than w_1 $((0, 0), 0) \in \overline{X}$ in $(G^3 \square G^3) \square G^3$. Consider $a[X] = \{(0, (0, 1)), (2, (2, 1)), (2, (2, 0)), (0, (1, 2)), (1, (0, 2)), (1, (1, 2))\}$. Let $(x_0, (y_0, z_0)) \in w_2 a[X] - a[X]$. Then there exists $A \in \mathcal{S}$ such that $(x_0, (y_0, z_0)) \in \overline{B}_A - B_A$ in $G^3 \square A$ where $B_A =$
$= (1 \square p)^{-1}[a[X]] \cap (G^3 \square A)$. Then by 1.19. $x_0 = 0$ or $(y_0, z_0) =$

$(x, y) \in (G^{\xi} \times G^{\zeta}) - \{(0_{\xi}, 0_{\zeta})\}$ and $M \subset G^{\xi} \times G^{\zeta}$ $(x, y) \in uM$ iff $(x, y) \in wM$. Therefore $G^{\xi} \square G^{\zeta} \neq G^{\xi} \otimes G^{\zeta}$ iff there exists $M \subset G^{\xi} \times G^{\zeta}$ with $wM - uM = \{(0_{\xi}, 0_{\zeta})\}$.

<u>1.21. Lemma.</u> Let $(\mathcal{C}, \square, H)$ be a closed category and there exist $\xi', \zeta' \geqq 2$ such that $(0_{\xi'}, 0_{\zeta'}) \notin \overline{A_{\xi'} \times A_{\zeta'}}$ in $G^{\xi'} \square G^{\zeta'}$. Then $\square = \otimes$.

<u>Proof.</u> Consider the extremal $\mathcal{C}$-epimorphisms $h: G^{\xi'} \longrightarrow G^2$; $0_{\xi'} \longmapsto 0$, $h[A_{\xi'}] = \{1\}$, $k: G^{\zeta'} \longrightarrow G^2$; $0_{\zeta'} \longmapsto 0$, $k[A_{\zeta'}] = \{1\}$. Since $h \square k$ is an extremal $\mathcal{C}$-epimorphism and $(h \square k)^{-1}(1, 1) = A_{\xi'} \times A_{\zeta'}$ we obtain $(0, 0) \notin \overline{\{(1, 1)\}}$ in $G^2 \square G^2$ so that (it can be easily shown that) $G^2 \square G^2 =$ $= G^2 \otimes G^2$. Now suppose $\xi, \zeta \geqq 2$ to be arbitrary cardinals. We need to show (by 1.20.) that for each $M \subset G^{\xi} \square G^{\zeta}$ $(0_{\xi}, 0_{\zeta}) \in \overline{M} - M$ in $G^{\xi} \square G^{\zeta}$ implies $0_{\xi} M = A_{\zeta}$ or $M 0_{\zeta} = A_{\xi}$. Let $(0_{\xi}, 0_{\zeta}) \in \overline{M} - M$ in $G^{\xi} \square G^{\zeta}$ and there exist some elements $(0_{\xi}, y_0)$, $y_0 \neq 0_{\zeta}$ and $(x_0, 0_{\zeta})$, $x_0 \neq 0_{\xi}$ for which $(0_{\xi}, y_0) \notin M$ and $(x_0, 0_{\zeta}) \notin M$. Define the $\mathcal{C}$-morphisms $f: G^{\xi} \longrightarrow G^2 \sqcup \{*\}$ by $0_{\xi} \longmapsto 0$, $x_0 \longmapsto 1$, $f[A_{\xi} - \{x_0\}] \subset \{*\}$ and $g: G^{\zeta} \longrightarrow G^2 \sqcup \{*\}$ by $0_{\zeta} \longmapsto 0$, $y_0 \longmapsto 1$, $g[A_{\zeta} - \{y_0\}] \subset \{*\}$. For the $\mathcal{C}$-morphism $f \square g: G^{\xi} \square G^{\zeta} \longrightarrow (G^2 \sqcup \{*\}) \square (G^2 \sqcup \{*\}) = (G^2 \square G^2) \sqcup (G^2 \square \{*\}) \sqcup$ $\sqcup (\{*\} \square G^2) \sqcup (\{*\} \square \{*\})$ we have $(f \square g)[M] \cap (G^2 \square G^2) =$ $= \{(1, 1)\}$ so that $(f \square g)(0_{\xi}, 0_{\zeta}) = (0, 0) \notin$ $\notin \overline{(f \square g)[M] \cap (G^2 \square G^2)} = \overline{(f \square g)[M]} \cap (G^2 \square G^2)$. Hence $(f \square g)(0_{\xi}, 0_{\zeta}) \notin \overline{(f \square g)[M]}$ - a contradiction. Thus $G^{\xi} \square G^{\zeta} =$ $= G^{\xi} \otimes G^{\zeta}$ for any cardinals $\xi, \zeta \geqq 2$ so that $\square = \otimes$.

<u>1.22. Lemma.</u> If $(\mathcal{C}, \square, H)$ is a closed category and $\square \neq \otimes, \boxtimes$, then $G^3 \boxtimes G^3 < G^3 \square G^3$.

<u>Proof.</u> Evidently, it follows from 1.21. that $\square \neq \otimes$ implies $G^{\xi} \boxtimes G^{\zeta} \leqq G^{\xi} \square G^{\zeta}$ for each $\xi, \zeta \geqq 2$. Let $\square \neq \boxtimes$.

Then there exist $\xi, \zeta > 3$ such that $G^\xi \boxtimes G^\zeta < G^\xi \square G^\zeta$ (by 1.18). Therefore there exists $M \subset G^\xi \times G^\zeta$ with $(0_\xi, 0_\zeta) \in vM - uM$ where u is the topology of $G^\xi \boxtimes G^\zeta$ and v is the topology of $G^\xi \square G^\zeta$. $(0_\xi, 0_\zeta) \in uM$ implies $M \not\supset A_\xi \times A_\zeta$ and there exist $x_0 \in A_\xi$ and $y_0 \in A_\zeta$ such that $(0_\xi, y_0) \notin M$ and $(x_0, 0_\zeta) \notin M$. Since $(0_\xi, 0_\zeta) \in vM$ it follows that $(x_0, y_0) \in M$ (by 1.19.). Let $(x_1, y_1) \in (A_\xi \times A_\zeta) - M$. Then $(x_1, y_1) \neq (x_0, y_0)$ so that $x_1 \neq x_0$ or $y_1 \neq y_0$. Denote by w the topology of $G^3 \boxtimes G^3$ and by $^-$ the topology of $G^3 \square G^3$.

1. Let $x_0 = x_1$ and $y_0 \neq y_1$. Define $f\colon G^\xi \longrightarrow G^3$ by $f(0_\xi) = 0$, $f(x_0) = 1$, $f[G^\xi - \{0_\xi, x_0\}] = \{2\}$ and $g\colon G^\zeta \longrightarrow G^3 \sqcup \{*\}$ by $g(0_\zeta) = 0$, $g(y_0) = 1$, $g(y_1) = 2$ and $g(y) = *$ otherwise. Clearly, f, g are $\mathcal{C}$-morphisms so that $f \square g\colon G^\xi \square G^\zeta \longrightarrow G^3 \square (G^3 \sqcup \{*\}) =$
$= (G^3 \square G^3) \sqcup (G^3 \square \{*\})$ is a $\mathcal{C}$-morphism. Therefore

$(f \square g)(0_\xi, 0_\zeta) = (0, 0) \in \overline{(f \square g)[M]} \cap (G^3 \square G^3) =$
$= \overline{(f \square g)[M] \cap (G^3 \square G^3)} \subset \overline{\{(0, 2), (2, 0), (1, 1), (2, 1), (2,2)\}}$.

But (0, 0) does not belong to
$w\{(0, 2), (2, 0), (1, 1), (2, 1), (2, 2)\}$. Thus $G^3 \boxtimes G^3 < G^3 \square G^3$.

2. $y_0 = y_1$ and $x_0 \neq x_1$ - similarly as 1..

3. Let $x_0 \neq x_1$, $y_0 \neq y_1$. Define $f\colon G^\xi \longrightarrow G^3 \sqcup \{*\}$ by $f(0_\xi) =$
$= 0$, $f(x_0) = 1$, $f(x_1) = 2$ and $f(x) = *$ otherwise and g as in 1.. By considering the $\mathcal{C}$-morphism $f \square g$ we similarly as in 1. obtain that $G^3 \boxtimes G^3 < G^3 \square G^3$.

Now denote by f the $\mathcal{C}$-morphism $G^3 \longrightarrow G^3$; $0 \longmapsto 0$, $1 \longmapsto 2$, $2 \longmapsto 1$. Evidently, f is a $\mathcal{C}$-isomorphism and therefore $f_1 =$
$= f \square 1_{G^3}$, $f_2 = 1_{G^3} \square f$ and $f_3 = f \square f$ are $\mathcal{C}$-isomorphisms.

Throughout the remainder of this section let $(\mathcal{C}, \square, H)$ be a closed category with $\square \neq \otimes, \boxtimes$. Then (1.22.) $G^3 \boxtimes G^3 < G^3 \square G^3$. Denote the C-topologies of $G^3 \otimes G^3$, $G^3 \boxtimes G^3$

and $G^3 \square G^3$ by u, v and w respectively. Let $M \subset G^3 \times G^3$ such that $(0, 0) \in wM - vM$. Since $(0, 0) \in wM - M$, M must fulfil 1.19.(iv). Since $(0, 0) \notin vM$ it follows that card $MO \leqq 1$ and card $OM \leqq 1$ ($MO = \{x \in G^3: (x, 0) \in M\}$, $OM = \{y \in G^3: (0, y) \in M\}$; if card $OM > 1$ or card $MO > 1$, then $(0, 0) \in uM \subset vM$ - a contradiction). $(0, 0) \notin vM$ implies $\{1, 2\} \times \{1, 2\} \not\subset M$ and therefore $MO \neq \emptyset$ or $OM \neq \emptyset$. (Suppose the contrary. Then 1.19.(iv) implies $\{1, 2\} \times \{1, 2\} \subset M$.) If $(0, i)$ and $(j, 0)$ do not belong to M for $i, j \in \{1, 2\}$, then $(j, i) \in M$ (1.19.(iv)). Put $C_1 = \{(0, 1), (1, 0), (2, 2)\}$, $C_{i+1} = f_i[C_1]$, $1 \leqq i \leqq 3$, $D_1 = \{(0, 1), (1, 2), (2, 2)\}$, $D_{i+1} = f_i[D_1]$ for $1 \leqq i \leqq 3$, $D_{4+j} = c[D_j]$ for $1 \leqq j \leqq 4$ where $c: (x, y) \longmapsto (y, x)$. From the considerations above it easily follows that $(0, 0) \in wM - vM$ implies $C_i \subset M$ for some $i \in \{1, 2, 3, 4\}$ or $D_j \subset M$ for some j, $1 \leqq j \leqq 8$ for each $M \subset G^3 \times G^3$. Put $A_1 = \{(0, 1), (1, 0), (1, 1), (1, 2), (2, 2)\}$, $A_{1+i} = f_i[A_1]$ where $1 \leqq i \leqq 3$, $A_{4+j} = c[A_j]$ where $1 \leqq j \leqq 4$, $B_1 = \{(0, 1), (1, 0), (1, 2), (2, 1), (2, 2)\}$, $B_{1+i} = f_i[B_1]$ where $1 \leqq i \leqq 3$. Let $\tilde{\varphi} = \{A_i: 1 \leqq i \leqq 8\} \cup \{B_j: 1 \leqq j \leqq 4\}$. Evidently, $(0, 0) \notin vX$ for each $X \in \tilde{\varphi}$. On the other hand all $X \in \tilde{\varphi}$ fulfil the condition 1.19.(iv). Moreover, every $M \subset G^3 \times G^3$ satisfying 1.19.(iv) and such that $(0, 0) \notin vM$ is a subset of some $X \in \tilde{\varphi}$. Evidently, $G^3 \boxtimes G^3 < G^3 \square G^3$ implies that there exists some $X \in \tilde{\varphi}$ with $(0, 0) \in wX$. Since $(0, 0) \in wA_i$ for $i \in \{1, 2, \ldots, 8\}$ iff $(0, 0) \in wA_1$ and $(0, 0) \in wB_j$ for $j \in \{1, 2, 3, 4\}$ iff $(0, 0) \in wB_1$ we have:

<u>1.23. Lemma.</u> If $G^3 \boxtimes G^3 < G^3 \square G^3$, then $(0, 0) \in wA_1$ or $(0, 0) \in wB_1$.

Now suppose the topology w of $G^3 \square G^3$ to fulfil $(0, 0) \in wB_1$.

Let $\mathcal{Y}_1$ be the family of all subsets M of $G^3 \times G^3$ with $(0, 0) \in wM - vM$. For each $M \in \mathcal{Y}_1$ define the C-space G^m_M on the set $M \cup \{(0, 0)\}$ by $\overline{M} = M \cup \{(0, 0)\}$ and $\overline{K} = K$ for all $K \subset M \cup \{(0, 0)\}$, $K \neq M$ ($m = 1 + \operatorname{card} M$ and evidently G^m_M is isomorphic with G^m). The family $\{G^m_M : M \in \mathcal{Y}_1\}$ denote by $\mathcal{Y}_2$. Analogously define G^5_K on $K \cup \{(0, 0)\}$ where $K = \{1, 2\} \times \{1, 2\}$. Put $\mathcal{Y} = \{\{k\} \,\square\, G^3 : 0 \leq k \leq 2\} \cup \{G^3 \,\square\, \{k\} : 0 \leq k \leq 2\} \cup \mathcal{Y}_2 \cup \{G^5_K\}$. For each $D \in \mathcal{Y}$ $j_D : D \longrightarrow G^3 \,\square\, G^3 : z \longmapsto z$ is a $\mathcal{C}$-morphism and the family $(j_D)_{D \in \mathcal{Y}}$ uniquely determines the $\mathcal{C}$-morphism $p: \bigsqcup_{D \in \mathcal{Y}} D \longrightarrow G^3 \,\square\, G^3$ which is evidently an extremal $\mathcal{C}$-epimorphism. Then $p \,\square\, 1_{G^3}$ and $1_{G^3} \,\square\, p$ are extremal $\mathcal{C}$-epimorphisms which uniquely determine the C-topologies of $(G^3 \,\square\, G^3) \,\square\, G^3$ and $G^3 \,\square\, (G^3 \,\square\, G^3)$ respectively. Next we show that the map $a: (G^3 \,\square\, G^3) \,\square\, G^3 \longrightarrow G^3 \,\square\, (G^3 \,\square\, G^3)$; $((x, y), z) \longmapsto (x, (y, z))$ is not a $\mathcal{C}$-isomorphism. Consider the set $X = \{((0, 0), 1), ((0, 1), 0), ((0, 1), 2), ((1, 0), 1),$ $((1, 1), 1), ((2, 1), 1), ((2, 2), 1), ((1, 0), 2), ((1, 1), 2),$ $((2, 1), 2), ((2, 2), 2)\} \subset (G^3 \,\square\, G^3) \,\square\, G^3$. Then $((0, 0), 0) \in \overline{X}$ in $(G^3 \,\square\, G^3) \,\square\, G^3$. In fact, $(p \,\square\, 1_{G^3})^{-1}[X] \cap (G^6_{B_1} \,\square\, G^3) = X$ and $((0, 0), 0) \in \overline{X}$ in $G^6_{B_1} \,\square\, G^3$ because $f: G^6_{B_1} \longrightarrow G^3$; $(0, 0) \longmapsto 0$, $(0, 1) \longmapsto 1$, $(1, 0) \longmapsto 2$, $(1, 1) \longmapsto 2$, $(2, 1) \longmapsto 2$, $(2, 2) \longmapsto 2$ is an extremal $\mathcal{C}$-epimorphism and therefore $f \,\square\, 1$ (we write 1 instead of 1_{G^3}) is also an extremal $\mathcal{C}$-epimorphism, $(f \,\square\, 1)^{-1}[B_1] = X$ and $(f \,\square\, 1)^{-1}(0, 0) = \{((0, 0), 0)\}$ (and $(0, 0) \in wB_1$). Now it suffices to show that $(0, (0, 0)) \notin \overline{a[X]}$ in $G^3 \,\square\, (G^3 \,\square\, G^3)$. Clearly, $a[X] = \{(0, (0, 1)), (0, (1, 0)),$ $(0, (1, 2)), (1, (0, 1)), (1, (1, 1)), (2, (1, 1)), (2, (2, 1)),$ $(1, (0, 2)), (1, (1, 2)), (2, (1, 2)), (2, (2, 2))\}$.

$(0, (0, 0)) \in \overline{a[X]}$ iff there exists $D \in \dot{\mathcal{P}}$ such that $(0, (0, 0)) \in \overline{(1 \square p)^{-1}[a[X]] \cap (G^3 \square D)}$ in $G^3 \square D$. Denote $(1 \square p)^{-1}[a[X]] \cap (G^3 \square D)$ by H_D. We want to show that for each $D \in \mathcal{P}$ $(0, (0, 0)) \notin H_D$ in $G^3 \square D$. Obviously, it suffices to consider the following cases:

(1) Let $D = \{0\} \square G^3$. Then $H_D = \{(0, (0, 1)), (1, (0, 2)), (1, (0, 1))\}$ and evidently $(0, (0, 0)) \notin \bar{H}_D$ (because $(0, 0) \notin \overline{\{(0, 1), (1, 2), (1, 1)\}}$ in $G^3 \square G^3$ by 1.19.(iv)).

(2) If $D = G^3 \square \{0\}$, then $H_D = \{(0, (1, 0))\}$ and clearly $(0, (0, 0)) \notin \bar{H}_D$.

(3) If $D = G^5_K$, then $H_D = \{(0, (1, 2)), (1, (1, 1)), (2, (1, 1)), (2, (2, 1)), (1, (1, 2)), (2, (1, 2)), (2, (2, 2))\}$. Let $(0, (0, 0)) \in \bar{H}_D$. Since $H_D(0, 0) = \emptyset$ we have $H_D(2, 1) = \{1, 2\}$ by 1.19.(iv) but this fails to be true. Hence $(0, (0, 0)) \notin \bar{H}_D$.

(4) Let $D \in \mathcal{P}_2$. Because $H_D(0, 0) = \emptyset$, $(0, (0, 0)) \in \bar{H}_D$ implies that $1H_D \cup 0H_D = D - \{(0, 0)\}$ and $2H_D \cup 0H_D = D - \{(0, 0)\}$. But $1H_D \cap 2H_D \subset \{(1, 1), (1, 2)\}$, $0H_D \subset \{(0, 1), (1, 0), (1, 2)\}$ so that $D - \{(0, 0)\} \subset \{(0, 1), (1, 2), (1, 0), (1, 1)\}$. Since $D - \{(0, 0)\}$ belongs to $\mathcal{P}_1$ it follows that $D - \{(0, 0)\}$ has to fulfil 1.19.(iv). But $\{(0, 1), (1, 0), (1, 2), (1, 1)\}$ does not fulfil it. Hence $(0, (0, 0)) \notin \bar{H}_D$.

Thus for each $D \in \mathcal{P}$ $(0, (0, 0)) \notin \bar{H}_D$ in $G^3 \square D$ so that $(0, (0, 0)) \notin \overline{a[X]}$ in $G^3 \square (G^3 \square G^3)$ and therefore a is not continuous.

So, we have proved that $(0, 0)$ cannot belong to wB_1 and therefore also to wB_i for each $i \in \{1, 2, 3, 4\}$. Obviously, $(0, 0) \notin wM$ if $M \subset B_i$ for some $i \in \{1, 2, 3, 4\}$. This implies that if $G^3 \boxtimes G^3 < G^3 \square G^3$ ($\neq G^3 \bigcirc G^3$), then (by 1.23.) $(0, 0) \in wA_1$. Put $\mathcal{S}_3 = \{M: M \subset G^3 \times G^3$ and $(0, 0) \in wM - vM\}$ (clearly, $M \in \mathcal{S}_3$ implies $M \subset A_j$ for some $j \in \{1, 2, \ldots, 8\}$). For each $M \in \mathcal{S}_3$ define G^m_M analogously as in the previous case and put $\mathcal{S}_4 = \{G^m_M: M \in \mathcal{S}_3\}$. Let $\mathcal{S}' = \mathcal{S}_4 \cup \{\{k\} \square G^3: 0 \leq k \leq 2\} \cup \cup \{G^3 \square \{k\}: 0 \leq k \leq 2\} \cup \{G^6_K\}$. The $\mathcal{C}$-morphisms $(j_D: D \longrightarrow G^3 \square G^3;\ z \longmapsto z)_{D \in \mathcal{S}'}$ uniquely determine the $\mathcal{C}$-morphism $p: \bigsqcup_{D \in \mathcal{S}'} D \longrightarrow G^3 \square G^3$ which is evidently an extremal $\mathcal{C}$-epimorphism. Again the extremal $\mathcal{C}$-epimorphisms $p \square 1$ and $1 \square p$ uniquely determine $(G^3 \square G^3) \square G^3$ and $G^3 \square (G^3 \square G^3)$ respectively. By considering the set $X = \{((0, 0), 1), ((1, 1), 0), ((1, 1), 1), ((1, 1), 2), ((1, 2), 2), ((2, 2), 2), ((0, 1), 2), ((1, 0), 2)\} \subset (G^3 \square G^3) \square G^3$ it can be easily shown that the map $a: (G^3 \square G^3) \square G^3 \longrightarrow G^3 \square (G^3 \square G^3);\ ((x, y), z) \longmapsto (x, (y, z))$ is not a $\mathcal{C}$-isomorphism. In fact, it is easy to check (analogously as in the previous case) that $((0, 0), 0) \in \overline{X}$ in $(G^3 \square G^3) \square G^3$ but $(0, (0, 0)) \notin \overline{a[X]}$ in $G^3 \square (G^3 \square G^3)$. Hence the hypothesis $G^3 \boxtimes G^3 < G^3 \square G^3$ yields a contradiction so that it holds:

<u>1.24. Theorem.</u> If $(\mathcal{C}, \square, H)$ is a closed category, then $\square$ is naturally isomorphic with $\otimes$ or with $\boxtimes$.

2. Tensor products in the category $\mathcal{F}$

Recall (see [2]) that an F-space is a C-space (P, u) for which $u(uM) = uM$ for each $M \subset P$. The subcategory $\mathcal{F}$ of $\mathcal{C}$ consisting of all F-spaces is a bireflective subcategory of $\mathcal{C}$ (see [3]) and therefore (by [8]) it (together with the forgetful functor) is an initially structured category. If (P, u) is a

C-space, v is the F-modification of u (i.e. $v = u^{\lambda}$ where λ is the smallest ordinal such that $u^{\lambda+1}M = u^{\lambda}M$ for all $M \subset P$), then $(id_P, (P, v))$ is the $\mathfrak{F}$-reflection of (P, u).

Let (P, u), (Q, v) be F-spaces and $(id_{P \times Q}, (P, u) \otimes_{\mathfrak{F}} (Q, v))$ is the $\mathfrak{F}$-reflection of $(P, u) \otimes (Q, v)$. Then, obviously, $\otimes_{\mathfrak{F}} : \mathfrak{F} \times \mathfrak{F} \longrightarrow \mathfrak{F}$; $(P, Q) \longmapsto P \otimes_{\mathfrak{F}} Q$, $(f, g) \longmapsto f \otimes_{\mathfrak{F}} g = f \times g$ is a functor.

<u>2.1. Proposition.</u> Let P, Q, S be F-spaces. Then a map $h: P \otimes_{\mathfrak{F}} Q \longrightarrow S$ is continuous if and only if for each $a \in P$ and $b \in Q$ h_a and h^b are continuous.

<u>Proof.</u> Immediate from 1.3. and from the properties of the $\mathfrak{F}$-reflection $(id_{P \times Q}, P \otimes_{\mathfrak{F}} Q)$.

Since $\mathfrak{F}$ is bireflective it is closed under the formation of $\mathcal{C}$-products and subspaces. Therefore if P, Q are F-spaces, then [P, Q] is also an F-space so that $[-, -]_{\mathfrak{F}} = [-, -]|_{\mathfrak{F}^{op} \times \mathfrak{F}} : \mathfrak{F}^{op} \times \mathfrak{F} \longrightarrow \mathfrak{F}$ is a functor and by [4, Theorem 1.9.] we have:

<u>2.2. Theorem.</u> $(\mathfrak{F}, \otimes_{\mathfrak{F}}, [-, -]_{\mathfrak{F}})$ is a closed category.

In the following we shall prove that it is (considering Convention 1.16.) the only closed structure on $\mathfrak{F}$.

<u>2.3. Proposition.</u> If $(\mathfrak{F}, \Box, H)$ is a closed category, then $id_{P \times Q}: (P, u) \otimes_{\mathfrak{F}} (Q, v) \longrightarrow (P, u) \Box (Q, v)$ is an $\mathfrak{F}$-morphism for all F-spaces (P, u), (Q, v).

<u>Proof.</u> Immediate from [4, Proposition 1.12.].

Recall (see the first section) that for any cardinal $\alpha \geq 2$ the coreflective hull of $\mathcal{G}_{\alpha} = \{G^{\xi} : \xi \geq \alpha\}$ in $\mathfrak{F}$ coincides with $\mathfrak{F}$. Hence, if $(\mathfrak{F}, \Box, H)$ is a closed category, then $\Box$ is uniquely determined by its values on $\mathcal{G}_{\alpha} \times \mathcal{G}_{\alpha}$

Let $\xi, \zeta \geq 2$ be cardinals and u be a C-topology of $G^\xi \otimes G^\zeta$. It is easy to verify that u^2 is the F-modification of u, i.e. u^2 is the topology of $G^\xi \otimes_{\mathcal{F}} G^\zeta$.

2.4. Proposition. Let $(\mathcal{F}, \square, H)$ be a closed category, $G^3 \square G^3 = G^3 \otimes_{\mathcal{F}} G^3$. Then $\square = \otimes_{\mathcal{F}}$.

Proof. Suppose $\square \neq \otimes_{\mathcal{F}}$. Then there exist $\xi, \zeta \geq 3$ such that $G^\xi \otimes_{\mathcal{F}} G^\zeta < G^\xi \square G^\zeta$ $(\leq G^\xi \bigcirc G^\zeta)$. This implies that there exists a set $M \subset G^\xi \times G^\zeta$ for which $(0_\xi, 0_\zeta) \in wM - u^2M$ where w denotes the F-topology of $G^\xi \square G^\zeta$ and u is the C-topology of $G^\xi \otimes G^\zeta$. $(0_\xi, 0_\zeta) \notin u^2M$ implies that there exist $y_0 \in A_\zeta$ and $x_0 \in A_\xi$ with $(0_\xi, y_0) \in uM$ and $(x_0, 0_\zeta) \in uM$. But then $My_0 \subsetneqq A_\xi$ and $x_0M \subsetneqq A_\zeta$. From 1.19.(iv) it follows that $(x_0, y_0) \in M$. Therefore the maps $f: G^\xi \longrightarrow G^3 \sqcup \{*\}$; $0_\xi \longmapsto 0$, $f[A_\xi - My_0] \subset \{1\}$, $x_0 \longmapsto 2$ and $f(x) = *$ otherwise, $g: G^\zeta \longrightarrow G^3 \sqcup \{*\}$; $0_\zeta \longmapsto 0$, $g[A_\zeta - x_0M] \subset \{1\}$, $y_0 \longmapsto 2$ and $g(y) = * $ otherwise are defined correctly and evidently they are continuous. Then $f \square g$: $G^\xi \square G^\zeta \longrightarrow (G^3 \sqcup \{*\}) \square (G^3 \sqcup \{*\}) = (G^3 \square G^3) \sqcup (\{*\} \square G^3) \sqcup (G^3 \square \{*\}) \sqcup (\{*\} \square \{*\})$ is also continuous and $(f \square g)(0_\xi, 0_\zeta) = (0, 0) \in \overline{(f \square g)[M]}$ in $(G^3 \sqcup \{*\}) \square (G^3 \sqcup \{*\})$ iff $(0, 0) \in \overline{(f \square g)[M] \cap (G^3 \square G^3)}$ in $G^3 \square G^3 = G^3 \otimes_{\mathcal{F}} G^3$. But $(f \square g)[M] \cap (G^3 \square G^3) \subset \{(0, 1), (1, 0), (1, 1), (2, 2)\} = D$ and $(0, 0) \notin \overline{D}$ in $G^3 \otimes_{\mathcal{F}} G^3$ - a contradiction. Thus $G^\xi \square G^\zeta = G^\xi \otimes_{\mathcal{F}} G^\zeta$ for all cardinals $\xi, \zeta \geq 3$, i.e. $\square = \otimes_{\mathcal{F}}$.

Throughout the remainder of this section, let $(\mathcal{F}, \square, H)$ be a closed category with $\square \neq \otimes_{\mathcal{F}}$. Then by 2.4. and 1.19. $G^3 \otimes_{\mathcal{F}} G^3 < G^3 \square G^3 \leq G^3 \bigcirc G^3$. Recall that $f: G^3 \longrightarrow G^3$; $0 \longmapsto 0$, $1 \longmapsto 2$, $2 \longmapsto 1$ is an $\mathcal{F}$-isomorphism and $f_1 = f \square 1$, $f_2 = 1 \square f$, $f_3 = f \square f$ are also $\mathcal{F}$-isomorphisms (1 is always written instead of 1_{G^3}). Put $M_1 = \{(0, 1), (1, 2), (2, 2)\}$, $M_2 = f_2[M_1]$, $M_3 = c[M_1]$, $M_4 = c[M_2]$ and $K = \{1, 2\} \times \{1, 2\}$. Denote by w the F-topology of $G^3 \square G^3$, by u the C-topology of $G^3 \otimes G^3$. Then $v = u^2$ is the topology of $G^3 \otimes_{\mathfrak{F}} G^3$. It is easy to see that $(0, 0) \in vM = u^2M$ iff $(0, 0) \in M$ or $(0, 0) \in uM$ or there exists $i \in \{1, 2, 3, 4\}$ such that $M_i \subset M$ or $K \subset M$. In fact, $(0, 0) \in u^2M - uM$ iff $\{0\} \times \{1, 2\} \subset uM$ or $\{1, 2\} \times \{0\} \subset uM$ and simultaneously $\{0\} \times \{1, 2\} \not\subset M$ and $\{1, 2\} \times \{0\} \not\subset M$ and this holds iff there exists $i \in \{1, 2, 3, 4\}$ with $M_i \subset M$ or $K \subset M$.

Now let $D \subset G^3 \times G^3$ with $(0, 0) \in wD - vD$. Then card DO = card $OD = 1$ (otherwise 1.19. implies that there exists $i \in \{1, 2, 3, 4\}$ such that $M_i \subset D$ or $K \subset D$ or $(0, 0) \in uD$). It can be easily checked that then D is one of the following sets: $B_1 = \{(0, 1), (1, 0), (2, 2)\}$, $B_{i+1} = f_i[B_1]$ where $1 \leq i \leq 3$, $H_1 = \{(0, 1), (1, 0), (1, 1), (2, 2)\}$, $H_{i+1} = f_i[H_1]$ where $1 \leq i \leq 3$ (observe that for each i, $B_i \subset H_i$). It is obvious that $G^3 \otimes_{\mathfrak{F}} G^3 < G^3 \square G^3$ iff $(0, 0) \in wH_1$. Hence suppose $(0, 0) \in wH_1$. Let $\mathcal{G}_1 = \{\{k\} \square G^3 : 0 \leq k \leq 2\} \cup \{G^3 \square \{k\} : 0 \leq k \leq 2\} \cup \cup \{G^4_{M_i} : 1 \leq i \leq 4\} \cup \{G^5_K\}$ where $G^4_{M_i}$ is the F-space defined on $M_i \cup \{(0, 0)\}$ by $\overline{M}_i = M_i \cup \{(0, 0)\}$ and $\overline{X} = X$ otherwise (isomorphic with G^4), G^5_K is defined on $K \cup \{(0, 0)\}$ similarly. Put $\mathcal{G}'_3 = \{H_i : 1 \leq i \leq 4\}$, $\mathcal{G}'_4 = \{B_i : 1 \leq i \leq 4\}$. Let $\mathcal{G}'$ be the family of all subsets D of $G^3 \square G^3$ with $(0, 0) \in wD - vD$. Then, clearly, $\mathcal{G}'_3 \subset \mathcal{G}' \subset \mathcal{G}'_3 \cup \mathcal{G}'_4$. Let $\mathcal{G}_2 = \{G^d_D : D \in \mathcal{G}'\}$ where $d = 1 + \text{card } D$ and G^d_D is defined analogously as $G^4_{M_i}$. Put $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$. For each $A \in \mathcal{G}$ define the $\mathfrak{F}$-morphism $j_A : A \longrightarrow G^3 \square G^3$ by $j_A(z) = z$ for all $z \in A$. Then the family

$= (0, 0)$.

1. If $A = \{k\} \square G^3$ or $A = G^3 \square \{k\}$ for $0 \leq k \leq 2$, then it can be easily verified that $\overline{B}_A = B_A$ in $G^3 \square A$.

2. Let $A \in \mathcal{S}$ and A be different from the spaces considered in 1. Let $q_1 : G^3 \times (G^3 \times G^3) \longrightarrow G^3$ be the projection $(x, (y, z)) \longmapsto x$, $q_{23} : G^3 \times (G^3 \times G^3) \longrightarrow G^3 \times G^3$ be the projection $(x, (y, z)) \longmapsto (y, z)$. Since for each $j \in \{0, 1, 2\}$ card $jB_A \leq$ $\leq$ card $j(a[X]) = 2$ ($jB_A = \{(y, z) \in A : (j, (y, z)) \in B_A\}$) we obtain $(j, (0, 0)) \notin \overline{B}_A$ for each $j \in \{0, 1, 2\}$. Let $(y, z) \in q_{23}[a[X]]$. Then $0 \in B_A(y, z)$ or card $B_A(y, z) \leq$ $\leq$ card $a[X](y, z) = 1$ ($B_A(y, z) = \{j \in G^3 : (j, (y, z)) \in B_A\}$) so that $(0, (y, z)) \in \overline{B}_A - B_A$ in $G^3 \square A$ for each $A \in \mathcal{S}$ excluding the spaces considered in 1.; card $0B_A \leq$ card $0(a[X]) =$ $= 2$ and $(1B_A \cap 2B_A) \subset (1a[X] \cap 2\, a[X]) = \emptyset$ so that by 1.19. $(0, (0, 0)) \in \overline{B}_A$ in $G^3 \square A$. Hence for each $A \in \mathcal{S}$ $\overline{B}_A = B_A$ in $G^3 \square A$ and therefore $w_2 a[X] = a[X]$. But then for any ordinal λ $(w_2)^\lambda a[X] = a[X]$ and therefore $\overline{a[X]} = a[X]$ in $G^3 \square (G^3 \square G^3)$. Thus a is not continuous, i.e. the hypothesis $G^3 \otimes_{\mathfrak{F}} G^3 < G^3 \square G^3$ yields a contradiction so that $G^3 \square G^3 = G^3 \otimes_{\mathfrak{F}} G^3$. Then by 2.4. it holds:

2.5. Theorem. If $(\mathfrak{F}, \square, H)$ is a closed category, then $\square$ is isomorphic with $\otimes_{\mathfrak{F}}$.

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Received: 22. 8. 1977

S ú h r n

TENZOROVÉ SÚČINY V JEDNEJ KATEGÓRII
ZOVŠEOBECNENÝCH TOPOLOGICKÝCH PRIESTOROV

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Topológia u topologického priestoru (P, u) v Čechovom zmysle z roku 1937 spĺňa nasledujúce axiomy: 1. $u\emptyset = \emptyset$, 2. $M \subset uM$ pre každé $M \subset P$, 3. $uL \subset uM$, ak $L \subset M \subset P$. Zobrazenie

f: $(P, u) \longrightarrow (Q, v)$ sa nazýva spojité, ak $f[uM] \subset vf[M]$ pre každé $M \subset P$. V práci je dokázané, že v kategórii $\mathcal{C}$ topologických priestorov a spojitých zobrazení v horeuvedenom zmysle existujú (až na prirodzený izomorfizmus) práve dva tenzorové súčiny. Naviac je dokázané, že v (plnej) podkategórii $\mathfrak{F}$ kategórie $\mathcal{C}$, ktorá pozostáva zo všetkých C-priestorov (P, u) s vlastnosťou $u(uM) = uM$ pre každé $M \subset P$, existuje práve jeden tenzorový súčin.

Р е з ю м е

ТЕНЗОРНЫЕ ПРОИЗВЕДЕНИЯ В ОДНОЙ КАТЕГОРИИ ОБОБЩЕННЫХ ТОПОЛОГИЧЕСКИХ ПРОСТРАНСТВ

Юрай Чинчура, Братислава

Топология топологического пространства (P, u) в смысле Чеха из 1937 года исполняет следующие аксиомы: 1. $u\emptyset = \emptyset$, 2. $M \subset uM$ для каждого $M \subset P$, 3. $uL \subset uM$, если $L \subset M \subset P$. Отображение f: $(P, u) \longrightarrow (Q, v)$ называется непрерывным, если $f[uM] \subset vf[M]$ для каждого $M \subset P$. В работе доказывается, что в категории $\mathcal{C}$ топологических пространств и непрерывных отображений в приведенном выше смысле можно определить с точностью до изоморфизма точно два тензорных произведения. Более того доказано, что в (полной) подкатегории $\mathfrak{F}$ всех пространств (P, u) исполняющих $u(uM) = uM$ для каждого $M \subset P$ категории $\mathcal{C}$ можно определить с точностью до изоморфизма единственное тензорное произведение.