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ON SIMPLEXES IN THE HERMITIAN SPACE H_n WITH CERTAIN PROPERTIES OF REAL SIMPLEXES

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The purpose of this paper which refers directly to [3] is to formulate the necessary and sufficient condition for a simplex in the space H_n to have an obvious property of real simplexes, namely that the "orthogonal bisectors" of its edges have exactly one point in common. More-over, it will be shown that this case occurs iff there exists a point in the H_n such that one of the two bisectors of the angle of each pair of distinct, (n-1)-dimensional faces of the simplex passes through this point. The mathematical apparatus developed in [3] will be currently used throughout this paper. The part of this work showing the criteria of orthocentrity of real simplexes, formulated by M. Fiedler in [2], to be also the criteria of orthocentrity of simplexes in H_n is not without interest too.

All undefined concepts and unproved statements or relations used in this paper are defined or proved in [3].

Before introducing the problem itself we will prove three lemmas containing the most important relations which determine the distances and the angles (the orthogonality particularly) of the basic figures of the space H_n by means of the barycentric coordinates of their elements and (e+g)-norms of a certain simplex.

Let Σ be a fixly chosen n-simplex in the space H_n with the vertices A^i, $i \in N = \{0,1,\dots,n\}$ and let e_{ij} $[g_{ij}]$, $i,j \in N$ are its e-norms [g-norms].

When speaking about the barycentric coordinates of an element $\xi \in H_n$ we always consider its barycentric coordinates with respect to this n-simplex. Moreover, it is always assumed that the indices i, j, k, p, ... run over the set N, the indices α, β, γ, ... run over its subset $N_1 = \{1,2,\dots,n\}$ and the indices r, s, t, ... over the hyperset $N_2 = \{0,1,\dots,n,n+1\}$.

<u>Lemma 1.</u> The numbers $x_i = \sum_k g_{ki} \bar{\alpha}_k$, $i \in N$ may be considered to be the barycentric coordinates of a vector perpendicular to the hyperplane α with the equation $\sum_i \alpha_i z_i = 0$.

<u>Proof.</u> At first it is evident that not all x_i are equal to zero. Otherwise the (n+1)-tuple $(\alpha_0, \alpha_1, \dots, \alpha_n)$ would be a solution of the homogeneous system of linear equations:

$$\sum_k g_{ik} z_k = 0 , \quad i = 0, 1, \dots, n$$

i. e. it would be a multiple of the solution $(1,1,\dots,1)$ [since for all $i \in N$ $\sum_k g_{ik} = 0$ and rank $(g_{ij}) = n$]. This contradicts the properties of the coefficients of the hyperplane equation. The fact, that the numbers x_i may be considered as the barycentric coordinates of a vector $\xi \in H_n$ is a consequence of the following calculation:

$$\sum_i x_i = \sum_{i,k} g_{ki} \bar{\alpha}_k = \sum_k \Big(\sum_i g_{ki} \Big) \bar{\alpha}_k = 0$$

Since

$$[ex\bar{z}] = \sum_{i,j} e_{ij} x_i \bar{z}_j = \sum_{i,j,k} e_{ij} g_{ki} \bar{\alpha}_k \bar{z}_j = \sum_{j,k} \Big(\sum_i e_{ij} g_{ki} \Big) \bar{\alpha}_k \bar{z}_j =$$

$$= \sum_{j,k} (-2\,\delta_{jk} - g_{k,n+1})\,\bar{\alpha}_k \bar{z}_j =$$

$$= -2\,\overline{\sum_j \alpha_j z_j} - \sum_j \bar{z}_j \sum_k g_{k,n+1}\,\bar{\alpha}_k = 0$$

for each direction vector $\xi = (z_i)$ of the hyperplane α, the vector ξ is perpendicular to the hyperplane α and the proof is complete.

L e m m a 2. Let φ be the angle of the hyperplanes $\alpha \equiv \sum_i \alpha_i z_i = 0$ and $\beta \equiv \sum_i \beta_i z_i = 0$. Then:

$$\cos\varphi = \frac{|[g\bar{\alpha}\beta]|}{\sqrt{[g\bar{\alpha}\alpha]}\,\sqrt{[g\bar{\beta}\beta]}} \quad \text{where} \quad [g\bar{\alpha}\beta] = \sum_{i,j} g_{ij}\,\bar{\alpha}_i\,\beta_j \qquad (1)$$

and analogically for $[g\bar{\alpha}\alpha]$ and $[g\bar{\beta}\beta]$.

P r o o f. Let $p \perp \alpha$, $q \perp \beta$ be arbitrary lines. According to Lemma 1 the numbers $x_i = \sum_k g_{ki}\,\bar{\alpha}_k$ $\left[y_i = \sum_k g_{ki}\,\bar{\beta}_k \right]$

$i = 0, 1, \ldots, n$ are the barycentric coordinates of the direction vector of the line p $[q]$. Then:

$$\cos\varphi = \cos\sphericalangle pq = \frac{|[ex\bar{y}]|}{\sqrt{[exx]}\,\sqrt{[eyy]}} \quad \left[0 \leq \sphericalangle pq \leq \pi/2\right] \qquad (2)$$

Moreover:

$$[ex\bar{y}] = \sum_{i,j} e_{ij} \left(\sum_p g_{pi}\,\bar{\alpha}_p\right) \left(\overline{\sum_k g_{kj}\,\bar{\beta}_k}\right) =$$

$$= \sum_{i,k,p} \left(\sum_j e_{ij} g_{jk}\right) g_{pi}\,\bar{\alpha}_p\,\beta_k =$$

$$= \sum_{i,k,p} \left(-2\,\delta_{ik} - g_{n+1,k}\right) g_{pi}\,\bar{\alpha}_p\,\beta_k =$$

$$= -2 \sum_{i,p} g_{pi}\,\bar{\alpha}_p\,\beta_i - \sum_{p,k} \left(\sum_i g_{pi}\right) g_{n+1,k}\,\bar{\alpha}_p\,\beta_k = -2\,[g\bar{\alpha}\beta]$$

It can be shown analogically that

$$[ex\bar{x}] = -2[g\bar{\alpha}\alpha] \quad \text{and} \quad [ey\bar{y}] = -2[g\bar{\beta}\beta]$$

Substituting these results into (2) we get (1) .

Lemma 3. The distance from the point A, which homogeneous barycentric coordinates are (a_i), to the hyperplane α with the equation $\sum_i \alpha_i z_i = 0$ is given by the formula:

$$d^2(A,\alpha) = \frac{\left|\sum_i \alpha_i a_i\right|^2}{\left|\sum a_i\right|^2 [g\bar{\alpha}\alpha]} \tag{3}$$

Proof. Let (a'_i) be nonhomogeneous barycentric coordinates of the point A and let (b'_i) be nonhomogeneous barycentric coordinates of the orthogonal projection A^* of the point A on the hyperplane α. Since the vector $A^* - A$ is perpendicular to the hyperplane α, there exist a complex number $\mu \neq 0$ such that

$$\forall\, i \in N \quad b'_i = a'_i + \mu\left(\sum_k g_{ki}\bar{\alpha}_k\right)$$

Substituting these values in the equality $\sum_i \alpha_i b'_i = 0 \quad [A^* \in \alpha]$ we obtain the equality:

$$\mu \sum_{i,k} g_{ki}\bar{\alpha}_k \alpha_i = -\sum_i \alpha_i a'_i \quad \text{i.e.} \quad \mu = -\frac{\sum_i \alpha_i a'_i}{[g\bar{\alpha}\alpha]}$$

consequently

$$|\mu|^2 = \frac{\left|\sum_i \alpha_i a_i\right|^2}{\left|\sum_i a_i\right|^2 [g\bar{\alpha}\alpha]^2}$$

Since the distance from the point A to the hyperplane α is equal to the distance between the points A, A^*, we have:

$$d^2(A,\alpha) = |A^* - A|^2 = -\frac{1}{2}\sum_{i,j} e_{ij}\left(\mu\sum_k g_{ki}\bar{\alpha}_k\right)\overline{\left(\mu\sum_p g_{pj}\bar{\alpha}_p\right)} =$$

$$= -\frac{1}{2}|\mu|^2 \sum_{j,k,p}\left(\sum_i e_{ij} g_{ki}\right)\bar{\alpha}_k g_{jp}\alpha_p =$$

$$= -\frac{1}{2}|\mu|^2 \sum_{j,k,p} (-2\,\delta_{jk} - g_{k,n+1})\,\bar{\alpha}_k g_{jp} \alpha_p =$$

$$= |\mu|^2 \sum_{k,p} g_{kp} \bar{\alpha}_k \alpha_p + \frac{1}{2}|\mu|^2 \sum_{k,p} \Big(\sum_j g_{jp} \Big) g_{k,n+1} \bar{\alpha}_k \alpha_p =$$

$$= |\mu|^2 [g\bar{\alpha}\alpha] = \frac{\left| \sum_i \alpha_i a_i \right|^2}{\left|\sum_i a_i\right|^2 [g\bar{\alpha}\alpha]}$$

L e m m a 4. The e-norms of the simplex Σ are real iff the e-norms of all its two-dimensional subsimplexes are real.

P r o o f. Let α, β be any two distinct indices of the set N_1. Let us consider two-dimensional subsimplex $\varphi = \{A^0, A^\alpha, A^\beta\}$ of the simplex Σ. It follows from the assumption of its e-norms being real $\big[[3]$, $(19)\big]$ that $c_{\alpha\beta}$ is real. The rest of the statement follows from the relations (17), (30) of the paper [3].

R e m a r k 1. It is evident that a sufficient condition for the e-norms of the simplex Σ to be real is the e-norms of all its two-dimensional subsimplexes with one common vertex are real.

R e m a r k 2. It follows from the above mentioned relations (17), (19) of the [3], from the matrix $(g_{\alpha\beta})$ being the inverse of $(c_{\alpha\beta})$ and from the equalities: $g_{0\alpha} = -\sum_\gamma g_{\gamma\alpha}$; $g_{\alpha 0} = -\sum_\gamma g_{\alpha\gamma}$ valid for each $\alpha \in N_1$, that the e-norms of the simplex Σ are real iff its g-norms are real.

C o r o l l a r y. The g-norms of the simplex Σ are real iff the g-norms of all its two-dimensional subsimplexes are real.

L e m m a 5. All g-norms of the simplex Σ are real iff the product:

$$s_{i_1 i_2 \ldots i_q} = g_{i_1 i_2} g_{i_2 i_3} \ldots g_{i_{q-1} i_q} g_{i_q i_1} \qquad (4)$$

is real for each variation $i_1, i_2, \ldots, i_q$; $2 < q \leq n+1$ of numbers $0, 1, \ldots, n$.

P r o o f. Let $\sum_h$ be the (n-1)-dimensional subsimplex of the simplex $\sum$, which does not contain the vertex A^h. Let us consider the matrix G_h with the elements:

$$g'_{\lambda\varphi} = \frac{1}{g_{hh}} \begin{vmatrix} g_{\lambda\varphi} & g_{\lambda h} \\ g_{h\varphi} & g_{hh} \end{vmatrix} \qquad \lambda, \varphi \in N - \{h\} \tag{5}$$

If $h \neq 0$, its submatrix $G_h(0/0)$ is the inverse of the Gram matrix of the system of vectors $\{A^1 - A^0, A^2 - A^0, \ldots, A^n - A^0\} \setminus \{A^h - A^0\}$. This follows from the following equalities:

$$\sum_{\xi \in N'} c_{\lambda\xi}\, g'_{\xi\varphi} = \frac{1}{g_{hh}} \left[g_{hh} \sum_{\xi \in N'} c_{\lambda\xi}\, g_{\xi\varphi} - g_{h\varphi} \sum_{\xi \in N'} c_{\lambda\xi}\, g_{\xi h} \right] =$$
$$= \frac{1}{g_{hh}} \left[g_{hh}(\delta_{\lambda\varphi} - c_{\lambda h} g_{h\varphi}) + g_{h\varphi} c_{\lambda h} g_{hh} \right] = \delta_{\lambda\varphi} ,$$

where $\lambda, \varphi \in N' = N_1 - \{h\}$.

Let us assume now that $h = 0$ and denote $a_{\alpha\beta} = (A^\alpha - A^1, A^\beta - A^1)$, where $\alpha, \beta \in N'' = N \setminus \{0,1\}$. Then:

$$\sum_{\xi \in N''} a_{\lambda\xi}\, g'_{\xi\varphi} = \frac{1}{g_{oo}} \sum_{\xi \in N''} (c_{\lambda\xi} - c_{1\xi} - c_{\lambda 1} + c_{11})\,(g_{\xi\varphi} g_{oo} - g_{\xi o} g_{o\varphi}) =$$
$$= \frac{1}{g_{oo}} \Big[(\delta_{\lambda\varphi} - c_{\lambda 1} g_{1\varphi} + c_{11} g_{1\varphi} + c_{\lambda 1} g_{o\varphi} + c_{\lambda 1} g_{1\varphi} - c_{11} g_{o\varphi} -$$
$$- c_{11} g_{1\varphi})\, g_{oo} + (1 + c_{\lambda 1} g_{1o} - 1 - c_{11} g_{1o} - c_{\lambda 1} g_{oo} -$$
$$- c_{\lambda 1} g_{1o} + c_{11} g_{oo} + c_{11} g_{1o})\, g_{o\varphi} \Big] = \delta_{\lambda\varphi}$$

for all $\lambda, \varphi \in N''$.

Thus we have shown that the submatrix $G_h(1/1)$ is the inverse of the matrix $(a_{\alpha\beta})$, $\alpha, \beta \in N''$ for $h = 0$.

Moreover, it can be easily shown that the sum of all elements of any row or column of the matrix G_h is equal to zero.

Subsequently, it follows from the above stated that the numbers $g'_{\lambda\gamma}$ are g-norms of the (n-1)-simplex Σ_h.

Let now $i_1, i_2, \dots, i_p$ be any distinct elements of the set $N - \{h\}$. If we suppose that all numbers $s_{i_1 i_2 \dots i_q}$ are real, then the following product is evidently real, too:

$$g'_{i_1 i_2} g'_{i_2 i_3} \cdots g'_{i_p i_1} = g_{i_1 i_2} g_{i_2 i_3} \cdots g_{i_p i_1} + \sum_{k=1}^{p} \left(-\frac{1}{g_{hh}}\right)^k \sum_{S_k} \left[\sum_{(P)} (g_{hi_1} \cdots g_{i_{s_1} h})(g_{hi_{s_1+1}} \cdots g_{i_{s_2} h}) \cdots (g_{hi_{s_{k-1}+1}} \cdots g_{i_{s_k} h}) \right]$$

where

1/ S_k is a set consisting of all k-tuples $(s_1, s_2, \dots, s_k)$, defined as follows: $s_\alpha = \sum_{\beta=1}^{\alpha} d_\beta - \alpha$, where $d_1, d_2, \dots, d_k$ is an arbitrary set of natural numbers satisfying the conditions:

(i) $2 \leq d_1 \leq d_2 \leq \dots \leq d_k \leq p+1$

(ii) $\sum_{\alpha=1}^{k} d_\alpha = p + k$

2/ P is a set consisting of a p-tuple $(i_1, i_2, \dots, i_p)$ and of all its cyclic permutations

3/ (P) is an arbitrary maximal subset of the set P, having the property that the members belonging to its elements are distinct.

Of this is applied on each m-dimensional subsimplex Σ' $(3 \leq m \leq n-1)$ of the simplex Σ (i.e. the n-simplex Σ is replaced by m-simplex Σ') we get that the product $g^*_{ij} g^*_{jk} g^*_{ki}$ is real for each two-dimensional face $\Sigma^* = \{A^i, A^j, A^k\}$ of the simplex Σ. That means the equality $g^*_{ij} g^*_{jk} g^*_{ki} = g^*_{ik} g^*_{kj} g^*_{ji}$ holds. This equality may be rewritten as:

$$(g^*_{ij} - g^*_{ji}) \begin{vmatrix} g_{ii} & g_{ij} \\ g_{ji} & g_{jj} \end{vmatrix} = 0$$

from where it follows that g^*_{ij} is real. But then also the numbers $g^*_{ik} = - g^*_{ii} - g^*_{ij}$ and $g^*_{jk} = - g^*_{ji} - g^*_{jj}$ must be real. Now the reality of the g-norms of the simplex Σ follows from the corollary of Lemma 4. The converse statement is evident.

Similarly as in E_n the hyperplane passing through the midpoint of a line segment and perpendicular to the line segment will be called the perpendicular bisector of the line segment.

<u>Lemma 6.</u> Let s_{ij} be the perpendicular bisector of the edge A^iA^j of the simplex Σ. Then

$$\operatorname{card}\Big(\bigcap_{\substack{i,j\in N \\ i<j}} s_{ij} \Big) \leq 1 \tag{6}$$

<u>Proof.</u> Let $A, B \in \bigcap_{\substack{i,j\in N \\ i<j}} s_{ij}$. Then:

$$\forall\, \alpha \in N_1 \qquad B - A \perp \gamma_\alpha = A^\alpha - A^0$$

From these relations and the fact that $\{\gamma_\alpha\}$ is the basis of the space V_n it follows that $B - A \perp V_n$, particularly $B - A \perp B - A$ i.e. $(B - A, B - A) = 0$. The latter equation and the well known properties of the inner (scalar) produkt in unitary vector space give $B - A$ to be the zero vector i.e. the points A, B are identical. This completes the proof.

<u>Lemma 7.</u> The perpendicular bisector s_{ij} of the edge A^iA^j of the n-simplex Σ has the following equation in the barycentric coordinates:

$$\sum_k (e_{kj} - e_{ki})\, z_k = \tfrac{1}{2}(e_{ij} - e_{ji}) \tag{7}$$

P r o o f. The point $Z = (z_i) \in s_{ij}$ iff the vectors: $Z - \frac{1}{2}(A^i + A^j)$ and $A^j - A^i$ are perpendicular, i.e. $\sum_{k,p} e_{kp}\left[z_k - \frac{1}{2}(\delta_{ik} + \delta_{jk})\right]\left[\delta_{jp} - \delta_{ip}\right] = 0$. After a minor rearrangement of this equality the (7) is obtained.

C o r o o o o a r y. The perpendicular bisectors $s_{o\alpha}$ and $s_{\alpha\beta}$ are given by the following equations:

$$s_{o\alpha} \equiv \sum_{\gamma} c_{\gamma\alpha} z_{\gamma} - \frac{1}{2} c_{\alpha\alpha} = 0, \quad \alpha = 1,2,\dots,n \tag{8}$$

$$s_{\alpha\beta} \equiv \sum c_{\gamma\alpha} z_{\gamma} - \frac{1}{2} c_{\alpha\alpha} - \left(\sum_{\gamma} c_{\gamma\beta} z_{\gamma} - \frac{1}{2} c_{\beta\beta}\right) = \frac{1}{2}(c_{\beta\alpha} - c_{\alpha\beta}) \tag{9}$$

where $c_{\alpha\beta} = \frac{1}{2}(e_{\alpha o} + e_{o\beta} - e_{\alpha\beta})$.

D e f i n i t i o n 1. The simplex Σ will be called edgecentric if $\operatorname{card}\left(\bigcap_{\substack{i,j\in N\\ i<j}} s_{ij}\right) = 1$ i.e. if the perpendicular bisectors of all its edges have exactly one point in common. This point will be called the edgecenter of the simples Σ.

T h e o r e m 1. The simplex Σ is edgecentric iff all its e-norms are real.

P r o o f. As the matrix $(c_{\alpha\beta})$ is not singular the system of linear equations (8) has exactly one solution i.e. the hyperplanes $s_{o\alpha}$, $\alpha = 1, 2, \dots, n$ have exactly one point in common. It follows from (8) and (9) that $\bigcap_{\gamma\in N_1} s_{o\gamma} \in s_{\alpha\beta}$ iff $c_{\beta\alpha} - c_{\alpha\beta} = 0$ i.e. if $c_{\alpha\beta}$ is real. The rest of the proof follows from the relations (17) and (19) of [3].

C o r o l l a r y 1. The simplex Σ is edgecentric iff all its g-norms are real.

C o r o l l a r y 2. The edgecenter of the simplex Σ (if it exists) has the barycentric coordinates:

$$\left(-\frac{1}{2}g_{n+1,0}, -\frac{1}{2}g_{n+1,1}, \ldots, -\frac{1}{2}g_{n+1,n}\right) \qquad (10)$$

P r o o f. The statement is evident because if the matrices (e_{ij}), (g_{ij}) are real then the n-tuple $\left(-\frac{1}{2}g_{n+1,1}, -\frac{1}{2}g_{n+1,2}, \ldots, -\frac{1}{2}g_{n+1,n}\right)$ is the solution of the system of linear equations (8).

C o r o l l a r y 3. Every k-dimensional subsimplex $(1<k<n)$ of an edgecentric n-simplex Σ is an edgecentric k-simplex.

P r o o f. The statement follows from Theorem 8 of [3].

C o r o l l a r y 4. The e-norms of an edgecentric simplex are the squares of the lengths of its edges.

R e m a r k. Using the above corollary and corollary to Lemma 6 of [3] we may verify analogicaly as in E_n the validity of the following statements:

1/ Every set $\binom{n+1}{2}-1$ independent inner angles of an edgecentric simplex uniquely determines all remaining inner angles.

2/ Two edgecentric simplexes are congruent if at least $\nu = \binom{n+1}{2}-1$ independent inner angles of one simplex are equal, respectively to ν independent inner angles of another simplex and if there exists at least one pair of distinct indices $i,j \in N$ such that $e'_{ij} = e_{ij}$ (see [3], Lemma 7, Corollary 2).

L e m m a 8. Let ω_i, ω_j, $i \neq j$ be two (n-1)-dimensional nonperpendicular faces of the simplex Σ. Then there exist exactly two hyperplanes

$$^{(\alpha)}\varphi_{ij} \equiv g_{ij}\sqrt{g_{jj}}\, z_i \pm |g_{ij}|\sqrt{g_{ii}}\, z_j = 0\,, \quad \alpha = 1,2 \qquad (11)$$

such that:

$$^{(\alpha)}\sigma(\omega_i) = \omega_j \quad , \quad \alpha = 1,2$$

where $^{(\alpha)}\sigma$ is the symmetry determined by $^{(\alpha)}\varphi_{ij}$.

Proof. Every hyperplane $\varphi_{ij} \supseteq \omega_i \cap \omega_j$ distinct from ω_i and ω_j has an equation $z_i = \alpha z_j$ where α is a nonzero complex number. According to Lemma 1 the vector ξ with the barycentric coordinates $x_k = g_{ik} - g_{jk}\bar{\alpha}$, $k = 0, 1, \ldots, n$ is perpendicular to the hyperplane φ_{ij}. Let us choose a point $A = (a_i)$ in the set $\omega_i \setminus (\omega_i \cap \omega_j)$ and denote by A' its mirror image in the hyperplane φ_{ij}. Since $A' - A \perp \varphi_{ij}$ there exists a complex number β such that $A' = A + \beta\xi$. Further, it follows from $\frac{1}{2}(A + A') \in \varphi_{ij}$ that:

$$\beta = \frac{2\alpha a_j}{g_{ii} - g_{ij}\alpha - g_{ji}\bar{\alpha} + g_{jj}|\alpha|^2}$$

The following equation for α follows from the condition $A' \in \omega_j$:

$$g_{ii} + g_{ij}\alpha - g_{ji}\bar{\alpha} - g_{jj}|\alpha|^2 = 0$$

This equation has exactly two solutions

$$\alpha_{1,2} = \pm \frac{g_{ij}}{|g_{ij}|}\sqrt{\frac{g_{ii}}{g_{jj}}}$$

These solutions determine two hyperplanes with the equations (11).

Definition 2. Each of the hyperplanes $^{(1)}\varphi_{ij}$, $^{(2)}\varphi_{ij}$ will be called bisector of the angle of the faces ω_i, ω_j. The bisector of the angle of two perpendicular faces ω_k, ω_p will be any of the pair of hyperplanes:

$$\sqrt{g_{kk}}\, z_p \pm \sqrt{g_{pp}}\, z_k = 0 \tag{12}$$

Definition 3. The simplex Σ will be called anglecentric if there exists a point O (anglecenter) in the space H_n that

$$\forall\, i,j \in N,\ i \neq j \quad \exists!\ \alpha \in \{1,2\} \qquad O \in {}^{(\alpha)}\varphi_{ij}$$

As an anglecenter cannot lie on any face of the simplex Σ, all its barycentric coordinates are nonzero (complex numbers).

Theorem 2. The simplex Σ is anglecentric iff all its g-norms (e-norms) are real.

Proof. Let $O = (o_i)$ be anglecenter of the simplex Σ. Then for each pair of distinct indices $i,j \in N$ there exists a real number $\varepsilon_{ij} = \varepsilon_{ji} \in \{1,-1\}$ such that:

$$g_{ij}\sqrt{g_{jj}}\, o_i - \varepsilon_{ij}\,|g_{ij}|\sqrt{g_{ii}}\, o_j = 0 \tag{13}$$

Let now $\{i_1, i_2, \ldots, i_p\}$ be any sequence of distinct elements from N. If the pairs (i_p, i_1), (i_{k-1}, i_k), $k = 2, 3, \ldots, p$ are substituted into (13) instead of (i,j) the following system of equalities will be obtained:

$$g_{i_p i_1} = \varepsilon_{i_p i_1}\,|g_{i_p i_1}|\,\frac{o_{i_1}}{o_{i_p}}\sqrt{\frac{g_{i_p i_p}}{g_{i_1 i_1}}}$$

$$g_{i_{k-1} i_k} = \varepsilon_{i_{k-1} i_k}\,|g_{i_{k-1} i_k}|\,\frac{o_{i_k}}{o_{i_{k-1}}}\sqrt{\frac{g_{i_{k-1} i_{k-1}}}{g_{i_k i_k}}}$$

It follows immediately from these equalities that the product $g_{i_1 i_2} \cdots g_{i_{p-1} i_p} g_{i_p i_1}$ is real. From this fact according to Lemma 5 we get that all g-norms of the simplex Σ are real.

Conversely, if the g-norms of the simplex Σ are real, then

$$\sqrt{g_{jj}}\, z_i \pm \sqrt{g_{ii}}\, z_j = 0$$

are the equations of the bisectors ${}^{(1)}\varphi_{ij}$, ${}^{(2)}\varphi_{ij}$. Consequently, a point with the homogeneous barycentric coordinates $(\sqrt{g_{ii}})$ is the anglecenter of the simplex Σ and the proof is over.

It is known that for a real n-simplex Σ (i.e. $\Sigma \subset E_n$) a point Q with the above described barycentric coordinates is the center of the inscribed hypersphere. Therefore a question may be put whether this is not valid for the simplexes in H_n, too.

T h e o r e m 3. Let Σ be an arbitrary simplex in H_n. Then the point Q with the homogeneous barycentric coordinates

$$\left(\sqrt{g_{00}}\,,\ \sqrt{g_{11}}\,,\dots,\ \sqrt{g_{nn}}\right) \tag{14}$$

with respect to the simplex Σ is the center of the minimal hypersphere inscribed into the simplex Σ; more exactly

a/ $\forall i \in N \quad d(Q,\omega_i) = \dfrac{1}{\sum_k \sqrt{g_{kk}}}$

b/ for each point $M \in H_n$ satisfying the condition

$$\exists r \in R \quad \forall i \in N \quad d(M,\omega_i) = r \tag{15}$$

the following relation holds:

$$r \geqslant \frac{1}{\sum_k \sqrt{g_{kk}}}$$

P r o o f. Let (m_i) be the homogeneous barycentric coordinates of a point M which satisfies (15). It is obvious from Lemma 3 that the ratio $\dfrac{m_i}{\sqrt{g_{ij}}}$ is independent on i, hence it may be denoted e.g. by v.

Then:

$$r = \frac{v}{\left|\sum_i m_i\right|} \geqslant \frac{v}{\sum_i |m_i|} = \frac{1}{\sum_i \sqrt{g_{ii}}}$$

The rest of the statement follows from Lemma 3.

D e f i n i t i o n 4. The simplex Σ is called orthocentric if all its altitudes have exactly one point in common. This point is called orthocenter of the simplex Σ.

Theorem 5. All e-norms (g-norms) of an orthocentric simplex are real and form a symmetric matrix.

Proof. Let H be the orthocenter of the simplex Σ. It is obvious that the vector $\beta_i = A^i - H$, $i \in N$ is either zero or perpendicular to the face ω_i i.e.

$$\forall j,k \in N - \{i\},\ j \neq k \quad \beta_i \perp A^k - A^j = \beta_k - \beta_j$$

resp.

$$(\beta_i, \beta_k) = (\beta_i, \beta_j) \tag{16}$$

If in (16) the vector β_i is replaced by the vector β_j and β_k respectively, the following equalities are obtained:

$$(\beta_j, \beta_k) = (\beta_j, \beta_i)\ , \quad (\beta_j, \beta_k) = (\beta_i, \beta_k)$$

and consequently $(\beta_j, \beta_i) = (\beta_i, \beta_k)$. This equality and (16) show that (β_i, β_j) is real. Since $\gamma_\alpha = \beta_\alpha - \beta_0$ the numbers $c_{\alpha\beta} = (\gamma_\alpha, \gamma_\beta)$ are real too and according to [3] - (17) the same is valid for the e-norms of the simplex Σ.

Theorem 6. Let Σ be an orthocentric simplex. Then:

1/ There exist real numbers $\pi_0, \pi_1, \ldots, \pi_n$ satisfying one of the conditions:

a/ $\forall\ i \in N \quad \pi_i > 0$

b/ $(\exists!\ \pi_i = 0) \wedge (\forall j \in N - \{i\} \quad \pi_j > 0)$

c/ $(\exists!\ \pi_i < 0) \wedge (\forall j \in N - \{i\} \quad \pi_j > 0) \wedge \sum_k \frac{1}{\pi_k} < 0$

such that

$$i, j \in N\ , \quad i \neq j \qquad e_{ij} = \pi_i + \pi_j \tag{17}$$

2/ The numbers $h_i = \prod_{\substack{k=0 \\ k \neq i}}^{n} \pi_k$, $i = 0, 1, \ldots, n$ are the homogeneous barycentric coordinates of orthocenter H. If neither one of the numbers π_i is equal to zero then also the numbers $\frac{1}{\pi_i}$, $i = 0, 1, \dots, n$ are the homogeneous barycentric coordinates of the point H.

P r o o f. 1/ Let us consider the numbers:

$$\pi_0 = -(\beta_0, \gamma_\alpha), \quad \pi_\alpha = (\beta_\alpha, \gamma_\alpha), \quad \alpha = 1, 2, \dots, n \qquad (18)$$

where $\beta_i = A^i - H$ and $\gamma_\alpha = A^\alpha - A^0$. Definition of the number π_0 is correct because

$$\forall \alpha, \beta \in N_1 \qquad \beta_0 \perp \gamma_\beta - \gamma_\alpha \quad \text{i.e.} \quad (\beta_0, \gamma_\beta) = (\beta_0, \gamma_\alpha) .$$

It is known from the proof of the preceding theorem that the numbers (β_i, β_j) are real and $(\beta_\alpha, \gamma_\beta) = 0$ for $\alpha \neq \beta$. Because of $(\beta_i, \gamma_\alpha) = (\beta_i, \beta_\alpha) - (\beta_i, \beta_0)$ all the numbers π_i defined in (18) are real.

Besides, it holds for all $\alpha, \beta \in N_1$, $\alpha \neq \beta$:

$$e_{0\alpha} = d^2(A^0, A^\alpha) = (\gamma_\alpha, \gamma_\alpha) = (\beta_\alpha - \beta_0, \gamma_\alpha) = \pi_0 + \pi_\alpha$$

$$e_{\alpha\beta} = d^2(A^\alpha, A^\beta) = (\gamma_\beta - \gamma_\alpha, \gamma_\beta - \gamma_\alpha) = (\beta_\beta - \beta_\alpha, \gamma_\beta - \gamma_\alpha) = \pi_\alpha + \pi_\beta$$

i.e. the conditions (17) are satisfied. Simultaneously we have proved that

$$\forall\, i, j \in N \qquad i \neq j \qquad \pi_i + \pi_j > 0$$

This result leads to the conclusion that at most one number of π_i's is not positive, i.e. the numbers π_i satisfy one of the conditions a/, b/, c/.

It may be easily shown that for any two vectors $\xi = (x_i)$, $\gamma = (y_i) \in H_n$ $\left[\sum_i x_i = \sum_i y_i = 0 \right]$:

$$(\xi, \gamma) = -\frac{1}{2} \sum_{i,j} e_{ij} x_i \bar{y}_j = \sum_i \pi_i x_i \bar{y}_i \qquad (19)$$

so that:

$$\sum_i \pi_i \, | z_i |^2 > 0 \qquad (20)$$

for every set of complex numbers z_i , $i = 0, 1, \ldots, n$, not all zero, such that $\sum_i z_i = 0$. Let us assume now that for example π_0 is negative. If we put

$$z_0 = - \sum_\alpha \frac{1}{\pi_\alpha} \ ; \quad z_\alpha = \frac{1}{\pi_\alpha} \ , \quad \alpha = 1, 2, \ldots, n$$

in (20) the following inequality is obtained:

$$\pi_0 \sum_\alpha \frac{1}{\pi_\alpha} \sum_k \frac{1}{\pi_k} > 0$$

As $\pi_0 < 0$ and $\sum_\alpha \frac{1}{\pi_\alpha} > 0$ it follows from the above inequality that $\sum_k \frac{1}{\pi_k} < 0$. This inequality may be proved analogically for $\pi_\alpha < 0$, $\alpha \in N_1$.

2/ It is evident from the definition of numbers π_i and equalities $(\beta_\alpha, \gamma_\beta) = 0$ valid for each $\alpha \neq \beta$ that $\pi_i = 0$ iff the vector β_i is perpendicular to each vector γ_φ, where $\varphi \in N_1$. As $\langle \gamma_\varphi \rangle$ is the basis of the space H_n, $\pi_i = 0 \Longleftrightarrow H = A^i$ (i.e. $\beta_i = 0$). In the case b/ the validity of the second part of the theorem is evident from the above remarks.

Let us consider now that neither one of the numbers π_i equals to zero and denote by M the point with the barycentric coordinates $\left[\frac{1}{\varrho \pi_i} \right]$ where $\varrho = \sum_i \frac{1}{\pi_i}$. Let us choose three distinct indices i, j, k in the set N and form the vectors $\xi = A^i - M$ and $\gamma = A^k - A^j$. It may be easily shown by using the expression (19) that their scalar product is equal to zero and therefore the vector $A^i - M$ is perpendicular to the face ω_i for each $i \in N$. Thus the point M is an orthocenter of the simplex Σ and the proof is completed.

T h e o r e m 7. Let $\pi_0, \pi_1, \dots, \pi_n$ be a sequence of real numbers, satisfying one of the conditions a/, b/, c/ of the preceding theorem. Then there exists an orthocentric simplex Σ such that (17) holds for its e-norms.

P r o o f. It is sufficient to prove that the numbers e_{ij} defined as

$$e_{ii} = 0 \,, \quad e_{ij} = \pi_i + \pi_j \quad \text{for } i \neq j$$

satisfy the assumptions of the Theorem 3 of [3] . The fulfillment of the conditions (i) and (ii) is evident. Also (iii) is satisfied for the cases a/, b/ because according to (19) it holds, for every nonzero (n+1)-tupel (z_i) of complex numbers the sum of which is zero

$$\sum_{i,j} e_{ij} z_i \bar{z}_j = -2 \sum_i \pi_i \, |z_i|^2 < 0$$

Now we show the fulfillment of the (iii) under the assumption that $\pi_0 < 0$. Let (z_i) be again any nonzero sequence of complex numbers satisfying the condition $\sum_i z_i = 0$. After writing the Schwarz inequality for the vectors $(z_\alpha \sqrt{\pi_\alpha}), (\frac{1}{\sqrt{\pi_\alpha}}) \in V_n(C)$ [unitary] and multiplying it by the number π_0 we get the inequality:

$$\pi_0 \sum_\alpha \frac{1}{\pi_\alpha} \sum_\alpha \pi_\alpha |z_\alpha|^2 \leq \pi_0 \Big|\sum_\alpha z_\alpha\Big|^2$$

such that:

$$\sum_{i,j} e_{ij} z_i \bar{z}_j = -2\Big(\sum_\alpha \pi_\alpha \,|z_\alpha|^2 + \pi_0 \Big|\sum_\alpha z_\alpha\Big|^2\Big) \leq$$

$$\leq -2\Big(\sum_\alpha \pi_\alpha |z_\alpha|^2 + \pi_0 \sum_\alpha \frac{1}{\pi_\alpha} \sum_\alpha \pi_\alpha |z_\alpha|^2\Big) =$$

$$= 2\,|\pi_0| \sum_\alpha \pi_\alpha |z_\alpha|^2 \sum_k \frac{1}{\pi_k} < 0$$

The proof is done analogicaly for the case when one of the numbers

π_α is negative.

Theorem 8. Every simplex to which a sequence of real numbers $\pi_0, \pi_1, \ldots, \pi_n$ satisfying some of the conditions a/, b/, c/ of Theorem 6 may be assigned is orthocentric if (17) is valid for its e-norms.

Proof. Let $\pi_k = 0$. Then for each pair of distinct indices $i, j \in N \setminus \{k\}$ the following relation holds:

$$\sum_{p,m} e_{pm}(\delta_{pi} - \delta_{pk})(\delta_{mj} - \delta_{mk}) = -2 \sum_p \pi_p (\delta_{pi} - \delta_{pk})(\delta_{pj} - \delta_{pk}) =$$

$$= -2 \sum_p \pi_p \delta_{pk} = -2 \pi_k = 0$$

It follows from the above calculation that the vector $A^i - A^k$ is perpendicular to the (n-1)-dimensional face ω_i for each $i \in N$ ($A^j - A^k$, $j \in N \setminus \{i,k\}$ is the basis in the direction ω_i). By this it was proved that A^k is the orthocenter of the simplex Σ. The proof is not done for the case of all numbers π_i different from zero as this is identical with the last part of the proof of Theorem 6.

Theorem 9. Every k-dimensional subsimplex Σ' of an orthocentric n-simplex Σ is an orthocentric k-simplex.

Proof. Let π_k, $k = 0, 1, \ldots, n$ be the real numbers belonging to the simplex Σ according to Theorem 6 and $N' = \{ i \in N \mid A^i \in \Sigma' \}$. It follows from Theorem 8 of [3] :

$$\forall\, i, j \in N', \quad i \neq j \qquad e'_{ij} = e_{ij} = \pi_i + \pi_j$$

The statement is evident on the basis of this relation and the previous theorem.

According to Theorems 1 and 5 every orthocentric simplex Σ is necessarily edgecentric (and anglecentric too, naturally).

Therefore we may ask whether the mutual position of the center of gravity, orthocenter, and edgecenter of the simplex Σ is the same as in E_n. The following, last theorem of this work answers the given question.

T h e o r e m 10. The center of gravity T, edgecenter S, and orthocenter H of an orthocentric simplex Σ lie on the same line and:

$$T = \frac{2}{n+1} S + \frac{n-1}{n+1} H \tag{21}$$

P r o o f. The elements of the matrix $G_2 = (g_{rs})$ belonging to the orthocentric simplex Σ may be expressed by means of numbers π_i, $i = 0, 1, \ldots, n$ as follows:

$$\varrho g_{ii} = \frac{1}{\pi_i}\left(\varrho - \frac{1}{\pi_i}\right), \quad \varrho g_{ij} = -\frac{1}{\pi_i \pi_j} \text{ where } i \neq j$$
$$\varrho g_{n+1,i} = \frac{n-1}{\pi_i} - \varrho, \quad \varrho g_{n+1,n+1} = \varrho \sum_k \pi_k - (n-1)^2, \tag{22}$$
$$\varrho = \sum_k \frac{1}{\pi_k}$$

if $\pi_i \neq 0$ for each $i \in N$ and

$$g_{ii} = -g_{ik} = \frac{1}{\pi_i}, \quad g_{ij} = 0, \quad g_{kk} = \sum_{\substack{j=0 \\ j\neq k}}^{n} \frac{1}{\pi_j}, \quad i \neq j \neq k \neq i$$
$$g_{n+1,i} = -1 \text{ for } i \neq k, \quad g_{n+1,k} = n-2, \quad g_{n+1,n+1} = \sum_{\substack{j=0 \\ j\neq k}}^{n} \frac{1}{\pi_j} \tag{23}$$

if $\pi_k = 0$.

We will not confirm these relations as it is more or less mechanical calculation and similar to that in E_n. It follows from the above relations, Theorem 6 and relations (10) that the barycentric coordinates of the points S, H are either the numbers

$$s_i = -\frac{1}{2}\left[\frac{n-1}{\varrho \pi_i} - 1\right], \quad h_i = \frac{1}{\varrho \pi_i}, \quad i = 0, 1, \ldots, n$$

or the numbers:

$$s'_i = \frac{1}{2} \text{ for } i \neq k , \quad s'_k = 1 - \frac{n}{2} , \quad h'_i = \delta_{ik}$$

according to whether all numbers π_i are nonzero or whether there exists an index k (unique) in the set N such $\pi_k = 0$. Since $t_i = \frac{1}{n+1}$, $i = 0, 1, \ldots, n$ are the barycentric coordinates of the point T in both cases, the validity of (21) is obvious and the proof is complete.

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S ú h r n

O SIMPLEXOCH HERMITOVSKÉHO PRIESTORU H_n S URČITÝMI VLASTNOSŤAMI REÁLNYCH SIMPLEXOV

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Táto práca bezprostredne naväzuje na prácu [3], v ktorej sú zavedené všetky používané pojmy a dokázané tvrdenia na ktoré sa v tejto práci odvolávame. Hlavným cieľom práce bolo dokázať, že každý z výrokov:

"Symetrálne nadroviny všetkých hrán n-simplexu Σ prechádzajú jedným bodom"

"V hermitovskom priestore H_n existuje bod taký, že pre každé $(i,j) \in N \times N$, $i \neq j$ je obsiahnutý v jednej z osových nadrovín stien ω_i, ω_j n-simplexu Σ",

je ekvivalentný s tým, že e_{ij} je reálne číslo pre každé $i,j \in N$.

Okrem toho je v práci dokázané, že bod s homogénnymi barycentrickými súradnicami $(\sqrt{g_{ii}})$ je stredom minimálnej guľovej nadplochy vpísanej do Σ.

V poslednej časti práce sa ukazuje, že v orthocentrickom simplexe Σ sú všetky čísla e_{ij}, $i,j \in N$ reálne, vytvárajú symetrickú maticu a že podmienky pre orthocentricitu simplexov v H_n sú také isté ako v euklidovskom priestore E_n.

Резюме

ОБ СИМПЛЕКСАХ В ПРОСТРАНСТВЕ ЭРМИТТА H_n С НЕКОТОРЫМИ СВОЙСТВАМИ РЕАЛЬНЫХ СИМПЛЕКСОВ

Валент Затько, Братислава

Эта работа является непосредственным продолжением работы [3] в которой можно найти определения всех нужных понятий и доказательства всех использованых теорем. Главным результатом работы является доказательство, что каждое из следующих утверждений:

"Гиперплоскости симметри всех ребр n-симплекса Σ проходят через одну точку"

"В пространстве H_n существует такая точка, что для каждого $(i,j) \in N \times N$, $i \neq j$ она находится в одной из бисектральных гиперплоскостей угла $(n-1)$-мерных граней ω_i, ω_j симплекса Σ", эквивалентно тому, что для каждого $i, j \in N$ число e_{ij} есть вещественное.

Кроме того в работе показано, что точка, которая имеет барицентрические координаты $(\sqrt{g_{ii}})$, является центром минимального гипершара, касающегося всех $(n-1)$-мерных граней n-симплекса Σ.

В заключительной части работы показывается, что у ортоцентрического n-симплекса Σ все числа e_{ij} ; $i,j \in N$ действительны, они составляют кососиметрическую матрицу и, что условия для того, чтобы Σ являлся ортоцентрическим одни и те же как и в пространстве Эвклида.