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CALCULATION OF DERIVATIVES OF SOLUTIONS OF THE n-th ORDER ORDINARY DIFFERENTIAL EQUATION

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At the numerical treatment of a differential equation $y^{(n)}=f(x,y,y',\dots,y^{(n-1)})$ it is often necessary to determine derivatives of $y(x)$ up to $(n+k)$-th order for some natural number k as a function of $f(x,y,y',\dots,y^{(n-1)})$. In this paper the recurrence relation which enables to calculate the $(n+k)$-th derivative of $y(x)$ using derivatives of lower order is proved.

1. Preliminary. Let k be a natural number. Let a function $y(x)$, which is the solution of

$$y^{(n)}=f(x,y,y',\dots,y^{(n-1)}) \tag{1.1}$$

has all derivatives up to the $(n+k)$-th order.

Let us define differential operators D_{in}, $i=1,2,\dots,k$:

$$D_{1n}f = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}y' + \frac{\partial}{\partial y'}y'' + \dots + \frac{\partial}{\partial y^{(n-1)}}y^{(n)}\right)f \tag{1.2}$$

$$D_{in}f = \left(\frac{\partial}{\partial y}y^{(i)} + \frac{\partial}{\partial y'}y^{(i+1)} + \dots + \frac{\partial}{\partial y^{(n-1)}}y^{(i+n-1)}\right)f \ , \quad i=2,3,\dots,k \tag{1.3}$$

For simplicity we write f instead of $f(x,y,y',\dots,y^{(n-1)})$. The product of two operators D_{i_1n}, D_{i_2n}; $i_1,i_2=1,2,\dots,k$, we understand the operator $D_{i_1n}D_{i_2n}$ which is the formal product (as polynomials) of D_{i_1n} and D_{i_2n}. In this formal product a product of partial derivatives means mixed second order partial derivative and total derivatives of $y(x)$ we multiply in usual way. Analogically we can define product $D_{i_1 n} D_{i_2 n} \dots D_{i_p n}$ for some natural $p \geq 2$; $i_1, i_2, \dots, i_p = 1, 2, \dots, k$. When $i_1 = i_2 = \dots = i_p = i$ for some natural p, we shall talk about the p-th power of D_{in} and write D^p_{in}. In addition we define D^0_{in} as the identical operator.

R e m a r k : It is evident, that $D^p_{in} f \neq (D_{in} f)^p$ for $p \geq 2$.

L e m m a : Let $r, i_j \geq 2$ $(j=2,3,\dots,r)$, p_j $(j=1,2,\dots,r)$ be natural numbers. Then

$$\frac{d}{dx}\left(D^{p_1}_{1n} \prod_{j=2}^{r} D^{p_j}_{i_j n} f\right) = D^{p_1+1}_{1n} \prod_{j=2}^{r} D^{p_j}_{i_j n} f + p_1 D^{p_1-1}_{1n} D_{2n} \prod_{j=2}^{r} D^{p_j}_{i_j n} f + \sum_{s=2}^{r} p_s D^{p_1}_{1n} \frac{\prod_{j=2}^{r} D^{p_j}_{i_j n}}{D_{i_s n}} D_{i_s+1,n} f \qquad (1.4)$$

R e m a r k : The expression in the sum in (1.4) means that the s-th term has the form

$$p_s D^{p_1}_{1n} D^{p_2}_{i_2 n} \dots D^{p_{s-1}}_{i_{s-1},n} D^{p_s-1}_{i_s n} D_{i_s+1,n} D^{p_{s+1}}_{i_{s+1},n} \dots D^{p_r}_{i_r n} f .$$

P r o o f : If we write out all operators on the left-hand side (1.4) according to the definition, we have

$$\frac{d}{dx}\left(D^{p_1}_{1n} \prod_{j=2}^{r} D^{p_j}_{i_j n} f\right) = \frac{d}{dx}\left[\sum_{M_1} \sum_{M_2} \dots \sum_{M_r} \left(\prod_{j=1}^{r} K_j\right) A \left(\prod_{j=1}^{r} B_j\right)\right] \qquad (1.5)$$

where

$$K_1 = \frac{p_1!}{\prod_{q=0}^{n} (p_{1q}!)} \quad (1.6), \quad K_j = \frac{p_j!}{\prod_{q=1}^{n} (p_{jq}!)}, \; j=2,3,\dots,r \qquad (1.7)$$

$$M_1 = \left\{(p_{10}, p_{11}, \dots, p_{1n}),\ \sum_{i=0}^{n} p_{1i} = p_1\right\} \qquad (1.8)$$

$$M_j = \left\{ (p_{j1}, p_{j2}, \ldots, p_{jn}) \; , \; \sum_{i=1}^{n} p_{ji} = p_j \right\} \; , \; j=2,3,\ldots,r \qquad (1.9)$$

$$A = \frac{\partial^{L_0} f}{\partial x^{p_{10}} \partial y^{L_1} \partial (y')^{L_2} \ldots \partial (y^{(n-1)})^{L_n}} \qquad (1.10)$$

$$L_0 = \sum_{i=1}^{r} p_i \quad (1.11) \; , \; L_j = \sum_{i=1}^{r} p_{ij} \; , \; j=1,2,\ldots,n \qquad (1.12)$$

$$B_1 = \prod_{i=1}^{n} (y^{(i)})^{p_{1i}} \qquad (1.13)$$

$$B_j = \prod_{s=1}^{n} (y^{(i_j+s-1)})^{p_{js}} \; , \quad j=2,3,\ldots,r \qquad (1.14)$$

Differentiation of the right-hand side (1.5) as a product of functions gives

$$\sum_{M_1} \sum_{M_2} \ldots \sum_{M_r} \left(\prod_{j=1}^{r} K_j\right) \left(\frac{d}{dx} A\right) \left(\prod_{j=1}^{r} B_j\right) +$$

$$\sum_{M_1} \sum_{M_2} \ldots \sum_{M_r} \left(\prod_{j=1}^{r} K_j\right) A \left(\frac{d}{dx} B_1\right) \left(\prod_{j=2}^{r} B_j\right) +$$

$$\sum_{s=2}^{r} \sum_{M_1} \sum_{M_2} \ldots \sum_{M_r} \left(\prod_{j=1}^{r} K_j\right) A \left(\frac{d}{dx} B_s\right) \left(\prod_{\substack{j=1 \\ j\neq s}}^{r} B_j\right) \qquad (1.15)$$

Now let us consider the first term in (1.15) . After short calculation we have

$$\sum_{M_1} \sum_{M_2} \ldots \sum_{M_r} \left(\prod_{j=1}^{r} K_j\right) \left(\frac{d}{dx} A\right) \left(\prod_{j=1}^{r} B_j\right) =$$

$$= \left\{ \left[\sum_{M_1} K_1 \left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} y' + \ldots + \frac{\partial}{\partial y^{(n-1)}} y^{(n)} \right) \frac{\partial^{p_1}}{\partial x^{p_{10}} \partial y^{p_{11}} \ldots \partial (y^{(n-1)})^{p_{1n}}} \right] \times \right.\right.$$

$$\left. \times B_1 \right\} \left(\prod_{j=2}^{r} D_{i_j n}^{p_j} \right) \Big\} f = D_{1n} D_{1n}^{p_1} \left(\prod_{j=2}^{r} D_{i_j n}^{p_j} \right) f$$ and this is the first

term on the right-hand side (1.4) .

In the second term in (1.15) is

$$\frac{d}{dx}B_1 = \sum_{i=1}^{n} p_{1i}\,(y^{(i)})^{p_{1i}-1}\, y^{(i+1)} \prod_{\substack{j=1\\ j\neq i}}^{n} (y^{(j)})^{p_{1j}}$$ and then

$$\sum_{M_1}\sum_{M_2}\dots\sum_{M_r}\Big(\prod_{j=1}^{r}K_j\Big)\,A\,\Big(\frac{d}{dx}B_1\Big)\Big(\prod_{j=2}^{r}B_j\Big) = \Big[\sum_{M_1}\sum_{i=1}^{n}K_1\,p_{1i}\times$$

$$\times\frac{\partial^{p_1}}{\partial x^{p_{10}}\partial y^{p_{11}}\dots\partial(y^{(n-1)})^{p_{1n}}}(y^{(i)})^{p_{1i}-1}\,y^{(i+1)}\prod_{\substack{j=1\\ j\neq i}}^{n}(y^{(j)})^{p_{1j}}\Big]\times$$

$$\times\prod_{j=2}^{r}D_{i_j n}^{p_j}\,f = \Big\{\Big[\sum_{M_1'}\frac{p_1!}{\prod_{q=0}^{n}(t_{1q}!)}\,\frac{\partial^{p_1-1}}{\partial x^{t_{10}}\partial y^{t_{11}}\dots\partial(y^{(n-1)})^{t_{1n}}}\times$$

$$\times\prod_{j=1}^{n}(y^{(j)})^{t_{1j}}\Big]\Big(\sum_{i=1}^{n}\frac{\partial}{\partial y^{(i-1)}}\,y^{(i+1)}\Big)\Big\}\prod_{j=2}^{r}D_{i_j n}^{p_j}\,f =$$

$$= p_1\,D_{1n}^{p_1-1}\,D_{2n}\prod_{j=2}^{r}D_{i_j n}^{p_j}\,f$$, and this is the second term

on the right-hand side (1.4) . In the last derivation

the substitution $p_{1i}-1=t_{1i}$, $p_{1j}=t_{1j}$, $j\neq i$ for all $i=1,2,\dots,n$

was used, $M_1'=\{(t_{10},t_{11},\dots,t_{1n}),\ \sum_{i=0}^{n}t_{1i}=p_1-1\}$.

Analogously to that we can show for every $s=2,3,\dots,r$ that

$$\sum_{M_1}\sum_{M_2}\dots\sum_{M_r}\Big(\prod_{j=1}^{r}K_j\Big)\,A\,\Big(\frac{d}{dx}B_s\Big)\Big(\prod_{\substack{j=1\\ j\neq s}}^{r}B_j\Big) =$$

$$= p_s\,D_{1n}^{p_1}\,\frac{\prod_{j=2}^{r}D_{i_j n}^{p_j}}{D_{i_s n}}\,D_{i_s+1,n}\,f$$ and summing through all s we have

the third term on the right-hand side (1.4), what completes the proof.

2. Recurrence relation.

Theorem : Let k be a natural number. Let

$$P_k = \{(p_1,p_2,\ldots p_k), p_i \text{ - non-negative integers}, \sum_{i=1}^{k} ip_i = k\}$$

and let function y(x) satisfy assumptions from the previous section. Then

$$y^{(n+k)} = \sum_{P_k} \frac{k!}{\prod_{i=1}^{k}(p_i!)} \frac{1}{\prod_{i=1}^{k}(i!)^{p_i}} \left(\prod_{i=1}^{k} D_{in}^{p_i}\right) f \tag{2.1}$$

Remark : From the definition of P_k follows, that if $(p_1,p_2,\ldots,p_k) \in P_k$, then $p_k=1$ if $(p_1,p_2,\ldots,p_{k-1}) = (0,0,\ldots,0)$, or $p_k=0$ otherwise.

Proof : We shall accomplish the proof using iduction with respect to k.

I. Let k=1. Differentiation of (1.1) with respect to x gives

$$y^{(n+1)} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \ldots + \frac{\partial f}{\partial y^{(n-1)}}y^{(n)}\right) = D_{1n} f \tag{2.2}$$

The same result we shall get from (2.1), because $P_1 = \{(1)\}$.

II. Let the theorem holds for m = 1,2,...,k. Let now m = k+1. The proof of this part we shall accomplish in two steps.

1. We shall show that, if we do not take coefficients into consideration, then all terms of $y^{(n+k+1)}$ have the form

$$\left(\prod_{i=1}^{k+1} D_{in}^{p_i}\right) f \tag{2.3}$$

Really, differentiating of every term in (2.1) without respect to coefficients we have (according to lemma)

$$\frac{d}{dx}\left[\left(\prod_{i=1}^{k} D_{in}^{p_i}\right) f\right] = \left(D_{1n}^{p_1+1} \prod_{i=2}^{k} D_{in}^{p_i}\right) f + \sum_{\substack{j=1 \\ p_j \neq 0}}^{k} p_j \left(\frac{\prod_{i=1}^{k} D_{in}^{p_i}}{D_{jn}} D_{j+1,n}\right) f \tag{2.4}$$

Denoting in the first term on the right-hand side (2.4) $(t_1,t_2,\dots,t_k,t_{k+1}) = (p_1+1,p_2,p_3,\dots,p_k,0)$ and in the second term $(t_1,t_2,\dots,t_k,t_{k+1}) = (p_1,p_2,\dots,p_{j-1},p_j-1,p_{j+1}+1,p_{j+2},\dots,p_k,0)$ for $j=1,2,\dots,k-1$ and $(t_1,t_2,\dots,t_k,t_{k+1}) = (0,0,\dots,0,1)$ for $j=k$ (the last expression arised from $(p_1,p_2,\dots,p_k) = (0,0,\dots,0,1)$), it is easy to see, that in every case is $(t_1,t_2,\dots,t_{k+1}) \in P_{k+1}$ and that the differentiation of every term of $y^{(n+k)}$ gives terms of the form (2.3) . And because every term of $y^{(n+k+1)}$ we can get only by differentiation of $y^{(n+k)}$, there are no terms of other form in $y^{(n+k+1)}$.

2. We shall show that the term of the form (2.3) has the coefficient

$$\frac{(k+1)!}{\prod_{i=1}^{k+1}(p_i!)}\;\frac{1}{\prod_{i=1}^{k+1}(i!)^{p_i}} \tag{2.5}$$

Let $(p_1,p_2,\dots,p_{k+1}) \in P_{k+1}$. There are two alternatives :

a/ $p_{k+1} = 0$.

Then $\sum_{i=1}^{k} ip_i = k+1$ and according to lemma we can term (2.3) get from these terms of $y^{(n+k)}$:

$$D_{1n}^{p_1-1}\prod_{i=2}^{k} D_{in}^{p_i} f \tag{2.6}$$

which contribution to the coefficient of (2.3) is

$$\frac{k!}{(p_1-1)!\prod_{i=2}^{k}(p_i!)}\;\frac{1}{(1!)^{p_1-1}\prod_{i=2}^{k}(i!)^{p_i}} \tag{2.7}$$

and

$$D_{jn}^{p_j+1} \; D_{j+1,n}^{p_{j+1}-1} \prod_{\substack{i=1 \\ i\neq j, i\neq j+1}}^{k} D_{in}^{p_i} f \;, \quad j=1,2,\dots,k-1 \tag{2.8}$$

which contribution is for every j

$$\frac{k!}{(p_j+1)!(p_{j+1}-1)! \prod\limits_{\substack{i=1 \\ i\neq j, i\neq j+1}}^{k} (p_i!)} \; \frac{p_j+1}{(j!)^{p_j+1} \left[(j+1)!\right]^{p_{j+1}-1} \prod\limits_{\substack{i=1 \\ i\neq j, i\neq j+1}}^{k} (i!)^{p_i}} \tag{2.9}$$

Summing (2.9) through all $j = 1,2,\dots,k-1$ and adding (2.7) we have

$$\frac{k!}{\prod\limits_{i=1}^{k}(p_i!)} \; \frac{1}{\prod\limits_{i=1}^{k}(i!)^{p_i}} \left[\sum_{j=1}^{k} \frac{j!}{(j-1)!} p_j\right] = \frac{(k+1)!}{\prod\limits_{i=1}^{k+1}(p_i!)} \; \frac{1}{\prod\limits_{i=1}^{k+1}(i!)^{p_i}} \tag{2.10}$$

and this is (2.5) .

It is still necessary to consider that it can be $p_j=0$ for some j. Of course, there is no term of the form (2.8) in $y^{(n+k)}$ for such j and also the corresponding term in the square bracket on the left-hand side (2.10) is missing. But this fact has no influence neither upon the value of this bracket nor upon the final expression.

b/ $p_{k+1} = 1$.

Then $p_i = 0$, $i=1,2,\dots,k$, and (2.3) has the form

$$D_{k+1,n} \, f \tag{2.11}$$

which we can get according to lemma only differentiating the term $D_{kn} f$ of $y^{(n+k)}$ and this one has coefficient

$$\frac{k!}{1!} \, \frac{1}{k!} = 1 = \frac{(k+1)!}{1!} \, \frac{1}{(k+1)!} \tag{2.12}$$

what is again (2.5). This completes the proof of the theorem.

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Súhrn

Výpočet derivácií riešenia diferenciálnej rovnice obyčajnej n-tého rádu

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Pri vyšetrovaní obyčajnej diferenciálnej rovnice $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$ je často potrebné určiť derivácie funkcie $y(x)$ až do rádu $n+k$ pre nejaké prirodzené číslo k ako funkciu pravej strany tejto diferenciálnej rovnice.
V článku je dokázaný rekurentný vzťah pre výpočet $(n+k)$-tej derivácie funkcie $y(x)$ pomocou nižších derivácií tejto funkcie.

Резюме

Вычисление производных решения обыкновенного дифференциального уравнения n-того порядка

Ян Штекауер

Рассматривая обыкновенное дифференциальное уравнение $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$ часто надо определить производные функции $y(x)$ до $(n+k)$-того порядка для какого-нибудь натурального числа k как функцию правой стороны этого уравнения. В статье доказана рекурентная формула для вычисления производной $(n+k)$-того порядка функции $y(x)$ при помощи низших производных этой функции.