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SOME PROPERTIES OF TRIANGLES IN THE HERMITIAN PLANE

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It is assummed that readers of this paper are familiar with the terms, notation and results given in [2] and [3] . However, the symbols used in [2] to denote certain facts are again used in [3] for different facts to emphasize an analogy between the results of the both papers.As the results of [2] and [3] are now studied together it was necessary to change a part of the previously used notation. For that reason the results of [2] and [3] will be recalled first.

Let Σ be a triangle in the Hermitian plane H_2 with vertices A^o, A^1, A^2. All points and vectors of the plane H_2 are determined by homogeneous barycentric coordinates with respect to the triangle Σ . The vectors δ_o, δ_1, δ_2 defined by the conditions:

a/ $\delta_o + \delta_1 + \delta_2 = 0$

b/ for all $i \in N = \{0, 1, 2\}$ the vectors δ_j, δ_k form a biorthogonal basis associated with the basis $\langle \gamma_j = A^j - A^i$, $\gamma_k = A^k - A^i \rangle$, where $j, k \in N$ and $i \neq j \neq k \neq i$.

are called direction vectors of external normals of the triangle Σ . By means of them numbers $g_{ij} = (\delta_i, \delta_j)$ are defined which are called g-norms of the triangle Σ .

If the number

$$g = -\frac{(g_{ij}-g_{ji})^2}{4G}, \text{ where } G = \begin{vmatrix} g_{ii} & g_{ji} \\ g_{jj} & g_{jj} \end{vmatrix},$$

is positive, then the triangle $\Sigma = {}^{(o)}\Sigma = {}^{(o)}\Sigma^*$ may be included into two sequences of triangles $\{{}^{(n)}\Sigma\}_{-\infty}^{\infty}$ and $\{{}^{(n)}\Sigma^*\}_{-\infty}^{\infty}$. It is characteristic for the first of them that edges of its arbitrary term lie on the symmetry lines of edges of the preceding one.

The vertices of its terms are as follows:

$$^{(1)}A^j = [\,^{(1)}a_i^j = -\frac{1}{2G}\, g_{ii}g_{ki},\quad ^{(1)}a_j^j = \frac{1}{2G}\,(g_{ij}g_{ki} + g_{kj}g_{ik})\,]$$

$$^{(2m)}A^j = S + (-q)^m\,[A^j - S\,];\quad ^{(2m+1)}A^j = S + (-q)^m\,[\,^{(1)}A_j - S\,],$$

where $S = [\,s_i = -\frac{g_{ii}(g_{jk}+g_{ki})}{4(1+q)G}\,]$ is the centre of a circle with the minimal radius circumscribed to the triangle Σ and consequently to all other terms of the sequence $\{{}^{(n)}\Sigma\}_{-\infty}^{\infty}$. The points $^{(2m)}A^j$, $m = 0, \pm1, \pm2, \dots$ are now colinear for each $j \in N$ and the corresponding lines p_j pass through the point S. The points $^{(2m+1)}A^j$, $m = 0, \pm1, \pm2, \dots$ lie on a line q_j perpendicular to the line p_j through the point S. The g-norms of the triangle $^{(n)}\Sigma$ are connected with the g-norms of the triangle Σ by the relations:

$$^{(n)}g_{ij} = \frac{1}{q^n}\, g_{ij} \quad \text{for } n \text{ even and}$$

$$^{(n)}g_{ij} = \frac{1}{q^n}\, g_{ji} \quad \text{for } n \text{ odd}$$

so that $^{(n)}G = \frac{1}{q^{2n}}\, G$, $^{(n)}q = q$.

The second sequence has the property that the edges of its arbitrary term lie on the altitudes of the preceding one. Vertices of its terms are as follows:

$$^{(1)}B^j = [\,^{(1)}b_i^j = \frac{1}{G}\, g_{ij}g_{ki},\quad ^{(1)}b_j^j = \frac{1}{G}\, g_{ij}g_{kj}\,]$$

$^{(2m)}B^j = H + (-h)^m [A^j - H]$, $^{(2m+1)}B^j = H + (-h)^m [^{(1)}B^j - H]$ where $h = 4q$ and $H = [h_i = \frac{g_{ij}g_{ki}}{(1+h)G}]$. The vertices of all its terms, similarly to the preceding case, lie on the three pairs of perpendicular lines passing through the point H. Sometimes this point is called orthocentre of the triangle Σ . The g-norms of the triangle $^{(n)}\Sigma^*$ are:

$$^{(n)}g^*_{ij} = \frac{1}{h^n} g_{ij} \text{ for n even and } ^{(n)}g^*_{ij} = \frac{1}{h^n} g_{ji} \text{ for n odd}$$

so that $^{(n)}G^* = \frac{1}{h^{2n}} G$ and $^{(n)}h^* = h$.

It follows from the above facts that the same coefficients q and h belong to all triangles of both the sequences $\{^{(n)}\Sigma\}_{-\infty}^{\infty}$, $\{^{(n)}\Sigma^*\}_{-\infty}^{\infty}$. Moreover, $^{(n)}g_{ij} = 4^{n(n)}g^*_{ij}$ and $^{(n)}G = 4^{2n(n)}G^*$ for every integer n.

The purpose of the present paper is to investigate properties of centres of circumcircles with minimal radius, orthocentres and centres of gravity of triangles belonging to the sequences $\{^{(n)}\Sigma\}_{-\infty}^{\infty}$ and $\{^{(n)}\Sigma^*\}_{-\infty}^{\infty}$. Let us first consider a triple of points S, T, H where T is the centre of gravity of the triangle Σ . The mutual position of those points is shown by the following examples.

The triangle $\mathcal{A} = \{A^o = (0,0,1), A^1 = \frac{1}{2}(1+i, 1-i, 2), A^2 = \frac{1}{2}(1-i, -1+i, 2)\}$ has g-norms:

$$g_{oo} = 6,\ g_{11} = g_{22} = 2,\ g_{o1} = -3-i,\ g_{o2} = -3+i,\ g_{12} = 1-i$$

because $\alpha_o = (-2, -1-i, 0)$, $\alpha_1 = (1,1,0)$, $\alpha_2(1,i,0)$ are the direction vectors of its external normals. It follows from the above relations that the points $S^{\mathcal{A}}$, $T^{\mathcal{A}}$, $H^{\mathcal{A}}$ have the following barycentric coordinates with respect to the triangle $\mathcal{A}$:

$[-1,1,1]$, $\frac{1}{3}[1,1,1]$, $\frac{1}{3}[5, -(1+2i), -1+2i]$ so that

$$S^{\mathcal{A}} = (1,0,1)\ ,\ T^{\mathcal{A}} = \frac{1}{3}(1,0,3),\ H^{\mathcal{A}} = \frac{1}{3}(1,-2-2i,3).$$

As the point $H^{\mathcal{A}}$ does not lie on the line $z_2 = 0$ those points are linearly independent i.e. they form a triangle. The orthocentre, the centre of the circumcircle with minimal radius, and the centre of gravity of the triangle T_A described in [2] have the same property.

The case of the triangle $\mathcal{B} = \{B^0 = (0,0,1), B^1 = (0,i,1), B^2 = (1,-i,1)\}$ is somewhat different. The points $S^{\mathcal{B}}$, $T^{\mathcal{B}}$, $H^{\mathcal{B}}$ have barycentric coordinates with respect to $\mathcal{B}$:

$$\frac{1}{2}[-5,4,3]\ ,\ \frac{1}{3}[1,1,1]\ ,\ [6,-3,-2]$$

because the g-norms of the triangle $\mathcal{B}$ have the values:

$$g_{oo} = 5,\ g_{11} = 2,\ g_{22} = 1,\ g_{o1} = g_{1o} = -3,\ g_{o2} = g_{2o} = -2$$
$$g_{12} = g_{21} = 1.$$

Then:

$$S^{\mathcal{B}} = \frac{1}{2}(3,i,2),\ T^{\mathcal{B}} = \frac{1}{3}(1,0,3),\ H^{\mathcal{B}} = (-2,-i,1).$$

It can be easily proved that the points lie on one line and it holds:

$$3T^{\mathcal{B}} - 2S^{\mathcal{B}} - H^{\mathcal{B}} = 0.$$

In the triangle of the next example the orthocentre, centre of gravity, and centre of circumcircle with minimal radius coincide:

$$\mathcal{C} = \{C^0 = (1-i,-1+i,1), C^1 = \frac{1}{2}(3-2i,-3+2i,2),\ C^2 = \frac{1}{2}(3-2i,-2+i,2)\}.$$

Its g-norms are given by the relations:

$$g_{oo} = g_{11} = g_{22} = 4,\ g_{o1} = g_{12} = g_{2o} = -2+2i,\ g_{o2} = g_{21} = g_{1o} =$$
$$= -2-2i.$$

Before the general theory will be discussed let us introduce the notion of the degree of generality of a triangle. Every triple of different collinear points is considered to be a triangle with the degree of generality equal 0. Degree of generality of triangles isometric with real triangles is considered to be one. To a triplet of coincident points degree of generality zero or one is assigned. All other triangles in the plane H_2 have degree of generality two. However, the triangle Σ is assumed to have the degree of generality greater or equal to one.

Since for each triple of different indices $i, j, k \in N$ there is $|A^j - A^i| = \frac{1}{G} g_{kk}$ (see the relation (27) in [2]), the triangle Σ is equilateral if and only if

$$g_{oo} = g_{11} = g_{22}.$$

Now, $g_{oi} + g_{1i} + g_{2i} = g_{io} + g_{i1} + g_{i2} = 0$ for all $i \in N$ so that the above condition may be replaced by any of the following ones:

$$g_{o1} = g_{12} = g_{2o} \quad \text{resp.} \quad g_{o2} = g_{21} = g_{1o}.$$

It may be shown by mechanical but somewhat complicated calculation that the determinant the rows of which are formed by barycentric coordinates of the points S, T, H is equal to:

$$D = \frac{(g_{12} - g_{21})R}{12G^2(1+q)(1+h)}, \quad \text{where} \tag{1}$$

$$R = (g_{o1} + g_{1o})(g_{o2} + g_{2o})(g_{12} + g_{21}) + g_{oo}g_{11}g_{22}.$$

Further it follows that the points S, T, H are collinear (linearly dependent) if and only if at least one of the following two equations is satisfied: $g_{12} = g_{21}$, $R = 0$. The condition $g_{12} = g_{21}$ was proved in [2] to be a necessary and sufficient condition for the triangle Σ to be isometric with a real triangle, e.g. to have degree of generality equal to one. It may be easily shown that in this case the points S, T, H satisfy the condition: $3T - 2S - H = 0$.

Using the relation (7) from [2] it may be proved that

$$|S - H| = \frac{1}{2G}\sqrt{\frac{R}{(1+q)(1+h)}}\,, \quad |S - T| = \frac{1}{6G}\sqrt{\frac{R}{1+q}}\,, \qquad (2)$$

$$|T - H| = \frac{1}{3G}\sqrt{\frac{R}{1+h}}$$

The verification of these relations is again rather complicated technically.

From (2) it follows that the points S, T, H coincide if and only if $R = 0$. Besides, it follows from (2) that two those points cannot coincide with the third being different from them.

Further, a connection between equilaterality of the triangle Σ and the value R will be shown. If we assume the triangle Σ to be equilateral it follows that $g_{ij} + g_{ji} = - g_{oo} = - g_{11} = - g_{22}$, for any two different indices $i, j \in N$ so that R must be equal to zero. On the other hand it follows from the assumption $R = 0$ that for example $H = T$. It is obvious from the last equality that all barycentric coordinates of the point H are equal and real so that $h_i = \bar{h}_j$, $g_{ki} = g_{jk}$ resp. for any two different indices $i, j \in N$ where $k \in N - \{i, j\}$. The last equality may be modified as: $g_{ii} = g_{jj}$ what leads to the conclusion that the triangle Σ is equilateral. We have thus proved the following theorem.

<u>T h e o r e m 1.</u> For the points S, T, H exactly the following cases may occur:

1/ $S = T = H$;

2/ $S \neq T \neq H \neq S$ and $3T - 2S - H = 0$;

3/ S, T, H are linearly independent, i.e. they form a triangle.

The first case takes place if and only if the triangle Σ is equilateral, i.e. $R = 0$ or three of the following equations are fulfilled: $g_{oo} = g_{11} = g_{22}$, $g_{o1} = g_{12} = g_{2o}$. The second case takes place if and only if Σ has the degree of generality equal to one and is not equilateral. The necessary and sufficient condition for the case three is that the triangle Σ is not equilateral and has the degree of generality equal to two.

Remark. In the case of an equilateral triangle Σ the point $S = T = H$ will be called centre of the triangle Σ and denoted by S. This name and notation will be naturally used for other equilateral triangles, too.

Theorem 2. If Σ is an equilateral triangle with the degree of generality equal to two then all triangles of the sequences $\{{}^{(n)}\Sigma\}_{-\infty}^{\infty}$, $\{{}^{(n)}\Sigma^*\}_{-\infty}^{\infty}$ are equilateral and their centres coincide with the centre of the triangle Σ . For every integer $n \neq 0$, the triangle ${}^{(n)}\Sigma^*$ is homothetic with the triangle ${}^{(n)}\Sigma$. The point S is the centre of this homothety and the number $4^{\frac{n}{2}}$ is the coefficient for n even, the number $-2.4^{\frac{n-1}{2}}$ for n odd.

Proof. Since for each pair of indices $i, j \in N$ ${}^{(n)}g_{ii} = \frac{1}{q^n} g_{ii} = \frac{1}{q^n} g_{jj} = {}^{(n)}g_{jj}$, ${}^{(n)}g^*_{ii} = \frac{1}{h^n} g_{ii} = \frac{1}{h^n} g_{jj} = {}^{(n)}g^*_{jj}$ the triangles ${}^{(n)}\Sigma$, ${}^{(n)}\Sigma^*$ are equilateral, i.e. ${}^{(n)}S = {}^{(n)}T = {}^{(n)}H$ and ${}^{(n)}S^* = {}^{(n)}T^* = {}^{(n)}H^*$. Because of the relations ${}^{(n)}S = S$ and ${}^{(n)}H^* = H$ the first part of assertion is obvious. If n is even then the poiints ${}^{(n)}A^j$, ${}^{(n)}B^j$ lie on the line $\overline{SA^j} = \overline{HA^j}$ and we have

$$^{(n)}A^j = S + (-q)^{\frac{n}{2}} [A^j - S] , \quad ^{(n)}B^j = S + (-h)^{\frac{n}{2}} [A^j - S] .$$

Then

$$^{(n)}B^j - S = 4^{\frac{n}{2}} [{}^{(n)}A^j - S] . \qquad (3)$$

For n odd the points ${}^{(n)}A^j$, ${}^{(n)}B^j$ lie on the line passing through the point S perpendicular to the line $\overline{SA^j}$ and we have

$$^{(n)}A^j = S + (-q)^{\frac{n-1}{2}} [{}^{(1)}A^j - S] , \quad {}^{(n)}B^j = S + (-h)^{\frac{n-1}{2}} [{}^{(1)}B^j - S].$$

On the other hand it follows from the equality

$$^{(1)}b_i^j - \frac{1}{3} = - \frac{1}{3G} (3g_{ii}g_{ij} + g_{ii}g_{jj} + 2g_{ij}g_{ji}) = -2 [{}^{(1)}a_i^j - \frac{1}{3}]$$

valid for each $i \neq j$ that

$$^{(1)}B^j - S = -2 [{}^{(1)}A^j - S].$$

Then:

$$^{(n)}B^j - S = - 2.4^{\frac{n-1}{2}} [{}^{(n)}A^j - S]. \qquad (4)$$

The second part of the theorem follows immediately from (3) and (4).

Theorem 3. If $h = 2$, i.e. $q = \frac{1}{2}$ in an equilateral triangle Σ then

$$^{(-n)}\Sigma^* = {}^{(n)}\Sigma \quad \text{and} \quad {}^{(-n)}\Sigma = {}^{(n)}\Sigma^*$$

for every positive integer n.

Proof. The assertion is obvious as there holds for each $j \in N$, n even:

$$^{(-n)}B^j = S + (-2)^{-\frac{n}{2}} [A^j - S] = S + (- \frac{1}{2})^{\frac{n}{2}} [A^j - S] = {}^{(n)}A^j,$$

$$^{(-n)}A^j = S + (- \frac{1}{2})^{-\frac{n}{2}} [A^j - S] = S + (-2)^{\frac{n}{2}} [A^j - S] = {}^{(n)}B^j,$$

and for n odd:

$$^{(-n)}B^j = S + (-2)^{-\frac{n+1}{2}} [{}^{(1)}B^j - S] = S - 2 (- \frac{1}{2})^{\frac{n+1}{2}} [{}^{(1)}A^j - S] =$$

$$= S + (- \frac{1}{2})^{\frac{n-1}{2}} [{}^{(1)}A^j - S] = {}^{(n)}A^j$$

$$^{(-n)}A^j = S + (- \frac{1}{2})^{-\frac{n+1}{2}} [{}^{(1)}A^j - S] = S - \frac{1}{2} (-2)^{\frac{n+1}{2}} [{}^{(1)}B^j - S] =$$

$$= S + (-2)^{\frac{n-1}{2}} [{}^{(1)}B^j - S] = {}^{(n)}B^j.$$

Some properties of equilateral triangles were shown in the above theorems. The next part will be devoted to nonequilateral triangles.

L e m m a 1. Direction vectors of external normals in a triangle with vertices $P^i = \sum_{i=0}^{2} p^i_j A^j$, i = 0, 1, 2 are given by the relations:

$$\Delta_i = \frac{1}{P} \sum_{k=0}^{2} \bar{P}^i_k \delta_k \tag{5}$$

where P^i_k is the cofactor of the element p^i_k in the determinant $P = |p^i_k|$.

Its g-norms are.

$$g'_{ij} = \frac{1}{|P|^2} \sum_{k,s=0}^{2} \bar{P}^i_k P^j_s g_{ks}. \tag{6}$$

P r o o f . As $\sum_{j=0}^{2} p^i_j = 1$ for i = 0, 1, 2, P has the same value as the determinant with all elements in one column equal to one and in all other columns the same as in the determinant P. If we expand the determinant with respect to this unit column, say i-th, we obtain:

$P = P^0_i + P^1_i + P^2_i$ or $\frac{1}{P}(\bar{P}^0_i + \bar{P}^1_i + \bar{P}^2_i) = 1$ for each i = 0, 1, 2.

Then:

$$\sum_{i=0}^{2} \Delta_i = \frac{1}{P} \sum_{k=0}^{2} \left[\sum_{i=0}^{2} \bar{P}^i_k \right] \delta_k = \sum_{k=0}^{2} \delta_k = 0.$$

From the relation (6) of [2] it follows:

$$(P^i - P^j, \Delta_i) = \frac{1}{P} \sum_{k=0}^{2} (p^i_k - p^j_k) P^i_k = 1 \quad \text{and}$$

$$(P^k - P^k, \Delta_i) = \frac{1}{P} \sum_{r=0}^{2} (p^k_r - p^j_r) P^i_r = 0$$

for each $i, j, k \in N$; $i \neq j \neq k \neq i$.

By that we have proved that $\langle \Delta_i, \Delta_k \rangle$ is biorthogonal basis to $\langle p^i - p^j, p^k - p^j \rangle$. The second part of the assertion is obvious. Let now Σ be a triangle with degree of generality equal to two. Its points S, T, H form a triangle Σ' g-norms of which may be found according to Lemma 1. If we put $p_i^o = -\frac{g_{ii}(g_{jk}+h_{kj})}{4(1+q)G}$, $p_i^1 = \frac{1}{3}$, $p_i^2 = \frac{g_{ji}g_{ki}}{(1+h)G}$ where $i, j, k \in N$; $i \neq j \neq k \neq i$ then $P = D$.

Cofactors of the elements of the given determinant are:

$$D_k^o = \frac{1}{3G(1+h)} (g_{ij}g_{kj} - g_{ji}g_{ki})$$

$$D_k^1 = \frac{1}{4G^2(1+q)(1+h)} [g_{ii}g_{ij}g_{kj}(g_{jk} + g_{kj}) - g_{ji}g_{jj}g_{ki}(g_{ij} + g_{ki})]$$

$$D_k^2 = \frac{1}{12G(1+q)} (g_{ii} - g_{jj})(g_{ij} + g_{ji})$$

where (0, 1, 2) and all its cyclic permutations should be substituted for (k, k, j). If the values D_k^i are substituted into (6) for P_k^i we obtain:

$$g'_{oo} = 4(1+q)M \ , \ g'_{o1} = g'_{1o} = -6M \ , \ g'_{o2} = g'_{2o} = (2-h)M \ , \ g'_{11} = 9M,$$

$$g'_{12} = g'_{21} = -3M \quad , \quad g'_{22} = (1+h)M \qquad (7)$$

where $M = \frac{4G^2(1+q)(1+h)}{hR}$

The calculation is rather long and complicated and not of much interest for the reader as only adding and multiplying of polynomials is used.

There exists another way how to obtain the results of (7). From the equations of edges of the triangle {S, T, H} we determine barycentric coordinates of vectors perpendicular to them using Lemma 1 from [3] . After multiplying the vectors by convenient scalars we get direction vectors of external normals of the triangle {S, T, H} . Their scalar products obtained by using

(7) from [2] are the g-norms in the triangle $\{S, T, H\}$. This way we used to check the correctness of the result (7).

The next theorem follows from the above:

Theorem 4. If Σ is a triangle with degree of generality $\alpha \in N_1 = \{1, 2\}$ then the points S, T, H form a triangle Σ' with degree of generality $\alpha - 1$. For $\alpha = 2$ the triangle Σ' has the orthocentre $H' = \frac{1}{3h}[2(h-2)S + 6T + (h-2)H]$ and centre of its circumcircle is the point $S' = \frac{1}{3h}[2(1+q)S + 3(2q-1)T + (1+h)H]$. The point $T' = \frac{2}{3}S' + \frac{1}{3}H'$ is the centre of gravity of the triangle Σ'. If $\alpha = h = 2$ then the triangle Σ' is restangular at the vertex T.

Let us now consider that the triangle Σ has the degree of generality equal to two. We shall characterize in more detail the sequences of triangles $\{{}^{(n)}\Sigma\}_{-\infty}^{\infty}$, $\{{}^{(n)}\Sigma^*\}_{-\infty}^{\infty}$. It is obvious that all terms of both the sequences have degree of generality equal to two. The point S is the centre of circumcircle with the minimal radius for all terms of the first sequence, i.e. ${}^{(n)}S = S$ for each $n = 0, \pm 1, \pm 2, \ldots$. Likewise the point H belongs to all terms of the second sequence, i.e. ${}^{(n)}H = H$ for each $n = 0, \pm 1, \pm 2, \ldots$. Let us first consider the triangles ${}^{(1)}\Sigma$ and ${}^{(1)}\Sigma^*$.

Triple of points ${}^{(1)}S = S$, ${}^{(1)}T$, ${}^{(1)}H$ forming a triangle with the degree of generality equal to one belongs to the triangle (1)

The barycentric coordinates of the point ${}^{(1)}H$ with respect to the triangle ${}^{(1)}\Sigma$ are as follows:

$$\frac{{}^{(1)}g_{ji}\,{}^{(1)}g_{ki}}{(1+h)\,{}^{(1)}G} = \frac{g_{ij}g_{ik}}{(1+h)G} = \bar{h}_i$$

and its barycentric coordinates with respect to the triangle Σ are:

$$^{(1)}h_i = \sum_\alpha {}^{(1)}a_i^\alpha \bar{h}_\alpha = \frac{1}{2G}(g_{ji}g_{kj} + g_{ki}g_{jk})\frac{g_{ij}g_{ik}}{(1+h)G} - \frac{1}{2G}g_{ii}g_{kj}\cdot$$

$$\cdot\frac{g_{ji}g_{jk}}{(1+h)G} - \frac{1}{2G}g_{ii}g_{jk}\frac{g_{ki}g_{kj}}{(1+h)G} = \frac{1}{2(1+H)G^2}(g_{ii}^2 g_{jk}g_{kj} +$$

$$+ g_{ij}g_{ik}g_{ji}g_{kj} + g_{ij}g_{ik}g_{jk}g_{ki}) = \frac{1}{2(1+H)G}(g_{ii}g_{ji} - g_{ij}^2 +$$

$$+ g_{ij}g_{ji} + g_{ii}g_{jj}) = \frac{1}{2}\frac{g_{ii}g_{jj} + g_{ij}g_{ji} - g_{ij}^2 - g_{ji}^2}{(1+h)G} -$$

$$- \frac{1}{2}\frac{g_{ji}g_{ki}}{(1+h)G} = \frac{1}{2}(1 - h_i).$$

Thus

$$^{(1)}H = \frac{1}{2}\sum_i (1 - h_i)A^i = \frac{3}{2}T - \frac{1}{2}H. \tag{8}$$

For the point $^{(1)}T$, we have

$$^{(1)}T = \frac{1}{3}\sum_j {}^{(1)}A^j = \frac{1}{3}\sum_j\left[\sum_i {}^{(1)}a_i^j A^i\right] = \frac{1}{3}\sum_i\left[\sum_j {}^{(1)}a_i^j\right]A^i =$$

$$= \frac{1}{6G}\sum_i (g_{ji}g_{kj} + g_{ki}g_{jk} - g_{ii}g_{kj} - g_{ii}g_{jk})A^i =$$

$$= \frac{1}{6G}\sum_i (2g_{ii}g_{jj} + g_{ii}g_{ij} + g_{ii}g_{ji} + g_{ii}g_{jj} - g_{ij}g_{ji} + g_{ji}^2 + g_{ii}g_{ji})A^i =$$

$$= \frac{1}{6G}\sum_i (2g_{ii}(g_{jk} + g_{kj}) + G - g_{ji}g_{ki})A^i = \frac{1}{6G}\sum_i (4(1+q)Gs_i +$$

$$+ G - (1+h)Gh_i)A^i = \frac{2}{3}(1+q)S + \frac{1}{2}T - \frac{1}{6}(1+h)H.$$

The triangle with the vertices $^{(1)}S^*$, $^{(1)}T^*$, $^{(1)}H^* = H$ belonging to the triangle $^{(1)}\Sigma^*$ has the degree of generality one too. The barycentric coordinates of the point $^{(1)}S^*$ with respect to the triangle $^{(1)}\Sigma^*$ are as follows:

$$-\frac{^{(1)}g_{ii}^*(^{(1)}g_{jk}^* + {}^{(1)}g_{kj}^*)}{4(1+q)\,^{(1)}G^*} = -\frac{g_{ii}(g_{jk} + g_{kj})}{4(1+q)G} = s_i.$$

Consequently, its barycentric coordinates with respect to the triangle Σ' are:

$$ {}^{(1)}s_i^* = \sum_\alpha {}^{(1)}b_i^\alpha s_\alpha = -\frac{1}{G} g_{ji}g_{ki}\frac{g_{ii}(g_{jk}+g_{kj})}{4(1+q)G} - \frac{1}{G} g_{ij}g_{ki}\frac{g_{jj}(g_{ik}+g_{ki})}{4(1+q)G} - $$

$$ -\frac{1}{G} g_{ik}g_{ji}\frac{g_{kk}(g_{ij}+g_{ji})}{4(1+q)G} = -\frac{1}{4(1+q)G^2}[g_{ii}g_{ji}(g_{ii}+g_{ji})\cdot $$

$$ \cdot(g_{ij}+g_{ji}+2g_{jj}) + g_{ij}g_{jj}(g_{ii}+g_{ji})(2g_{ii}+g_{ij}+g_{ji}) - $$

$$ -(g_{ij}g_{ji}+g_{ji}^2)(g_{ii}+g_{ij}+g_{ji}+g_{jj})(g_{ii}+g_{ij})] = $$

$$ = -\frac{1}{4(1+q)G^2}[2g_{ii}^2g_{ji}g_{jj} + g_{ii}g_{ji}^2g_{jj} + 2g_{ii}^2g_{ij}g_{jj} + 2g_{ii}g_{ij}g_{ji}g_{jj} + $$

$$ + g_{ii}g_{ij}^2g_{jj} - 2g_{ii}g_{ij}^2g_{ji} - 2g_{ii}g_{ij}g_{ji}^2 - 2g_{ij}^2g_{ji}^2 - g_{ij}^3g_{ji} - $$

$$ - g_{ij}g_{ji}^3] = \frac{1}{4(1+q)G}[-2g_{ii}g_{ij} - 2g_{ii}g_{ji} - 2g_{ij}g_{ji} - g_{ij}^2 - $$

$$ - g_{ji}^2] = \frac{1}{4(1+q)G}[4G(1+q) - 2g_{ii}g_{ij} - 2g_{ii}g_{ji} - 4g_{ii}g_{jj}] = $$

$$ + 1 + 2\frac{g_{ii}(g_{jk}+g_{kj})}{4(1+q)G} = 1 - 2s_i. $$

Thus

$$ {}^{(1)}S^* = \sum_i (1-2s_i)A^i = 3T - 2S. \tag{9} $$

For the point ${}^{(1)}T^*$, we have

$$ {}^{(1)}T^* = \frac{1}{3}\sum_j {}^{(1)}B^j = \frac{1}{3}\sum_j\Big[\sum_i {}^{(1)}b_i^jA^i\Big] = \frac{1}{3}\sum_i\Big[\sum_j {}^{(1)}b_i^j\Big]A^i = $$

$$ = \frac{1}{3G}\sum_i[g_{ji}g_{ki} + g_{ij}g_{ki} + g_{ik}g_{ji}]A^i = -\frac{1}{3G}\sum_i[g_{ii}g_{ji} + $$

$$ + g_{ji}^2 + g_{ii}g_{ij} + 2g_{ij}g_{ji} + g_{ii}g_{ji}]A^i = -\frac{1}{3G}\sum_i[g_{ii}(g_{ij} + $$

$$+ g_{ji} + 2g_{jj}) - 2g_{ii}g_{jj} + 2g_{ij}g_{ji} + g_{ji}(g_{ii} + g_{ji})\,]\; A^i =$$

$$= -\frac{1}{3G} \sum_i [-g_{ii}(g_{jk} + g_{kj}) - 2G - g_{ji}g_{ki}]\; A^i =$$

$$= -\frac{1}{3} \sum_i [4(1+q)s_i - 2 - (1+h)h_i]\; A^i = -\frac{4}{3}(1+q)S + 2T +$$

$$+ \frac{1}{3}(1+h)H.$$

The following theorem is a certain generalization of the well-known theorem about the centre of gravity, the centre of circumcircle and the orthocentre of a triangle from the Euclidean geometry.

Theorem 5. The triples $[{}^{(1)}A^j, {}^{(1)}B^j, T]$, $j = 0, 1, 2$; $[{}^{(1)}H, H, T]$, $[S, {}^{(1)}S^*, T]$, $[{}^{(1)}T, {}^{(1)}T^*, T]$ in the set of vertices of the triangles ${}^{(1)}\Sigma$, ${}^{(1)}\Sigma^*$ and the points S, T, H, ${}^{(1)}H$, ${}^{(1)}T$, ${}^{(1)}S^*$, ${}^{(1)}T^*$ are collinear and their affine ratios in the given sequence are $-\frac{1}{2}$.

Proof. The equality $G = -g_{ii}g_{kj} + g_{ij}g_{ki}$ may be written as

$$1 = 2\,[-\frac{1}{2G} g_{ii}g_{kj}] + \frac{1}{G} g_{ij}g_{ki} \quad \text{or} \quad \frac{1}{3} = \frac{2}{3}\,{}^{(1)}a_i^j + {}^{(1)}b_i^j.$$

Consequently,

$$T = \frac{2}{3}\,{}^{(1)}A^j + \frac{1}{3}\,{}^{(1)}B^j \quad \text{for all} \quad j \in N. \qquad (10)$$

Further, if we multiply the first of the above proved equations:

$${}^{(1)}T = \frac{2}{3}(1+q)S + \frac{1}{2}T - \frac{1}{6}(1+h)H$$

$${}^{(1)}T^* = -\frac{4}{3}(1+q)S + 2T + \frac{1}{3}(1+h)H$$

by $\frac{2}{3}$, the second by $\frac{1}{3}$, the sum of them is the equality

$$T = \frac{2}{3}\,{}^{(1)}T + \frac{1}{3}\,{}^{(1)}T^*. \qquad (11)$$

The assertion follows immediately from the relations (8), (9), (10), (11).

T h e o r e m 6. The triangle $\Sigma' = \{S, T, H\}$ is orthogonal to both triangles $^{(1)}\Sigma' = \{^{(1)}S, {}^{(1)}T, {}^{(1)}H\}$ and $^{(1)}\Sigma^* = \{^{(1)}S^*, {}^{(1)}T^*, {}^{(1)}H^*\}$, e.g. its edges are perpendicular to the corresponding edges of these triangles.

P r o o f . Using Lemma 1, we obtain from [3]
$(g'_{00}S + g'_{01}T + g'_{02}H) \perp \overline{HT}$, $(g'_{10}S + g'_{11}T + g'_{12}H) \perp \overline{HS}$,
$g'_{20}S + g'_{21}T + g'_{22}H) \perp \overline{ST}$.

The assertion of the theorem follows then from the following relations:

$$^{(1)}T^* - H = -2\,[\,^{(1)}T - {}^{(1)}H\,] = -\frac{1}{3}\,[\,4(1+q)S - 6T + (2-h)H\,] =$$
$$= -\frac{1}{3M}\,[\,g'_{00}S + g'_{01}T + g'_{02}H\,]$$
$$^{(1)}S^* - H = 2\,[\,^{(1)}H - S\,] = -2S + 3T - H = \frac{1}{3M}\,[\,g'_{10}S + g'_{11}T + g'_{12}H\,]$$

$$^{(1)}T^* - {}^{(1)}S^* = -2\,[\,^{(1)}T - S\,] = \frac{1}{3}\,[\,(2-h)S - 3T + (1+h)H\,] =$$
$$= \frac{1}{3M}\,[\,g'_{20}S + g'_{21}T + g'_{22}H\,]\,.$$

R e m a r k . It is obvious from the preceding theorems that the triangle $^{(1)}\Sigma^*$ is the image of the triangle $^{(1)}\Sigma$ in the homothety with the centre T and the coefficient -2.

The following theorems concern mutual positions of the centres of circumcircles with minimal radius, orthocentres, and centres of gravity of the sequences $\{^{(n)}\Sigma\}_{-\infty}^{\infty}$, $\{^{(n)}\Sigma^*\}_{-\infty}^{\infty}$.

T h e o r e m 7. Orthocentres of all the triangles $^{(2m)}\Sigma$, where $m = 0, \pm 1, \pm 2, \ldots$, and the centres of circles with the minimal radiuc circumscribed to the triangles $^{(2m)}\Sigma^*$ lie on the line $\overline{SH}$. The orthocentres of the triangles $^{(2m+1)}\Sigma$ lie on the line perpendicular to $\overline{SH}$ through the point S. The centres of circles with the minimal radiuc circumscribed to the triangles ${}^{(2m+1)}\Sigma^*$ lie on the perpendicular to the line $\overline{SH}$ through the point H.

P r o o f . Regarding that the barycentric coordinates of the ${}^{(2m)}H$ with respect to the triangle ${}^{(2m)}\Sigma$ are $[h_i]$, its barycentric coordinates with respect to the triangle Σ are the numbers:

$$x_i = \sum_\alpha {}^{(2m)}a_i^\alpha h_\alpha = \sum_\alpha \{(-q)^m \delta_{i\alpha} + [1-(-q)^m]\, s_i\}\, h_\alpha = $$
$$= (-q)^m h_i + [1-(-q)^m]\, s_i$$

so that

$$ {}^{(2m)}H = (-q)^m H + [1-(-q)^m]\, S = S + (-q)^m [H - S]. \qquad (12)$$

The barycentric coordinates of the point ${}^{(2m+1)}H$ with respect to the triangle ${}^{(2m+1)}\Sigma$ are $[\bar{h}_i]$. Consequently, its barycentric coordinates with respect to the triangle Σ are the numbers:

$$y_i = \sum_\alpha {}^{(2m+1)}a_i^\alpha \bar{h}_\alpha = \sum_\alpha \{s_i + (-q)^m [{}^{(1)}a_i^\alpha - s_i]\}\, \bar{h}_\alpha =$$
$$= s_i + (-q)^m [\sum_\alpha {}^{(1)}a_i^\alpha \bar{h}_\alpha - s_i] = s_i + (-q)^m [{}^{(1)}h_i - s_i]$$

Thus

$$ {}^{(2m+1)}H = S + (-q)^m [{}^{(1)}H - S]. \qquad (13)$$

Further it is obvious that the barycentric coordinates of the point ${}^{(m)}S^*$ with respect to the triangle ${}^{(m)}\Sigma^*$ are $[s_i]$ regardless whether m is even or odd. Then:

$$s_i = \sum_\alpha {}^{(2m)}b_i^\alpha s_\alpha = \sum_\alpha \{(-h)^m \delta_{i\alpha} + [1-(-h)^m]\, h_i\}\, s_\alpha =$$
$$= (-h)^m s_i + [1-(-h)^m]\, h_i$$

are the barycentric coordinates of the point ${}^{(2m)}S^*$ and

$$t_i = \sum_\alpha {}^{(2m+1)}b_i^\alpha s_\alpha = \sum_\alpha \{ (-h)^{m(1)}b_i^\alpha + [1-(-h)^m] h_i\} s_\alpha =$$

$$= (-h)^m \sum_\alpha {}^{(1)}b_i^\alpha s_\alpha + [1-(-h)^m] h_i = (-h)^{m(1)}s_i^* + [1-(-h)^m] h_i$$

are the barycentric coordinates of the point ${}^{(2m+1)}S^*$ with respect to the triangle Σ .

Thus

$$ {}^{(2m)}S^* = H + (-h)^m [S - H] \quad \text{and} \tag{14}$$

$$ {}^{(2m+1)}S^* = H + (-h)^m [{}^{(1)}S^* - H] . \tag{15}$$

The statement of the theorem follows immediately from the relations (12), (13), (14), (15) and Theorem 6.

T h e o r e m 8. The centres of gravity of all terms of the sequence $\{{}^{(n)}\Sigma\}_{-\infty}^{\infty}$ lie on the two perpendicular lines $\overline{TS}$ and ${}^{(1)}\overline{TS}$. The centres of gravity of the even terms lie on the first, centres of gravity of the odd terms on the latter of these lines. Also the centers of gravity of the sequence $\{{}^{(n)}\Sigma^*\}_{-\infty}^{\infty}$ lie on two perpendiculars, namely on the line $\overline{TH}$ for even terms, on the line ${}^{(1)}\overline{T^*H}$ for odd terms.

P r o o f . The theorem is verified by the following relations:

$$ {}^{(2m)}T = \frac{1}{3} \sum_j {}^{(2m)}A^j = \frac{1}{3} \sum_j [S + (-q)^m (A^j - S)] = \tag{16}$$

$$= S + (-q)^m [T - S] , \quad {}^{(2m+1)}T = \frac{1}{3} \sum_j {}^{(2m+1)}A^j =$$

$$= \frac{1}{3} \sum_j [S + (-q)^m ({}^{(1)}A^j - S)] = S + (-q)^m [{}^{(1)}T - S]$$

$$ {}^{(2m)}T^* = \frac{1}{3} \sum_j {}^{(2m)}B^j = \frac{1}{3} \sum_j [H + (-h)^m (A^j - H)] =$$

$$= H + (-h)^m [T - H]$$

$$^{(2m+1)}T^* = \frac{1}{3} \sum_j {}^{(2m+1)}B^j = \frac{1}{3} \sum_j [\, H + (-h)^m ({}^{(1)}B^j - H) \,] =$$

$$= H + (-h)^m [\, {}^{(1)}T^* - H \,] .$$

<u>T h e o r e m 5.</u> For any nonzero integer n the triangle $^{(n)}\Sigma^*$ is homothetic to the triangle $^{(n)}\Sigma$. For $n = 2m$, $m = \pm 1, \pm 2, \dots$ the centre of this homothety is the point $Q_{2m} = \frac{1}{1-4^m} \{ -4^m [\, 1-(-q)^m \,] S + [\, 1-(-h)^m \,] H \}$ and the coefficient equals 4^m. For $n = 2m+1$ (m integer) the centre is the point $Q_{2m+1} = \frac{1}{1+2.4^m} \{ 2.4^m S + H + (-h)^m [\, 3T - 2S - H \,] \}$ and the coefficient is -2.4^m.

<u>P r o o f .</u> It may be easily shown by substituting for $^{(n)}A^j$, $^{(n)}B^j$ and using (10) that for each $m = 0, \pm 1, \pm 2, \dots$ and $j \in N$

$$Q_{2m} = \frac{1}{1-4^m} [\, -4^m \, {}^{(2m)}A^j + {}^{(2m)}B^j \,] , \quad (m \neq 0)$$

$$Q_{2m+1} = \frac{1}{1+2.4^m} [\, 2.4^m \, {}^{(2m+1)}A^j + {}^{(2m+1)}B^j \,] . \tag{17}$$

Modifying them we obtain the relations:

$$^{(2m)}B^j - Q_{2m} = 4^m [\, {}^{(2m)}A^j - Q_{2m} \,] ,$$

$$^{(2m+1)}B^j - Q_{2m+1} = -2.4^m [\, {}^{(2m+1)}A^j - Q_{2m+1} \,] ,$$

from which the statement of the theorem is obvious.

<u>C o r o l l a r y 1.</u> The affine ratio of the point Q_{2m} with respect to the pair of points $^{(2m)}A^j$, $^{(2m)}B^j$ is equal to $\frac{1}{4^m}$. The point Q_{2m+1} has the affine ratio with respect to the pair of points $^{(2m+1)}A^j$, $^{(2m+1)}B^j$ equal $-\frac{1}{2.4^m}$.

<u>C o r o l l a r y 2.</u> For any nonzero integer m the point Q_{2m} lies on the line determined by the points S, H. Its affine

ratio with respect to these points is equal to $\frac{1}{4^m}\frac{1-(-h)^m}{1-(-q)^m}$.

<u>Corollary 3.</u> If in the triangle Σ :

1/ $q = 1$, then all points Q_{2m+1} lie on two perpendicular lines $\overline{HT}$ and $\overline{H^{(1)}T^*}$; or more exactly for every integer p, $Q_{4p+1} \in \overline{HT}$ and $Q_{4p+3} \in \overline{H^{(1)}T^*}$.

2/ $h = 1$, then $Q_{4p+1} \in \overline{ST}$ and $Q_{4p+3} \in \overline{S^{(1)}T} \perp ST$.

3/ $h = 2$, then all points Q_{2m+1} (m integer) lie on the circumcircle to the triangle $\Sigma' = \{S, T, H\}$.

<u>Proof.</u> Let us consider the case $q = 1$, i.e. $h = 4$. Then ${}^{(1)}T^* = \frac{1}{3}[-8S + 6T + 5H]$, consequently the lines $\overline{HT}$, $\overline{H^{(1)}T^*}$ have the following equations in the barycentric coordinates with respect to the triangle Σ': $x_o = 0$, resp. $3x_o + 4x_1 = 0$.
The first is satisfied by the triple $[0, 3.4^{2p}, 1-4^{2p}]$, the second by the triple $[4.4^{2p+1}, -3.4^{2p+1}, 1+4^{2p+1}]$ of homogeneous barycentric coordinates of the points Q_{4p+1}, Q_{4p+3} with respect to the triangle Σ' . That proves the first statement. The second statement is proved analogically. Equations of the lines $\overline{ST}$, $\overline{S^{(1)}T}$ with respect to the triangle Σ' for $h = 1$ are the following: $x_2 = 0$, $2x_1 + 3x_2 = 0$, respectively. The homogeneous barycentric coordinates of the points Q_{4p+1}, Q_{4p+3}, ${}^{(1)}T$ are the following: $[2(4^{2p}-1), +3, 0]$, $[2(1+4^{2p+1}), -3, 2]$, $[\frac{5}{6}, \frac{1}{2}, -\frac{1}{3}]$. Using (8) from [2] we may prove easily that the numbers:

$$e'_{o1} = \frac{1+h}{3hM} \quad , \quad e'_{o2} = \frac{3}{hM} \quad , \quad e'_{12} = \frac{4+h}{3hM}$$

are the nonzero e-norms of the triangle Σ' . The circumcircle k to Σ' has the equation:

$$\sum_{i,j=0}^{2} e'_{ij} z_i \bar{z}_j = 0, \quad \text{resp.}$$

$$(1+h)(z_0\bar{z}_1+\bar{z}_0 z_1) + 9(z_0\bar{z}_2+\bar{z}_0 z_2) + (4+h)(z_1\bar{z}_2+\bar{z}_1 z_2) = 0 \quad (18)$$

in the barycentric coordinates with respect to Σ'.

If $h = 2$ in the triangle Σ, then the point Q_{2m+1} (m integer) has the following homogenous barycentric coordinates with respect to Σ':

$$z_0 = 2\left[4^m - (-2)^m\right], \; z_1 = 3(-2)^m, \; z_2 = 1-(-2)^m.$$

It may be easily shown that this triple of numbers satisfies the equation (18), so that $Q_{2m+1} \in k$. The proof is complete.

<u>T h e o r e m 10.</u> For every nonzero integer n the point Q_n is an intersection point of the lines determined by the pairs of points $[{}^{(n)}T, {}^{(n)}T^*]$, $[{}^{(n-1)}T, {}^{(n-1)}T^*]$. Its affine ratio being the same with respect to the both pairs is equal to the affine ratio of the point Q_n with respect to the pairs of points ${}^{(n)}A^j$, ${}^{(n)}B^j$, where j is an arbitrary index from the set N.

<u>P r o o f .</u> The assertion follows from (17) and the following relations:

$$Q_{2m} = \frac{1}{1-4^m}\left[-4^m\,{}^{(2m-1)}T + {}^{(2m-1)}T^*\right] = \frac{1}{1-4^m}\left[-4^m\,{}^{(2m)}T + {}^{(2m)}T^*\right],$$

$$m \neq 0$$

$$Q_{2m+1} = \frac{1}{1+2.4^m}\left[2.4^m\,{}^{(2m)}T + {}^{(2m)}T^*\right] = \frac{1}{1+2.4^m}\left[2.4^m\,{}^{(2m+1)}T + {}^{(2m+1)}T^*\right]. \quad (19)$$

Their validity can be easily verified by means of (11) and (16).

In the last part we shall describe in more detail the sequence of triangles with the terms ${}^{(n)}\Sigma' = \{{}^{(n)}S = S, {}^{(n)}H, {}^{(n)}T\}$ and ${}^{(n)}\Sigma^{*\prime} = \{{}^{(n)}S^*, {}^{(n)}T^*, {}^{(n)}H^* = H\}$.

Let us denote the homothety with the centre S and doefficient $-q$

by φ and the homothety with the centre H and coefficient -h by ψ.

Then the following theorem holds:

Theorem 11. The triangle ${}^{(n)}\Sigma'$ is similar either to the triangle Σ' or ${}^{(1)}\Sigma'$, according to whether n is even or odd. The coefficient of similarity is $(-q)^m$ for both $n = 2m$ and $n = 2m+1$. Analogously, the triangle ${}^{(n)}\Sigma^{*\prime}$ is similar to the triangle Σ' for n even and to ${}^{(1)}\Sigma^{*\prime}$ for n odd. These similarities have the same coefficient $(-h)^m$ for both $n = 2m$ and $n = 2m+1$. More exactly:

$$ {}^{(2m)}\Sigma' = \varphi^m(\Sigma') \quad , \quad {}^{(2m+1)}\Sigma' = \varphi^m({}^{(1)}\Sigma'), $$
$$ {}^{(2m)}\Sigma^{*\prime} = \psi^m(\Sigma') \quad , \quad {}^{(2m+1)}\Sigma^{*\prime} = \psi^m({}^{(1)}\Sigma^{*\prime}) $$

(m, n integers).

Proof. The theorem is obvious by relations (12), (13), (14), (15) and (16).

Corollary. Each of the triangles ${}^{(n)}\Sigma'$, ${}^{(n)}\Sigma^{*\prime}$, $n \neq 0$ is perpendicular to both triangles ${}^{(n+1)}\Sigma'$ and ${}^{(n+1)}\Sigma^{*\prime}$.

Theorem 12. For every $n \neq 0$ the triangle ${}^{(n)}\Sigma^{*\prime}$ is homothetic to the triangle ${}^{(n)}\Sigma'$. Centre of this homothety is the point Q_n and the coefficient is equal to $4^{\frac{n}{2}}$ for n even, $-2.4^{\frac{n-1}{2}}$ for n odd.

Proof. The statement of Theorem 12 follows from Theorem 9, the relations (17), (19) and the following relations:

$$ Q_{2m} = \frac{1}{1-4^m}\left[-4^{m}\,{}^{(2m)}S + {}^{(2m)}S^*\right] = \frac{1}{1-4^m}\left[-4^{m}\,{}^{(2m)}H + {}^{(2m)}H^*\right] $$

$$ Q_{2m+1} = \frac{1}{1+2.4^m}\left[2.4^{m}\,{}^{(2m+1)}S + {}^{(2m+1)}S^*\right] = $$

$$= \frac{1}{1+2.4^m} [2.4^{m(2m+1)}H + {}^{(2m+1)}H^*]$$

Their validity is obvious from the definitions of the points Q_{2m}, Q_{2m+1} and relations (12) - (15).

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S Ú H R N

NIEKTORÉ VLASTNOSTI TROJUHOLNÍKOV HERMITOVSKEJ ROVINY

Valent Zaťko, Bratislava

V tejto práci sa zisťuje vzájomná poloha význačných bodov trojuholníka Hermitovskej roviny i niektorých ním určených postupností trojuholníkov, ktoré boli zavedené v prácach [2] a [3] . Dosiahnuté výsledky sú zovšeobecnením známych viet z geometrie trojuholníka Euklidovskej roviny.

РЕЗЮМЕ

НЕКОТОРЫЕ СВОЙСТВА ТРЕУГОЛЬНИКОВ ПЛОСКОСТИ ЭРМИТА

Валент Затько, Братислава

В этой работе рассматривается взаимное расположение замечательных точек треугольника, принадлежащего плоскости Эрмита, и некоторых им определенных последовательностей треугольников, которые были введены в работах [2] и [3]. Полученные результаты являются обобщениями известных теорем из геометрии треугольника евклидовой плоскости.