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GENERALIZATION OF A THEOREM OF S. PICCARD

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In 1942 S. Piccard has proved (see [5]) a theorem, which is a topological analogue of the classical theorem of Steinhaus (see [6]): If $A \subset R$ is a second category Baire set, then the set $A + A$ contains an interval.

In 1973 Z. Kominek in [3] attained more general result. He has shown, that if $f(x, y)$ is a function from $R \times R$ to R, which is a homeomorphism with respect to each variable separately and if $A, B \subset R$ are second category Baire sets, then the set $f(A \times B)$ contains an interval.

Kominek has inspirated by the paper M. E. Kuczma and M. Kuczma [4] , which generalizes the theorem of Steinhaus.

In the present note we give the following generalization of the Kominek's result:

T h e o r e m. Let $f(x, y) : R \times R \longrightarrow R$ be a function, which is continuous with respect to each variable separately and which is constant in no subinterval I of R. If $A, B \subset R$ are second category Baire sets, then the set $f(A \times B)$ contains an interval.

R e m a r k 1. The function f may be discontinuous, but in the Kominek's result the continuity of f is assumed.

In the sequel we shall use the following notation. By symbol f_x (for every $x \in R$) we denote the function $R \longrightarrow R$ such, that for every $y \in R$ is $f_x(y) = f(x, y)$. Similarly $f^y(x) = f(x, y)$

for every $x \in R$. Interval will be always a non-degenerate interval.

<u>Remark 2.</u> The theorem does not hold if both the functions f_x, f^y are continuous but only the functions f^y are assumed to be non-constant everywhere, as is shown by the following example:

$$f(x, y) = x$$

and

$A = B = J$, where J is the set of irrational numbers.

<u>Remark 3.</u> Similarly, the theorem does not hold if the functions f_x, f^y are assumed to be non-constant everywhere but only f^y are continuous, as is shown by the following example:

$$f(x,y) = \begin{cases} x & \text{for rational } y \\ 1 + x & \text{for irrational } y \end{cases}$$

and

$A = B = J$, where J is the set of irrational numbers

For the proof of the theorem we shall need the following auxiliary results.

<u>Lemma 1.</u> Let $f(x, y) : R \times R \longrightarrow R$ be a function such, that for every x, y the functions f_x, f^y are continuous. Then there exist first category sets A, B of the type F_σ such, that the function f is continuous on the set $(R \times R) \setminus (A \times B)$.

<u>Proof.</u> See [1] or [2] , page 337.

<u>Lemma 2.</u> Let $g : R \longrightarrow R$ be a continuous function, let I be an interval and let $v \in \text{int } g(I)$. Then there exists some $u \in \text{int } I$ such, that $g(u) = v$ and u is not a point of local extreme of function g.

<u>Proof.</u> Let assumptions of the Lemma be fulfiled. Since $\text{int } g(I)$ is an open interval, there exist numbers $v_1, v_2 \in \text{int } g(I)$

such, that $v_1 < v < v_2$. Hence there exist numbers $u_1, u_2 \in I$ such, that $g(u_1) = v_1$, $g(u_2)=v_2$. Without loss of generality can be assumed, that $u_1 < u_2$ (in the case $u_1 > u_2$ the proof is analogous).

Let

$$M = \{ a \in \langle u_1, u_2 \rangle : \text{for every } x \in \langle u_1, a \rangle \text{ is } g(x) \leq v\}.$$

Clearly the set M is non-empty and upper bounded. Let u be the least upper bound of M. It is easy to verify that $g(u) = v$ and that g has no local extreme at u, q. e. d.

L e m m a 3. Let f satisfy the assumptions of the theorem. Let $I, J \subset R$ be closed intervals and let $\alpha = (u, v)$ be an interior point of the interval $I \times J$ such, that $f(\alpha) = t$. Let at least one of functions f_u, f^v has not local extreme in point α. 1/

Let A, B be nowhere dense sets. Then there exist closed intervals $I_1 \subset I \setminus A$, $J_1 \subset J \setminus B$ and a point $\alpha_1 = (u_1, v_1) \in \in \text{int } (I_1 \times J_1)$ such, that the function f is in the point α_1 continuous, $f(\alpha_1) = t$ and at least one of functions f_{u_1}, f^{v_1} has not local extreme in α_1.

P r o o f . Without loss of generality can be assumed, that the function f_u has not local extreme in the point α. Since $v \in \text{int } J$, there exist points c, $d \in J$ such, that $f_u(c) < t$, $f_u(d) > t$. Without loss of generality can be assumed, that $c < d$. The functions f^c, f^d are continuous and $f^c(u) < t$, $f^d(u) > t$, hence there exists $\varepsilon > 0$, such, that $(u - \varepsilon, u + \varepsilon) \subset I$, and $f^c(x) < t$ and $f^d(x) > t$ for all $x \in (u - \varepsilon, u + \varepsilon)$.

1/ The fact, that the function f_u has not local extreme in point $\alpha = (u, v)$, means, that f_u has not local extreme in point v. Similarly for f^v.

Since A is a nowhere dense set, there exists closed interval $I_1 \subset (u - \varepsilon, u + \varepsilon) \setminus A \subset I \setminus A$. For every $x \in I_1$ put

$$M_x = \{ y \in \langle c, d \rangle : f_x(y) > t \}$$

Clearly $d \in M_x$ and $c \notin M_x$. Thus for every $x \in I_1$ there is the greatest lower bound of M_x, $\inf M_x \in (c, d)$. Moreover, one can easily verify that for each $x \in I_1$ we have

$$f_x(\inf M_x) = t$$

Since the functions f^y are constant on no interval there exist numbers $a, b \in I_1$ such, that $\inf M_a \neq \inf M_b$. Let e.g. $\inf M_a < \inf M_b$. From the definition of M_x it follows, that there exists a point from interval $(\inf M_a, \inf M_b)$, where the function f_a has a value greater than t. But then there is an interval $J_+ \subset (\inf M_a, \inf M_b)$ such, that $f_a(y) > t$ for each $y \in J_+$. On the other hand, for all $y \in \langle c, \inf M_b \rangle$ we have $f_b(y) \leqq t$. Since f_b is not constant in $J_+ \subset (c, \inf M_b)$, there is a subinterval J_+^- of J_+ such that $f_b(y) < t$ for each $y \in J_+^-$. Since the set B is nowhere dense, there exists a closed interval $J_1 \subset J_+^- \setminus B$. Obviously $J_1 \subset J \setminus B$. According to the lemma 1 the set of points of discontinuity of function $f(x, y)$ is a subset of Cartesian produkt $D_x \times D_y$, where D_x, D_y are first category sets of the type F_σ. First category set, however, contains no interval, thus there exists $v_1 \in \operatorname{int} J_1 \setminus D_y$.
As we know, $f^{v_1}(a) > t$, $f^{v_1}(b) < t$ and f^{v_1} is continuous function. According to the lemma 2 thus there exists $u_1 \in \operatorname{int} I_1$ (note, that $a, b \in I_1$) such, that $f^{v_1}(u_1) = t$ and u_1 is not a point of local extreme of function f^{v_1}. Thus the lemma has been proved, because the point $\alpha_1 = (u_1, v_1) \in \operatorname{int} (I_1 \times J_1)$ is a point of continuity of the function $f(x, y)$ (since $v_1 \notin D_y$).

<u>P r o o f of the theorem.</u> $A, B \subset R$ are by assumption a second category Baire sets. Then

$$A = (G_1 \setminus P_1) \cup Q_1$$

$$B = (G_2 \setminus P_2) \cup Q_2$$

for some non empty open sets G_1 and G_2, where P_1, Q_1, P_2, Q_2 are of the first category. Let us denote by I_0 and J_0 closed intervals contained in G_1 and G_2, respectively. It suffices to prove, that $f\,[(I_0 \setminus P_1) \times (J_0 \setminus P_2)]$ contains an interval. Let us take arbitrarily but fixed point $u_0 \in \operatorname{int} I_0$. The function $f(x, y)$ maps interval $I_H = \{u_0\} \times J_0$ onto some interval I_V, because function f_{u_0} is continuous and not constant on any interval. Let $I^* = \operatorname{int} I_V$.
We shall show that

$$I^* \subset f\,[(I_0 \setminus P_1) \times (J_0 \setminus P_2)]$$

Since P_1, P_2 are first category sets, we have

$$P_1 = \bigcup_{n=1}^{\infty} P_{1n}$$

$$P_2 = \bigcup_{n=1}^{\infty} P_{2n},$$

where P_{1n} and P_{2n} are nowhere dense sets.

Let $t \in I^*$ be an arbitrary but fixed point from I^*. According to the Lemma 2 there exists $v_0 \in \operatorname{int} J_0$ such, that $f_{u_0}(v_0) = t$, and such, that v_0 is not a point of local extreme of function f_{u_0}. The point $\alpha_0 = (u_0, v_0)$ is interior point of the closed rectangle $F_0 = I_0 \times J_0$ such, that $f(\alpha_0) = t$. The function f_{u_0} has not local extreme in point α_0. The sets P_{11}, P_{21} are nowhere dense. According to the lemma 3 thus there exist closed intervals $I_1' \subset I_0 \setminus P_{11}$, $J_1' \subset J_0 \setminus P_{21}$ and point $\alpha_1 = (u_1, v_1) \in$

$\in$ int $(I_1' \times J_1')$, which is a point of continuity of the function f such, that $f(\alpha_1) = t$ and at least one of functions f_{u_1}, f^{v_1} has not local extreme in α_1. Since the function f is continuous in the point α_1, there exists closed rectangle

$$F_1 = (I_1 \times J_1) \subset (I_1' \times J_1') \subset (I_0 \setminus P_{11}) \times J_0 \setminus P_{21})$$

such that

$$\alpha_1 = (u_1, v_1) \in \text{int } F_1$$

and

$$\text{for all } (x, y) \in F_1 \text{ is } |f(x, y) - t| < 1$$

Moreover, $f(\alpha_1) = t$ and at least one of functions f_{u_1}, f^{v_1} has not local extreme in α_1.

In a similar way we can find a closed rectangle $F_2 \subset F_1 = I_1 \times J_1$ such that

$$F_2 = (I_2 \times J_2) \subset (I_1 \setminus P_{12}) \times (J_1 \setminus P_{22}) \subset$$

$$\subset (I_0 \setminus (P_{11} \cup P_{12})) \times (J_0 \setminus (P_{21} \cup P_{22}))$$

and such that there is a point

$$\alpha_2 = (u_2, v_2) \in \text{int } F_2$$

with the following properties: $f(\alpha_2) = t$, at least one of the functions f_{u_2}, f^{v_2} has not local extreme at α_2, and

$$|f(x, y) - t| < \frac{1}{2} \text{ for all } (x, y) \in F_2$$

Into rectangle F_2 similarly we inscribe rectangle F_3, etc. By the induction so we may construct a sequence of closed non empty rectangles $\{F_n\}_{n=1}^{\infty}$ such that

$$F_1 \supset F_2 \supset F_3 \supset \dots \supset F_k \supset \dots \tag{1}$$

$$F_k \subset (I_0 \setminus \bigcup_{i=1}^{k} P_{1i}) \times (J_0 \setminus \bigcup_{i=1}^{k} P_{2i}); \; k = 1, 2, \dots \tag{2}$$

and

for every $(x, y) \in F_k$ is $|f(x, y) - t| < \frac{1}{k}$;

$k = 1, 2, \dots$ (3)

The rectangles F_n form a decreasing sequence of non empty compact sets. Hence there is a point $\alpha = (u, v) \in \bigcap_{n=1}^{\infty} F_n$.

By (2) we have

$$\alpha = (u, v) \in (I_0 \setminus P_1) \times (J_0 \setminus P_2) \tag{4}$$

By (3)

$$|f(u, v) - t| < \frac{1}{k} \quad \text{for every } k = 1, 2, \dots$$

hence

$$f(u, v) = t \tag{5}$$

With respect to (4) and (5) we have $t \in f[(I_0 \setminus P_1) \times (J_0 \setminus P_2)]$.

The point $t \in I^*$ has been chosen arbitrarily, hence

$$I^* \subset f[(I_0 \setminus P_1) \times (J_0 \setminus P_2)], \quad \text{q.e.d.}$$

<u>R e m a r k 4.</u> The above quoted theorem is a special case of the following more general result: Under the assumptions of the theorem the set $f[(G_1 \setminus P_1) \times (G_2 \setminus P_2)]$ is an union of a countable family of intervals. Moreover,

$$f[(G_1 \setminus P_1) \times (G_2 \setminus P_2)] = f(G_1 \times G_2) \setminus S,$$

where S is a countable nowhere dense set.

The proof is more complicated, but involves the same ideas as the proof of the theorem.

R E F E R E N C E S

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S Ú H R N

ZOVŠEOBECNENIE JEDNEJ VETY S. PICCARDOVEJ

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Nech $f(x, y)$ je funkcia $R \times R \longrightarrow R$. Nech pre každé $x \in R$ sú funkcie $f(x, -)$ a $f(-, x)$ spojité a nekonštantné na žiadnom intervale. Nech $A, B \subset R$ sú množiny druhej Baireovej kategórie s Baireovou vlastnosťou. Potom množina $f(A, B)$ obsahuje nejaký interval.

Р Е З Ю М Е

ОБОБЩЕНИЕ ОДНОЙ ТЕОРЕМЫ С. ПИКАРДОВОЙ

ЯРОСЛАВ СМИТАЛ, ЛЮБОМИР СНОХА, БРАТИСЛАВА

Пусть $f(x, y)$ - функция $R \times R \longrightarrow R$. Пусть для всякого $x \in R$ функции $f(x, -)$ и $f(-, x)$ непрерывные и не постоянные ни на каком отрезке. Пусть $A, B \subset R$ множества Бера второй категории Бера. Тогда множество $f(A, B)$ содержит какой-нибудь отрезок.