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# ON THE CONSTRUCTION OF THE MEASURE FROM A CONTENT

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The method stated in the title is very well known (see e.g. [5], [6], [7]). We want to present here a generalization of the method applicable also to integrals of the Daniell type. Hence we shall consider functions $J : S \to R$, where $S$ is a convenient lattice. If $S$ is a lattice of sets then the classical measure construction theorem can be obtained. If $S$ is a lattice of real-valued functions (on a compact space) then an analogy for Daniell integral case is obtained.

Of course, similar lattice generalizations have appeared (see [1], [2], [3], [8], [9], [10], [11], [12], [14], [15]) with various constructions. (A review of the constructions is contained in [13]). The construction of the measure from a content uses the Carathéodory measurability theorem. Hence we shall use the results of the paper [12], where an abstract lattice generalization of this method is studied.

## 1. Content

We shall define here the generalized content, a function defined on a sublattice of a given lattice H. First the assmmptions stated on H:

Let H be a lattice with the least element $o$ and with a binary operation + satisfying the following condition:

1. If $a_n \nearrow a$ and $b_n \nearrow b$ then $a_n + b_n \nearrow a + b$ and

$a_n \wedge b_n \nearrow a \wedge b$.

Besides the usual ordering $\leqq$ induced by the lattice operations, there is given a relation $\prec$ on H satisfying the following conditions:

2. If $a \prec b$ then $a \leqq b$.
3. $\sigma \prec \sigma$.
4. If $a \prec b$ and $b \leqq c$ then $a \prec c$.
5. If $a \prec b$ and $c \prec d$ then $a \vee c \prec b \vee d$, $a \wedge c \prec b \wedge d$ and $a + c \prec b + d$.

Further let C, B be sublattices of H containing $\sigma$, closed under + and satisfying the following conditions:

6. If $b_n \in B$ $(n = 1, 2, \ldots)$, $b_n \nearrow b$, then $b \in B$.
7. To every $b \in B$ there are $\{c_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, such that $c_n \in C$, $b_n \in B$, $b_n \leqq c_n \prec b$ and $b_n \nearrow b$.
8. If $b_n \nearrow b$, $c \prec b$, $c \in C$, $b_n \in B$ $(n = 1, 2, \ldots)$, then there is $n_0$ such that $c \prec b_{n_0}$.

Example 1. Let X be a locally compact Hausdorff topological space satisfying the second axiom of countability, $H = 2^X$, C be the family of all compact sets, B be the family of all open Borel sets, $\leqq$ and $\prec$ coincide with the inclusion.

Example 2. Let H be the set of all non-negative real-valued functions defined on a compact set Q, C be the family consisting of all positive continuous functions and the zero function, B be the family of corresponding lower semicontinuous functions, $\leqq$ usual ordering and $f \prec g$ iff either $f(x) < g(x)$ for all $x \in Q$ or $f(x) = g(x) \equiv 0$ for all $x \in Q$.

Now, we can define a content:

D e f i n i t i o n 1. A real-valued function $\lambda : C \to R$ is called a content, if it satisfies the following conditions:

9. $\lambda(\sigma) = 0$.

10. If $c \leqq d$, then $\lambda(c) \leqq \lambda(d)$.

11. $\lambda(c) + \lambda(d) \geqq \lambda(c \vee d) + \lambda(c \wedge d)$ for every $c, d \in C$.

12. $\lambda(c + d) \leqq \lambda(c) + \lambda(d)$ for every $c, d \in C$.

D e f i n i t i o n 2. For every content $\lambda : C \to R$ we define a mapping $J_0 : B \to R \cup \{\infty\}$ by the formula:

$$J_0(b) = \sup \{ \lambda(c);\ c \prec b,\ c \in C \}.$$

L e m m a 1. Let $b_n \leqq c_n \prec b$, $b_n \in B$, $c_n \in C$ $(n = 1, 2, \ldots)$, $b_n \nearrow b$. Then $J_0(b) = \lim \lambda(c_n)$.

P r o o f . Put $c \in C$, $c \prec b$. By (8) there is $n_0$ such that $c \prec b_{n_0}$. Then $c \prec b_{n_0} \leqq b_n \leqq c_n \prec b$ for every $n \geqq n_0$. Hence, (2) and (10) imply $\lambda(c) \leqq \lambda(c_n)$ for $n \geqq n_0$ and therefore $\lambda(c) \leqq \liminf \lambda(c_n) \leqq \limsup \lambda(c_n) \leqq J_0(b)$.

T h e o r e m 1. The mapping $J_0$ has the following properties:

13. $J_0(\sigma) = 0$

14. If $u, v \in B$, $u \leqq v$, then $J_0(u) \leqq J_0(v)$.

15. $J_0(u) + J_0(v) \geqq J_0(u \wedge v) + J_0(u \vee v)$ for every $u, v \in B$.

16. $J_0(u+v) \leqq J_0(u) + J_0(v)$ for every $u, v \in B$.

17. If $u_n \in B$ $(n + 1, 2, \ldots)$ and $u_n \nearrow u$, then $\lim J_0(u_n) = J_0(u)$.

P r o o f . If $c \prec \sigma$, then $c \leqq o$. Hence $c = \sigma$ and therefore $J_0(o) = \lambda(o) = 0$. The property (13) is proved. The property

(14) is evident. To prove (15) and (16) take $u_n, v_n \in B$, $c_n, d_n \in C$ such that $u_n \leqq c_n \prec u$, $v_n \leqq d_n \prec v$ $(n = 1, 2, \dots)$ and $u_n \nearrow u$, $v_n \nearrow v$. By Lemma 1 $J_0(u) = \lim \lambda(c_n)$, $J_0(v) = \lim \lambda(d_n)$. (1) and (5) imply:

$$u_n \wedge v_n \leqq c_n \wedge d_n \prec u \wedge v, \quad u_n \wedge v_n \nearrow u \wedge v$$

$$u_n \vee v_n \leqq c_n \vee d_n \prec u \vee v, \quad u_n \vee v_n \nearrow u \vee v$$

$$u_n + v_n \leqq c_n + d_n \prec u + v, \quad u_n + v_n \nearrow u + v;$$

hence by Lemma 1, (11) and (12)

$$J_0(u \vee v) + J_0(u \wedge v) = \lim \lambda(c_n \vee d_n) + \lim \lambda(c_n \wedge d_n) \leqq$$
$$\lim \lambda(c_n) + \lim \lambda(d_n) = J_0(u) + J_0(v),$$

$$J_0(u + v) = \lim \lambda(c_n + d_n) = \lim \lambda(c_n) + \lim \lambda(d_n) =$$
$$J_0(u) + J_0(v).$$

Finally we prove (17). Let $u_n \in B$, $u_n \nearrow u$, $c \in C$, $c \prec u$. Then (8) implies the existence of $n_0$ such that $c \prec u_{n_0}$. Hence $\lambda(c) \leqq J_0(u_{n_0}) \leqq \lim J_0(u_n) \leqq J_0(u)$ and therefore $J_0(u) \leqq$ $\leqq \lim J_0(u_n) \leqq J_0(u)$.

## 2. Measurability

First we define $J^*: H \rightarrow R \cup \{\infty\}$ by the formula:

$$J^*(x) = \inf \{ J_0(b);\ b \geqq x,\ b \in B \}.$$

The notion of the Caratheodory measurability needs another binary operation $\setminus$ on $H$.

In Example 1 the operation $\setminus$ can be understood as the set theoretic difference, in Example 2 it can be defined by $f \setminus g =$ $= f \dot{-} \min(f, g)$.

In the general case the following definition is possible:

<u>Definition 3.</u> An element $x \in H$ is called to be measurable, if $J_0(u) \geq J^*(u \setminus x) + J^*(u \wedge x)$ for every $u \in B$. Denote by $M$ the set of all measurable elements.

If we assume some properties of the operation $\setminus$ and the initial mapping $J_0$(in our case these properties of $J_0$ are satisfied - see Theorem 1), we obtain good properties of $M$ (see [12] ). Here we are interested in the inclusion $C \subset M$ only.

<u>Theorem 2.</u> Let $H$ satisfy (1) - (5), $B$ and $C$ satisfy (6) - (8) and the following two conditions:

18. If $d \prec u \setminus c$, $u \in B$, $d, c \in C$ then $u \wedge c \leq u \setminus d$.

19. If $u \in B$, $c \in C$ then $u \setminus c \in B$.

Let $\lambda : C \to R$ be a content satisfying the following condition

20. If $d \prec u \setminus c$, $e \prec u \setminus d$; $c, e, d \in C$, $u \in B$, then $e + d \prec u$ and $\lambda(e + d) = \lambda(e) + \lambda(d)$.

Then $C \subset M$.

<u>Proof.</u> Let $c \in C$. Take $e, d \in C$ and $u \in B$ satisfying (20). Then $J_0(u) \geq \lambda(e + d) = \lambda(e) + \lambda(d)$.
Hence $J_0(u) \geq \sup \{ \lambda(e); e \prec u \setminus d \} + \lambda(d) = J_0(u \setminus d) + \lambda(d) = J^*(u \wedge c) + \lambda(d)$ by (18) and (19).
Since $d \prec u \setminus c$ was arbitrary, we obtain

$$J_0(u) \geq J^*(u \wedge c) + J_0(u \setminus c) = J^*(u \wedge c) + J^*(u \setminus c).$$

## 3. Extendability

Finally we give a sufficient condition for the equality $J^*(c) = \lambda(c)$ for every $c \in C$.

<u>Theorem 3.</u> Let $\lambda(c) = \sup \{ \lambda(d);\ d \prec u \leqq e,\ d \in C,\ u \in B \} = \inf \{ \lambda(e);\ c \leqq v \leqq e,\ e \in C,\ v \in B \}$ for every $c \in C$. Then $\lambda(c) = J^*(c)$ for every $c \in C$.

<u>Proof.</u> If $u \leqq c$, $u \in B$, $c \in C$ and $d \prec u$, $d \in C$, then $d \prec c$ by (4). Henc $\lambda(d) \leqq \lambda(c)$ by (2) and (10), and therefore

21. $J_0(u) = \sup \{ \lambda(d);\ d \prec u \} \leqq \lambda(c)$.

To every $\varepsilon > 0$ and $c \in C$ there are $e, d \in C$, $u, v \in B$ such that $d \prec u \leqq c$, $c \leqq v \leqq e$ and

22. $\lambda(c) - \lambda(d) < \varepsilon$ , $\lambda(e) - \lambda(c) < \varepsilon$ .

Then (21) and (22) give

$$J^*(c) \geqq J^*(u) = J_0(u) \geqq \lambda(d) > \lambda(c) - \varepsilon$$
$$J^*(c) \leqq J^*(v) = J_0(v) \leqq \lambda(e) < \lambda(c) + \varepsilon \quad \text{for every} \quad \varepsilon > 0.$$

Hence $J^*(c) = \lambda(c)$.

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## S Ú H R N

### O KONŠTRUKCII MIERY Z OBJEMU

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V práci sa študujú zobrazenia $J : S \rightarrow R$, kde $S$ je vhodný zväz. V prípade, že $S$ je zväz množín, dostávame klasickú konštrukciu miery z objemu. V prípade, že $S$ je zväz reálnych funkcií (na kompaktnej množine), dostávame analógiu tejto konštrukcie pre Daniellov integrál.

## Р Е З Ю М Е

### О КОНСТРУКЦИИ МЕРЫ ИЗ ОБъЕМА

БЕЛОСЛАВ РИЕЧАН - МИХАЛ ШАБО, БРАТИСЛАВА

В работе изучаются отображения $J : S \rightarrow R$, где $S$ подходящая структура. Если в качестве $S$ взять структуру множеств, то получается классическая конструкция меры из объема. В случае, когда $S$ - структура действительных функций (на компактном множестве), получается аналог этой конструкции для интеграла Даниеля.