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ON ALMOST CONTINUOUS FUNCTIONS

JAROSLAV SMÍTAL, EVA STANOVÁ, Bratislava

D. B. Smith [5] has shown that a real function $f : R \longrightarrow R$ is continuous iff (a) it is almost continuous in the sense of Husain, (b) it is almost continuous in the sense of Stallings, and (c) it is not of the Cesàro type. He also gives examples to show that conditions (a) and (b) are not redundant. R. J. Fleissner [1] has given an example to show that also the condition (c) is not redundant.

In the present note there are given some more general results. It is shown that a function which is almost continuous in the sense of Stallings and not of the Cesàro type, is cliquish (Theorem 1), but not necessarily quasicontinuous (Theorem 2). Next result states that a function which is cliquish and almost continuous in the sense of Husain, is continuous (Theorem 3). As a consequence we obtain the Smith's result quoted above. All results are stated in a more general case for functions from a topological space to the set R of reals equipped with the usual topology.

First we recall definitions of the generalized types of continuity which are involved in the sequel. Here X and Y denote topological spaces.

D e f i n i t i o n 1. A function $f : X \longrightarrow Y$ is almost continuous in the sense of Stallings iff any open set $H \subset X \times Y$ containing the graph of f contains the graph of a continuous function $g : X \longrightarrow Y$ (cf. [6]).

D e f i n i t i o n 2. A function $f : X \longrightarrow Y$ is almost continuous in the sense of Husain iff for each $x \in X$, if $V \subset Y$ is a neighbourhood of $f(x)$ then $f^{-1}(V)$ is dense in a certain neighbourhood of x (cf. [2]).

D e f i n i t i o n 3. A function $f : X \longrightarrow Y$ is of the Cesàro type iff there exist nonempty open sets, $U \subset X$ and $V \subset Y$, such that, for each $y \in V$, $f^{-1}(y)$ is dense in U (cf. [5]).

D e f i n i t i o n 4. A function $f : X \longrightarrow Y$ is quasi-continuous iff for each $x \in X$, if $U \subset X$ and $V \subset Y$ are neighbourhoods of x and $f(x)$, respectively, then there is a nonempty open set $G \subset U$ such that $f(G) \subset V$ (cf. [3] or [4]).

D e f i n i t i o n 5. Let (Y, ρ) be a metric space. A function $f : X \longrightarrow Y$ is cliquish iff for each $\varepsilon > 0$, every nonempty open set $G \subset X$ contains a nonempty open set H such that $\rho(f(x), f(y)) < \varepsilon$ whenever $x, y \in H$ (cf. [4]).

Clearly each function with values in a metric space, which is quasicontinuous, is cliquish (cf. [4]).

To prove Theorem 1 we need the following three lemmas.

L e m m a 1. Let X be a T_1 topological space, let $f : X \longrightarrow R$ be continuous in the sense of Stallings. Assume that there is a closed connected set $F \subset X$ and some $a \in R$ such that $a \notin f(F)$. Then f is upper- or lower-bounded on F by a.

P r o o f . Assume that there exist $b, c \in F$ such that $f(b) < a$ and $f(c) > a$. Let $P = F \times \{a\} \cup \{b\} \times [a, \infty) \cup \{c\} \times (-\infty, a]$. Then clearly P is a closed subset of $X \times R$ not intersecting the graph of f, hence $G = X \times R \setminus P$ is an open neighbourhood of the graph of f. Thus G contains the graph of a continuous function g. Since $g(F)$ is connected and $a \notin g(F)$, we have $g(F) \subset (-\infty, a)$ or $g(F) \subset (a, \infty)$, hence $\{c\} \times \{g(c)\} \notin G$ or $\{b\} \times \{g(b)\} \notin G$, respectively, and this is a contradiction.

Lemma 2. Let X be a locally connected T_3 Baire topological space. Let $f : X \longrightarrow R$ be almost continuous in the sense of Stallings and not of the Cesàro type. Then each nonempty open set G contains a nonempty open subset H such that f is bounded on H.

Proof. Let $G \neq \emptyset$ be an open set. Put $A_n = \overline{f^{-1}((-n, n))}$, for $n = 1, 2, \ldots$. Since $\bigcup_{n=1}^{\infty} A_n = X$ and G is of the second category, there is some k and an open subset $H_1 \neq \emptyset$ of G such that A_k is dense in H_1. Since f is not of the Cesàro type and X is T_3 and locally connected, there is some $p \in (k, \infty)$ and a connected closed subset H_2 of H_1 with nonempty interior such that $p \notin f(H_2)$. Since A_k is dense in H_2 we have, by Lemma 1, $f(t) < p$ for $t \in H_2$. Similarly there is some $q \in (-\infty, -k)$ and a connected closed subset H_3 of H_2 with nonempty interior such that $f(t) > q$ for $t \in H_3$. Now it suffices to put $H = \operatorname{int} H_3$, q.e.d.

Lemma 3. Let X be a T_3 locally connected topological space. Let $f : X \longrightarrow R$ be almost continuous in the sense of Stallings and not of the Cesàro type. Let G be a nonempty open set such that $|f(x) - f(y)| < K$ for $x, y \in G$. Then there is a nonempty open set $H \subset G$ such that $|f(x) - f(y)| < \frac{2}{3} K$, for $x, y \in H$.

Proof. Let G, f satisfy the assumptions. Then there is some $a \in R$ such that $f(x) \in [a - K/2, a + K/2]$ for each $x \in G$. Let $I = (a - K/6, a + K/6)$. Since f is not of the

Cesàro type there is some $p \in I$, and a nonempty open $G_1 \subset G$ such that $p \notin f(G_1)$. Let F be a closed connected subset of G_1 with nonempty interior. Then by Lemma 1, either

(1) $f(x) > p$, for each $x \in F$, or

(2) $f(x) < p$, for each $x \in F$.

In the case (1) we have $f(x) \in (a - K/6, a + K/2]$, and hence

$$|f(x) - f(y)| < \frac{2}{3} K,$$

for each $x, y \in F$. The same is true if (2) holds. To finish the proof it suffices to put $H = \text{int } F$.

<u>Theorem 1.</u> Let X be a T_3 locally connected Baire topological space. Let $f : X \to R$ be almost continuous in the sense of Stallings and not of the Cesàro type. Then f is cliquish.

<u>Proof.</u> Let $a \in X$, let G be an open set containing a, and let $\varepsilon > 0$. By Lemma 2, there is a nonempty open set $G_1 \subset G$ and some $K > 0$ such that $|f(x) - f(y)| < K$ for $x, y \in G_1$. Choose a positive integer n such that $(2/3)^n K < \varepsilon$. Now using n-times Lemma 3 we obtain a nonempty open subset H of G_1 such that $|f(x) - f(y)| < \varepsilon$ for $x, y \in H$, q.e.d.

The following theorem shows that in the preceeding theorem the cliquishness cannot be replaced by the more strong quasicontinuity.

<u>Theorem 2.</u> There is a function $h : R \to R$ almost continuous in the sense of Stallings and not of the Cesàro type, which is not quasicontinous.

First we introduce notation which will be used throughout the proof. Let K_n $(n = 1, 2, \ldots)$ be the intervals contiguous to the Cantor set in the interval $[0,1]$, and put

$M = \bigcup_{n=1}^{\infty} \overline{K}_n \cup \{0, 1\}$, where $\overline{A}$ denotes as usually, the closure of A. Let D be the set of all numbers of the form $a + n$ where $a \in M$ and n is an integer. Moreover, Bd A denotes the boundary of a set A, and if A is a subset of the plane $R \times R$ then A_X denotes its projection to the x-axis.

We begin the proof of our theorem with the following

L e m m a 4. Let $G \subset R \times R$ be an open set such that $D \times \{0\} \subset G$. Assume that G contains the graph of a function $f : R \to R$, and that the cardinality of the set $(\mathrm{Bd}\ G)_X \cap (R \setminus D)$ is less then the cardinality of the continuum. Then G contains the graph of a continuous function $g : R \to R$.

Proof of the lemma. If $R \times \{0\} \subset G$ the Lemma is true. Hence we may assume that there is some point $a \in R \setminus D$ such that $\{a\} \times \{0\} \notin G$. Since G is open and $f(a) \in G$ there are bounded open intervals I_a and J_a in R such that $I_a \times \{f(a)\} \subset G$, and both $0 \in J_a$ and $f(a) \in J_a$. Put $F = (\mathrm{Bd}\ G) \cap (I_a \times J_a)$. Then $\overline{F} \subset \mathrm{Bd}\ G$ and $\overline{F}$ is a compact. Hence $\overline{F}_X$ is a compact and therefore $I_a \setminus \overline{F}_X$ is open. Denote by a_1, a_2 the left - and right-hand end point of I_a, respectively. Then $(a_1, a) \setminus \overline{F}_X \neq \emptyset$ and $(a, a_2) \setminus \overline{F}_X \neq \emptyset$. Indeed, if e.g. $(a_1, a) \subset \overline{F}_X$, then

$$(\mathrm{Bd}\ G)_X \cap (R \setminus D) \supset \overline{F}_X \cap (R \setminus D) \supset (a_1, a) \cap (R \setminus D)$$

has the power of the continuum by the well-known property of the Cantor set (each point of $R \setminus D$ is a bilateral point of condensation of $R \setminus D$). Let $L_1(a)$, $L_2(a)$ be open intervals such that $L_2(a) \subset (a_1, a) \setminus \overline{F}_X$, and $L_2(a) \subset (a, a_2) \setminus \overline{F}_X$. We may assume that $L_1(a), L_2(a) \subset D$. The rectangular $L_1(a) \times J_a$ does not contain any point of Bd G, hence there is a surjective nondecreasing continuous function $\varphi_a: L_1(a) \to [0, f(a)]$, whose graph lies in G. Similarly we can find a surjective non-increasing function $\psi_a: L_2(a) \to [0, f(a)]$, whose graph lies in G. Now let S_a be the least open interval containing $L_1(a)$ and $L_2(a)$, and let g_a be a function $S_a \to [0, f(a)]$ such that $g_a(x) = \varphi_a(x)$ for $x \in L_1(a)$, $g_a(x) = \psi_a(x)$ for $x \in L_2(a)$, and $g_a(x) = f(a)$ otherwise. Clearly g_a is continuous.

If $a \in R$ is such a point that $\{a\} \times \{0\} \in G$ let S_a be an open interval containing a such that $S_a \times \{0\} \subset G$, and let $g_a(x) = 0$ for $x \in S_a$.

Now the intervals $\{S_a, a \in R\}$ form an open cover of R hence we may choose a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers such that the corresponding intervals S_{a_n} cover R and such, that no three such intervals overlap. Let $g: R \to R$ be defined as follows: If there is only one S_{a_n} containing x, let $g(x) = g_{a_n}(x)$, and if $x \in S_{a_n} \cap S_{a_m}$ let $g(x) = \max\{g_{a_m}(x), g_{a_n}(x)\}$. It is easy to see that g is continuous and that its graph is contained in G.

Now we may proceed with

<u>Proof of the theorem.</u> For any two points $u, v \in R \setminus D$, $u < v$, let $H(u, v) = R \times R \setminus (\{u\} \times (-\infty, 1] \cup [u, v] \times \{1\} \cup \{v\} \times [1, \infty))$. Every such set is called spetial set (cf. [1]). Let Ω denote the first ordinal of the power of the continuum, and let $\{G_\alpha\}_{\alpha < \Omega}$ be a transfinite sequence of the open sets in the plane which (a) contain $D \times \{0\}$, (b) contain the graph of a function $R \to R$, and (c) do not contain the graph of any continuous function $R \to R$. It is easy to see that each spetial set occur in the sequence.

Using Lemma 4 we may define by the transfinite induction sequences $\{a_\alpha\}_{\alpha<\Omega}$ and $\{b_\alpha\}_{\alpha<\Omega}$ such that, for each α, $a_\alpha \in R \setminus D$, $\{a_\alpha\} \times \{b_\alpha\} \in \mathrm{Bd}\, G_\alpha$, and $a_\alpha \neq a_\beta$ for $\alpha \neq \beta$. Moreover, since the spetial sets occur in the sequence, we may assume that for some α, $b_\alpha = 1$.

Now let $h(x) = b_\alpha$ if $x = a_\alpha$ for some α, and let $h(x) = 0$ otherwise. Clearly h is almost continuous in the sense of Stallings since if an open set G contains the graph of h then $G \neq G_\alpha$, for each α, hence G contains the graph of a continuous function. Since h is zero in the whole R except of a nowhere dense set, h is not of the Cesàro type. And finally, it is easy to see that h is not quasicontinuous at any point, a, for which $h(a) \neq 0$ and this finishes the proof of our theorem.

The following theorem is stated for functions with more general range. May be it is known but we are not able to give any references.

<u>Theorem 3.</u> Let $f : X \to Y$, where X is a topological space, and Y a metric space with a metric ρ. Then f is continuous iff f is both cliquish and almost continuous in the sense of Husain.

From Theorems 1 and 3 we obtain immediately

<u>Corollary.</u> Let X be a T_3 locally connected Baire topological space. A function $f : X \to R$ is continuous iff it is almost continuous in the sense of Stallings, almost continuous in the sense of Husain, and not of the Cesàro type.

<u>Proof of Theorem 3.</u> Let f be almost continuous in the sense of Husain and not continuous at a point a. Choose $\varepsilon > 0$ such that

$$\limsup_{x \to a} \rho(f(x), f(a)) > 3\varepsilon .$$

Let V_a be the ε-ball centred at $f(a)$; then the set $f^{-1}(V_a)$ is dense in a certain neighbourhood G of a. Let $b \in G$ such that

$$\rho(f(a), f(b)) > 3\varepsilon .$$

If V_b is the ε-ball centred at $f(b)$, then the set $f^{-1}(V_b)$ is dense in a certain neighbourhood $H \subset G$ of b. But in this case $osc_f(x) > \varepsilon$, for each $x \in H$, and hence f is not cliquish, q.e.d.

Author's address: Jaroslav Smítal a Eva Stanová
Katedra teórie pravdepodobnosti a matematickej štatistiky PFUK,
Matematický pavilón, Mlynská dolina
816 31 Bratislava

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R E F E R E N C E S

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S Ú H R N

O SKORO SPOJITÝCH FUNKCIÁCH

JAROSLAV SMÍTAL A EVA STANOVÁ, BRATISLAVA

V práci sa skúmajú vzťahy medzi týmito typmi zovšeobecnenej spojitosti reálnych funkcií definovaných na topologických priestoroch: Spojitosť, skoro spojitosť v zmysle Stallingsa, skoro spojitosť v zmysle Husaina, kvázispojitosť a "cliquish" - spojitosť.

Р Е З Ю М Е

О ПОЧТИ НЕПРЕРЫВНЫХ ФУНКЦИЯХ

ЯРОСЛАВ СМИТАЛ И ЕВА СТАНОВА, БРАТИСЛАВА

В работе исследуются некоторые связи между следующими типами обобщенной непрерывности вещественных функций на топологических пространствах: Непрерывность, почти-непрерывность в смысле Сталлингса, почти-непрерывность в смысле Хусайна, квазинепрерывность и "кликвиш"-непрерывность.