UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XXXVII - 1980

DIAMETER STRONGLY CRITICAL GRAPHS

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Under a graph we mean throughout the paper an undirected finite graph without loops and multiple lines. This paper presents the continuation of the series of papers [2] - [7] about e-critical graphs. A graph G is said to be e-critical if the deleting of an arbitrary line from G increases its diameter. Many results of these papers were formulated for the graphs of the diameter $d \geq 2$ with a girth at least $d + 2$ called δ_d-graphs. The class of diameter strongly critical graphs examined in this paper is a subclass of the class of e-critical graphs.

Our terminology as well as denotation is based on [1] except for the given here. A graph $G = (V, E)$ has the point set $V = V(G)$ and the set of lines $E = E(G)$. By $d_G(x, y)$ we denote the distance between two points $x, y \in V(G)$. If $v \in V(G)$, then by $N_G(v)$ we denote the neigborhood of v (i.e. $N_G(v) = \{u/d_G(u, v) = 1\}$). The lenght of a walk P we denote by $\lambda(P)$. Let k be a natural number. Let for $i = 1, 2, \ldots, k$ P_i be a $v_{n_{i-1}} - v_{n_i}$ walk in G with the points $v_{n_{i-1}}, v_{n_{i-1}+1}, \ldots, v_{n_i}$. The concatenation of P_i (in a given ordering) we define as the $v_{n_0} - v_{n_k}$ walk with the points $v_{n_0}, v_{n_0+1}, \ldots, v_{n_1}, \ldots, v_{n_{k-1}+1}, \ldots, v_{n_k}$. Let P be a $v_0 - v_n$ walk with points $v_0, v_1, \ldots, v_n$. By P^{-1} we denote the $v_n - v_0$ walk with the points $v_n, v_{n-1}, \ldots, v_0$.

Further, for any integers k, m with $1 \leqq k \leqq m \leqq n$ we denote by $P(v_k - v_m)$ the $v_k - v_m$ subwalk of P with the points $v_k, v_{k+1}, \ldots, v_m$.

<u>D e f i n i t i o n 1.</u> A graph $G = (V, E)$ is said to be diameter strongly critical (in the next strongly critical only) with the diameter d, if the following holds:

1/ $d(G) = d < +\infty$

2/ for every line $xy \in E(G)$ and every two points $t, s \in V(G)$ the inequality $d_{G-xy}(t,s) > d$ holds if and only if $ts \equiv xy$.

In Fig. 1 we have shown two examples of strongly critical graphs with diameter two.

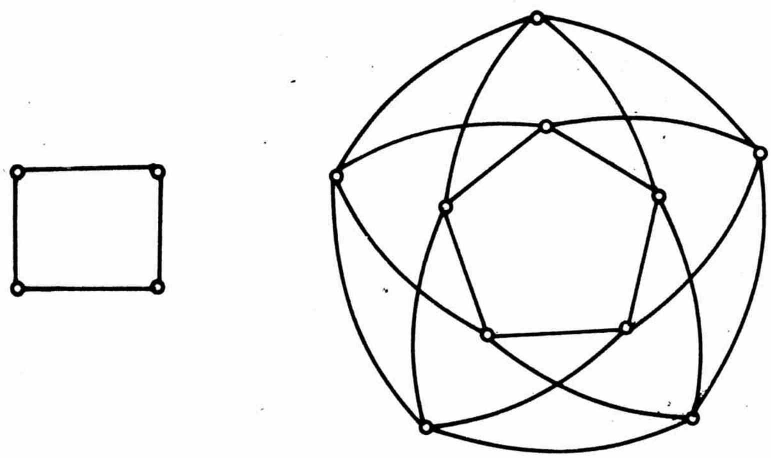

Fig. 1

The next three assertions follow immediately from the definition 1.

A s s e r t i o n 1. The complete graph K_n, $n \geq 2$, is strongly critical with the diameter one.

A s s e r t i o n 2. Every strongly critical graph with a diameter $d \geq 2$ is a δ_d-graph.

A s s e r t i o n 3. The only strongly critical graph with an endpoint is the complete graph K_2.

In the next theorem we introduce an operation for extending of strongly critical graphs with the diameter two by one point.

T h e o r e m 1. Let $G = (V, E)$ be strongly critical graph with the diameter two. Let $x \in V(G)$, $y \notin V(G)$. Then the graph $G_1 = (V_1, E_1)$, where $V_1 = V \cup \{y\}$ and $E_1 = E \cup \{yt / t \in N_G(x)\}$ is strongly critical with diameter two.

P r o o f . Let us denote $N_G(x) = \{t_1, t_2, \ldots, t_k\}$ (fig. 2).

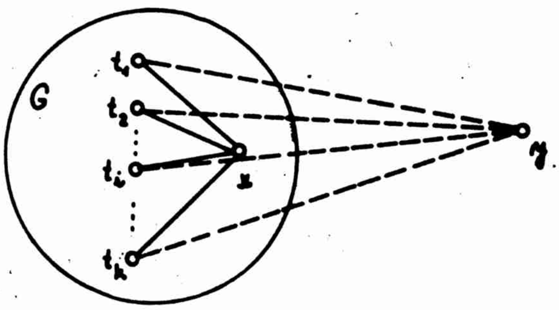

Fig. 2

By the assertion 3 we have $k \geq 2$. Obviously $d(G) = d(G_1)$ and G_1 is a δ_2-graph. Thus for every line $uv \in E_1$ we have $d_{G_1 - uv}(u,v) > 2$. For every two points $m,n \in V_1$ with $d_{G_1}(m,n) = 2$

we shall prove that $d_{G_1-uv}(m,n) \leqq 2$. Clearly $d_{G_1-uv}(m,n) \leqq 2$ for every two points $m, n \in V$, because G is strongly critical with the diameter two. Let $m \in V$, $n = y$, $m \notin N_G(x)$, $m \neq x$. As $d_G(m,x) = 2$, there exists i such that $1 \leqq i \leqq k$ and $mt_i \in E$. From the properties of G we have $d_{G-mt_i}(m,x) = 2$. Hence there exists j with $i \neq j$, $1 \leqq i, j \leqq k$ and $mt_j \in E$. Therefore $d_{G_1-uv}(m,y) = 2$. If $m = x$, we have $d_{G_1-uv}(x,y) = 2$, because $k \geqq 2$. Hence G_1 is strongly critical with the diameter two.

Further we bring out some necessary and sufficient conditions for a graph to be strongly critical.

<u>Theorem 2.</u> Let $G = (V, E)$ be a σ_d-graph ($d \geqq 2$). Let for every two points $x, y \in V$ with $d_G(x,y) > 1$ there be two line-disjoint x-y paths P_1, P_2 in G such that $\lambda(P_i) \leqq d$ for $i = 1, 2$. Then G is strongly critical graph with the diameter d.

<u>Proof.</u> Let $uv \in E$. Then $d_{G-uv}(u,v) > d$, because G is a σ_d-graph. Let $t,q \in V$, $t \neq q$, $\{t,q\} \neq \{u,v\}$. If $d_G(t,q) = 1$, then also $d_{G-uv}(t,q) = 1$. Let $d_G(t,q) > 1$. By the assumption, there are two t-q paths P_1, P_2 of the length $\lambda(P_i) \leqq d$ for $i = 1, 2$. Therefore $d_{G-uv}(t,q) \leqq d$ and G is strongly critical graph. Hence the theorem holds.

<u>Corollary 1.</u> Let $G = (V, E)$ be a σ_d-graph. Let for every two points $x, y \in V$ with $d_G(x,y) > 1$ there be a cycle C containing x and y such that $\lambda(C) \leqq d + d_G(x,y)$. Then G is a strongly critical graph.

<u>Proof.</u> There are two line disjoint paths $C(x-y)$ and $C(y-x)$ in G both of the length less than or equal to d. The proof immediately follows from the theorem 2.

T h e o r e m 3. Let G be a strongly critical graph with a diameter $d \geqq 2$. Then for every two points $x, y \in V$ with $d_G(x,y) > 1$ there are at least two point-disjoint $x - y$ paths P_1, P_2 of the length not exceeding d.

P r o o f . Let $x, y \in V$ with $d_G(x,y) > 1$. Put $h = d_G(x,y)$. There is an x-y path X of the length h. We denote the points of this path by $x = x_0, x_1, \ldots, x_{h-1}, x_h = y$ (Fig. 3). Further by $\mathcal{P}$ is denoted the set of all x-y paths with length not exceeding d and not containing the line xx_1. Analogously by $\mathcal{R}$ we denote the set of all y-x paths of the length not exceeding d and not containing the line $x_{h-1}y$. As G is a $\mathcal{S}_d$-graph we have $d_{G-xx_1}(x,y) \leqq d$ and also $d_{G-x_{h-1}y}(x,y) \leqq d$. Hence $\mathcal{P} \neq \emptyset$, $\mathcal{R} \neq \emptyset$. Let P be a shortest path from $\mathcal{P}$ and Q be a shortest path from $\mathcal{R}$. Let us denote the points of P by $p_0 = x, p_1, p_2, \ldots, p_n = y$; the points of Q by $q_0 = y, q_1, q_2, \ldots, q_m = x$. Certainly $n, m \leqq d$. Let s be the minimal index such that $s > 0$ and $p_s \in X$. To prove the theorem in the case $s = n$ it is sufficient to set $P_1 = X$, $P_2 = P$. Analogously let r be the minimal index such that $q_r \in X$ and $r > 0$. For the proof of the theorem in the case $r = m$ it is enough to set $P_1 = Q^{-1}$ $P_2 = X$. Let $0 < s < n$ and $0 < r < m$. There must exist the natural numbers i, j $1 \leqq i, j \leqq h$ such that $p_s = x_i$ and $q_r = x_j$. We will distinguish two cases:

A/ $q_c \notin P(x-p_s)$, for every $c < r$, (Fig. 3).

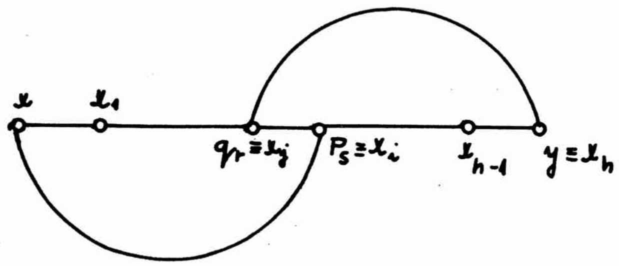

Fig. 3

By the concatenation of $Q(y-q_r)$ and $X(x_j-y)$ we get a cycle. Since G is a δ_d-graph, we have:

$$r + h - j \geq d + 2 \qquad (1)$$

Analogously by the concatenation of $P(x-p_s)$ and $X^{-1}(x_i-x_0)$ we get a cycle. Hence

$$s + i \geq d + 2 \qquad (2)$$

If $n - s < h - i$ then the concatenation $X(x-p_s)$ and $P(p_s-y)$ is an x-y walk of the length $i + n - s < h$, which is a contradiction. Thus

$$n - s \geq h - i \qquad (3)$$

If $m - r < j$, then the concatenation of $Q^{-1}(q_m-q_r)$ and $X(q_r-y)$ is an x-y walk of the length $m - r + h - j < h$, which is a contradiction. Hence

$$j + r \leq m \leq d \qquad (4)$$

Summing up (1) and (2), (3) and (4) we get two inequalities which immediately give $i \geq j + 2$. We have $\lambda(P_1) = s + h - i \leq n \leq d$ by (3) and $\lambda(P_2) = j + r \leq d$ by (4). To prove the theorem in this case, it is sufficient to put the concatenation of $P(x-p_s)$ and $X(p_s-y)$ for P_1, the concatenation of $X(x-x_j)$ and $Q^{-1}(q_r-y)$

for P_2.

B/ There is c, $c < r$, such that $q_c \in P(x-p_s)$, (Fig. 4).

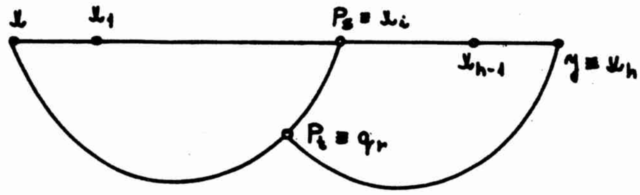

Fig. 4

Let k be the minimal index for which $q_k \in P(x-p_s)$. There is t such that $0 < t < s$ and $p_t = q_k$.

First we show that $Q(x-q_k)$ does not contain the line xx_1. Let the opposite be true. Then $xx_1 = q_m q_{m-1}$, and $xx_1 \notin Q(q_k-q_{m-1})$. From the properties of P follows $xx_1 \notin P(x-p_s)$. The concatenation of $Q(q_k-x_1)$, xx_1 and $P(x-q_k)$ is a closed walk of the length $m - k + t$. Then there exists a cycle F containing xx_1 such that $\lambda(F) \leqq m - k + t$. Hence

$$m - k + t \geqq d + 2 \qquad (5)$$

because G is a δ_d-graph. The concatenation of $Q(y-q_k)$, $P(q_k-p_s)$ and $X(p_s-y)$ is a cycle, and we have also

$$k + s - t + h - i \geqq d + 2 \qquad (6)$$

The inequality (3) holds in this case too. A comparison of (3) and the sum of (5) and (6) gives $m \geqq d + 4$, a contradiction. Thus, $xx_1 \notin Q$.

If $m - k < t$, then the concatenation of $Q^{-1}(q_m-q_k)$ and

$P(q_k - y)$ is an x-y walk U of the length $\lambda(U) = m - k + n - t < n$. Evidently $xx_1 \notin U$. Then there is an x-y path Z in G such that $\lambda(Z) < n$ and $xx_1 \notin Z$. This path belongs to $\mathcal{P}$, a contradiction. Hence $m \geqq k + t$. The concatenation of $P(x-p_t)$ and $Q^{-1}(q_k - q_o)$ is an x-y path W. The paths X and W are point-disjoint. The length of W is $\lambda(W) = k + t \leqq m \leqq d$. For the proof in this case it is sufficient to put $P_1 = W$ and $P_2 = X$.

This completes the proof.

<u>C o r o l l a r y 2.</u> Every strongly critical graph has no cutpoint.

<u>C o r o l l a r y 3.</u> A graph $G = (V, E)$ with a diameter d is strongly critical if and only if G is a δ_d-graph and for every two points $x,y \in V$ with $d_G(x,y) > 1$ there are at least two line-disjoint x-y paths P_1, P_2 in G such that $\lambda(P_i) \leqq d$ for $i = 1, 2$.

<u>P r o o f .</u> The proof immediately follows from the Theorems 2 and 3.

<u>T h e o r e m 4.</u> Let $G = (V, E)$ be a δ_d-graph with $2 \leqq d \leqq 4$. Then G is strongly critical if and only if for every two points $x,y \in V$ with $d_G(x,y) > 1$ there is a cycle C such that $x,y \in C$ and $\lambda(C) \leqq d + d_G(x,y)$.

<u>P r o o f .</u> The sufficient condition follows immediately from the Corollary 1.

Let $d_G(x,y) = h$, $2 \leqq h \leqq 3$. There is an x-y path X of the length h in G. Because G is strongly critical, there exists a y-x path Q, with $\lambda(Q) \leqq d$, which does not contain $x_{h-1}y$. As G cannot contain a cycle of the length less than d + 2, the concatenation of Q and X form the required cycle C.

The existence of C in the case $h = 4$ follows from the Theorem 3.

T h e o r e m 5. Let $G = (V, E)$ be a δ_2-graph. Then there exists a strongly critical graph $F = (U, W)$ with the diameter two, containing G as an induced subgraph.

P r o o f . Let $|V| = n$, $V = \{x_1, x_2, \ldots, x_n\}$. Let $S = \{y_1, y_2, \ldots, y_n\}$ be a set, for which $S \cap V = \emptyset$. We define the graph F as follows:

$U = V \cup S$, $W = E \cup E_1 \cup E_2$, where

$$E_1 = \bigcup_{i=1}^{n} \{y_i x_j \mid x_i x_j \in E \text{ for } j = 1, 2, \ldots, n\}.$$

$$E_2 = \{y_i y_j \mid x_i x_j \in E \text{ for } i,j = 1, 2, \ldots, n\}.$$

By this definition, G is an induced subgraph of F, and G does not contain any cycle of the length 3. Let $p,q \in U$, $pq \notin W$. We shall show, that in F there exists a cycle of the length 4, which contains p and q. It is sufficient to consider the following cases:

1/ $p \in V$, $q \in V$, $p = x_i$, $q = x_j$, $i \neq j$, $1 \leq i,j \leq n$. Then $d_G(x_i,x_j) = d_F(x_i,x_j) = 2$ and there is $x_k \in V$, $k \neq i, j$ such that $x_i x_k \in E$ and $x_j x_k \in E$. Then also $y_k x_i \in E_1$ $y_k x_j \in E_1$, and $x_i y_k x_j x_k x_i$ is the required cycle.

2/ $p \in U - V$, $q \in V$, $p = y_i$, $q = x_j$, $1 \leq i,j \leq n$. If $i = j$, then there exists $x_k \in V$, $x_k \neq x_i$ such that $x_k x_i \in E$. Then also $y_i x_k \in E_1$. Simultaneously $y_i y_k \in E_2$ and $y_k x_i \in E_1$ by definition of F. So, $x_k x_i y_k y_i x_k$ is the required cycle.

If $i \neq j$, then $d_G(x_i,x_j) = 2$ because in the opposite case we have $y_i x_j \in E_1$ which is a contradiction. Hence, there is a point $x_k \in V$, $x_k \neq x_i,x_j$ such that $x_j x_k \in E$ and $x_i x_k \in E$. Then also $y_i y_k \in E_1$. As $x_i x_k \in E$, there must be $y_i y_k \in E_2$.

Simultaneously, $y_k x_j \in E_1$. Thus $x_j x_k y_i y_k x_j$ is the required cycle.

The proof follows from the Corollary 1.

Theorem 6. Let $G = (V, E)$ be a graph without triangles. Then there exists a strongly critical graph with the diameter two, containing G as an induced subgraph.

Proof. By the Theorem 1 from the paper [7] there is a δ_2-graph $G_1 = (V_1, E_1)$, containing G as an induced subgraph. The proof then follows from the Theorem 5.

Further we are going to present two constructions of strongly critical graphs with the diameter four.

Construction 1. (Fig. 5).

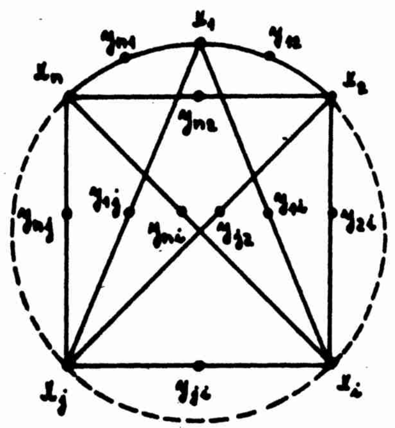

Fig. 5

Let $K_n = (U_n, H_n)$ be the complete graph, with $|U_n| \geq 4$. Let $U_n = \{x_1, x_2, \ldots, x_n\}$. To construct a strongly critical graph G with the diameter four, we insert into every line x_ix_j from H_n $(i \neq j)$ exactly one new point $y_{ij}(y_{ij} = y_{ji}, y_{ij} \notin U_n)$. So, we get a graph G with the diameter four. As every two line of H_n must be in some 3 or 4-angle in K_n, the assumptions of the Theorem 4 for G are satisfied. So, G is a strongly critical graph.

<u>Construction 2.</u> (Fig. 6).

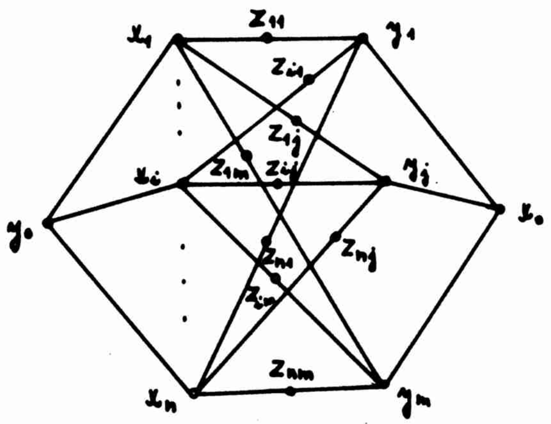

Fig. 6

Let $K_{n+1,\ m+1} = (V, E)$ be a complete bigraph. There are the sets $V_1 = \{x_0, x_1, \ldots, x_n\}$, $V_2 = \{y_0, y_1, \ldots, y_m\}$ such that $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and $E = \{x_iy_j \ / \ i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, m\}$. To construct the graph G in Fig. 6 we insert into every line x_iy_j, with $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$ of the graph $K_{n+1,m+1} - x_oy_o$ exactly one new point $z_{ij}/z_{ij} = z_{ji}$, $z_{ij} \notin V/$. It is easy to verify, that the assumptions of the Theorem 4 are satisfied and $d(G) = 4$. So, G is strongly critical with the diameter four.

We have presented some constructions of strongly critical graphs with the diameter two and four. In the isolated cases we know examples of strongly critical graphs with another diameters. For example in Fig. 7 there are two strongly critical graphs, first with the diameter three and the second with the diameter six.

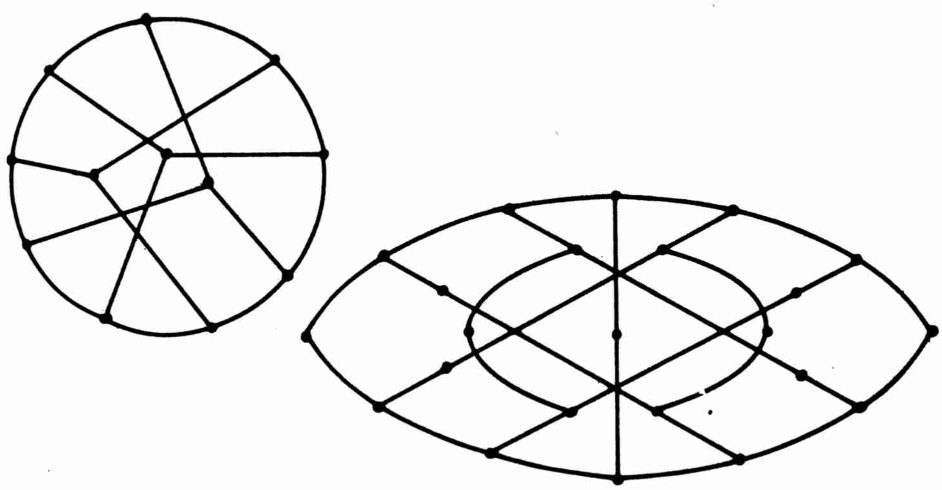

Fig. 7

We conjecture that for every natural number $d \geqq 2$ there is a strongly critical graph with the diameter d.

R E F E R E N C E S

[1] Harary,F.: Graph theory. 1. ed. Reading Massachusetss, 1969.

[2] Glivjak, F., Kyš,P., Plesník, J.: On the extension of graphs with a given diameter without superfluous edges. Mat. Časop. 19, 1969, 92 - 101.

[3] Glivjak, F., Kyš, P., Plesník, J.: On irreducible graphs of diameter two without triangles. Mat. časop. 19, 1969, 149-157.

[4] Glivjak, F., Plesník, J.: On the existence of certain graphs with diameter two. Mat. časop. 19, 1969, 276-282.

[5] Plesník, J.: Critical graphs of given diameter., Acta Fac. R.N. Univ. Comen. Math., 30, 1975, 71-93.

[6] Glivjak, F., Plesník, J.: On the impossibility to construct certain classes of graphs by extensions. Acta Math. Acad. Sc. Hungar. 22, 1971, 5-10.

[7] Glivjak, F.: On certain edge critical graphs of a given diameter Mat. časop. 25, 1975, 249-263.

[8] Chomenko, N.P., Ostroverchij, N.A.: Diametraľno-kritičeskije grafy. Ukrain. Matem. Žurnal 22, 1970, 637-646.

[9] Ore, O.: Diameters in graphs. J. Comb. Theory, 5, 1968 75-81.

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Received: 28.4.1977

SÚHRN

SILNE KRITICKÉ GRAFY S DANÝM PRIEMEROM

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Autor v práci skúma grafy, ktorých priemer sa zväčší, keď odstránime ľubovoľnú hranu, pričom jedinou dvojicou vrcholov, ktorých vzdialenosť bude väčšia ako priemer pôvodného grafu, je dvojica vrcholov incidentných s vynechanou hranou. Sú dané niektoré vlastnosti tejto triedy grafov, nutné a postačujúce podmienky na to, aby graf patril do tejto triedy, ako aj niektoré konštrukcie takýchto grafov.

Р Е З Ю М Е

СИЛЬНО КРИТИЧЕСКИЕ ГРАФЫ ДАННОГО ДИАМЕТРА

ПЕТЕР КЫШ, БРАТИСЛАВА

В работе исследуются графы, диаметр которых увеличиваетсяа удалив любое ребро, причем единственной парой вершин, которых расстояние станет большим диаметра основного графа, будет пара вершин инцидентных с удаленным ребром. Даны некоторые свойства у конструкции этицх графов, как и необходимие и достаточные условия для того, чтобы граф принадлежал к этомы классу графов.