ACTA FACULTATIS
RERUM NATURALIUM
UNIVERSITATIS COMENIANAE

MATHEMATICA SPECIAL NUMBER

1975
SLOVENSKÉ PEDAGOGICKÉ NAKLADATEĽSTVO BRATISLAVA

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE

MATHEMATICA

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PREFACE

This volume contains abstracts of the lectures held at the Summer Session on the Theory of Ordered Sets and General Algebra, at Horní Lipová in the days from the 3rd till 11th of September, 1972.

These Summer Sessions have regularly been organized since 1962 jointly by the Departments of Mathematics of the Faculty of Science of the J. E. Purkyně University in Brno, the Comenius University in Bratislava and the Technical College in Košice.

(ACTA F. R. N. UNIV. COMEN.-MATHEMATICA, SPECIAL NUMBER, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA – SPECIAL NUMBER – 1975

ON A REPRESENTATION OF LATTICES BY CONGRUENCE RELATIONS

H. DRAŠKOVIČOVÁ

The proofs of the results of this abstract can be found in [3].

In this note under a lattice $\mathcal{L}$ it is always meant a lattice with the least element 0 and the greatest element 1. Given a set A, E(A) denotes the lattice of all equivalence relations on A and Δ its least element.

Definition 1 [4]. A lattice $\mathcal{L}$ $(0,1 \in \mathcal{L})$ is said to be strongly representable as a congruence lattice if whenever $\mathcal{L}$ is isomorphic to a sublattice $\mathcal{L}'$ of E(A) for some A, where Δ, $A \times A \in \mathcal{L}'$, then there is an algebra based on A whose congruence lattice is $\mathcal{L}'$.

In [4] it is shown that every finite ditributive lattice is strongly representable.

The above notion of strong representability seems to be designed for finite lattices. The class of infinite strongly representable lattices is relatively small. E.g. the infinite chain $a_0 < a_1 < < a_2 < \dots < u$ is not strongly representable because it suffices to find a chain of equivalence relations $\Delta < \alpha_1 < \alpha_2 < \dots < A \times A$ on a set A, such that $\bigvee_{i=1}^{\infty} \alpha_i \neq A \times A$. There is no algebra based on A having this chain as a congruence lattice. Moreover the stronger assertion holds:

Theorem 1. No infinite distributive lattice $\mathcal{L}$ is strongly representable.

The following definition seems to be useful.

Definition 2. A complete algebraic lattice [2] $\mathcal{L}$ is said to be quasi strongly representable as a congruence lattice if whenever $\mathcal{L}$ is isomorphic to a closed sublattice [2] $\mathcal{L}'$ of E(A) for some A, where Δ and $A \times A$ belong to $\mathcal{L}'$, then there is an algebra based on A whose congruence lattice is $\mathcal{L}'$.

Lemma 1. Every strongly representable lattice is quasi strongly representable.

Remark 1. The class of quasi strongly representable lattices is larger than that of strongly representable lattices.

Using results of the paper [1] we get the following theorem:

Theorem 2. Let $\mathcal{L}$ be a completely distributive [2] closed sublattice of $E(A)$ containing Δ and $A \times A$. Then there is an algebra $\mathcal{A}$ based on A whose congruence lattice $C(\mathcal{A}) = \mathcal{L}$. The algebra $\mathcal{A}$ can be choosen in such a way that all its operations are unary and two-valued [1].

Corollary 1. Every complete algebraic and completely distributive lattice is quasi strongly representable.

Remark 2. If $\mathcal{L}$ is a closed sublattice of $E(A)$ containing Δ and $A \times A$ which is distributive but not completely distributive then it is possible that there is no algebra $\mathcal{A}$ based on A having only unary and two-valued operations such that the congruence lattice $C(\mathcal{A}) = \mathcal{L}$. Such an example is the congruence lattice $C(\mathcal{A})$ on the ring $\mathcal{A}$ of all integers.

Remark 3. It can be easily shown by an example that the lattice of all congruence relations of an algebra with only unary two-valued operations need not be distributive.

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(ACTA F. R. N. UNIV. COMEN.-MATHEMATICA, SPECIAL NUMBER, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA - SPECIAL NUMBER - 1975

ON A CONSTRUCTION OF COVERING GROUP FOR AN N-GROUP

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In 1929 Dörnte introduced the notion of an n-group [1], which is a natural generalization of the notion of a group. They were investigated by Post [7], who called them polyadic groups.

By an n-group we mean a set A with an n-ary operation defined in G and which satisfies the following conditions:

1^o for all $x_1, \ldots, x_{2n-1} \in A$,

$$f(f(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}) = f(x_1, f(x_2, \ldots, x_{n+1}), \ldots, x_{2n-1}) = \ldots = f(x_1, \ldots, x_{n-1}, f(x_n, \ldots, x_{2n-1}));$$

2^o for all $x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in A$ there exists precisely one element $x_i \in A$ $(i=1, \ldots, n)$ such that

$$f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = x_0.$$

We shall call condition 1^o, which is a natural generalization of the associativity in a group, the associative law in an n-group. Condition 2^o is a generalization of the solvability of equations $ax = b$ and $ya = b$ in the group, and it defines n operations g_i $(i=1,2,\ldots,n)$ inverse relative to f in A. It follows from the definition of n-group that it may be conceived as a universal algebra $\mathfrak{A} = (A; f, g_1, \ldots, g_n)$ with n+1 fundamental n-ary operations.

In particular, a 2-group is just a group. A special case of n-groups form also Prüfer's Schar [8].

An element $\bar{x}$ which satisfies the equation

$$f(x, x, \ldots, x, \bar{x}) = x$$

is called skew to x. One can prove that

$$f(x, \ldots, x, \bar{x}) = f(x, \ldots, \bar{x}, x) = \ldots = f(\bar{x}, x, \ldots, x),$$

and that

$$f(y,x,\ldots,\bar{x},\ldots,x) = f(x,\ldots,\bar{x},\ldots,x,y) = y$$

for all x, $y \in A$, where $\bar{x}$ can appear at any place under the sign of the function f.

In 1966 I proved together with B. Gleichgewicht that the following theorem is true

Theorem 1. A set with an n-ary operation $(n > 2)$ satisfying the associative law is an n-group if and only if for every $x \in A$ there exists an element $\bar{x} \in A$ such that for any $y \in A$ the following conditions are satisfied:

$$f(\bar{x},x,\ldots,x,y) = f(y,x,\ldots,x,\bar{x}) = y$$
$$f(x,\bar{x},\ldots,x,y) = f(y,x,\ldots,\bar{x},x) = y.$$

Theorem 1 gives a possibility to define, for $n > 2$, an n-group with the aid of another set of axioms. Thus the n-group $(n > 2)$ can be also conceived as universal algebra $\mathfrak{A} = (A;\ f,\ ^{-})$ with two fundamental operations: an n-ary operation f and an unary operation $^{-}$.

Corollary. The class of n-groups is a primitive class of algebras.

The group $(G;\ o)$ is a covering group for an n-group $(A;\ f,\ ^{-})$, if there exists an injection $\tau : A \longrightarrow G$ such that

$$\tau(x_1) \circ \tau(x_2) \ldots \circ \tau(x_n) = \tau(f(x_1,\ldots,x_n))$$

and $\tau(A)$ is a set of generators of G.

In particular, an n-group is reducible to group if the mapping τ is "onto". The covering group G is a free covering group if for every covering group G^* and for any injection $\tau_* : A \longrightarrow G$ satisfying the same conditions as the mapping τ, there exists epimorphism φ such that $\varphi \tau_* = \tau$.

Let the n-group $(A;\ f,\ ^{-})$ have a covering group $(G;\ o)$, and let $G_o = (\tau(A))^{n-1} = \{\tau(x_1) \circ \tau(x_2) \circ \ldots \circ \tau(x_{n-1}) : x_i \in A, i = 1,\ldots,n-1\}$.

Post [7] (for finite n-groups, Miller [5]) has proved that then G_o is a normal subgroup of G and $\tau(A)$ is a coset of G with respect to G_o. Moreover G/G_o is a finite cyclic group, whose the rank divides n-1.

Theorem 2. (Post [7], Miller [5]) Every n-group has a covering group.

A similary theorem: "every n-semigroup is a subreduct of a semigroup" is also true (see Gluskin [4], Monk and Sioson [6]).

I obtained another proof of Theorem 2. My construction is more simple then the Post's construction. For simplicity I shall describe one for 3-groups (see [3]). In general it is more complicated.

Let $(A: f, ^{-})$ be a 3-group, thus

$$f(f(x,y,z),u,v) = f(x,f(y,z,u),v) = f(x,y,f(z,u,v)),$$

$$f(\bar{x},x,y) = f(y,x,\bar{x}) = y = f(x,\bar{x},y) = f(y,\bar{x},x) \text{ and } \bar{\bar{x}} = x.$$

On the set $A^2 \cup A$ we define the operation o as follows:

(i) $x \circ y = \langle x,y \rangle$,
(ii) $x \circ \langle y,z \rangle = f(x,y,z)$,
(iii) $\langle x,y \rangle \circ z = f(x,y,z)$,
(iv) $\langle x,y \rangle \circ \langle z,u \rangle = \langle f(x,y,z),u \rangle$.

Further we define the relation $\sim$ on the set $A^2 \cup A$ by:

$x \sim y$ if and only if $x = y$,

$\langle x,y \rangle \sim \langle x',y' \rangle$ if and only if there exist $a,b \in A$ such that

$$f(x',a,b) = x, \; f(a,b,y) = y'$$

Lemma 1. $\langle f(x,y,z),u \rangle \sim \langle x,f(y,z,u) \rangle$.

Lemma 2. $\langle u,\bar{u} \rangle \sim \langle v,\bar{v} \rangle$ for every $u,v \in A$.

Corollary. $\langle u,\bar{u} \rangle \sim \langle \bar{u},u \rangle$ for any $u \in A$.

Lemma 3. The relation $\sim$ is a congruence relation in the algebra $(A^2 \cup A; o)$.

Theorem 3. $(A^2 \cup A/_{\sim} : \breve{o})$ is a covering free group for 3-group $(A; f, ^{-})$.

Here $\breve{o}$ is the operation induced by o in the factor algebra. In this group the element $[\langle u,\bar{u} \rangle]_{\sim}$ is a unit.

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(ACTA F. R. N. UNIV. COMEN.-MATHEMATICA, SPECIAL NUMBER, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA - SPECIAL NUMBER - 1975

QUASI-CONSTANTS IN UNIVERSAL ALGEBRAS AND INDEPENDENT SUBALGEBRAS

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1. Introduction

In this report I shall use the terminology and notations of papers by Professor E. Marczewski [6] and [5] (see also [1]). By an algebra $\mathfrak{A}$ I shall mean a pair $\mathfrak{A} = (A; \mathbf{F})$, where A is a non--empty set, and $\mathbf{F}$ is a class of fundamental operations consisting of A-valued functions of several variables running over A. $\mathbf{A}$ (or $\mathbf{A}(\mathfrak{A})$) is the class of all algebraic operations, i.e. the smallest class containing trivial operations

$$e_k^{(n)}(x_1,\dots,x_n) = x_k \qquad (k = 1,2,\dots,n;\ n=1,2,\dots)$$

and closed under superpositions with the fundamental operations. $\mathbf{A}^{(n)}$ $\mathbf{A}^{(n)}(\mathfrak{A})$) is the subclass of $\mathbf{A}$ consisting of all n-ary algebraic operations of $\mathfrak{A}$. $\mathbf{A}^{(0)}(\mathfrak{A}) = C(\emptyset)$ denotes the class of constant algebraic operations of $\mathfrak{A}$ as well as their values (the set of algebraic constants).

The sense of notions: homomorphism, isomorphism, subalgebra, subalgebra $C(T)$ generated by a set $T \subset A$ is clear. $2^{\mathfrak{A}}$ denotes the set of all subalgebras of the algebra $\mathfrak{A}$.

Let $\mathbf{M}(A)$ (or shortly $\mathbf{M}$) denote the family of all mappings $p: T \to A$ from any $T \subset A$ and let $\mathbf{Q} \subset \mathbf{M}$, finally let $\mathfrak{A} = (A; \mathbf{F})$ be a given algebra. Further $\mathbf{H}(\mathfrak{A})$(shortly $\mathbf{H}$) will denote the set of such mappings p: $T \to A$ (for $T \subset A$ which have an extension to the homomorphisms $\bar{p}$: $C(T) \to A$. Then set $I \subset A$ will be said independent with respect to the family $\mathbf{Q}$ or, shortly, $\mathbf{Q}$-independent (in the algebra $\mathfrak{A}$), if $\mathbf{Q} \subset A^I \cap \mathbf{H}(\mathfrak{A})$. Ind $(\mathbf{Q})$ denotes the family of all $\mathbf{Q}$ - independent sets of $\mathfrak{A}$ (see [1]).
If we put $\mathbf{Q} = \mathbf{M}$, we obtain the notion of "algebraic independence" introduced by E. Marczewski in 1958 (see [4]). $I \in$ Ind $(\mathbf{M})$ if and only if for any system of different elements $a_1,\dots,a_n \in I$ and for every pair operations $f, g \in \mathbf{A}^{(n)}$ the equality $f(a_1,\dots,a_n) = g(a_1,\dots,a_n)$ implies $f = g$ in $\mathfrak{A}$.

Recently, in February 1972, Professor Marczewski sugested the notion of independence of subalgebras of a universal algebra. Two subalgebras B_1 and B_2 of an algebra $\mathfrak{A} = (A; \mathbf{F})$ are independent if for every pair of homomorphisms $h_i: B_i \longrightarrow A (i=1,2)$ there exists a homomorphisms $h: C(B_1 \cup B_2) \longrightarrow A$, such that $h|_{B_i} = h_i (i=1,2)$. Then one can verify that $h_1(a) = h_2(a) = a$ for any $a \in B_1 \cap B_2$. It leads to a new notion of quasi-constant elements with respect to a subalgebra B of $\mathfrak{A}$ (see the following section).

Majority of results presented here I obtained together with Anzelm Iwanik (see [2] and [3]). A part of the theorems arose from questions posed to us by Professor Edward Marczewski.

2. Quasi-constants

An element a of a subalgebra B of $\mathfrak{A}$ is a quasi-constant with respect to B if $h(a) = a$ for every homomorphism $h: B \longrightarrow A$. We denote by $Q(B)$ the set of all quasi-constants with respect to B in the algebra $\mathfrak{A}$. It seems worth to notice that any element of $Q(B)$ is a fully invariant element and that the sets $Q(B)$ are fully invariant subsets in the sense of [11]. It is easy to see that

(i) If $Q(B) \neq \emptyset$ then $Q(B)$ is a subalgebra of B
(ii) $Q(B_1) \subset Q(B_2)$ for subalgebras $B_1 \subset B_2 \subset A$.
(iii) Every element $a \in Q(B)$ is self-dependent, ([7]).
(iv) $Q(B) \subset h(B)$ for every homomorphism $h: B \longrightarrow A$.

Theorem 1. If a is the only fix-point of a unary algebraic operation f of $\mathfrak{A}$, then $a \in Q(C(\{a\}))$.

Consequently

(v) $c \in Q(C(\{c\}))$ for every $c \in C(\emptyset)$.
(vi) $C(\emptyset) = Q(C(\emptyset)) \subset Q(B)$ for every subalgebra B of $\mathfrak{A}$.

Note that, in general, the inclusion in (vi) is proper, which shows the example: $\mathfrak{A} = (\{a,b,c\} ; f)$ where $f(a) = b$, $f(b) = c$ and $f(c) = c$. Here $C(\emptyset) = \emptyset$ and $c \in Q(C(\{c\}))$.

From statement (iv) follows that, if $\{c\}$ is a subalgebra of $\mathfrak{A}$, then $Q(A) \subset \{c\}$. Hence, if an algebra has exactly one the one-element subalgebra (e.g. the algebraic constant) $\{c\}$, then $Q(A) = \{c\}$. Consequently $Q(A) = \{0\} = C(\emptyset)$ for groups, for rings (without unit as the algebraic constant) or for modules. It is also clear that $Q(A) = \emptyset = C(\emptyset)$ whenever $\mathfrak{A}$ has at least two idempotent elements, which is valid in idempotent (at least two-element) algebras, e.g. in lattices and diagonal algebras ([8]). Another class of algebras with at most one the quasi-constant we obtain from

Theorem 2. If an algebra $\mathfrak{A}$ is weakly commutative ([1], p. 10) then there is at most one the quasi-constant (respect to A) in $\mathfrak{A}$.

In particular, in groupoids in which $(xy)(uv) = (xu)(yv)$ (moreover in groupoids with $(xy)^2 = x^2y^2$) and thus in medial semigroups the set $Q(A)$ is at most one-element and $c \in Q(A)$ is only idempotent element. It is also true in generalized diagonal algebras (for definition see [8]).

From the following theorem we obtain a class of algebras in which $Q(A) = C(\emptyset)$.

Theorem 3. Let $I \in \mathrm{Ind}(\mathbf{M})$ and $c \in C(I)$ in an algebra $\mathfrak{A}$. Then c is a quasi- constant with respect to $C(I)$ if and only if c is an algebraic constant in $\mathfrak{A}$.

Corollary. If $\mathfrak{A}$ has a basis then $Q(A) = C(\emptyset)$.

It would be interesting to know the answer to the following

Problem 1. For which algebras $Q(A) = C(\emptyset)$?

Problem 2. For which algebras $Q(A) = \emptyset$?

The following example shows that, in general, $Q(Q(B)) \neq Q(B)$. Let $\mathfrak{A} = (A; f, g)$, $B = A = \{a, b, c, d, e\}$ and

$$f(x) = \begin{cases} b & \text{for } x = a \\ c & \text{for } x = b \\ a & \text{for } x = c, d, e \end{cases} \qquad g(x) = \begin{cases} c & \text{for } x = a \\ b & \text{for } x = c \\ a & \text{for } x = b, d, e. \end{cases}$$

Then $Q(B) = \{a, b, c\}$, $Q(Q(B)) = \emptyset$.

Denote $Q^0(B) = B$ and $Q^n(B) = Q(Q^{n-1}(B))$ for $n = 1, 2, \ldots$. Observe that if $Q^k(B) = Q^m(B)$ for some $m > k$, then $Q^k(B) = Q^{k+1}(B) = \ldots$ If $Q^n(A)$ is one-element for some $n \geqslant 1$ then it is obviously $Q^1(A) = \ldots = Q^n(A) = Q^{n+1}(A) = \ldots$ Other possible sequences $Q^n(B)$ are given by the following theorem, which answers the question posed by Jerzy Płonka:

Theorem 4. Let n be a natural number and $\varkappa \neq 1$ be a cardinal number. Then the following three cases can be realized in unary algebras:

(1) $Q(B) \supset Q^2(B) \supset \ldots \quad Q^n(B) = Q^{n+1}(B), \quad |Q^n(B)| = \varkappa,$

(2) $Q(B) \supset Q^2(B) \supset \ldots \quad \left|\bigcap_{k=1}^{\infty} Q^k(B)\right| = \varkappa,$

(3) $Q(B) = Q^2(B) = \ldots, \quad |Q(B)| = \varkappa$ or 1,

where the inclusions in (1) and (2) are proper.

The proof is complicated. We use the following lemma:

Lemma. Let $\mathfrak{A}_0 = (A_0; \mathbf{F}_0)$ be a unary algebra without one-element subalgebras. Then there exists a unary algebra $\mathfrak{A}_1 = (A_1; \mathbf{F}_1)$ such that A_0 is a proper subalgebra of $\mathfrak{A}_1$ and $Q(A_1) = A_0$.

Put $Q^{\nu}(B) = Q(\bigcap_{\tau<\nu} Q^{\tau}(B))$ for a limit ordinal number ν.

Problem 3. (M. Novotný) Is there the subalgebra B of an algebra $\mathfrak{A}$ and is there the ordinal number μ such that $Q^{\mu}(B) = Q^{\mu+1}(B) = \ldots$ and $Q^{\tau}(B) \supset Q^{\mu}(B)$ for an ordinal number $\tau < \mu$ and $|Q^{\mu}(B)| = \varkappa$ for an arbitrary cardinal number $\varkappa$?

In the set $Q(A)$ in an algebra $\mathfrak{A} = (A; \mathbf{F})$ we shall distinguish elements a such that $a \in Q(C(\{a\}))$. These elements we shall call semi-constants. By (v) all constants are semi-constants. Note that Theorem 1 is concerned with semi-constants.

In general, semi-constants do not form a subalgebra. For example $\mathfrak{A} = (\{a,b,c\}; f, g)$, where $f(a) = g(b) = b, f(b) = g(a) = g(c) = a$ and $f(c) = c$. Here $g(c) = a \notin Q(C(\{a\}))$, while $c \in Q(C(\{c\}))$.

Theorem 5. Let $a_1, \ldots, a_n$ be semi-constants in $\mathfrak{A}$ and let an arbitrary $b \in C(\{a_1, \ldots, a_n\})$. Then b is a quasi-constant with respect to $C(\{a_1, \ldots, a_n\})$. Moreover, if $C(\{b\}) = C(\{a_1, \ldots, a_n\})$ then b is also a semi-constant.

Problem 4. In which algebras the set of all semi-constant forms a subalgebras.

3. Independence of subalgebras

A set $\mathcal{Y}$ of subalgebras of the algebra $\mathfrak{A} = (A; \mathbf{F})$ is independent if for every family of homomorphisms $h_B : B \to A$, $B \in \mathcal{Y}$, there exists a homomorphism $h : C(\bigcup_{B \in \mathcal{Y}} B) \to A$ such that $h|_B = h_B$ for $B \in \mathcal{Y}$. We use the notation $\mathcal{Y} \in \operatorname{ind} 2^{\mathfrak{A}}$ (or shortly $\mathcal{Y} \in \operatorname{ind}$) if $\mathcal{Y}$ is an independent set of subalgebras of $\mathfrak{A}$. It follows immediately from the definition of independence of subalgebras that the homomorphism h is unique. It is also obvious that the independence of set $\mathcal{Y}$ of subalgebras of $\mathfrak{A}$ is hereditary property with respect to subsets of $\mathcal{Y}$.

Example 1. The following notion of independence (which we shall call $\mathcal{B}$-independence) of subalgebras of a Boolean algebra is well known: The indexed set $\{B_t\}_{t \in T}$ of subalgebras of Boolean algebras $\mathfrak{A} = (A; \cup, \cap, ', 0, 1)$ is $\mathcal{B}$-independent if $\bigcap_{t \in T_0} b_t \neq 0$ for arbitrary finite $T_0 \subset T$ and $b_t \neq 0$, $b_t \in B_t$ (see [10], § 13). Theorem 13.1 of [10] shows that any $\mathcal{B}$-independent set of subalgebras is independent in our sense. One can show that the converse is also true.

Example 2. If $\mathfrak{A}$ is a unary algebra, then any pairwise disjoint set of subalgebras of $\mathfrak{A}$ is independent. The following example shows that, in general, it is not true in non-unary algebras. Viz. $A = \{a, b, c\}$ and o be a binary fundamental operation, such that $x \circ x = c \circ x = c$ for every $x \in A$ and $x \circ y = x$ for $x \neq c$ and $y \neq c$. Let $h_a: \{a\} \longrightarrow \{c\}$ and $h_b: \{b\} \longrightarrow \{b\}$. If there were a homomorphism $h: \{a, b\} \longrightarrow A$ such that $h(a) = c$ and $h(b) = b$, then $b = h(b) = h(b \circ a) = h(b) \circ h(a) = h(b) \circ c = c$, which is a contradiction.

Theorem 6. The following conditions are equivalent:

(1) $\mathcal{Y} \in \operatorname{ind} 2^{\mathfrak{A}}$,

(2) The subalgebra $C(\bigcup_{B \in \mathcal{Y}} B)$ of the algebra $\mathfrak{A}$ is a $\mathcal{K}$-free product of subalgebras $B \in \mathcal{Y}$ (see [9] and [4]) for any class $\mathcal{K}$ of algebras such that $\mathfrak{A} \in \mathcal{K} \subset 2^{\mathfrak{A}}$.

(3) $\{h(B): B \in \mathcal{Y}\} \in \operatorname{ind} 2^{\mathfrak{A}}$ for every meromorphism (i.e. an injective homomorphism) $h: C(\bigcup_{B \in \mathcal{Y}} B) \longrightarrow A$.

(4) For any family of homomorphisms $h_B: B \longrightarrow A$, $B \in \mathcal{Y}$, if f, g are n-ary and m-ary algebraic operations in $\mathfrak{A}$ and $a_i \in A_i \in \mathcal{Y}$, $b_j \in B_j \in \mathcal{Y}$ ($i = 1, \ldots, n$, $j = 1, \ldots, m$; $n, m = 1, 2, \ldots$) then the equality $f(a_1, \ldots, a_n) = g(b_1, \ldots, b_n)$ implies $f(h_{A_1}(a_1), \ldots, h_{A_n}(a_n)) = g(h_{B_1}(b_1), \ldots, h_{B_m}(b_m))$.

(5) For any family of homomorphisms $h_B: B \longrightarrow A$ $(B \in \mathcal{Y})$ if f is an n-ary algebraic operation in $\mathfrak{A}$ and $a_i \in B_i \in \mathcal{Y}$ ($i = 1, \ldots, n$; $n = 1, 2, \ldots$) then the implication

(+) $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n) \Longrightarrow$

$\Longrightarrow f(h_{B_1}(a_1), \ldots, h_{B_n}(a_n)) = g(h_{B_1}(a_1), \ldots, h_{B_n}(a_n))$

holds.

(6) For any family of homomorphisms $h_B\colon B \longrightarrow A$ $(B \in \mathfrak{I})$ if f is an n-ary algebraic operation in $\mathfrak{A}$ $(n=1,2,\ldots)$ and $a_1,\ldots,a_n$ are different elements such that $a_i \in B_i \in \mathfrak{I}$ $(i=1,\ldots,n)$ then the implication (+) holds.
(7) $\bigcup_{B \in \mathfrak{I}} B \in \mathrm{Ind}\,(\mathbf{H}_*^{\mathfrak{I}})$, where $\mathbf{H}_*^{\mathfrak{I}}$ is a subfamily of the family $\mathbf{H}_*$ ($\mathbf{H}_*$ is the smallest family of mappings from subsets of algebra $\mathfrak{A} = (A;\ \mathbf{F})$ into A, containing $\mathbf{H}$ and closed with respect to restrictions and "sticking together" of mappings on disjoint subsets; see [1], p. 28) such that for every $T \subset A$ and $B \in \mathfrak{I}$ if $p \in \mathbf{H}_*^{\mathfrak{I}} \cap A^T$ then $p|_{T\cap B} = h_B|_{T \cap B}$ for some homomorphism $h_B\colon B \longrightarrow A$.

The equivalence $(1)\Longleftrightarrow(2)$ shows that the notion of independence of subalgebras is connected with the notion of free product of subalgebras ([9], [4]) similarly as the algebraic independence of elements is connected with the notion of free generated algebra.

From the equivalence $(1)\Longleftrightarrow(4)$ we have

Corollary 1. The independence of a set of subalgebras is a property of finitary character (i.e. $\mathfrak{I} \in$ ind if and only if $\mathfrak{I}_0 \in$ ind for any finite subset $\mathfrak{I}_0$ of the set $\mathfrak{I}$).

Corollary 2. Every independent set of subalgebras is contained in some maximal independent set of subalgebras.

Corollary 3. Let $\mathfrak{I} \in \mathrm{ind}\ 2^{\mathfrak{A}}$ and $|\mathfrak{I}| \geqslant 2$ and $b \in B$ for every $B \in \mathfrak{I}$. Then b is a quasi-constant with respect to each subalgebra $B \in \mathfrak{I}$.

In particular in algebras with exactly one constant 0 (e.g. in groups, rings and modules) the intersection of independent subalgebras is always equal to $\{0\}$. In algebras with the basis this intersection is equal to $C(\emptyset)$, and in idempotent algebras it is empty set.

From the equivalence $(1)\Longleftrightarrow(7)$ we have several corollaries

Corollary 4. Let $\{I_t\}_{t \in T}$ be a family of pairwise disjoint subsets of an algebra $\mathfrak{A}$. Then

$\mathfrak{I} = \{C(I_t)\colon t \in T\} \in \mathrm{ind}\ 2^{\mathfrak{A}}$ if and only if $(\bigcup_{t \in T} I_t)$ Ind $(\mathbf{H}_*^{\mathfrak{I}})$.

Thus

Corollary 5. Let $\{I_t\}_{t \in T}$ be a family of pairwise disjoint subsets of an algebra $\mathfrak{A}$ and $I_t \in \mathrm{Ind}\,(\mathbf{M})(t \in T)$. Then

$\{C(I_t), t \in T\} \in \mathrm{ind}\ 2^{\mathfrak{A}}$ if and only if $(\bigcup_{t \in T} I_t) \in \mathrm{Ind}\,(\mathbf{M})$.

Since the notions of $\mathbf{H}_*$- and $\mathbf{H}_*^{\mathfrak{I}}$-indepence coincide for one-element sets, hence, in virtue of Corollary 4, we obtain

Corollary 6. $\{C(\{a\}) : a \in I\} \in \operatorname{ind} 2^{\mathfrak{A}}$ if and only if $I \in \operatorname{Ind}(\mathbf{H}_*)$

Hence we have two corollaries, which answer one of the questions posed to us by Professor Marczewski:

Corollary 7. If I is an algebraic independent subset of A in the algebra $\mathfrak{A}$, then $\{C(\{a\}) : a \in I\} \in \operatorname{ind} 2^{\mathfrak{A}}$.

Corollary 8. Let I be a set of non-self-dependent elements ([7]) of the algebra $\mathfrak{A}$. Then, if $\{C(\{a\}) : a \in I\}$ ind $2^{\mathfrak{A}}$ then $I \in \operatorname{Ind}(\mathbf{M})$.

The next theorem is analogous to the "exchange theorem" from [5] ((ii), p. 58).

Theorem 7. Let $(\mathcal{Y}_0 \cup \mathcal{Y}_1) \in \operatorname{ind} 2^{\mathfrak{A}}$, $\mathcal{Y}_0 \cap \mathcal{Y}_1 = \emptyset$, $\mathcal{Y}_2 \in \operatorname{ind} 2^{\mathfrak{A}}$ and let there exists the set $\mathcal{Y}_1^0 \subset \mathcal{Y}_1$ such that $C(\bigcup_{B \in \mathcal{Y}_1^0} B) = C(\bigcup_{B \in \mathcal{Y}_2} B)$. Then $(\mathcal{Y}_0 \cup \mathcal{Y}_2) \in \operatorname{ind} 2^{\mathfrak{A}}$.

The following problems are connected with the study of the notion of independence of subalgebras:

Problem 5. Given a pair A and B, where B is a subalgebra of $(A; \mathbf{F})$, describe the congruences Θ of B such that Θ coresponds to some homomorphism $h: B \longrightarrow A$.

Problem 6. The same in the case $B = A$. (Added in proof: the congruences in this case are called nuclear, see [11])

Problem 7. Which algebras $\mathfrak{A} = (A; \mathbf{F})$ have the property that for every subalgebra B there exists a homomorphism $h: A \longrightarrow B$?

Problem 8 (J. Ježek). Let A be an arbitrary set and $\{B_t\}_{t \in T}$ be a family of subsets of A. Under which assumptions $\{B_t : t \in T\} = \{h(A) ; h \text{ is an endomorphism of } \mathfrak{A}\}$ for some algebra $\mathfrak{A}$ on the set A?

Problem 9. Let B be a subalgebra of an algebra $\mathfrak{A}$. Under which assumptions on the algebra $\mathfrak{A}$ and the subalgebra B, every homomorphism $h: B \longrightarrow A$ has an extention to some endomorphism of the algebra $\mathfrak{A}$?

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(ACTA F. R. N. UNIV. COMEN.-MATHEMATICA, SPECIAL NUMBER, 1975)

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ALGEBRAICITY OF ENDOMORPHISMS OF SOME STRUCTURES

JAROSLAV JEŽEK, Praha

Let a transformation monoid H on a set A be given, i.e. a set of transformations of A which is closed under composition and contains the identical transformation. H is called algebraic if H is just the set of all endomorphisms of some universal algebra (with the underlying set A). B. JÓNSSON raised the following problem: give a necessary and sufficient condition for a transformation monoid to be algebraic.

The following theorem, proved by M. SEKANINA, implies that there are many transformation monoids which are not algebraic:

Theorem 1. Let $\langle A, F \rangle$ be an algebra with finitary operations such that any isotone mapping of a given ordered set $\langle A, r \rangle$ into itself is an endomorphism of $\langle A, F \rangle$. If $\langle A, r \rangle$ is not linearly ordered, then any transformation of A is an endomorphism of $\langle A, F \rangle$.

I have two remarks to this theorem:

1. Theorem 1 holds if we replace "ordered set" by "yuasiordered set". (The proof for quasiordered sets that are not ordered is not difficult.) As there is a well-known correspondence between finite quasiordered sets and finite topological spaces (and between isotone and continuous mappings), this shows that that a re-formulation of Theorem 1 could be given for finite algebras and finite topological spaces.

2. Theorem 1 holds if "finitary" is replaced by "infinitary". The proof in the infinitary case differs essentially from the original Sekanina´s proof. In fact, Sekanina got the result by showing that any operation of F is trivial; this is not true if $\langle A, F \rangle$ is infinitary.

The following theorem gives an answer to the problem formulated above, in one special case:

Theorem 2. Let H be a transformation monoid on A such that every $h \in H$ is either injective or constant. H is algebraic iff it satisfies the following three conditions:

(1) If h is an injective transformation of A, then $h \in H$ iff for every finite $X \subseteq A$ there exist some $h_1, \dots, h_{2n} \in H$ such that $h_{2n}^{-1} \cdot h_{2n-1} \cdot \dots \cdot h_2^{-1} \cdot h_1$ is defined on X and coincides with h on X;

(2) If $h \in H$ is injective, $a \in A$ and the constant transformation with value $h(a)$ belongs to H, then the constant transformation with value a belongs to H, too;

(3) If $a \in A$ and the constant transformation with value a does not belong to H, then there exists a $b \in A$ different from a such that whenever $f_1, \dots, f_{2n}, g_1, \dots, g_{2n} \in H$ are injective and

$$(f_{2n}^{-1} \cdot f_{2n-1} \cdot \dots \cdot f_2^{-1} \cdot f_1)(a) = (g_{2n}^{-1} \cdot g_{2n-1} \cdot \dots \cdot g_2^{-1} \cdot g_1)(a)$$

(so that a belongs to the domain of these two mappings), then

$$(f_{2n}^{-1} \cdot f_{2n-1} \cdot \dots \cdot f_2^{-1} \cdot f_1)(b) = (g_{2n}^{-1} \cdot g_{2n-1} \cdot \dots \cdot g_2^{-1} \cdot g_1)(b)$$

(so that a belongs to the domains, too).

The problem formulated above is closely related to the following problem: given a relational structure $\langle A, R \rangle$, decide whether the transformation monoid of all its endomorphisms is algebraic. The following theorem gives an answer in the special case of $\langle A, R \rangle$ being unary (i.e. R being a set of subsets of A and endomorphisms of $\langle A, R \rangle$ being transformations h such that $h(r) \subseteq r$ for any $r \in R$).

Theorem 3. Let $\langle A, R \rangle$ be a unary relational structure. For every $a \in A$ denote by $\bar{a}$ the set of all $b \in A$ such that $a \in r \in R$ implies $b \in r$; denote by B the set of all $a \in A$ such that $\bar{a} = \{a\}$. The transformation monoid of endomorphisms of $\langle A, R \rangle$ is algebraic iff the following four conditions are satisfied:

(1) If B is empty, then every $r \in R$ is either empty or equal to A;

(2) If Card $(B)=1$, then $B \subseteq r$ for any non-empty $r \in R$;

(3) Whenever $a, b \in A-B$ and $b \in \bar{a}$, then $\bar{a} = \bar{b}$;

(4) Whenever $a_1, \dots, a_n$ (where $n \geq 2$) are pairwise different elements of $A-B$ and $u_1, \dots, u_n$ pairwise different elements of B such that $u_1 \in \bar{a}_1 \cap \bar{a}_2$, $u_2 \in \bar{a}_2 \cap \bar{a}_3, \dots, u_n \in \bar{a}_n \cap \bar{a}_1$, then $\bar{a}_1 = \dots = \bar{a}_n$.

(ACTA F. R. N. UNIV. COMEN.-MATHEMATICA, SPECIAL NUMBER, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE MATHEMATICA - SPECIAL NUMBER - 1975

EQUATIONAL CLASSES OF MODULAR P-ALGEBRAS

TIBOR KATRIŇÁK

A universal algebra $\langle L; \cup, \cap, {}^*, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ is called a (modular) p-algebra iff $\langle L; \cup, \cap, 0, 1\rangle$ is a bounded (modular) lattice and where, for every $a \in L$, the element a^* is a pseudocomplement of a, i. e. $x \leqslant a^*$ iff $a \cap x = 0$. The class of all (modular) p-algebras is equationally definable (see [1]). The standard results on p-algebras may be found in [1].

For a p-algebra L define the set $B(L) = \{x \in L;\ x = x^{**}\}$ of closed elements. The partial ordering of L partially orders $B(L)$ and makes the latter into a Boolean algebra.

$$\langle B(L); \vee, \cap, {}^*, 0, 1\rangle \text{ for which}$$
$$a \vee b = (a \cup b)^{**}$$

holds. Another significant subset of a p-algebra L is the set of dense elements $D(L) = \{x \in L;\ x^* = 0\}$. $D(L)$ is a filter in L.

For a p-algebra L define the relation Θ by

$$x \equiv y(\Theta) \quad \text{iff } x^* = y^*.$$

It is easy to see that Θ is a congruence relation on L and $L/\Theta \cong B(L)$. Each congruence class of Θ contains exactly one element of $B(L)$, which is the largest element in the congruence class. Hence, Θ partitions L into $\{F_a;\ a \in B(L)\}$, where $F_a = \{x \in L;\ x^{**} = a\}$, $a \in B(L)$. Obviously, $F_0 = \{0\}$ and $F_1 = D(L)$.

An element x of a p-algebra L is said to be a Stone element if it satisfies the identity

$$x^* \cup x^{**} = 1.$$

Theorem 1 ([3]). In a modular p-algebra L the subset $S(L) = \{x \in L;\ x \text{ is a Stone element}\}$ is a subalgebra of L.

Theorem 2 ([3]). Let L be a modular p-algebra. Then L is subdirectly irreducible iff L satisfies the following conditions:

(i) D(L) is a subdirectly irreducible lattice;

(ii) for each closed Stone element $0 < a < 1$ of L we have card $(F_a) \geqslant 2$.

Theorem 3 ([4]). There are $2^{\aleph_0}$ distinct equational classes of modular p-algebras.

Theorem 3 gives a solution of [1, Problem 65].

We shall study now the equational classes of modular p-algebras generated by subdirectly irreducible modular p-algebras L satisfying $D(L) \cong M_n$ $(n \geqslant 3)$, where M_n is the modular lattice of dimension 2 and order $n+2$ (see the diagram of M_3 in Fig. 1). Let M_1 is the the one-element, and M_2 the two-element lattice.

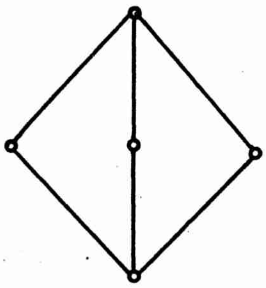

Fig. 1

By B. Jónsson [2], a modular subdirectly irreducible lattice L satisfies

(J) $x_1 \cap [x_2 \cup (x_3 \cap x_4)] \cap (x_3 \cap x_4) \leqslant x_2 \cup (x_1 \cap x_3) \cup (x_1 \cap x_4)$

iff $L \cong M_n$ for some cardinal number $n \geqslant 1$.

First we characterize the subdirectly irreducible modular p-algebras L with $D(L) \cong M_n$ $(n \geqslant 3)$. (For $D(L) \cong M_n$, $n \leqslant 2$, see [6]). We construct a map $G^L\colon D(L) \longrightarrow F(B(L))$ ($F(B(L))$ is the lattice of all filters of $B(L)$ ordered under set inclusion in the following way: $G^L(1) = [1)$, $G^L(s) = B(L)$, where s denotes the smallest element of $D(L)$, and for $s < d < 1$ set

$$G^L(d) = \{x \in B(L);\ x \cup s = d\} \cup \{1\}.$$

Theorem 4 ([5]). Let L be a subdirectly irreducible modular p-algebra with $D(L) \cong M_n$ $(n \geq 3)$. Then G^L is a dual $(e,\cup)$-homomorphism of $D(L)$ into $F(B(L))$, i.e. $G^L(1) = [1)$, $G^L(s) = B(L)$ and $G^L(d \cup e) = G^L(d) \cap G^L(e)$.
The triple $\langle B(L), D(L), G^L \rangle$ characterizes L up to isomorphism.

Theorem 5 ([5]). Given a Boolean algebra B, a modular lattice M_n $(n \geq 3)$ and a dual $(e,\cup)$-homomorphism $G: M_n \longrightarrow F(B)$, there exists a subdirectly irreducible modular p-algebra L such that $\langle B(L), D(L), G^L \rangle = \langle B, M, G \rangle$.

Finally, there is given in terms of identities a characterization of all equational classes of modular p-algebras satisfying (J) and containing subdirectly irreducible p-algebras L with $G^L(d) = [1)$ for $s \neq d \in D(L)$ only. The lattice of all such equational classes is also described.

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(ACTA F. R. N. UNIV. COMEŇ.-MATHEMATICA, SPECIAL NUMBER, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
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DECOMPOSITIONS OF N-ANGLES ONTO THE 2-COMPONENT SUMS

A. KOŚLIŃSKI, Wrocław

We accept the definition of a graph from [1]. A graph L is a pair $\langle X; R\rangle$ where X is a non-empty set (the set of vertices) and R is the binary relation in X. The pairs $\langle a, b\rangle \in R$ are called edges. Here we consider only symmetrical unigraphs without loops (see [1]), therefore we denote by (a, b) the undirected edge. A finite graph L will be called an n-angle if there exists a simple cycle in it containing all its vertices. We shall numerate the vertices of the n-angle by $a_1, a_2, \dots, a_n$ according to their order on a chosen simple cycle containing all vertices of L. An n-angle we can consider as a convex geometrical n-angle containing some diagonals or not, where the simple cycle is the circuit.

In [2] the definition of the sum of a direct system of graphs was given.

In [3] one proved that if a graph L is decomposable into the sum of a direct system of graphs then L is decomposable into the sum of a direct system of two graphs. Here we recall the notion of the sum of a direct system of two graphs called further the 2-component sum. Take two graphs $L_1 = \langle X_1; R_1\rangle$, $L_2 = \langle X_2; R_2\rangle$. The mapping $h: X_1 \longrightarrow X_2$ is a homomorphism of L_1 into L_2 iff for any $a, b \in X_1$ we have $a R_1 b \Longrightarrow h(a) R_2 h(b)$. Assume $X_1 \cap X_2 = \emptyset$ and there exist a homomorphism h of L_1 into L_2. We define the two component sum of L_1 and L_2 as a new uni graph L in the following way:

$$L = (X_1 \cup X_2; R); \quad R = R_1 \cup R_2 \cup R_0 \quad \text{where}$$

$$R_0 = \{(a,b) : a \in X_1, b \in X_2, h(a) R_2 b\} \cup$$

$$\cup \{(c,d) : c \in X_2, d \in X_1, c R_2 h(d)\}$$

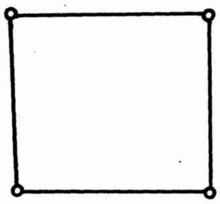
We say (see [2]), that the graph is decomposable into the 2-component sum iff it can be represented as a 2-component sum.

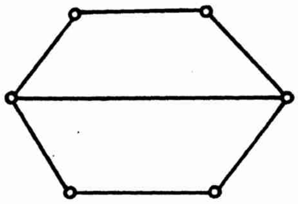
In [3] one proved that the an angle without diagonals is decomposable iff n=4.

In [2] one can find that the a-angle with all diagonals is not decomposable.

In this paper we consider the case when an n-angle has some but not all diagonals.

Theorem 1. An n-angle with exactly one diagonal is decomposable iff n=4 or n=6 where in the last case the diagonal is main.





An n-angle is called (n-1)-replete iff it has all diagonals without those which go out from the fixed vertex a_1.

Theorem 2. An n-angle (n-1)-replate where $n > 2$ is decomposable iff n=4.

Theorem 3. The n-angle having all diagonals going out from the fixed vertex and having no others diagonals is decomposable iff n=4 or n=5.

Theorem 4. For any pair n, k of integers where $n > 4$, $I < k < \frac{n(n-3)}{2}$ there exists an n-angle having exactly k-diagonals which is decomposable.

Theorem 5. From all 5-angles we have four which are decomposable and four which are not decomposable.

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(ACTA F. R. N. UNIV. COMEN.-MATHEMATICA, SPECIAL NUMBER, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA - SPECIAL NUMBER - 1975

SOME REPRESENTATIONS OF ORTHOMODULAR AND SIMILAR POSETS*)

M. J. MĄCZYŃSKI - T. TRACZYK

Let us consider a partially ordered set $(L; \leqslant)$. A mapping $m: L \longrightarrow [0,1]$ is said to be a state on L iff

$$a \leqslant b \quad \text{implies} \quad m(a) \leqslant m(b)$$

for every $a, b \in L$. A set M of states on L is said to be full iff

$$\left[\underset{m \in M}{\forall} \; m(a) \leqslant m(b) \right] \quad \text{implies} \quad a \leqslant b$$

Using the notion of principal filters one can easily prove the following theorem:

Theorem 1. Every poset admits a full set of states.

Now let us suppose an unary operation $' : L \longrightarrow L$ to be defined in L. The algebraic system $(L; \leqslant, ')$ is said to be an orthocomplemented poset iff for every $a, b, a_1, \ldots, a_n$ in L the following axioms are satisfied:

(A1) $a \leqslant b$ implies $b' \leqslant a'$;

(A2) $a'' = a$;

(A3) if $a_i \leqslant a_j'$ for $i \neq j$, then the least upper bound $a_1 \cup \ldots \cup a_n$ exists in $(L; \leqslant)$;

(A4) $a \cup a' = b \cup b'$.

It evidently follows by (A1) - (A4) that the orthocomplemented poset $(L; \leqslant, ')$ has two algebraic constants; the greatest element 1 which is equal to $a \cup a'$ for every $a \in L$, and the least element 0 which is equal to $1'$. Note that the existence of the least upper bound $a \cup a'$ follows by (A2) and (A3).

Any two elements a, b of an orthocomplemented poset are said to be orthogonal, $a \perp b$, if $a \leqslant b'$. We have $a \perp a'$ by A2 . It is also easy to whow that

$$a \perp b \quad \text{implies} \quad a \cap b = 0$$

It means that for every pair of orthogonal elements of an orthocomplemented poset the greatest lower bound exists and equals 0.

An orthocomplemented poset $(L; \leqslant, ')$ is said to be orthomodular iff the additional axiom (A5) is satisfied:

(A5) $a \leqslant b$ implies $b = a \cup (a \cup b')'$ for every $a, b \in L$.

The axiom (A4) can be left out if one suppose (A5) and the existence of the greatest element in $(L; \leqslant)$.

In an orthomodular poset we can now define a state to be a mapping $m: L \longrightarrow [0,1]$ such that

(s) $m(1) = 1$,

(ss) $m(a_1 \cup \ldots \cup a_n) = m(a_1) + \ldots + m(a_n)$ whenever $a_i \perp a_j$ for $i \neq j$.

Each state on the orthomodular poset $(L; \leqslant, ')$ is a state on the poset $(L; \leqslant)$ in the above mentioned sense, as well. There are orthomodular posets which neither admit a full set of states nor even a single nontrivial state, [ ] . On the other hand there are some important examples of orthomodular posets which do admit a full set of states, e.g. Boolean algebra and the lattice of all closed subspaces of a Hilbert space. Another example will be shown later.

2. Let F be a set of functions from a set A into a set B. Let $\underline{a}$ be a mapping from F into B defined by

$\underline{a}(f) = f(a)$ for every $f \in F$ and for a fixed $a \in A$. The set

$$F' = \{\underline{a} : a \in A\}$$

is said to be the dual of F.

<u>Theorem 2</u>. If M is a full set of states of an orthomodular poset $(L; \leqslant, ')$ and M' is the dual of M, then M' is an orthomodular poset with respect to the natural order of real functions and the complementation $f' = 1 - f$, i.e. $f'(m) = 1 - f(m)$ for every $m \in M$. Moreover the orthomodular posets $(L; \leqslant, ')$ and $(M'; \leqslant, ')$ are isomorphic, and the mapping $a \longrightarrow \underline{a}$ is the isomorphism in question.

One can prove this theorem by rather easy verification. Now we will formulate the main result of the paper, which implies an important characterization of any orthomodular poset admitting a full set of states.

<u>Theorem 3</u>. Let L be a set of functions from a set M into the closed interval $[0,1]$ and such that

(i) the zero function belongs to L;
(ii) $f \in L$ implies $1 - f \in L$;
(iii) if $f_1, f_2, f_3 \in L$ and $f_i + f_j \leqslant 1$ for $i \neq j$, then $f_1 + f_2 + f_3 \in L$.

Then L is an orthomodular poset with respect to the natural order of real functions and with the complementation $f' = 1 - f$. Moreover, if $f_1, \dots, f_n \in L$ and $f_i \perp f_j$ for $i \neq j$ then the least upper bound $f_1 \cup \dots \cup f_n$, which exists on account of (A4), is equal to the arithmetical sum $f_1 + \dots + f_n$. $(L; \leqslant, ')$ admits a full set of states.

Conversely, if $(L; \leqslant, ')$ is an abstract orthomodular poset admitting a full set of states M, then the dual M' satisfies (i) - (iii).

Corollary. An orthomodular poset L admits a full set of states if and only if it can be represented as a set of functions satisfying (i) - (iii).
The proof of theorem 3 can be found in [ ].

3. The orthomodular poset $(L, \leqslant, ')$ is called Boolean iff

$$a \cap b = 0 \quad \text{implies} \quad a \perp b.$$

Note that the converse implication holds in every orthomodular poset.

Theorem 4. Every Boolean orthomodular poset admits a full set of states.

One can build up the proof of this theorem (see [ ]) by taking in account the notion of maximal filters in Boolean orthomodular posets and its properties, which has been found by Barros, [ ] .

Let M be a full set of states on a Boolean orthomodular poset $(L; \leqslant, ')$. Let M' be the dual of M. By theorem 3 it satisfies (i) - (iii). Since $(M'; \leqslant, ')$ is Boolean, it also satisfies the following condition

(iv) If for some $f, g \in M'$ the only function $h \in M'$ such that both, $h \leqslant f$ and $h \leqslant g$, is the zero function, then $f + g \in 1$.

Hence we have the following characterization

Corollary. Every Boolean orthomodular poset can be isomorphically represented as a set of functions satisfying (i) - (iv). Conversely, every set of functions satisfying (i) - (iv) is a Boolean orthomodular poset with respect ot the natural order and the complementation $f' = 1 - f$.

If instead of (iv) we assume the stronger condition

(iv') for every $f, g \in M'$ there are $h_1, h_2, h_3 \in M'$ such that $h_i + h_j \leqslant 1$ for $i \neq j$ and $f = h_1 + h_2$, $g = h_2 + h_3$, then M', and consequently L, appear to be Boolean algebras. Since the converse statement is also true, we then have the following important characterization of Boolean algebras, which has been found recently by M. Maczynski, [ ].

Theorem 5. Every Boolean algebra can be isomorphically represented by a set of functions satisfying (i) - (iii) and (iv').

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This paper has been presented by the second author at the Summer School in Horná Lipová.

(ACTA F. R. N. UNIV. COMEN.-MATHEMATICA, SPECIAL NUMBER, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA - SPECIAL NUMBER - 1975

ON SOME OPERATORS REDUCING GENERALIZED GRAMMARS

M. NOVOTNÝ

If V is a set then we denote by V^* the free monoid over V, i.e. the set of all finite sequences of elements of V - including the empty sequence - provided by the binary operation of concatenation. The elements of V^* are called strings. We identify $x \in V$ with $(x) \in V^*$ for each $x \in V$. Thus, $V \leqslant V^*$. If $x \in V^*$ then there is an integer $p \geqslant 0$ and some elements $x_1, x_2, \ldots, x_p \in V$ such that $x = x_1 x_2 \ldots x_p$. We put $|x| = p$.

If V is a set and $L \leqslant V^*$ then the ordered pair (V, L) is called a language.

If $R \leqslant V^* x V^*$ then the elements of R are called rules or productions. We denote by $\xRightarrow{*}(R)$ the reflexive, transitive and stable closure of R in V^*. If $x, y \in V^*$ and $x \xRightarrow{*} y\ (R)$ then we define the norm in R of the ordered pair (x,y) which is denoted by $\|(x,y)\|_R$ to be the least nonnegative integer N such that there is $R' \leqslant R$ with the following properties:

(1) $\max\{|s|, |t|\} \leqslant N$ for each $(s,t) \in R'$;
(2) $x \xRightarrow{*} y\ (R')$.

Let V be a set, $S \leqslant V^*$, $R \leqslant V^* x V^*$ sets. Then the ordered triple $G = \langle V, S, R \rangle$ is called a generalized grammar. We denote by $\mathbf{Z}$ the class of all generalized grammars.

If $G \in \mathbf{Z}$, $G = \langle V, S, R \rangle$ then we put
$\mathscr{L}(G) = \{x;\ x \in V^*$, there is $s \in S$ such that $s \xRightarrow{*} x(R)\}$;
$(V, \mathscr{L}(G))$ is called the language generated by G.

If $G \in \mathbf{Z}$, $G = \langle V, S, R \rangle$ and the sets V, S, R are finite then G is called a grammar. We denote by $\mathbf{G}$ the class of all grammars.

We are interested in languages generated by grammars. Thus, the following problem is natural:

<u>Problem</u>. Let an arbitrary $G \in \mathbf{Z}$, $G = \langle V, S, R \rangle$ be given. Find conditions for the existence of $G' \in \mathbf{G}$, $G' = \langle V', S', R' \rangle$ such that $V' \leqslant V$, $S' \leqslant S$, $R' \leqslant R$ and that the languages generated by G and G' are equal.

If such a G' exists then, clearly, $V' = V$. This problem leads to following definitions:

Definition 1. Let $G, G' \in \mathbf{Z}$, $G = \langle V,S,R \rangle$, $G' = \langle V',S',R' \rangle$ be given. We put $G' \leqslant G$ if $V' = V$ and $S' \leqslant S$, $R' \leqslant R$; further, we put $G' \equiv G$ if $V' = V$ and $\mathscr{L}(G') = \mathscr{L}(G)$,

Definition 2. Let $\alpha : \mathbf{Z} \longrightarrow \mathbf{Z}$ be an operator assigning, to each $G \in Z$ an element $\alpha G \in Z$ such that $\alpha G \equiv G$, $\alpha G \leqslant G$. Then α is called a reducing operator.

Clearly, the operator obtained by superposition of two reducing operators is a reducing one.

We denote by Δ the class of all reducing operators provided by the operation of superposition.

Definition 3. Let us have $G \in \mathbf{Z}$ and $\alpha \in \Delta$. Then we say that G is well reducible by means of α if $\alpha G \in \mathbf{G}$. We say that $G \in \mathbf{Z}$ is well reducible if there is $\alpha \in \Delta$ such that G is well reducible by means of α.

It is easy to see that our Problem can be reformulated in the following way:

Problem: Let an arbitrary $G \in \mathbf{Z}$ be given. Find conditions for G to be well reducible, i.e. for the existence of a reducing operator $\alpha \in \Delta$ such that G is well reducible by means of α.

Definition 4. Let us have $G \in \mathbf{Z}$, $G = \langle V,S,R \rangle$. Then we put $B(S,R) = \{s;\ s \in S$ and the conditions $t \in S$, $t \overset{*}{\Longrightarrow} s$ (R) imply $|t| \geqslant |s|\}$, $\beta G = \langle V, B(S,R), R \rangle$.

Definition 5. Let us have $G \in \mathbf{Z}$, $G = \langle V,S,R \rangle$. Then for each $z \in L(G)$, we put $\|z\|^S_R = \min \{\|(s,z)\|_R;\ s \in S,\ s \Longrightarrow z(R)\}$. Further, we put
$Z(S,R) = \{(y,x);\ (y,x) \in R$ and there is $z \in \mathscr{L}(G)$ such that $\max\{|y|, |x|\} \leqslant \|z\|^S_R\}$, $\xi G = \langle V,S,Z(S,R) \rangle$.

Definition 6. We denote by Γ the monoid of all transformations of the class Z generated by the set $\{\beta, \xi\}$ provided by the operation of superposition.

Theorem 1. Γ has precisely five elements $\varepsilon = \mathrm{id}_{\mathbf{Z}}$, β, ξ, $\gamma = \beta\xi$, $\delta = \xi\beta$; the operation of superposition on Γ is given by the following table

	ε	β	ξ	γ	δ
ε	ε	β	ξ	γ	δ
β	β	β	γ	γ	δ
ξ	ξ	δ	ξ	γ	δ
γ	γ	δ	γ	γ	δ
δ	δ	δ	γ	γ	δ

<u>Theorem 2</u>. If $\xi \in \Gamma$, $G \in \mathbf{Z}$ then $\xi G \equiv G$, $\xi G \leqslant G$.

<u>Theorem 3</u>. $\Gamma \leqslant \Delta$ and the operation on Γ is the restriction of the operation on Δ.

<u>Theorem 4</u>. Let $G \in \mathbf{Z}$ be arbitrary. Then it is well reducible iff it is well reducible by means of δ.

<u>Definition 7</u>. Let us have $\xi, \eta \in \Delta$. We put $\xi \longrightarrow \eta$ if the following condition is satisfied: Each $G \in \mathbf{Z}$ which is well reducible by means of η is well reducible by means of ξ.

<u>Theorem 5</u>. (A) $\delta \longrightarrow \xi$ for each $\xi \in \Gamma$.

(B) $\xi \longrightarrow \delta$, $\xi \in \Gamma$ imply $\xi = \delta$.

<u>Definition 8</u>. Let (V, L) be a language. For $x, y \in V^*$, we put $(y, x) \in {>}(V, L)$ if $u, v \in V^*$, $uyv \in L$ imply $uxv \in L$.

<u>Theorem 6</u>. Let (V, L) be a language, $G = \langle V, S, R \rangle$ a generalized grammar generating (V, L). Then $G \leqslant \langle V, L, {>}(V, L) \rangle$.

<u>Definition 9</u>. Let (V, L) be a language, $R \leqslant V^* x V^*$ a set. Then R is called sufficient for (V, L) if there is $G = \langle V, S, R \rangle \in \mathbf{Z}$ such that $\mathcal{L}(G) = L$.

<u>Corollary</u>. Let (V, L) be a language, $R \leqslant V^* x V^*$. Then R is sufficient for (V, L) iff $R \leqslant {>}(V, L)$.

<u>Definition</u> 10. Let (V, L) be a language, R a sufficient set for (V, L). Then (V, L) is called R-bounded if $\delta \langle V, L, R \rangle \in \mathbf{G}$.

<u>Theorem 7</u>. Let (V, L) be a language, R a sufficient set for (V, L). Then the following two assertions are equivalent:

(A) (V, L) is R-bounded.
(B) There is $G \in \mathbf{G}$, $G = \langle V, S, R' \rangle$ generating (V, L) such that $R' \leqslant R$.

Theorem 8. Let (V, L) be a language, $R_1 \leqslant R_2$ sufficient sets for (V, L). If (V, L) is R_1-bounded then it is R_2-bounded.

Definition 11. A language (V, L) is called bounded[x/] if there is a sufficient set R for (V, L) such that (V, L) is R-bounded.

Theorem 9. Let (V, L) be a language. Then the following two assertions are equivalent:

(A) (V, L) is bounded.
(B) (V, L) is (V, L)-bounded.

Definition 12. A language (V, L) is called grammatizable if there is $G = \langle V, S, R \rangle \in \mathbf{G}$ generating (V, L).

Theorem 10. Let (V, L) be a language. Then the following two assertions are equivalent:

(A) (V, L) is grammatizable.
(B) (V, L) is bounded.

Definition 13. Let (V, L), (U, M) be languages. We put $(V, L) \cap (U, M) = (V \cap U, L \cap M)$.

Definition 14. A language (V, L) is called constructive if there is a grammatizable language (U, M) such that $(V, L) = (U, M) \cap (V, V^*)$.

Theorem 11. Let (V, L) be a language. Then the following two assertions are equivalent:

(A) (V, L) is constructive.
(B) There is a bounded language (U, M) such that $(V, L) = (U, M) \cap (V, V^*)$.

The definition of a grammatizable language (V, L) operates with the existence of a grammar generating this language. According to our results, the existence of such a grammar can be re-

[x/] We use the expression "bounded language" in another sense than in the book S. GINSBURG, The Mathematical Theory of Context-Free Languages. New York, 1966.

cognized on the basis of information given by (V, L) only, i.e. by interior means of the language. It is sufficient to know the relation $>(V, L)$ and $\delta\langle V, L, >(V, L)\rangle$, especially the cardinalities of the sets V, $B(L, >(V, L))$, $Z(B(L, >(V, L)), >(V, L))$ which are constructed on the basis of $>(V, L)$.

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(ACTA F. R. N. UNIV. COMEN.-MATHEMATICA, SPECIAL NUMBER, 1975)

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ON RELATIONS BETWEEN DECOMPOSITIONS OF AN ALGEBRA INTO THE SUMS OF DIRECT SYSTEMS OF SUBALGEBRAS

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In [1] the notion of the sum of a direct system of algebras and of the partition function were given; we shall use results of [1]. For this reason in this paper we consider only algebras of a fixed type and without nullary fundamental operations. If $\mathfrak{A}$ is an algebra, $\mathfrak{U}$ is a direct system of subalgebras of $\mathfrak{A}$ and $\mathfrak{A} = S(\mathfrak{U})$, we say that $\mathfrak{U}$ is a decomposition of $\mathfrak{A}$. It was proved in [1] (theorem 2) that there exists a 1-1 correspondence between the decompositions of $\mathfrak{A}$ and the partition functions of $\mathfrak{A}$. We consider relations between different decompositions of an algebra $\mathfrak{A}$, in particular we are looking for a decomposition with smallest components.

1. Let $\mathfrak{U}_1$ and $\mathfrak{U}_2$ be two decompositions of an algebra $\mathfrak{A}$. Let

$$(1)\quad \mathfrak{U}_1 = \langle I; \{X_i\}_{i \in I}, \{h_i^j\}_{i \leqslant j,\, i,j \in I} \rangle$$

$$\mathfrak{U}_2 = \langle K, \{Y_k\}_{k \in K}, \{g_k^l\}_{k \leqslant l,\, k,l \in K} \rangle .$$

We say that $\mathfrak{U}_1 = \mathfrak{U}_2$ if for all $a \in X_i$, $b \in X_j$, $b \in Y_l$, $a \in Y_k$ we have:

$$X_i = Y_k; \quad i \leqslant j \Longleftrightarrow k \leqslant l;$$

$$h_i^j(a) = b \Longleftrightarrow g_k^l(a) = b.$$

Let $\mathbb{D}(\mathfrak{A})$ be the set of all decompositions of $\mathfrak{A}$. $\mathbb{D}(\mathfrak{A})$ is not empty because the trivial direct system $\langle \{1\}, \{X_1\}, \{h_1^1(X)\} \rangle$ with the unique component $X_1 = X$ is a decomposition of $\mathfrak{A}$. For $\mathfrak{U}_1$, $\mathfrak{U}_2$ where $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are defined by (1), we define a relation $\leqslant_C$ as follows: $\mathfrak{U}_1 \leqslant_C \mathfrak{U}_2$ iff for $a \in X_i$, $a \in Y_k$ we have $X_i \subset Y_k$. Obviously, the relation $\leqslant_C$ is a quasiorder.

Lemma 1. If $\mathfrak{A}$ is a non-unary algebra, $\mathbb{U}_1$ and $\mathbb{U}_2 \in \mathbb{D}(\mathfrak{A})$ $\mathbb{U}_2$ and $\mathbb{U}_2$ are defined by (1), $\mathbb{U}_1 \leqslant_C \mathbb{U}_2$ then for all a, b,

$$(a \in X_i,\ b \in X_j,\ a \in Y_k,\ b \in Y_l) \qquad i \leqslant j \Longrightarrow k \leqslant l.$$

It means that if $\mathfrak{A}$ is not a unary algebra then the relation $\leqslant_C$ preserves not only components but also the relation $\leqslant$. Let us define in $\mathbb{D}(\mathfrak{A})$ the relation $\leqslant_n$ as follows: $\mathbb{U}_1 \leqslant_n \mathbb{U}_2$ iff for $a \in X_i$, $b \in X_j$, $a \in Y_k$, $b \in Y_l$, $g^i_k(a) = b$ implies $h^j_i(a) = b$.

Lemma 2. If $\mathfrak{A}$ is a non-unary algebra then for $\mathbb{U}_1, \mathbb{U}_2 \in \mathbb{D}(\mathfrak{A})$ we have:

$$\mathbb{U}_1 \leqslant_n \mathbb{U}_2 \Longrightarrow \mathbb{U}_1 \leqslant_C \mathbb{U}_2.$$

The meaning of the relation $\leqslant_n$ is obvious by Lemma 2. If $\mathbb{U}_1 \leqslant_n \mathbb{U}_2$ then $\mathbb{U}_2$ preserves inclusion of components, order of indices and homomorphisms of $\mathbb{U}_1$. From Lemma 1 and Lemma 2 we get:

Corollary 1. The relation $\leqslant_n$ is a partial order in $\mathbb{D}(\mathfrak{A})$. The binary term which is a p-function in $\mathfrak{A}$ will be called algebraic p-function. Such p-function is unique, see [2].

Lemma 3. If x.y is the algebraic p-function in $\mathfrak{A}$ and $\mathbb{U}_0$ is the decomposition corresponding to x.y, then $\mathbb{U}_0$ is the smallest element in the relational system $(\mathbb{D}(\mathfrak{A}); \leqslant_n)$.

Lemma 4. If x.y is an idempotent term in $\mathfrak{A}$ then we have $\leqslant_C = \leqslant_n$.

Lemma 5. If there exists an idempotent binary term x.y in $\mathfrak{A}$ and $\mathbb{U}_1$ and $\mathbb{U}_2$ are two decompositions of $\mathfrak{A}$ with the corresponding p-functions o_1 and o_2 then there exists in $\mathfrak{A}$ a decomposition $\mathbb{U}$ such that the partition of $\mathfrak{A}$ defined by $\mathbb{U}$ is exactly the intersection of the partitions defined by $\mathbb{U}_1$ and $\mathbb{U}_2$. In other words, $\mathbb{U} = \mathbb{U}_1 \cap \mathbb{U}_2$ in $(\mathbb{D}(\mathfrak{A}); \leqslant_C)$.

The relational system $(\mathbb{D}(\mathfrak{A}); \leqslant_C)$ in general is not closed under join.

Theorem 1. If there exists in $\mathfrak{A}$ the algebraic partition function then the relational system $(\mathbb{D}(\mathfrak{A}); \leqslant_C)$ is a meet semilattice with 0 and 1 which is isomorphic with some subsemilattice of the semilattice of congruences of $\mathfrak{A}$ with meet operation.

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ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
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SUBLATTICES OF THE LATTICE OF TOPOLOGIES

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By a topology $\mathcal{T}$ on a set E is meant a system $\mathcal{T}$ of subsets of T closed under finite intersections and arbitrary unions. The system $\underline{B}$ (E) of all topologies on a set E ordered by the set-inclusion forms a complete lattice. The least element of $\underline{B}$ (E) is the indiscrete topology $\{\phi, E\}$, the greatest element the discrete topology exp E. An important complete sublattice of $\underline{B}$ (E) is the lattice $\underline{K}$ (E) of all T_1-topologies on E. The least element of $\underline{K}$ (E) is the cofinite topology $K(E) = \{X \,/\, E - X \text{ is finite}\} \cup \{\phi\}$ and $\underline{K}(E) = [K(E), \exp E]$.

I. ROMAN DUDA has put the problem (see Coll. Math. XXIII (1971), 2, P 749) whether any lattice is isomorphic to a sublattice of $\underline{B}$ (E) (or even of $\underline{K}$ (E)) for a certain set E. Let m be a cardinal number. A topology is called m-resolvable if it contains m disjoint dense sets. A topology $\mathcal{T}$ on E is defined to be m-generated if $X \in \mathcal{T} \Longleftrightarrow X \cap A \in \mathcal{T}/A$ for every $A \subseteq E$ with card $A \leqslant m$.

<u>Theorem 1</u>. Let $\underline{C}$ be a class of topologies with the properties: 1° $\mathcal{T} \in \underline{C} \cap \underline{B}(E), \mathcal{T}' \in \underline{B}(E), \mathcal{T} \subseteq \mathcal{T}' \Longrightarrow \mathcal{T}' \in \underline{C}$
2° $\underline{C}$ contains an m-resolvable topology for any cardinal number m.

Then for any lattice L there exists a set E and an embedding $\psi : L \longrightarrow \underline{B}(E)$ with $\psi L \leqslant \underline{C}$.

<u>Corollary 1</u>. For any lattice L there exists a set E and an embedding $\psi : L \longrightarrow \underline{B}(E)$ such that $\psi(x)$ is a T_2-topology for any $x \in L$.

<u>Theorem 2</u>. Let $\underline{C}$ be a class of T_2-topologies such that for any lattice L there exists a set E and an embedding $\psi : L \longrightarrow \underline{B}(E)$ with $\psi L \leqslant \underline{C}$. Let m be a cardinal number. Then there exists a topology $\mathcal{T} \in \underline{C}$ which is not m-generated.

Corollary 2. There exists a lattice L with no embedding $\psi : L \longrightarrow \underline{B}(E)$ such that $\psi(x)$ is metrizible for any $x \in L$.

It is a problem whether any lattice can be represented by means of regular topologies.

II. The lattice $\underline{B}(E)$ is not modular for card $E \geqslant 3$ (see [5]) and the lattice $\underline{K}(E)$ is not modular for an infinite set E (see [1]). It arises a question whether an interval $[\mathcal{T}_1, \mathcal{T}_2]$ of $\underline{B}(E)$ is distributive resp. modular.

Theorem 3. Let E be a set, $\mathcal{T}_1, \mathcal{T}_2 \in \underline{B}(E)$ and $\mathcal{T}_1 \leqslant \mathcal{T}_2$. Let $\mathcal{T}_1/M - \mathrm{Int}_{\mathcal{T}_1}(M) = \mathcal{T}_2/M - \mathrm{Int}_{\mathcal{T}_1}(M)$ for every $M \in \mathcal{T}_2$. Then $[\mathcal{T}_1, \mathcal{T}_2]$ is a distributive lattice.

Any finite distributive lattice is isomorphic to an interval of $\underline{K}(E)$ for some E (see [7]).

Corollary 3: Let L be a finite lattice. Then L is distributive iff it is isomorphic to an interval of $\underline{K}(E)$ for some E.

The following theorem shows that the condition from Theorem 3 is not necessary.

Theorem 4. Let $\mathcal{T} \in \underline{B}(E)$. The following conditions are equivalent:

(i) $(\mathcal{T}]$ is modular
(ii) $(\mathcal{T}]$ is distributive
(iii) $\mathcal{T}$ is a chain in exp E or $\mathcal{T} = \{\phi, X, E - X, E\}$ for $X \leqslant E$, $\phi \neq X \neq E$.

Theorem 5. Let $\mathcal{T} \in \underline{B}(E)$. The following conditions are equivalent:

(i) $[\mathcal{T})$ is modular
(ii) $[\mathcal{T})$ is distributive
(iii) $\mathcal{T}/M - \mathrm{Int}_{\mathcal{T}}(M)$ is discrete for every $M \leqslant E$.

A.K. STEINER has proved that the lattice $\underline{B}(E)$ is complemented (see [5]) and $\underline{K}(E)$ not (see [6]). A topology is called submaximal if any dense set is open.

Theorem 6. Let $\mathcal{T} \in \underline{K}(E)$. The following conditions are equivalent:

(i) $[\mathcal{T})$ is complemented
(ii) $[\mathcal{T})$ is a complete atomic Boolean algebra
(iii) $\mathcal{T}$ is submaximal and for any infinite system $\mathfrak{a} \leqslant \exp E$ of mutually disjoint sets it holds $\bigcap_{A \in \mathfrak{a}} \mathrm{Cl}_{T}(A) = \phi$.

Theorem 7. Let $\mathcal{T} \in \underline{K}(E)$. The following conditions are equivalent:

(i) $[\underline{K}(E), \mathcal{T}]$ is complemented

(ii) $[\underline{K}(E), \mathcal{T}]$ is a finite Boolean algebra.

(iii) There exists $F \leq E$ with $\mathcal{T} = \underline{K}(E) \cup \exp F$.

The given results can be found in [2], [3] and [4].

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(ACTA F. R. N. UNIV. COMEN.-MATHEMATICA, SPECIAL NUMBER, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA – SPECIAL NUMBER – 1975

BEITRAG ZUM PROBLEM DER INNEREN DIREKTEN PRODUKTE VON GRUPPOIDEN

LADISLAV SEDLÁČEK, Olomouc

In diesem Beitrag werden nur ganz kurz einige Fragen aus der erwähnten Problematik behandelt. Dabei werden manche Begriffe und Ergebnisse aus der Arbeit [1] benutzt, besonders der Begriff des relativ konvexen und des konvexen Gruppoides. Es wird die Verwendbarkeit dieser Begriffe in der erwähnten Theorie gezeigt. Vorher es ist aber nötig die nowendigsten Begriffe und derer Eigenschaften aus [1] anzuführen.

D* 1. Es sei α eine natürliche Zahl, $\mathcal{A}_1, \dots, \mathcal{A}_\alpha$, $\mathcal{A}$ Gruppoide. Gibt es eine isomorphe Abbildung d des Gruppoides $\mathcal{A}$ auf das Gruppoid $\mathcal{A}_1 \times \dots \times \mathcal{A}_\alpha = \mathcal{A}^{[\alpha]}$, sagen wir, daß das Gruppoid $\mathcal{A}$ das direkte Produkt von Gruppoiden $\mathcal{A}_1, \dots, \mathcal{A}_\alpha$ ist und wir schreiben auch $\mathcal{A} \longrightarrow \mathcal{A}_1 \times \dots \times \mathcal{A}_\alpha$ oder kürzer $\mathcal{A} \longleftrightarrow \mathcal{A}^{[\alpha]}$. Wenn für $a \in \mathcal{A}$, $(a_1, \dots a_\alpha) \in \mathcal{A}^{[\alpha]}$ $d(a) = (a_1, \dots, a_\iota, \dots, a_\alpha)$ gilt, schreiben wir auch $a \longleftrightarrow (a_1, \dots, a_\iota, \dots, a_\alpha)$. Das Element a_ι $(\iota = 1, \dots, \alpha)$ heißt die ι-te Komponente des Elements a oder auch die Projektion von a in $\mathcal{A}_\iota$.

D* 2. Ein Untergruppoid $\mathcal{S}$ des Gruppoids $\mathcal{G} \longleftrightarrow \mathcal{A}^{[\alpha]}$ heißt in bezug auf $\mathcal{A}^{[\alpha]}$ konvex, wenn es die folgende Eigenschaft besitzt: Seien $a, b \in \mathcal{S}$, $(a_1, \dots, a_\alpha)$, $(b_1, \dots, b_\alpha) \in \mathcal{A}^{[\alpha]}$, $a \longleftrightarrow (a_1, \dots, a_\alpha)$, $b \longrightarrow (b_1, \dots b_\alpha)$ und x_ι ein beliebiges der Elemente a_ι, $b_\iota \in \mathcal{A}_\iota$ $(\iota = 1, \dots, \alpha)$, $x \longleftrightarrow (x_1, \dots, x_\alpha)$. Dann gehört auch das Element x in $\mathcal{S}$.
Ist $\mathcal{S}$ in bezug auf jedes direkte Produkt $\mathcal{B}^{[\beta]} = \mathcal{B}_1 \times \dots \times \mathcal{B}_\beta \longleftrightarrow \mathcal{G}$ konvex, so sagt man, daß $\mathcal{S}$ ein konvexes Untergruppoid in $\mathcal{G}$ ist.

S* 1. Es seien $\mathcal{G} \longrightarrow \mathcal{A}_1 \times \dots \times \mathcal{A}_\alpha = \mathcal{A}^{[\alpha]}$, $\mathcal{G} \longrightarrow \mathcal{B}_1 \times \dots \times \mathcal{B}_\beta = \mathcal{B}^{[\beta]}$, e das idempotente Element aus $\mathcal{G}$, $e \longleftrightarrow (e_1, \dots, e_\alpha) \in \mathcal{A}^{[\alpha]}$, $e \longleftrightarrow (e'_1, \dots, e'_\beta) \in \mathcal{B}^{[\beta]}$, $\mathcal{A}^0_\iota \longleftrightarrow (\{e_1\} \times \dots \times \mathcal{A}_\iota \times \dots \times \{e_\alpha\}$, $\mathcal{B}^0_\varkappa \leftrightarrow (\{e'_1\} \times \dots \times \mathcal{B}_\varkappa \times \dots \times \{e'\})$.
a) Es sei $\mathcal{A}^*_{\iota\varkappa}$ die Projektion von $\mathcal{A}^0_\iota$ in $\mathcal{B}_\varkappa$ und $\mathcal{B}^*_{\varkappa\iota}$ die Projektion von $\mathcal{B}^0_\varkappa$ in $\mathcal{A}_\iota$ für $\iota = 1, \dots, \alpha$, $\varkappa = 1, \dots, \beta$. Dann gilt

1. $\mathcal{M}_\iota^0$ ist in bezug auf $\mathcal{L}^{[\beta]}$ und gleichzeitig $\mathcal{L}_\varkappa^0$ in bezug auf $\mathcal{M}^{[\alpha]}$ dann und nur dann konvex, wenn

2. $\mathcal{M}_\iota \longleftrightarrow \mathcal{L}_{1\iota}^* \times \ldots \times \mathcal{L}_{\beta\iota}^*$ und gleichzeitig $\mathcal{L}_\varkappa \longleftrightarrow \mathcal{M}_{1\varkappa}^* \times \ldots \times \mathcal{M}_{\alpha\varkappa}^*$.

b) Ist $\mathcal{M}_\iota^0$ in bezug auf $\mathcal{L}^{[\beta]}$ und $\mathcal{L}_\varkappa^0$ in bezug auf $\mathcal{L}^{[\beta]}$ konvex für $\iota = 1, \ldots, \alpha$, $\varkappa = 1, \ldots, \beta$, besitzen die betrachteten direkten Produkte isomorphe Verfeinerungen und das Gruppoid $\mathcal{G}$ ist das direkte Produkt von Gruppoiden $\mathcal{L}_{\iota\varkappa}$, die mit $\mathcal{M}_\iota^0 \cap \mathcal{L}_\varkappa^0$ isomorph sind.

Jetzt wollen wir schon zur Frage der inneren direkten Produkte übergehen.

D 1. Es seien $\mathcal{R}_\iota$ Untergruppoide des Gruppoides $\mathcal{G}$ ($\alpha \geqq 2$ eine natürliche Zahl, $\iota = 1, \ldots, \alpha$). Es sei $\mathcal{G}$ das direkte Produkt von diesen Gruppoiden, dh. es sei $\mathcal{G} \longleftrightarrow \mathcal{R}_1 \times \ldots \times \mathcal{R}_\alpha = \mathcal{R}^{[\alpha]}$, $r \in \mathcal{G}$, $r_\iota \in \mathcal{R}_\iota$. Ist die isomorphe Abbildung d von $\mathcal{G}$ auf $\mathcal{R}^{[\alpha]}$ durch die Vorschrift $d(r) = (r_1, \ldots, r_\alpha)$ gegeben dann und nur dann, wenn $r = r_1 r_2 \ldots r_\alpha$, so sagt man, daß $\mathcal{G}$ das innere direkte Produkt seiner Untergruppoide $\mathcal{R}_\iota$ ist und man schreibt auch $\mathcal{G} = \mathcal{R}_1 \times \ldots \times \mathcal{R}_\alpha = \mathcal{R}^{[\alpha]}$.

Mit Rücksicht auf D 1 folgt unmittelbar aus [1].

S 1. $\mathcal{Z}$ sei das Zentrum der Halbgruppe $\mathcal{G} = \mathcal{R}_1 \times \ldots \times \mathcal{R}_\alpha$ und $\mathcal{Z}_\iota$ das Zentrum in $\mathcal{R}_\iota$ ($\iota = 1, \ldots, \alpha$). Dann ist $\mathcal{Z}$ in $\mathcal{G}$ konvex.

Folgerung 1. $\mathcal{Z}$ ist das innere direkte Produkt der Zentren $\mathcal{Z}_\iota$, dh. $\mathcal{Z} = \mathcal{Z}_1 \times \ldots \times \mathcal{Z}_\alpha$.

Weiter konnen wir mit Rucksicht auf D 1 die folgenden Sätze beweisen.

S 2. $\mathcal{G}$ sei eine Gruppe ohne nichtriviales Zentrum und $\mathcal{G} = \mathcal{R}_1 \times \ldots \times \mathcal{R}_\alpha = \mathcal{R}^{[\alpha]}$, $\mathcal{G} = \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_\beta = \mathcal{Y}^{[\beta]}$. Dann ist $\mathcal{R}_\iota$ in bezug auf $\mathcal{Y}^{[\beta]}$ und $\mathcal{Y}_\varkappa$ ist in bezug auf $\mathcal{R}^\alpha$ für $\iota = 1, \ldots, \alpha$, $\varkappa = 1, \ldots, \beta$ konvex.

Folgerung 2. Die betrachteten inneren direkten Produkte besitzen isomorphe Verfeinerungen und $\mathcal{G}$ ist das innere direkte Produkt genau aller Untergruppen $\mathcal{L}_{\iota\varkappa} = \mathcal{R}_\iota \cap \mathcal{Y}_\varkappa$.

S 3. Es sei $\mathcal{L}$ die Kommutatorgruppe in der Gruppe $\mathcal{G} = \mathcal{R}_1 \times \ldots \times \mathcal{R}_\alpha$ und $\mathcal{L}_\iota$ die Kommutatorgruppe in $\mathcal{R}_\iota$ ($\iota = 1, \ldots, \alpha$). Dann ist $\mathcal{L}$ in $\mathcal{G}$ konvex.

Folgerung 3. $\mathcal{L}$ ist das innere direkte Produkt von $\mathcal{L}_\iota$, dh. $\mathcal{L} = \mathcal{L}_1 \times \ldots \times \mathcal{L}_\alpha$.

Weiter könnten wir zum Problem der direkt zerlegbaren Gruppen und des Isomorphismus derer Zerlegungen, der mit dem Begriff des vollständigen Systems von Homomorphismen zusammenhängt, übergehen und noch manche andere Fragen lösen. Diese Problematik soll in meiner Arbeit, "Zum Problem der inneren direkten Produkte von Gruppoiden" in ACTA Un. Pal. 01., F.R.N.-Mathematica im Jahre 1973 erscheinen. Darum wird auch nicht die Literatur angegeben.

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(ACTA F. R. N. UNIV. COMEN.-MATHEMATICA, SPECIAL NUMBER, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA – SPECIAL NUMBER – 1975

DISTRIBUTIVE SEMILATTICES AND BOOLEAN LATTICES

J. C. VARLET

The notions of filter and ideal can be defined in a $\wedge$-semilattice (henceforth semilattice) as follows.

A filter of a semilattice S is a non-empty subset F of S such that $x \wedge y \in F$ iff $x \in F$ and $y \in F$. An ideal I of a semilattice S is a nonempty subset of S such that

(I_1) $y \leqslant x$ and $x \in I$ imply $y \in I$;

(I_2) for any $x, y \in I$, there exists $z \in I$ such that $z \geqslant x$ and $z \geqslant y$.

A filter F of S is prime if, whenever two filters F_1 and F_2 are such that $\phi \neq F_1 \cap F_2 \subseteq F$, then F_1 or F_2 belongs to F.

A filter F is maximal if the only filter strictly containing F is S.

The concept of distributive semilattice is due to G. Grätzer and E. T. Schmidt. A semilattice is distributive if $c \geqslant a \wedge b$ $(a, b, c \in S)$ implies the existence of $a_1, b_1 \in S$ such that $a_1 \geqslant a$, $b_1 \geqslant b$ and $a_1 \wedge b_1 = c$. In 1968 we introduced the notion of O-distributive lattice, in order to generalize that of pseudo-complemented lattice. As a matter of fact the concept of O-distributivity applies to semilattices bounded below. A semilattice S with least element O will be said O-distributive if, for any $a \in S$, the subset $I = \{x \in S : x \wedge a = 0\}$ is an ideal. It is easy to prove that a distributive semilattice with O is O-distributive.

A semilattice S with O is weakly complemented if for any pair a, b of distinct elements of S, there exists an element c disjoint from one of these elements but not from the other (or, equivalently, there is a maximal filter containing one of them but not the other).

Finally we make use of another notion introduced by G. Grätzer and E. T. Schmidt: a lattice with O is very weakly complemented if the zero ideal is the kernel of a unique congruence, the identity.

First the classical definition of a Boolean lattice can be improved, as shown by

Theorem 1. A bounded distributive semilattice is a Boolean lattice iff it is complemented.

The four steps in the proof of the "if" part are:

1/ the complement a' of any element a of S is unique;
2/ complementation is order-reversing: $a \geqslant b \Longrightarrow a' \leqslant b'$;
3/ S is a lattice: the l.u.b. of any two elements a and b is $(a' \wedge b')'$;
4/ since S is distributive as a semilattice, it is distributive as a lattice.

Then the meaning of O-distributivity in an up-directed semilattice is clarified by

Theorem 2. An up-directed semilattice with O is O-distributive iff any maximal filter is prime.

Stone characterized distributive lattices by means of the following separation property: a lattice is distributive iff when a filter F and an ideal I are disjoint, there exists a prime filter containing F and disjoint from I. This result can be generalized to semilattices as follows.

Theorem 3. An up-directed semilattice is distributive iff for any filter F and any ideal I such that $F \cap I = \phi$, there exists a prime filter containing F and disjoint from I.

While the necessity of the condition has already been mentioned in the literature, its sufficiency seems to be established for the first time and in fact we only use the weaker assumption: a filter and an element not belonging to it can be separated by a prime filter. Hence the question arises whether the still weaker condition: any two distinct elements can be separated by a prime filter, would suffice to ensure distributivity. We were unable to answer this question and were obliged to deviate from semilattices to lattices in our last theorem.

Theorem 4. A bounded very weakly complemented lattice L is Boolean if prime and maximal filters of L coincide.

This theorem is to be compared with Nachbin's classical result: a bounded distributive lattice is Boolean iff every prime filter is maximal.

The operation $\wedge$ we have dealt with so far is idempotent, associative and commutative. We deliberately abandon these restrictions and consider a partially order groupoid, i.e. a p.o.

set G with a binary multiplication satisfying the isotonicity condition

$$a \leqslant b \Longrightarrow ax \leqslant bx \text{ and } xa \leqslant xb \quad \text{for every } a,b,x \in G.$$

Distributivity is then defined by

$$z \geqslant xy \Longrightarrow \exists x_1 y_1 \in G : x_1 \geqslant x; \; y_1 \geqslant y; \; x_1 y_1 = z,$$

a condition which can take the concise form

$$[x) \; [y) = [xy) \quad \text{for every } x,y \in G$$

where $[x) = \{g : g \in G, \; g \geqslant x\}$ and the multiplication in the first member has to be interpreted as in the calculus of complexes.

Among the numerous examples of distributive p. o. groupoids let us just mention the following ones:
the negative cone of any p. o. group having the Riesz Interpolation Property (in particular, any lattice-ordered group) is a distributive p. o. semigroup; the real numbers in $[0,1]$ under the usual multiplication or under the operation $a \circ b = a + b - ab$ form a distributive integral (i. e. with a neutral element which is at the same time the greatest element) totally ordered semigroup.

A sample of the various properties of distributive p. o. groupoids is: in a distributive integral p. o. groupoid, if b and c are both coprime to a, then so is bc.

As in the case of lattices and semilattices but under a slight supplementary hypothesis, the distributive character of the distributive groupoid is reflected upon its filter lattice.

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(ACTA F. R. N. UNIV. COMEN.-MATHEMATICA, SPECIAL NUMBER, 1975)
**ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA – SPECIAL NUMBER – 1975**

A REGULAR VARIETY OF TYPE «2,2,1,1,0,0»

J. C. VARLET

An algebra $\mathcal{A} = \langle A; F \rangle$ is regular if any two congruences of $\mathcal{A}$ are necessarily equal when they have a class in common. A variety is named regular if all algebras belonging to it are regular. The nicest examples of regular varieties are the varieties of quasi-groups, groups, rings, modules and Boolean algebras; among these no one has the type indicated in the title. We fill up the gap by the variety of double pseudo-complemented lattices satisfying the condition

(M) if $a^* = b^*$ and $a^+ = b^+$, then $a = b$.

The binary operations are the lattice operations $\vee$ and $\wedge$, the unary ones are pseudo-complementation $(*)$ and its dual $(^+)$ while the distinguished elements are the least and greatest elements of the lattice, 0 and 1 respectively. But let us make this clear.

A pseudo-complemented lattice is an algebra $\mathcal{L} = \langle L; \vee, \wedge, *, 0, 1 \rangle$ satisfying the following axioms:

(1) $\langle L; \vee, \wedge \rangle$ is a lattice;
(2) for any $a \in L$, $0 \vee a = 1 \wedge a = a$;
(3) for any $a \in L$, there exists $a^* \in L$ such that $a \wedge x = 0$ iff $x \leqslant a^*$, i.e. any element a has the pseudo-complement a^*.

When (3) is replaced by
(3') for any $a \in L$, there exists $a^+ \in L$ such that $a \vee x = 1$ iff $x \geqslant a^+$, S is said to be dually pseudo-complemented.

When L at the same time satisfies (1), (2), (3) and (3'), it will be named double pseudo-complemented.

A Stone algebra $\mathcal{S} = \langle S; \vee, \wedge, *, 0, 1 \rangle$ is a distributive pseudo-complemented lattice satisfying the additional axiom
(4) for any $a \in S$, $a^* \vee a^{**} = 1$.

A Stone algebra satisfying (3') and

(4') for any $a \in S$, $a^+ \wedge a^{++} = 0$,
is a double Stone algebra.

We point out that pseudo-complemented lattices as well as Stone algebras are equationally definable. The three-valued Lukasięwicz algebras, introduced and deeply investigated by G. Moisil, are the double Stone algebras which satisfy the condition (M).

In a pseudo-complemented lattice $\mathscr{L}$, an element a is dense if $a^* = 0$. The dense set $D(L)$ is the filter of dense elements of L. The definitions of dually dense element and dual dense set (denoted by $\bar{D}(L)$) are obvious.

In any pseudo-complemented lattice $\mathscr{L}$, the relation $\sim_1$ defined by

$$a \sim_1 b \quad \text{iff} \quad a^* = b^*$$

is a congruence, that is an equivalence relation preserving the operations $\vee$, $\wedge$, $*$. Its kernel is $\{0\}$ while its antikernel, i.e. $[1] \sim_1$, is $D(L)$. Similarly, in any dually pseudo-complemented lattice $\mathscr{L}$, the relation $\sim_2$ defined by

$$a \sim_2 b \quad \text{iff} \quad a^+ = b^+$$

is a congruence with kernel $\bar{D}(L)$ and antikernel $\{1\}$. Not so trivial is the fact that in any double pseudo-complemented lattice $\sim \; = \; \sim_1 \wedge \sim_2$ (i.e., $a \sim b$ if $a^* = b^*$ and $a^+ = b^+$ is also a congruence. The axiom (M) can thus be rephrased as follows: $\sim = \omega$, the equality relation.

In a pseudo-complemented lattice, $\sim_1 = \omega$ iff the lattice is Boolean. This implies that a pseudo-complemented lattice is a regular algebra iff it is Boolean. On the contrary, a double pseudo-complemented lattice can be a regular algebra without being Boolean. In fact we have

Theorem 1. The double pseudo-complemented lattice $\mathscr{L} = \langle L; \vee, \wedge, *, +, 0, 1 \rangle$ is a regular algebra iff $\sim = \omega$.

Under the assumption of distributivity we can provide two other forms of this last condition.

Theorem 2. In a distributive double pseudo-complemented lattice $\mathscr{L}$, the following conditions are equivalent:

(i) $\sim = \omega$;
(ii) any chain of prime ideals has at most two elements;

(ii') any chain of prime filters has at most two elements;
(iii) any dense element of L is an upper bound of $D(L)$.

The kernel I of any congruence of a pseudo-complemented lattice enjoys the following property: the ideal I contains a^{**} whenever it contains a. Such ideals will be called $**$-closed ideals.

We now introduce three derived binary operations:

$$p_1(x,y) = (x^* \wedge y^{**}) \vee (x^{**} \wedge y^*),$$
$$p_2(x,y) = (x^+ \wedge y^{++}) \vee (x^{++} \wedge y^+),$$
$$p(x,y) = p_1(x,y) \vee p_2(x,y).$$

Theorem 3. In a double Stone algebra $\mathcal{S} = \langle S; \vee, \wedge, *, +, 0, 1 \rangle$, for any $**$-closed ideal I the binary relation Θ_I defined by $a \equiv b(\Theta_I)$ iff $p(a,b) \in I$ is a congruence.

Finally the correspondence between the congruences and the $**$-closed ideals is made clearer by

Theorem 4. In a distributive double pseudo-complemented lattice $\mathcal{L} = \langle L; \vee, \wedge, *, +, 0, 1 \rangle$, there is a bijection between the congruences and the $**$-closed ideals iff $\mathcal{L}$ is a three-valued Lukasiewicz algebra.

Theorems 3 and 4 enable us to claim that in a three-valued Lukasiewicz algebra $p(x,y)$ determines the form of any congruence, generalizing the well-known fact (due to J. Slomiński) that any Boolean algebra is a regular algebra in which, for every congruence Θ, $x \equiv y\ (\Theta)$ iff $(x \wedge y') \vee (x' \wedge y) \equiv 0(\Theta)$. In fact, when $x^* = x^+$ for an element x, clearly $x^* = x'$. Then $p(x,y)$ takes the preceding form and any ideal is $**$-closed.

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(ACTA F. R. N. UNIV. COMEN.-MATHEMATICA, SPECIAL NUMBER, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE MATHEMATICA – SPECIAL NUMBER – 1975

ON EQUATIONALLY DEFINABLE CLASSES OF QUASI-ALGEBRAS

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Let G be a set such that for every element $g \in G$ is subordinated an ordinal number $n(g)$, called arrity of an element g. Quasi-algebra of type G is called a sequence $\underline{A} = \langle A, (g_{\underline{A}}, g \in G)\rangle$, where A is a set, called support of $\underline{A}$, and $g_{\underline{A}}$ for every $g \in G$, $n(g)$-ary quasi-operation determined on a set A, that is a function $g_{\underline{A}} : D(g_{\underline{A}}) \longrightarrow A$, where $D(g_{\underline{A}}) \subseteq A^{n(g)}$. The set $D(g_{\underline{A}})$ is called a domain of quasi-operation $g_{\underline{A}}$. A quasi-operation $g_{\underline{A}}$ is called an operation, if $D(g_{\underline{A}}) = A^{n(g)}$. If for every $g \in G$ a quasi-operation $g_{\underline{A}}$ is an operation, then a quasi-algebra $\underline{A}$ of type G is an algebra $\underline{A}$ of type G.

Let $\underline{P^{\alpha}(G)} = \langle P^{\alpha}(G), (g_{\underline{P^{\alpha}(G)}}, g \in G)\rangle$ be Peano-algebra of type G generated by set $X = \{x_{\lambda}, \lambda < \alpha\}$, i.e.:

1. $X \subseteq P^{\alpha}(G)$,
2. the elements of X are not values of the operations $g_{\underline{P^{\alpha}(G)}}$, $g \in G$, for elements of $P^{\alpha}(G)$,
3. for all $g, g' \in G$ and all sequences $(w_{\sigma}, \sigma < n(g)) \in P^{\alpha}(G)^{n(g)}$ and $(w'_{\sigma}, \sigma < n(g')) \in P^{\alpha}(G)^{n(g')}$, if $g_{\underline{P^{\alpha}(G)}}(w_{\sigma}, \sigma < n(g)) = g'_{\underline{P^{\alpha}(G)}}(w'_{\sigma}, \sigma < n(g'))$, then $g = g'$ and $w_{\sigma} = w'_{\sigma}$ for $\sigma < n(g)$,
4. the set X generates the algebra $\underline{P^{\alpha}(G)}$.

The elements of the set X we will call the variables and the elements of the set $P^{\alpha}(G)$ are called the terms.

Let $\underline{A}$ be a quasi-algebra of type G. Every term $p \in P^{\alpha}(G)$ induces in quasi-algebra $\underline{A}$ α-ary quasi-operation ${}_{\underline{A}}p$ defined in the following way:

1. an element $\underline{a} = (a_{\lambda}, \lambda < \alpha) \in A^{\alpha}$ belongs to $D({}_{\underline{A}}p)$ iff $p \in D(\bar{\varphi})$, where $\varphi : X \longrightarrow A$ is such mapping that $\varphi(x_{\lambda}) = a_{\lambda}$ for $\lambda < \alpha$, and $\bar{\varphi}$ denotes a partial homomorphism of Peano-algebra $\underline{P^{\alpha}(G)}$ in quasi-algebra $\underline{A}$ induced by φ;
2. ${}_{\underline{A}}p(\underline{a}) = \bar{\varphi}(p)$ for $\underline{a} \in D({}_{\underline{A}}p)$.

The pairs of terms $\langle p, q\rangle$, $p, q \in P^{\alpha}(G)$, are called the equations. Farther on we will write $\ulcorner p = q \urcorner$ instead of $\langle p, q\rangle$.

In the case of quasi-algebras there are posible three different kinds of validity of equations (J. Słomiński - Peano-algebras and quasi-algebras, Dissertationes Math. 57 (1968)).

Definition 1. Equation $\ulcorner p = q \urcorner$ is said to be weakly valid in the quasi-algebra $\underline{A}$ if for all sequences $\underline{a} = (a_\lambda, \lambda < \alpha)$ in A^α we have $_{\underline{A}}p(\underline{a}) = {}_{\underline{A}}q(\underline{a})$ provided $\underline{a} \in D(_{\underline{A}}p) \cap D(_{\underline{A}}q)$.

Definition 2. Equation $\ulcorner p = q \urcorner$ is said to be valid in the quasi-algebra $\underline{A}$ if $D(_{\underline{A}}p) = D(_{\underline{A}}q)$ and $_{\underline{A}}p(\underline{a}) = {}_{\underline{A}}q(\underline{a})$ for each sequence $\underline{a} \in D(_{\underline{A}}p) = D(_{\underline{A}}q)$.

Definition 3. Equation $\ulcorner p = q \urcorner$ is said to be strongly valid in the quasi-algebra $\underline{A}$ if we have $p \sim_{\underline{a}} q$ for each sequence $\underline{a} \in A^\alpha$, where $\sim_{\underline{a}}$ denotes the least congruence relation $\sim$ of the Peano-algebra $P^\alpha(G)$ such that for all $r, s \in P^\alpha(G)$ we have:

$$r \sim s \text{ if } \underline{a} \in D(_{\underline{A}}r) \cap D(_{\underline{A}}s) \text{ and } {}_{\underline{A}}r(\underline{a}) = {}_{\underline{A}}s(\underline{a}).$$

Every equation strongly valid in quasi-algebra $\underline{A}$ is in this quasi-algebra valid. Every equation valid in quasi-algebra $\underline{A}$ is weakly valid in $\underline{A}$. In the case of algebras all three kinds of validity are equivalent, but in the case of quasi-algebras they are different, which is proved by the following examples. Let $\underline{Ad}$ be a discrete quasi-algebra of type G (i. e. for every $g \in G$ $D(g_{\underline{Ad}}) = \emptyset$) such that $G \neq \emptyset$ and $Ad = \{a\}$. Then the equation $\ulcorner p = q \urcorner$, where $p \neq q$ and p, q are not variables, is valid, but it is not strongly valid in quasi-algebra $\underline{Ad}$. Whereas the equation $\ulcorner x = p \urcorner$, where x is a variable and p is a term different than a variable, is weakly valid but not valid in $\underline{Ad}$.

Let E be any set of equations of type G. Denote by $Cn_*(E)$ the least congruence relation of Peano-algebra $\underline{P^\alpha(G)}$ containing the set E, and by $Cn(E)$ - the least full invariant congruence relation of $\underline{P^\alpha(G)}$ containing E. By (weak, strong) E-quasi-algebra we will understand such a quasi-algebra in which all equations from the set E are (weakly, strongly) valid. Farther on denote by $wG(E)$, $G(E)$, $sG(E)$ and $G^*(E)$ the classes of all weak E-quasi-algebras, E-quasi-algebras, strong E-quasi-algebras and E-algebras, respectively.

The question arises: for which sets E the following relations take place (the subject commenced by J. Słomiński - above):

1. $G^*(E) = sG(Cn(E))$,
2. $G^*(E) = G(Cn(E))$,
3. $G^*(E) = wG(Cn(E))$,
4. $G^*(E) = sG(Cn_*(E))$,
5. $G^*(E) = G(Cn_*(E))$,
6. $G^*(E) = wG(Cn_*(E))$,
7. $G^*(E) = sG(E)$,
8. $G^*(E) = G(E)$,
9. $G^*(E) = wG(E)$,
10. $sG(Cn(E)) = G(Cn(E))$,

. . . .

45. $G(E) = wG(E)$.

Examples of theorems:

<u>Theorem 1</u>. For any set E $sG(Cn_*(E)) = sG(E)$ and $G(Cn_*(E)) = G(E)$.

<u>Theorem 2</u>. There is no set of equations E for which the equations 3, 6 and 9 would take place.

<u>Theorem 3</u>. If equations 2, 5 or 8 take place, then the sets $Cn(E)$, $Cn_*(E)$ or E recpectively contain the equation $\ulcorner x = p \urcorner$, where x is a variable.

<u>Theorem 4</u>. Let E be an arbitrary set of equations of type G. Then each weak E-quasi-algebra $\underline{A}$, where $\overline{\overline{A}} \geqslant 2$, is E-quasi-algebra (problem 45) if and only if the set E is empty, or E contains only equations of the form $\ulcorner p = p \urcorner$.

<u>Theorem 5</u>. Each weak E-quasi-algebra is a $Cn(E)$-quasi-algebra if and only if the set E of equations is empty or contains only equations of the form $\ulcorner p = p \urcorner$.

<u>Theorem 6</u>. The sufficient condition for every E-quasi-algebra to be $Cn(E)$-quasi-algebra is that the set E should contain only such equations $\ulcorner p = q \urcorner$ in which the terms p and q have the same variables.

It seems that these results may find the application in the investigations connected with the formation of axiomatic systems for certain classes of quasi-algebras of the fixed type G.

(ACTA F. R. N. UNIV. COMEN.-MATHEMATICA, SPECIAL NUMBER, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE MATHEMATICA - SPECIAL NUMBER - 1975

CAYLEY COLOR GRAPHS OF GROUPS AND THEIR GENERALISATIONS

BOHDAN ZELINKA

1. Introduction

In this lecture we use the graph-teoretical terminology of ORE [2] and the translation of the algebraic terminology of BELOUSOV [1]. The term "loop" will be used here in two quite different senses: in the algebraic sense (a quasigroup with a two-side unit element) and in the graph-theoretical sense (an edge joining a vertex with this vertex itself). For avoiding misunderstandings due to this homonymy, after the word "loop" we shall always put either "a.s." (algebraic sense), or "g.s." (graphtheoretical sense) in brackets.

Proofs are omitted; they can be found in [4] and [5].

One of the earliest problems which were met at studying interrelations between the graph theory and the abstract algebra was the problem to characterize automorphism groups of graphs. The results of CAYLEY and FRUCHT (quoted in [2]) showed that every at most countable group is isomorphic with the group of all automorphisms of some graph.

To any group H we can assign the so-called Cayley color graph C(H). Its vertices are the elements of H. The edges of C(H) are directed and colored so that there exists a one-to-one correspondence between the set of colors of edges and the set of elements of H. If $x \in H$, $y \in H$, then in C(H) a directed edge of the color corresponding to y goes from x into xy. It can be proved that the group of all automorphisms of C(H) which preserve colors of edges is isomorphic with H. But the group of all automorphisms of G is a symmetric group, because C(H) is a complete digraph with loops (g.s.). Therefore the Cayley color graph is not a complete solution of the problem.

If H is at most countable, we adapt C(H) to obtain the so-called Frucht graph F(H) of H. We number the elements of H by positive integers (quite arbitrarily). For each positive integer n by G_n we denote the graph on Fig. 1. In C(H) we substitute any edge of the color corresponding to the n-th element of H by G_n so that the initial vertex of this edge is identified with the vertex a of G_n and the terminal one with b. The graph F(H) thus obtained has the automorphism group isomorphic with H.

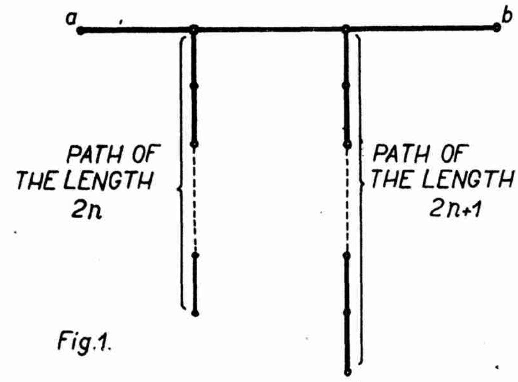

Fig.1.

We shall study the above defined concepts in the case when H is not a group, but a quasigroup, a loop (a.s.) or a semigroup.

2. Quasigroups and loops (a.s).

A quasigroup is a groupoid in which the equations $ax = b$ and $ya = b$ have unique solutions x and y for any a and b.

A loop (a.s.) is a quasigroup with a two-side unit element.

An isotopy of a quasigroup Q_1 onto a quasigroup Q_2 is an ordered triple $\langle \alpha, \beta, \gamma \rangle$ of one-to-one mappings of Q_1 onto Q_2 such that for any three elements x, y, z of Q_1 the equality $\alpha(x).\beta(y) = \gamma(z)$ in Q_2 is equivalent to the equality $xy = z$ in Q_1. An isotopy of a quasigroup Q onto itself is called an autotopy of Q.

Let G_1 and G_2 be two digraphs whose edges are colored by some way. A color-preserving isotopy of G_1 onto G_2 is an ordered triple $\langle f_1, f_2, \varphi \rangle$, where f_1 and f_2 are one-to-one mappings of the wertex set V_1 of G_1 onto the vertex set V_2 of G_2 and φ is a one-to-one mapping of the set of colors of edges of G_1 onto the set of colors of edges of G_2 such that for any two vertices u, v of G_1 the existence of the edge $\overrightarrow{uv}$ in G_1 is equivalent to the existence of the edge $\overrightarrow{f_1(u)f_2(v)}$ in G_2 and if $\overrightarrow{uv}$ in G_1 exists and has the color c, then $\overrightarrow{f_1(u)f_2(v)}$ in G_2 has the color $\varphi(c)$. (Compare [3]).

The Cayley color graph C(Q) can be considered as an ordered pair $\langle F, \xi \rangle$, where F is a decomposition of the complete digraph $\vec{K}_n$ with n vertices with loops (g.s.) into edge-disjoint linear factors and ξ is a one-to-one mapping of the vertex set of $\vec{K}_n$ onto the set of factors of F (as well-known, if $\vec{K}_n$ is decomposed into linear factors, the number of these factors is exactly n). The decomposition F corresponds here to the coloring of edges and ξ to the correspondence between elements of Q and the colors of edges. (From the definition of the quasigroup it follows that C(Q) is again a complete digraph with loops (g.s.).)

Now we shall introduce some theorems. Their proofs can be found in [4].

Theorem 1. Let $\vec{K}_n$ be the complete graph with n vertices with loops (g.s.), where n is a finite or infinite cardinal number. Any ordered pair $\langle F, \xi \rangle$, where F is a decomposition of $\vec{K}_n$ into edge-disjoint linear factors and ξ is a one-to-one mapping of the vertex set of K_n onto the set of factors of F, determines a quasigroup Q such that the Cayley color graph C(Q) of Q can be considered as $\langle F, \xi \rangle$ as described above.

Theorem 2. Let Q_1 and Q_2 be two quasigroups on the same set M of n elements. The following two assertions are equivalent:

(1) The Cayley color graphs of Q_1 and Q_2 can be considered as pairs $\langle F, \xi_1 \rangle$ and $\langle F, \xi_2 \rangle$ respectively, F being the same in both pairs.
(2) There exists an isotopy of Q_1 onto Q_2 of the form $\langle \varepsilon, \beta, \varepsilon \rangle$ where ε is the identical mapping of the set M.

Theorem 3. Let Q_1, Q_2 be two quasigroups, let there exist an isotopy of Q_1 onto Q_2. Then there exists a color-preserving isotopy of $C(Q_1)$ onto $C(Q_2)$ and vice versa.

Theorem 4. Let Q_1, Q_2 be two quasigroups, let there exist an isotopy of Q_1 onto Q_2 of the form $\langle \alpha, \beta, \alpha \rangle$. Then there exists a color-preserving isomorphism of $C(Q_1)$ onto $C(Q_2)$.

Theorem 5. Let Q be a quasigroup, let C(Q) be its Cayley color graph. The group of strongly color-preserving automorphisms of C(Q) is isomorphic to the group of all isotopies of Q having the form $\langle \alpha, \varepsilon, \alpha \rangle$ where ε is the identical mapping of Q.

A strongly color-preserving automorphism of a graph is a color-preserving autotopy $\langle f_1, f_2, \varphi \rangle$, where $f_1 \equiv f_2$ and φ is the identical mapping of the set of colors of edges of this graph.

Theorem 6. Let Q be a quasigroup, let C(Q) be its Cayley color graph. The group of color-preserving automorphisms of C(Q) is isomorphic to the group of all isotopies of Q having the form $\langle \alpha, \beta, \alpha \rangle$.

Theorem 7. Let L be a loop (a.s.). Then its Cayley color graph C L can be considered as a pair $\langle F, \xi \rangle$, where F is a decomposition of $\vec{K}_n$ into edge-disjoint linear factors and ξ is a one-to-one mapping of the vertex set of $\vec{K}_n$ onto the set of factors of F and which has the following properties:
(1) One of the factors of F is formed by all loops (g.s.) of $\vec{K}_n$.
(2) There exists a vertex of $\vec{K}_n$ such that any edge outgoing from it belongs to the factor $\xi(v)$, where v is its terminal vertex.

Any pair $\langle F, \xi \rangle$ with the described properties determines a Cayley color graph C(L) of some loop (a.s.) L.

Corollary. To any decomposition F_0 of a complete digraph with n vertices without loops (g.s.) into pairwise disjoint linear factors and for any arbitrarily chosen vertex v of it there exists a loop (a.s.) L such that the pair $\langle F, \xi \rangle$, where F is obtained from F_0 by adjoining a linear factor consisting of loops (g.s.) at each vertex and ξ is a suitable one-to-one mapping of the vertex set of this graph onto the set of factors of F, is its Cayley color graph and the vertex v corresponds to the unit element of L.

Theorem 8. The group of strongly color-preserving automorphisms of the Cayley color graph C(L) of a loop (a.s.) L is isomorphic to the left kernel of L.

As defined in [1], the left kernel of a quasigroup Q is the set of elements a of Q such that (ax)y = a(xy) for any two elements x and y of Q. The left kernel of a loop (a.s.) L is a group under the multiplication in L.

Theorem 9. Let H be a group. Then its Cayley color graph C(H) can be considered as a pair $\langle F, \xi \rangle$, where F is a decomposition of $\vec{K}_n$ into edge-disjoint linear factors and ξ is a one-to-one mapping of the vertex set of $\vec{K}_n$ onto the set of factors of F and which has the properties (1) and (2) from Theorem 7 and a further property:
(3) In each acyclically directed triangle T of $\vec{K}_n$ the factors of F to which two edges of T belong determine uniquely the factor of F to which the third edge of T belongs.

Any pair $\langle F, \xi \rangle$ with the described properties determines a Cayley color graph C(H) of some group H.

3. Semigroups and their meromorphisms

A meromorphic mapping, or shortly meromorphism, of a graph G is an isomorphic mapping of G into itself.

By other words, it is a one-to-one mapping m which maps the vertex set of G into itself and the edge set of G into itself, such that the vertices m(u) and m(v) are joined by the edge m(h) if and only if the vertices u and v are joined by the edge h.

Evidently automorphisms are particular cases of meromorphisms. They are such meromorphisms which map G onto G (this means the vertex set of G onto itself and the same for the edge set). A meromorphism of G which is not an automorphism of G is called a proper meromorphism of G. Obviously only infinite graphs can have proper meromorphisms.

We can prove following two theorems.

Theorem 10. Let M(G) be the semigroup of all meromorphisms of a graph G. Then M(G) is left-cancellative and $M(G) = A(G) \cup M_0(G)$, where $A(G) \cap M_0(G) = \emptyset$, A(G) is a subgroup of M(G) and $M_0(G)$ is either a torsion-free subsemigroup of M(G) which is an ideal of M(G), or is an empty set. The unit element of A(G) is a unit element for whole M(G).

Theorem 11. Let M be an at most countable left-cancellative semigroup and $M = A \cup M_0$, where $A \cap M = \emptyset$, A is a group, M_0 is either a torsionfree semigroup which is an ideal in M, or an empty set. Let the unit element of A be a unit element for whole M. Then there exists a graph G whose meromorphism semigroup is isomorphic to M.

A subgroup of a semigroup is a subsemigroup of this semigroup which is a group. The meromorphisms of a graph evidently form a semigroup.

The proof of theorem 11 is based also on an analogon of the Cayley color graph and of the Frucht graph. For a semigroup satisfying the assumptions of the theorem we construct such an analogon by the same way as we made it for a group. (The analogon of the Cayley color graph will not be complete digraph in general.) Then we prove that the semigroup of all meromorphisms of the analogon of the Frucht graph is isomorphic with the given semigroup.

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03/2 - Schválené výmerom SUKK č. 1197/I-OR-1973 - Prvé vydanie - Náklad 874 - Rukopis zadaný v januári 1973 - Vytlačené v júni 1974 - Papier 2 122 3,80 g - Tlačené ofsetom - Písané strojom - Vytlačili Nitrianske tlačiarne, n.p., Nitra

Strán 64 - AH 3,03 - VH 3,53

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