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ON TRANSFINITE CONVERGENCE IN MEASURE SPACES

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The notion of the transfinite convergence was introduced by Sierpinski in [1]. He proved several results on transfinite pointwise limits of Baire functions. Other types of transfinite convergences were studied e.g. in [2] and [3]. This is a note on transfinite convergence in measure spaces.

Notations and notions. In what follows Ω denotes the first uncountable ordinal number. Transfinite sequences of the type Ω will be considered. If nothing else is said, $\{a_\xi\}_{\xi<\Omega}$ means a transfinite sequence of real numbers. A transfinite sequence of real valued functions defined on a set X is denoted as $\{f_\xi\}_{\xi<\Omega}$.

In accordance with [1], given $\{a_\xi\}_{\xi<\Omega}$ and a real number a, we say that $\{a_\xi\}_{\xi<\Omega}$ is convergent to a or $\lim\limits_{\xi<\Omega} a_\xi = a$ provided that for any $\varepsilon > 0$ there is $\lambda < \Omega$ such that for $\xi \geq \lambda$, $|a_\xi - a| < \varepsilon$.

Given a sequence $\{f_\xi\}_{\xi<\Omega}$ of real functions defined on X, then it is said to be (pointwise) convergent to a function f defined on X if $\lim\limits_{\xi<\Omega} f_\xi(x) = f(x)$ for every $x \in X$.

The definition of the uniform convergence of $\{f_\xi\}_{\xi<\Omega}$ is natural and we shall not introduce it. It was studied in relation with transfinite convergence of infinite series. In an obvious way it is posible to formulate, for transfinite sequences, various types of convergences used in the measure theory. We shall give some of them, leaving the formulation of others to the reader.

The notion of a measure space $(X, \mathcal{S}, m)$ and the related notions are used according to [4] .

A transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of real functions defined on X is said to be almost uniformly convergent to a function f if for any $\varepsilon > 0$ there exists $E \in \mathcal{S}$ such that $m(E) < \varepsilon$ and $\{f_\xi\}_{\xi<\Omega}$ converges uniformly to f on $X - E$.

A sequence $\{f_\xi\}_{\xi<\Omega}$ of measurable functions is said to be convergent in measure to a measurable function f if for any $\varepsilon > 0$ there is $\lim_{\xi<\Omega} m(\{x : |f_\xi(x) - f(x)| \geqq \varepsilon\} = 0$.

A sequence $\{f_\xi\}_{\xi<\Omega}$ of measurable functions is said to be fundamental in the measure m if for any $\varepsilon > 0$, $\delta > 0$ there exists $\lambda < \Omega$ such that for $\xi, \eta > \lambda$ we have $m(\{x: |f_\xi(x) - f_\eta(x)| \geqq \varepsilon\}) < \delta$.

The definitions of the convergence almost everywhere and the convergence uniform almost everywhere may be formulated in a natural way.

<u>Results and proofs.</u> The following two lemmas were used in the above mentioned papers (see e.g. [1] and [3]). Their proofs will be ommited.

<u>L e m m a 1.</u> If $\{a_\xi\}_{\xi<\Omega}$ is a transfinite sequence of real numbers such that $\lim_{\xi<\Omega} a_\xi = a$ then there exists an ordinal number $\lambda < \Omega$ such that $a_\xi = a$ for $\xi \geq \lambda$.

<u>L e m m a 2.</u> If $\{f_\xi\}_{\xi<\Omega}$ is a transfinite sequence of real functions defined on a set X such that $\{f_\xi\}_{\xi<\Omega}$ is uniformly convergent to f then there exists $\lambda < \Omega$ such that $f_\xi = f$ for $\xi \geq \lambda$, $\xi < \Omega$.

The following theorem in which $\xrightarrow{c}$ denotes any of the mentioned types of convergences on a measure space is also evident and the proof is omitted.

<u>Theorem 1.</u> Let $(X, \mathcal{S}, m)$ be a measure space. If a transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of measurable functions defined on X is such that $f_\xi \xrightarrow{c} f$, $f_\xi \xrightarrow{c} g$, where f and g are measurable then $f = g$ almost everywhere.

There exists a sequence $\{f_\xi\}_{\xi<\Omega}$ of measurable functions such that $f_\xi \xrightarrow{c_1} f$, $f_\xi \xrightarrow{c_2} g$ where $\xrightarrow{c_1}$ and $\xrightarrow{c_2}$ are two different type of the mentioned convergences and f is not equal to g almost everywhere.

<u>Example 1.</u> Let $(X, \mathcal{S}, m)$ be as follows: $X = \{\eta : \eta < \Omega\}$, $\mathcal{S}$ is the collection of all subsets of X which are countable or have a countable complement. The measure m is defined as $m(E) = 0$ if E is countable and $m(E) = 1$ if E is not countable.

For any $\xi < \Omega$, let

$$f_\xi(\eta) = \begin{cases} 0 & \text{if } \xi < \eta \\ 1 & \text{if } \xi \geqq \eta \end{cases}$$

The sequence $\{f_\xi\}_{\xi<\Omega}$ converges pointwise to the measurable function f which equals 1 in any $\eta \in X$. It converges in measure to the function g which equals 0 in any $\eta \in X$.

The following theorem shows that for the transfinite sequences of functions it is useless to introduce the almost uniform convergence.

T h e o r e m 2. A sequence $\{f_\xi\}_{\xi<\Omega}$ converges almost uniformly to f if and only if, it is convergent to f uniformly almost everywhere.

P r o o f . Since the suficiency is trivial, let us prove the necessity. For the numbers $\frac{1}{n}$, $n = 1, 2 \ldots$ there exist $E_n \in \mathcal{S}$ such that $m(E_n) < \frac{1}{n}$ and the convergence is uniform in $X - E_n$. Hence there exists $\lambda_n < \Omega$ such that $f_\xi(x) = f(x)$ for $\xi \geq \lambda_n$ and all $x \in X - E_n$ (see lemma 2.). Let $\lambda < \Omega$ be such that $\lambda \geq \lambda_n$ for $n = 1, 2, \ldots$ and let $E = \bigcap_{n=1}^{\infty} E_n$.

We have $m(E) = 0$ and the convergence of $\{f_\xi\}_{\xi<\Omega}$ is uniform to $f = f_\lambda$ on $X - E$. The proof is finished.

The following two theorems characterize the convergence in measure and the fundamentalness in measure.

T h e o r e m 3. A transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of measurable functions is convergent in measure to a measurable function f if and only if there exists $\lambda < \Omega$ such that for any $\xi \geq \lambda$, $\xi < \Omega$ there is $E_\xi \in \mathcal{S}$ with $m(E_\xi) = 0$ and $f_\xi(x) = f(x)$ for $x \in X - E_\xi$.

P r o o f . The sufficiency is obvious. So let $\{f_\xi\}_{\xi<\Omega}$ be convergent to f in measure. Since for any $\varepsilon > 0$ $\lim_{\xi<\Omega} m(\{x : |f_\xi(x) - f(x)| \geq \varepsilon\}) = 0$; there exists, according to Lemma 1, $\lambda_\varepsilon < \Omega$ such that $m(\{x : |f_\xi(x) - f(x)| \geq \varepsilon\}) = 0$ for $\xi \geq \lambda_\varepsilon$, $\xi < \Omega$.

Hence there exists for any $\xi \geq \lambda_\varepsilon$ a set $E_\xi^\varepsilon \in \mathcal{S}$ with $m(E_\xi^\varepsilon) = 0$ and $|f_\xi(x) - f(x)| < \varepsilon$ for $x \in X - E_\xi^\varepsilon$. Put successively $\varepsilon = \varepsilon_n = \frac{1}{n}$, $n = 1, 2, 3, \ldots$, and denote $E_\xi^{\varepsilon_n} =$

$= E^n_\xi$, $\lambda_{\xi_n} = \lambda_n$. Let $\lambda < \Omega$ be such that $\lambda > \lambda_n$ for $n = 1, 2, \ldots$

Further let $E_\xi = \bigcup_{n=1}^{\infty} E^n$. We have $E_\xi \in \mathcal{S}$, $m(E_\xi) = 0$ for any $\xi > \lambda$ and

$$0 \leqq |f_\xi(x) - f(x)| < \frac{1}{n} \text{ for } x \in X - E^n_\xi$$

Hence for $x \in X - E_\xi$ we have $f_\xi(x) = f(x)$.

T h e o r e m 4. A transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of measurable functions is fundamental in measure if and only if there exists $\lambda < \Omega$ such that for any two ξ, $\eta > \lambda$ there is $E_{\xi\eta} \in \mathcal{S}$ with $m(E_{\xi\eta}) = 0$ and $f_\xi(x) = f_\eta(x)$, for $x \in X - E_{\xi\eta}$.

P r o o f. The sufficiency is again evident. Necessity. For any $n, k = 1, 2, 3, \ldots$ there exist $\lambda_{nk} < \Omega$ such that for $\xi, \eta \geqq \lambda_{nk}$

$$m(\{x : |f_\xi(x) - f(x)| \geqq \frac{1}{n}\}) < \frac{1}{k}$$

For every n let $\lambda_n < \Omega$ be such that $\lambda_n > \lambda_{nk}$ for $k = 1, 2, \ldots$. For $\xi, \eta > \lambda_n$ we have

$$m(\{x : |f_\xi(x) - f_\eta(x)| \geqq \frac{1}{n}\}) = 0$$

Putting $E^n_{\xi\eta} = \{x : |f_\xi(x) - f_\eta(x)| \geqq \frac{1}{n}\}$ we have $|f_\xi(x) - f_\eta(x)| < \frac{1}{n}$ for any $\xi, \eta \geqq \lambda_n$ and any $x \in X - E^n_{\xi\eta}$. Further $m(E^n_{\xi\eta}) = 0$. Let $\lambda > \lambda_n$ for $n = 1, 2, \ldots$ be such that $\lambda < \Omega$. Putting $E_{\xi\eta} = \bigcup_{n=1}^{\infty} E^n_{\xi\eta}$, we have $m(E_{\xi\eta}) = 0$ and for $\xi, \eta > \lambda$, $|f_\xi(x) = f_\eta(x)|$ for any $x \in X - E_{\xi\eta}$.

T h e o r e m 5. A transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ converges in measure to a measurable function f if and only if it is fundamental in measure.

P r o o f . Let $\{f_\xi\}$ be convergent to f in measure. According to Theorem 3 there is $\lambda < \Omega$ such that for any $\xi > \lambda$ there exists $E_\xi \in \mathcal{S}$ with $m(E_\xi) = 0$ and $f_\xi(x) = f(x)$ for $x \in X - E$. Let $\xi, \eta > \lambda$. Put $E_{\xi\eta} = E_\xi \cup E_\eta$. We have $m(E_{\xi\eta}) = 0$ and $f_\xi(x) = f_\eta(x)$ for $x \in X - E_{\xi\eta}$. The result follows from Theorem 4.

Conversely, let $\{f_\xi\}$ be fundamental in measure. There is $\lambda < \Omega$ such that for $\xi, \eta \geqq \lambda$ exists $E_{\xi\eta} \in \mathcal{S}$ with $m(E_{\xi\eta}) = 0$ and $f_\xi(x) = f_\eta(x)$ for $x \in X - E_{\xi\eta}$. Hence $f_\xi(x) = f_\lambda(x)$ for any $x \in E_{\xi\lambda}$. So $\{f_\xi\}_{\xi<\Omega}$ converges in measure to f_λ.

One of the well known theorems on the convergence in measure spaces is the Jegoroff's theorem (see [4] p. 88) affirming that the almost everywhere convergence of measurable functions on a set of a finite measure implies the almost uniform convergence on that set. It is also known that the theorem is not necessarily valid if instead of the sequences $\{f_n\}_{n=1}^{\infty}$ the convergence os systems $f(t, x)$ of real functions, where t is a real parameter, is considered (see [5], [6]). The problem was considered also for functions with values in more general spaces (see [7], [8], [9]). When transfinite sequences $\{f_\xi\}_{\xi<\Omega}$ are considered then Jegoroff's theorem in general is not valid. The sequence in Example 1. serves as an counterexample. But the following is a necessary condition for validity of Jegoroff's theorem for transfinite sequences of real valued measurable functions.

T h e o r e m 6. A necessary condition for validity of Jegoroff's theorem for transfinite sequences in a measure space $(X, \mathcal{S}\ m)$ is:

a) For any set $E \in \mathcal{S}$ of finite measure and any transfinite sequence $\{E_\xi\}_{\xi<\Omega}$ of measurable sets such that $E_\mu \subset E_\nu$ for $\mu < \Omega$, $\nu < \Omega$ and such that $\bigcup_{\xi<\Omega} E_\xi = E$, $\lim_{\xi<\Omega} m(E) = m(E)$ holds.

P r o o f . Let the condition a) be not valid. Take a sequence $E_\xi \in \mathcal{S}$, $\xi < \Omega$ such that $\bigcup_{\xi<\Omega} E_\xi = E \in \mathcal{S}$ and $E_\mu \subset E_\nu$ for $\mu < \nu$, $\mu, \nu < \Omega$ with $\lim_{\xi<\Omega} m(E) = m(E)$.

There exists λ, $\lambda < \Omega$ such that $\lim m(E_\xi) = m(E_\lambda)$ and $m(E_\xi) = m(E_\lambda)$ for $\xi \geq \lambda$. Hence $m(E - E_\lambda) = m(E) - m(E_\lambda) > 0$. For any $\xi < \Omega$ put $f_\xi(x) = \chi_{E_\xi}(x)$ and $f(x) = \chi_E(x)$ for any $x \in X$. (χ_A denotes the characteristic function of the set A.)

The functions f_ξ are measurable and $\lim f_\xi(x) = f(x)$ in any point $x \in E$. But the convergence is not uniform almost everywhere. In fact take $N \in \mathcal{S}$ such that $m(N) = 0$. Let $\varepsilon = \frac{1}{2}$ and $\lambda < \Omega$ any ordinal number. There exists $x_0 \notin E_{\lambda+1}$, $x_0 \notin N$, $x_0 \in E$. Hence $|f_{\lambda+1}(x_0) - f(x_0)| = 1 > \frac{1}{2} = \varepsilon$.

Note. The convergence almost everywhere of $\{f_\xi\}_{\xi<\Omega}$ to f in general does not imply the convergence in measure to f. (see example 1).

It may occur that a sequence $\{f_\xi\}_{\xi<\Omega}$ of measurable functions converges in measure to a measurable function f but does not converge almost everywhere to any measurable function.

<u>E x a m p l e 2.</u> Let $(X, \mathcal{S}, m)$ have the same meaning as in Example 1. Let $E \subset X$ be such that $E \notin \mathcal{S}$. Define a sequence $\{f_\xi\}_{\xi < \Omega}$ as follows:

Given $\xi < \Omega$

$$f_\xi(\eta) = 0 \quad \text{if} \quad \eta \geqq \xi$$

$$\text{if } \eta < \xi \quad \text{then } f_\xi(\eta) = \begin{cases} 0 & \text{if } \eta \notin E \\ 1 & \text{if } \eta \in E \end{cases}$$

The sequence $\{f_\xi\}_{\xi < \Omega}$ is a sequence of measurable functions which converges in measure to the measurable function $f = 0$. But the sequence is pointwise convergent to the nonmeasurable function χ_E. Hence $\{f_\xi\}_{\xi < \Omega}$ does not converge to any measurable function (see Theorem 1).

R E F E R E N C E S

[1] SIERPINSKI V.: Sur les suites transfinies convergentes de fonctions de Baire, Fund. Math., 1, 1920, p. 132.

[2] MALCHAIR. H.: Queques nouvelles considerations relatives aux suites et séries transfinies. Bulletin de la Societé Royale des Sciences Liege, 3, 1934, p. 133-140

[3] KOSTYRKO P.: On convergence of transfinite sequences. Mat. Čas. 21, 1971, p. 233-239

[4] HALMOS P.R.: Measure theory. New York 1950

[5] TOLSTOFF G.: Une remarque sur le theoreme de Th. Egoroff. Dokl. Akad. Nauk ZSSR N.S., 22, 1939, p.305-307

[6] WESTON J.D.: A counterexample concerning Egoroff's theorem. J. London Math. Soc., 31, 1959, p. 139-140

[7] NEUBRUNN T.: Poznámka k meratelným transformáciam. Acta Fac.

Rer. nat. Univ. Commenianae, 5, 1959, p. 287 - 289

[8] ŠTERBOVÁ, D.: A note on the generalization of Jegoroff's theorem. Mat. čas. SAV, 1968, p. 83 - 89

[9] ZAKON E.: On almost uniform convergence of families of functions. Canad. Math. Bull., 7, 1964, p. 43 - 48.

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S ú h r n

O TRANSFINITNEJ KONVERGENCII V PRIESTOROCH S MIEROU

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V priestoroch s mierou možno definovať rôzne typy konvergencie aj pre transfinitné postupnosti. Vzťahy medzi jednotlivými typmi konvergencií nie sú vždy analogické vzťahom platným pre konvergenciu obyčajných postupností. V práci sa skúmajú rôzne typy konvergencie transfinitných postupností na priestore s mierou. Okrem iného sa tiež udáva nutná podmienka k tomu, aby pre transfinitné postupnosti platila veta analogická Jegorovovej vete.

Р Е З Ю М Е

О ТРАНСФИНИТНОЙ КОНВЕРГЕНЦИИ В ПРОСТРАНСТВАХ С МЕРОЙ

ТИБОР НОЙБРУН, БРАТИСЛАВА

В пространстве с мерой можно определить различные понятия конвергенции, также для трансфинитных последовательностей. Не все связи между этими типами аналогичны связям между обыкновенными последовательностями. В работе исследуется ситуация в случае различных типов трансфинитной конвергенции. Также установлено необходимое условие для того, чтобы теорема аналогичная теореме Егорова была верна.

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PAIRING FORMULAS FOR TRIPOLYKAYS

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1. Introduction

a) The polynomials of the present paper, called "tripolykays" are symmetric functions in the sense that they are invariant under permutations of any of the subscripts. This family of polynomials is an extension, from sets to matrices, of the family of symmetric polynomials k_1, k_2, k_{11}, k_3, k_{12}, ... etc., defined by Tukey [3] and Hook [4] called "bipolykays" and which are a generalization of the family k_1, k_2, ..., etc., defined by R. A. Fisher [1].

This paper defines the tripolykays, shows that they are inherited on the average, develops the formulas for use in random pairing.

b) Generalized symmetric mean is the function which to each system of real numbers $x_{111}, \dots, x_{rst}$ asociates the number

$$\frac{1}{M} \sum x^a_{i_1 j_1 k_1} x^b_{i_2 j_2 k_2} \dots$$

where M is the number of all summands $i_1, i_2, \dots \in \{1, 2, \dots, r\}$, $i_2 \neq i_1$, $i_3 \neq i_2$, $i_3 \neq i_1$, ... and similarly for j_u, k_n.

For example $\frac{1}{M} \sum x^2_{i_1 j_1 k_1} x_{i j_2 k_2}$

denotes $\frac{1}{M} \sum_{i_1}^{r} \sum_{j_1}^{s} \sum_{j_2}^{s} \sum_{k_1}^{t} \sum_{k_2}^{t} x^2_{i_1 j_1 k_1} x_{i j_2 k_2}$

$$j_1 \neq j_2,\ k_1 \neq k_2$$

Obviously $M = rs(s-1)\,t\,(t-1)$. Let the sum of exponents be n. All terms of the sum can be expressed in the form of product of n coefficients, e.g.

$$x^2_{i_1 j_1 k_1} x_{i_1 j_2 k_2} \doteq x_{i_1 j_1 k_1} x_{i_1 j_1 k_1} x_{i_1 j_2 k_2}$$

Consider n-tuple of symbols $\{p_1, \dots, p_n\}$ in our case it is a triple $\{pqr\}$. Let as define the partition α of this set so that two symbols p_i, p_j are in the same class exactly, when the first indices in i-th or j-th coefficient are equal. In our example all three indices are equal, therefore all pqr belong to the same class. Then partition α consists of only one class $\{pqr\}$. Similarly we define partition β of the set $\{q_1, q_2, \dots, q_n\}$ in dependence on the second indices. In our example the second indices of the first and the second member are equal but the third member has the second index different. So β is combined of two classes $\{pq\}$, $\{t\}$ expressed

$$\beta = \{pq,\ t\}$$

Analogicaly we construct partition γ which in our case has the following form

$$\gamma = \{pq,\ r\}$$

Generalized symmetric mean is characterized by the abovementioned triple of patritions α, β, γ, therefore we denote it by $\langle \alpha / \beta / \gamma \rangle$. From geometrical point of view even the matrix of entries is suitably applied in [4].

For example:

$$\langle pqsv, u/pqsv, u/pq, svu \rangle =$$

$$= \frac{1}{r(r-1)\, s(s-1)\, t(t-1)} \sum x^2_{i_1 j_1 k_1} x^2_{i_1 j_1 j_2} x_{i_2 j_2 k_2}$$

or

$$\left[\begin{array}{cc|cc} 2 & - & 2 & - \\ - & - & - & 1 \end{array}\right] =$$

$$= \frac{1}{r(r-1)\, s(s-1)\, t(t-1)} \sum x^2_{i_1 j_1 k_1} x^2_{i_1 j_1 k_2} x_{i_2 j_2 k_2}$$

Sings $\langle \alpha \rangle$, $\langle \beta \rangle$ and similar, denote symmetric means of partition α , β of a set $\{p_1, p_2, \dots, p_m\}$. If α is the minimal partition, e.i. its classes are $\{p_1\}$, $\{p_2\}$, ..., $\{p_m\}$ then we define the polykay (α) formally as symmetric mean in the following way:

$$(\alpha) = \langle \alpha \rangle$$

For arbitrary symmetric mean $\langle \alpha \rangle$ we define the polykay (α) of the degree of m with the help of recurent formula by equation

$$\langle \alpha \rangle = (\alpha) + \sum_{\beta \in \Gamma_\alpha} (\beta)$$

where Γ_α is the set of all propre subpartitions β of partition of α . E.g. $\alpha = \{pq\}$ consist of the only class,

$$(p, q) = \langle p, q \rangle$$

$$\langle pq \rangle = (pq) + (p, q) = (pq) + \langle p, q \rangle$$

then

$$(pq) = \langle pq \rangle - \langle p, q \rangle$$

Thus each polykay (α) can be expressed as a linear combination of symmetric means $\langle \alpha_i \rangle$, i = 1, 2, ..., m.

$$(\alpha) = \sum_{t} a_i \langle \alpha_i \rangle$$

similar is true for the rest.

Let G be a set containing M elements. Let f be a real function defined on the set C_m of all subsets of G containing m elements. Then we define ave $\{f\}$ as an aritmetic mean of all values of $f(\mu)$, $\mu \in C_m$, e.i.

$$\text{ave}\ \{f\} = \binom{M}{m}^{-1} \sum_{\mu \in C_m} f(\mu)$$

Let the real function g be defined on the set of all permutations P_m of a set containing m elements. Then aver g is an arithmetic mean of all its values, e.i.

$$\text{aver}\ \{g\} = \frac{1}{m!} \sum_{\tau \in P_m} g(\tau)$$

The mentioned definition is related to the case when we consider two distinct sets $(x_1, x_2, \ldots, x_m)$, $(y_1, y_2, \ldots, y_m)$ and we form a set of type

$$z_i = (x_i + y_{\varepsilon(i)})$$

where $\varepsilon(1)$, $\varepsilon(2)$, ..., $\varepsilon(m)$ is a permutation of numbers 1, 2, ..., m.

2. Pairing formulas for tripolykays

We define a "dot-multiplication" for symmetric means as follows:

$$\langle \alpha \rangle \langle \beta \rangle \langle \gamma \rangle = \begin{cases} \langle \alpha / \beta / \gamma \rangle & \text{if } \alpha, \beta, \gamma \text{ consist of the same symbols} \\ 0 & \text{otherwise} \end{cases}$$

Definition 1. The tripolykays $(\alpha / \beta / \gamma)$ are defined as

$$(\alpha/\beta/\gamma) = (\alpha)(\beta)(\gamma)$$

where α, β and γ are partitions of the same set of symbols, where it is understood that (α), (β) and (γ) are expresed as sums of symmetric means before the dot-product is taken.

E x a m p l e . Consider the tripolykay $\left(\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right)$:

$$\left(\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right) = (q,\ t/qt/q,\ t) =$$

$$= (q,\ t)(qt)(q,\ t) = \langle q,\ t\rangle\ [\langle qt\rangle - \langle q,\ t\rangle]\ \langle q\ t\rangle =$$

$$= \langle q,\ t\rangle\ \langle qt\rangle\ \langle q,\ t\rangle - \langle q,\ t\rangle\ \langle q,\ t\rangle\ \langle q,\ t\rangle =$$

$$= \langle q,\ t/qt/q,\ t\rangle - \langle q,\ t/q,\ t/q,\ t\rangle$$

$$\left(\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right) = \left[\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right] - \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right]$$

D e f i n i t i o n 2. In dealing with two different populations, we define

$$[\langle\alpha\rangle_*\langle\beta\rangle_{**}]\cdot[\langle\gamma\rangle_*\langle\delta\rangle_{**}]\cdot[\langle\varepsilon\rangle_*\langle\xi\rangle_{**}] = [\langle\alpha\rangle\langle\gamma\rangle\langle\varepsilon\rangle]_*\cdot[\langle\beta\rangle\langle\delta\rangle\langle\xi\rangle]_{**}$$

and by extension this provides a meaning for any expression which is formally written as a dot product of linear combinations of terms of the type $\langle\alpha\rangle_*\langle\beta\rangle_{**}$.

L e m m a 1.

ave aver $\{\langle\alpha\rangle\langle\beta\rangle\langle\gamma\rangle\}$ = {ave aver $\langle\alpha\rangle$}{ave aver $\langle\beta\rangle$}{ave aver $\langle\gamma\rangle$}

ave aver $\{(\alpha)(\beta)(\gamma)\}$ = {ave aver (α)} {ave aver (β)}{ave aver (γ)}

(Dot product the expressions ave aver α, ..., etc. having meaning only from the definition just above. Similarly for the tripolykays, which must be expressed as sums of g.s.m. before the above definition gives them meaning).

Proof is evident.

<u>Lemma 2.</u> $(\alpha)_0(\beta)_{00}^{\cdot}(\gamma)_0(\delta)_{00}^{\cdot}(\varepsilon)_0(\zeta)_{00} = \{(\alpha)(\gamma)(\varepsilon)\}_0\{(\beta)(\delta)(\zeta)\}_{00}$

(Each side of this equation has the meaning, each polykay has been written as a linear combination of symmetric means). Proof is evident.

<u>Definition 3.</u> A simple dichotomy of a partition α is a set $\{\alpha_1, \alpha_2\}$ of two partitions α_1 and α_2, such that it has the property that each part of α belongs entirely to α_1 or to α_2.

<u>Definition 4.</u> The tripolykay $(\alpha/\beta/\gamma)$ is said to be decomposable if there exist simple dichotomies $\{\alpha_1, \alpha_2\}$, $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ of partitions α, β, γ such that α_1, β_1 and γ_1 consist of the same symbols and neither α_1, β_1 nor γ_1 is null.

<u>Theorem 1.</u> If α, β and γ are arbitrary partitions. If a tripolykay $(\alpha/\beta/\gamma)$ is indecomposable then its pairing formula is

$$\text{ave aver } \{(\alpha/\beta/\gamma)\} = (\alpha/\beta/\gamma)_0 + (\alpha/\beta/\gamma)_{00}$$

<u>Proof.</u>

$$\text{ave aver } \{(\alpha/\beta/\gamma)\} = \text{ave aver } \{(\alpha)(\beta)(\gamma)\} =$$

$$= \{\text{ave aver } (\alpha)\}\{\text{ave aver } (\beta)\}\{\text{ave aver } (\gamma)\} =$$

$$= \sum (\alpha_1)_0(\alpha_2)_{00} \cdot \sum (\beta_1)_0(\beta_2)_{00} \cdot \sum (\gamma_1)_0(\gamma_2)_{00}$$

by definitions, by Lemma 1, where the first sum extends over all simple dichotomies $\{\alpha_1, \alpha_2\}$ of α and similarly for the second and for the third sum. Hence by Lemma 2

$$\text{ave aver } \{(\alpha/\beta/\gamma)\} =$$

$$= \sum\{(\alpha_1)_0(\beta_1)_0(\gamma_1)_0\}\cdot\{(\alpha_2)_{00}(\beta_2)_{00}(\gamma_2)_{00}$$

Since $\{\alpha_1, \alpha_2\}$ is a simple dichotomy of α, then α_1 is a partition consisting of some of the parts of α. Similarly β_1, γ_1 consists of some of the parts of β and γ. The expression $(\alpha_1)_0(\beta_1)_0(\gamma_1)_0$ vanishes unless α_1, β_1 and γ_1 comprise exactly the same symbols. Thus the nonvanishing terms arise when

1. $\alpha_1 = \alpha$, $\beta_1 = \beta$, $\gamma_1 = \gamma$, $\alpha_2 = \beta_2 = \gamma_2 = \emptyset$

then

$$\{(\alpha_1)_0(\beta_1)_0(\gamma_1)_0\}\cdot\{(\alpha_2)_{00}(\beta_2)_{00}(\gamma_2)_{00}\} =$$

$$= \{(\alpha)_0(\beta)_0(\gamma)_0\}\cdot\{(\emptyset)_{00}(\emptyset)_{00}(\emptyset)_{00}\} = (\alpha)_0(\beta)_0(\gamma)_0 =$$

$$= (\alpha/\beta/\gamma)_0$$

2. $\alpha_1 = \beta_1 = \gamma_1 = \emptyset$, $\alpha_2 = \alpha$, $\beta_2 = \beta$, $\gamma_2 = \gamma$

then

$$(\alpha)_{00}(\beta)_{00}(\gamma)_{00} = (\alpha/\beta/\gamma)_{00}$$

<u>Example.</u> If $\alpha = \{pq, t\}$, $\beta = \{tq, p\}$, $\gamma = \{tq, p\}$ then

$$\text{ave aver } \{(\alpha/\beta/\gamma)\} =$$

$$= \{(pq, t)_0(tq, p)_0(tq, p)_0\} \ \{(\emptyset)_{00}(\emptyset)_{00}(\emptyset)_{00}\} +$$

$$+ \{ (\emptyset)_0 (\emptyset)_0 (\emptyset)_0 \} \{ (pq, t)_{00} (tq, p)_{00} (tq, p)_{00} \} =$$

$$= (pq, t/tq, p/tq, p)_0 + (pq, t/tq, p/tq, p)_{00}$$

$$\text{ave aver } \{ (\alpha / \beta / \gamma) \} = (\alpha / \beta / \gamma)_0 + (\alpha / \beta / \gamma)_{00}$$

T h e o r e m 2. If α , β and γ are arbitrary partitions. If a tripolykay $(\alpha / \beta / \gamma)$ is decomposable and is the product of indecomposable components $(\alpha_i / \beta_i / \gamma_i)$, then the pairing formula is

$$\text{ave aver } \{ (\alpha / \beta / \gamma) \} = (\alpha / \beta / \gamma)_0 + (\alpha / \beta / \gamma)_{00} +$$

$$+ \sum (\alpha_1 / \beta_1 / \gamma_1)_0 (\alpha_2 / \beta_2 / \gamma_2)_{00}$$

where the summation extends over all simple dichotomies of α , β , γ .

P r o o f . First part of the proof of this theorem, being virtualy identical with that of Theorem 1.

Thus the only nonvanishing terms arise when

1. $\alpha_1 = \alpha$, $\beta_1 = \beta$, $\gamma_1 = \gamma$, $\alpha_2 = \beta_2 = \gamma_2 = \emptyset$
2. $\alpha_1 = \beta_1 = \gamma_1 = \emptyset$, $\alpha_2 = \alpha$, $\beta_2 = \beta$, $\gamma_2 = \gamma$,
3. $\alpha_1 = \beta_1 = \gamma_1$, $\alpha_2 = \beta_2 = \gamma_2$ and none of this is null.

The tripolykay is decomposable, case 3., gives exactly the various terms that correspond to the splitting of the tripolykay into indecomposable components.

E x a m p l e . If $\alpha = \{ pq, s \}$, $\beta = \{ p, q, s \}$, $\gamma = \{ pq, s \}$

then

$$\text{ave aver } \{(\alpha/\beta/\gamma)\} = (pq,\ s/p,\ q,\ s/pq,\ s)_0 +$$

$$+ (pq,\ s/p,\ q,\ s/pq,\ s)_{00} + (pq/p,\ q/pq)_0 (s/s/s)_{00} +$$

$$+ (s/s/s)_0 (pq/p,\ q/pq)_{00} = (\alpha/\beta/\gamma)_0 + (\alpha/\beta/\gamma)_{00} +$$

$$+ \sum (\alpha_1/\beta_1/\gamma_1)_0 (\alpha_2/\beta_2/\gamma_2)_{00}$$

R E F E R E N C E S

[1] FISHER R.A.: Moments and product moments of sampling distributions, Math. Soc. 2, Vol. 30, 1928 - 1929

[2] TUKEY J. W.: Keeping moment-like sampling computations simple. Ann. Math. Stat., Vol. 27, 1956

[3] TUKEY J. W.: Some sampling simplified. J. Amer. Stat. Assn., Vol. 45, 1950

[4] HOOKE R.: Symmetric functions of a two-way array, Ann. Math. Stat. Vol. 27, 1956

[5] LAMOŠ F.: Generalized symmetric means of a three-way array, Acta F.R.N.U.Comen. - Math. XXVII, 1972.

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S ú h r n

VZORCE NA VYTVÁRANIE DVOJÍC PRE TRIPOLYŠTATISTIKY

FRANTIŠEK LAMOŠ, BRATISLAVA

V práci sa vyšetrujú funkcie, ktoré sa nazývajú tripolyštatistiky a ktoré sú definované ako lineárna kombinácia zovšeobecnených symetrických priemerov. Nadväzuje na práce, ktoré sa zaoberajú náhodným tvorením dvojíc dvoch rôznych množín. Hlavný výsledok možno uviesť v nasledovnej vete:

Nech α, β, a γ sú ľubovoľné rozklady nejakej množiny znakov. Nech tripolyštatistika $(\alpha/\beta/\gamma)$ je

1. nerozložiteľná, potom platí

$$\text{ave aver } \{(\alpha/\beta/\gamma)\} = (\alpha/\beta/\gamma)_0 + (\alpha/\beta/\gamma)_{00}$$

2. rozložiteľná, potom platí

$$\text{ave aver } \{(\alpha/\beta/\gamma)\} = (\alpha/\beta/\gamma)_0 + (\alpha/\beta/\gamma)_{00} + $$

$$+ \sum (\alpha_1/\beta_1/\gamma_1)_0 (\alpha_2/\beta_2/\gamma_2)_{00}$$

kde ave znamená aritmetický priemer cez výbery a aver znamená aritmetický priemer cez permutácie prvkov množiny.

Р Е З Ю М Е

ФОРМУЛЫ ДЛЯ ОВРАЗОВАНИЯ ПАР ТРИПОЛИСТАТИСТИК

ФРАНТИШЕК ЛАМОШ, БРАТИСЛАВА

В стате иследуются функции, которме называются триполистатистики и которые определены как линейная комбинация обобщенных симметрических средних. Эта работа является продолжением работ, которые занимаются случайным образованием пар двух различных множеств. Главный результат стати задан в следующей теореме:

Пусть α , β , γ произвольные разбиения некоторого множества знаков. Пусть триполистатистика

1. неразложимая, тогда имеет место

$$\text{ave aver } \{(\alpha/\beta/\gamma)\} = (\alpha/\beta/\gamma)_0 + (\alpha/\beta/\gamma)_{00}$$

2. разложимая, тогда

$$\text{ave aver } \{(\alpha/\beta/\gamma)\} = (\alpha/\beta/\gamma)_0 + (\alpha/\beta/\gamma)_{00} + \\ + \sum (\alpha_1/\beta_1/\gamma_1)_0 (\alpha_2/\beta_2/\gamma_2)_{00}$$

где ave означает среднее арифметическое выборки и aver означает среднее арифметическое перестановок елементов множества.

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SOME FORMULAS FOR GENERALIZED SYMMETRIC MEANS (G.S.M.'S) AND TRIPOLYKAYS

FRANTIŠEK LAMOŠ, Bratislava

1. Introduction

The distinct generalized symmetric means of degree 1, 2 and 3 have been listed in [2] and for degree 3 the notations have been use $u_1, u_2, \dots, u_{37}$. For example generalized symmetric mean of degree 1 over a trisample is

$$\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right] = \sum_{\neq} x_{ijk}/rst$$

The tripolykays are linear combinations of g.s.m.'s. Those of degree 3 can be expressed in similar notation and they will be denoted by $U_1, U_2, \dots, U_{37}$, (cf. [3]).

2. Conversion formulas for g. s. m's and tripolykays

The following conversion formulas apply to the tripolykays of degree 1 and 2:

Degree 1: $\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right) = \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right]$

Degree 2: $\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right) = \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right]$.

$$\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \end{array}\right) = \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \end{array}\right] - \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right]$$

$$\left(\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right) = \left[\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right] - \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right]$$

$$\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \end{array}\right) = \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \end{array}\right] - \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right]$$

$$\left(\begin{array}{cc|cc} 1 & - & 1 & - \\ - & - & - & - \end{array}\right) = \left[\begin{array}{cc|cc} 1 & - & 1 & - \\ - & - & - & - \end{array}\right] - \left[\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right] - \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \end{array}\right] + \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right]$$

$$\left(\begin{array}{cc|cc} 1 & - & - & - \\ 1 & - & - & - \end{array}\right) = \left[\begin{array}{cc|cc} 1 & - & - & - \\ 1 & - & - & - \end{array}\right] - \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \end{array}\right] - \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \end{array}\right] + \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right]$$

$$\left(\begin{array}{cc|cc} 1 & 1 & - & - \\ - & - & - & - \end{array}\right) = \left[\begin{array}{cc|cc} 1 & 1 & - & - \\ - & - & - & - \end{array}\right] - \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \end{array}\right] - \left[\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right] + \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right]$$

$$\left(\begin{array}{cc|cc} 2 & - & - & - \\ - & - & - & - \end{array}\right) = \left[\begin{array}{cc|cc} 2 & - & - & - \\ - & - & - & - \end{array}\right] - \left[\begin{array}{cc|cc} 1 & 1 & - & - \\ - & - & - & - \end{array}\right] - \left[\begin{array}{cc|cc} 1 & - & - & - \\ 1 & - & - & - \end{array}\right] - \left[\begin{array}{cc|cc} 1 & - & 1 & - \\ - & - & - & - \end{array}\right] + \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \end{array}\right] + \left[\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right] + \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \end{array}\right] - \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right] = \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right)$$

$$\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \end{array}\right] = \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right)$$

$$\left[\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right] = \left(\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right)$$

$$\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \end{array}\right] = \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right)$$

$$\left[\begin{array}{cc|cc} 1 & - & 1 & - \\ - & - & - & - \end{array}\right] = \left(\begin{array}{cc|cc} 1 & - & 1 & - \\ - & - & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right)$$

$$\left[\begin{array}{cc|cc} 1 & - & - & - \\ 1 & - & - & - \end{array}\right] = \left(\begin{array}{cc|cc} 1 & - & - & - \\ 1 & - & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right)$$

$$\left[\begin{array}{cc|cc} 1 & 1 & - & - \\ - & - & - & - \end{array}\right] = \left(\begin{array}{cc|cc} 1 & 1 & - & - \\ - & - & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right)$$

$$\left[\begin{array}{cc|cc} 2 & - & - & - \\ - & - & - & - \end{array}\right] = \left(\begin{array}{cc|cc} 2 & - & - & - \\ - & - & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & 1 & - & - \\ - & - & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & - \\ 1 & - & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & 1 & - \\ - & - & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \end{array}\right) + \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right)$$

The distinct tripolykeys of degree 3 will be denoted by U's.

$U_1 = u_1$

$U_2 = u_1 - u_1$

$U_3 = u_3 - u_1$

$U_4 = u_4 - u_1$

$U_5 = u_5 - u_3 - u_2 + u_1$

$U_6 = u_6 - u_4 - u_2 + u_1$

$U_7 = u_7 - u_4 - u_3 + u_1$

$U_8 = u_8 - u_3 - u_2 + u_1$

$U_9 = u_9 - u_4 - u_2 + u_1$

$U_{10} = u_{10} - u_4 - u_3 + u_1$

$U_{11} = u_{11} - u_7 - u_6 - u_5 + u_4 + u_3 + u_2 - u_1$

$U_{12} = u_{12} - u_8 - u_7 - u_6 + u_4 + u_3 + u_2 - u_1$

$U_{13} = u_{13} - u_9 - u_7 - u_5 + u_4 + u_3 + u_2 - u_1$

$U_{14} = u_{14} - u_{10} - u_6 - u_5 + u_4 + u_3 + u_2 - u_1$

$U_{15} = u_{15} - u_{10} - u_9 - u_8 + u_4 + u_3 + u_2 - u_1$

$U_{16} = u_{16} - 3u_2 + 2u_1$

$U_{17} = u_{17} - 3u_3 + 2u_1$

$U_{18} = u_{18} - 3u_4 + 2u_1$

$U_{19} = u_{19} - u_{16} - u_8 - 2u_5 + 2u_3 + 3u_2 - 2u_1$

$U_{20} = u_{20} - u_{17} - u_8 - 2u_5 + 2u_3 + 2u_2 - 2u_1$

$U_{21} = u_{21} - u_{16} - u_9 - 2u_6 + 2u_4 + 3u_2 - 2u_1$

$U_{22} = u_{22} - u_{17} - u_{10} - 2u_7 + 2u_4 + 3u_3 - 2u_1$

$U_{23} = u_{23} - u_{18} - u_9 - 2u_6 + 3u_4 + 2u_2 - 2u_1$

$U_{24} = u_{24} - u_{18} - u_{10} - 2u_7 + 3u_4 + 2u_3 + 2u_1$

$U_{25} = u_{25} - u_{21} - u_{19} + u_{16} - u_{13} - u_{12} - u_{11} + u_9 + u_8 + 2u_7 +$
$+ 2u_6 + 2u_5 - 2u_4 - 2u_3 - 3u_2 + 2u_1$

$U_{26} = u_{26} - u_{22} - u_{20} + u_{17} - u_{14} - u_{12} - u_{11} + u_{10} + u_8 +$
$+ 2u_7 + 2u_6 + 2u_5 - 2u_4 - 3u_3 - 2u_2 + 2u_1$

$U_{27} = u_{27} - u_{24} - u_{23} + u_{18} - u_{14} - u_{13} - u_{11} + u_{10} + u_9 +$
$+ 2u_7 + 2u_6 + 2u_5 - 3u_4 - 2u_3 - 2u_2 + 2u_1$

$U_{28} = u_{28} - u_{21} - u_{19} + u_{16} - u_{15} - 2u_{14} + 2u_{10} + u_9 + u_8 +$
$+ 2u_6 + 2u_5 - 2u_4 - 2u_3 - 3u_2 + 2u_1$

$U_{29} = u_{29} - u_{22} - u_{20} + u_{17} - u_{15} - 2u_{13} - u_{10} + 2u_9 + u_8 +$
$+ 2u_7 + 2u_5 - 2u_4 - 3u_3 - 2u_2 + 2u_1$

$U_{30} = u_{30} - u_{24} - u_{23} + u_{18} - u_{15} - 2u_{12} + u_{10} + u_9 + 2u_8 +$
$+ 2u_7 + 2u_6 - 3u_4 - 2u_3 - 2u_2 + 2u_1$

$U_{31} = u_{31} - 3u_{20} - 3u_{19} + 2u_{17} + 2u_{16} + 3u_8 + 6u_5 - 6u_3 -$
$- 6u_2 + 4u_1$

$U_{32} = u_{32} - 3u_{23} - 3u_{21} + 2u_{18} + 2u_{16} + 3u_9 + 6u_6 - 6u_4 -$
$- 6u_2 + 4u_1$

$U_{33} = u_{33} - 3u_{24} - 3u_{22} + 2u_{18} + 2u_{17} + 3u_{10} + 6u_7 - 6u_4 -$
$- 6u_3 + 4u_1$

$U_{34} = u_{34} - u_{31} - u_{29} - u_{28} - 2u_{26} - 2u_{25} + 2u_{22} + 2u_{21} +$

$+ 3u_{20} + 3u_{19} - 2u_{17} - 2u_{16} + u_{15} + 2u_{14} + 2u_{13} + 2u_{12} +$

$+ 2u_{12} + 2u_{11} - 2u_{10} - 2u_{9} - 3u_{8} - 4u_{7} - 4u_{6} - 6u_{5} + 4u_{4} +$

$+ 6u_{3} + 6u_{2} - 4u_{1}$

$U_{35} = u_{35} - u_{32} - u_{30} - u_{28} - 2u_{27} - 2u_{25} + 2u_{24} + 3u_{23} +$

$+ 3u_{21} + 2u_{19} - 2u_{18} - 2u_{16} + u_{15} + 2u_{14} + 2u_{13} + 2u_{12} +$

$+ 2u_{11} - 2u_{10} - 3u_{9} - 2u_{8} - 4u_{7} - 6u_{6} - 4u_{5} + 6u_{4} + 4u_{3} +$

$+ 6u_{2} - 4u_{1}$

$U_{36} = u_{36} - u_{33} - u_{30} - u_{29} - 2u_{27} - 2u_{26} + 3u_{24} + 2u_{23} +$

$+ 3u_{22} + 2u_{20} - 2u_{18} - 2u_{17} + u_{15} + 2u_{14} + 2u_{13} + 2u_{12} +$

$+ 2u_{11} - 3u_{10} - 2u_{9} - 2u_{8} - 6u_{7} - 4u_{6} - 4u_{5} + 6u_{4} + 6u_{3} +$

$+ 4u_{2} - 4u_{1}$

$U_{37} = u_{37} - 3u_{36} - 3u_{35} - 3u_{34} + 2u_{33} + 2u_{32} + 2u_{31} + 3u_{30} +$

$+ 3u_{29} + 3u_{28} + 6u_{27} + 6u_{26} + 6u_{25} - 6u_{24} - 6u_{23} - 6u_{22} -$

$- 6u_{21} - 6u_{20} - 6u_{19} + 4u_{18} + 4u_{17} + 4u_{16} - 3u_{15} - 6u_{14} -$

$- 6u_{13} - 6u_{12} - 6u_{11} + 6u_{10} + 6u_{9} + 6u_{8} + 12u_{7} + 12u_{6} +$

$+ 12u_{5} - 12u_{4} - 12u_{3} - 12u_{2} + 8u_{1}$

The conversion formulas of degree 3:

$u_1 = U_1$

$u_2 = U_2 + U_1$

$u_3 = U_3 + U_1$

$u_4 = U_4 + U_1$

$u_5 = U_5 + U_3 + U_2 + U_1$

$u_6 = U_6 + U_4 + U_2 + U_1$

$u_7 = U_7 + U_4 + U_3 + U_1$

$u_8 = U_8 + U_3 + U_2 + U_1$

$u_9 = U_9 + U_4 + U_2 + U_1$

$u_{10} = U_{10} + U_4 + U_3 + U_1$

$u_{11} = U_{11} + U_7 + U_6 + U_5 + U_4 + U_3 + U_2 + U_1$

$u_{12} = U_{12} + U_8 + U_7 + U_6 + U_4 + U_3 + U_2 + U_1$

$u_{13} = U_{13} + U_9 + U_7 + U_5 + U_4 + U_3 + U_2 + U_1$

$u_{14} = U_{14} + U_{10} + U_6 + U_5 + U_4 + U_3 + U_2 + U_1$

$u_{15} = U_{15} + U_{10} + U_9 + U_8 + U_4 + U_3 + U_2 + U_1$

$u_{16} = U_{16} + 3U_2 + U_1$

$u_{17} = U_{17} + 3U_3 + U_1$

$u_{18} = U_{18} + 3U_4 + U_1$

$u_{19} = U_{19} + U_{16} + U_8 + 2U_5 + U_3 + 3U_2 + U_1$

$u_{20} = U_{20} + U_{17} + U_8 + 2U_5 + 3U_3 + U_2 + U_1$

$u_{21} = U_{21} + U_{16} + U_9 + 2U_6 + U_4 + 3U_2 + U_1$

$u_{22} = U_{22} + U_{17} + U_{10} + 2U_7 + U_4 + 3U_3 + U_1$

$u_{23} = U_{23} + U_{18} + U_9 + 2U_6 + 3U_4 + U_2 + U_1$

$u_{24} = U_{24} + U_{18} + U_{10} + 2U_7 + 3U_4 + U_3 + U_1$

$$u_{25} = U_{25} + U_{21} + U_{19} + U_{16} + U_{13} + U_{12} + U_{11} + U_9 + U_8 + $$
$$+ U_7 + 2U_6 + 2U_5 + U_4 + U_3 + 3U_2 + U_1$$

$$u_{26} = U_{26} + U_{22} + U_{20} + U_{17} + U_{14} + U_{12} + U_{11} + U_8 + 2U_7 + $$
$$+ U_6 + 2U_5 + U_4 + 3U_3 + U_2 + U_1$$

$$u_{27} = U_{27} + U_{24} + U_{23} + U_{18} + U_{14} + U_{13} + U_{11} + U_9 + 2U_7 + $$
$$+ 2U_6 + U_5 + 2U_4 + U_3 + U_2 + U_1$$

$$u_{28} = U_{28} + U_{21} + U_{19} + U_{16} + U_{15} + 2U_{14} + U_{10} + U_9 + U_8 + $$
$$+ 2U_6 + U_4 + U_3 + 3U_2 + U_1$$

$$u_{29} = U_{29} + U_{22} + U_{20} + U_{17} + U_{15} + 2U_{13} + U_{10} + U_9 + U_8 + $$
$$+ 2U_7 + 2U_5 + U_4 + 3U_3 + U_2 + U_1$$

$$u_{30} = U_{30} + U_{24} + U_{23} + U_{18} + U_{15} + 2U_{12} + U_{10} + U_9 + U_8 + $$
$$+ 2U_7 + 2U_6 + 3U_4 + U_3 + U_2 + U_1$$

$$u_{31} = U_{31} + 3U_{20} + 3U_{19} + U_{17} + U_{16} + 3U_8 + 6U_5 + 3U_3 + $$
$$+ 3U_2 + U_1$$

$$u_{32} = U_{32} + 3U_{23} + 3U_{21} + U_{18} + U_{16} + 2U_9 + 6U_6 + 3U_4 + $$
$$+ 3U_2 + U_1$$

$$u_{33} = U_{33} + 3U_{24} + 3U_{22} + U_{18} + U_{17} + 3U_{10} + 6U_7 + 3U_4 + $$
$$+ 3U_3 + U_1$$

$$u_{34} = U_{34} + U_{31} + U_{29} + U_{28} + 2U_{26} + 2U_{25} + U_{22} + U_{21} + $$
$$+ 3U_{20} + 3U_{19} + U_{17} + U_{16} + U_{15} + 2U_{14} + 2U_{13} + 2U_{12} + $$

$+ 2U_{11} + U_{10} + U_9 + 3U_8 + 2U_7 + 2U_6 + 6U_5 + U_4 + 3U_3 + 3U_2 +$

$+ U_1$

$u_{35} = U_{35} + U_{32} + U_{30} + U_{28} + 2U_{27} + 2U_{25} + U_{24} + 3U_{23} + 3U_{21} +$

$+ U_{19} + U_{18} + U_{16} + U_{15} + 2U_{14} + 2U_{13} + 2U_{12} + 2U_{11} + U_{10} +$

$+ 3U_9 + U_8 + 2U_7 + 6U_6 + 2U_5 + 3U_4 + U_3 + 3U_2 + U_1$

$u_{36} = U_{36} + U_{33} + U_{30} + U_{29} + 2U_{27} + 2U_{26} + 3U_{24} + U_{23} + 3U_{22} +$

$+ U_{20} + U_{18} + U_{17} + U_{15} + 2U_{14} + 2U_{13} + 2U_{12} + 2U_{11} + 3U_{10} +$

$+ U_9 + U_8 + 6U_7 + 2U_6 + 2U_5 + 3U_4 + 3U_3 + U_2 + U_1$

$u_{37} = U_{37} + 3U_{36} + 3U_{35} + 3U_{34} + U_{33} + U_{32} + U_{31} + 3U_{30} + 3U_{29} +$

$+ 3U_{28} + 6U_{27} + 6U_{26} + 6U_{25} + 3U_{24} + 3U_{23} + 3U_{22} + 3U_{21} +$

$+ 3U_{20} + 3U_{19} + U_{18} + U_{17} + U_{16} + 3U_{15} + 6U_{14} + 6U_{14} + 6U_{13} +$

$+ 6U_{12} + 6U_{11} + 3U_{10} + 3U_9 + 3U_8 + 6U_7 + 6U_6 + 6U_5 + 3U_4 +$

$+ 3U_3 + 3U_2 + U_1$

3. Multiplication formulas for tripolykays

The usefulness of the property of inheritance on the average is pretty well limited to the case where functions having this property occur linearly. Any polynomial in tripolykays for one trisample, however, can be expressed as a linear combination of tripolykays for that trisample, given the proper multiplication formulas. We give below the multiplication formulas for tripolykays.

$$\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right)\cdot\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right)=\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right]\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right]=$$

$$= \frac{1}{rst}\sum_{\neq} x_{ijk}\frac{1}{rst}\sum_{\neq} x_{ijk} = \frac{1}{r^2s^2t^2}\left[\sum_{\neq} x^2_{i_1j_1k_1} +\right.$$

$$+ \sum_{\neq} x_{i_1j_1k_1}x_{i_1j_1k_2} + \sum_{\neq} x_{i_1j_1k_1}x_{i_1j_2k_1} + \sum_{\neq} x_{i_1j_1k_1}x_{i_2j_1k_1} +$$

$$+ \sum_{\neq} x_{i_1j_1k_1}x_{i_1j_2k_2} + \sum_{\neq} x_{i_1j_1k_1}x_{i_2j_1k_2} + \sum_{\neq} x_{i_1j_1k_1}x_{i_2j_2k_1} +$$

$$\left. + \sum_{\neq} x_{i_1j_1k_1}x_{i_2j_2k_2}\right]$$

$$rst\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right)^2=\left[\begin{array}{cc|cc} 2 & - & - & - \\ - & - & - & - \end{array}\right]+(t-1)\left[\begin{array}{cc|cc} 1 & - & 1 & - \\ - & - & - & - \end{array}\right]+$$

$$+(s-1)\left[\begin{array}{cc|cc} 1 & - & - & - \\ 1 & - & - & - \end{array}\right]+(r-1)\left[\begin{array}{cc|cc} 1 & 1 & - & - \\ - & - & - & - \end{array}\right]+$$

$$+(s-1)\;(t-1)\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \end{array}\right]+(r-1(t-1)\left[\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right]+$$

$$+(r-1)(s-1)\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \end{array}\right]$$

$$+(r-1)(s-1)(t-1)\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right]$$

Then after a little manipulation we obtain

$$rst\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right)^2=\left(\begin{array}{cc|cc} 2 & - & - & - \\ - & - & - & - \end{array}\right)+st\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \end{array}\right)+$$

$$+rs\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \end{array}\right)+rt\left(\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right)+t\left(\begin{array}{cc|cc} 1 & - & 1 & - \\ - & - & - & - \end{array}\right)+$$

$$+ s\left(\begin{array}{cc|cc} 1 & - & - & - \\ 1 & - & - & - \end{array}\right) + r\left(\begin{array}{cc|cc} 1 & 1 & - & - \\ - & - & - & - \end{array}\right) + rst\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right)$$

Products of tripolykays of degree 1 and 2 are more complicated, therefore we introduce only results.

$$rst\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right).\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right) = 2U_{15} + 2U_{14} + 2U_{13} + 2U_{12} + 2rU_{10} + 2sU_9 + 2tU_8 + 2rU_7 + 2sU_6 + 2tU_5 + 2rsU_4 + 2rtU_3 + 2stU_2 + rstU_1$$

$$rst\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right).\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \end{array}\right) = 2U_{30} + 2U_{27} + 2rU_{24} + 2sU_{23} + rsU_{18} + 2tU_{12} + 2tU_{11} + 2rtU_7 + 2stU_6 + rstU_4$$

$$rst\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right).\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \end{array}\right) = 2U_{28} + 2U_{25} + 2sU_{21} + 2tU_{19} + stU_{16} + 2rU_{14} + 2rU_{11} + 2rsU_6 + 2rtU_5 + rstU_2$$

$$rst\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right).\left(\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right) = 2U_{29} + 2U_{26} + 2rU_{22} + 2tU_{20} + rtU_{17} + 2sU_{13} + 2sU_{11} + 2rsU_7 + 2stU_5 + rstU_3$$

$$rst\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right).\left(\begin{array}{cc|cc} 1 & - & 1 & - \\ - & - & - & - \end{array}\right) = 2U_{34} + tU_{31} + 2rU_{26} + 2sU_{25} + rtU_{20} + stU_{19} + 2rsU_{12} + rstU_8$$

$$rst\left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right).\left(\begin{array}{cc|cc} 1 & 1 & - & - \\ - & - & - & - \end{array}\right) = 2U_{36} + rU_{33} + 2sU_{27} + 2tU_{26} + rsU_{24} + rtU_{22} + 2stU_{14} + rstU_{10}$$

$$rst \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right) . \left(\begin{array}{cc|cc} 1 & - & - & - \\ 1 & - & - & - \end{array}\right) = 2U_{35} + sU_{32} + 2rU_{27} + 2tU_{25} + $$
$$+ rsU_{23} + stU_{21} + 2rtU_{13} + rstU_{9}$$

$$rst \left(\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right) . \left(\begin{array}{cc|cc} 2 & - & - & - \\ - & - & - & - \end{array}\right) = U_{37} + rU_{36} + sU_{35} + tU_{34} + rsU_{30} + $$
$$+ rtU_{29} + stU_{28} + rstU_{15}$$

The multiplication results of this section enables us to find the variances and covariances, in taking trisamples from a population matrix, of tripolykays of degree 1, 2 or 3.

R E F E R E N C E S

[1] HOOK R.: Symmetric function of a two-way array. Ann. Math. Stat., vol. 27, 1956

[2] LAMOŠ F.: Generalized symmetric means of a three-way array. Acta Fac. Rer. Nat. Univ. Com., Mathematica, XXVII, 1972

[3] LAMOŠ F.: Pairing formulas for tripolykays. Acta Fac. Rer. Nat. Univ. Com., in the press.

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S ú h r n

NIEKOĽKO VZŤAHOV PRE VŠEOBECNÉ SYMETRICKÉ PRIEMERY A TRIPOLYŠTATISTIKY

FRANTIŠEK LAMOŠ, BRATISLAVA

Článok hovorí o všeobecných symetrických priemeroch a o tripolyštatistikách. Opisujú sa obrátené vzťahy medzi tripolyštatistikami a všeobecnými symetrickými priemermi. V závere sú odvodené vzťahy pre súčin tripolyštatistík 1 a 2 stupňa.

Р Е З Ю М Е

НЕСКОЛЬКО ОТНОШЕНИЙ ДЛЯ ОБОБЩЕННЫХ СИММЕТРИЧЕСКИХ СРЕДНИХ И ТРИПОЛИСТАТИСТИК

ФРАНТИШЕК ЛАМОШ, БРАТИСЛАВА

В статьи изучаются обобщенные симметрические средние и триполистатистики. Описываются соотношения между триполистатистиками и обобщенными симметрическими средними.

В конце статьи изложены соотношения для произведения триполистатистик 1-ой и 2-ой степеней.

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CLASSICAL RUNGE-KUTTA FORMULAS OF THE FIFTH ORDER WITH RATIONAL COEFFICIENTS FOR AN ORDINARY DIFFERENTIAL AQUATION OF THE FOURTH ORDER

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The contents of this paper is the numerical solution of an ordinary differential equation of the fourth order by means of classical Runge-Kutta method. The requirement all coefficients to be rational numbers is given. Conditional equations are then identically fulfilled in contrast to the case, when the coefficients are rounded decimal numbers. The formulas of the 5th order are derived and in addition weights are Newton-Cotes numbers.

1. Formulation of the problem. Let us consider the differential equation

$$y^{(4)} = f(x, y, y', y'', y''') \tag{1.1}$$

with the initial contitions

$$y^{(i)}(x_0) = y_0^{(i)}, \ i = 0, 1, 2, 3 \tag{1.2}$$

Denoting $y = u_0$ we can transform problem (1.1), (1.2) to the problem for the system of four differential equations of the 1st order

$$u_i' = u_{i+1}, \ u_3' = f(x, u_0, u_1, u_2, u_3), \ i = 0, 1, 2 \tag{1.3}$$

with the initial conditions

$$u_i(x_0) = u_{i0}, \ i = 0, 1, 2, 3 \tag{1.4}$$

which is equivalent to the problem (1.1), (1.2).

Let the solution

$$u_i = F_i(x),\ i = 0,\ 1,\ 2,\ 3 \tag{1.5}$$

of the problem (1.3), (1.4) exist. Then our task is to determine the increments K_i of functions $F_i(x)$, for $i = 0, 1, 2, 3$, if the increment of the independent variable is h:

$$u_{i0} + K_i = F_i(x_0 + h),\ i = 0,\ 1,\ 2,\ 3 \tag{1.6}$$

2. The Taylor series method. Expending the right sides of (1.6) in Taylor series with respect of (1.3) and (1.5) we have:

$$K_i = \sum_{j=1}^{3-i} \frac{h^j}{j!}\, u_{j+i,0} + \sum_{j=4-i}^{\infty} \frac{h^j}{j!}\, f^{(j+i-4)}(x_0, u_{00}, u_{10}, u_{20}, u_{30}), \tag{2.1}$$

$$i = 0,\ 1,\ 2$$

$$K_3 = \sum_{j=1}^{\infty} \frac{h^j}{j!}\, f^{(j-1)}(x_0,\ u_{00},\ u_{10},\ u_{20},\ u_{30}) \tag{2.2}$$

For the simplification of the following derivation it is useful to introduce some symbols. If we denote

$$f(x,\ u_0,\ u_1,\ u_2,\ u_3) = f \tag{2.3}$$

$$\frac{\partial^{\,p+q+r+s+t}}{\partial x^p \partial u^q \partial u^r \partial u^s \partial u^t}\, f = {}^{p}_{qr}f_{st} \tag{2.4}$$

we can introduce the operator

$$D^k{}_{i_1 i_2} f_{i_3 i_4} = \sum_{p+q+r+s+t=k} \binom{k}{p,\ q,\ r,\ s,\ t} \tag{2.5}$$

$${}^{p}_{i_1+q,\,i_2+r}f_{i_3+s,\,i_4+t} \cdot u_1^q u_2^r u_3^s f^t$$

where $k, p, q, r, s, t, i_1, i_2, i_3, i_4$ are non-negative integers and $\binom{k}{p,\ q,\ r,\ s,\ t} = \frac{k!}{p!q!r!s!t!}$ is the polynomial coefficient. In the denotation (2.4) we shall omit "unnecesary" zeros. We shall also write only x, u_i instead of x_0, u_{i0}, for $i = 0, 1, 2, 3$.

It is easy to show, that

$$(2.6)\quad (D^k{}_{i_1 i_2}f_{i_3 i_4})' = D^{k+1}{}_{i_1 i_2}f_{i_3 i_4} + k(u_2 D^{k-1}{}_{i_1+1, i_2}f_{i_3 i_4} + u_3 D^{k-1}{}_{i_1, i_2+1}f_{i_3 i_4} + fD^{k-1}{}_{i_1 i_2}f_{i_3+1, i_4} + DfD^{k-1}{}_{i_1 i_2}f_{i_3, i_4+1})$$

Using the relations (2.3) - (2.6) and denoting

$$(2.7)\qquad A_0 = f$$

$$(2.8)\qquad A_1 = Df$$

$$(2.9)\qquad A_2 = D^2f + u_2\,{}_{10}f + u_3\,{}_{01}f + ff_{10} + f_{01}Df$$

$$(2.10)\qquad A_3 = D^3f + f_{01}D^2f + f_{01}^2Df + f_{10}Df + 3(u_2D_{10}f + u_3D_{01}f + fDf_{10} + DfDf_{01}) + u_3\,{}_{10}f + f\,{}_{01}f + (u_2\,{}_{10}f + u_3\,{}_{01}f + ff_{10})f_{01}$$

$$(2.11)\qquad A_4 = D^4f + f_{01}D^3f + f_{01}^2D^2f + f_{10}D^2f + 4Df_{01}D^2f + 6(u_2D^2\,{}_{10}f + u_3D^2\,{}_{01}f + fD^2f_{10} + DfD^2f_{01}) + 3f_{02}(Df)^2 + f_{01}^3Df + 4Df_{10}Df + 2f_{10}\,f_{01}Df + {}_{01}fDf + 7f_{01}Df_{01}Df + 6(u_2\,{}_{10}f_{01}Df + u_3\,{}_{01}f_{01}Df + ff_{11}Df + u_2u_3\,{}_{11}f + u_2\,f\,{}_{10}f_{10} + u_3\,f\,{}_{01}f_{10}) + 3(u_2D_{10}f + u_3D_{01}f + fDf_{10})\,f_{01} + 4(u_2\,{}_{10}fDf_{01} + u_3\,{}_{01}fDf_{01} + ff_{10}Df_{01} + u_3D_{10}f + fD_{01}f) + 3(u_2^2\,{}_{20}f + u_3^2\,{}_{02}f + f^2\,f_{20}) + (u_2\,{}_{10}f + u_3\,{}_{01}f + ff_{10})\,f_{01}^2 +$$

$$+ (u_3\ {}_{10}f + f\ {}_{01}f)f_{01} + (u_2\ {}_{10}f + u_3\ {}_{01}f + ff_{10})\ f_{10} +$$
$$+ f\ {}_{10}f$$

we can (2.1) and (2.2) rewrite:

(2.12) $$K_i = \sum_{j=1}^{3-i} \frac{h^j}{j!} u_{j+i} + \sum_{j=4-i}^{5} \frac{h^j}{j!} A_{j+i-4} + \ldots,\ i = 0, 1, 2$$

(2.13) $$K_3 = \sum_{j=1}^{5} \frac{h^j}{j!} A_{j-1} + \ldots$$

3. Runge-Kutta method - the 5th order formulas. Now we shall look for the approximate solutions k_i of K_i, for $i = 0, 1, 2, 3$.

Let

$$m_{i0} = u_{i+1},\ m_{30} = f(x, u_0, u_1, u_2, u_3)$$

(3.1) $$m_{iq} = u_{i+1} + w_{iq}h,\ m_{3q} = f(x + a_q h,\ u_0 + i_{0q}h,\ u_1 +$$
$$u_1 + l_{1q}h,\ u_2 + l_{2q}h,\ u_3 + l_{3q}h),\ i = 0, 1, 2;$$
$$q = 1, 2, \ldots, 5$$

where

(3.2) $$w_{iq} = \sum_{j=0}^{q-1} B_{iqj}\, m_{i+1,j},\ q = 1, 2, \ldots, 5;\ i = 0, 1, 2$$

(3.3) $$l_{iq} = \sum_{j=0}^{q-1} b_{iqj}\, m_{ij},\ q = 1, 2, \ldots, 5;\ i = 0, 1, 2, 3$$

Let us still denote:

(3.4) $$k_{ij} = m_{ij}h,\ i = 0, 1, 2, 3;\ j = 0, 1, \ldots, 5$$

Then we shall look for k_i in the form:

(3.5) $$k_i = \sum_{j=0}^{5} p_{ij}k_{ij},\ i = 0, 1, 2, 3$$

Our aim is to determine coefficients a_q, B_{iqj}, b_{iqj} and weights p_{ij} so that expansions k_i and K_i in Taylor series, for $i = 0, 1, 2, 3$, conform one another up to the terms with h^5. In addition we require all coefficients to be rational numbers. Let the coefficients B_{iqj}, b_{iqj} satisfy so called Runge's conditions:

$$(3.6) \qquad \sum_{k=0}^{q-1} B_{iqk} = \sum_{k=0}^{q-1} b_{iqk} = a_q, \; q = 1, 2, \ldots, 5;$$

$$i = 0, 1, 2; \; j = 0, 1, 2, 3$$

Extending (3.1) in Taylor series, introducing into (3.5), with respect of (3.4), and comparing with the corresponding terms of (2.12) and (2.13) we can get following system of nonlinear equations:

$$(3.7) \qquad \sum_{i=k}^{5} p_{ji} a_i^k = \frac{1}{k+1}, \; k = 0, 1; \; j = 0, 1, 2, 3$$

$$(3.8) \qquad \sum_{i=2}^{5} p_{ji} c_{ji1}^{(0)} = \frac{1}{6}, \; j = 0, 1, 2$$

$$(3.9) \qquad \sum_{i=3}^{5} p_{ji} d_{j+1,i1}^{(0)} = \frac{1}{24}, \; j = 0, 1$$

$$(3.10) \qquad \sum_{i=4}^{5} p_{ki} r_{ji1}^{(k)} = \frac{1}{120}, \; k = 1, 2; \; j = 0, 1, \ldots, 4 - k$$

$$(3.11) \qquad \sum_{i=3}^{5} p_{ji} d_{j+1,i2}^{(j-1)} = \frac{1}{60}, \; j = 1, 2$$

$$(3.12) \qquad \sum_{i=2}^{5} p_{ji} c_{jik}^{(j-2)} = \frac{1}{(k+1)(k+2)}, \; j, k = 2, 3$$

(3.13) $$\sum_{i=3}^{5} p_{k+1,i} f_{ji1}^{(k)} = \frac{1}{24}, \quad k = 1, 2; \; j = 0, 1, \ldots, 4 - k$$

(3.14) $$\sum_{i=3}^{5} p_{k+2,i} g_{ji1}^{(k)} = \frac{1}{40}, \quad k = 0, 1; \; j = 0, 1, \ldots, 3 - k$$

(3.15) $$\sum_{i=4}^{5} p_{2i} e_{ji1}^{(3)} = \frac{1}{120}, \quad j = 0, 1, 2, 3$$

(3.16) $$\sum_{i=1}^{5} p_{3i} a_i^j = \frac{1}{j+1}, \quad j = 2, 3, 4$$

(3.17) $$\sum_{i=2}^{5} p_{3i} a_i^k c_{ji1}^{(1)} = \frac{1}{2(k+3)}, \quad j = 0, 1, 2, 3; \; k = 0, 1, 2$$

(3.18) $$\sum_{i=3}^{5} p_{3i} d_{ji1}^{(3)} = \frac{1}{24}, \quad j = 0, 1, 2, 3$$

(3.19) $$\sum_{i=3}^{5} p_{3i} d_{ji2}^{(j)} = \frac{1}{60}, \quad j = 2, 3$$

(3.20) $$\sum_{i=2}^{5} p_{3i} c_{ji1}^{(1)} c_{ki1}^{(1)} = \frac{1}{20}, \quad j = 0, 1, 2, 3; \; k = j, \ldots, 3$$

(3.21) $$\sum_{i=3}^{5} p_{3i} a_i d_{ji1}^{(k)} = \frac{1}{30}, \quad j = 0, 1, 2; \; k = 2, 3$$

(3.22) $$\sum_{i=4}^{5} p_{3i} r_{ji1}^{(0)} = \frac{1}{120}, \quad j = 1, 2$$

(3.23) $$\sum_{i=4}^{5} p_{3i} r_{ji1}^{(2k+1)} = \frac{1}{120}, \quad k = 0, 1; \; j = 0, 1, \ldots, k + 2$$

(3.24) $$\sum_{i=4}^{5} p_{3i} r_{ji1}^{(2)} = \frac{1}{120}, \quad j = 0, 1$$

$$(3.25)\qquad \sum_{i=4}^{5} p_{0i}e_{2i1}^{(0)} = \frac{1}{120}$$

$$(3.26)\qquad \sum_{i=2}^{5} p_{3i}a_i c_{3i2}^{(1)} = \frac{1}{15}$$

$$(3.27)\qquad \sum_{i=3}^{5} p_{3i}(g_{3i1}^{(1)} + a_i d_{3i1}^{(3)} = \frac{7}{120}$$

$$(3.28)\qquad \sum_{i=4}^{5} p_{3i}(r_{3i1}^{(1)} + r_{2i1}^{(2)} = \frac{1}{60}$$

In this system particular symbols have following meaning:

$$c_{ijk}^{(0)} = \sum_{q=1}^{j-1} a_q^k B_{ijq},\ j = 2, 3, 4, 5; \begin{cases} i = 0, 1;\ k = 1 \\ i = 2;\ k = 1, 2, 3 \end{cases}$$

$$d_{ijk}^{(0)} = \sum_{q=2}^{j-1} c_{iqk}^{(0)} B_{i-1,jq},\ j = 3, 4, 5; \begin{cases} i = 1;\ k = 1 \\ i = 2;\ k = 1, 2 \end{cases}$$

$$e_{2j1}^{(0)} = \sum_{q=3}^{j-1} d_{2q1}^{(0)} B_{0jq},\ j = 4, 5$$

$$c_{ijk}^{(1)} = \sum_{q=1}^{i-1} a_q^k b_{ijq},\ j = 2, 3, 4, 5; \begin{cases} i = 0, 1, 2;\ k = 1 \\ i = 3;\ k = 1, 2, 3 \end{cases}$$

$$d_{ijk}^{(1)} = \sum_{q=2}^{j-1} c_{iqk}^{(1)} B_{2jq},\ j = 3, 4, 5; \begin{cases} i = 0, 1, 2;\ k = 1 \\ i = 3;\ k = 1, 2 \end{cases}$$

$$e_{ij1}^{(k)} = \sum_{q=3}^{j-1} d_{iq1}^{(k)} B_{kjq},\ k = 1, 2;\ j = 4, 5;\ i = 0, 1, \ldots, 4 - k$$

$$d_{ijk}^{(2)} = \sum_{q=2}^{j-1} c_{iqk}^{(0)} b_{ijq},\ j = 3, 4, 5; \begin{cases} i = 0, 1;\ k = 1 \\ i = 2;\ k = 1, 2 \end{cases}$$

$$d_{ijk}^{(3)} = \sum_{q=2}^{j-1} c_{iqk}^{(1)} h_{3jq},\ j = 3, 4, 5; \begin{cases} i = 0, 1, 2;\ k = 1 \\ i = 3;\ k = 1, 2 \end{cases}$$

$$e^{(3)}_{ij1} = \sum_{q=3}^{j-1} d^{(3)}_{iq1}B_{2jq}, \; j = 4, 5; \; i = 0, 1, 2, 3$$

$$g^{(0)}_{ij1} = \sum_{q=2}^{j-1} a_q c^{(1)}_{iq1}B_{2jq}, \; j = 3, 4, 5; \; i = 0, 1, 2, 3$$

$$g^{(1)}_{ij1} = \sum_{q=2}^{j-1} a_q c^{(1)}_{iq1}b_{3jq}, \; j = 3, 4, 5; \; i = 0, 1, 2, 3$$

$$r^{(0)}_{ij1} = \sum_{q=3}^{j-1} a^{(0)}_{iq1}b_{i-1,jq}, \; j = 4, 5; \; i = 1, 2$$

$$r^{(k)}_{ij1} = \sum_{q=3}^{j-1} d^{(k)}_{iq1}b_{k+1,jq}, \; k = 1, 2; \; j = 4, 5; \; i = 0, 1, \ldots, 4-k$$

$$r^{(3)}_{ij1} = \sum_{q=3}^{j-1} d^{(3)}_{iq1}b_{3jq}, \; j = 4, 5; \; i = 0, 1, 2, 3$$

The system (3.7) - (3.28) is the system of 96 equations containing 99 unknows: B_{ijk}, for $i = 0, 1, 2$; $j = 2, 3, 4, 5$; $k = 1, 2, \ldots, j - 1$, b_{ijk}, for $i = 0, 1, 2, 3$; $j = 2, 3, 4, 5$; $k = 1, 2, \ldots, j - 1$, p_{ij}, for $i = 0, 1, 2, 3$; $j = 0, 1, \ldots, 5$, a_i, for $i = 1, 2, \ldots, 5$.

Solving this system analogically to Huťa [3] we can get the following formulas of the fifth order (in the variables of the original problem (1.1) and (1.2):

(3.29) $N_0 = 0, \; N_1 = \frac{1}{6}, \; N_i = \frac{i-1}{4}, \; i = 2, 3, 4, 5$

(3.30) $M_{0i} = 0$

(3.31) $M_{1i} = \frac{1}{6}k_{i0}$

(3.32) $M_{2i} = \frac{1}{16}(k_{i0} + 3k_{i1})$

(3.33) $$M_{3i} = \frac{1}{4}(k_{i0} - 3k_{i1} + 4k_{i2})$$

(3.34) $$M_{4i} = \frac{3}{16}(k_{i0} + 3k_{i3})$$

(3.35) $$M_{5i} = \frac{1}{7}(-4k_{i0} + 3k_{i1} + 12k_{i2} + 8k_{i4})$$

all for $i = 0, 1, 2, 3$

(3.36) $$k_{ij} = (y_0^{(i+1)} + M_{j,i+1})h, \quad i = 0, 1, 2; \; j = 0, 1, \ldots, 5$$

(3.37) $$k_{3j} = f(x_0 + N_j, y_0 + M_{j0}, y_0' + M_{j1}, y_0'' + M_{j2}, y_0''' + M_{j3}), \quad j = 0, 1, \ldots, 5$$

(3.38) $$k_i = \frac{1}{90}(7k_{i0} + 32k_{i2} + 12k_{i3} + 32k_{i4} + 7k_{i5}),$$

$i = 0, 1, 2, 3.$

We can see that the requirement given above is fulfilled and in addition the weights in (3.38) are Newton-Cotes numbers. From the form of the formula (3.29) - (3.38) follows their easy using on digital computers.

REFERENCES

[1] HUŤA A.: Une amélioration de la méthode de Runge-Kutta-Nyström pour la résolution numérique des équations différentielles du premier ordre. Acta F.R.N.Univ. Comen., Mathematica IV-VI, Bratislava 1956

[2] HUŤA A.: Contribution á la formule de sixiéme ordre dans la méthode de Runge-Kutta-Nyström. Acta F.R.N.Univ. Comen., Mathematica I-II, Bratislava 1957

[3] HUŤA A.: Contribution to the Numerical Solution of Differential Equations by Means of Runge-Kutta Formulas with Newton-Cotes Numbers Weights. Acta F.R.N.Univ. Comen., Mathematica

XXVIII, Bratislava 1972.

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S ú h r n

KLASICKÉ VZORCE RUNGE-KUTTA PIATEHO RÁDU S RACIONÁLNYMI KOEFICIENTMI PRE OBYČAJNÚ DIFERENCIÁLNU ROVNICU ŠTVRTÉHO RÁDU

JÁN ŠTEKAUER, BRATISLAVA

Obsahom článku je numerické riešenie obyčajnej diferenciálnej rovnice štvrtého rádu klasickou metódou Runge-Kutta. V práci je vyslovená požiadavka, aby všetky koeficienty boli racionálne čísla. Sú odvodené vzorce piateho rádu, pričom navyše váhy sú Newtonove-Cotesove čísla.

Р Е З Ю М Е

КЛАССИЧЕСКИЕ ФОРМУЛЫ РУНГЕ-КУТТА ПЯТОГО ПОРЯДКА С РАЦИОНАЛЬНЫМИ КОЭФФИЦИЕНТАМИ ДЛЯ ОБЫКНОВЕННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ЧЕТВЕРТОГО ПОРЯДКА

ЯН ШТЕКАУЕР, БРАТИСЛАВА

Содержанием этой статьи является численное решение обыкновенного дифференциального уравнения четвертого порядка классическим

методом Рунге-Кутта. Притом предъявлено требование, чтобы все коэффициенты были рациональные числа. Б статье выведены формули пятого порядка и вдобавок к этому требованию весы являются чицлами Ньютона-Котеса.

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ON A NONLINEARY STATIONARY PARABOLIC BOUNDARY VALUE PROBLEM

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1. Introduction

In the widely worked classic and non-classical theory of parabolic differential equations (see for instance [4], [5], [6]), the different mixed problems are studied on the finite time-cylinder $Q = \Omega \times \langle 0, T\rangle$, where Ω is a domain of the Euclidean space R_m and T is a fixed positive number. In the paper [8] an initial-boundary value problem for the system

$$(A_1) \qquad D_t u - \sum_{|k|=2b} A_k(x, t)\, D_x^k u = F(x, t)u$$

with the nonlinear differential operator $F(x, t)$ of the order $\leqq 2b - 1$ was considered on Q.

The present paper deals with a stationary mixed problem for the equation of the type (A_1) on the infinite cylinder $\Omega \times \langle 0, \infty)$. This problem is studied in a complete topological space of locally bounded and locally Hölder continuous functions. Using the S. D. E'jdel'man's and S. D. Ivasišen's estimation of the Green function from [1] we establish sufficient conditions for the existence of solution and investigate its structure. These conditions restrict the growth of $F(x, t)$ much more than the conditions (17) formulated in [8], however they express the dependence of the growth of F by the selection of the time cylinder.

By the below described method we may solve the nonlinear Cauchy problem for the equation (A_1) on the unbounded sets $R_m \times \langle 0, T\rangle$ or $R_m \times \langle 0, \infty)$ in a class of the locally Hölder continuous functions.

2. The formulation of problem.

First of all we introduce notions and notations which will be used throughout this paper.

The symbol Ω means a bounded domain of R_m for $m \geqq 1$ with the boundary $\partial\Omega$ and the diameter diam Ω. For $0 < T < \infty$ we define the bounded cylindrical domain $Q_T = \Omega \times \langle 0, T\rangle$ with the lateral surface $\Gamma_T = \partial\Omega \times \langle 0, T\rangle$ and for $T = \infty$ we put $Q_\infty = \Omega \times \langle 0, \infty)$ with $\Gamma_\infty = \partial\Omega \times \langle 0, \infty)$.

For any integer r the symbol $t(r)$ means Card $\{k = (k_1, \ldots, k_m) : |k| = r\}$

and $s = \sum_{r=0}^{2b-1} t(r)$ for $b \geqq 1$. The Cartesian product

$Q_\infty \times \prod_{i=1}^{s} \prod_{j=1}^{p} \{-\infty < u_j^i < \infty\}$ for $p \geqq 1$ will be denoted by H_∞.

By $\mathcal{J}$ and $\mathcal{O}$ the $(p \times 1)$ - unit vector and the $(p \times 1)$ - zero vector is denoted respectively. E_1 means the $(p \times p)$ - matrix whose all elements are equal to 1 and E means the $(p \times p)$ - unit matrix.

The class of the holder continuous vector functions with respect to x on Ω_1 with exponent $0 < \rho \leqq 1$ and uniformly with respect to λ on Ω_2 will be denoted by

$H_\rho(x, \Omega_1; \lambda, \Omega_2)$. The class of locally Hölder continuous vector functions with respect to x $H_{loc,\rho}(x, \Omega_1; \lambda, \Omega_2) =$ $= \{u \in H_\rho(x, \Delta_1; \lambda, \Omega_2)$: for any subset Δ_1 of the domain Ω_1 such that $\bar{\Delta}_1 \subset \Omega_1\}$. The system of Hölder continuous vector functions with respect to x and locally uniformly with respect to λ $H_\rho^{loc}(x, \Omega_1; \lambda, \Omega_2) = \{u \in H_\rho(x, \Omega_1; \lambda, \Delta_2)$: for any subset Δ_2 of the domain Ω_2 such that $\bar{\Delta}_2 \subset \Omega_2\}$.

Consider the system of $p \geqq 1$ differential equations of the 2b-th order ($b \geqq 1$) with p unknown functions

$$(1)\quad L(x, D_x, D_t)\, u \overset{df}{=} D_t u - \sum_{|k|=2b} A_k(x)\, D_x^k u =$$

$$= F(x, t, \ldots, D_x^\gamma u, \ldots), \; (x, t) \in \Omega_\infty$$

where $\gamma = (\gamma_1, \ldots, \gamma_m)$ is a multiindex such that $0 \leqq |\gamma| \leqq$ $= 2b - 1$.

The solution of (1) is required to fulfil the initial condition

$$(2)\qquad u(x, 0) = \sigma, \; x \in \Omega$$

and the boundary conditions

$$(3)\qquad B_q(x, D_x)u|_{\Gamma_\infty} \overset{df}{=} \sum_{|k| \leqq r_q} (B_k^{(q)}, D_x^k u)|_{\Gamma_\infty} = 0$$

for $r_q \leqq 2b - 1$ and $q = 1, \ldots, bp$; $A_k(.) = (a_k^{hj}(.))_{h,j=1}^{p}$ is a matrix function and $B_k^{(q)}(.) = (b_k^{q1}(.), \ldots, b_k^{qp}(.))$ is a vector function on Ω and $F = (f_1, \ldots, f_p)$ on H_∞.

The just formulated problem (1), (2), (3), will be solved in the following class of Hölder continuous functions:

Consider the nondecreasing parametric function $f_{B,\sigma}$ mapping the interval $\langle 0, \infty)$ into $\langle 0, \infty)$ with the properties

$$(B_1) \qquad f_{B,\sigma}(t) \geqq \int_0^t z^{-\sigma} e^{Bz} dz$$

for $t \in \langle 0, \infty)$ and $B > 0$ and $0 < \sigma < 1$ and

$$(B_2) \qquad f_{B,\sigma}(0) > 0.$$

Further, for a real function v on R_{m+1} and for $0 < a < 1$ we put

$$\langle v(x, t)\rangle_{a,x} = |v(x, t) - v(y, t)| \; |x - y|^{-a}$$

and

$$\langle v(x, t)\rangle_{a,t} = |v(x, t) - v(x, t')| \; |t - t'|^{-a}.$$

Then for $b \geqq 1$ and $0 < \alpha < 1$ we define the linear space $C^{2b-1+\alpha,\,(2b-1+\alpha)/2b}_{x,t,f(B,\varkappa,\mu,\nu)}(Q_\infty)$ of the vector functions $u(x, t) = (u_1(x, t), \ldots, u_p(x, t))$ from Q_∞ into R_p by the equality

$$(4) \qquad \|u\|^{f(B,\varkappa,\mu,\nu)}_{2b-1+\alpha,\, Q_\infty} \overset{df}{=}$$

$$\overset{df}{=} \max_{j=1,\ldots,p} \Big\{ \sum_{i=0}^{2b-1} \sum_{|k|=i} \sup_{Q_\infty} \left[|D_x^k u_j(x, t)| \, f^{-1}_{B,\varkappa}(t) \right] +$$

$$+ \sum_{|k|=2b-1} \sup_{\substack{(x, t), (y, t) \in Q_\infty \\ x \neq y}} \left[\langle D_x^k u_j(x, t)\rangle_{\alpha,x} \, f^{-1}_{B,\mu}(t) \right] +$$

$$+ \sum_{i=0}^{2b-1} \sum_{|k|=i} \sup_{\substack{(x, t), (x, t') \in Q_\infty \\ t \neq t'}}$$

$$[\langle D_x^k u_j(x, t)\rangle_{(2b-1+\alpha-|k|)/2b,\, t} \cdot$$

$$\cdot\, f_{B,\nu}^{-1}(|t-t'|)\, f_{B,\nu}^{-1}(t^*)]\} < \infty$$

where the parameters $\varkappa$, μ, and ν belong to the interval $(0, 1)$ ant $t^* = \max(t, t')$.

R e m a r k 1. a) We immediately see that if the function $u \in C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,t,f(B,\varkappa,\mu,\nu)}(Q_\infty)$ then $D_x^k u \in H_\alpha^{loc}(x, \Omega;\, t, \langle 0, \infty))$ for $|k| = 2b - 1$ and $D_x^k u \in H_{loc,(2b-1+\alpha-|k|)/2b}(t, \langle 0, \infty);\, x, \Omega)$ for $|k| = 0, 1, \ldots, 2b - 1$.

b) The derivatives $D_x^k u$ for $|k| = 2b - 1$ may be continuously proceeded on $R_m \times \langle 0, T\rangle$ for any $T \in (0, \infty)$ and then using the mean value theorem and the relation

$$(5) \qquad K_m \sum_{i=1}^m |x_i| \leqq |x| \leqq \sum_{i=1}^m |x_i|$$

for $x \in R_m$ and $K_m \in (0, (1/\sqrt{2})^{m-1})$ one obtains: $D_x^k u \in H_1^{loc}(x, \Omega;\, t, \langle 0, \infty))$ for $|k| = 0, 1, \ldots, 2b - 2$.

The operator L and B_q from (1) and (3) and the boundary $\partial\Omega$ satisfy the following assumptions:

(A) The system (1) is uniformly parabolic in the sense of I. G. Petrovskij ([5]).

(B) The operator B_q and the system (1) are connected by the "uniform supplementary" condition ([5] , [8]).

($D_{\ell+\alpha}$) The coefficients A_k and $B_k^{(q)}$ and the boundary $\partial\Omega$ satisfy the condition (C_ℓ) from [1] (or the modified condition

($C_{\ell+\alpha}$) for the domain Ω from [8]) for $0 < \alpha < 1$ and an integer $\ell \geqq 0$.

The estimations of the Green's matrix of the operator L and its derivatives on the infinite cylinder Q_∞ are established by

T h e o r e m 1. (S. D. E'idel'man and S. D. Ivasišen [1].) Let the assumptions (A), (B), $D_{\ell+\alpha}$) be fulfilled. Then there exists the Green's matrix function $G(x, t; \xi, \tau)$ of the problem (1), (2), (3), (with $F = 0$). For $0 \leqq \tau < t_0 < t < \infty$ and $x, y, \xi \in R_m$ ($r = 1/(2b-1)$) we have

$$(6) \qquad |D_t^{k_0} D_x^k G(x, t; \xi, \tau)| \leqq$$

$$\leqq C(t-\tau)^{-(m+2bk_0+|k|)/2b} \times$$

$$\times \exp\{A(t-\tau) - c|x-\xi|^{2br}/(t-\tau)^r\} E_1$$

if $2bk_0 + |k| \leqq 2b + \ell$;

$$(7) \qquad |D_t^{k_0} D_x^k G(x, t; \xi, \tau) - D_t^{k_0} D_x^k G(y, t; \xi, \tau)| \leqq$$

$$\leqq C|x-y|^\alpha (t-\tau)^{-(m+2bk_0+|k|+\alpha)/2b} \cdot$$

$$\cdot \exp\{A(t-\tau) - c|x^* - \xi|^{2br}/(t-\tau)^r\} E_1$$

if $2bk_0 + |k| = 2b + \ell$ and $|x^* - \xi| = \min(|x-\xi|, |y-\xi|)$;

$$(8) \quad |D_t^{k_0} D_x^k G(x, t; \xi, \tau) - D_t^{k_0} D_x^k G(x, t_0; \xi, \tau)| \leqq$$

$$\leqq C(t-t_0)^{\{2b(1-k_0)+\ell-|k|+\alpha\}/2b} (t_0-\tau)^{-(m+2b+\ell+\alpha)/2b} \cdot$$

$$\cdot \exp A\{(t-\tau) - c|x-\xi|^{2br}/(t-\tau)^r\} E_1$$

if $\ell < 2bk_0 + |k| \leqq 2b + \ell$; A, C, c are positive constants independing of x, y, t, t_0 and ξ, τ.

R e m a r k 2. In our considerations we shall often use instead of estimation (6) its modified form

$$(6') \ |D_t^{k_0} D_x^k G(x, t; \xi, \tau)| \leqq C(t-\tau)^{-\mu} |x-\xi|^{2b\mu - (m+2bk_0+|k|)} \cdot$$

$$\cdot \left[|x-\xi|^{2b}/(t-\tau)\right]^{(m+2bk_0+|k|-2b\mu)/2b} \cdot$$

$$\cdot \exp\{-c|x-\xi|^{2br}/(t-\tau)^r\} \, e^{A(t-\tau)} E_1 \leqq$$

$$\leqq K(t-\tau)^{-\mu}|x-\xi|^{2b\mu-(m+2bk_0+|k|)} \, e^{A(t-\tau)} E_1$$

for $0 \leqq \tau < t < \infty$ and $x, \xi \in R_m$, $\xi \neq x$ and $\mu \leqq$ $\leqq (m + 2bk_0 + |k|)/2b$, where $K > 0$ does not depend of x, t, ξ and τ. If the fraction $|x-\xi|^{2b}/(t-\tau) \geqq \varepsilon > 0$ then this estimation holds for any $\mu \in (-\infty, \infty)$.

An obvious consequence of Remark 2 and Theorem 3 from [8] is

T h e o r e m 2. Let the assumptions (A), (B), $(D_{\ell+\alpha})$ be satisfied and let $\Phi \in C^0(Q_T) \cap H_\alpha^{loc}(x, \Omega; t, \langle 0, \infty))$ be bounded (p x 1) - vector function in the norm $\|\cdot\|_{0,Q_T}$ ([8]) on any cylinder Q_T, $T \in (0, \infty)$. Then the function

$$u(x, t) = \int_0^t d\tau \int_\Omega G(x, t; \xi, \tau) \, \Phi(\xi, \tau) \, d\xi$$

is a solution of the linear equation $L(x; D_x, D_t) u = \Phi(x, t)$ on Q_∞ satisfying data (2) and (3).

For brevity in the following text we shall denote arbitrary positive constants by L.

3. The existence of solution

To derive the fundamental theorem we introduce some properties of the space $P(Q_\infty) =$

$= C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,\ t,f(B,\varkappa,\mu,\nu)}(Q_\infty)$ and the Green's function G and the integro-differential operator

$$(9)\quad A(x,t)u = \int_0^t d\tau \int_\Omega G(x,t;\xi,\tau)\,F[\xi,\tau,\ldots$$

$$\ldots, D_x^{\gamma} u(\xi,\tau),\ldots]\,d\xi\ .$$

It is obvious that the space $P(Q_\infty)$ with the norm defined in (4) does not form the Banach space. The sequence of the functional $\{\sigma_n\}_{n=1}^{\infty}$ defined by

$$\sigma_n(u) = \|u\|^{f(B,\varkappa,\mu,\nu)}_{2b-1+\alpha,Q_n}\ ,\ n = 1,2,\ldots$$

determines a countable, monotone family of seminorms on $P(Q_\infty)$ satisfying the axiom of separation, that is, for any $u_0 \in P(Q_\infty)$, $u_0 \neq \mathcal{O}$ there is n_0 such that $\sigma_{n_0}(u_0) \neq 0$. The linear space $P(Q_\infty)$, topologized by the family of seminorms $\{\sigma_n(u)\}_{n=1}^{\infty}$ in such way that an arbitrary neighbourhood $N(\mathcal{O}, n, \varepsilon)$ of the zero element $\mathcal{O} \in P(Q_\infty)$ is determined by

$$N(\mathcal{O}, n, \varepsilon) = \{u \in P(Q_\infty) : \sigma_n(u) < \varepsilon\}\ ,\ n = 1,2,\ldots,\ \varepsilon > 0,$$

is a locally convex, linear topological Hausdorff space. Denote it by $(P(Q_\infty), \tau)$, where τ is the above constructed topology.

L e m m a 1. The locally convex, linear topological space $(P(Q_\infty), \tau)$ is complete.

P r o o f . Since the topology τ is defined by the countable family of seminorms $\{\sigma_n\}_{n=1}^{\infty}$ it is sufficient to show that the space $(P(Q_\infty), \tau)$ is sequentially complete for $p = 1$. Let $\{u_s(x, t)\}_{s=1}^{\infty}$ be a fundamental sequence of real functions of $(P(Q_\infty), \tau)$, that is, $u_\ell - u_s \in N(\sigma, n, \varepsilon)$ for any ℓ, $s > s_0(n, \varepsilon)$ (s_0 is a fixed positive integer) and any neighbourhood $N(\sigma, n, \varepsilon)$. Hence

$$| D_x^k u_\ell(x, t) - D_x^k u_s(x, t) | \leqq \varepsilon f_{B,\varkappa}(t) \leqq \varepsilon f_{B,\varkappa}(n)$$

for $|k| = 0, 1, \ldots, 2b-1$ on any finite cylinder Q_n. Consequently there is a function $u \in C^{2b-1}(Q_\infty)$ such that $\lim_{s\to\infty} D_x^k u_s(x, t) = D_x^k u(x, t)$ at every point $(x, t) \in Q_\infty$ for $|k| = 0, 1, \ldots, 2b-1$. Letting $s \to \infty$ in the relations

$$|D_x^k u_s(x, t)| \, f^{-1}_{B,\varkappa}(t) \leqq L \quad \text{for } |k| = 0, 1, \ldots, 2b-1$$

$$|D_x^k u_s(x, t) - D_x^k u_s(y, t)| \, f^{-1}_{B,\mu}(t) \leqq L\, |x-y|^{\alpha} \quad \text{for } |k| = 2b-1$$

$$|D_x^k u_s(x, t) - D_x^k u_s(x, t')| \, f^{-1}_{B,\nu}(|t-t'|)\, f^{-1}_{B,\nu}(t^*) \leqq$$

$$\leqq L\, |t-t'|^{(2b-1+\alpha-|k|)/2b} \quad \text{for } |k| = 0, 1, \ldots, 2b-1$$

we get $u \in P(Q_\infty)$. From the inequality $\sigma_n(u_\ell - u_s) < \varepsilon$ we easily obtain that $u_s - u \in N(\sigma, n, L\varepsilon)$ for all $s > s_0(n, \varepsilon)$ what guarantees the convergence of the sequence $\{u_s(x, t)\}_{s=1}^{\infty}$ to $u(x, t)$ in the topology τ.

<u>Lemma 2.</u> Let (x, t), (y, t), (x, t'), $t < t'$ be points of Q_∞ and $|k| = 0, 1, \ldots, 2b-1$ and $\beta \in (0, 1)$. If the hypotheses (A), (B), (D_α) hold then

$$(10)\quad I_{1,k}(x, t) = \int_0^t d\tau \int_\Omega |D_x^k G(x, t; \xi, \tau)|\, d\xi \leqq L\, f_{A,\varkappa}(t) E_1$$

for $0 < \varkappa < (m + |k|)/2b$ $(\varkappa < 1)$ and

$$(11)\quad I_{2,k}(x, y, t) =$$

$$= \int_0^t d\tau \int_\Omega |D_x^k G(x, t; \xi, \tau) - D_x^k G(y, t; \xi, \tau)|\, d\xi \leqq$$

$$\leqq L|x - y|^\beta f_{A,\mu}(t) g(|x - y|)\, E_1 \leqq L|x - y|^\beta f_{a,\mu}(t)\, E_1$$

for $\mu \in (0, 1)$ such that $(|k| + 1)/2b \leqq \mu \leqq (m + |k|)/2b$ if $0 \leqq |k| \leqq 2b - 2$ and $(2b - 1 + \beta)/2b \leqq \mu$ if $|k| = 2b - 1$, where $g(z) = z^{1-\beta + 2b(\mu - 1)[|k|/(2b-1)]}$. (The expression $[x]$ in the exponent denotes the integer for which $[x] \leqq x < [x] + 1$.)

If the conditions (A), (B) and $(D_{2b-1+\alpha})$ are satisfied then

$$(12)\quad I_{3,k}(x, t, t') =$$

$$= \int_0^t d\tau \int_\Omega |D_x^k G(x, t; \xi, \tau) - D_x^k G(x, t'; \xi, \tau)|\, d\xi +$$

$$+ \int_t^{t'} d\tau \int_\Omega |D_x^k G(x, t'; \xi, \tau)|\, d\xi \leqq L(t' - t)^{(2b-1+\beta-|k|)/2b} \cdot$$

$$\cdot f_{A,\nu}(t' - t) f_{A,\nu}(t') h(t' - t)\, E_1 \leqq$$

$$\leqq L(t' - t)^{(2b-1+\beta-|k|)/2b} f_{A,\nu}(t' - t) f_{A,\nu}(t')\, E_1$$

for $(4b - 1 + \beta)/4b \leqq \nu < 1$, where $h(z) = z^{(2b\nu - 2b + 1 - \beta)/2b} \cdot f_{B,\nu}^{-1}(z)$.

P r o o f . The estimation (10) follows directly by (6'). For $0 \leqq |k| \leqq 2b - 2$ the inequality (11) may be established by (6') using the mean value theorem. Really, there is
$x_i^* = (y_1, \ldots, y_{i-1}, \xi_i, x_{i+1}, \ldots, x_m) \in R_m$ $(y = (y_1, \ldots, y_m))$
such that

$$D_x^k G(x, t; \xi, \tau) - D_x^k G(y, t; \xi, \tau) \leqq$$

$$= \sum_{i=1}^{m} |x_i - y_i| \, |D_x^{k(i)} G(x_i^*, t; \xi, \tau)|$$

where $k(i) = (k_1, \ldots, k_{i-1}, k_i+1, k_{i+1}, \ldots, k_m)$ and ξ_i lies between x_i and y_i adn $|x - y| > |x_i^* - x|$. Hence, in view of (5) we obtain for $(|k| + 1)/2b < \mu \leqq (m + |k|)/2b$ $(\mu < 1)$

$$I_{2,k}(x, y, t) \leqq L|x - y| \; f_{A,\mu}(t).$$

$$\cdot \max_{i=1,\ldots,m} \sup_{x_i^*} \{ \int_{\Omega} |x_i^* - \xi|^{2b\mu - (m+|k(i)|)} d\xi \} \, E_1.$$

which proves (11).

Let now $|k| = 2b - 1$. Divide the domain Ω into two subset $S_1 = \{ \xi \in \Omega : |x - \xi| > 2|x - y| \}$ and $S_2 = \Omega - S_1$. From the inequality $|\xi - x| > 2|x - y|$ we get $|x - \xi| < 2|x_i^* - \xi|$

whence $I_{2,k}(x, y, t) \leqq L \, f_{A,\mu}(t) \{ 2^{m+2b-2b\mu} \int_{S_1} |x - y| \, .$

$$. \, |x - \xi|^{2b\mu - (m+2b)} d\xi + \int_{S_2} [\, |x - \xi|^{2b\mu - (m+2b-1)} +$$

$$+ |y - \xi|^{2b\mu - (m+2b-1)}] \, d\xi \} \, E_1.$$

The both integrals converge for $(2b - 1 + \beta)/2b \leqq \mu < 1$ and so (11) is true for $|k| = 2b - 1$ too.

To prove the third estimation we put $S_3 = \{ \xi \in \Omega : |\xi - x| >$

$> |t' - t|^{1/2b}\}$ and $S_4 = \Omega - S_3$. By the mean value theorem we find $\tilde{t} \in (t, t')$ such that for $|k| = 0, 1, \ldots, 2b - 1$ and $(2b - 1 + \beta)/2b < (4b - 1 + \beta)/4b \leqq \nu < 1$ and $0 < \lambda < |k|/2b$

$$I_{3,k}(x, t, t') \leqq \int_0^t d\tau \int_{S_3} |D_x^k G(x, t; \xi, \tau) - D_x^k G(x, t'; \xi, \tau)| d\xi +$$

$$+ \int_t^{t'} d\tau \int_{S_3} |D_x^k G(x, t'; \xi, \tau)| \; d\xi + \int_0^t d\tau \int_{S_4} |D_x^k G(x, t; \xi, \tau)| \; d\xi +$$

$$+ \int_0^{t'} d\tau \int_{S_4} |D_x^k G(x, t'; \xi, \tau) \quad d\xi \leqq$$

$$\leqq L \{(t' - t)(t' - t)^{(2b\nu - 2b - |k|)/2b} \int_0^t (\tilde{t} - \tau)^{-\nu} e^{A(\tilde{t} - \tau)} d\tau +$$

$$+ (t' - t)^{(2b\lambda - |k|)/2b} f_{A,\lambda}(t' - t) + (t' - t)^{(2b\nu - |k|)/2b} f_{A,\nu}(t) +$$

$$+ (t' - t)^{(2b\nu - |k|)/2b} f_{A,\nu}(t')\} E_1.$$

Since $\lambda < \nu < 1$ we easily see that $(t' - t)^{\lambda - \nu} f_{A,\lambda}(t' - t) < f_{A,\nu}(t' - t)$. Hence and by the monotonicity of $f_{A,\nu}$ one obtains

$$I_{3,k}(x, t, t') \leqq L (t' - t)^{(2b\nu - |k|)(2b} f_{A,\nu}(t') E_1.$$

Finally the relation (12) follows from the boundedness of the function

$$(t' - t)^{(2b\nu - 2b + 1 - \beta)/2b} \left(\int_0^{t' - t} z^{-\nu} e^{Az} dz \right)^{-1}$$

for $(t, t') \in \langle 0, \infty) \times \langle 0, \infty)$.

L e m m a 3. Let the conditions (A), (B), $(D_{2b-1+\alpha})$ be fulfilled and let the (p x 1) - vector function $F : H_\infty \to R_p$ be continuous and bounded in the norm $\| \cdot \|_{0,H_\infty}$. Then there is a real number $R > 0$ such that $A(x, t)\, P(Q_\infty) \subset S_R$, where the sphere $S_R = \{u \in P(Q_\infty) : \| u \|^{f(A,\varkappa,\mu,\nu)}_{2b-1+\alpha, Q_\infty} \leqq R\}$. (The values of parameters $\varkappa$, μ and ν are defined as in Lemma 2.)

P r o o f . Let $u \in P(Q_\infty)$ and $\| F \|_{Q,H_\infty} \leqq L$. Using (9) and Lemma 2 for $\beta = \alpha$ we have for $(x, t), (y, t), (x, t') \in Q_\infty$ and $t < t'$

$$|D_x^k A(x, t)\, u| \leqq L I_{1,k}(x, t)\, J \leqq pLf_{A,\varkappa}(t) J$$

and

$$|D_x^k A(x, t)\, u - D_x^k A(x, t')\, u| \leqq L I_{3,k}(x, t, t') J \leqq$$

$$\leqq pL(t' - t)^{(2b-1+\alpha-|k|)/2b} f_{A,\nu}(t' - t) f_{A,\nu}(t') J$$

for $|k| = 0, 1, \ldots, 2b - 1$ and

$$|D_x^k A(x, t)\, u - D_x^k A(y, t)\, u| \leqq L I_{2,k}(x, y, t) J \leqq$$

$$\leqq pL\, |x - y|^{\alpha} f_{A,\mu}(t) J$$

for $|k| = 2b - 1$. Thus it is sufficient to take $R \geqq$ $\geqq pL\,\{2s + t(2b - 1)\}$.

Now we may formulate the existence theorem.

T h e o r e m 3. Let the conditions (A), (B), $(D_{2b-1+\alpha})$ be satisfied and let the right hand side $F : \widetilde{H}_\infty \to R_p$ of (1) be the continuous and bounded (p x 1) - vector function in the norm $\| \cdot \|_{0,\widetilde{H}_\infty}$, where $\widetilde{H}_\infty = Q_\infty \times \prod_{i=1}^{s} \prod_{j=1}^{p} \{-Rf_{A,\varkappa}(t) \leqq u_j^i \leqq Rf_{A,\varkappa}(t)\} \subset$ $\subset H_\infty$. (R is the constant of Lemma 3.) Further, let the Hölder condition

(13) $|F(x, t, \ldots, u^{\gamma}, \ldots) - F(y, t, \ldots, v^{\gamma}, \ldots)| \leqq$

$$\leqq \{q^{(t)} |x - y|^{\beta} + \sum_{i=0}^{2b-1} \sum_{|\gamma|=i} (q^{\gamma}, |u^{\gamma} - v^{\gamma}|^{\beta_{\gamma}}\} J$$

be fulfilled for $\beta, \beta_{\gamma} \in (0, 1\rangle$ and $(x, t, \ldots, u^{\gamma}, \ldots)$, $(y, t, \ldots, v^{\gamma}, \ldots) \in \tilde{H}_{\infty}$, where $q^{\gamma}(t) = (q_1^{\gamma}(t), \ldots, q_p^{\gamma}(t))$ and $q_j^{\gamma}(t) \geqq 0$, $q(t) > 0$ for $j = 1, \ldots, p$ and $|\gamma| = 0, 1, \ldots, 2b-1$ are bounded and integrable real functions on $\langle 0, T\rangle$ for every $T > 0$. Then the problem (1), (2), (3) has at least one solution u belonging to

$C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_{\infty})$ for which $\|u\|^{f(A,\varkappa,\mu,\nu)}_{2b-1+\alpha,Q_{\infty}} \leqq R_0$

where $R_0 \geqq R$. Here the parameter $\varkappa < 1$ such that $0 < \varkappa < (m + |k|)/2b$ for $|k| = 0, 1, \ldots, 2b - 1$ and $\mu \in (0, 1)$ such that $(|k| + 1)/2b \leqq \mu \leqq (m + |k|)/2b$ for $|k| = 0, 1, \ldots, 2b-2$ and $\mu \geqq (2b - 1 + \beta)/2b$ for $|k| = 2b - 1$ and $(4b - 1 + \beta)/4b \leqq \nu < 1$.

<u>P r o o f .</u> According to Lemma 1, $(C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_{\infty}), \tau)$ is the locally convex complete linear topological Hausdorf space. The sphere S_R is bounded, closed and convex set in the topology τ and the operator $A(x, t)$ from (9) maps S_R into itself (see Lemma 3). In virtue of (13) and Remark 1b) for any $v \in S_R$ the vector function $F_v(x, t) = F[x, t, \ldots, D_x^{\gamma} v(x, t), \ldots]$ satisfies the inequality

$$|F_v(x, t) - F_v(y, t)| \leqq$$

$$\leqq \{q(t) |x - y|^{\beta} + \sum_{i=0}^{2b-2} \sum_{|\gamma|=i} \sum_{j=1}^{p} q_j^{\gamma}(t) L^{\beta_{\gamma}}(t) |x - y|^{\beta_{\gamma}} +$$

$$+ \sum_{|\gamma|=2b-1} \sum_{j=1}^{p} q_j^{\gamma}(t) \, [\, Rf_{A,\mu}(t) \, |x - y|^{\alpha} \,]^{\beta_\gamma} \} \, J$$

where $L(t) > 0$ is a bounded function on every interval $\langle 0, T \rangle$ for $T \in (0, \infty)$. Consequently $F_v \in H_\rho^{loc}(x, \Omega; t, \langle 0, \infty))$, where $\rho = \min(\beta, \beta_\gamma, \alpha\beta_\gamma) < 1$. Moreover $F_v \in C^0(Q_\infty)$ and so the problems (1), (2), (3) and the operator equation $A(x, t)\, u = u$ are mutually equivalent on S_R (Theorem 2).

The existence of solution of $A(x, t)\, u = u$ will be proved by the Tychonoff fixed point theorem (see [2]).

First of all we establish the continuity of the operator $A(x, t)$.

Let $\{u_s(x, t)\}_{s=1}^{\infty}$ be a sequence of elements $u_s(x, t) = (u_1^s(x, t), \ldots, u_p^s(x, t))$ of S_R such that $u_s \to u_0$ in the topology τ; $u_0(x, t) = (u_1^0(x, t), \ldots, u_p^0(x, t)) \in S_R$. Then to any neighbourhood $N(\sigma, n, \varepsilon) \in \tau$ there is a positive integer $s_0(n, \varepsilon)$ such that for all $s > s_0$ the relation $u_s - u_0 \in N(\sigma, n, \varepsilon)$ holds. Hence

$$(14) \qquad |D_x^k u_s(x, t) - D_x^k u_0(x, t)| \leqq f_{A,\varkappa}(n)\, \varepsilon\, J$$

for $(x, t) \in Q_n$ and $|k| = 0, 1, \ldots, 2b - 1$. From the hypothesis (13) we get for $(x, t), (y, t), (x, t') \in Q_n$ and $|k| = 2b - 1$

$$|D_x^k A(x, t)\, u_s - D_x^k A(x, t) u_0 - D_x^k A(y, t) u_s + D_x^k A(y, t) u_0| \leqq$$

$$\leqq I_{2,k}(x, y, t)\, C(n) J$$

and for $|k| = 0, 1, \ldots, 2b - 1$

$$|D_x^k A(x, t) u_s - D_x^k A(x, t) u_0| \leqq I_{1,k}(x, t)\, C(n) J$$

and

$$|D_x^k A(x,t)u_s - D_x^k A(x,t)u_0 - D_x^k A(x,t')u_s + D_x^k A(x,t')u_0| \leqq$$

$$\leqq I_{3,k}(x, t, t')\, C(n) J$$

where $C(n) =$

$$= \sup_{Q_n} \sum_{i=0}^{2b-1} \sum_{|\gamma|=i} (q^{\gamma}(t),\ |D_x^{\gamma} u_s(\xi, \tau) - D_x^{\gamma} u_0(\xi, \tau)|^{\beta_\gamma}).$$

Using Lemma 2 for $\beta = \alpha$ and the estimation (14) we have

$$\| A(x,t)u_s - A(x,t)u_0 \|^{f(A,\varkappa,\mu,\nu)}_{2b-1+\alpha, Q_n} < L(n)\varepsilon ,$$

i.e. $A(x,t)u_s - A(x,t)\, u_0 \in N(\mathcal{O}, n, L(n)\varepsilon)$ for $s > s_0(n, \varepsilon)$. The constant $L(n) > 0$ dependes only of n and so the operator $A(x, t)$ is continuous in the topology τ .

To prove the relative compactness of $A(x,t)\, S_R$ in τ we use the well known N. Dunford's lemma (see [3]).

Put $v_s(x,t) = (v_1^s(x,t), \dots, v_p^s(x,t)) \in A(x,t)S_R \subset S_R$ for $s = 1, 2, \dots$. Then there exists a sequence of elements $u_s(x,t) = (u_1^s(x,t), \dots, u_p^s(x,t)) \in S_R$ such that $v_s = A(x,t)u_s$ for $s = 1, 2, \dots$.

By $\| v_s \|^{f(A,\varkappa,\mu,\nu)}_{2b-1+\alpha, Q_\infty} \leqq R$, the sequence of derivatives

$\{ D_x^k v_j^s(x,t) \}_{s=1}^{\infty}$ is uniformly bounded on Q_n for any $n = 1, 2, \dots$ and $j = 1, \dots, p$ and $|k| = 0, 1, \dots, 2b - 1$. With respect to the assumption $\| F \|_{0, \tilde{H}_\infty} < L$ for $|k| = 0, 1, \dots, 2b - 1$

$$|D_x^k v_s(x,t) - D_x^k v_s(y,t')| \leqq |D_x^k A(x,t)u_s - D_x^k A(y,t)u_s| +$$

$$+|D_x^k A(y, t)u_s - D_x^k A(y, t')u_s| \leqq$$

$$\leqq L[I_{2,k}(x, y, t) + I_{3,k}(y, t, t')] J.$$

Thus Lemma 2 guarantees the equicontinuity of the sequence $\{D_x^k v_j^s(x, t)\}_{s=1}^{\infty}$. Then there is a subsequence $\{v_{s_\ell}(x, t)\}_{\ell=1}^{\infty} =$

$= \{A(x, t)u_{s_\ell}\}_{\ell=1}^{\infty}$ of the sequence $\{v_s(x, t)\}_{s=1}^{\infty}$ and $v_0(x, t) =$

$= (v_1^0(x, t), \ldots, v_p^0(x, t))$ such that $\| D_x^k v_{s_\ell} - D_x^k v_0 \|_{0,Q_n} \longrightarrow 0$

as $\ell \to \infty$ for $n = 1, 2, \ldots$ and $|k| = 0, 1, \ldots, 2b - 1$. Consequently the sequence $\{D_x^k v_{s_\ell}(x, t)\}_{\ell=1}^{\infty}$ converges to $D_x^k v_0(x, t)$ at every point $(x, t) \in Q_\infty$.

Letting for fixed (x, t), (y, t), $(x, t') \in Q_\infty$, $\ell \to \infty$ in the inequalities

$$|D_x^k v_0(x, t) - D_x^k v_0(y, t)| \leqq$$

$$\leqq |D_x^k v_0(x, t) - D_x^k v_{s_\ell}(x, t)| + L I_{2,k}(x, y, t) J +$$

$$+ |D_x^k v_{s_\ell}(y, t) - D_x^k v_0(y, t)|$$

for $|k| = 2b - 1$ and

$$|D_x^k v_0(x, t)| \leqq |D_x^k v_0(x, t) - D_x^k v_{s_\ell}(x, t)| + L I_{1,k}(x, t) J,$$

$$|D_x^k v_0(x, t) - D_x^k v_0(x, t')| \leqq$$

$$\leqq |D_x^k v_0(x, t) - D_x^k v_{s_\ell}(x, t)| + L I_{3,k}(x, t, t') J +$$

$$+ |D_x^k v_{s_\ell}(x, t') - D_x^k v_0(x, t')|$$

for $|k| = 0, 1, \ldots, 2b - 1$ we immediately obtain that v_0 belongs to $C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$. Moreover we must show the convergence of the sequence $\{v_{s_\ell}(x, t)\}_{s=1}^{\infty}$ to $v_0(x, t)$ in the topology τ.

Denote $S^*_{R,n}$ the completion of the set $A(x, t)S_R$ in the norm

$$\|u\|^t_{2b-1,Qn} = \max_{j=1,\dots,p} \{\sum_{i=0}^{2b-1} \sum_{|k|=i} \sup_{Q_n} |D^k_x u_j(x, t)|\}$$

for $n = 1, 2, \dots$. The sequence $\{S^*_{R,n}\}^{\infty}_{n=1}$ possesses the following properties:

a) $v_0 \in S^*_{R,n}$ for any $n = 1, 2, \dots$

b) The non-void intersection $S^*_R = \bigcap\limits_{n=1}^{\infty} S^*_{R,n}$ is a subset of $C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$. [Really for $v(x, t) = (v_1(x,t), \dots, v_p(x,t)) \in$ $\in S^*_R$ there is a sequence $\{w_s(x, t)\}^{\infty}_{s=1} \subset A(x, t)S_R$ such that $\|w_s - v\|^t_{2b-1,Q_n} \to 0$ as $s \to \infty$ for any $n = 1, 2, \dots$. Thus $\{D^k_x w_s\}^{\infty}_{s=1}$ converges to $D^k_x v$ at every point $(x, t) \in Q_\infty$. Hence using the some considerations as above for v_0 we obtain that $v \in$ $\in C^{2b-1+\alpha,\ (2b-1+\alpha)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$ $(\supset S^*_R)$.]

c) By Lemma 2 for $v \in S^*_R$

$$\lim_{x \to y} |D^k_x v_j(x, t) - D^k_x v_j(y, t)| f^{-1}_{A,\mu}(t) |x - y|^{-\alpha} = 0$$

uniformly with respect to $t \in \langle 0, \infty)$ for $|k| = 2b - 1$ and

$$\lim_{t \to t'-} |D^k_x v_j(x, t) - D^k_x v_j(x, t')| f^{-1}_{A,\nu}(t' - t) f^{-1}_{A,\nu}(t') \cdot$$

$$\cdot (t' - t)^{-(2b-1+\alpha-|k|)/2b} = 0$$

uniformly with respect to $x \in \Omega$ for $|k| = 0, 1, \dots, 2b-1$ and $j = 1, \dots, p$.

In view of c) to each $\varepsilon > 0$ and $n = 1, 2, \dots$ we find $\delta(\varepsilon, n) > 0$ such that for every $j = 1, \dots, p$ and $\ell = 1, 2, \dots$ and $(x, t), (x, t') \in Q_n$, $t < t'$

$$\langle D_x^k v_j^{s_\ell}(x, t) - D_x^k v_j^0(x, t) \rangle_{\alpha, x} f_{A,\mu}^{-1}(t) < \varepsilon$$

if $|k| = 2b - 1$ and $0 < |x - y| < \delta$ and

$$\langle D_x^k v_j^{s_\ell}(x, t) - D_x^k v_j^0(x, t) \rangle_{(2b-1+\alpha - k)/2b, t} \cdot$$

$\cdot\, f_{A,\nu}^{-1}(t' - t)\, f_{A,\nu}^{-1}(t') < \varepsilon$ if $|k| = 0, 1, \ldots, 2b - 1$ and $0 < t' - t < \delta$.

From the relation $\| v_{s_\ell} - v_0 \|_{2b-1, Q_n}^{t} \longrightarrow 0$ if $\ell \to \infty$ follows the existence of a positive integer $\ell_0(\varepsilon, n)$ such that for $\ell > \ell_0(n, \varepsilon)$ and $|x - y| \geqq \delta$ on Q_n

$$\langle D_x^k v_j^{s_\ell}(x, t) - D_x^k v_j^0(x, t) \rangle_{\alpha, x} f_{A,\mu}^{-1}(t) \leqq$$

$$\leqq f_{A,\mu}^{-1}(0)\,\delta^{-\alpha} \max_{j=1,\ldots,p} \Big\{ \sup_{(x,\, t) \in Q_n} | D_x^k v_j^{s_\ell}(x, t) - D_x^k v_j^0(x, t) | +$$

$$+ \sup_{(y,t) \in Q_n} | D_x^k v_j^{s_\ell}(y, t) - D_x^k v_j^0(y, t) | \Big\} < \varepsilon$$

if $|k| = 2b - 1$. For $t' - t \geqq \delta$

$$\langle D_x^k v_j^{s_\ell}(x, t) - D_x^k v_j^0(x, t) \rangle_{(2b-1+\alpha-|k|)/2b, t} \cdot$$

$$\cdot\, f_{A,\nu}^{-1}(t' - t)\, f_{A,\nu}^{-1}(t') \leqq f_{A,\nu}^{-2}(0)\,\delta^{-(2b-1+\alpha - |k|)/2b}$$

$$\cdot \max_{j=1,\ldots,p} \Big\{ \sup_{(x,t) \in Q_n} | D_x^k v_j^{s_\ell}(x, t) - D_x^k v_j^0(x, t) | +$$

$$+ \sup_{(x,t') \in Q_n} | D_x^k v_j^{s_\ell}(x, t') - D_x^k v_j^0(x, t') | \Big\} < \varepsilon$$

if $|k| = 0, 1, \ldots, 2b - 1$. Finally

$$\| v_{s_\ell} - v_0 \|^{f(A,\varkappa,\mu,\nu)}_{2b-1+\alpha,Q_n} \leqq$$

$$\leqq \max_{j=1,\dots,p} \{ \sum_{i=0}^{2b-1} \sum_{|k|=i} \sup_{Q_n} | D_x^k v_j^{s_\ell}(x,t) - D_x^k v_j^0(x,t) | f^{-1}_{A,\varkappa}(t) +$$

$$+ \sum_{|k|=2b-1} \max [\sup_{\substack{(x,t),(y,t)\in Q_n \\ 0<|x-y|<\delta}} \langle D_x^k v_j^{s_\ell}(x,t) - D_x^k v_j^0(x,t) \rangle_{\alpha,x} f^{-1}_{A,\mu}(t),$$

$$\sup_{\substack{(x,t),(y,t)\in Q_n \\ |x-y|\geqq\delta}} \langle D_x^k v_j^{s_\ell}(x,t) - D_x^k v_j^0(x,t) \rangle_{\alpha,x} f^{-1}_{A,\mu}(t)] +$$

$$+ \sum_{i=0}^{2b-1} \sum_{|k|=i} \max [\sup_{\substack{(x,t),(x,t')\in Q_n \\ 0<t'-t<\delta}} \langle D_x^k v_j^{s_\ell}(x,t) -$$

$$- D_x^k v_j^0(x,t) \rangle_{(2b-1+\alpha-|k|)/2b,t} f^{-1}_{A,\nu}(t'-t) f^{-1}_{A,\nu}(t'),$$

$$\sup_{\substack{(x,t),(t,t')\ Q_n \\ t'-t\geqq\delta}} \langle D_x^k v_j^{s_\ell}(x,t) - D_x^k v_j^0(x,t) \rangle_{(2b-1+\alpha-|k|)/2b,t} \cdot$$

$$\cdot f^{-1}_{A,\nu}(t'-t) f^{-1}_{A,\nu}(t')] \} <$$

$$< \varepsilon [f^{-1}_{A,}(0)s + t(2b-1) + s]$$

for $\ell > \ell_0(n, \varepsilon)$ and arbitrary $n = 1, 2, \dots$. Hence $v_{s_\ell} - v_0 \in$ $\in N(\sigma, n, \varepsilon[f^{-1}_{A,\varkappa}(0)s + t(2b-1) + s])$ for $\ell > \ell_0(n,\varepsilon)$ which concludes the proof of this theorem.

R E F E R E N C E S

[1] E'JDEĹMAN S. D. - IVASIŠEN S. D.: The investigation of the Green's matrix for a homogeneous boundary value problem of parabolic type. Trudy Moskov. Mat. Obšč. 23, 1970, p. 179 - 234.

[2] TICHONOV A. N.: Ein Fixpunktsatz. Mat. Ann. 111, 1935, p. 767-776.

[3] NEIMARK M. A.: Normed rings. P. Noordhoff 1959.

[4] FRIEDMAN A.: Partial equations of parabolic type. Prentice Hall 1964. (The Russian translation, Moscow, izd. Mir, 1968).

[5] E'JDEĹMAN S. D.: Parabolic systems, Moskow, Nauka 1964.

[6] GAJEWSKI H. - GRÖGER K. - ZACHARIAS K.: Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen. Akademie-Verlag, Berlin 1974.

[7] YÓSIDA K.: Functional analysis. Springer-Verlag, Berlin, Heidelberg, New York 1966.

[8] ĎURIKOVIČ V.: An initial-boundary value problem for quasilinear parabolic system of higher order, Ann. Polon. Math. XXX, 1974, p. 145-164.

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S ú h r n

O NELINEÁRNEJ STACIONÁRNEJ PARABOLICKEJ OKRAJOVEJ ÚLOHE

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V práci sa vyšetruje existencia riešenia nelineárnej zmiešanej úlohy parabolického typu (1), (2), (3) metódou Greenových funkcií a jeho štruktúra na nekonečnom valci $\Omega \times \langle 0, \infty)$. Táto úloha sa rieši v lokálne konvexných topologických priestoroch dostatočne hladkých hölderovských funkcií.

Р Е З Ю М Е

О НЕЛИНЕЙНОЙ СТАЦИОНАРНОЙ КРАЕВОЙ ЗАДАЧИ ПАРАБОЛИЧЕСКОГО ТИПА

ВЛАДИМИР ДЮРИКОВИЧ, БРАТИСЛАВА

В этой статье исследуется существование решения нелинейной смешанной задачи (1), (2), (3) методом функций Грина и его структура в бесконечном цилиндре $\Omega \times \langle 0, \infty)$. Эта задача решенна в локально выпуклых топологических пространствах достаточно гладких функций Гелгера.

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ON SOME PROPERTIES OF STATIONARY PARABOLIC MIXED PROBLEM

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1. Introduction

In the paper [2] the existence of solution of the stationary parabolic initial-boundary value problem was studied for the system of $p \geqq 1$ differential equations with p unknown functions

$$D_t - \sum_{|k|=2b} A_k(x)D_x^k u = F(x, t, \ldots, D_x^{\gamma} u, \ldots) \tag{1}$$

for $0 \leqq |\gamma| \leqq 2b - 1$, $b \geqq 1$ on the infinity cylinder $Q_\infty = \Omega \times \langle 0, \infty)$ with data

$$u\big|_{t=0} = \vartheta, \quad x \in \Omega \tag{2}$$

$$\sum_{|k| \leqq r_q} (B_k^{(q)}, D_x^k u)\Big|_{\Gamma_\infty} = 0 \tag{3}$$

for an integer $r_q \leqq 2b - 1$ and $q = 1, \ldots, bp$; Ω is a bounded domain of the Euclidean space R_m with the boundary $\partial\Omega$ and $\Gamma_\infty = \partial\Omega \times \langle 0, \infty)$.

The present paper is a continuation of the paper [2] and it deals with some further fundamental questions of the problem (1), (2), (3). Namely, the regularity, uniqueness and asymptotic behaviours of the solution and the convergence of successive approximations are treated. In the quoted paper [2] the reader can find all notations, notions, assumptions (A), (B), ($D_{\ell+\alpha}$) and statements used in this paper and not defined here (for brevity). Therefore it is suitable first to acquaint oneself with the paper [2] .

Our considerations will be carried out in the following space of Hölder continuous vector functions:

With the help of real functions $g_{i,k_0,k} : \langle 0, \infty) \to (0, \infty)$ for $i = 1, 2$, and $g_{3,k_0,k} : \langle 0, \infty) \times \langle 0, \infty) \to (0, \infty)$ (k_0 is a non-negative integer and k is a multiindex $(k_1, \ldots, k_m)$) which are (j) bounded and integrable on $\langle 0, T\rangle$ and $\langle 0, T\rangle \times \langle 0, T\rangle$, respestively for any real $T > 0$, we define the linear space $C^{\ell+\alpha, (\ell+\alpha)/2b}_{x,t,g}(Q_\infty)$ of vector functions $u(x, t) = (u_1(x, t), \ldots, u_p(x, t))$ from Q_∞ into R_p for an integer $\ell \geqq 0$ and $b \geqq 1$ and $0 < \alpha < 1$ by the inequality

(4) $$\|u\|^{g}_{\ell+\alpha, Q_\infty} \overset{df}{=}$$

$$\overset{df}{=} \max_{j=1,\ldots,p} \sum_{i=0}^{\ell} \sum_{2bk_0+|k|=i} \sup_{Q_\infty} \left[|D_t^{k_0} D_x^k u_j(x, t)| \, g^{-1}_{1,k_0,k}(t) \right] +$$

$$+ \sum_{2bk_0+|k|=\ell} \sup_{\substack{(x, t), (y, t) \in Q_\infty \\ x \neq y}} \left[\langle D_t^{k_0} D_x^k u_j(x, t) \rangle_{\alpha, x} \, g^{-1}_{2,k_0,k}(t) \right] +$$

$$+ \sum_{0 < \ell+\alpha-2bk_0-|k| < 2b} \sup_{\substack{(x, t), (x, t') \in Q_\infty \\ t \neq t'}} H(x, t, t') < \infty,$$

where

$$H(x, t, t') = \langle D_t^{k_0} D_x^k u_j(x, t) \rangle_{(\ell+\alpha-2bk_0-|k|)/2b, t} \, g^{-1}_{3,k_0,k}(t, t')$$

and

$$\langle v(x,t)\rangle_{a,x} = |v(x,t) - v(y,t)|\,|x-y|^{-a},\ \langle v(x,t)\rangle_{a,t} =$$

$$= |v(x,t) - v(x,t')|\,|t-t'|^{-a}.$$

Note, the space $C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,t,f(B,\varkappa,\mu,\nu)}(Q_\infty)$ is a special case of the just defined space $C^{\ell+\alpha,(\ell+\alpha)/2b}_{x,\,t,\,g}(Q_\infty)$ for $\ell = 2b-1$ and $g_{1,k_0,k}(t) = f_{B,\varkappa}(t)$, $g_{2,k_0,k}(t) = f_{B,\mu}(t)$ and $g_{3,k_0,k}(t,t') = f_{B,\nu}(|t-t'|)\, f_{B,\nu}(t^*)$ ($t^* = \max(t,t')$; see [2], p. 3).

R e m a r k 1. If $0 < \rho < \alpha < 1$ then there is $\delta \in \langle 1 - (\alpha-\rho)/2b,\ 1)$ such that $C^{\ell+\alpha,(\ell+\alpha)/2b}_{x,\,t,\,g}(Q_\infty) \subseteq$ $\subset C^{\ell+\rho,(\ell+\rho)/2b}_{x,\,t,\,\tilde g}(Q_\infty)$, where $\tilde g_{1,k_0,k}(t) = g_{1,k_0,k}(t)$, $\tilde g_{2,k_0,k}(t) = g_{2,k_0,k}(t)$ and $\tilde g_{3,k_0,k}(t,t') = g_{3,k_0,k}(t,t')\,\cdot$ $\cdot\, f_{B,\delta}(|t-t'|)$.

In addition to the estimations of the Green's function (6), (7), (8) from Theorem 1 and (6') from Remark 2 of [2] we shall use the following modification of the quoted estimation (7):

$$(5)\quad |D_t^{k_0}D_x^{k}G(x,t;\xi,\tau) - D_t^{k_0}D_x^{k}G(y,t;\xi,\tau)| \leq$$

$$\leq K|x-y|^{\alpha}(t-\tau)^{-\mu}|x^*-\xi|^{2b\mu-(m+2bk_0+|k|+\alpha)}e^{A(t-\tau)}\ E_1$$

for $\mu \leq (m+2bk_0+|k|+\alpha)/2b$, $A>0$, $K>0$ and $0 \leq \tau < t < \infty$, $\xi \neq x$ and $|x^*-\xi| = \min(|x-\xi|, |y-\xi|)$.

T h e o r e m 1. (A. Deleanu - G. Marinescu [1].) Let (P, τ) be a locally convex, sequentially complete space and let F be a saturated family of seminorms defining the topology τ.

Let $\varphi : F \longrightarrow F$ be a mapping for which $\varphi^2(\sigma) = \varphi(\sigma)$ for any $\sigma \in F$ and let S be a closed subset of P and the operator $A : S \rightarrow S$ satisfying the following conditions:

a) For any $\sigma \in F$ there is $q_\sigma > 0$ such that

$$\| A u - A v \|_\sigma \leqq q_\sigma \| u - v \|_{\varphi(\sigma)}$$

for any $u, v \in S$;

b) $q_{\varphi(\sigma)} < 1$ for every $\sigma \in F$.

Then the operator A has a unique fixed point in S.

This theorem is a generalization of the Banach's principle of the contractive mappings for locally convex spaces. It will be applied only in the special case if φ is an identical mapping.

2. The smoothness of solution.

In the paper [2] (Theorem 3) we have proved that the solution of problem (1), (2), (3) possesses Hölder continuous derivatives according to x up to the order 2b - 1. The following theorem investigates the structure of the derivatives $D_x^{2b}u$ and $D_t u$ and shows that they are Hölder continuous in a subdomain of Ω.

T h e o r e m 2. Let the assumptions (A), (B), $(D_{2b+\alpha})$ be satisfied. Let $F : \widetilde{H}_\infty \rightarrow R_p$ be a continuous and bounded vector function in the norm $\| \cdot \|_{0,\widetilde{H}_\infty}$, where

$$\widetilde{H}_\infty = Q_\infty \times \prod_{i=1}^{s} \prod_{j=1}^{p} \{ -R\, f_{A,\varkappa}(t) \leqq u_j^i \leqq R\, f_{A,\varkappa}(t) \} \subset H_\infty$$

and let the Hölder condition

$$(6) \qquad |F(x, t, \ldots, u^{r}, \ldots) - F(y, t', \ldots, v^{r}, \ldots)| \leqq$$
$$\leqq \{q(t)\ |x - y|^{\beta} + p(t)\ |t - t'|^{\beta'/2b} +$$
$$+ \sum_{i=0}^{2b-1} \sum_{|r|=i} (q^{r}(t), |u^{r} - v^{r}|^{\beta_r})\} J$$

hold on $\widetilde{H}_{\infty}$, where $q^{r}(t) = (q_1^{r}(t), \ldots, q_p^{r}(t))$ and $q_j^{r}(t) \geqq$ $\geqq 0$ for $j = 1, \ldots, p$ and $q(t) > 0$, $p(t) > 0$ are bounded and integrable real functions on $\langle 0, T\rangle$ for any $T > 0$ (J is $1 \times p$-unit vector). Then to each β, β' and β_r belonging to the interval $(0,1\rangle$ and $0 < \varrho < \min(\beta, \beta', \alpha\beta_r)$ (< 1) there exist functions $g_{i,k_0,k}$ for $i = 1, 2, 3$ with the property (j) such that the solution of (1), (2), (3) belongs to $C_{x,\,t,\,g}^{2b+\varrho,(2b+\varrho)/2b}(Q_{\infty}^{*})$, where $Q_{\infty}^{*} = \langle 0, \infty) \times \Omega^{*}$ and Ω^{*} is an arbitrary subdomain of Ω which closure lies in Ω, i.e. $\overline{\Omega^{*}} \subset \Omega$.

<u>P r o o f .</u> According to Theorem 3 of [2] the solution u of (1), (2), (3) belongs to the space $C_{x,t,f(A,\varkappa,\mu,\nu)}^{2b-1+\alpha,(2b-1+\alpha)/2b}(Q_{\infty})$ and with respect to Remark 1 it appertains to $C_{x,t,\tilde{g}}^{2b-1+\varrho,(2b-1+\varrho)/2b}$ $(\dot{Q}_{\infty})$ and satisfies the operator equation $u = A(x, t)\, u$ on $\dot{Q}_{\infty}$. (The integral operator $A(x, t)$ is defined by the formula (9) from [2] .) To prove this theorem we have to show that the expression $\|u\|_{2b+\varrho,Q_{\infty}}^{g}$ defined by

$$(7) \qquad \max_{j=1,\ldots,p} \sum_{i=0}^{2b} \sum_{|k|=i} \sup_{Q_{\infty}} [\,|D_x^{k} u_j(x, t)|\ g_{1,0,k}^{-1}(t)] +$$

$$+ \sup_{Q^*_\infty} \left[|D_t\, u_j(x,\ t)|\ g^{-1}_{1,1,0}(t) \right] +$$

$$+ \sum_{|k|=2b}\ \sup_{\substack{(x,\ t),\ (y,\ t)\ \in\ Q^*_\infty \\ x \neq y}} \left[\langle D^k_x u_j(x,\ t) \rangle_{\varrho, x}\ g^{-1}_{2,0,k}(t) \right] +$$

$$+ \sup_{\substack{(x,t),(y,\ t)\ \in\ Q^* \\ x \neq y}} \left[\langle D_t u_j(x,\ t) \rangle_{\varrho, x}\ g^{-1}_{2,1,0}(t) \right] +$$

$$+ \sum_{i=1}^{2b} \sum_{k=i}\ \sup_{\substack{(x,\ t),\ (x,\ t')\ \in\ Q^* \\ t \neq t'}} \left[\langle D^k_x u_j(x,\ t) \rangle_{(2b+\varrho-|k|)/2b,\ t} \cdot \right.$$

$$\left. \cdot\ g^{-1}_{3,0,k}(t,\ t') \right] + \sup_{\substack{(x,\ t),\ (x,\ t')\ \in\ Q^*_\infty \\ t \neq t'}} \left[\langle D_t u_j(x,\ t) \rangle_{\varrho/2b,\ t} \cdot \right.$$

$$\left. \cdot\ g^{-1}_{3,1,0}(t,\ t') \right]$$

(see (4)) is finite.

Put

$$\inf_{\substack{\xi \in \partial\Omega \\ x \in \Omega^*}} |x - \xi| = \omega > 0 \text{ and } |x^* - \xi| = \min\left(|x - \xi|, |y - \xi|\right)$$

for $\xi \in \Omega$ and $(x,\ t),\ (y,\ t),\ (x,\ t') \in Q^*_\infty$, $t < t'$. Employing the Gren's formula and the estimation (6') of [2] and (5) (with the exponent $\alpha = \varrho$) and the assumption (6) we obtain for $|k| = 2b$

$$\left| D^k_x u(x,\ t) \right| \leq L \int_0^t d\tau\ \left| \int_\Omega D^k_x G(x,\ t;\ \xi,\ \tau)\ d\xi \right| J +$$

$$+ \int_0^t d\tau \int_\Omega \left| D^k_x G(x,\ t;\ \xi,\ \tau) \right|\ \left| F\left[\xi,\ \tau,\ \ldots,\ D^r_x u(\xi,\ \tau), \ldots\right] - \right.$$

$$\left. - F\left[x, \tau,\ \ldots,\ D^r_x\ u(x, \tau),\ \ldots\right] \right| d\xi \leq$$

$$\leqq L \{ \text{mes } (\partial\Omega)\, \omega^{2b\mu - m - 2b + 1} f_{A,\mu}(t) +$$

$$+ \int_0^t d\tau \int_\Omega (t-\tau)^{-\mu} e^{A(t-\tau)} |x - \xi|^{2b\mu - (m+2b)} P_\mu (x, \xi, \tau)\, d\xi \} J$$

and

$$|D_x^k u(x, t) - D_x^k u(y, t)| \leqq$$

$$\leqq L \int_0^t d\tau \left| \int_\Omega [D_x^k G(x, t; \xi, \tau) - D_x^k G(y, t; \xi, \tau)]\, d\xi \right| J +$$

$$+ \int_0^t d\tau \int_\Omega |D_x^k G(x, t; \xi, \tau) - D_x^k G(y, t; \xi, \tau)|\, |F[\xi, \tau, \ldots,$$

$$D_x^r u(\xi, \tau), \ldots] - F[x^*, \tau, \ldots, D_x^r u(x, \tau), \ldots]|\, d\xi \leqq$$

$$\leqq L |x - y|^\rho \{ \text{mes } (\partial\Omega)\, \omega^{2b\mu - (m+2b-1+\rho)} f_{A,\mu}(t) +$$

$$+ \int_0^t d\tau \int_\Omega (t-\tau)^{-\mu} e^{A(t-\tau)} |x^* - \xi|^{2b\mu - (m+2b+\rho)} P_\mu (x^*, \xi, \tau)\, d\xi \} J,$$

where

$$P_\mu(x, \xi, \tau) = q(\tau) |x - \xi|^\beta + \sum_{i=0}^{2b-2} \sum_{|r|=i} (q^r(\tau), f_{A,\mu}^{\beta_r}(\tau) |x - \xi|^{\beta_r} J) +$$

$$+ \sum_{|r|=2b-1} (q^r(\tau), f_{A,\mu}^{\beta_r}(\tau) |x - \xi|^{\alpha\beta_r} J).$$

Hence if we take μ such that $\max \{ [2b - (\beta - \rho)]/2b,$ $[2b - (\alpha\beta_r - \rho)]/2b < \mu < 1$ for $|r| = 0, 1, \ldots, 2b-1$ so

$$|D_x^k u(x, t)| \leqq L g_{1,0,k}(t)\, J.$$

$$|D_x^k u(x, t) - D_x^k u(y, t)| \leqq L |x - y|^\rho g_{2,0,k}(t)\, J,$$

where the functions $g_{1,0,k}$ and $g_{2,0,k}$ are uniquely determined by the preceding estimations.

The equation (1) and assumption (D_ς) and the last estimations yield

$$|D_t u(x, t)| \leq |[F x, t, \ldots, D_x^r u(x, t), \ldots]| +$$

$$+ \sum_{|k|=2b} |A_k(x)| \; |D_x^k u(x, t)| \leq L g_{1,1,0}(t) J$$

and

$$|D_t u(x, t) - D_t u(y, t)| \leq$$

$$\leq |F[x, t, \ldots, D_x^r u(x, t), \ldots] - F[y, t, \ldots, D_x^r u(y, t), \ldots]| +$$

$$+ \sum_{|k|=2b} |A_k(x)| \; |D_x^k u(x, t) - D_x^k u(y, t)| +$$

$$+ \sum_{|k|=2b} |A_k(x) - A_k(y)| \; |D_x^k u(x, t)| \leq L|x - y|^\rho g_{2,1,0}(t) J.$$

For $|k| = 1, \ldots, 2b$ we have

$$|D_x^k u(x, t) - D_x^k u(x, t')| \leq$$

$$\leq \Big\{ L \int_0^t d\tau \Big| \int_\Omega [D_x^k G(x, t; \xi, \tau) - D_x^k G(x, t'; \xi, \tau)] \, d\xi \Big| +$$

$$+ \int_t^{t'} d\tau \Big| \int_\Omega D_x^k G(x, t'; \xi, \tau) \, d\xi \Big| \Big\} J +$$

$$+ \int_0^t d\tau \int_\Omega |D_x^k G(x, t; \xi, \tau) - D_x^k G(x, t'; \xi, \tau)| \, .$$

$$\cdot |F[\xi, \tau, \ldots, D_x^r u(\xi, \tau), \ldots] - F[x, \tau, \ldots, D_x^r u(x, \tau), \ldots]| \, d\xi +$$

$$+ \int_t^{t'} d\tau \int_\Omega |D_x^k G(x, t'; \xi, \tau)| \, .$$

$$\cdot \left| F\left[\xi, \tau, \ldots, D_x^{r} u(\xi, \tau), \ldots - F\left[x, \tau, \ldots, D_x^{r} u(x, \tau), \ldots\right]\right| d\xi .$$

Denote gradually the integrals in the last inequality by I_1, I_2, I_3 and I_4. Using the Green's formula and the mean value theorem and assumption (6) one obtains

$$I_1 \leqq L(t' - t) f_{A,\mu}(t') E_1$$

$$I_2 \leqq L\, f_{A,\lambda}(t' - t)\, E_1$$

To estimate I_3 we put $S_3 = \{ \xi \in \Omega : |\xi - x| > (t' - t)^{1/2b} \}$ $S_4 = \Omega - S_3$. Then there is $\tilde{t} \in (t, t')$ for which

$$I_3 \leqq L \Big\{ (t' - t) \int_0^t d\tau \int_{S_3} e^{A(\tilde{t} - \tau)} (\tilde{t} - \tau)^{-\mu} |x - \xi|^{2b - (m + 2b + |k|)} \cdot$$

$$\cdot\, P_\mu(x, \xi, \tau)\, d\xi + \int_0^t d\tau\, (t - \tau)^{-\mu} e^{A(t - \tau)} |x - \xi|^{2b\mu - (m + |k|)} \cdot$$

$$\cdot\, P_\mu(x, \xi, \tau)\, d\xi + \int_0^t d\tau \int_{S_4} (t' - \tau)\, e^{A(t' - \tau)} |x - \xi|^{2b - (m + |k|)} \cdot$$

$$\cdot\, P_\mu(x, \xi, \tau)\, d\xi \Big\}\, J$$

Further

$$I_4 \leqq L \Big\{ \int_t^{t'} d\tau \int_\Omega (t' - \tau)^{-\lambda} e^{A(t' - \tau)} |x - \xi|^{2b - (m + |k|)} \cdot$$

$$\cdot\, P_\mu(x, \xi, \tau)\, d\xi \Big\}\, J.$$

If $\max \left[(2b + \varrho - |k|)/2b,\ (4b + \varrho - \beta_1)/4b,\ (|k| - \beta)/2b,\ (|k| - \alpha\beta_{\gamma})/2b \right] \leqq \mu < 1$ and $\max \left[(|k| - \beta)/2b,\ (|k| - \alpha\beta_{\gamma})/2b \right] < \lambda < (|k| - \varrho)/2b$ for $|r| = 0, 1, \ldots, 2b - 1$ and $|k| = 1, \ldots, 2b$ then the functions

$$(t'-t)^{(|k|-\varrho)/2b}\left(\int_0^{t'-t} z^{-\mu}e^{Az}\,dz\right)^{-1}$$

$$(t'-t)^{(2b\mu-2b-\varrho+\beta)/2b}\left(\int_0^{t'-t} z^{-\mu}e^{Az}\,dz\right)^{-1}$$

are bounded for $t\in\langle 0,\infty)$, $t'\in\langle 0,\infty)$ and $t<t'$ and the estimation

$$(t'-t)^{-(2b+\varrho-|k|)/2b}\int_0^{t'-t} z^{-\lambda}e^{Az}dz \leqq \int_0^{t'-t} z^{-\varkappa}e^{Az}\,dz$$

holds for $\varkappa = 1-(|k|-\varrho)/2b+\lambda$ and $t,\ t'\in(0,\infty)$, $t<t'$.

Hence for I_1, I_2, I_3 and I_4 we get

$$I_1 \leqq L(t'-t)^{(2b+\varrho-|k|)/2b} f_{A,\mu}(t')f_{A,\mu}(t'-t)\ E_1\ .$$

$$I_2 \leqq L(t'-t)^{(2b+\varrho-|k|)/2b} f_{A,\varkappa}(t'-t)\ E_1$$

$$I_3 = L(t'-t)^{(2b+\varrho-k)/2b}\ W(t,\ t')J$$

$$I_4 < \tilde{L}(t')(t'-t)^{(2b+\varrho-|k|)/2b}\left[1+f^{\beta_r}_{A,\mu}(t')+f^{\beta_r}_{A,\mu}(t')\right].$$

$$\cdot f_{A,\varkappa}(t'-t)\ J,$$

where W is a positive function with the property (j) and $\tilde{L}$ is a positive nondecreasing function on $\langle 0,\infty)$. Consequently

$$\left|D^k_x u(x,\ t)-D^k_x u(x,\ t')\right| \leqq L(t'-t)^{(2b+\varrho-|k|)/2b}\ g_{3,0,k}(t,\ t')\ J$$

which gives the required inequality. The function $g_{3,0,k}$ is determined by (8).

In conclusion of this proof we establish the estimation for the difference

$$|D_t u(x, t) - D_t u(x, t')| \leq$$

$$= |F[x, t, \dots, D_x^{\gamma} u(x, t), \dots] - F[x, t', \dots, D_x^{\gamma} u(x, t'), \dots]| +$$

$$+ \sum_{|k|=2b} |A_k(x)| \, |D_x^k(x, t) - D_x^k u(x, t')| .$$

Since $u \in C^{2b-1+\alpha, (2b-1+\alpha)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$, by (6) we can majorize the first member of the previous inequality by

$$\Big\{ p(t)\,(t'-t)^{\beta'/2b} +$$

$$+ L \sum_{i=0}^{2b-1} \sum_{|\gamma|=2b-1} (t'-t)^{\beta_\gamma (2b-1+\alpha-|\gamma|)/2b} \, .$$

$$. \,(q^{\gamma}(t), f_{A,\nu}(t'-t)\, f_{A,\nu}(t')\, J) \Big\}\, J.$$

We easily see that for $\max\left[1-(\beta-\varrho)(2b)^{-1},\ 1-(2b-1-|\gamma|)\beta_\gamma(2b)^{-1},\ 1-(\alpha\beta_\gamma-\varrho)(2b)^{-1}\right] \leq \mu < 1$ and $t, t' \in (0, \infty)$, $t < t'$, the functions $(t'-t)^{(\beta'-\varrho)/2b}$.

$$\cdot \Big(\int_0^{t'-t} z^{-\mu} e^{Az}\, dz\Big)^{-1} \quad \text{and} \quad (t'-t)^{\beta_\gamma(2b-1-|\gamma|)/2b} \Big(\int_0^{t'-t} z^{-\mu} e^{Az}\, dz\Big)^{-1}$$

for $|\gamma| = 0, 1, \dots, 2b-2$, $(t'-t)^{(\alpha\beta_\gamma-\varrho)/2b} \Big(\int_0^{t'-t} z^{-\mu} e^{Az}\, dz\Big)^{-1}$

for $|\gamma| = 2b-1$ are bounded in t and t'.

Hence analogically to the difference $|D_x^k u(x, t) - D_x^k(x, t')|$, we may conclude that

$$|D_t u(x, t) - D_t u(x, t')| \leq L(t'-t)^{\varrho/2b}\, g_{3,1,0}(t, t')\, J.$$

where

$$g_{3,1,0}(t, t') = p(t) f_{A,\mu}(t'-t) +$$

$$+ f^2_{A,\mu}(t'-t)\sum_{i=1}^{2b-2}\sum_{|\tau|=i}(q^{\tau}(t),\, f^{\beta_{\tau}}_{A,\nu}(t'-t)\, f^{\beta_{\tau}}_{A,\nu}(t')J) +$$

$$+ f_{A,\mu}(t'-t)\sum_{|\tau|=2b-1}(q^{\tau}(t),\, f^{\beta_{\tau}}_{A,\nu}(t'-t) f^{\beta_{\tau}}_{A,\nu}(t')J) +$$

$$+ g_{3,0,k}(t,\, t').$$

This finishes the proof.

R e m a r k 2. If there exists the solution $u \in C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_{\infty})$ of (1), (2), (3) then for the smoothness of this solution it is sufficient to assume that $|F| \leqq g(t)J$ on $\widetilde{H}_{\infty}$, where q is a bounded and integrable real function on every interval $\langle 0,\, T\rangle$ for any $T > 0$, instead of the assumption $\|F\|_{0,\widetilde{H}_{\infty}} \leqq L$ in Theorem 2.

3. The uniqueness

Using the Theorem 1 we get the following uniqueness

T h e o r e m 3. Let the assumptions (A), (B), $(D_{2b-1+\alpha})$ be fulfilled and $F : \widetilde{H}_{\infty} \to R_p$ be continuous and bounded vector function in the norm $\|\cdot\|_{0,\widetilde{H}_{\infty}}$. Let the Lipschitz condition

$$(9)\qquad |F(x,\, t,\, \ldots,\, u^{\tau},\, \ldots) - F(x,\, t,\, \ldots,\, v^{\tau},\, \ldots)| \leqq$$
$$= \sum_{i=0}^{2b-1}\sum_{|\tau|=i}(|r^{\tau}(t)|,\, |u^{\tau} - v^{\tau}|)\, J$$

be satisfied on $\widetilde{H}_{\infty}$. The vector funcion $r^{\tau}(t) = (r^{\tau}_1(t), \ldots, r^{\tau}_p(t))$ is integrable on Q_n for $|\tau| = 0, 1, \ldots, 2b - 1$ and $(|r^{\tau}(t)|\, ,\, f_{A,\varkappa}(t)\, J) \leqq L_n$ for $t \in \langle 0,\, n\rangle$, where $L_n > 0$ and $n = 1, 2, \ldots$. Further let $B_n = LL_n p^2 [\, 2s + t(2b - 1)\,] < 1$

for n = 1, 2, ... (L is a constant from the estimations in Lemma 2 of [2] and s and t(r) are defined in [2] p. 2). Then the integro-differential equation $u = A(x, t)\, u$ has one and only one solution belonging to $C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$ and furthemore the successive approximations $u_\ell = A(x, t)\, u_{\ell-1}$ tend to this unique solution as $\ell \longrightarrow \infty$ in the norm $\| \cdot \|^{f(A,\varkappa,\mu,\nu)}_{2b-1+\alpha,Q_\infty}$ for any $u_0 \in C^{2b-1+\alpha,\ (2b-1+\alpha)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$.

P r o o f . Consider the locally convex space $(P(Q_\infty), \mathcal{T}) = (C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty); \mathcal{T})$ topologized by the family of seminorms $\sigma_n(u) = \|u\|^{f(A,\varkappa,\mu,\nu)}_{2b-1+\alpha,Q_n}$ for n = 1, 2, ... According to hypothesis (9) and Lemma 2 of [2] for $u, v \in P(Q_\infty)$ and $(x, t) \in Q_n$ we have

$$|D^k_x A(x, t)u - D^k_x A(x, t)\, v| \leq \Big\{ \int_0^t d\tau \int_\Omega |G(x, t; \xi, \tau)| \cdot$$

$$\cdot \sum_{i=0}^{2b-1} \sum_{|r|=i} (|r^r(\tau)|\, f_{A,\varkappa}(\tau),\ |D^r_x u(\xi, \tau) -$$

$$- D^r_x v(\xi, \tau)|\ f^{-1}_{A,\varkappa}(\tau))d\xi \Big\}\ J \leq L_n p\, I_{1,k}(x, t)\, \sigma_n(u - v)J \leq$$

$$\leq LL_n p^2 f_{A,\varkappa}(t)\, \sigma_n(u - v)J, \quad |k| = 0, 1, \ldots, 2b-1$$

and

$$|D^k_x A(x, t)\, u - D^k_x A(x, t)v - D^k_x A(y, t)u + D^k_x A(y, t)v| \leq$$

$$\leq LL_n p^2 f_{A,\mu}(t)\, \sigma_n(u - v)\, |x - y|^\alpha\, J, \quad |k| = 2b - 1$$

and

$$\left| D_x^k A(x, t)u - D_x^k A(x, t)v - D_x^k A(x, t')u + D_x^k A(x, t')v \right| \leq$$

$$\leq L_n p\, I_{3,k}(x, t, t')\; \sigma_n(u - v)\, J \leq$$

$$\leq LL_n p^2 f_{A,\nu}(t' - t) f_{A,\nu}(t')\, \sigma_n(u - v)(t' - t)^{(2b-1+\alpha - |k|)/2b}\; J$$

for $|k| = 0, 1, \ldots, 2b- 1$. Hence we obtain

$$\sigma_n \left[A(x, t)u - A(x, t)v \right] \leq LL_n p^2 \left[2s + t(2b - 1 \right] \sigma_n(u - v)$$

which gives the conditions a) and b) of Theorem 1 for the identical mapping .

The remained claims in Theorem 1 are evidently satisfied and so the uniqueness of solution of $u = A(x, t)u$ is proved.

Since the space $(P(Q_\infty), \tau)$ is complete (see Lemma 1 of [2]) and the sequence of the successive approximations $\{u_\ell\}_{\ell=1}^{\infty}$ is fundamental (i.e. $\sigma_n(u_\ell - u_{\ell+m}) = B_n^\ell (1 - B_n)^{-1} \cdot \sigma_n(u_0 - A(x, t)u_0)$ for $n = 1, 2, \ldots$ and $u_0 \in P(Q_\infty)$) there exists a vector function $u^* \in P(Q_\infty)$ such that $\lim_{\ell \to \infty} u_\ell = u^*$ in $(P(Q_\infty), \tau)$. For sufficiently large index $\ell > 0$ and $\varepsilon > 0$

$$\sigma_n(u^* - A(x, t)u^*) \leq$$

$$\leq \sigma_n(u^* - u_\ell) + B_n\, \sigma_n(u_{\ell-1} - u^*) < \varepsilon$$

whence $u^* = A(x, t)u^*$ on every Q_n, $n = 1, 2, \ldots$. The proof is complete.

4. Asymptotic behaviours

The aim of this part is to investigate the solution of problem (1), (2), (3) as $t \to \infty$.

Theorem 4. Let the condition

(10) $$\left|F(x, t, \ldots, u^{r}, \ldots) - F(x, t', \ldots, v^{r}, \ldots)\right| \leq$$
$$= L\, e^{-c_1 t^*} \left\{ \left|t - t'\right|^{\beta'/2b} + \right.$$
$$\left. + \sum_{i=0}^{2b-1} \sum_{|r|=i} e^{-A\beta_r |t-t'|}\, e^{-c_2\beta_r t^*} \left(J, \left|u^{r} - v^{r}\right|^{\beta_r}\right)\right\} J$$

hold on $\tilde{H}_{\infty}$. Here A is a positive constant from the estimation (6) of [2] and $L > 0$, $c_i > A$ for $i = 1, 2$ and β , $\beta_r \in (0, 1\rangle$ for

$r = 1, \ldots, 2b-1$ and $t^* = \max(t, t')$; moreover $F(x, 0, \ldots, 0, \ldots) = 0$ on Ω . Then to each solution $u \in C^{2b-1+\alpha,\ (2b-1+\alpha)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_{\infty})$ of the problem (1), (2), (3) there exists a real number $t_o(u) > 0$ such that

$$\left|D_x^k u(x, t)\right| = O(e^{-Bt})\ J \quad \text{for } x \in \Omega ,\ t > t_o(u)$$

and for $|k| = 0, 1, \ldots, 2b-1$, where $0 < B < \min(c_1 - A, c_2 - A)$.

Proof. For $u \in C^{2b-1+\alpha,(2b-1+\alpha)}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_{\infty})$ one obtains

$$D_x^k u(x, t) \leq \int_0^t d\tau \int_{\Omega} \left|D_x^k G(x, t; \xi, \tau)\right| \left| F\left[\xi, t, \ldots, D_x^{r} u(\xi, t), \ldots\right]\right| d\xi +$$
$$+ \int_0^t d\tau \int_{\Omega} \left|D_x^k G(x, t; \xi, \tau)\right| \left| F\left[\xi, \tau, \ldots, D_x^{r} u(\xi, \tau), \ldots\right] - \right.$$
$$\left. - F\left[\xi, t, \ldots, D_x^{r} u(\xi, t), \ldots\right]\right| d\xi \overset{df}{=} I_1 + I_2$$

on Q_{∞} for $|k| = 0, 1, \ldots, 2b-1$.

Using the hypothesis (10) and the estimation of the Green's function (6') from [2] we get

$$I_1 \leq L\, e^{-c_1 t} \left\{ t^{\beta'/2b} + \right.$$
$$\left. + p \sum_{i=0}^{2b-1} \sum_{|r|=i} e^{-(A+c_2)\beta_r t} \left(\int_0^t e^{Az} z^{-\varkappa} dz\right)^{\beta_r}\right\} \left(\int_0^t e^{Az} z^{-\mu} dz \right) J$$

and

$$I_2 \leqq \Big\{ L\, e^{-c_1 t} \int_0^t e^{Az} z^{-\mu} \Big[z^{\beta'/2b} +$$

$$+\, p \sum_{i=0}^{2b-1} \sum_{|\gamma|=i} e^{-(Az+c_2 t)\beta_\gamma} z^{\beta_\gamma (2b-1+\alpha-|\gamma|)/2b} \,.$$

$$.\Big(\int_0^z e^{Au} u^{-\nu} du \Big)^{\beta_\gamma} \Big(\int_0^t e^{Av} v^{-\nu} dv \Big)^{\beta_\gamma} \Big] dz \Big\}\, J.$$

For sufficiently large $t_0(u) > 0$ and $t > t_0(u)$ we easily obtain

$$(11) \qquad e^{-c_1 t}\, t^{\beta'/2b} \int_0^t e^{Az} z^{-\mu} dz \leqq L\, e^{-Bt}$$

$$e^{-c_2 t} \int_0^t e^{Az} z^{-\varkappa} dz \leqq L\, e^{-Bt}$$

and

$$(12) \qquad e^{-A\beta_\gamma t}\, e^{-c_1 t} \int_0^t e^{Az} z^{-\mu} dz \leqq L\, e^{-Bt}$$

similarly

$$(13) \qquad e^{-c_1 t} \int_0^t e^{Az} z^{\beta'/2b-\mu} dz \leqq L\, e^{-Bt}$$

and

$$(14) \qquad e^{-c_1 t} \int_0^t e^{Az(1-\beta_\gamma)} z^{\beta_\gamma (2b-1+\alpha-\gamma)/2b-\mu} \,.$$

$$.\Big(\int_0^z e^{Au} u^{-\nu} du \Big)^{\beta_\gamma} dz \leqq L\, e^{-Bt}$$

where L is a positive constant.

Hence the assertion of this theorem is true.

If at least one constant c_i, $i = 1, 2$, from hypothesis (10) is equal to the constant A we get a weaker result than the one in Theorem 4. The following theorem is true:

T h e o r e m 5. Let the assumptions of Theorem 4 be satisfied with the constants $c_i \geqq A$ for i = 1, 2. Then for the solution $u \in C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$ of the problem (1), (2), (3) with $2b-1+\alpha-|r| < (\mu/\beta_r)+\nu$ we have

$$|D^k_x u(x,t)| = \mathcal{O}(1/t^{a\beta_0})J,\ x \in \Omega,\ t > t_0,\ t_0 > 0,$$ where

$|k| = 0, 1, \ldots, 2b-1$ and $0 < a < \min(\mu - \beta/2b, \nu)$ and

$$\beta_0 = \min_{|r|=0,1,\ldots,2b-1} \beta_r .$$

The proof is the same as in Theorem 4.

R E F E R E N C E S

[1] DELEANU A. - MARINESCU G.: Fixed point and implicit functions theorem in locally convex spaces. Revue de Math. pures et appliq., 8, 1963, p. 91 - 99.

[2] ĎURIKOVIČ V.: On a nonlinear stationary parabolic boundary value problem. Acta Fac. Rerum Natur. Univ. Comenian. Math. (to appear).

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S ú h r n

O NIEKTORÝCH VLASTNOSTIACH STACIONÁRNEJ PARABOLICKEJ ZMIEŠANEJ ÚLOHY

VLADIMÍR ĎURIKOVIČ, BRATISLAVA

V práci sa vyšetruje hladkosť a jednoznačnosť riešenia úlohy (1), (2), (3), jeho asymptotické vlastnosti a konvergencia postupných aproximácií v priestore hölderovských funkcií

$C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$.

Р Е З Ю М Е

О НЕКОТОРЫХ СВОЙСТВАХ СТАЦИОНАРНОЙ СМЕШАННОЙ ЗАДАЧИ ПАРАБОЛИЧЕСКОГО ТИПА

ВЛАДИМИР ДЮРИКОВИЧ, БРАТИСЛАВА

В этой статье исследуется однозначность и регуларность задачи (1), (2), (3), асимптотическое поведение решения и сходимость последовательных приближений в пространствах функций Гельдера

$C^{2b-1+\alpha,\ (2b-1+\alpha)/2b}_{x,t,\ f(A,\varkappa,\mu,\nu)}(Q_\infty)$.

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THE ESTIMATIONS OF THE LIMIT OF THE GEOMETRIC MEANS OF LÜROTH'S DIGITS

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It is well-known, that each real number $x \in (0, 1)$ can be uniquely expressed in the form

$$(1)\quad x = \frac{1}{d_1+1} + \frac{1}{s_1}\frac{1}{d_2+1} + \frac{1}{s_1 s_2}\frac{1}{d_3+1} + \dots + \frac{1}{s_1 s_2 \dots s_n}\frac{1}{d_n+1} + \dots$$

where $d_i = d_i(x)$ are positive integers and $s_i = d_i(d_i + 1)$ $(i = 1, 2, \dots)$. The expression (1) is called thě Lüroth's expansion of x and the numbers d_i digits of x (see [1], p. 116-122).

It is proved in [2] (see also [5]), that for almost all $x \in (0, 1)$ the equality

$$\lambda = \lim_{n\to\infty} \sqrt[n]{d_1(x)\, d_2(x) \dots d_n(x)} = \prod_{k=1} k^{1/k(k+1)}$$

hodls.

This result is analogous to a well-known Chinčin's result on continued fractions (see [6], p. 110-111).

We shall deal in this paper with numerical calculation of λ. The calculation of λ leads to the calculation of then sum of infinite series for $\ln \lambda$. We shall estimate the remainder of this series in an elementary way, what allows to estimate its sum with arbitrarily high accuracy. Following there is given the way, how to correct this estimation by using of higher differences. The estimation of λ by Euler's summation formula is given at the end of this paper.

1. Elementary estimations for λ.

We consider the convergent series

$$\ln \lambda = \sum_{k=2}^{\infty} \frac{\ln k}{k(k+1)} = \sum_{k\geq 2} f(k) ,$$

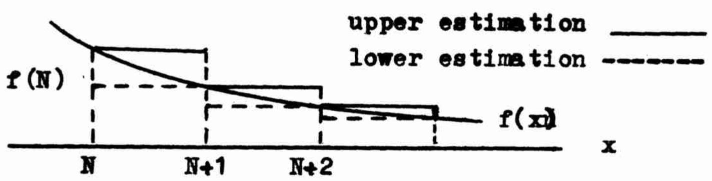
where $f(x) = \ln x / x(x + 1)$ is decreasing function for $x \geq 2$. Hence for each integer $k \geq 2$ we have

$$f(k) > \int_{k}^{k+1} f(x)\, dx > f(x + 1)$$

and so we get

$$f(N) + f(N + 1) + \ldots > \int_{N}^{\infty} f(x)\, dx > f(N + 1) + f(N + 2) + \ldots$$

by summation for $k = N + 1, \ldots$.



Hence

$$S = \ln \lambda = \sum_{k=2} f(k) = \sum_{k=2}^{N} f(k) + \sum_{k=N+1} f(k) < S_N + \int_{N}^{\infty} f(x)\, dx$$

and similary

$$S = \ln \lambda = \sum_{x=2}^{\infty} f(k) = \sum_{k=2}^{N} f(k) + \sum_{k=N+1} f(k) > S_N + \int_{N+1}^{\infty} f(x)\, dx,$$

where S_N is a N-th partial sum of the series $\sum_{k\geq 2} f(k)$.

Hence

$$(2) \quad S_N + \int_{N+1}^{\infty} f(x)\, dx < S < S_N + \int_{N}^{\infty} f(x)\, dx$$

and if E_* (E^*) is a lower (upper) estimation of the integral in (2), then it is possible to write

(3) $S_N + E_*(N+1) < S < S_N + E^*(N)$.

Some "symmetrization" of estimations from [3] is shown in (3), to get an improvement of lower estimation. We can get the estimations of the integral $\int_a^\infty \frac{\ln x}{x(x+1)}\,dx$ for $a \geq N$ in the following way: we write

$$\frac{1}{x(x+1)} = \frac{1}{x^2} \cdot \frac{1}{1+1/x} = \frac{1}{x^2} \cdot (1 - 1/x + 1/x^2 - \ldots) .$$

We get from this

$$\int_a^\infty (1/x^2 - 1/x^3) \ln x\,dx < \int_a^\infty \frac{\ln x}{x(x+1)}\,dx < \int_a^\infty (1/x^2 - 1/x^3 + 1/x^4) \cdot$$

$$\cdot \ln x\,dx .$$

Since

$$\int_a^\infty \frac{\ln x}{x^{s+1}}\,dx = \int_{\ln a}^\infty t.e^{-st}\,dt = \frac{1}{s^2} \cdot \frac{1 + s\ln a}{a^s} \quad \text{for} \quad s > 0$$

we get following estimations from previous inequalities:

(4) $$E_*(a) = \frac{1+\ln a}{a} - \frac{1}{4} \cdot \frac{1 + 2\ln a}{a^2} ,$$

$$E^*(a) = E_*(a) + \frac{1}{9} \cdot \frac{1 + 3\ln a}{a^3} .$$

It is possible to improve these estimations. The improvement would have sense only with adequate improvement of the estimation (2). It could be done in the following way (what is an intuitive geometrical interpretation):

Let us suppose that $\sum_1^\infty f(k)$ is a convergent series, $f(x)$ is positive and decreasing for $x \geq x_0$, $f(x) \to 0$ monotonically $(x \to \infty)$. Let there exist derivatives f' and f'' for $x \geq x_0$ and they monotonically tend to zero as $x \to \infty$.

Hence $f'(x) < 0$, $f''(x) > 0$ for $x \geqq x_0$. Then for every $x \in (k, k + 1)$ $(k \geqq x_0)$ there exists an $s_x \in (k, k + 1)$ such that $f(x) - f(k) = f'(s_x).(x - k) < f'(x) \cdot (x - k)$. Hence

$$\int_k^{k+1} f(x)\, dx < \int_k^{k+1} \left[f(k) + f'(x)\ (x - k)\right] dx =$$

$$= \left[f(k).x + f(x)\ (x - k)\right]_k^{k+1} - \int_k^{k+1} f(x)\, dx,$$

This yields

$$(5) \qquad \int_k^{k+1} f(x)\, dx < \frac{f(k) + f(k + 1)}{2}\ .$$

Since there is $r_x \in (x, k + 1)$, such that $f(x) = f(k + 1) - (k + 1 - x) \cdot (f'(r_x)$ and $f' < 0$ is increasing, we get $f(x) = f(k + 1) - (k + 1 - x) \cdot f'(r_x) > f(k + 1) -(k + 1 - x) \cdot f'(k + 1)$. Since there is some $t \in (k + 1, k + 2)$, such that $f(k + 2) - f(k + 1) = f'(t) > f'(k + 1)$, we get for $x \in (k, k + 1)$ the following result:

$$f(k + 1) + (k + 1 - x)\left[f(k + 1) - f(k + 2)\right] < f(x)$$

hence

$$(6) \qquad \frac{3\, f(k + 1) - f(k + 2)}{2} < \int_k^{k+1} f(x)\, dx.$$

We get for $k = N, N + 1, \ldots$ by summation of both inequalities (5), (6):

$$(7)\ \frac{1}{2} f(N + 1) + \sum_{k=N+1}^{\infty} f(k) < \int_N^{\infty} f(x)\, dx < -\frac{1}{2} f(N) + \sum_{k=N}^{\infty} f(k)$$

and from (7) by replacing N into $N + 1$ we obtain

$$\int_{N+1}^{\infty} f(x)\, dx + \frac{1}{2} f(N + 1) < \sum_{k=N+1}^{\infty} f(k) < \int_N^{\infty} f(x)\, dx - \frac{1}{2} f(N+1).$$

Hence, if $E_*(N)$ $(E^*(n))$ is lower (upper) estimation for the integral in (7), then we have

$$(8)\quad S_N + E_*(N+1) + \frac{1}{2} f(N+1) < S < S_N + E^*(N) - \frac{1}{2} f(N+1).$$

This inequality was used for the computation of λ . The following table gives some values:

N	100	1000	4000	7000
S_N	0.7329597935	0.7806299615	0.7862075874	0.7871230852
$\frac{1}{2}f(N+1)$	0.0002239915	0.0000034440	0.0000002590	0.0000000903
$E_*(N+1)$	0.0553445360	0.0078971569	0.0023227195	0.0014073904
$E^*(N)$	0.0557980895	0.0079040538	0.0023232376	0.0014075711
$S_*(N)$	0.7885283210	0.7885305624	0.7885305658	0.7885305659
$S^*(N)$	0.7885338915	0.7885305712	0.7885305660	0.7885305659
$\exp S_*(N)$	2.2001561189	2.2001610504	2.2001610579	2.2001610580
$\exp S^*(N)$	2.2001683750	2.2001610697	2.2001610582	2.2001610580

where

$$S_*(N) = E_*(N+1) + \frac{1}{2} f(N+1) + S_N,$$

$$S^*(N) = S_N + E^*(N) - \frac{1}{2} f(N+1).$$

Hence

$$\ln \lambda = 0.7885305659 \pm \varepsilon$$

$$\lambda = 2.2001610580 \pm \delta, \quad |\delta| < 1{,}8.10^{-10}$$

where $|\varepsilon| < 9.10^{-11}$. In order to eliminate rounding errors the sum $S_N = \sum_{k=2}^{N} f(k)$ was calculated in double precision (23 digits).

2. Estimations of remainder by using of higher derivatives

Previouos estimations are associated with the approximation of $f(x)$ on the interval $[k, k+1]$ by a constant or a linear function. It is natural to try to approximate f by a polynomial of higher degree. This method, used on finite interval, gives Gregory's formula or Laplace's formula for numerical quadrature. We shall deal with an arrangement for the estimation of the remainder of the series $\sum_{k=1}^{\infty} f(k)$.

Let us suppose that the function f is defined on the interval $[x_0, \infty)$, it has all derivatives and $f^{(i)}(x) \to 0$ $(i = 0, 1, \ldots)$ monotonically as $x \to \infty$. Suppose, that f is positive on the interval $[x_0, \infty)$ and all its derivatives are integrable on $[x_0, \infty)$.

We seek for $f(x)$ an interpolation polynomial $L_s(x)$ (of degree s) at tabular points $x = k, k+1, \ldots, k+s$.

Let previouos assumptions for $f(x)$ hold. Then for $x \in [k, k+1]$ we have

$$f(x) = L_s(x) + \frac{1}{(s+1)!} f^{(s+1)}(\xi_x).(x-k)\,(x-k-1)\ldots(x-k-s).$$

We shall consider $L_s(x)$ in the form

$$(9) \quad L_s(x) = f(k) + (x-k)\,\Delta f(k) + \frac{1}{2!}(x-k)(x-k-1)\,\Delta^2 f(k) + \\ + \ldots + \frac{1}{s!}(x-k)\ldots(x-k-s+1)\,\Delta^s f(k).$$

Then we get for $p = 0, 1, \ldots, s$

$$L_s(k+p) = f(k) + p\,\Delta f(k) + \frac{1}{2!}p(p-1)\,\Delta^2 f(k) +$$

$$+ \dots + \frac{p(p-1)\dots(p-s+1)}{s!}\Delta^s f(k) =$$

$$= \sum_{j=0}^{p} \binom{p}{j} \Delta^j f(k) = \sum_{j=0}^{p} \binom{p}{j} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} f(k+i) =$$

$$= \sum_{i=0}^{p} (-1)^i f(k+i) \sum_{j=i}^{p} (-1)^j \binom{p}{j}\binom{j}{i} = \sum_{i=0}^{p} (-1)^i f(k+i) \sum_{j=i}^{p} (-1)^i \binom{p}{i}\binom{p-i}{p-j} =$$

$$= \sum_{i=0}^{p} (-1)^i f(k+i) \binom{p}{i} \delta_{pi} = f(k+p)$$

or, in an operator form by using $\Delta = e^D - 1$, $D = \frac{d}{dx}$,

$$L_s(k+p) = \sum_{j=0}^{p} \binom{p}{j} \Delta^j f(k) = (1+\Delta)^p f(k) = e^{Dp} f(k) =$$

$$= e^{pD} f(k) = f(k+p).$$

Hence (9) is the interpolation polynomial indeed and in the notation

$$A_0 = 1,\ A_p = \frac{1}{p!}\int_k^{k+1} (x-k)(x-k-1)\dots(x-k-p+1)\,dx =$$

$$= \frac{1}{p!}\int_0^1 t(t-1)\dots(t-p+1)\,dt = \int_0^1 \binom{t}{p}\,dt$$

we get after the integration on $[k, k+1]$:

$$\int_k^{k+1} f(x)\,dx = A_0 f(k) + A_1 \Delta f(k) + \dots + A_s \Delta^s f(k) + A_{s+1} f^{(s+1)}(\eta_k),$$

where $\eta_k \in (k, k+1)$.

By the summation for $k = N, N+1, \dots$ we get

$$\int_N^{\infty} f(x)\,dx = A_0 \sum_{k=N}^{\infty} f(k) - A_1 f(N) - A_2 \Delta f(N) - \dots - A_s \Delta^{s-1} f(N) +$$

$$+ A_{s+1} \sum_{k=N}^{\infty} f^{(s+1)}(\eta_k).$$

We have to estimate the sum

$$\sigma = \sum_{k \geq N} f^{(s+1)}(\eta_k), \quad \eta_k \in (k, k+1)$$

in the last relation.

1. If $s + 1$ is even then $f^{(s+1)}$ is a monotonically decreasing positive function, thus

$$\int_{N-1}^{\infty} f^{(s+1)}(x)\,dx > f^{(s+1)}(N) + f^{(s+1)}(N+1) + \dots > \sigma >$$

$$> f^{(s+1)}(N+1) + f^{(s+1)}(N+2) + \dots > \int_{N+1}^{\infty} f^{(s+1)}(x)\,dx.$$

Hence

$$-f^{(s)}(N-1) > \sigma > -f^{(s)}(N+1),$$

Since $\operatorname{sgn} A_{s+1} = (-1)^s = -1$ in this case, we get for $s + 1$ even

$$-A_{s+1}f^{(s)}(N-1) < A_{s+1}\sigma < -A_{s+1}f^{(s)}(N+1).$$

2. If $s + 1$ is odd then $f^{(s+1)}$ is a monotonically increasing negative function, thus

$$\int_{N-1}^{\infty} f^{(s+1)}(x)\,dx < f^{(s+1)}(N) + f^{(s+1)}(N+1) + \dots < \sigma <$$

$$< f^{(s+1)}(N+1) + f^{(s+1)}(N+2) + \dots < \int_{N+1}^{\infty} f^{(s+1)}(x)\,dx.$$

Hence

$$-f^{(s)}(N-1) < \sigma < -f^{(s)}(N+1).$$

Since $\operatorname{sgn} A_{s+1} = (-1)^s = 1$ in this case, we have again

$$-A_{s+1}f^{(s)}(N-1) < A_{s+1}\sigma < -A_{s+1}f^{(s)}(N+1).$$

So we get

$$(10)\quad \int_N^{\infty} f(x)\,dx - (1 - A_1)f(N) + A_2\,\Delta f(N) + \dots + A_s\,\Delta^{s-1}f(N) +$$

$$+ A_{s+1}f^{(s)}(N-1) > R_N > \int_N^{\infty} f(x)\,dx - (1 - A_1)\,f(N) + A_2\,\Delta f(N) +$$

$$+ \dots + A_s\,\Delta^{s-1}\,f(N) + A_{s+1}\,f^{(s)}(N+1).$$

If we denote by E_{*} (E^{*}) the lower (upper) estimation of the integral in (10) and if

$$(11)\quad W_{*}(N) = E_{*}(N) - (1 - A_1)f(N) + A_2\,\Delta f(N) + \dots + A_s\,\Delta^{s-1}f(N),$$

$$W^{*}(N) = E^{*}(N) - (1 - A_1)\,f(N) + A_2\,\Delta f(N) + \dots + A_s\,\Delta^{s-1}\,f(N),$$

then we get

$$(12)\quad S_N + W_{*}(N) + A_{s+1}\,f^{(s)}(N+1) < S < S_N + W^{*}(N) + A_{s+1}f^{(s)}(N-1),$$

where $S_N = \sum_1^N f(k)$ is again the N-th partial sum of the series $\sum_{k\geq 1} f(k)$.

The relations (11), (12) give estimations of the sum of the series $\sum_{k\geq 1} f(k)$ and they allow to take far less terms in S_N necessary for the estimation of S with given accuracy.

It is necessary to know koefficients $A_p = \int_0^1 \binom{t}{p}\,dt$ for the using of these estimations. The simpliest way how to define numbers A_p is to use the recurrent relation. We get this relation in following way [4] :

The generating function A_p is

$$\sum_{p\geq 0} A_p\,x^p = \sum_{p\geq 0} x^p \int_0^1 \binom{t}{p} dt = \int_0^1 \sum_{p\geq 0} \binom{t}{p} x^p\,dt = \int_0^1 (1 + x)^t\,dt = \frac{x}{\ln(1+x)}.$$

We get

$$1 = \sum_{p \geq 0} A_p x^p \sum_{q \geq 0} (-1)^q \frac{x^q}{q+1} = \sum_{r \geq 0} x^r \sum_{q=0}^{r} \frac{(-1)^q}{q+1} A_{r-q}.$$

Hence

$$A_0 = 1, \; A_r = \sum_{q=1}^{r} \frac{(-1)^{q-1}}{q+1} A_{r-q}.$$

it means

$A_0 = 1$, $A_1 = 1/2$, $A_2 = -1/12$, $A_3 = 1/24$, $A_4 = -19/720$, $A_5 = 3/160, \ldots$

Given assumptions are mostly fulfilled in practice, or it is possible to transform the series to fulfill given assumptions.

3. Application of Euler summation formula (ESF)

We transform the given problem to get an easy estimation of the integral in ESF. We have

$$S = \ln \lambda = \sum_{k=2}^{\infty} \frac{\ln k}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) \cdot \ln k = \sum_{k=1}^{\infty} \frac{1}{k+1} \ln \frac{k+1}{k} =$$

$$= \sum_{k=2}^{\infty} \frac{1}{k} \ln \frac{k}{k-1} = - \sum_{k=2}^{\infty} \frac{1}{k} \ln \left(1 - \frac{1}{k}\right).$$

Hence

$$S = -\sum_{k=2}^{\infty} \Psi(k), \text{ where } \Psi(x) = \frac{1}{x} \ln\left(1 - \frac{1}{x}\right).$$

Now we use ESF on $\Psi(x)$ to compute S (see [4] :

$$- S = \sum_{k=2}^{N} \Psi(k) + \int_{N+1}^{\infty} \Psi(x)\, dx + \frac{1}{2} \Psi(N+1) - \frac{1}{12} \Psi'(N+1) +$$

$$+ \frac{1}{720} \Psi'''(N + 1) + R_1,$$

where

$$|R_1| < \frac{1}{30240} \left|\Psi^{(5)}(N + 1)\right|.$$

It is easy to compute derivatives of Ψ and we get ($q = (N + 1)^{-1}$)

$$(13) \quad \int_{N+1}^{\infty} \Psi(x)\, dx = \int_0^q \frac{\ln(1 - w)}{w} dw - \int_0^q \frac{w + w^2/2 + w^3/3 + \dots}{w} dw =$$

$$= -\sum_{k=1}^{\infty} \frac{q^k}{k^2},$$

therefore

$$\int_{N+1}^{\infty} \Psi(x)\, dx = - \sum_{k=1}^{M} \frac{q^k}{k^2} + R_2, \text{ where } |R_2| < \frac{q^{M+1}}{(M+1)^2} (1+q+\dots) =$$

$$= \frac{q^{M+1}}{(M+1)^2} \cdot \frac{1}{1 - q} .$$

Hence we get

$$(13) \quad \ln \lambda = - \sum_{k=2}^{N} \Psi(k) + \sum_{k=1}^{M} \frac{q^k}{k^2} - \frac{1}{2} \Psi(N + 1) + \frac{1}{12} \Psi'(N + 1) -$$

$$- \frac{1}{720} \Psi'''(N + 1) + R.$$

where $|R| = |R_1 + R_2| \leqq |R_1| + |R_2|$.

The application of (13) for $N = 50$, $M = 20$ gave the same λ value as estimations with linear approximation for $N = 7000$. It is possible to use ESF for original series too. For example formula (4) gives estimations of $f(x) = \ln x/(x(x + 1))$.

4. Conclusion

By given methods we are able to compute the sum of series on computer, or to seek lower and upper estimations for these series. The application of ESF is most effective, but it is necessary to know the derivatives. The application of finite difference formulas is more simple. There is a strong relation between ESF and the estimation (12), evocated by changing of derivatives with differences. The computation of λ , resp. $\ln \lambda$ with derived relations was made on computer GE 427 in VS Tesla Rožnov.

R E F E R E N C E S

[1] PERRON O.: Irrationalzahlen. Berlin-Leipzig 1921.

[2] ŠALÁT T.: Zur metrischen Theorie der Lürothschen Entwicklungen der reellen Zahlen. Czechoslovak Mathematical Journal, 18(93), 1968, p. 489-522.

[3] DÁVID A.: Berechnung einer bestimmten Konstante aus der Theorie der Lürothschen Reihen. Acta F.R.N. Univ. Comen., Mathematica XXI, 1968, p. 15 - 20.

[4] BEREZIN I. S., ŽIDKOV N.P.: Metody vyčislenij I. Moskva 1959.

[5] JAGER H. - DE VROEDT C.: Lüroth series and their ergodic properties, Indag. Math. 31, 1961, p. 31 - 42.

[6] CHINČIN A.: Cepnyje droby. Moskva, 1961.

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S ú h r n

ODHADY GEOMETRICKÝCH PRIEMEROV LÜROTHOVYCH CIFIER

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Článok sa zaoberá numerickým výpočtom súčtu pomaly konvergentného radu $S = \sum_{k=1}^{\infty} \ln k/(k(k+1))$, ktorý hrá dôležitú úlohu v Lürothovych rozvojoch. V prvej časti sú odvodené odhady zvyšku príslušného radu, ktoré dovoľujú určiť jeho súčet s ľubovoľnou presnosťou. V druhej časti je ukázaná metóda korekcií týchto odhadov pomocou derivácií vyššieho rádu. V poslednej časti je odvodený odhad súčtu radu pomocou Eulerovej sumačnej formuly. Metódy uvedené v tomto článku sú vhodné aj pre iné pomaly konvergentné rady.

Р Е З Ю М Е

ОЦЕНКИ ПРЕДЕЛА ГЕОМЕТРИЧЕСКИХ СРЕДНИХ ЦИФР ЛЮРОТА

А. ДАВИД - С. ДВОРЖАК, БРАТИСЛАВА - РОЖНОВ

Б этой статье приводятся вычисления медленно сходьящегося ряда $S = \sum_{k=1}^{\infty} \ln k/(k(k+1))$, который играет важную роль в развитьях Люрота. Б первой части оценки остатка данного ряда, позволяющее вычислить его сумму с произвольной точностью. Во второй части показан метод дальнейших исправлений этих оценок за помощи производных высшего порядка. В последней части статьи дана оценка ряда при помощи суммационной формулы Ейлера. Методы этой статьи удобны и для других медленно сходьящихся рядов.

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CATEGORIES OF KLEIN SPACES

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In authors paper [3] several definitions of "(in)dependency" in Klein spaces are described and mutually compared. The main tool for this investigation is one special kind of a closure operation.

This paper gives a categorical approach to both Klein spaces and the mentioned closure operation.

1. Preliminary

Let M be a nonempty set of points. A bijective map $f : M \longrightarrow M$ is said to be a transformation of M. The set $T(M)$ of all transformations of M has a natural group structure given by the composition $(f, g) \longmapsto f \circ g$, where $(f \circ g)(x) = f(g(x))$. An alternative notation for $f \circ g$ is fg. The full transformation group $(T(M), \circ)$ will be denoted briefly by $T(M)$. As usually 1 or 1_M is the identity in $T(M)$; the inverse transformation to $f \in T(M)$ is denoted by f^{-1}.

<u>Definition 1.</u> Let G be a subgroup of the full transformation group $T(M)$ of M. Then $\mathbf{M} = (G, M)$ is Klein space (briefly k-space) with the underlying set M and the transformation group G. If G is trivial then $\mathbf{M}$ is called trivial.

The simplest k-space is the singleton $* = (\{1\}, \{*\})$ whose underlying set $\{*\}$ consists of exactly one point.

It is occasionally convenient to introduce a k-space via representation or action. Let $(G, \cdot)$ be an "abstract" group with the identity e and let $M \neq \emptyset$ be a set (of points).
A representation (of G on M) is any homomorphism

$$\Delta : G \longrightarrow T(M),\ g \longmapsto \Delta_g$$

An action (of G on M) is map

$$\alpha : G \times M \longrightarrow M,\ (g, x) \longmapsto \alpha(g, x)$$

such that for all $x \in M$ and all $f, g \in G$ it holds

$$\alpha(e, x) = x \quad \text{and} \quad \alpha(g, \alpha(f, x)) = \alpha(g \cdot f, x)$$

There is only a formal difference between a representation and an action. In fact

if Δ is a representation then $\alpha : (g, x) \longmapsto \Delta_g(x)$ is an action,

if α is an action then $\alpha : g \longmapsto \alpha(g, -)$ is a representation.

Here $\alpha(g, -) : M \longrightarrow M,\ x \longmapsto \alpha(g, x)$ is a transformation of M, A k-space M described by a representation Δ or an action α will be sometimes written (Δ, G, M) or (α, G, M). This notation will be used even in the case $G \subset T(M)$ where both Δ and α are canonical: $\Delta_g(x) = g(x)$ and $\alpha(g, x) = g(x)$.

Given a k-space $M = (G, M)$, we recall some terms:

$G(a) = \{g(a) \mid g \in G\}$ is the orbit of a point $a \in M$,

$M/G = \{G(x) \mid x \in M\}$ is the orbital decomposition of M,

$M : G \subset M \times M$ is the equivalence relation on M given by the orbital decomposition M/G: $(x, y) \in M : G \Longleftrightarrow G(x) = G(y)$,

$\pi_M : M \longrightarrow M/G$, $x \longmapsto G(x)$ is the natural projection,

Orb. $(M) = M/G = (\{1\}, M/G)$ is the orbital k-space (trivial, of course),

$M \mid N = (\alpha, G, N)$ is the restriction of a k-space M to a subset N where $N \neq \emptyset$ is a union of orbits of M and $\alpha(g, x) = g(x)$.

A k-space $M = (G, M)$ is homogenous (i.e. G acts transitively on M) if card $M/G = 1$.

E x a m p l e 1. Let $V^n(K)$ be an n-dimensional vector space over a field K and $GV(n, K)$ its group of all automorphisms. The k-space $V^n(K) = (GV(n, K), V^n(K))$ has exactly two orbits, namely $\{0\}$ and ${}^*V^n(K) = V^n(K) - \{0\}$. The definition of the projective k-space $P^{n-1}(K)$ started from the k-space ${}^*V^n(K) =$ $= V^n(K) \mid {}^*V^n(K)$ which is the restriction of the k-space $V^n(K)$ to the orbit ${}^*V^n(K)$, see Section 5. We will write ${}^*V^n(K) = (G\overset{*}{V}(n,K), {}^*V^n(k))$. The groups $G\overset{*}{V}(n, K)$ and $GV(n, K)$ are isomorphic.

2. The Klein Kategory

Our first task is to introduce the category point of view into the class of all k-spaces.

D e f i n i t i o n 2. A pair of maps

$$\Omega : G_1 \longrightarrow G_2, \qquad \omega : M_1 \longrightarrow M_2$$

is said to be a k-morphism from a k-space $M_1 = (G_1, M_1)$ to a k-space $M_2 = (G_2, M_2)$ if the square

$$\begin{tikzcd} M_1 \arrow[r, "\omega"] \arrow[d, "g"'] & M_2 \arrow[d, "\Omega(g) = \Omega_g"] \\ M_1 \arrow[r, "\omega"] & M_2 \end{tikzcd}$$

commutes for all $g \in G_1$. If in addition Ω is a group homomorphism then the k-morphism (Ω, ω) is called the strong k-morphism. A k-morphism (Ω, ω) will be briefly denoted by ω.

There is a very simple example of a k-morphism which is not a strong k-morphism. In fact, let M be a set consisting of at least three points and $m \in M$ a point. Choose $f \in T(M)$ such that $f \neq 1_M$ and $f(m) = m$. The k-morphism $\omega : * \longmapsto (T(M), M)$ given by $\omega : * \longmapsto m$ and $\Omega : 1 \longmapsto f$ is not a strong k-morphism. However the lack of "strong" in this example seems to be very unsubstantial, for the k-morphism ω can be turn to a strong k-morphism by replacing $\Omega \rightsquigarrow \Omega'$ where $\Omega' : 1 \longmapsto 1$. The following example shows that such a replacing does not exists in general.

<u>E x a m p l e 2.</u> Let us define k-spaces $M_i = (G_i, M_i)$, $i = 1, 2$ as follows: $M_1 = \{a, b\}$, $G_1 = T(M_1) = \{1, g\}$, $M_2 = \{a', b', c_i \mid i \in Z\}$ and G_2 is generated by the transformation $f \in T(M_2)$, $f(a') = b'$, $f(b') = a'$, $f(c_i) = c_{i+1}$, $i \in Z$. The k-morphism $\omega : M_1 \longrightarrow M_2$ is given by $\omega(a) = a'$, $\omega(b) = b'$, $\Omega(1) = 1$, $\Omega(g) = f$. Since $G_1 \cong Z_2$ and $G_2 \cong Z$ there is no homomorphism $\Omega' : G_1 \longrightarrow G_2$ except the trivial one.

<u>D e f i n i t i o n 3.</u> Each of the following four categories will be called the Klein category:

K the category of k-spaces and k-morphisms

K_s the category of k-spaces and strong k-morphisms

K^M the full subcategory of K whose objects have a given set $M \neq \emptyset$ as a common underlying set

K_s^M the intersection of categories K^M and K_s.

The categories K^M and K_s^M are small and it is obvious that

$$\begin{array}{ccc} K_s & \subset & K \\ \cup & & \cup \\ K_s^M & \subset & K^M \end{array}$$

where each inclusion means "the subcategory" and those two vertical mean even "the full subcategory".

At first sight it seems that K_s^M is "equal" to the small category of all subgroups of $T(M)$ and homomorphisms. However this is true only if card $M \leqq 3$.

<u>Example 3.</u> Let card $M = 4$, say $M = \{a, b, c, d\}$ and define transformations $f : (a, b, c, d) \longmapsto (b, a, c, d)$ and $g: (a, b, c, d) \longmapsto (b, a, d, c)$. The k-spaces $M_1 = (G_1, M)$, $G_1 = \{1, f\}$ and $M_2 = (G_2, M)$, $G_2 = \{1, g\}$ are different objects of K_s^M but $G_1 \cong G_2$. Moreover the isomorphism $\Omega: G_1 \longrightarrow G_2$, $\Omega(1) = 1$, $\Omega(f) = g$ can not be extended to a k-morphism $(\Omega, \omega) : M_1 \longrightarrow M_2$.

3. Similarity

Let $M \neq \emptyset$ be a set and $t \in T(M)$ a transformation. The inner automorphism $\Sigma_t: T(M) \longrightarrow T(M)$, $g \longmapsto tgt^{-1}$ or more precisely its restriction to a subgroup $G \subset T(M)$, leads to the concept of "similarity" in the small Klein category K^M.

<u>Definition 4.</u> Let $M = (G, M)$ be a k-space and $t \in T(M)$ a transformation. The k-space $\mathrm{Sim}_t M = (\Sigma_t, G, M)$, called the similar k-space to M, is given by the representation

$$\Sigma_t: G \longrightarrow T(M), \qquad \Sigma_t(g) : x \longmapsto (tgt^{-1})(x) \;;$$

this k-space will be frequently denoted by $(tgt^{-1})(x)$

this k-space will be frequently denoted by $(tGt^{-1}, M))$

Two k-spaces $\mathbf{M}_1 = (G_1, M_1)$ and $\mathbf{M}_2 = (G_2, M_2)$ are said to be similar, $\mathbf{M}_1$ sim $\mathbf{M}_2$, if $M_1 = M_2$ and $\mathbf{M}_2 = \mathrm{Sim}_t \mathbf{M}_1$ for some $t \in T(M_1)$.

Following three propositions can be easily verified.

P r o p o s i t i o n 1. Let $\mathbf{M} = (G, M)$ be a k-space and $t, s \in T(M)$ transformations. Then

$$\mathrm{Sim}_t\ \mathrm{Sim}_s\ \mathbf{M} = \mathrm{Sim}_{ts}\ \mathbf{M}$$

and

$$\mathrm{Sim}_1\ \mathbf{M} = \mathbf{M}.$$

Hence sim is the equivalence relation on the set of all objects of the Klein category K^M.

P r o p o s i t i o n 2. Let $\mathbf{M} = (G, M)$ be a k-space and $t, s \in T(M)$ transformations. Then there exists a strong k-morphism

$$\boldsymbol{\sigma}_t = (\Sigma_t, \sigma_t) : \mathbf{M} \longrightarrow \mathrm{Sim}_t\ \mathbf{M}$$

given by

$$\Sigma_t : G \longrightarrow tGt^{-1},\ g \longmapsto tgt^{-1},\ \ \sigma_t : M \longrightarrow M,\ x \longmapsto tx$$

Moreover

a) $\boldsymbol{\sigma}_t\ \boldsymbol{\sigma}_s = \boldsymbol{\sigma}_{ts}$

and

b) $\boldsymbol{\sigma}_1 = 1$

Hence $\boldsymbol{\sigma}_t$ is an isomorphism in the category K^M.

R e m a r k . Slightly confusing symbolic, Σ_t is a map $T(M) \longrightarrow T(M)$ or $G \longrightarrow T(M)$ or $G \longrightarrow tGt^{-1}$, does not leads to misunderstanding.

P r o p o s i t i o n 3. Let $\omega = (\Omega, \omega) : M_1 =$ $= (G_1, M) \longrightarrow M_2 = (G_2, M)$ be a k-morphism and $t \in T(M)$ a transformation. Then there is a k-morphism

$$Sim_t\ \omega : Sim_t\ M_1 \longrightarrow Sim_t\ M_2$$

given by the components

$$tG_1t^{-1} \longrightarrow tG_2t^{-1},\ tgt^{-1} \longrightarrow t\Omega_g t^{-1}$$

and

$$M \longrightarrow M,\ x \longmapsto (t\,\omega\, t^{-1})(x).$$

Hence Sim_t is the covariant functor from the Klein category K^M to itself.

4. M o n o m o r p h i s m a n d E p i m o r p h i s m.

The main result of this section is

T h e o r e m 1. Let be given a k-morphism or a strong k-morphism

$$\omega = (\Omega, \omega): M_1 \longrightarrow M_2, \quad M_i = (G_i, M_i),\ i = 1, 2$$

regarded as a morphism of the Klein category K or K_s. Then

a) ω is a monomorphism $\Longleftrightarrow$ both Ω and ω are injective,

b) ω is an epimorphism $\Longleftrightarrow$ both Ω and ω are surjective.

Before to prove Theorem 1 we have to solve the problem of dependancy between the maps Ω and ω.

P r o p o s i t i o n 4. Let $\omega = (\Omega, \omega) : M_1 \longrightarrow M_2$ be a k-morphism. Then

c) ω is injective $\Longrightarrow$ Ω is injective

d) ω is surjective $\Longrightarrow$ Ω is a group homomorphism.

P r o o f . Choose $f, g \in G_1$, $f \neq g$ and a point $x \in M_1$ such that $f(x) \neq g(x)$. Since ω is injective, it is $\omega f(x) \neq$ $\neq \omega g(x)$, hence $\Omega_f(\omega x) \neq \Omega_g(\omega x)$. Therefore $\Omega_f \neq \Omega_g$, and c) is proved.

To prove the second implication let us denote $h = gf$. The commutativity of the diagram

$$\begin{tikzcd}
M_2 \arrow[dd, "\Omega_h"'] & M_1 \arrow[l, "\omega"'] \arrow[rrr, "\omega"] \arrow[dr, "f"] \arrow[dd, "h"'] & & & M_2 \arrow[dd, "\Omega_g \circ \Omega_f"] \arrow[dl, "\Omega_f"'] \\
& & M_1 \arrow[r, "\omega"] \arrow[dl, "g"] & M_2 \arrow[dr, "\Omega_g"] & \\
M_2 & M_1 \arrow[l, "\omega"] \arrow[rrr, "\omega"] & & & M_2
\end{tikzcd}$$

yields $\Omega_h \omega = \omega h = \Omega_g \Omega_f \omega$, hence (ω is surjective) $\Omega_h = \Omega_g \Omega_f$.

Now the proof of Theorem 1 is divided into three parts:

1. (a), $\Rightarrow$) , 2. (b), $\Rightarrow$) , 3. (a) and b), $\Leftarrow$).

1. If ω is not injective, say $\omega x = \omega y$, $x \neq y$, $x, y \in M_1$ then there are two different strong k-morphisms ω_1, ω_2 from the singelton $*$ into the k-space M_1, given by $\omega_1(*) = x$ and $\omega_2(*) = y$. Thus $\omega_1 \neq \omega_2$ and $\omega \omega_1 = \omega \omega_2$, hence ω is not monomorphism. So if ω is monomorphism then ω and hence Ω (see Proposition 4) are injective.

2. Suppose by contradiction that at least one of the maps ω , Ω is not surjective. Then we shall find two strong k-morphisms ω_1 ω_2: $M_2 \longrightarrow M = (T(M), M)$ such that $\omega_1 \omega = \omega_2 \omega$ but $\omega_1 \neq \omega_2$.

Case A. ω is not surjective. Denote $L = \omega(M_1)$, $N = M_2 - L$. Thus $M_2 = L \cup N$ is the nontrivial decomposition of the set M_2. Let us now define the set $M = L \cup (N \times \{1, 2\})$,

two injections $\omega_i : M_2 \longrightarrow M$, $x \longmapsto \begin{cases} x & \text{for } x \in L \\ (x,i) & \text{for } x \in N \end{cases}$

and two homomorphisms

$$\Omega_i : G_2 \longrightarrow T(M),\ \Omega_i(g) : x \longmapsto \begin{cases} (\omega_i g \omega_i^{-1})(x) & \text{for } x \in \omega_i M_2 \\ x & \text{for } x \notin \omega_i M_2. \end{cases}$$

It is easy to verify that M, ω_1 and ω_2 are the required k-space and strong k-morphisms.

Case B: ω is surjective and Ω is not surjective. It is known (see [5] pg 38) that there exists a group G and two homomorphisms $\Omega_i : G_2 \longrightarrow G$, $i = 1, 2$ such that $\Omega_1\Omega = \Omega_2\Omega$ but $\Omega_1 \neq \Omega_2$. Define

the set $M = M_2 \cup G$,

two injections $\omega_1 = \omega_2 : M_2 \longrightarrow M$, $x \longmapsto x$

and two homomorphisms

$$\Omega_i : G_2 \longrightarrow T(M),\ \Omega_i(g) : x \longmapsto \begin{cases} g(x) & \text{for } x \in M_2 \\ \Omega_i(g).x & \text{for } x \in G. \end{cases}$$

Both $\boldsymbol{\omega}_i = (\Omega_i, \omega_i)$, $i = 1, 2$, are the strong k-morphisms, and $\boldsymbol{\omega}_1\boldsymbol{\omega} = \boldsymbol{\omega}_2\boldsymbol{\omega}$. If $\Omega_1(g) \neq \Omega_2(g)$ then also $\Omega_1(g) . x \neq \Omega_2(g) . x$, hence $\boldsymbol{\omega}_1 \neq \boldsymbol{\omega}_2$.

3. This part by the direct computation.

As a corollary of Theorem 1 we obtain

<u>Proposition 5.</u> The Klein categories K and K_s are balanced i.e. if a k-morphism $\boldsymbol{\omega}$ is both mono and epi then it is isomorphism.

Similar results about two small Klein categories are given in Proposition 6 and Example 4.

<u>Proposition 6.</u> After the replacement K or K_s

by K^M or K_s^M Theorem 1 remains true with one only exception, namely the implication (b),$\Longrightarrow$). Neither K^M nor K_s^M is balanced.

P r o o f . The proof of the part 1 is the same as in Theoreme 1; instead of the singelton $*$ we can take the trivial k-space $(\{1\}, M)$. Part 3 is proved by the direct computation. The rest of Proposition 6 is proved via Example 4.

E x a m p l e 4. Let $M = \{a, b, c\}$ be a set and $f : (a, b, c) \longmapsto (b, a, c)$ an involution. Define two groups $G_1 = \{1, f\}$ and $G_2 = T(M)$. The k-morphism $(\Omega, \omega):(G_1, M) \longrightarrow (G_2, M)$ uniquely given by its part $\omega : (a, b, c) \longmapsto (a, a, c)$ is the strong k-epimorphism, in K^M and K_s^M, but neither Ω nor ω is surjective.

The k-morphism $(\Omega', \omega'):(G_1, M) \longrightarrow (G_2, M)$ uniquely given by its part $\omega': (a, b, c) \longmapsto (a, b, c)$ is the strong k-morphism, which is both mono and epi in K^M and K_s^M; however, (Ω', ω') is not the isomorphism.

5. F a c t o r i s a t i o n

The factorisation process is a valuable tool for a production of new k-spaces from a given one. A well known example is the factorisation in V^{n+1} which leads to the k-space P^n(see Example 1) or the antipodal glueing of the sphere k-space $(O(n+1), S^n)$ which leads to the n-dimensional Riemannian k-space.

We recall that an equivalence relation $\sim$ on $M \neq \emptyset$ is said to be a congruence relation on $M = (G, M)$ if it is compatible with the action G i.e. if

$$x \sim y \Longrightarrow gx \sim gy \qquad \forall\, x, y \in M, \quad \forall\, g \in G.$$

In such a case we define the k-quotient space

$M/\sim\ = (\Pi, G, M/\sim)$ whose underlying set $M/\sim$ consists of all cosets $[a] = \{x \in M | x \sim a\}$, $a \in M$ and whose representation $\Pi : G \longrightarrow T(M/\sim)$ is defined by

$$\Pi(g) : [x] \longmapsto [gx]$$

Moreover the natural projection $\pi : M \longrightarrow M/\sim$ together with the homomorphism Π formed the strong k-morphism

$$\pi = (\Pi, \pi) : M \longrightarrow M/\sim$$

called the natural k-projection.

There are several "canonical" congruences on a given k-space M. To find these we shall first express a congruence on M in terms of subgroups of $T(M)$.

A subgroup H of $T(M)$ is said to be compatible with a k-space $M = (G, M)$ if the equivalence relation $M : H$ is a congruence on M i.e. if

$$H(x) = H(y) \Longrightarrow H(gx) = H(gy) \quad \forall\, x, y \in M, \ \forall\, g \in G$$

<u>P r o p o s i t i o n 7.</u> If $\sim$ is a congruence on M then there is a subgroup $H \subset T(M)$ such that $\sim$ is equal to $M : H$ and H is generated only by involutions.

<u>P r o o f .</u> To each couple of congruent points $a \sim b$ we assign the involution $f \in T(M)$ given by $f(a) = b$, $f(b) = a$, $f(x) = x$ for all $x \in M$, $x \neq a, b$. The subgroup of $T(M)$ generated by all just described involutions is the required group H.

<u>P r o p o s i t i o n 8.</u> Let $M = (G, M)$ be a k-space and let $N_F(H)$ means the normalizer of H in F. The sufficient condition for $H \subset T(M)$ to be compatible with M is $G \subset N_{T(M)}(H)$.

<u>P r o o f .</u> Let be $g \in G \subset N_{T(M)}(H)$ and $H(x) = H(y)$. Then we have $H(gx) = gHg^{-1}(gx) = g(H(x)) = g(H(y)) = gHg^{-1}(gy) = H(gy)$.

E x a m p l e 5. The sufficient condition $G \subset N_{T(M)}(H)$ for H to be compatible with M is not necessary.

Let $M = \{a_i, b_i, c_i \mid i = 1, 2, 3\}$ be a set and let $G \subset T(M)$ be a group generated by the following two transformations

$$f:(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) \longmapsto$$
$$\longmapsto (b_1, b_2, b_3, c_1, c_2, c_3, a_1, a_2, a_3)$$

$$g:(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) \longmapsto$$
$$\longmapsto (a_2, a_3, b_1, b_2, b_3, a_1, c_2, c_3, c_1)$$

Define $H = \{1, f, f^2\}$, then $gfg^{-1}(a_1) = c_1$ and $gfg^{-1}(a_2) = b_2$, hence $gfg^{-1} \notin H$. Nevertheless $M : H$ is a congruence on M.

Now the "canonical" congruences on M claimed above are described by

C o r o l l a r y . Let $\mathcal{M} = (G, M)$ be a k-space. Each of the groups

$$(*) \qquad T(M), \quad G, \quad C_{T(M)}(G) \quad \text{and} \quad C_G = C_G(G)$$

is compatible with M . Here $C_F(G)$ is the centre of a group G in a group F.

Associated k-quotient spaces of the first two of the groups (*) are well known k-spaces: $M/T(M)$ is the singelton $*$ and M/G is the orbital k-space of M. In addition both

$$M \to M/T)M) \quad \text{and} \quad M \longrightarrow M/G$$

have the functorial properties, as the functor from K to K_s. The second of these functors is frequnetly denoted by Orb. If $\omega: M_1 \longrightarrow M_2$, is a k-morphism then $\mathrm{Orb}\,\omega = (1, \mathrm{Orb}\,\omega)$ is the k-morphism $M_1/G_1 \longrightarrow M_2/G_2$ uniquely defined by the commutative square

$$\begin{tikzcd} M_1 \arrow[r, "\pi_1"] \arrow[d, "\omega"'] & M_1/G_1 \arrow[d, "\mathrm{Orb}\,\omega"] \\ M_2 \arrow[r, "\pi_2"] & M_2/G_2 \end{tikzcd}$$

6. Projectivisation

More interesting factorisations, as those two mentioned in Section 5 arise from the last two of the groups (*). Because of the inclusion $C_G \lhd C_{T(M)}$ there is the strong k-epimorphism

$$M/C_G \longrightarrow M/C_{T(M)}$$

We shall restricted our attention only to M/C_G, the "finest" of these two k-quotient spaces.

<u>Definition 5.</u> Let $M = (G, M)$ be a k-space and let C be the centre of the group G. Then the k-quotient space $M/C = (\Pi, G, M/C)$ is called the projectivisation of a k-space M.

We have already seen that the projectivisation of the k-space ${}^*V^{n+1}(K)$ is the projective k-space $P^n(K)$ and the projectivisation of the sphere k-space $(\sigma(n+1), S^n)$ is the n-dimensional Riemann-k-space.

<u>Remark.</u> Rather more general situation, which will be mentioned just in this short remark, comes from replacement of C by some its proper subgroup. For example if L is a subfield of a field K and E the unit matrix in $GL(n + 1, K)$ then $W = \{kE \mid k \in L\}$ is a subgroup of the centre $C = \{kE \mid k \in K\}$ of $GL(n + 1, K)$. The strong k-morphisms

$${}^*V^{n+1}(K) \longrightarrow {}^*V^{n+1}(K)/W \longrightarrow {}^*V^{n+1}(K)/C = P^n(K)$$

are similar to projections in fibre bundles.

Next we are going to prove the following three results concerning the projectivisation:

1. $C \subset \mathrm{Ker}\,\Pi$ but in general $C \neq \mathrm{Ker}\,\Pi$.

2. Each k-space can be obtain as the projectivisation M/C of a suitable k-space M.

3. The natural projection $M \longrightarrow M/C$ has functórial properties if and only if we restricted ourself to the category K_{epi} of all k-spaces and k-epimorphisms.

Proposition 9. Let $M = (G, M)$ be a k-space and $M/C = (\Pi, G, M/C)$ its projectivisation. Then $C \subset \mathrm{Ker}\,\Pi$, but in general $C \neq \mathrm{Ker}\,\Pi$.

Proof. The first assertion is obvious, the second one follows from

Example 6. Let $M = \{a, b, c, d\}$ be a set and $G \subset T(M)$ a group generated by the transformations $f : (a, b, c, d) \longmapsto (b, c, d, a)$ and $g: (a, b, c, d) \longmapsto (c, b, a, d)$. Since $C = \{1, f^2\}$ is a centre of G, M/C consists of two cosets $\{a, c\}$ and $\{b, d\}$. Now $g \in \mathrm{Ker}\,\Pi$ but $g \notin C$.

Theorem 2. Each k-space M is k-isomorphic to the projectivisation of a suitable k-space $\bar{M}$.

Proof. The construction of the k-space $\bar{M} = (\bar{G}, \bar{M})$ started with the underlying set $\bar{M} = M \times \{0, 1\}$. Let now $\bar{G}$ be the set of all transformations $\bar{g} \in T(\bar{M})$ for which $\pi\bar{g} = g\pi$ where g is an element of G and $\pi : \bar{M} \longrightarrow M, (x, i) \longmapsto x$ is a natural projection, $i = 1, 2$.

The proof is given in four steps.

1. $\bar{G}$ is the group. If $\bar{f}, \bar{g} \in G$ then there are $f, g \in G$ such that $\pi\bar{f} = f\pi$ and $\pi\bar{g} = g\pi$. Since $\pi\bar{f}\bar{g} = f\pi\bar{g} = fg\pi$ and $\pi(\bar{g})^{-1} = g^{-1}\pi(\bar{g})^{-1} = g^{-1}\pi\,\bar{g}(\bar{g})^{-1} = g^{-1}\pi$, it is $\bar{f}\bar{g} \in \bar{G}$ and $(\bar{g})^{-1} \in \bar{G}$.

2. The transformation $\bar{t} : (x, i) \longmapsto (x, 1 - i)$ belongs to the centre $\bar{C}$ of $\bar{G}$. Since $\pi \bar{t} = \pi = 1_M \pi$ it is $\bar{t} \in \bar{G}$. Moreover $\bar{g}\,(x, i) = (y, j)$ yields $g(x, 1 - i) = (y, 1 - j)$; therefore

$\overline{tg}(x, i) = (y, 1 - j) = \bar{g}(x, 1 - i) = \overline{gt}(x, i)$, hence $\overline{tg} =$
$= \overline{gt}$, $\forall\, \bar{g} \in \bar{G}$.

3. If $\bar{h} \in \bar{C}$ then $\pi \bar{h} = \pi$. Suppose $\pi \bar{h} = h\, \pi$, $h \neq 1_M$, say $h(a) = b \neq a$ and denote $c = h^{-1}(a)$. Then $\bar{h}(a, 0) = (b, i)$, $(\bar{h})^{-1}(a, 0) = (c, j)$ and we can define $\bar{f} \in T(\bar{M})$ by

$$\bar{f}(a, 0) = (c, 1 - j),\ \bar{f}(a, 1) = (c, j),\ \bar{f}(b, 0) =$$
$$= (a, i),\ \bar{f}(b, 1) = (a, 1 - i)$$

and $\bar{f}(x, 0) = (h^{-1}(x), 0)$, $\bar{f}(x, 1) = (h^{-1}(x), 1)$ for $x \in M$, $x \neq a, b$. Since $\pi \bar{f} = h^{-1} \pi$ it is $\bar{f} \in G$. Now $\overline{fh}(a, 0) =$
$= \bar{f}(b, i) = (a, 0)$ and $\overline{hf}(a, 0) = \bar{h}(c, 1 - j) = (a, 1)$ hence $\overline{fh} \neq \overline{hf}$ - contrradiction.

4. K-spaces M and M/C are isomorphic. According to steps 2 and 3 $\bar{C} = \{1, \bar{t}\}$ and M/C consists of all cosets $[x] =$
$= \{(x, 0), (x, 1)\}$. To each $g \in G$ there exists at least one $\bar{g} \in \bar{G}$ (say $\bar{g} : (x, i) \longmapsto (g(x), i)$ such that $\pi \bar{g} = g\, \pi$. Therefore $\omega : \bar{M}/\bar{C} \longrightarrow M$, $[x] \longmapsto X$, $\Omega : \Pi(\bar{G}) \longrightarrow G$, $\Pi_{\bar{g}} \longmapsto g$ is the required isomorphism.

<u>Proposition 10.</u> Let K_{epi} be the category of all k-speces and k-epimorphisms. Then K_{epi} is a subcategory of the strong Klein category K_s and there is a functor $K_{epi} \longrightarrow K_{epi}$ given by

$M = (G, M) \longrightarrow M/C = (\Pi, G, M/C)$ for objects and

$$\begin{array}{ccccc} M_1 & & (\Pi_1, G_1, M_1/C_1) & & C_1(x) \quad \Pi_1(g) \\ \omega\big\downarrow & \longrightarrow & \omega_*\big\downarrow & \text{where} & \big\downarrow\omega_* \qquad \big\downarrow\Omega_* \quad \text{for morphisms.} \\ M_2 & & (\Pi_2, G_2, M_2/C_2) & & C_2(\omega x) \quad \Pi_2(\Omega g) \end{array}$$

P r o o f . The first assertion follows directly from Theorem 1. To prove the second result we recall that for a group epimorphism $\Omega : G_1 \longrightarrow G_2$ it holds $\Omega(C_1) \subset C_2$ where C_i is the centre of G_i. If $C_1(x) = C_1(y)$ then there is $f \in C_1$ such that $y = f(x)$, hence $\Omega_f \in C_2$ and $\omega y = \Omega_f(\omega x)$; therefore $C_2(\omega x) = C_2(\omega y)$ and ω_* is well defined. The definition of Ω_* is justified by the commutative square

$$\begin{array}{ccc} C_1(x) & \xrightarrow{\ \omega_*\ } & C_2(\omega x) \\ \Pi_1(g)\big\downarrow & & \big\downarrow \Pi_2(\Omega_g) \\ C_1(gx) & \xrightarrow{\ \omega_*\ } & C_2(\omega gx) = C_2(\Omega_g\, \omega x) \end{array}$$

Since both ω_* and Ω_* are surjective, ω_* is an epimorphism.

E x a m p l e 7. Proposition 10 does not hold beyond the category K_{epi}. Let us define a strong k-morphism $\omega : M_1 \longrightarrow M_2$ as follows

$M_1 = M_2 = \{a, b, c\}$, $G_1 \cong Z_3$, $G_2 = T(M)$,

Ω is the inclusion map and ω is the identity 1_M.

Here M_1/C_1 is the singelton and $M_2/C_2 = M_2$ becourse C_2 is trivial. Thus there is no "canonical" k-morphism $\omega_* : M_1/C_1 \longrightarrow M_2/C_2$ which can be assign to the k-monomorphism ω.

7. Galois Connection

This Section induces the (categorial) concept of the Galois connection, which, beeing applied, is of fundamental importance in our approach to the geometry of Klein spaces.

D e f i n i t i o n 6. The Galois-connection category GC is the category whose:

1. Objects are sequences $\Gamma = (A, B, \Gamma, \alpha, \beta) = (\alpha, \beta)$ where

A, B, $\Gamma \subset A \times B$ are nonempty sets,

$\alpha : P(B) \longrightarrow P(A)$, $Y \longmapsto \{x \in A \mid (x, y) \in \Gamma, \forall y \in Y\}$

and $\beta : P(A) \longrightarrow P)B)$, $X \longmapsto \{y \in B \mid (x, y) \in \Gamma, \forall x \in X\}$

are maps called the first and the second polar respectively;

2. Morphisms $\lambda : \Gamma_1 \longrightarrow \Gamma_2$, $\Gamma_i = (A_i, B_i, \Gamma_i, \alpha_i, \beta_i)$ are pairs of maps

$$\Lambda : A_1 \longrightarrow A_2, \qquad \lambda : B_1 \longrightarrow B_2$$

with the property

(1) $(\Lambda \times \lambda)(\Gamma_1) \subset \Gamma_2$ i.e. $(x, y) \in \Gamma_1 \Longrightarrow (x, \lambda y) \in \Gamma_2$

The following three statements are well known, see for example [1] Chapter V.

<u>Proposition 11.</u> Let (P, ξ) means either $(P(A), \beta)$ or $(P(B), \alpha)$.

Then

$$X \subset Y \Longrightarrow \xi X \supset \xi Y$$
$$\xi(X \cup Y) = \xi X \cap \xi Y$$
$$\xi(X \cap Y) \supset \xi X \cup \xi Y$$

for all $X, Y \in P$.

<u>Proposition 12.</u> The maps $\gamma = \beta\alpha$ and $\delta = \alpha\beta$ are closure operators on $P(B)$ and $P(A)$ respectively; this means that both γ and δ are extensive, idempotent and isotone 1/.

1/ A map $\xi : P(A) \longrightarrow P(A)$ is

a/ extensive iff $X \subset \xi X$ for all $X \in P(A)$

b/ idempotent iff $\xi(\xi X) = X$ for all $X \in P(A)$,

c/ isotone iff $X \subset Y \Longrightarrow \xi X \subset \xi Y$ for all $X, Y \in P(A)$

Moreover it is $\alpha = \alpha\gamma = \delta\alpha$, $\beta = \beta\delta = \gamma\beta$ and $X \in P(A)$ or $Y \in P(B)$ is closed if and only if $X \in \mathrm{Im}\,\alpha = \mathrm{Im}\,\delta$ or $Y \in \mathrm{Im}\,\beta = \mathrm{Im}\,\delta$.

Corollary. Let (P, ξ) means either $(P)A), \delta)$ or $(P(B), \gamma)$.

Then

$$\xi(X \cup Y) = \xi(\xi X \cup \xi Y)$$
$$X \subset \xi Y \Longrightarrow \xi(X \cup Y) = \xi Y$$

for all $X, Y \in P$.

Proposition 13. There are complete lattices $(\mathrm{Im}\,\alpha, \subset)$ and $(\mathrm{Im}\,\beta, \subset)$ in which great lower bound means intersection. The Galois connection (α, β) gives a dual isomorphism between the lattices $(\mathrm{Im}\,\alpha, \subset)$ and $(\mathrm{Im}\,\beta, \subset)$.

Proposition 14. Let $\lambda = (\Lambda, \lambda): \Gamma_1 \longrightarrow \Gamma_2$, be a morphism in the Galois category GC, $\Gamma_i = (A_i, B_i, \Gamma_i, \alpha_i, \beta_i)$, $i = 1, 2$.

(2) $\quad (\Lambda\, \alpha_1)(Y) \subset (\alpha_2\, \lambda)(Y)$ for all $Y \in P(B_1)$

and

(3) $\quad (\lambda\, \beta_1)(X) \subset (\beta_2\, \Lambda)(X)$ for all $X \in P(A_1)$

Moreover the equation in (1) implies equations in (2) and (3), as well as

(4) $\quad (\lambda\, \gamma_1)(Y) = (\gamma_2\, \lambda)(Y)$ for all $Y \in P(B_1)$

and

(5) $\quad (\Lambda\, \delta_1)(X) = (\delta_2\, \Lambda)(X)$ for all $X \in P(A_1)$

whereas $\gamma_i = \beta_i \alpha_i$, $\delta_i = \alpha_i\, \beta_i$ for $i = 1, 2$.

Proof. (1) $\Longrightarrow$ (2). For $(\Lambda\, \alpha_1)(Y) = \emptyset$ there is nothing to be proved. Assume $(\Lambda\, \alpha_1)(Y) \neq \emptyset$ and choose $v \in (\Lambda\, \alpha_1)(Y)$. Then there exists $u \in \alpha_1(Y)$ such that $v = \Lambda(u)$. Since $(u, y) \in \Gamma_1$ for each $y \in Y$, it is $(v, \lambda y) = (\Lambda u, \lambda y) \in (\Lambda \times \lambda)(\Gamma_1) \subset \Gamma_2$ and therefore $v \in \alpha_2(\lambda Y) = (\alpha_2\, \lambda)(Y)$. Similarly (or using the duality functor $D : GC \longrightarrow GC$ we can now show (1) $\Longrightarrow$ (3). The proof of the first assertion is finished.

Suppose now $(\Lambda \times \lambda)(\Gamma_1) = \Gamma_2$ and pick a point $v \in (\beta_2 \Lambda)(X)$; since $z \in \Lambda(X)$ implies $(z, v) \in \Gamma_2 = (\Lambda \times \lambda)(\Gamma_1)$ there exists $u \in \lambda^{-1}(v)$. It is $x \in X$ $\Longrightarrow (\Lambda x, \lambda u) \in \Gamma_2 \Longrightarrow (x, u) \in \Gamma_1 \Longrightarrow u \in \beta_1(X) \Longrightarrow v = \lambda u \in (\lambda \beta_1)(X)$. Therefore the equation in (1) yields the equation in (3) and, similarly, in (2) . The relations (4) and (5) are consequences of the facts just proved. For example

$$\lambda \gamma_1 = \lambda \beta_1 \alpha_1 = \beta_2 \Lambda \alpha_1 = \beta_2 \alpha_2 \Lambda = \gamma_2 \lambda$$

<u>Corollary.</u> If both Λ and λ are bijective and $(\Lambda \times \lambda)(\Gamma_1) = \Gamma_2$ then

$$\alpha_2 = \Lambda \alpha_1 \lambda^{-1}, \quad \gamma_2 = \lambda \gamma_1 \lambda^{-1}$$
$$\beta_2 = \lambda \beta_1 \Lambda^{-1}, \quad \delta_2 = \Lambda \delta_1 \Lambda^{-1}$$

<u>Remark.</u> In general nothing can be proved about the inclusion relations between $(\lambda \gamma_1)(Y)$ and $(\gamma_2 \lambda)(Y)$ or between $(\lambda \delta_1)(X)$ and $(\delta_2 \Lambda)(X)$.

8. Stabilizer

Previous Section will be applied to k-spaces. The most interesting object which we shall regard, is the closure operation:

$$\varkappa : P(M) \longrightarrow P(M)$$

described as follows. If A is a subset of an underlying set M then $\varkappa(A) \subset M$ is the set of all points $x \in M$ which are invariant under each of the transformations $g \in G$ with the property: $g(a) = a$ for all $a \in A$.

<u>Definition 7.</u> Let $M = (G, M)$ be a k-spece. The set

$$SM = \{(f, x) \in G \times M \mid f(x) = x\}$$

is called the stabilizer of a k-space M.

Nonempty sets G, M and SM, together with the maps

$$\varphi_M: P(M) \longrightarrow P(G),\ Y \longmapsto \{x \in G \mid (x, y) \in SM \text{ for all } y \in Y\}$$

$$\psi_M: P(G) \longrightarrow P(M),\ X \longmapsto \{y \in M \mid (x, y) \in SM \text{ for all } x \in X\}$$

built up an object of the Galois connection category GC. Index M in the symbols φ_M and ψ_M will be frequently omitted.

<u>Proposition 15.</u> There is a covariant functor $\Gamma: K \longrightarrow GC$ defined by

$$M \longrightarrow \Gamma M = (G, M, SM, \varphi_M, \psi_M) \qquad \text{for objects}$$

and

$$\boldsymbol{\omega} \longrightarrow \Gamma \boldsymbol{\omega} = (\Omega, \omega) \qquad \text{for morphisms.}$$

<u>Proof.</u> Given a k-morphism $\boldsymbol{\omega} = (\Omega, \omega): M_1 \longrightarrow M_2$, $M_i = (G_i, M_i)$, $i = 1, 2$ we have to show $(\Omega \times \omega)(SM_1) \subset SM_2$. In fact if $(f, x) \in SM_1$, i.e. $f(x) = x$ then $\Omega_f(\omega x) = \omega x$, hence $(\Omega \times \omega)(f, x) = (\Omega_f, \omega x) \in SM_2$.

<u>Remark.</u> Another way of introducing the Galois connection structure into a k-space M can be establish by the functor $M \longrightarrow (G, P(M), S'(M, \ldots)$ where $S'M \subset G \times P(M)$ consists of exactly those (f, X) for which $f(X) = X$.

<u>Proposition 16.</u> Let $M = (G, M)$ be a k-space and $t \in T(M)$ a transformation. The k-isomorphism

$$\boldsymbol{\sigma}_t = (\Sigma_t, \sigma_t): M \longrightarrow Sim_t M = (tGt^{-1}, M)$$

induces the GC - isomorphism

$$\Gamma_{\sigma_t}: \Gamma M = (G, M, SM, \varphi, \psi) \longrightarrow \Gamma(Sim_t M) =$$

$$= (tGt^{-1}, M, S(Sim_t M), \varphi', \psi')$$

where

$$(\Sigma_t \times \sigma_t): SM \longrightarrow S(Sim_t M),\ (g, x) \longmapsto (tgt^{-1}, tx),$$

$$\varphi' = \Sigma_t\, \varphi\, \sigma_t^{-1} \quad \text{and} \quad \psi' = \sigma_t\, \psi\, \Sigma_t^{-1}.$$

If $t \in G$ then $\varphi = \varphi'$ and $\psi = \psi'$.

P r o o f follows directly from Propositions 2, 14, 15 and Corollary of Proposition 14.

C o r o l l a r y . By the above notation, completed by

$$\varkappa = \psi\varphi \qquad \varkappa' = \psi'\varphi'$$
and
$$\mu = \varphi\psi \qquad \mu' = \varphi'\psi'$$

it holds

$$t(\varphi X)t^{-1} = \varphi'(tX), \quad t(\varkappa X) = \varkappa'(tX) \quad \text{for all } X \in P(M)$$
$$t(\psi X) = \psi'(tXt^{-1}), \quad t(\mu X)t^{-1} = \mu'(tXt^{-1}) \text{ for all } X \in P(G).$$

If, in addition, $t \in G$ then

$$t(\varphi X) = (\varphi(tX))t, \quad t(\varkappa X) = \varkappa(tX) \quad \text{for all } X \in P(M)$$
$$t(\psi X) = \psi(tXt^{-1}), \quad t(\mu X)t^{-1} = \mu(tXt^{-1}) \text{ for all } X \in P(G).$$

Regarding the classical geometries (Euclidean, Affine, Projextive, Vectorspaces,...) seven concepts of (in) dependency have been introduced and studyied in [3] .

R E F E R E N C E S

[1] BIRKHOFF G.: Lattice theory. New York 1948.

[2] COHN P.M.: Universal Algebra. Harper Row, New York 1965.

[3] HEJNÝ M.: Dependency in Klein spaces, Mathematica Slovaca (to appear).

[4] KLEIN F.: Vergleichende Betrachtungen ubeŕ neuere Geometrische Forshungen. Erlangen 19872.

[5] MITCHELL: Theory of Categories. New York 1965.

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S ú h r n

KATEGÓRIA KLEINOVÝCH PRIESTOROV

M. HEJNÝ, BRATISLAVA

Z hľadiska teórie kategórií analyzujú sa pojmy "nezávislosti" v kleinových priestoroch; tieto boli definované v práci [3] .

Р Е З Ю М Е

КАТЕГОРИЯ ПРОСТРАНСТВ КЛЕЙНА

МИЛАН ГЕЙНЫ, БРАТИСЛАВА

С точки зрения теории категории обсуждается подход к понятиям независимости в пространствах Клейна, определенным в статье [3] .

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QUALITATIVE EIGENSCHAFTEN EINES TYPS VON RANDPROBLEMEN I.

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In der Arbeit werden die Eigenschaften der Operatoren des Systems von quasilinearen partiellen Differentialgleichungen des Kármánschen Typs untersucht. Mit diesem System können schlanke Wände oder flache Schalen, sowie auch Fälle von anfänglichen Imperfektionen modeliert werden. Die Belastungen können quer oder der Länge nach gehen und die erwogenen Randbedingungen können auch nicht homogen sein. Angewandt wird derselbe mathematische Apparat wie auch in [1].

1. Grundlegende Begriffe

Ω sei ein begrenztes Gebiet im zweidimensionalen Euklidischen Raum und dessen Grenze bezeichnen wir mit $\partial\Omega$. In Ω erwägen wir das System der Gleichungen

$$(1)\qquad \begin{aligned} \frac{D}{t}\,\Delta\Delta w &= \lambda\,[F_0, w] + [k, F] + [F, w] + q_1 \\ -\frac{1}{E}\,\Delta\Delta F &= \frac{1}{2}\,[w, w] + [k, w] - q_2 \end{aligned}$$

Im System (1) sind D, t, E Konstanten. $F_0 = F_0(x, y)$, $k = k(x, y)$, $q_1 = q_1(x, y)$, $q_2 = q_2(x, y)$ sind genügend reguläre Funktionen $F_0 = F_0(x, y) \neq 0$. Weiter bezeichnen wir

$$[A, B] = \frac{\partial^2 A}{\partial x^2}\frac{\partial^2 B}{\partial y^2} + \frac{\partial^2 A}{\partial y^2}\frac{\partial^2 B}{\partial x^2} - 2\,\frac{\partial^2 A}{\partial x \partial y}\,\frac{\partial^2 B}{\partial x \partial y}$$

Δ ist ein Laplacescher Operator. Alle Funktionen in (1) setzen wir als real voraus und λ ist ein realer Parameter.

Die Eigenschaften des Systems (1) werden wir im Raum $W = H_1 \times H_2$ untersuchen, wo wir die Räume H_1, H_2 (Räume der Funktionen w, F) in Abhängigkeit zu den Randwertproblemen wählen.

1. Im Falle der Grenze $\partial\Omega$ mit stetigsichverändernder Normale setzen wir voraus, dass $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, dass $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$ und dass mes $\partial\Omega_1 > 0$. In diesem Falle nehmen wir die Randbedingungen der Form

$$
(2)\quad
\begin{aligned}
& w\Big|_{\partial\Omega_1} = \frac{\partial w}{\partial n}\Big|_{\partial\Omega_1} = F\Big|_{\partial\Omega} = \frac{F}{\partial n}\Big|_{\partial\Omega} = 0 \\
& w\Big|_{\partial\Omega_2} = \mu \Delta w + (1-\mu)\left(n_x^2 \frac{\partial^2 w}{\partial x^2} + 2n_x n_y \frac{\partial^2 w}{\partial x\,\partial y} + \right. \\
& \qquad \left. + n_y^2 \frac{\partial^2 w}{\partial y^2}\right)\Big|_{\partial\Omega_2} = 0
\end{aligned}
$$

In (2) ist $n = (n_x, n_y)$ die äussere Normale zu Ω und $0 \leq \mu < 1$ ist eine Konstante. Bezeichnen wir

$$
V(\Omega) = \left\{ u : w \in W_2^{(2)}(\Omega),\ u|_{\partial\Omega} = \frac{\partial u}{\partial n}\Big|_{\partial\Omega_1} = 0 \right\}
$$

Weiter erwägen wir fur $u, v \in W_2^{(2)}(\Omega)$ über die bilineare Form

$$
A(u, v) = \int_\Omega \Big[\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + 2(1-\mu)\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} +
$$
$$
+ \mu\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}\right)\Big]\, d\Omega
$$

<u>H i l f s s a t z 1.</u> Für $u \in V(\Omega)$ gilt

$$
(3)\quad C_1 \|u\|^2_{W_2^{(2)}} \leq A(u, u) \leq C_2 \|u\|^2_{W_2^{(2)}}
$$

wo C_1, C_2 von u unabhängige positive Konstanten sind.

<u>B e w e i s .</u> Fur $0 \leq \mu < 1$ und die beliebigen Zahlen a, b

gilt die Ungleichheit

$$a^2 + 2\mu ab + b^2 \geqq (1 - \mu)\ (a^2 + b^2)$$

Wenn wir

$$a = \frac{\partial^2 u}{\partial x^2},\quad b = \frac{\partial^2 u}{\partial y^2}$$

bezeichnen, dann haben wir

$$A(u, u) \geqq (1 - \mu) \int_{\Omega} \left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 \right] d\Omega ,$$

woraus wir durch Verwendung der Ungleichheit von Poincaré und Friedrichs die erste Ungleichheit in (3) erhalten.

Die zweite Ungleichheit folgt direkt aus der Verwendung der Ungleichheit von Buňakovskij.

Durch einfache Berechnungen und durch die Verwendung des Hilfssatzes 1 erhalten wir, dass

(4) $$(u, v)_V = A(u, v)$$

alle Eigenschaften eines Skalarproduktes hat.

Vervollständigen wir $V(\Omega)$ in der generierten Norm mit dem Skalarprodukt (4) und den gewonnenen vollständigen Raum bezeichnen wir wieder mit $V(\Omega)$. Aus der Konstruktion des Raumes ist ersichtlich, dass

$$\overset{o}{W}{}_2^{(2)}(\Omega) \subset V(\Omega) \subset W_2^{(2)}(\Omega)$$

Mit vollkommen denselben Erwägungen und Verfahren wie in [1] können wir zeigen, dass jedem $w \in V(\Omega)$ gerade eine schwache Lösung $F(w) \in \overset{o}{W}{}_2^{(2)}(\Omega)$ der zweiten Gleichung von (1) entspricht und es gilt die Abschätzung

(5) $$\|F\|_{W_2^{(2)}} \leqq C_3 (\|w\|_{W_2^{(2)}} + C_4)\ \|w\|_{W_2^{(2)}} + C_5$$

wo C_3, C_4, C_5 von w unabhängige positive Konstanten sind.

2. $\partial\Omega$ sei ein rechtwinkeliges Viereck. Nehmen wir die

Randbedingungen der Form

$$w|_{\partial\Omega} = \frac{\partial^2 w}{\partial x^2}\Big|_{\partial\Omega} = \frac{\partial^2 w}{\partial y^2}\Big|_{\partial\Omega} = 0$$

(6)

$$F|_{\partial\Omega} = \frac{\partial^2 F}{\partial x^2}\Big|_{\partial\Omega} = \frac{\partial^2 F}{\partial y^2}\Big|_{\partial\Omega} = 0$$

Den Raum konstruieren wir in diesem Falle genau so wie in [1]

Das System (1) untersuchen wir bei beliebiger Kombination der Randbedingungen (2), (6). Die entsprechenden Räume für w und F wählen wir nach den Randbedingungen und den erwähnten Konstruktionen. Die entsprächenden Räume werden wir mit H_1, H_2 bezeichnen.

Bezeichnen wir

$$W(\Omega) = H_1 \times H_2$$

Wenn $h_i \in W(\Omega)$ sind

$$h_i = \langle w_i, F_i \rangle \quad , \quad w_i \in H_1, \ F_i \in H_2, \ i = 1, 2$$

ist $W(\Omega)$ ein Hilbert Raum mit skalaren Produkt

$$(h_1, h_2)_W = (w_1, w_2)_{H_1} + (F_1, F_2)_{H_2}$$

2. Schwache Lösung des Systems (1)

<u>Definition 1.</u> Das Paar $\langle w, F \rangle \in W(\Omega)$ werden wir eine schwache Lösung der Aufgaben (1), (2) bezw. (6) nennen, wenn folgendes gilt:

a) $\forall \varphi \in H_1$: $\frac{D}{t} A(w, \varphi) = \lambda L(F, w, \varphi) + L(F, w, \varphi) +$

$$+ L(k, F, \varphi) + \int_\Omega q_1(x, y)\, \varphi(x, y)\, d\Omega$$

b) $\forall \gamma \in H_2$: $\frac{1}{E} (F, \gamma)_{H_2} = - \frac{1}{2} L(w, w, \gamma) - L(k, w, \gamma) +$

$$+ \int_\Omega q_2(x, y)\, \gamma(x, y)\, d\Omega$$

wo

$$L(\varphi,\psi,\xi) = \int_{\Omega} \Big[\Big(\frac{\partial^2\varphi}{\partial x\,\partial y}\,\frac{\partial\psi}{\partial y} - \frac{\partial^2\varphi}{\partial y^2}\,\frac{\partial\psi}{\partial x}\Big)\frac{\partial\xi}{\partial x} + \Big(\frac{\partial^2\varphi}{\partial x\,\partial y}\,\frac{\partial\psi}{\partial x} - \frac{\partial^2\varphi}{\partial x^2}\,\frac{\partial\psi}{\partial y}\Big)\frac{\partial\xi}{\partial y}\Big]\, d\Omega$$

Aus der Definition 1 ist ersichtlich, dass wenn das Paar $\langle w, F\rangle \in W(\Omega)$ eine schwache Lösung der Aufgaben (1), (2) bezw. (6) ist, dann ist für alle Paare $\langle\varphi,\psi\rangle \in W(\Omega)$:

$$\begin{aligned}
G(\langle w, F\rangle, \langle\varphi,\psi\rangle) &= \frac{D}{t}A(w,\varphi) - \frac{1}{E}(F,\psi)_{H_2} - \\
&- \lambda\int_{\Omega}\Big[\Big(\frac{\partial^2 F_0}{\partial x\,\partial y}\,\frac{\partial w}{\partial y} - \frac{\partial^2 F_0}{\partial y^2}\,\frac{\partial w}{\partial x}\Big)\frac{\partial\varphi}{\partial x} + \Big(\frac{\partial^2 F_0}{\partial x\,\partial y}\,\frac{\partial w}{\partial x} - \\
&- \frac{\partial^2 F_0}{\partial x^2}\,\frac{\partial w}{\partial y}\Big)\frac{\partial\varphi}{\partial y}\Big]\, d\Omega - \int_{\Omega}\Big[\Big(\frac{\partial^2 k}{\partial x\,\partial y}\,\frac{\partial F}{\partial y} - \frac{\partial^2 k}{\partial y^2}\,\frac{\partial F}{\partial x}\Big)\frac{\partial\varphi}{\partial x} + \\
&+ \Big(\frac{\partial^2 k}{\partial x\,\partial y}\,\frac{\partial F}{\partial x} - \frac{\partial^2 k}{\partial x^2}\,\frac{\partial F}{\partial y}\Big)\frac{\partial\varphi}{\partial y}\Big]\, d\Omega - \int_{\Omega}\Big[\Big(\frac{\partial^2 F}{\partial x\,\partial y}\,\frac{\partial w}{\partial y} - \frac{\partial^2 F}{\partial y^2}\,\frac{\partial w}{\partial x}\Big)\frac{\partial\varphi}{\partial x} + \\
&+ \Big(\frac{\partial^2 F}{\partial x\,\partial y}\,\frac{\partial w}{\partial x} - \frac{\partial^2 F}{\partial x^2}\,\frac{\partial w}{\partial y}\Big)\frac{\partial\varphi}{\partial y}\Big]\, d\Omega + \int_{\Omega}\Big(\frac{\partial w}{\partial x}\,\frac{\partial^2 w}{\partial y^2}\,\frac{\partial\psi}{\partial x} - \\
&- \frac{\partial w}{\partial x}\,\frac{\partial^2 w}{\partial x\,\partial y}\,\frac{\partial\psi}{\partial y}\Big)\, d\Omega - \int_{\Omega}\Big[\Big(\frac{\partial^2 k}{\partial x\,\partial y}\,\frac{\partial w}{\partial y} - \frac{\partial^2 k}{\partial y^2}\,\frac{\partial w}{\partial x}\Big)\frac{\partial\psi}{\partial x} + \\
&+ \Big(\frac{\partial^2 k}{\partial x\,\partial y}\,\frac{\partial w}{\partial x} - \frac{\partial^2 k}{\partial x^2}\,\frac{\partial w}{\partial y}\Big)\frac{\partial\psi}{\partial y}\Big]\, d\Omega - \int_{\Omega} q_1(x,y)\,\varphi(x,y)\, d\Omega + \\
&+ \int_{\Omega} q_2(x,y)\,\psi(x,y)\, d\Omega = 0
\end{aligned}$$

Nehmen wir die bilineare Form

$$(7) \quad G_1(\langle w, F\rangle, \langle\varphi,\psi\rangle) = \frac{D}{t}A(w,\varphi) - \frac{1}{E}(F,\psi)_{H_2}$$

Für jedes feste $\langle w, F\rangle \in W(\Omega)$ ist die Form G_1 ein lineares Funktional auf $W(\Omega)$. Aus der Verwendung der Hölderischen Ungleichheit folgt, dass dieses Funktional begrenzt ist. Aus dem Representationssatz von Riesz folgt dann, dass ein einziges solches Element aus $W(\Omega)$ (bezeichnen wir es mit $G_1 \langle w , F\rangle$) existiert, dass

(8) $$G_1(\langle w , F\rangle , \langle \varphi , \psi\rangle) = (G_1\langle w , F\rangle , \langle \varphi , \psi\rangle) ,$$
$$\forall \langle \varphi , \psi \rangle \in W(\Omega)$$

Damit erhielten wir einen linearen Operator G_1 welcher in $W(\Omega)$ operiert (wir schreiben $G_1 : W(\Omega) \rightarrow W(\Omega)$).

Weiter bezeichnen wir

(9) $$G_2(\langle w , F\rangle , \langle \varphi , \psi\rangle) = G(\langle w , F\rangle , \langle \varphi , \psi\rangle) -$$
$$- G_1(\langle w , F\rangle , \langle \varphi , \psi\rangle) + \int_\Omega q_1 \varphi \, d\Omega - \int_\Omega q_2 \psi \, d\Omega$$

Alle Ableitungen der ersten Ordnung aus $G_2(\langle w , F\rangle , \langle\varphi,\psi\rangle)$ sind nach den Einbettungssätzen von Sobolev aus $L_4(\Omega)$. Wenn wir also für jeden aus (9) die Hölderscher Ungleichheit verwenden, erhalten wir durch Benutzung der Sobolevschen Einbettungssätze, dass

$$|G_2(\langle w , F\rangle , \langle \varphi , \psi\rangle)| \leqq C_6 \, \|\langle w , F\rangle\|_W^2 \cdot \|\langle\varphi,\psi\rangle\|_W$$

ist, wo C_6 eine positive Konstante ist. Also ist bei festem $\langle w, F\rangle \in W(\Omega)$ die Form $G_2(\langle w , F\rangle , \langle\varphi , \psi\rangle)$ ein linear begrenztes Funktional auf $W(\Omega)$. Gemäss des Sätzes von Riesz existiert ein einziges Element $G_2\langle w , F\rangle \in W(\Omega)$ derart, dass

(10) $$\forall \langle\varphi, \psi\rangle \in W(\Omega) :$$
$$G_2(\langle w , F\rangle , \langle \varphi, \psi\rangle) = (G_2 \langle w, F\rangle , \langle\varphi, \psi\rangle)_W$$

Damit **erhielten** wir einen nichtlinearen Operator $G_2 : W(\Omega) \longrightarrow W(\Omega)$.

Aus den Sobolevschen Sätzen über die Einbettung

$$\left|\int_\Omega q_1 \varphi \, d\Omega\right| = C_7 \|\varphi\|_{H_1}$$

Also existiert ein einziges Element $q_1 \in H_1$ derart, dass

$$\forall \quad \varphi \in H_1 : \quad \int_\Omega q_1 \varphi \, d\Omega = (q_1, \varphi)_{H_1}$$

Was das selbe ist wie

$$\forall \quad \langle \varphi, \psi \rangle \in W(\Omega): \int_\Omega q_1 \varphi \, d\Omega = (\langle q_1, \theta_{H_2} \rangle, \langle \varphi, \psi \rangle)_W$$

Ähnlich für q_2,

$$\forall \quad \langle \varphi, \psi \rangle \in W(\Omega): \int_\Omega q_2 \psi \, d\Omega = (\theta_{H_1}, q_2 \rangle, \langle \varphi, \psi \rangle)_W$$

Aus den vorhergegangenen Ervägungen folgt die Gültigkeit von

S a z t 1. Mit der bilinearen Form $G(\langle w, F \rangle, \langle \varphi, \psi \rangle)$ wurde der Operator

$$(11) \quad G\langle w, F \rangle = G_1 \langle w, F \rangle + G_2 \langle w, F \rangle - \langle q_1, \theta_{H_2} \rangle + $$
$$+ \langle \theta_{H_1}, q_2 \rangle$$

definiert, welcher in $W(\Omega)$ operiert.

D e f i n i t i o n 2. Es sei Φ ein Funktional, $\Phi : W(\Omega) \longrightarrow R^1$. Es existiere das Gâteauxsches Differential $D\Phi(\langle w, F \rangle, \langle \varphi, \psi \rangle)$ welches linear und stetig in Hinsicht auf $\langle \varphi, \psi \rangle$ ist. Dann bezeichnen wir mit $\operatorname{grad} \Phi \langle w, F \rangle$ ein derartiges Element von $W(\Omega)$, fur welches

$$(12) \quad \forall \langle \varphi, \psi \rangle \in W(\Omega): D\Phi(\langle w, F \rangle, \langle \varphi, \psi \rangle) =$$
$$= (\operatorname{grad} \Phi \langle w, F \rangle, \langle \varphi, \psi \rangle)_W$$

gilt.

Der Operator $\operatorname{grad} \Phi \langle w, F \rangle$ aus (12) wird potentiell genannt und $\Phi \langle w, F \rangle$ ist seine Potential.

S a t z 2. Der Operator $G \langle w, F \rangle$ ist potential auf $W(\Omega)$.

B e w e i s . Bei Verwendung der Ervägungen aus [1] (Hilfssatz 1, Hilfssatz 2, Beweis des Satzes 1) genügt es die bilineare Form $A(w, \varphi)$ zu untersuchen. Aber

$DA(w, \bar{w}, \tilde{w}) = A(\bar{w}, \tilde{w}); \quad w, \bar{w}, \tilde{w} \in H_1,$

womit der Beweis des Satzes beendet ist.

Das entsprechende Funktional wir dann

$$(13) \quad \Phi\langle w, F\rangle = \frac{D}{2t} A(w, w) - \frac{1}{2E} \|F\|^2_{H_2} - \frac{\lambda}{2} ([F_0, w], w)_{L_2} - \frac{1}{2}([F, w], w)_{L_2} - ([k, F], w)_{L_2} - \int_\Omega q_1 w d\Omega + \int_\Omega q_2 F d\Omega$$

sein.

3. Eigenschaften des Operators und des Funktionals der schwachen Lösung

S a t z 3. Das Funktional $\Phi\langle w, F\rangle$ ist auf $W(\Omega)$ schwach halbstätig von unten.

B e w e i s . Durch Verwendung des Hilfssatzes 3 aus [1] ist der Beweis, genau derselbe wieder Beweis des Satzes 2 aus [1] .

D e f i n i t i o n 3. Der nichtlineare Operator wird begrenzt genannt, wenn er die begrenzte Menge auf die begrenzte Menge abbildet.

S a t z 4. Der Operator $G_1\langle w, F\rangle + G_2\langle w, F\rangle$ aus (11) ist begrenzt und stetig auf $W(\Omega)$.

B e w e i s . Aus dem schon Bewiesenen haben wir, dass

$$|(G_2\langle w, F\rangle, \langle\varphi, \psi\rangle)_W| \leq C_6 \|\langle w, F\rangle\|^2_W \cdot \|\langle\varphi, \psi\rangle\|_W$$

für ein beliebiges $\langle\varphi, \psi\rangle \in W(\Omega)$ ist. Nehmen wir

$$\langle\varphi, \psi\rangle = G_2\langle w, F\rangle.$$

Dann ist

$$\|G_2 \quad w, F \quad \|_W^2 \leqq c_6 \| \langle w, F \rangle \|_W^2 \; \|G_2 \langle w, F \rangle \|_W$$

woraus die Begrenztheit und die Stetigkeit des Operators G_2 ersichtlich ist. Für den Operator G_1 ist der Beweis analog.

S a t z 5. Der Operator G_2 ist auf $W(\Omega)$ vollstetig.

B e w e i s . Es sei $\{h_n\}_1^\infty \subset W(\Omega)$ und es sei $h_n \xrightarrow[n\to\infty]{} h_0$, wo $h_n = \langle w_n, F_n \rangle$ ist. Infolge der Eigenschaft der schwachen Konvergenz, existiert $M > 0$ derart, dass

$$\underset{n}{\forall} \; \|h_n\| \leqq M$$

$W(\Omega)$ ist ein Hilbertscher Raum, wo gemäss Satz 4 $\{G_2 h_n\}_1^\infty$ schwach zu irgendeinen Element $Z = \langle Z^{(1)}, Z^{(2)} \rangle$ konvergiert. Es sei also die Konvergenz der Folge $\{G_2 h_n\}_1^\infty$ nich stark. Dann würde gemäss des Satzes von Hahn-Banach eine solche Folge

$$\{v_n\}_1^\infty \subset W(\Omega), \; (v_n = \langle v_n^{(1)}, v_n^{(2)} \rangle \;)$$

existieren, dass

$$\|v_n\|_W = 1, \quad \langle G_2 h_n - Z, \; v_n \rangle = \|G_2 h_n - Z\|_W \geqq \gamma > 0.$$

Dabei folgt aus den Einbettungssätzen, dass $\{v_n\}_1^\infty$ zu irgendeinen Element v_0 in der Norm des Raumes $C(\bar{\Omega})$ konvergiert. Weiter gilt

(14) $(G_2 h_n - Z, \; v_n)_W = (G_2 h_n - Z, \; v_n - v_0)_W + (G_2 h_n - Z, \; v_0)_W$

Das zweite Glied auf der rechten Seite von (14) konvergiert für $n \to \infty$ zu Null. Das erste Glied können wir durch integrieren per partes, durch Verwendung der Randbedingungen und durch die Stetigkeit des Skalarproduktes überführen in die Form

(15) $$(-\tfrac{1}{2}[w_n, w_n] - [k, w_n] - Z^{(2)}, \; v_n^{(2)} - v_0^{(2)})_{L_2} -$$
$$-(\lambda[F_0, w_n] + [k, F_n] + [F_n, w_n] - Z^{(1)}, v_n^{(1)} - v_0^{(1)})_{L_2}$$

Die Ausdrücke in (15) sind Funktionale auf $W(\Omega)$. Es ist bekannt, dass eine starke Konvergenz in $C(\bar{\Omega})$ eine starke Konvergenz in $L_2(\Omega)$ impliziert. Durch Verwendung der bekannten Eigenschaften schwacher und starker Konvergenz in den Hilbertschen Räumen erhalten wir also, dass (15) für $n \to \infty$ zu Null konvergiert. Damit ist der Beweis beendet.

Es seien $F_1(x, y)$, $F_2(x, y)$ solche Funktionen, dass bei den Randbedingungen für F aus (2), oder (6) für $w \in H_1$ in schwachen Sinne

$$\frac{1}{E} \Delta\Delta F_1 = -\frac{1}{2}[w, w], \quad \frac{1}{E} \Delta\Delta F_2 = -[k, w] + q_2$$

gilt. Es ist leicht zu beweisen, dass F_1, F_2 existieren und eindeutig gegeben sind. Mit Hilfe dieser Funktionen können wir das Funktional (13) in die Form

$$\begin{aligned}(16) \qquad \Phi(w) = {} & \frac{D}{2t} A(w,w) - \frac{1}{2E} \| F_1 \|^2_{H_2} - \frac{1}{2E} \| F_2 \|^2_{H_2} + \\ & + \frac{1}{4E} ([w,w], F_2)_{L_2} + \frac{1}{2E} ([k,w], F_1)_{L_2} - \\ & - \frac{1}{2} ([F_1,w], w)_{L_2} - \frac{1}{2} ([F_2,w], w)_{L_2} - \\ & - \frac{\lambda}{2} ([F_0,w], w)_{L_2} - ([k,w], F_1 + F_2)_{L_2} - \\ & - \int_\Omega q_1 w \, d\Omega + \int_\Omega q_2 F_2 \, d\Omega\end{aligned}$$

schreiben.

<u>S a t z 6.</u> Das Funktional (16) ist auf H_1 wachsend, d.h. es gilt

$$\lim_{\|w\|_{H_1} \to \infty} \Phi(w) = \infty$$

<u>B e w e i s .</u> Ersichtlich gilt

$$A(w,w) \geq C_8 (\Delta w)^2_{L_2}$$

wo C_8 eine positive, von w unabhängige Konstante ist. Weiter verläuft der Beweis ähnlich, wie der Beweis des Satzes 9 aus [1] .

4. Die Existenz der Lösung

H i l f s s a t z 2. Das Funktional $\Phi(w)$ sei schwach halbstetig von unten und wachsend auf H_1. Dann ist

$$\inf_{w \in H_1} \Phi(w) > -\infty$$

und es existiert ein solches $w_0 \in H_1$, dass

$$\Phi(w_0) = \inf_{w \in H_1} \Phi(w)$$

B e w e i s diese Hilfssatzes ist in [2] .

D e f i n i t i o n 4. Es sei M eine offene Untermenge von H_1. Der Punkt $w_0 \in M$ wird kritischer Punkt des Funktionals heissen, wenn

$$\forall \varphi \in H_1 : D\Phi(w_0, \varphi) = \theta_{H_1}$$

H i l f s s a t z 3. M sei aus der Definition 4. Es sei Φ ein solches Funktional, dass für jedes $w \in M$ ein grad $\Phi(w)$ existiert. Dann ist jeder extremale Punkt des Funktionals Φ auch dessen kritischer Punkt.

B e w e i s dieses Hilfssatzes ist in [2] .

S a t z 7. Es seien $q_1, q_2 \in L_1(\Omega)$; $F_0, k \in W_2^{(2)}(\Omega)$. In H_1 existiert wenigstens eine schwache Lösung der Aufgabe (1), (2) bezw. (6) im Sinne der Definition 1.

B e w e i s . Aus dem Satz 3, dem Satz 6 und dem Hilfssatz 2 folgt, dass das Funktional (16) in H_1 einen extremalen Punkt besitz (genauer einen Punkt des Minimums). Aus dem Beweis der Potentialität des Operators G (Satz 2) geht die Existenz von grad Φ hervor. Also folgt aus dem Hilfssatz 4, dass im Punkt des Minimums

$$\forall \varphi \in H_1 : (\operatorname{grad} \Phi(w_0), \varphi)_{H_1} = 0$$

ist und das Paar $\langle w_0, F = F_1(w_0) + F_2(w_0)\rangle$ ist eine schwache Lösung der Aufgabe (1), (2) bezw. (6).

5. Die Verwendbarkeit der Annäherungsmethoden

Aus der Definition 1 ist ebenfalls ersichtlich, dass wenn das Paar $\langle w, F\rangle \in W(\Omega)$ eine schwache Lösung der Aufgabe (1), (2) bezw. (6) ist, dann gilt für alle Paare $\langle \varphi, \psi\rangle \in W(\Omega)$

$$T(\langle w, F\rangle, \langle \varphi, \psi\rangle) = \frac{D}{t} A(w, \varphi) + \frac{2}{E}(F, \psi)_{H_2} -$$

$$- \lambda L(F_0, w, \varphi) - L(F, w, \varphi) - L(k, F, \varphi) + L(w, w, \psi) +$$

$$+ 2L(k, w, \psi) - \int_\Omega q_1 \varphi \, d\Omega - \int_\Omega q_2 \psi \, d\Omega \equiv 0.$$

Auf dieselbe Art wie wir den Operator in Satz 1 konstruierten, erhielten wir den Operator

$$T\langle w, F\rangle = T_1\langle w, F\rangle + T_2\langle w, F\rangle - \langle q_1, \theta_{H_2}\rangle - \tag{17}$$

$$- \langle \theta_{H_1}, q_2\rangle.$$

Der Operator T operiert von $W(\Omega)$ bis $W(\Omega)$, ist auf $W(\Omega)$ begrenzt und stetig und der Operator T_2 is vollstetig. Die Beweise dieser Behauptungen sind genau dieselben, wie die Beweise der entsprechenden Behauptungen für den Operator G.

<u>Satz 8.</u> Der Operator T, der mit dem Ausdruck (17) gegeben ist, hat diese Eigenschaft:

Für eine beliebige Folge $\{h_n\}_1^\infty \subset W(\Omega)$, welche schwach zum Element h_0 konvergiert, und für welche gilt, dass

$$\lim_{n\to\infty} (Th_n, h_n - h_0)_W \leqq 0$$

gilt

$$h_n \xrightarrow[n\to\infty]{} h_0$$

B e w e i s . Es sei $\{h_n\}_1^\infty \subset W(\Omega)$ und es sei $h_n \xrightarrow[n\to\infty]{} h_0$

Es gilt

$$\lim_{n\to\infty} (Th_n, h_n - h_0)_W = \lim_{n\to\infty} (T_1 h_n, h_n - h_0)_W +$$

$$+ \lim_{n\to\infty} (T_2 h_n, h_n - h_0)_W$$

Die übrigen Elemente von T sind gleich Null. Aus der Konstruktion des Operators T_2, welche analogisch ist wie im Falle G_2 und aus der Integration per partes ist ersichtlich, dass

(18) $$(T_2 h_n, h_n - h_0)_W = ([w_n, w_n] + 2[k, w_n], F_n - F_0)_{L_2} -$$

$$- (\lambda [F_0, w_n] + [k, F_n] + [F_n, w_n], w_n - w_0)_{L_2}$$

Durch Verwendung der Sätze über die Einbettung und aus den Eigenschaften dass der lineare vollstetige Operator verstärkt stetig ist, erhalten wir aus der Voraussetzung des Satzes, dass

$$\|h_n - h_0\|_{L_2 \times L_2} \xrightarrow[n\to\infty]{} 0$$

In $W(\Omega)$ ist die schwach konvergente Folge $\{h_n\}_1^\infty$ begrenzt in $W(\Omega)$, deshalb erhalten wir durch die Verwendung der Ungleichheit von Cauchy-Schwarz aus (18)

$$(T_2 h_n, h_n - h_0)_W \xrightarrow[n\to\infty]{} 0 .$$

Aus der Definition der schwachen Konvergenz im Hilbertraum

$$\lim_{n\to\infty} (T_1 h_0, h_n - h_0)_W = 0$$

Wir können also endlich

$$\lim_{n\to\infty} (Th_n, h_n - h_0)_W = \lim_{n\to\infty} (T_1 h_n - T_1 h_0, h_n - h_0)_W =$$

$$= \lim_{n\to\infty} (T_1 (h_n - h_0), h_n - h_0)_W$$

schreiben. Aus dem Hilfssatz 1 und aus der Gestalt der bilinearen Form welche den Operator T_1 generiert erhalten wir

(19) $(T_1(h_n - h_0, h_n - h_0)_W \geqq C_9 \| h_n - h_0 \|_W^2$

wo C_9 eine von $\{ h_n \}_1^\infty$ unabhängige positive Konstante ist.

Aus der Voraussetzung des Satzes und aus (19) ist

$$0 \geqq \lim_{n \to \infty} (Th_n, h_n - h_0) \geqq C_9 \| h_n - h_0 \|_W^2,$$

also

$$\lim_{n \to \infty} \| h_n - h_0 \|_W^2 = 0 \equiv h_n \xrightarrow[n \to \infty]{} h_0$$

Es sei Q eine begrenztes Gebiet in $W(\Omega)$ und S ist die Grenze von Q . Es sei $\theta \in Q$.

S a t z 9. Es seien $k(x, y)$, $F_0(x, y)$ Elemente aus $W_2^{(2)}(\Omega)$ und zwar solche, dass für $h \in S$

$$(Th, h)_W \geqq 0$$

gelte. Dann hat die Aufgabe (1), (2) bezw. (6) wenigstens eine verallgemeinerte Lösung in Q für ein beliebiges q_1, $q_2 \in L_1(\Omega)$.

Wenn die verallgemeinerte Lösung die einzige ist, dann existieren Galerkinsche Annäherungen, welche zu der verallgemeinerten Lösung in der Norm des Raumes $W(\Omega)$ konvergieren.

B e w e i s . Dieser folgt aus der Verwendung des Satzes 8 und der Sätze 1, 2 aus [3] .

L I T E R A T U R

[1] KODNÁR R.: Bemerkungen zu einem System von nichtlinearen Gleichungen, Acta F.R.N. Univ. Comen., Mathematica XXVIII, 1972.

[2] VAJNBERG M. M.: Vyriacionnyje metody issledovanija nelinejnych operatorov. Moskva 1956.

[3] SKRYPNIK I.V.: O razrešimosti i obosnovaniji metoda Gelerkina v rade nelinejnych zadač mechaniki. Matematičeskaja fizika, 15, Kijev 1974.

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S ú h r n

KVALITATÍVNE VLASTNOSTI JEDNÉHO TYPU OKRAJOVÝCH ÚLOH I.

RUDOLF KODNÁR, BRATISLAVA

V práci sa skúmajú vlastnosti operátorov systému nelineárnych rovníc Kármánovho typu pri daných okrajových podmienkach. V priestore, ktorý je kartézskym súčinom priestorov typu $W_2^{(2)}$, je dokázaná existencia aspoň jedného slabého riešenia formulovanej úlohy. Dokázaná je tiež konvergencia Galerkinovej metódy v norme slabého riešenia.

Р Е З Ю М Е

КАЧЕСТВЕННЫЕ СВОЙСТВА ОДНОГО ТИПА КРАЕВЫХ ЗАДАЧ I.

РУДОЛФ КОДНАР, БРАТИСЛАВА

В статье рассматриваются свойства операторов системы нелинейных уравнений типа Кармана при однородных краевых условиях. В пространстве, которое является произведением пространств типа $W_2^{(2)}$ сделано доказательство существования по крайней мере одного обобщенного решения задачи. Доказана сходимость методы Галеркина в норме обобщенного решения.

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QUALITATIVE EIGENSCHAFTEN EINES TYPS VON RANDPROBLEMEN II.

RUDOLF KODNÁR, Bratislava

Knüpfen wir an die Arbeit [1] an. Durch Untersuchung des homogenen Systems [1] aus [1] lösen wir für dieses System bei den gegebenen Randbedingungen das Problem der Bifurkationspunkte. Für den ersten Bifurkationspunkt gilt der Satz aus [2] . Die Begriffe aus der Analyse in linearen Räumen nehmen wir aus [3] .

6. Die Eigenschaften des homogenen Systems aus (1)

Erwägen wir das System (1) [1] bei $q_1 = q_2 = 0$ und den Randbedingungen (2) bezw. (6). Wie wir im Teil 2 [1] zeigten, ist der Operator $G \langle w, F \rangle$ der schvachen Lösung dieser Aufgabe potentiell in $W(\Omega)$ und das entsprechende Funktional ist

$$(20) \qquad \Phi\langle w, F\rangle = \frac{D}{2t} A(w,w) - \frac{1}{2E} \| F \|^2_{H_2} - \frac{\lambda}{2} ([F_0, w], w)_{L_2} - \frac{1}{2} ([F, w], w)_{L_2} - ([k,F], w)_{L_2}$$

Aus dem Funktional (20) enthehmen wir das Funktional

$$(21) \qquad \mathcal{X}\langle w, F\rangle = \frac{D}{2t} A(w,w) - \frac{1}{2E} \| F \|^2_{H_2} - \frac{1}{2} ([F, w], w)_{L_2} - ([k, F], w)_{L_2}$$

Durch Berechnung und durch Verwendung des Hilfssatzes 2 [2] erhalten wir für ein beliebiges $\varphi \in H_1$

$$\lim_{r\to 0} \frac{\mathcal{X}\langle w + r\varphi, F\rangle - \mathcal{X}\langle w, F\rangle}{r} = \frac{D}{t} A(w, \varphi) -$$

$$- ([F, w], \varphi)_{L_2} - ([k, F], \varphi)_{L_2} = (\mathrm{grad}_w \mathcal{X}\langle w, F\rangle, \varphi)_{H_1}$$

Wobei wir bei der letzten Anordnung die Definition 2 [1] benutzen.

Durch denselben Vorgang für ein beliebiges $\psi \in H_2$ erhalten wir

$$\lim_{r\to 0} \frac{\mathcal{X}\langle w, F + r\varphi\rangle - \mathcal{X}\langle w, F\rangle}{r} = -\frac{1}{E}(F, \psi)_{H_2} -$$

$$-\frac{1}{2}([w, w], \psi)_{L_2} - ([k, w], \psi)_{L_2} = (\mathrm{grad}_F \mathcal{X}\langle w, F\rangle, \psi)_{H_2}$$

Bezeichnen wir weiter

$$\omega\langle w, F\rangle = \frac{1}{2}([F_0, w], w)_{L_2} \tag{22}$$

Dann ist

$$\lim_{r\to 0} \frac{\omega\langle w + r\varphi, F\rangle - \omega\langle w, F\rangle}{r} = ([F_0, w], \varphi)_{L_2} =$$

$$= (\mathrm{grad}_w\ \omega\langle w, F\rangle, \varphi)_{H_1}$$

$$\lim_{r\to 0} \frac{\omega\langle w, F + r\psi\rangle - \omega\langle w, F\rangle}{r} = 0 = (\mathrm{grad}_F\ \omega\langle w, F\rangle, \psi)_{H_2}$$

Durch Benutzung dieser Erwägungen und durch Integration per partes können wir das System von Gleichungen mit schwacher Lösung aus der Definition 1, bei $q_1 = q_2 \equiv 0$ in folgender Form schreiben:

$$\begin{aligned} &\forall_{\varphi\in H_1} (\mathrm{grad}_w \mathcal{X}\langle w, F\rangle, \varphi)_{H_1} = \lambda(\mathrm{grad}_w\ \omega\langle w, F\rangle, \varphi)_{H_1}; \\ &\forall_{\psi\in H_2} (\mathrm{grad}_F \mathcal{X}\langle w, F\rangle, \psi)_{H_2} = (\mathrm{grad}_F\ \omega\langle w, F\rangle, \psi)_{H_2}. \end{aligned} \tag{23}$$

<u>Hilfssatz 4.</u> Das Funktional $\omega\langle w, F\rangle$ ist schwach stetig und gleichmässig differenzierbar in $W(\Omega)$.

B e w e i s . Die schwache Stetigkeit geht aus der Folgerung 5 [2] hervor. Weiter gilt

$$\frac{\omega\langle w + r\varphi, F + r\psi\rangle - \omega\langle w, F\rangle}{r} = ([F_0, w], \varphi)_{L_2} +$$

$$+ \frac{r}{2}([F_0, \varphi], \varphi)_{L_2}$$

Also existiert die Gateaux-sche Ableitung des Funktional $\omega\langle w, F\rangle$ und es gilt

(24) $$\omega'(\langle w, F\rangle, \langle \varphi, \psi\rangle) = ([F_0, w], \varphi)_{L_2}$$

Durch Verwendung der Ungleichheit von Cauchy-Schwarz erhalten wir daraus sofort, dass auch die Ableitung von Fréchet existiert, welche der Gateaux-schen gleich ist. Aus der Formel für das Gateaux-sche Differenzial ist ersichtlich, dass $\omega\langle w, F\rangle$ gleichmässig differenzierbar ist, weil der Rest nicht von $\langle w, F\rangle$ abhängig ist.

Berechnen wir die zweite Ableitung des Funktionals $\omega\langle w, F\rangle$ im Punkt $\langle \theta, \theta\rangle$. Wir erhalten (verwendet wird der Hilfssatz 2 [2])

(25) $$\omega''(\langle \theta, \theta\rangle, \langle \varphi_1, \psi_1\rangle, \langle \varphi_2, \psi_2\rangle) =$$

$$= ([F_0, \varphi_1], \varphi_2)_{L_2}$$

Die zweite Gateaux-sche Ableitung in $\langle \theta, \theta\rangle$ ist stetig in $\langle \theta, \theta\rangle$ und ist also auch eine Fréchet-sche Ableitung. Aus der Gleichheit (25) ist ersichtlich, dass diese Fréchet-sche Ableitung ein symmetrischer am ganzen $W(\Omega)$ definierter Operator und daher auf $W(\Omega)$ ein selbstadjungierter Operator ist.

H i l f s s a t z 5. Das Funktional $\mathcal{X}\langle w, F\rangle$ ist in $W(\Omega)$ stetig differenzierbar.

B e w e i s . Durch Berechnung kann leicht festgestellt werden, dass

$$(26)\quad D\mathcal{X}\left(\langle w, F\rangle , \langle \varphi, \psi\rangle \right) = -\,($$
$$-\left([F,w], \varphi\right)_{L_2} - \frac{1}{2}\left([\psi,w], w\right)_{L_2} - \frac{1}{E}(F,\psi)_{H_2} +$$
$$+\frac{D}{t} A(w,\varphi) - \left([k,F], \varphi\right)_{L_2} - \left([k,\psi], w\right)_{L_2}$$

Dieser Ausdruck ist eine Form in den Veränderlichen $\langle w, F\rangle$, $\langle \varphi, \psi\rangle$. Durch Verwendung der Hölderschen Ungleichheit und der Sätze über die Einbettung ähnlich wie beim Beweis des Satzes 1 [1] würden wir erhalten, dass der Operator

$$\operatorname{grad} \mathcal{X} \langle w, F\rangle$$

definiert durch die Form (26) stetig ist.

Wählen wir F derart, dass es die Lösung der zweiten Gleichung aus (23) bei gegebenen $w \in H_1$ sei. Ein solches F existiert immer.

Dann ist

$$D\mathcal{X}\left(\langle w, F\rangle , \langle \varphi, \psi\rangle \right) = \frac{D}{t} A(w,\varphi) - \left([F,w], \varphi\right)_{L_2} -$$
$$-\left([k,F], \varphi\right)_{L_2}.$$

Für jedes feste $w \in H_1$ ist die bilineare Form $A(w, \varphi)$ ein lineares Funktional auf H_1. Bei Verwendung der Hölderschen Ungleichheit zeigt es sich, dass dieses lineare Funktional auch begrenzt ist. Dann folgt aus dem Satze von Riesz, dass ein einziges Element (bezeichnen wir es als Aw) aus H_1 derart existiert, dass

$$\frac{D}{t} A(w,\varphi) = \frac{D}{t}(Aw, \varphi)_{H_1}, \quad \forall \varphi \in H_1$$

Wir erhielten den linearen Operator A, welcher von H_1 bis H_1 operiert. Für ein festes $w \in H_1$ (also auch ein festes F) ist die Form

$$\left([F,w], \varphi\right)_{L_2} + \left([k,F], \varphi\right)_{L_2}$$

ein lineares Funktional auf H_1. Durch Integration per partes und mit dem gleichen Vorgang wie vor dem Satz 1 [1] würden wir einen Operator K erhalten, welcher von H_1 bis H_1 operiert und mit der Beziehung

$$\forall\ \varphi \in H_1 : ([F, w], \varphi)_{L_2} + ([k, F], \varphi)_{L_2} = (K w, \varphi)_{H_1}$$

definiert ist. Also können wir schreiben :

$$D\mathcal{X}(\langle w, F\rangle, \langle \varphi, \psi\rangle) = \frac{D}{t}(A w, \varphi)_{H_1} - (K w, \varphi)_{H_1},$$

$\forall\ \varphi \in H_1$

H i l f s s a t z 6. Der Operator K ist vollstetig.

B e w e i s . Zuerst wird die Begrenztheit und Stetigkeit des Operators bewiesen und zwar ähnlich wie im Satz 4 [1] für G_2. Der Beweis der Kompaktheit wird ähnlich durchgeführt, wie der Beweis des Satzes 5 [1] . Beim Beweis wird die Stetigkeit des Zuordnungsoperator F zu dem gegebenen w verwendet.

H i l f s s a t z 7. Für die beliebige Folge $\{w_n\}_1^\infty \subset H_1$, welche schwach zu irgendeinem $w_0 \in H_1$ konvergiert, gelte

$$\lim_{n\to\infty}\left\{\frac{D}{t}(A\ w_n, w_n - w_0)_{H_1} - (K w_n, w_n - w_0)_{H_1}\right\} \leqq 0.$$

Dann ist

$$w_n \xrightarrow[n\to\infty]{H_1} w_0$$

B e w e i s . Es sei $\{w_n\}_1^\infty \subset H_1$ und $w_n \xrightarrow[n\to\infty]{H_1} w_0$; $\{w_n\}_1^\infty$ ist also in H_1 begrenzt. Aus den Sätzen über die Einbettung folgt, dass $w_n \xrightarrow[n\to\infty]{L_2} w_0$ ist.

Aus der impliziten Konstruktion des Operators K und durch Anwendung der Cauchy-Schwarzschen Ungleichheit erhalten wir

$$(K\ w_n, w_n - w_0)_{H_1} \xrightarrow[n\to\infty]{} 0$$

Der weitere Verlauf des Beweises ist durch die Benutzung des Hilfssatzes 1 [1] genau derselbe, wieder Beweis des Satzes 8 [1] .

Bezeichnen wir mit U die Umgebung des Nullelements in H_1.

H i l f s s a t z 8. Es existiert eine positive Konstante γ derart, dass für $w \in U$

$$(27) \qquad D\mathcal{X}(\langle w, F\rangle, \langle w, F\rangle) \geq \gamma \|w\|^2_{H_1}$$

ist.

B e w e i s. Durch Anwendung der Cauchy-Schwarzschen Ungleichheit, der Einbettungssätze und der Ungleichheit (5) [1] erhalten wir die Abschätzungen

$$|([F, w], w)_{L_2}| \leq C_{11}\|w\|^2_{H_1} \|F\|_{H_2} \leq \|w\|^2_{H_1} C_{11}(C_{12}\|w\|^2_{H_1} + C_{13}\|w\|_{H_1})$$

$$|([k, F], w)_{L_2}| \leq C_{14}\|w\|^2_{H_1}(C_{12}\|w\|^2_{H_1} + C_{13}\|w\|_{H_1})$$

wo C_{11}, C_{12}, C_{13}, C_{14} positive, von w unabhängige Konstanten sind. Aus diesen Ungleichheiten erhalten wir für ein (hinreichend kleines) $w \in U$

$$(28) \qquad \begin{aligned} |([\dot{F}, w], w)_{L_2}| &\leq C_{15}\|w\|^2_{H_1} \\ |([k, F], w)_{L_2}| &\leq C_{16}\|w\|^2_{H_1} \end{aligned} .$$

Jetzt erhalten wir durch Ausnützung der Ungleichheit (28) und des Hilfssatzes 1 [1] die Ungleichheit (27).

Die existenz der zweiten Gateauxschen Ableitung des Funktionals $\mathcal{X}$ wurde schon im Satz 2 [1] gezeigt. Durch einfache Berechnung erhalten wir, dass

$$\mathcal{X}''(\langle w, F\rangle)\langle\varphi, \psi\rangle \langle\varphi_1, \psi_1\rangle = \frac{D}{t} A(\varphi, \varphi_1) -$$

$$- \frac{1}{E}(\psi, \psi_1)_{H_2} - ([k,\psi], \varphi_1)_{L_2} - ([\psi, w], \varphi_1)_{L_2} -$$

$$- ([k,\varphi], \psi_1)_{L_2} - ([w,\varphi], \psi_1)_{L_2} - ([F,\varphi], \varphi_1)_{L_2}$$

ist. Also

$$\mathcal{X}''(\langle\theta,\theta\rangle)\langle\varphi,\psi\rangle\langle\varphi_1,\psi_1\rangle = \frac{D}{t}A(\varphi, \varphi_1) -$$

$$- \frac{1}{E}(\psi, \psi_1)_{H_2} - ([k,\psi], \varphi_1)_{L_2} - ([k,\varphi], \psi_1)_{L_2}$$

Es sei $\{h_n\}_1^\infty \subset W(\Omega)$ eine solche Folge, dass $h_n \xrightarrow[n\to\infty]{} 0$.
Die Verwendung der Cauchy-Schwarzschen Ungleichheit ergibt dann gleich, dass

$$(29)\quad \mathcal{X}''(h_n)\langle\varphi,\psi\rangle\langle\varphi_1,\psi_1\rangle \xrightarrow[n]{} \mathcal{X}''(\theta)\langle\varphi,\psi\rangle\langle\varphi_1,\psi_1\rangle$$

7. Problem der Bifurkationspunkte

Es seien K_1, K_2 Operatoren auf irgendeinem Hilbertraum.

<u>Definition 5</u>. Die Zahl λ_0 nennt man einen Bifurkationspunkt der Gleichung

$$K_1h + \lambda K_2h = 0$$

wenn

$$\forall_{\varepsilon>0} \; \exists_{h_\varepsilon,\lambda_\varepsilon} \; (K_1h_\varepsilon + \lambda_\varepsilon K_2h = 0,$$

$$0 < |\lambda_0 - \lambda_\varepsilon| < \varepsilon, \; 0 < \|h_\varepsilon\| < \varepsilon \;) \text{ gilt.}$$

<u>Satz 10.</u> Dazu, dass die Zahl λ_0 ein Bifurkationspunkt der Aufgabe (23) sei, ist notwendig und hinreichend, dass die Aurgabe

$$(30)\quad \begin{array}{ll} \forall_{\varphi\in H_1} & \frac{D}{t}A(w,\varphi) - ([k,F],\varphi)_{L_2} = \lambda_0([F_0,w],\varphi)_{L_2} \\ \forall_{\psi\in H_2} & \frac{1}{E}(F,\psi)_{H_2} = -([k,w],\psi)_{L_2} \end{array}$$

die nichttrivial Lösung $\langle w, F\rangle \in W(\Omega)$ habe.

B e w e i s . Die Hilfssätze 4, 5, 7, 8 und die Eigenschaft (29) verbürgen die Erfüllung der Voraussetzungen des Satzes 1 [4]. Aus der Gültigkeit des Satzes 1 [4] für die Aufgabe (23) geht die Wahrhaftigkeit des Satzes hervor.

F o l g e r u n g . Wenn das Paar $\langle w, F\rangle \in W(\Omega)$ und λ_0 (30) erfüllen, sind diese auch eine Lösung der Aufgabe

(31) $\forall \langle \varphi, \psi\rangle \in W(\Omega)$: $\frac{D}{t} A(w, \varphi) - ([k, F], \varphi)_{L_2} +$
$+ \frac{1}{E}(F, \psi)_{H_2} + ([k, w], \psi)_{L_2} = \lambda_0([F_0, w], \varphi)_{L_2}$

B e m e r k u n g 1. Wenn das Paar $\langle w, F\rangle \in W(\Omega)$ und λ_0 (30) erfüllen und w, F hinreichend regulär sind, wird (30) äquivalent mit dem System

(32) $$\frac{D}{t} \Delta\Delta w - [k, F] = \lambda_0 [F_0, w]$$

$$\frac{1}{E} \Delta\Delta F + [k, w] = 0$$

sein, bei den Randbedingungen (2) bezw. (6). So war die Aufgabe für die Bifurkationspunkte des Systems (23) in [2] formuliert worden.

B e m e r k u n g 2. Im Falle einer idealen geraden schlanken Wand, würden wir für hinreichend reguläres w, F die Gleichung

(33) $$\frac{D}{t} \Delta\Delta w = \lambda [F_0, w]$$

erhalten, bei den Randbedingungen (2) bezw. (6). Die Gleichung für die Bifurkationspunkte einer idealen geraden schlanken Wand der Form (33) wurde bei den Dirichletschen Randbedingungen in mehreren Arbeiten abgeleitet.

B e m e r k u n g 3. Es gilt der Satz 12 [2].

L I T E R A T U R

[1] KODNÁR R.: Qualitative Eigenschaften eines Typs von Randwertproblemen I. Acta F.R.N.U.C., Mathematica (im Druck)

[2] KODNÁR R.: Bemerkungen zu einem Problem von nichtlinearen Gleichungen. Acta F.R.N.U.C., Mathematica XXVIII, 1972.

[3] VAJNBERG M. M.: Variacionnij metod i metod monotonnych operatorov. Moskva 1972.

[4] SKRYPNIK I. V.: Točky bifurkacie variacionnych zadač. Matematičeskaja fyzika, 9, 1971.

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S ú h r n

KVALITATÍVNE VLASTNOSTI JEDNÉHO TYPU OKRAJOVÝCH ÚLOH II

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Pre homogenný systém rovníc Kármánovho typu pri daných okrajových podmienkach je v práci riešený problém bodov bifurkácie. Systém rovníc pre body bifurkácie dostávame Fréchetovským derivovaním operátora slabého riešenia v nulovom prvku priestoru.

Р Е З Ю М Е

КАЧЕСТВЕННЫЕ СВОЙСТВА ОДНОГО ТИПА КРАЕВЫХ ЗАДАЧ II.

РУДОЛФ КОДНАР, ВРАТИСЛАВА

В статье решается проблема точек бифуркации для однородной системы уравнений типа Кармана при однородных краевых условиях. Система уравнений для точек бифуркации получается из производной Фреше оператора обобщенного решения.

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ON EXPONENTIAL INTERPOLATION

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A large number of events (processes) especially in economy, medicine, biology, chemistry etc. have an exponential character. At interpolating in numerical materials of these events exponential interpolation is more suitable than the interpolation based on the polynoms. The aim of this article is the derivation of formulae for exponential interpolation. In general, i.e. in the case of the arbitrary intervals formulae (35) and (36), were derived and in the case of interpolation with equal intervals formulae (67) and (68).

P r o b l e m . One has to find a real function

(1) $f(x_1, x_2, \dots, x_s)$ for $(x_1, x_2, \dots, x_s) \in E_s$

so that it has certain real values for prescribed fixed values at x. In this article we will limit ourselves to the case, that for the whole range of definition will always hold

$$f(x_1, x_2, \dots, x_s) > 0 \tag{2}$$

1. The general case of exponential interpolation

Introduction of the symbols

Sometimes it will be advantageous to work with vectors.

$\bar{a} = (a_1, a_2, \ldots, a_s)$, $\bar{a}_k = (a_{k,1}, a_{k,2}, \ldots, a_{k,s})$

$\bar{x} = (x_1, x_2, \ldots, x_s)$, $f(\bar{x}) = f(x_1, x_2, \ldots, x_s)$

$f(\bar{a}_k) = f(a_{k,1}, a_{k,2}, \ldots, a_{k,s})$

In the following considerations it will be usefull to introduce the concept of "rooted quotient".

N o t e 1. In further text, to accentuate, that a_2 is distinct from a_1 by the only value a_{i_1} we will sometimes write $\bar{a}_2(i_1)$. Similarly $\bar{a}_3$ distinguished from $\bar{a}_2$ only by the value a_{i_2} and thus from $\bar{a}_1$ by the values a_{i_1} and a_{i_2} we shall denote $\bar{a}_3(i_1, i_2)$ and finally for $\bar{a}_p$ we will introduce the denotation $\bar{a}_p(i_1, i_2, \ldots, i_{p-1})$.

D e f i n i t i o n 1. Partial rooted quotient with respect to i-th variable of the function $f(x)$ between arguments a_1 and a_2 is defined by the formula

$$(3)\quad Q_i[\bar{a}_1 | i | \bar{a}_2] = \sqrt[a_{2,i}-a_{1,i}]{\frac{f(\bar{a}_2)}{f(\bar{a}_1)}} = \sqrt[a_{2,i}-a_{1,i}]{\frac{f(a_{2,1}, a_{2,2}, \ldots, a_{2,i}, \ldots, a_{2,s})}{f(a_{1,1}, a_{1,2}, \ldots, a_{1,i}, \ldots, a_{1,s})}}$$

where $a_{2,\ell} = a_{1,\ell}$ for $\ell \neq i$ and $\ell = 1, 2, \ldots, s$

and $a_{2,i} \neq a_{1,i}$

D e f i n i t i o n 2. The second partial rooted quotient with respect to the i-th variable is defined by the relation

$$(4)\qquad Q_i^2 [\bar{a}_1 | i^2 | \bar{a}_3] = Q_i [Q_i [\bar{a}_1 | i | \bar{a}_2]]$$

etc. and finally

$$(5)\qquad Q_i^n [\bar{a}_1 | i^n | \bar{a}_{n+1}] = Q_i [Q_i^{n-1} [\bar{a}_1 | i^{n-1} | \bar{a}_n]]$$

D e f i n i t i o n 3. The second partial rooted quotient with respect to the i-th and j-th variable is

$$(6) \quad Q^2_{i,j}[\bar{a}_1 \mid i,\ j \mid \bar{a}_3] = Q_j[\ Q_i[\bar{a}_1 \mid i \mid \bar{a}_2]\ ] = $$
$$= Q_i[\ Q_j[\bar{a}_1 \mid j \mid \bar{a}_2'\ ]\ ]$$

where $\bar{a}_2' \neq \bar{a}_2$

L e m m a 1. In constructing the second partial rooted quotient of the same i-th variable between the values of arguments $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$ the ordering in which we construct the rooted quotients is of no importance.

P r o o f .

$$(7) \quad Q^2_i[\bar{a}_1 \mid i^2 \mid \bar{a}_3] = \sqrt[a_{3,i}-a_{1,i}]{\frac{Q_i[\bar{a}_2 \mid i \mid \bar{a}_3]}{Q_i[\bar{a}_1 \mid i \mid \bar{a}_2]}} =$$

$$= \left[\left(\frac{f(\bar{a}_3)}{f(\bar{a}_2)}\right)^{(a_{3,i}-a_{2,i})^{-1}} : \left(\frac{f(\bar{a}_2)}{f(\bar{a}_1)}\right)^{(a_{2,i}-a_{1,i})^{-1}} \right]^{(a_{3,i}-a_{1,i})^{-1}} =$$

$$= f(\bar{a}_1)^{[(a_{1,i}-a_{2,i})(a_{1,i}-a_{3,i})]^{-1}} \cdot$$

$$\cdot f(\bar{a}_2)^{[(a_{2,i}-a_{1,i})(a_{2,i}-a_{3,i})]^{-1}} \cdot$$

$$\cdot f(\bar{a}_3)^{[(a_{3,i}-a_{1,i})(a_{3,i}-a_{2,i})]^{-1}}$$

As follows from the formula (7), the second partial rooted quotient is a symmetrical function of its arguments and the validity of lemma 1 is thus demonstrated.

C o r o l l a r y 1. It is possible to prove that it holds generally

$$(8) \quad Q^n_i[\bar{a}_1 \mid i^n \mid \bar{a}_{n+1}] =$$

$$= \prod_{k=1}^{n} f(\bar{a}_k)^{[(a_{k,i}-a_{1,i})\dots(a_{k,i}-a_{k-1,i})(a_{k,i}-a_{k+1,i})\dots(a_{k,i}-a_{n,i})]^{-1}}$$

N o t e 2. In further text we shall omit the indices i, j etc. by Q.

L e m m a 2. The construction the second partial rooted quotient with respect to i-th and j-th variable does not depend on the order in which we construct the rooted quotient, thus it holds

$$(9) \qquad Q^2[\bar{a}_1 \mid i, j \mid \bar{a}_3] = Q^2[\bar{a}_1 \mid j, i \mid \bar{a}_3]$$

P r o o f . In the proof we must consider that the mediating variable between the variables $\bar{a}_1$ and $\bar{a}_3$ is in both cases different (viz. $\bar{a}_2$ or $\bar{a}_2'$).

Let us denote

$$(10) \qquad \begin{aligned} \bar{a}_1 &= (a_{1,1}, \ldots, a_{1,i_1}, \ldots, a_{1,j_1}, \ldots, a_{1,s}) \\ \bar{a}_2 &= (a_{2,1}, \ldots, a_{2,i_1}, \ldots, a_{2,j_1}, \ldots, a_{2,s}) \\ \bar{a}_2' &= (a_{1,1}, \ldots, a_{1,i_2}, \ldots, a_{1,j_2}, \ldots, a_{1,s}) \\ \bar{a}_3 &= (a_{2,1}, \ldots, a_{2,i_2}, \ldots, a_{2,j_2}, \ldots, a_{2,s}) \end{aligned}$$

Now let us assume that it holds

$$(11) \quad a_{1,i_1} = a_{1,i_2}, \; a_{2,i_1} = a_{2,i_2}, \; a_{1,j_1} = a_{2,j_1}, \; a_{1,j_2} = a_{2,j_2}$$

then we have

$$(12) \qquad \begin{aligned} Q[\bar{a}_1 \mid i \mid \bar{a}_2] &= \left(\frac{f(\bar{a}_2)}{f(\bar{a}_1)}\right)^{(a_{2,i_1} - a_{1,i_1})^{-1}} \\ Q[\bar{a}_2' \mid i \mid \bar{a}_3] &= \left(\frac{f(\bar{a}_3)}{f(\bar{a}_2')}\right)^{(a_{2,i_2} - a_{1,i_2})^{-1}} \\ Q[\bar{a}_1 \mid j \mid \bar{a}_2'] &= \left(\frac{f(\bar{a}_2')}{f(\bar{a}_1)}\right)^{(a_{1,j_2} - a_{1,j_1})^{-1}} \\ Q[\bar{a}_2 \mid j \mid \bar{a}_3] &= \left(\frac{f(\bar{a}_3)}{f(\bar{a}_2)}\right)^{(a_{2,j_2} - a_{2,j_1})^{-1}} \end{aligned}$$

and finally the second partial rooted quotient are

$$(13)\quad Q^2[\bar{a}_1 \mid i,\ j \mid \bar{a}_3] = \left(\frac{Q[\bar{a}_2' \mid i \mid \bar{a}_3]}{Q[\bar{a}_1 \mid i \mid \bar{a}_2]}\right)^{(a_{2,j_2} - a_{2,j_1})^{-1}}$$

$$(14)\quad Q^2[\bar{a}_1 \mid j,\ i \mid \bar{a}_3] = \left(\frac{Q[\bar{a}_2 \mid j \mid \bar{a}_3]}{Q[\bar{a}_1 \mid j \mid \bar{a}_2']}\right)^{(a_{2,i_2} - a_{1,i_2})^{-1}}$$

By substituting from (12) in (13) we obtain

$$(15)\left[\frac{f(\bar{a}_3)^{(a_{2,i_2} - a_{1,i_1})^{-1}}}{f(\bar{a}_2')^{(a_{2,i_2} - a_{1,i_1})^{-1}}} \cdot \frac{f(\bar{a}_1)^{(a_{2,i_1} - a_{1,i_1})^{-1}}}{f(\bar{a}_2)^{(a_{2,i_1} - a_{1,i_1})^{-1}}}\right]^{(a_{2,j_2} - a_{2,j_1})^{-1}}$$

and by substituting in (14) we obtain

$$(16)\left[\frac{f(\bar{a}_3)^{(a_{2,j_2} - a_{2,j_1})^{-1}}}{f(\bar{a}_2)^{(a_{2,j_2} - a_{2,j_1})^{-1}}} \cdot \frac{f(\bar{a}_1)^{(a_{1,j_2} - a_{1,j_1})^{-1}}}{f(\bar{a}_2')^{(a_{1,j_2} - a_{1,j_1})^{-1}}}\right]^{(a_{2,i_2} - a_{1,i_2})^{-1}}$$

With respect to (11) one can easily show that the relations (15) and (16) are identical q.e.d.

<u>C o r o l l a r y 2.</u> By repeating the procedure from lemma 2 we can prove the validity of the following formula

$$(17)\quad Q^n[\bar{a}_1 \mid 1,\ 2,\ \ldots,\ n \mid \bar{a}_{n+1}] = Q^n[\bar{a}_1 \mid i_1,\ i_2, \ldots,\ i_n \mid \bar{a}_{n+1}]$$

where the group of numbers $\{i_1,\ i_2,\ \ldots,\ i_n\}$ is an arbitrary permutation of numbers $1,\ 2,\ \ldots,\ n$.

<u>D e f i n i t i o n 4.</u> The partial rooted quotient (in further text only p.r.q.) of the n-th order can be written in this way

$$(18)\qquad Q^n\left[\bar{a}_1 \mid j_1^{k_1^{(n)}},\ j_2^{k_2^{(n)}},\ \ldots,\ j_n^{k_n^{(n)}}\right]$$

where the arguments $j_1^{k_1^{(n)}}$ etc. till $j_n^{k_n^{(n)}}$ denote, that the p.r.q.

is constructed $k_1^{(n)}$-times with respect to a_{j_1} etc. till $k_n^{(n)}$-times with respect to a_{j_n}, where holds

$$\sum_{t=1}^{n} k_t^{(n)} = n \tag{19}$$

Formula (18) can be written in an abbreviated form

$$Q^n [\bar{a}_1 \mid \bar{j}\ \bar{k}^{(n)}] \tag{20}$$

L e m m a 3. The number of all n-th p.r.q. of the function $f(x)$ is

$$\binom{s + n - 1}{n} \tag{21}$$

and the sum of the first p.r.q. till n-th is

$$\binom{s + n}{n} \tag{22}$$

P r o o f . From the values $\bar{a}_1$ and $\bar{a}_2(i_j)$ for $j = 1, 2, \ldots, s$ we can form the s first p.r.q. From the each first p.r.q. we can again form s second p.r.q., but because with respect to (9) the second p.r.q. do not depend on the ordering of variables with respect of which they were formed, the number of all possible p.r.q. of the 2. order is equal $\binom{s+1}{2}$. One can simplely prove that the number of p.r.q. of n-th order is $\binom{s + n - 1}{n}$ (i.e. the number of n-combination from s elements with repetition). The first part of the lemma 3 is thus demonstrated. Further the sum of the first n p.r.q. as it is well-known from combinatorial analysis is

$$\sum_{r=0}^{n} \binom{s + r - 1}{r} = \binom{s + n}{n} \tag{23}$$

Thus it is demonstrated also the second part of the lemma 3.

In accordance with (18) a fixed p.r.q. of the n-order Q^n depends on the elements $j_1, j_2, \ldots, j_n$. By application of all non negative integer solutions of condition (19) we get the number of all p.r.q. of the n-th order Q^n depending on the j_i for $i = 1, 2, \ldots, n$, as a number of the n-combinations of n elements with repetition nameli

$$(24) \qquad c'(n, n) = \binom{2n - 1}{n}$$

If the formula (18) is changed as follows

$$(25) \qquad Q^n \left[\bar{a}_1 \mid j_{1,\ell}^{k_{1,\ell}^{(n)}}, j_{2,\ell}^{k_{2,\ell}^{(n)}}, \ldots, j_{n,\ell}^{k_{n,\ell}^{(n)}} \right]$$

and thus if we change also the formula (20) into the following form

$$(26) \qquad Q^n \left[\bar{a}_1 \mid \bar{j}_\ell^{\bar{k}_\ell^{(n)}} \right]$$

where ℓ denote the number order of any n-combinations from s elements (i.e. of c(s, n)), then gradually for $\ell = 1, 2, \ldots, c(s, n)$ we get all n-combinations.

In total number

$$(27) \qquad c'(n, n) \cdot c(s, n) = \binom{2n - 1}{n}\binom{s}{n}$$

occur some combinations several times. Therefore by summation with respect to ℓ we must these combinations take only once. In this way we obtain as a total number

$$(28) \qquad c'(s, n) = \binom{s + n - 1}{n}$$

in accordance with (21).

D e f i n i t i o n 5. Let us introduce the function G by following relation

$$(29) \qquad G(\bar{x};\ 1^{m_1},\ 2^{m_2},\ \ldots,\ q^{m_q},\ \ldots,\ n^{m_n}) = $$

$$= C\,(1^{m_1},\ 2^{m_2},\ \ldots,\ q^{m_q},\ \ldots,\ n^{m_n}) \prod_{i=1}^{n} \prod_{j_i=1}^{m_i} (x_i - a_{i,j_i})$$

where the $C(1^{m_1},\ 2^{m_2},\ \ldots\ q^{m_q},\ \ldots,\ n^{m_n})$ is a constant belonging to the function G.

<u>Lemma 4.</u> If we construct the p.r.q. of the function G of the variable X between the values $a_{q,r}$ and $a'_{q,r}$ we obtain the function

$$(30) \qquad G(\bar{x};\ 1^{m_1},\ 2^{m_2},\ \ldots,\ q^{m_q-1},\ \ldots,\ n^{m_n})$$

In the exponent of this function does not occur already the factor $(x_q - a_{q,r})$.

<u>Proof.</u> Let us denote the function from (29) with the symbol G, further the one from (30) as G_2 and finally the function which differ from G only by parameter $a'_{q,r}$ which appear instead of $a_{q,r}$, as G_1. On the bases of the (3) holds

$$(31) \qquad \left(\frac{G_1}{G}\right)^{|a'_{q,r} - a_{q,r}|^{-1}} = G_2$$

<u>Corollary 3.</u> Analogously in accordance with lemma 4, if we construct the p.r.q. with respect to p_k values of variable x_k for $k = 1, 2, \ldots, n$ we get the function

$$(32) \quad G(\bar{x};\ 1^{m_1-p_1},\ 2^{m_2-p_2},\ \ldots,\ q^{m_q-p_q},\ \ldots,\ n^{m_n-p_n})$$

Now we prove the following theorem.

<u>Theorem 1.</u> Let the arguments (in the sense of notice 1)

(33) $\bar{x}_t(i_1, i_2, \ldots, i_{t-1})$, $t = 1, 2, \ldots, n+1$, $i_t = 1, 2, \ldots, s$ correspond the values

$$f(\bar{a}_t) > 0 \tag{34}$$

(which can be considered) the values of some function $f(x)$).
The general exponential interpolational function of the p-th order

(35) $F(\bar{x}) =$

$$= f(\bar{a}_1) \prod_{n=1}^{p} \prod_{l=1}^{c(s,n)} \prod_{i=1}^{s} \prod_{j_{i,l}=1}^{k_{i,l}^{(n)}} Q^n[\bar{a}_1 \mid \bar{j}_l^{\bar{k}_l^{(n)}}]^{\prod_{l=1}^{c(s,n)} \prod_{i=1}^{s} \prod_{j_{i,l}=1}^{k_{i,l}^{(n)}} (x_i - a_{i,j_{i,l}})}$$

or partially written

$$(36)\ F(\bar{x}) = f(\bar{a}_1) \prod_{i=1}^{s} Q[\bar{a}_1 \mid i]^{(x_i - a_{i,1})} \cdot$$

$$\cdot \prod_{l=1}^{c(s,2)} \prod_{i=1}^{s} \prod_{j_{i,l}=1}^{k_{i,l}^{(2)}} Q^2[\bar{a}_1 \mid j_{1,l}^{k_{1,l}^{(2)}}, j_{2,l}^{k_{2,l}^{(2)}}]^{\prod_{i=1}^{s} \prod_{j_{i,l}=1}^{k_{i,l}^{(2)}} (x_i - a_{i,j_{i,l}})} \cdot \ldots$$

$$\ldots \cdot \prod_{l=1}^{c(s,p)} \prod_{i=1}^{s} \prod_{j_{i,l}=1}^{k_{i,l}^{(p)}} Q^p[\bar{a}_1\ j_{1,l}^{k_{1,l}^{(p)}}, j_{2,l}^{k_{2,l}^{(p)}}, \ldots, j_{p,l}^{k_{p,l}^{(p)}}]^{\prod_{i=1}^{s} \prod_{j_{i,l}=1}^{k_{i,l}^{(p)}} (x_i - a_{i,j_{i,l}})}$$

where

$$\sum_{t=1}^{n} k_{t,l}^{(n)} = n \tag{37}$$

has for arguments in (33) the values in (34). The product $\prod_{l=1}^{c(s,n)}$ must be constructed in the sense, that the factors occuring several times must taken only once.

P r o o f . Let us assume, that the formulae for exponential interpolation (35) resp. (36) one can write by means of G defined by equation (29) in the following way

$$(38) \qquad F(\bar{x}) = f(\bar{a}_1) \cdot \prod_{n=1}^{p} \prod_{\ell=1}^{c(s,n)} G(\bar{x};\, \bar{j}_\ell^{\,\bar{k}_\ell^{(n)}})$$

resp.

$$(39) \qquad F(\bar{x}) = f(\bar{a}_1) \prod_{\ell=1}^{c(s,1)} G(\bar{x};\, j_{1,\ell}^{k_{1,\ell}^{(1)}}) \cdot \prod_{\ell=1}^{c(s,2)} G(\bar{x};\, j_{1,\ell}^{k_{1,\ell}^{(2)}}, j_{2,\ell}^{k_{2,\ell}^{(2)}}) \cdot \ldots$$

$$\ldots \cdot \prod_{\ell=1}^{c(s,p)} G(\bar{x};\, j_{1,\ell}^{k_{1,\ell}^{(p)}}, j_{2,\ell}^{k_{2,\ell}^{(p)}}, \ldots, j_{p,\ell}^{k_{p,\ell}^{(p)}})$$

If we denote the construction of p.r.q. of the function $F(x)$ $k_{t,\ell}^{(r)}$ -times with respect to x_t for $t = 1, 2, \ldots, r$ as

$$(40) \qquad Q\,[\,F(\bar{x};\, 1^{k_{1,\ell}^{(r)}}, 2^{k_{2,\ell}^{(r)}}, \ldots, r^{k_{r,\ell}^{(r)}})]$$

then by constructing of the p.r.q. of an arbitrary factor from the right side of (39) e.g.

$$(41) \qquad G(\bar{x};\, j_{1,\ell}^{k_{1,\ell}^{(p)}}, j_{2,\ell}^{k_{2,\ell}^{(p)}}, \ldots, j_{p,\ell}^{k_{p,\ell}^{(p)}}) \qquad \text{we obtain}$$

$$(42) \qquad G(\bar{x};\, j_{1,\ell}^{k_{1,\ell}^{(p)}-k_{1,\ell}^{(r)}}, j_{2,\ell}^{k_{2,\ell}^{(p)}-k_{2,\ell}^{(r)}}, \ldots, j_{p,\ell}^{k_{p,\ell}^{(p)}-k_{p,\ell}^{(r)}})$$

First of all let us investigate the case $r = p$. If it will hold

$$(43) \qquad k_{t,\ell}^{(r)} = k_{t,\ell}^{(p)} \qquad \text{for all } t = 1, 2, \ldots, r$$

then all factors of the exponent of the constant

$$(44) \qquad C(j_{1,\ell}^{k_{1,\ell}^{(p)}}, j_{2,\ell}^{k_{2,\ell}^{(p)}}, \ldots, j_{p,\ell}^{k_{p,\ell}^{(p)}}$$

will be equal to 1 and thus the term (40) will be the constant (44).

It is clear that for $r > p$ the term (40) will be 1, since the p.r.q. of an arbitrary constant is equal to 1. For $r < p$ the term (40) will be 1 again, because by the constructing of p.r.q. of the r-th order will remain in the exponent of the constant C yet $p - r$ factors which for the values $\bar{a}_k$ (in the sense of notice 1) will be equal to 0. Thus it will hold

$$(45)\quad Q^p[\bar{a}_1 \mid j_{1,\ell}^{k_{1,\ell}^{(p)}}, j_{2,\ell}^{k_{2,\ell}^{(p)}}, \ldots, j_{p,\ell}^{k_{p,\ell}^{(p)}}] = C(j_{1,\ell}^{k_{1,\ell}^{(p)}}, \ldots, j_{p,\ell}^{k_{p,\ell}^{(p)}})$$

and the validity of the theorem 1 is demonstrated.

<u>Specialization.</u> To make clear the construction of the products $\prod_{\ell=1}^{c(s,n)}$ we show two special cases: 1. $s = 3$, $n = 4$ and 2. $s = 4$, $n = 3$.

1. $s = 3$, $n = 4$ (the number of the factors is $\binom{3+4-1}{4} = 15$)

$$Q^4[\bar{a}_1 \mid 1^4]^{(x_1-a_{11})(x_1-a_{12})(x_1-a_{13})(x_1-a_{14})}.$$

$$Q^4[\bar{a}_1 \mid 1^3, 2]^{(x_1-a_{11})(x_1-a_{12})(x_1-a_{13})(x_2-a_{21})}.$$

$$Q^4[\bar{a}_1 \mid 1^2, 2^2]^{(x_1-a_{11})(x_1-a_{12})(x_2-a_{21})(x_2-a_{22})}.$$

$$Q^4[\bar{a}_1 \mid 1, 2^3]^{(x_1-a_{11})(x_2-a_{21})(x_2-a_{22})(x_2-a_{23})}.$$

$$Q^4[\bar{a}_1 \mid 2^4]^{x_2-a_{21})(x_2-a_{22})(x_2-a_{23})(x_2-a_{24})}.$$

$$Q^4[\bar{a}_1 \mid 1^3, 3]^{(x_1-a_{11})(x_1-a_{12})(x_1-a_{13})(x_3-a_{31})}.$$

$$Q^4[\bar{a}_1 \mid 1^2, 3^2]^{(x_1-a_{11})(x_1-a_{12})(x_3-a_{31})(x_3-a_{32})}.$$

$$Q^4\ [\bar{a}_1 \mid 1,\ 3^3]^{(x_1-a_{11})(x_3-a_{31})(x_3-a_{32})(x_3-a_{33})}\ .$$

$$Q^4\ [\bar{a}_1 \mid 3^4]^{(x_3-a_{31})(x_3-a_{32})(x_3-a_{33})(x_3-a_{34})}\ .$$

$$Q^4\ [\bar{a}_1 \mid 2^3,\ 3]^{(x_2-a_{21})(x_2-a_{22})(x_2-a_{23})(x_3-a_{31})}\ .$$

$$Q^4\ [\bar{a}_1 \mid 2^2,\ 3^2]^{(x_2-a_{21})(x_2-a_{22})(x_3-a_{31})(x_3-a_{32})}\ .$$

$$Q^4\ [\bar{a}_1 \mid 2,\ 3^3]^{(x_2-a_{21})(x_3-a_{31})(x_3-a_{32})(x_3-a_{33})}\ .$$

$$Q^4\ [\bar{a}_1 \mid 1^2,\ 2,\ 3]^{(x_1-a_{11})(x_1-a_{12})(x_2-a_{21})(x_3-a_{31})}\ .$$

$$Q^4\ [\bar{a}_1 \mid 1,\ 2^2,\ 3]^{(x_1-a_{11})(x_2-a_{21})(x_2-a_{22})(x_3-a_{31})}\ .$$

$$Q^4\ [\bar{a}_1 \mid 1,\ 2,\ 3^2]^{(x_1-a_{11})(x_2-a_{21})(x_3-a_{31})(x_3-a_{32})}\ .$$

2. $s = 4$, $n = 3$ (the number of the factors is $\binom{4+3-1}{3} = 20$)

$$Q^3\ [\bar{a}_1 \mid 1^3]^{(x_1-a_{11})(x_1-a_{12})(x_1-a_{13})}\ .$$

$$Q^3\ [\bar{a}_1 \mid 1^2,\ 2]^{(x_1-a_{11})(x_1-a_{12})(x_2-a_{21})}\ .$$

$$Q^3\ [\bar{a}_1 \mid 1,\ 2^2]^{(x_1-a_{11})(x_2-a_{21})(x_2-a_{22})}\ .$$

$$Q^3\ [\bar{a}_1 \mid 2^3]^{(x_2-a_{21})(x_2-a_{22})(x_2-a_{23})}\ .$$

$$Q^3\ [\bar{a}_1 \mid 1^2,\ 3]^{(x_1-a_{11})(x_1-a_{12})(x_3-a_{31})}\ .$$

$$Q^3\ [\bar{a}_1 \mid 1,\ 3^2]^{(x_1-a_{11})(x_3-a_{31})(x_3-a_{32})}\ .$$

$$Q^3\ [\bar{a}_1 \mid 2^2,\ 3]^{(x_2-a_{21})(x_2-a_{22})(x_3-a_{31})}\ .$$

$$Q^3\ [\bar{a}_1 \mid 2,\ 3^2]^{(x_2-a_{21})(x_3-a_{31})(x_3-a_{32})}\ .$$

$$Q^3\ [\bar{a}_1 \mid 3^3]^{(x_3-a_{31})(x_3-a_{32})(x_3-a_{33})}\ .$$

$$\cdot\ Q^3\ [\bar{a}_1 \mid 1^2, 4]^{(x_1-a_{11})(x_1-a_{12})(x_4-a_{41})}\ .$$

$$\cdot\ Q^3\ [\bar{a}_1 \mid 1, 4^2]^{(x_1-a_{11})(x_4-a_{41})(x_4-a_{42})}\ .$$

$$\cdot\ Q^3\ [\bar{a}_1 \mid 2^2, 4]^{(x_2-a_{21})(x_2-a_{22})(x_4-a_{41})}\ .$$

$$\cdot\ Q^3\ [\bar{a}_1 \mid 2, 4^2]^{(x_2-a_{21})(x_4-a_{41})(x_4-a_{42})}\ .$$

$$\cdot\ Q^3\ [\bar{a}_1 \mid 3^2, 4]^{(x_3-a_{31})(x_3-a_{32})(x_4-a_{41})}\ .$$

$$\cdot\ Q^3\ [\bar{a}_1 \mid 3, 4^2]^{(x_3-a_{31})(x_4-a_{41})(x_4-a_{42})}\ .$$

$$\cdot\ Q^3\ [\bar{a}_1 \mid 1, 2, 3]^{(x_1-a_{11})(x_2-a_{21})(x_3-a_{31})}\ .$$

$$\cdot\ Q^3\ [\bar{a}_1 \mid 1, 2, 4]^{(x_1-a_{11})(x_2-a_{21})(x_4-a_{41})}\ .$$

$$\cdot\ Q^3\ [\bar{a}_1 \mid 1, 3, 4]^{(x_1-a_{11})(x_3-a_{31})(x_4-a_{41})}\ .$$

$$\cdot\ Q^3\ [\bar{a}_1 \mid 2, 3, 4]^{(x_2-a_{21})(x_3-a_{31})(x_4-a_{41})}\ .$$

$$\cdot\ Q^3\ [\bar{a}_1 \mid 4^3]^{(x_4-a_{41})(x_4-a_{42})(x_4-a_{43})}\ .$$

2. Exponential interpolation with equal intervals

Introduction of symbols

In practice often occurs the problem to obtain the value of a given function tabulated in equal intervals. Therefore we also conform the symbols to this requirement. First of all it is clear that holds

(46) $\quad \bar{x} = \bar{a} + \bar{m}\bar{h}$ or $x_i = a_i + m_i h_i$ for $i = 1, 2, \ldots, s$

therefore it will be

(47) $F(\bar{a} + \bar{m}\bar{h}) = F(a_1 + m_1h_1,\ a_2 + m_2h_2,\ \dots,\ a_s + m_s\, h_s)$

where the h_i (for $i = 1, 2, \dots, s$) is the step of the i-th variable. Since the values a_i and h_i for $i = 1, 2, \dots, s$ are constants it suffices to write only the variables m_i and therefore instead of (47) we can write

(48) $$f(\bar{m}) = f(m_1, m_2, \dots, m_s)$$

If will be $m_i = 0$ for all $i = 1, 2, \dots, s$ we shall write instead of (48) simplely $f\,(\bar{0})$.

Further instead of "rooted quotients" we introduce by equal intervals only "quotients".

D e f i n i t i o n 6. The partial quotient with respect to the i-th variable of the function $f(\bar{m})$ is defined by relation

(49) $$Q[\,f(\bar{m}) \mid i\,] = \frac{f(a_1+m_1h_1,\ a_2+m_2h_2,\ \dots,\ a_i+m_ih_i+h_i,\ \dots, a_s+m_sh_s}{f(a_1+m_1h_1,\ a_2+m_2h_2,\ \dots,\ a_i+m_ih_i,\ \dots,\ a_s+m_sh_s}$$

One can derive all dependences and formulae occuring in the exponential interpolation with equal intervals by specialization of the general exponential interpolation by means of substitution

(50) $a_{i,j} = a_i + (j - 1)\, h_i$ for $i = 1, 2, \dots, s,\quad j = 1, 2, \dots,$

From it there follows that the theorems used in the derivation of the formulae of the exponential interpolation with equal intervals are special cases of theorems analogous for general exponential interpolation and therefore also hold the proofs. Therefore in the following we shall not prove the majority of the given theorems. From the lemma 2 follows immediately lemma 5.

L e m m a 5. In constructing the second partial quotient with respect to the i-th and j-th variable the ordering in which we construct the quotients is of no importance i. e. it holds

$$(51)\qquad Q^2 [\, f(\bar{m}) \mid i, j \,] = Q \{ Q [\, f(\bar{m}) \mid i \,] \mid j \} = Q \{ Q [\, f(\bar{m}) \mid j \,] \mid i \} = Q^2 [\, f(\bar{m}) \mid j, i \,]$$

From the lemma 5 follows corollary 4:

<u>C o r o l l a r y 4.</u> By repeating the procedure from lemma 5 we can prove the validity of the following formula

$$(52)\qquad Q^n [\, f(\bar{m}) \mid 1, 2, \ldots, n \,] = Q^n [\, f(\bar{m}) \mid i_1, i_2, \ldots, i_n \,]$$

where the group of numbers $\{ i_1, i_2, \ldots, i_n \}$ is an arbitrary permutation of numbers $1, 2, \ldots, n$. It is clear that the corollary 4 is a special case of the corollary 2.

<u>D e f i n i t i o n 7.</u> The partial quotient (in the further text only p. q.) of the n-th order of the function $f(\bar{m})$ can be written in the following way

$$(53)\qquad Q^n [\, f(\bar{m}) \mid j_1^{k_1^{(n)}}, j_2^{k_2^{(n)}}, \ldots, j_n^{k_n^{(n)}}$$

where the numbers $j_i^{k_i^{(n)}}$ denote that the p. q. is constructed $k_i^{(n)}$-times with respect to the j_i-th variable for $i = 1, 2, \ldots, n$, by condition

$$(54)\qquad \sum_{i=1}^{n} k_i^{(n)} = n$$

<u>L e m m a 6.</u> The number of the p. q. of the n-th order of the function $f(m)$ is from the formula (21) equal to $\binom{s+n-1}{n}$ and the number of all p. q. till n-th order is from the formula (22) equal

$$\sum_{r=0}^{n} \binom{s+r-1}{r} = \binom{s+n}{n}$$

D e f i n i t i o n 8. Let us introduce the function g by relation

$$(55)\quad g(m_1, m_2, \ldots, m_q, \ldots, m_n; 1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n}) =$$

$$= C(1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n}) \prod_{i=1}^{n} h_i^{k_i} (m_i)_{k_i}$$

where $C(1^{k_1}, 2^{k_2}, \ldots q^{k_q}, \ldots, n^{k_n})$ is a constant belonging to the function g defined in (55).

Briefly we can write this function vectorially

$$(56)\quad g(\bar{m}; 1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n}) \text{ or } g(\bar{m}; \bar{i}^{k_i})$$

It is clear that for $\bar{m} = \bar{0}$ i. e. $m_i = 0$ for $i = 1, 2, \ldots, n$ is

$$(57)\qquad g(\bar{0}; \bar{i}^{k_i}) = 1$$

L e m m a 7. The p. q. of the function g with respect to the variable m_q is

$$(58)\quad Q\,[g(\bar{m}; 1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n})\, \bar{m}_q\,] =$$

$$= \dot{g}(m; 1^{k_1}, 2^{k_2}, \ldots, q^{k_q - 1}, \ldots, n^{k_n})\, h_q k_q$$

where the point at g means that the constant of the function on the right side of (58) is

$C(1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n})$ and not $C(1^{k_1}, 2^{k_2}, \ldots, q^{k_q - 1}, \ldots, n^{k_n})$

P r o o f . From the formula (49) we have

$$(59)\quad Q\,[\,g(m_1, m_2, \ldots, m_q, \ldots, m_n; 1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n}) \mid m_q] =$$

$$= \frac{g(m_1, m_2, \ldots, m_q+1, \ldots, m_n; 1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n})}{g(m_1, m_2, \ldots, m_q, \ldots, m_n; 1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n})} =$$

$$= \frac{C(1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n})\, h_1^{k_1}(m_1)_{k_1} \ldots h_q^{k_q}(m_q+1)_{k_q} \ldots h_n^{k_n}(m_n)_{k_n}}{C(1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n})\, h_1^{k_1}(m_1)_{k_1} \ldots h_q^{k_q}(m_q)_{k_q} \ldots h_n^{k_n}(m_n)_{k_n}}$$

$$= \dot{g}(\bar{m}; 1^{k_1}, 2^{k_2}, \ldots, q^{k_q-1}, \ldots, n^{k_n})\, h_q k_q$$

q. e. d.

C o r o l l a r y 5. Analogously to the lemma 7, when we construct the p. q. r_i-times with respect to the variable m_i for $i = 1, 2, \ldots, n$ we get

$$(60) \quad Q^r [\, g(\bar{m}; 1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n}) \mid 1^{r_1}, 2^{r_2}, \ldots, q^{r_q}, \ldots, n^{r_n} \,] =$$
$$= \dot{g}(\bar{m}; 1^{k_1-r_1}, 2^{k_2-r_2}, \ldots, q^{k_q-r_q}, \ldots, n^{k_n-r_n}) \sum_{i=1}^{n} h_i^{r_i}(k_i)_{r_i}$$

where

$$(61) \qquad r = \sum_{i=1}^{n} r_i$$

or briefly

$$(62) \qquad \dot{g}(\bar{m}; j^{\bar{k}-r}) \prod_{i=1}^{n} h_i^{k_i}(k_i)_{r_i}$$

where the point at g draws again our attention to the constant C.

In the case $r_i = k_i$ for $i = 1, 2, \ldots, n$ we have

(63) $Q^n[g(\bar{m};\ 1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n}) | 1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n}] =$

$$= g^{\cdot}(\bar{m};\ j^0) \prod_{i=1}^{n} h_i^{k_i}(k_i)!$$

under the condition

(64)
$$\sum_{i=1}^{n} k_i = n$$

The right side from (63) is a constant.

N o t e 3. From the definition 8 follows

(65) $g^{\cdot}(\bar{m};\ \bar{j}^0) = C\ (1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n})$

and therefore

(66) $Q^n[g^{\cdot}(\bar{m};\ 1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n}) | 1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots n^{k_n}] =$

$$= C(1^{k_1}, 2^{k_2}, \ldots, q^{k_q}, \ldots, n^{k_n}) \prod_{i=1}^{n} h_i^{k_i}(k_i)!$$

T h e o r e m 2. Let us have an s-dimensional "table" of the function $f(\bar{x})$ with equidistant arguments

$\bar{x} = \bar{a} + \bar{t}\bar{h}$ where $\bar{t} = (t_1, t_2, \ldots, t_s)$ and $t_i \geq 0$ are integers for $i = 1, 2, \ldots, s$ and $\bar{h} = (h_1, h_2, \ldots, h_s)$ denotes the steps, then the value of the function $f(x)$ for $\bar{x} = \bar{a} + \bar{m}\bar{h}$ where $m_i \in (0, 1)$ is given by the exponential interpolational function of the p-th order with equal intervals, which according to (48) can be written in the following way

(67) $f(\bar{m}) =$

$$= f(\bar{0}) \cdot \prod_{n=1}^{p} \prod_{l=1}^{c(s,n)} \prod_{i=1}^{s} \prod_{j_{i,l}=1}^{k_{i,l}^{(n)}} Q^n[\bar{0}|\bar{j}^{\bar{k}_l^{(n)}}]^{\prod_{t=1}^{n} \binom{m_{j_{t,l}}}{k_{t,l}^{(n)}}}$$

or partially written

$$(68)\quad f(\bar{m}) =$$

$$= f(0) \prod_{i=1}^{s} Q[\,0|i\,]^{m_i} \prod_{l=1}^{c(s,2)} \prod_{i=1}^{s} \prod_{j_{i,l}=1}^{k_{i,l}^{(2)}} Q^2[\,0|j_{1,l}^{k_{1,l}^{(2)}}, j_{2,l}^{k_{2,l}^{(2)}}]^{\prod_{t=1}^{2}\binom{m_{j_{t,l}}}{k_{t,l}^{(2)}}} \,.$$

$$. \prod_{l=1}^{c(s,3)} \prod_{i=1}^{s} \prod_{j_{i,l}=1}^{k_{i,l}^{(3)}} Q^3[\,0|j_{1,l}^{k_{1,l}^{(3)}},\ j_{2,l}^{k_{2,l}^{(3)}},\ j_{3,l}^{k_{3,l}^{(3)}}]^{\prod_{t=1}^{3}\binom{m_{j_{t,l}}}{k_{t,l}^{(3)}}} \,. \ldots$$

$$\ldots \,. \prod_{l=1}^{c(s,p)} \prod_{i=1}^{s} \prod_{j_{i,l}=1}^{k_{i,l}^{(p)}} Q^p[\,0|j_{1,l}^{k_{1,l}^{(p)}}, \ldots , j_{p,l}^{k_{p,l}^{(p)}}]^{\prod_{t=1}^{p}\binom{m_{j_{t,l}}}{k_{t,l}^{(p)}}} \,,$$

under the condition

$$(69)\qquad \sum_{t=1}^{n} k_{t,l}^{(n)} = n$$

<u>Proof.</u> Let us assume that the formulae for exponential interpolation (67) resp. (68) can be written by means of function g defined by equation (55) in the following way

$$(70)\qquad f(\bar{m}) = f(\bar{0}) \,. \prod_{n=1}^{p} \prod_{\ell=1}^{c(s,n)} g(\bar{m};\ \bar{j}_{\ell}^{\bar{k}_{\ell}})$$

resp.

$$(71)\quad f(\bar{m}) = f(\bar{0}) \prod_{\ell=1}^{c(s,1)} g(\bar{m};\ j_{1,\ell}^{k_{1,\ell}^{(1)}}) \,. \prod_{\ell=1}^{c(s,2)} g(\bar{m};\ j_{1,\ell}^{k_{1,\ell}^{(2)}} j_{2,\ell}^{k_{2,\ell}^{(2)}}) \,. \ldots$$

$$\ldots \cdot \prod_{\ell=1}^{c(s,p)} g(\bar{m};\ j_{1,\ell}^{k_{1,\ell}^{(p)}},\ \ldots,\ j_{p,\ell}^{k_{p,\ell}^{(p)}})$$

In order to define the function g let us form the p. q. of the function $f(\bar{m})$ $k_{t,\ell}^{(r)}$-times with respect to $m_{j_{t,\ell}}$ for $t = 1, 2, \ldots, r$ where

$$\sum_{t=1}^{r} k_{t,\ell}^{(r)} = r \tag{72}$$

For $r \geq 1$ we obtain

$$Q^r \left[f(\bar{m}) \mid j_{1,\ell}^{k_{1,\ell}^{(r)}},\ \ldots,\ j_{r,\ell}^{k_{r,\ell}^{(r)}} \right] = \tag{73}$$

$$= \prod_{n=1}^{p} \prod_{\ell=1}^{c(s,n)} g(\bar{m};\ \bar{j}_{\ell}^{\bar{k}_{\ell}^{(n)}-\bar{k}_{\ell}^{(r)}}) \prod_{t=1}^{r} h_{t,\ell}^{k_{t,\ell}^{(r)}}(k_{t,\ell}^{(n)})_{k_{t,\ell}^{(r)}}$$

If now is $r = n$ then according to (63) and (66) holds for g

$$g^{\bullet}(\bar{m};\ \bar{j}^{0}) \prod_{t=1}^{n} h_t^{k_{t,\ell}^{(n)}}(k_{t,\ell}^{(n)})! \tag{74}$$

but according to (65) we have

$$g^{\bullet}(\bar{m};\ \bar{j}^{0}) = c(j_{1,\ell}^{k_{1,\ell}^{(n)}},\ \ldots,\ j_{n,\ell}^{k_{n,\ell}^{(n)}}) \tag{75}$$

and therefore

$$Q^n \left[g(\bar{m};\ j_{1,\ell}^{k_{1,\ell}^{(n)}},\ \ldots,\ j_{n,\ell}^{k_{n,\ell}^{(n)}} \right] = \tag{76}$$

$$= C\,(j_{1,\ell}^{k_{1,\ell}^{(n)}}, \ldots, j_{n,\ell}^{k_{n,\ell}^{(n)}}) \prod_{t=1}^{n} h_t^{k_{t,\ell}^{(n)}} (k_{t,\ell}^{(n)})!$$

what is a constant.

If $r > n$ then considering the fact that the p. q. of a constant equals to 1, we have

$$(77) \qquad Q^r\,[\,g(\bar{m};\; j_{1,\ell}^{k_{1,\ell}^{(n)}}, \ldots, j_{n,\ell}^{k_{n,\ell}^{(n)}})\,] = 1$$

Both formulae (76) and (77) hold naturally also for the case that

$$(78) \qquad \bar{m} = \bar{0}$$

If $r < n$ then under the condition (78) with respect to (57) holds

$$(79) \qquad Q^r\,[\,g(\bar{0};\; \bar{j}^k)\,] = 1$$

From the formulae (73), (76), (77), (78) and (79) follows

$$(80) \qquad C(j_{1,\ell}^{k_{1,\ell}^{(n)}}, \ldots, j_{n,\ell}^{k_{n,\ell}^{(n)}}) =$$

$$= Q^n\,[\,f(\bar{0}) \mid j_{1,\ell}^{k_{1,\ell}^{(n)}}, \ldots, j_{n,\ell}^{k_{n,\ell}^{(n)}}\,] \left[\prod_{t=1}^{n} h_t^{k_{t,\ell}^{(n)}} \prod_{t=1}^{n} (k_{t,\ell}^{(n)})!\,\right]^{-1}$$

this term together with (55) gives

$$(81) \qquad g(\bar{m};\; j_{1,\ell}^{k_{1,\ell}^{(n)}}, \ldots, j_{n,\ell}^{k_{n,\ell}^{(n)}}) =$$

$$= Q^n [f(\bar{0}) \mid j_{1,\ell}^{k_{1,\ell}^{(n)}}, \dots, j_{n,\ell}^{k_{n,\ell}^{(n)}}]^{\dfrac{\prod_{t=1}^{n} h_t^{k_{t,\ell}^{(n)}} \prod_{t=1}^{n} (m_{j_{t,\ell}})_{k_{t,\ell}^{(n)}}}{\prod_{t=1}^{n} h_t^{k_{t,\ell}^{(n)}} \prod_{t=1}^{n} (k_{t,\ell}^{(n)})!}}$$

or

$$g(\mathrm{m};\ j_{1,\ell}^{k_{1,\ell}^{(n)}},\ \dots,\ j_{n,\ell}^{k_{n,\ell}^{(n)}}) = \tag{82}$$

$$= Q^n [\ f(\bar{0}) \mid j_{1,\ell}^{k_{1,\ell}^{(n)}},\ \dots,\ j_{n,\ell}^{k_{n,\ell}^{(n)}}]^{\binom{m_{j_{t,\ell}}}{k_{t,\ell}^{(n)}}} .$$

The function g from (82) substituted in (71) gives the formula (68) q. e. d.

<u>Specialization.</u> To make clear the concrete construction of the products $\prod_{\ell=1}^{c(s,n)}$ we show again two special cases. From methodical reasons there are the same cases as in the preceding part:

1. $s = 3, \quad n = 4$ (the number of the factors are 15)
2. $s = 4, \quad n = 3$ (the number of the factors are 20)

1. $s = 3, \quad n = 4$

$$Q^4 [\ \bar{0} \mid 1^4\ ]^{\binom{m_1}{4}} \cdot Q^4 [\ \bar{0} \mid 1^3,\ 2\ ]^{\binom{m_1}{3}\binom{m_2}{1}} \cdot$$

$$\cdot\ Q^4 [\ \bar{0} \mid 1^2,\ 2^2\ ]^{\binom{m_1}{2}\binom{m_2}{2}} \cdot Q^4 [\ \bar{0} \mid 1,\ 2^3\ ]^{\binom{m_1}{1}\binom{m_2}{3}} \cdot$$

$$\cdot\ Q^4 [\ \bar{0} \mid 2^4\ ]^{\binom{m_2}{4}} \cdot Q^4 [\ \bar{0} \mid 1^3,\ 3\ ]^{\binom{m_1}{3}\binom{m_3}{1}} \cdot$$

$$\cdot\ Q^4 [\ \bar{0} \mid 1^2,\ 3^2\ ]^{\binom{m_1}{2}\binom{m_3}{2}} \cdot Q^4 [\ \bar{0} \mid 1,\ 3^3\ ]^{\binom{m_1}{1}\binom{m_3}{3}} \cdot$$

$$\cdot\ Q^4\,[\,\bar{0}\,|\,3^4\,]^{({}^{m}_{4}3)} \cdot Q^4\,[\,\bar{0}\,|\,2^3,\ 3\,]^{({}^{m}_{3}2)({}^{m}_{1}3)} \cdot$$

$$\cdot\ Q^4\,[\,\bar{0}\,|\,2^2,\ 3^2\,]^{({}^{m}_{2}2)({}^{m}_{2}3)} \cdot Q^4\,[\,\bar{0}\,|\,2,\ 3^3\,]^{({}^{m}_{1}2)({}^{m}_{3}3)} \cdot$$

$$\cdot\ Q^4\,[\,\bar{0}\,|\,1^2,\ 2,\ 3\,]^{({}^{m}_{2}1)({}^{m}_{1}2)({}^{m}_{1}3)} \cdot Q^4\,[\,\bar{0}\,|\,1,\ 2^2,\ 3\,]^{({}^{m}_{1}1)({}^{m}_{2}2)({}^{m}_{1}3)} \cdot$$

$$\cdot\ Q^4\,[\,\bar{0}\,|\,1,\ 2,\ 3^2\,]^{({}^{m}_{1}1)({}^{m}_{1}2)({}^{m}_{2}3)}$$

2. $s = 4$, $n = 3$

$$\cdot\ Q^3\,[\,\bar{0}\,|\,1^3\,]^{({}^{m}_{3}1)} \cdot Q^3\,[\,\bar{0}\,|\,1^2,\ 2\,]^{({}^{m}_{2}1)({}^{m}_{1}2)} \cdot$$

$$\cdot\ Q^3\,[\,\bar{0}\,|\,1,\ 2^2\,]^{({}^{m}_{1}1)({}^{m}_{2}2)} \cdot Q^3\,[\,\bar{0}\,|\,2^3\,]^{({}^{m}_{3}2)} \cdot$$

$$\cdot\ Q^3\,[\,\bar{0}\,|\,1^2,\ 3\,]^{({}^{m}_{2}1)({}^{m}_{1}3)} \cdot Q^3\,[\,\bar{0}\,|\,1,\ 3^2\,]^{({}^{m}_{1}1)({}^{m}_{2}3)} \cdot$$

$$\cdot\ Q^3\,[\,\bar{0}\,|\,2^2,\ 3\,]^{({}^{m}_{2}2)({}^{m}_{1}3)} \cdot Q^3\,[\,\bar{0}\,|\,2,\ 3^2\,]^{({}^{m}_{1}2)({}^{m}_{2}3)} \cdot$$

$$\cdot\ Q^3\,[\,\bar{0}\,|\,3^3\,]^{({}^{m}_{3}3)} \cdot Q^3\,[\,\bar{0}\,|\,1^2,\ 4\,]^{({}^{m}_{2}1)({}^{m}_{1}4)} \cdot$$

$$\cdot\ Q^3\,[\,\bar{0}\,|\,1,\ 4^2\,]^{({}^{m}_{1}1)({}^{m}_{2}4)} \cdot Q^3\,[\,\bar{0}\,|\,2^2,\ 4\,]^{({}^{m}_{2}2)({}^{m}_{1}4)} \cdot$$

$$\cdot\ Q^3\,[\,\bar{0}\,|\,2,\ 4^2\,]^{({}^{m}_{1}2)({}^{m}_{2}4)} \cdot Q^3\,[\,\bar{0}\,|\,3^2,\ 4\,]^{({}^{m}_{2}3)({}^{m}_{1}4)} \cdot$$

$$\cdot\ Q^3\,[\,\bar{0}\,|\,3,\ 4^2\,]^{({}^{m}_{1}3)({}^{m}_{2}4)} \cdot Q^3\,[\,\bar{0}\,|\,1,\ 2,\ 3\,]^{({}^{m}_{1}1)({}^{m}_{1}2)({}^{m}_{1}3)} \cdot$$

$$\cdot\ Q^3\,[\,\bar{0}\,|\,1,\ 2,\ 4\,]^{({}^{m}_{1}1)({}^{m}_{1}2)({}^{m}_{1}4)} \cdot Q^3\,[\,\bar{0}\,|\,1,\ 3,\ 4\,]^{({}^{m}_{1}1)({}^{m}_{1}3)({}^{m}_{1}4)} \cdot$$

$$. \; Q^3 [\, \bar{0} \mid 2,\ 3,\ 4\,]^{\binom{m}{1}2)\binom{m}{1}3)\binom{m}{1}4)} \quad . \; Q^3 [\, \bar{0} \mid 4^3\,]^{\binom{m}{3}4)}$$

Applications

1. Let us find a (exponential) function, the values of which satisfy to values of the Table I.

Table I

$x_2 \backslash x_1$	0,6	1,1	1,8	2,2
0,4	2,73577564	8,10962374	64,4366732	281,136723
0,9	3,70272405	12,5842549	121,087753	
1,4	6,23705696	24,3036670		
1,7	9,47246755			

Here is

$$a_{1,1} = 0{,}6, \quad a_{1,2} = 1{,}1, \quad a_{1,3} = 1{,}8, \quad a_{1,4} = 2{,}2$$

$$a_{2,1} = 0{,}4 \quad a_{2,2} = 0{,}9 \quad a_{2,3} = 1{,}4 \quad a_{2,4} = 1{,}7$$

$$f(0{,}6;\ 0{,}4) = 2{,}73577564$$

By computing we gain the values

$$Q\ (0{,}6;\ 0{,}4 \mid 1) = 8{,}78699648; \quad Q\ (0{,}6;\ 0{,}4 \mid 2) = 1{,}83181553$$

$$Q^2(0{,}6;\ 0{,}4 \mid 1^2) = 1{,}92775787; \quad Q^2(0{,}6;\ 0{,}4 \mid 1,2) = 1{,}72800000$$

$$Q^2(0{,}6;\ 0{,}4 \mid 2^2) = 1{,}54894141$$

$$Q^3(0{,}6;\ 0{,}4 \mid 1^3) = Q^3(0{,}6;\ 0{,}4 \mid 1^2,\ 2) = Q^3(0{,}6;\ 0{,}4 \mid 1,2^2) = $$

$$= Q^3(0{,}6;\ 0{,}4 \mid 2^3) = 1$$

Since all third and higher p.r.q. are 1 it suffices to take a formula of the second order.

$$F(x_1,\ x_2) = 2{,}73577564\ .\ 8{,}78699648^{(x_1-0{,}6)}\ .\ 1{,}83181553^{(x_2-0{,}4)}\ .$$

$$\cdot\ 1,92775787^{(x_1-0,6)(x_1-1,1)}\ \cdot\ 1,72800000^{(x_1-0,6)(x_2-0,4)}\ \cdot$$

$$\cdot\ 1,54894141^{(x_2-0,4)(x_2-0,9)}$$

or by modification we have finally

$$F(x_1,\ x_2) = 1,2^{3,6x_1^2 + 3x_1x_2 + 2,4x_2^2 + 4,6x_1 - 1,6x_2 + 1}$$

2. Let us find the value of the function given in Table II. for $x = 1,3$.

Table II

x	f(x)
0,7	3,28708122
1,2	14,01320367
1,6	64,07152267
1,9	247,15112700
2,1	671,82641989

a) The computing with a interpolation formula of the polynomial type of the 4-th order gives

$$f(1,\ 3) = 3,28708122 + 0,6\ .\ 10,72612245 + 0,6\ .\ 0,1\ .\ 39,33219655 -$$
$$- 0,6\ .\ 0,1\ .\ 0,3\ .\ 93,68908878 +$$
$$+ 0,6\ .\ 0,1\ .\ 0,3\ .\ 0,6\ .\ 14,88531445 = 10,5570443$$

b) The computing with the exponential formula of the 2-nd order gives

$$f(1,3) = 3,28708122\ .\ 18,17414539^{0,6}\ .\ 2,7182818^{0,06} = 19,8856822$$

c) The true value is 19,8856825.

(It was tabulated the function e^{x^2+x}).

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S ú h r n

O EXPONENCIÁLNEJ INTERPOLÁCII

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V ekonómii, lekárstve, biológii a technických vedách vyskytuje sa veľké množstvo javov, ktoré majú exponenciálny charakter. Pri interpolácii v numerických materiáloch týchto javov je výhodnejšia exponenciálna interpolácia, než interpolácia založená na polynómoch.

Cieľom tejto práce je odvodenie vzorcov pre exponenciálnu interpoláciu.

V prvej časti je dokázaná veta 1 pre exponenciálnu interpoláciu s neekvidistantnými argumentami (vzorce(35) a (36)) a v druhej časti veta 2 pre exponenciálnu interpoláciu s ekvidištantnými argumentmi.

Р Е З Ю М Е

ОБ ЭКСПОНЕНЦИАЛЬНОЙ ИНТЕРПОЛЯЦИИ

АНТОН ХУТЯ, БРАТИСЛАВА

В экономии, медицине, биологии и технических науках встречается большое множество явлений имеющих екпоненциальный характер. При вычислительной интерполяции этих явлений более выгодная экспоненциальная интерполяция, чем интерполяция основана на полиномах.

Целью этой работы является выведение формул для экспоненциальной интерполяции.

В первой части доказана теорема 1 для экспоненциальной интерполяции с неравнопромежуточными переменными [формулы (35) и (36)] и во второй части теорема 2 для экспоненциальной интерполяции с равнопромежуточными переменными.

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REFLECTIVE AND COREFLECTIVE SUBCATEGORIES OF SOME CATEGORY OF GENERALIZED TOPOLOGICAL SPACES

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0. Introduction

A $\mathscr{C}$ -space (see 1.1) is a topological space in Čech's sence from the year 1937. The category $\mathscr{T}$ of all topological spaces and continuous maps which is a full subcategory of the category $\mathscr{C}$ studied in this paper is neither reflective nor coreflective in $\mathscr{C}$. Therefore results of the theory of reflective and coreflective subcategories of $\mathscr{T}$ do not always give an immediate information about reflective and coreflective subcategories in $\mathscr{C}$ and there are some interesting differences between situations in $\mathscr{C}$ and in $\mathscr{T}$. The categorical terminology will be used according to [3] . The concepts separator, extremal separator (coseparator, extremal coseparator) used in [3] coincide with the concepts generator, extremal generator (cogenerator, extremal cogenerator) in [2] .

2. The category $\mathscr{C}$ - basic definitions and properties

In this section we shall prove only the facts needed for the study of reflective and coreflective subcategories of $\mathscr{C}$.

<u>1.1. Definition [5].</u> A $\mathscr{C}$ -space is a pair (P, u) where P is a set, u: $2^P \longrightarrow 2^P$ is a map and the following conditions are satisfied:

u_1) $u\emptyset = \emptyset$

u_2) $M \subset uM$ for each $M \in 2^P$

u_3) $uM \subset uN$ whenever $M \subset N$

The map u is called a $\mathscr{C}$-topology, uM is called the closure of M. A set $M \in 2^P$ is said to be closed provided that $uM = M$.

Let (P, u) be a $\mathscr{C}$-space and $p \in P$. A subset O of P is said to be a neighbourhood of the point $p \in P$ provided that $p \in P - u(P - O)$.

<u>R e m a r k .</u> It will be often written P instead of (P, u) and then $\overline{M}$ instead of uM.

Let (P, u) be a $\mathscr{C}$-space and $Q \subset P$. Put $vM = uM \cap Q$ for each $M \in 2^Q$. Then (Q, v) is a $\mathscr{C}$-space.

<u>1.2. D e f i n i t i o n [5] .</u> The $\mathscr{C}$-space (Q, v) is called a subspace of the $\mathscr{C}$-space (P, u).

<u>1.3. D e f i n i t i o n [5] .</u> Let (P, u), (Q, v) be $\mathscr{C}$-spaces. A map $f : P \longrightarrow Q$ is said to be continuous provided that for each $M \in 2^P$, $f[uM] \subset vf[M]$.

We shall always denote by $\mathscr{C}$ the category which is defined as follows:

ob $\mathscr{C}$ is the class of all $\mathscr{C}$-spaces,

mor $\mathscr{C}$ is the class of all continuous maps of $\mathscr{C}$-spaces (more exactly the class of all tripples $((P, u), f, (Q, v))$ where $(P, u), (Q, v) \in$ ob $\mathscr{C}$ and f is a continuous map $(P, u) \longrightarrow (Q, v)$).

It is clear that the category $\mathscr{T}$ of all topological spaces and continuous maps is a full subcategory of the category $\mathscr{C}$.

<u>1.4. T h e o r e m .</u> Let f be a $\mathscr{C}$-morphism.

1. f is a $\mathscr{C}$-monomorphism iff f is injective

2. f is a $\mathscr{C}$-epimorphism iff f is surjective

3. f is a $\mathscr{C}$-isomorphism iff f is bijective and f^{-1} is a continuous map

4. $\mathscr{C}$ is well-powered and co-well-powered

5. f is an extremal $\mathscr{C}$-monomorphism iff f is an embedding

6. $f: (P, u) \longrightarrow (Q, v)$ is an extremal $\mathscr{C}$-epimorphism (or a quotient map) iff f is a $\mathscr{C}$-epimorphism and for each $N \in 2^Q$, $vN = f[uf^{-1}[N]]$. The $\mathscr{C}$-space (Q, v) is called a quotient space of (P, u).

7. Let $\{(P_a, u_a)\}_{a \in A}$ be a set-indexed family of $\mathscr{C}$-objects. Let $(P, \{p_a\}_{a \in A})$ be the Set-product of $\{P_a\}_{a \in A}$. Define $u: 2^P \longrightarrow 2^P$ putting $uM = \bigcap_{a \in A} p_a^{-1}[u_a p_a[M]]$ for each $M \in 2^P$. Then (P, u) is a $\mathscr{C}$-space and $((P, u), \{p_a\}_{a \in A})$ is a $\mathscr{C}$-product of the family $\{(P_a, u_a)\}_{a \in A}$. We shall write $(P, u) = \prod_{a \in A} (P_a, u_a)$.

8. Let $\{(P_a, u_a)\}_{a \in A}$ be a set-indexed family of $\mathscr{C}$-objects. Let $(\{m_a\}_{a \in A}, P)$ be the Set-coproduct of $\{P_a\}_{a \in A}$. Define $u: 2^P \longrightarrow 2^P$ by $uM = \bigcup_{a \in A} m_a[u_a m_a^{-1}[M]]$ for each $M \in 2^P$. Then (P, u) is a $\mathscr{C}$-space and $(\{m_a\}_{a \in A}, (P, u))$ is the $\mathscr{C}$-coproduct of $\{(P_a, u_a)\}_{a \in A}$. We shall write $(P, u) = \coprod_{a \in A} (P_a, u_a)$.

9. If $r, s \in \hom_{\mathscr{C}}((P, u), (Q, v))$, then $Eq(r, s) = ((K, w), k)$ where (K, w) is a subspace of (P, u) such that $K = \{x \in P: r(x) = s(x)\}$ and $k: (K, w) \hookrightarrow (P, u)$ is an embedding.

10. If $r, s \in \hom_{\mathscr{C}}((P, u), (Q, v))$, then $Coeq(r, s) = (p, (E, w))$ where $E = Q/R$, R is the equivalence-relation on Q

generated by the relation S for which $(y, z) \in S$ iff there exists $x \in P$ such that $r(x) = y$ and $s(x) = z$, p is the projection $Q \longrightarrow Q/R$ and $(p,(E, w))$ is an extremal quotient of (Q, v).

11. $\mathscr{C}$ is complete and cocomplete.

12. Any non-empty $\mathscr{C}$-object is a separator.

R e m a r k s . The terms $\mathscr{C}$-product and $\mathscr{C}$-coproduct will be often (inaccurately) used also for their "object parts". The concepts extremal subobject and subspace are essentially identical in $\mathscr{C}$.

1.5. P r o p o s i t i o n . Let $(P, u) = \prod_{a \in A} (P_a, u_a)$. A subset $M \subset P$ is closed in (P, u) iff $M = \bigtimes_{a \in A} p_a[M]$ (a Set-product) and $p_a[M]$ is closed in (P_a, u_a) for each $a \in A$.

P r o o f . Immediate from the preceding theorem - (7).

1.6. D e f i n i t i o n [5] . A $\mathscr{C}$-space (P, u) is said to be an F-space provided that for each $M \in 2^P$, $u(uM) = uM$. A $\mathscr{C}$-space (P, u) is said to be an A-space provided that for each $M, N \in 2^P$, $u(M \cup N) = uM \cup uN$.

The full subcategory of the category $\mathscr{C}$ consisting of all F-spaces (A-spaces) will be denoted by $\mathscr{F}$ ($\mathfrak{A}$).

In the category $\mathscr{C}$ T_0-, T_1-, T_2-spaces can be defined analogously as in the category $\mathscr{T}$. Corresponding (full) subcategories will be denoted by $\mathscr{C}_0$, $\mathscr{C}_1$, $\mathscr{C}_2$, respectively.

R e m a r k . In $\mathscr{C}$ we can define also $\overline{T}_2$-spaces as follows: (P, u) is a $\overline{T}_2$-space iff for each $x, y \in P$, $x \neq y$ there exist neighbourhoods O_x, O_y of x, y, respectively such that $O_x \cap uO_y = uO_x \cap O_y = \emptyset$. In $\mathscr{C}$ T_2- and $\overline{T}_2$-spaces concide.

<u>1.7. Theorem.</u> Let X be a $\mathscr{C}_1$-object such that any non-constant $\mathscr{C}$-morphism $\varphi : X \longrightarrow X$ is an isomorphism. Let A be a non-empty set and $(X^A, \{p_a\}_{a \in A})$ be a corresponding $\mathscr{C}$-power. Then for any non-constant $\mathscr{C}$-morphism $f : X^A \longrightarrow X$ there exists $a_0 \in A$ such that $f = \varphi \circ p_{a_0}$ where φ is an (suitable) isomorphism $X \longrightarrow X$.

<u>Proof.</u> Let $f: X^A \longrightarrow X$ be a non-constant $\mathscr{C}$-morphism. Then $\varphi = f \circ \Delta : X \longrightarrow X$ where Δ is a diagonal morphism $X \longrightarrow X^A$ is non-constant. Indeed, if $f \circ \Delta (x) = x_0$ for each $x \in X$, then $\Delta [X] \subset f^{-1}[x_0]$. But $f^{-1}[x_0]$ is closed in X^A and $\overline{\Delta [X]} = X^A$. Thus f is constant - a contradiction. Hence φ is an isomorphism. Put $g = \varphi^{-1} \circ f$. Then for each $\xi \in \Delta [X]$ and $a \in A$ $g(\xi) = p_a(\xi)$. The family $\mathfrak{M} = \{g^{-1}[x] : x \in X\}$ is a decomposition of X^A and every element belonging to $\mathfrak{M}$ is a closed set. Thus, for each $x \in X$ $g^{-1}[x] = \underset{a \in A}{\times} B_a$ (a Set-product) where B_a is a closed subset of X, for each $a \in A$. Let $B_{a'} \neq X$ for some $a' \in A$. Let $t \in X - B_{a'}$ and $\eta \in X^A$ such that $p_{a'}(\eta) = t$ and $p_a(\eta) = x$ for each $a \in A - \{a'\}$. It is evident that $\eta \notin g^{-1}[x]$. Denote by $X_{a'}$ the set $\{\xi \in X^A : p_a(\xi) = x$ for each $a \in A - \{a'\}\}$. $X_{a'}$ is a subspace of X^A which is isomorphic with X. Hence all non-constant $\mathscr{C}$-morphisms $X_{a'} \longrightarrow X$ are isomorphisms. Put $g_{a'} = g|X_{a'}$. Then $g_{a'}$ is non-constant. In fact, consider $\xi \in \Delta [X] \cap g^{-1}[x]$. Then $g_{a'}(\xi) = x \neq t = g_{a'}(\eta)$. Hence $g_{a'}$ is an isomorphism and it implies $B_{a'} = \{x\}$. Thus, we have shown that $B_a = X$ or $B_a = \{x\}$ for each $a \in A$. Now, for each $x \in X$ there exists $a \in A$ such that $p_a[g^{-1}[x]] = \{x\}$. It follows from the fact that g is non-constant. For each $a \in A$ put $Y_a = \{x \in X : p_a[g^{-1}[x]] = \{x\}\}$. If $Y_a \neq X$ for each $a \in A$, then the point $\xi \in X^A$ for which $p_a(\xi) \in X - Y_a$ for

each $a \in A$ belongs to $X^A - \bigcup_{x \in X} g^{-1}[x]$ - a contradiction. Thus, there exists $a_0 \in A$ such that $Y_{a_0} = X$. Let $Y_{a_1} \neq \emptyset$ for some $a_1 \in A - \{a_0\}$ and $x_0 \in Y_{a_1}$. Then the point $\eta \in X^A$ for which $p_{a_0}(\eta) = x_0$ and $p_{a_1}(\eta) \neq x_0$ belongs to $X^A - \bigcup_{x \in X} g^{-1}[x]$ - a contradiction. Hence, for each $a \in A -$ $- \{a_0\}$ $Y_a = \emptyset$. Consequently, $g = p_{a_0}$ and $f = \varphi \circ p_{a_0}$.

1.8. Definition [5]. Let (P, u) be a $\mathscr{C}$-space, $M \subset P$.

Define: 1^0 $u^1 M = uM$,

2^0 $u^{\alpha+1} M = u(u^{\alpha} M)$ for any ordinal $\alpha \geq 1$,

3^0 $u^{\lambda} M = \bigcup_{\xi < \lambda} u^{\xi} M$ for any limit ordinal $\lambda \neq 0$.

1.9. Proposition. Let $(P, u) = \prod_{a \in A} (P_a, u_a)$ where $\{(P_a, u_a)\}_{a \in A}$ is a non-empty family of $\mathscr{C}$-objects. Let M be a subset of P. Then $u^n M = \bigcap_{a \in A} p_a^{-1}[u_a^n p_a[M]]$ for each positive integer n.

Proof. First we need to prove that $p_a[u^n M] = u_a^n p_a[M]$ holds for any positive integer and each $a \in A$. The case $n = 1$ is evident. Let it be true for $n \geq 1$. Then $p_a[u^{n+1} M] =$ $= p_a[u(u^n M)] = u_a p_a[u^n M] = u_a^{n+1} p_a[M]$. Now, we can prove our proposition. The case $n = 1$ directly follows from the definition of a $\mathscr{C}$-product. If it is true for $n \geq 1$, then $u^{n+1} M =$ $= u(u^n M) = \bigcap_{a \in A} p_a^{-1}[u_a p_a[u^n M]] = \bigcap_{a \in A} p_a^{-1}[u_a^{n+1} p_a[M]]$.

1.10. Definition. A $\mathscr{C}$-space (P, u) is said to be compact provided that for any filter $\mathscr{F}$ on P $\bigcap_{F \in \mathscr{F}} uF \neq \emptyset$.

Evidently it holds:

1.11. Proposition. (P, u) is compact iff for any ultrafilter $\mathscr{U}$ on P $\bigcap_{U \in \mathscr{U}} uU \neq \emptyset$.

1.12. T h e o r e m. Let $\{(P_a, u_a)\}_{a \in A}$ be a set-indexed family of compact $\mathscr{C}$-spaces. Then $(P, u) = \prod_{a \in A} (P_a, u_a)$ is compact.

P r o o f. Let $\mathscr{F}$ be a filter on P. Then $\mathscr{F}_a = \{p_a[F] : F \in \mathscr{F}\}$ is a filter-base on P_a. Therefore $\bigcap_{F \in \mathscr{F}} u_a p_a [F] \neq \emptyset$. Choose $x_a \in \overline{u_a p_a [U]}$ for each $a \in A$ and let $x \in P$ be such that $p_a(x) = x_a$ for each $a \in A$. Because $x_a \in u_a p_a [F]$ for each $a \in A$ and $F \in \mathscr{F}$, we have $x \in p_a^{-1}[u_a p_a [F]]$ for each $a \in A$. Therefore $x \in uF$ for each $F \in \mathscr{F}$ and $\bigcap_{F \in \mathscr{F}} uF \neq \emptyset$.

1.13. D e f i n i t i o n. Let (P, u) be a $\mathscr{C}$-space. A subset M of P is said to be semi-closed provided that for each $x \in P - M$ there exist neighbourhoods O_1, O_2 such that $O_1 \cap O_2 \cap M = \emptyset$.

1.14. P r o p o s i t i o n. If M is a semi-closed subset of a compact $\mathscr{C}$-space (P, u), then M is compact (i.e. the sub-space M together with the induced $\mathscr{C}$-topology is a compact space).

P r o o f. Let M be a semi-closed subset of (P, u) and $\mathscr{U}$ be an ultrafilter on M. Let $\mathscr{F}$ be a filter on P for which $\mathscr{U} \subset \mathscr{F}$. Then $\bigcap_{F \in \mathscr{F}} uF \neq \emptyset$. Choose $x \in \bigcap_{F \in \mathscr{F}} uF$. Then $x \in \bigcap_{B \in \mathscr{U}} uB$. Let O_1, O_2 be neighbourhoods of the point x. Clearly, $O_i \cap B \neq \emptyset$ for each $B \in \mathscr{U}$ and $i = 1, 2$ and

therefore $O_1 \cap M$ and $O_2 \cap M$ belong to $\mathscr{U}$. But it implies that $O_1 \cap O_2 \cap M = (O_1 \cap M) \cap (O_2 \cap M) \neq \emptyset$. Hence, $x \in M$ i.e. $x \in \bigcap_{B \in \mathscr{U}} (uB \cap M)$ so that the subspace M of (P, u) is compact.

1.15. T h e o r e m. (M, m) is an extremal subobject of an object (P, u) in the category $\mathscr{C}_2$ iff m is an embedding and $m[M]$ is a semi-closed subset of (P, u).

P r o o f . Sufficiency. Suppose that (M, m) satisfies the given conditions. Let $(\{j_1, j_2\}, (P, u) \sqcup (P, u))$ be a $\mathscr{C}_2$-coproduct (it coincides with $\mathscr{C}$-coproduct). Define the equivalence-relation E on $(P, u) \sqcup (P, u)$ as follows:
$(x, y) \in E$ iff $x = y$ or there exists $a \in m[M]$ such that $j_1(a) = x$ and $j_2(a) = y$. Denote by (Q, v) a quotient space (in $\mathscr{C}$) $(P, u) \sqcup (P, u)/E$.
Let $p: (P, u) \sqcup (P, u) \longrightarrow (Q, v)$ be the corresponding extremal $\mathscr{C}$-epimorphism. It can be easily shown that $(Q, v) \in \mathrm{ob}\ \mathscr{C}_2$ because $m[M]$ is semi-closed. Clearly, $(M, m) =$ $= \mathrm{Eq}\,(p \cdot j_1, p \cdot j_2)$ in $\mathscr{C}_2$. Hence (M, m) is a $\mathscr{C}_2$-extremal subobject of (P, u).

Necessity. If m is not an embedding, then $m = m' \cdot e$ where $e : M \longrightarrow m[M]$ defined by $e(x) = m(x)$ for each $x \in M$ is a $\mathscr{C}_2$-epimorphism which is not an isomorphism and $m': m[M] \hookrightarrow (P, u)$ is an embedding - a contradiction. If $m[M]$ is not semi-closed, then there exists $p \in P - m[M]$ such that for any neighbourhoods O_1, O_2 of p $O_1 \cap O_2 \cap m[M] \neq \emptyset$. Consider $A = m[M] \cup \{p\}$ as a subspace of (P, u). Then $e: M \longrightarrow A$ defined by $e(x) = m(x)$ for each $x \in M$ is a $\mathscr{C}_2$-epimorphism which is not an isomorphism and $m = j \cdot e$ where $j: A \hookrightarrow (P, u)$ is an embedding - a contradiction.

1.16. D e f i n i t i o n . Let (P, u) be a $\mathscr{C}$-space, $M \subset P$. Define $M^* = M \cup \{x \in P - M : u\,(u\{x\} \cap M) = u\{x\}\}$.

It is evident that $M^* \subset uM$ for each $M \in 2^P$. Denote by $\mathscr{F}_0$ the intersection $\mathscr{F} \cap \mathscr{C}_0$. Then it holds:

1.17. P r o p o s i t i o n . An $\mathscr{F}_0$-morphism $f: (P, u) \longrightarrow (Q, v)$ is an $\mathscr{F}_0$-epimorphism iff $f[P]^* = Q$.

P r o o f . Necessity. Suppose f is an $\mathcal{F}_0$-epimorphism and $f[P]^* \neq Q$. Let $y \in Q - f[P]^*$. Then $vM_y \neq v\{y\}$ where $M_y = v\{y\} \cap f[P]$. Put $C_3 = (K, t)$ where $K = \{0, 1, 2\}$, $t\emptyset = \emptyset$, $t\{0\} = \{0\}$, $t\{1\} = t\{0,1\} = \{0,1\}$ and $tM = K$ for all other subsets of K. Clearly, $C_3 \in \text{ob}\ \mathcal{F}_0$. Define $\mathcal{F}_0$-morphisms $h, g: (Q, v) \longrightarrow C_3$ by $h[vM_y] \subset \{0\}$, $h[v\{y\} - vM_y] = \{1\}$, $h[Q - v\{y\}] \subset \{2\}$ and $g[v\{y\}] = \{0\}$, $g[Q - v\{y\}] \subset \{2\}$. Then $h \neq g$ but $h \cdot f = g \cdot f$ - a contradiction.

Sufficiency. Suppose $h, g: (Q, v) \longrightarrow (S, w)$ are $\mathcal{F}_0$-morphisms for which $h \cdot f = g \cdot f$ i.e. $h|f[P] = g|f[P]$. Let $y \in Q - f[P]$. Evidently, $h|M_y = g|M_y$ $(M_y = f[P] \cap v\{y\})$. Since M_y is dense in $v\{y\}$ it implies that $h[M_y] = g[M_y]$ is dense in $h[v\{y\}]$ and in $g[v\{y\}]$. Therefore $wh[v\{y\}] = wh[M_y] = wg[M_y] = wg[v\{y\}]$. Since $h[v\{y\}] \subset w\{h(y)\}$ and $h(y) \in h[v\{y\}]$, it implies $w\{h(y)\} = wh[v\{y\}]$ and analogously for g. Hence, $w\{h(y)\} = wh[v\{y\}] = wh[M_y] = wg[M_y] = wg[v\{y\}] = w\{g(y)\}$. Because $(S, w) \in \text{ob}\ \mathcal{F}_0$, $h(y) = g(y)$. Hence, $h = g$ and f is an $\mathcal{F}_0$-epimorhism.

1.18. C o r o l l a r y . $\mathcal{F}_0$ is co-well-powered.

P r o o f . Let $f: (P, u) \longrightarrow (Q, v)$ be an $\mathcal{F}_0$-epimorphism. The map $\varphi: Q \longrightarrow 2^P$; $y \longrightarrow f^{-1}[v\{y\}]$ is injective so that card $Q \leq$ card 2^P.

1.19. P r o p o s i t i o n . $((Q, v), m)$ is an $\mathcal{F}_0$-extremal subobject of the $\mathcal{F}_0$-object (P, u) iff m is an embedding and $m[Q]^* = m[Q]$.

P r o o f . Necessity. Immediate from 1.17.

Sufficiency. For each $y \in P - m[Q]^*$ we can define $\mathcal{F}_0$-morphisms $h_y, g_y: (P, u) \longrightarrow C_3$ such that $h_y(y) \neq g_y(y)$ and $h_y/m[Q] =$

$= g_y/m[Q]$ (see Proof of 1.17). Let $h, g: (P, u) \to \prod_{y \in P} C_3^y$ be $\mathcal{F}_0$-morphisms induced by $\{h_y\}_{y \in P - m[Q]^*}$ $\{g_y\}_{y \in P - m[Q]^*}$ respectively. Clearly, $((Q, v), m) = Eq(h, g)$ and therefore $((Q, v), m)$ is an $\mathcal{F}_0$-extremal subobject of (P, u).

2. Reflective subcategories of the category $\mathcal{C}$

2.1. <u>Convention.</u> Throughout the remainder of this paper all subcategories will be assumed to be both full and isomorphism-closed.

We shall use standard symbols for the following $\mathcal{C}$-spaces for which the set $\{0, 1\}$ is an underlying set:

D_2 - the discrete space

A_2 - the antidiscrete space

C_2 - the space for which $\{\bar{0}\} = \{0\}$ and $\{\bar{1}\} = \{0, 1\}$.

Recall (see e.g. [3]) that if $\mathcal{A}$, $\mathcal{A}'$ are (full) reflective subcategories of a category $\mathcal{B}, \mathcal{A} \subset \mathcal{A}'$ and $\mathcal{A}$ is monoreflective, then $\mathcal{A}'$ is also monoreflective.

2.2. <u>Examples.</u> a) The category $\mathcal{F}$ is a bireflective subcategory of $\mathcal{C}$. If $(P, u) \in \mathrm{ob}\,\mathcal{C}$ and v is the F-modification of u (see [5]), then $(i, (P, v))$ where i is the identity map $P \longrightarrow P$ is the F-reflection of (P, u).

b) The category $\mathcal{A}_1$ consisting of all antidiscrete spaces is bireflective in $\mathcal{C}$.

c) The categories $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ are epireflective subcategories of $\mathcal{C}$.

d) The subcategory $\mathcal{D}_2$ of the category $\mathcal{C}$ consisting of all $\mathcal{C}$-powers of the space D_2 is a reflective subcategory of $\mathcal{C}$

which is not epireflective (see 2.12).

e) The category $\mathcal{T}$ is not a reflective subcategory of $\mathcal{C}$. In fact, $D_2 \in \text{ob}\,\mathcal{T}$ but $D_2 \sqcap D_2 \notin \text{ob}\,\mathcal{T}$, because $\{(0, 0)\} \cup \{(1, 1)\} = \{(0, 0), (1, 1)\} \neq \{0, 1\} \times \{0, 1\} = \overline{\{(0, 0), (1, 1)\}}$.

Next we show that there exists an injective map of the class of all cardinals into the conglomerate (the concept used in [3]) $\mathcal{R}$ of all reflective subcategories of $\mathcal{C}$.

Let ξ be a cardinal. Denote by $\mathcal{F}^{\xi}$ the subcategory of consisting of all $\mathcal{C}$-spaces (P, u) which satisfy the following condition: if $M \subset P$ for which card $M \leqq \xi$, then $u^2M = uM$.

2.3. Theorem. 1) For any cardinal ξ, $\mathcal{F}^{\xi}$ is a bireflective subcategory of the category $\mathcal{C}$.

2) For any cardinals ξ, ζ such that $\xi < \zeta$, $\mathcal{F}^{\xi} \supsetneqq \mathcal{F}^{\zeta}$.

Proof. 1. Let $(P, u) = \prod_{a \in A} (P_a, u_a)$ where $\{(P_a, u_a)\}_{a \in A}$ is a family of $\mathcal{F}^{\xi}$-objects. If $M \subset P$ such that card $M \leqq \xi$, then card $p_a[M] \leqq \xi$ for each $a \in A$. Therefore $u_a^2 p_a[M] = u_a p_a[M]$ for each $a \in A$. According to 1.9. it implies $u^2M = uM$. Hence, $\mathcal{F}^{\xi}$ is closed under the formation of $\mathcal{C}$-products. It is easy to verify that $\mathcal{F}^{\xi}$ is also closed under the formation of subspaces. Therefore $\mathcal{F}^{\xi}$ is epireflective in $\mathcal{C}$. Since $\mathcal{A}_1 \subset \mathcal{F}^{\xi}$ and $\mathcal{A}_1$ is bireflective (2.2. b)), $\mathcal{F}^{\xi}$ is bireflective.

2. Clearly, $\mathcal{F}^{\xi} \supset \mathcal{F}^{\zeta}$. Let P be a set for which card $P > \xi + 1$. Let $Q \subset P$ such that card $Q = \zeta$. Define $u: 2^P \longrightarrow 2^P$ as follows: If card $M \leqq \xi$, then $uM = M$. If card $M > \xi$, then $uM = Q$ whenever $M \subset Q$ and $M \neq Q$ and $uM = P$ whenever $M \not\subset Q$. Choose a fixed point $q \in P - Q$ and put $uQ = Q \cup \{q\}$. Clearly, (P, u) is a $\mathcal{C}$-space such that

$(P, u) \in \operatorname{ob} \mathcal{F}^{\xi}$. Since card $Q = \xi$ and $u^2Q \neq uQ$, $(P, u) \notin \operatorname{ob} \mathcal{F}^{\xi}$ i.e. $\mathcal{F}^{\xi} \neq \mathcal{F}^{\xi}$.

2.4. Proposition. The subcategory $\mathcal{K}_2$ of the category $\mathcal{C}$ consisting of all compact $\mathcal{C}_2$-spaces is reflective in $\mathcal{C}$.

Proof. $\mathcal{K}_2 \subset \mathcal{C}_2$. $\mathcal{K}_2$ is closed under the formation of products and extremal subobjects in $\mathcal{C}_2$ according to 1.12., 1.14. and 1.15. Therefore $\mathcal{K}_2$ is epireflective in $\mathcal{C}_2$. Since $\mathcal{C}_2$ is epireflective in $\mathcal{C}$, $\mathcal{K}_2$ is reflective in $\mathcal{C}$.

2.5. Example. Let α be an infinite cardinal. Denote by $\mathcal{K}^{\alpha}$ the subcategory of $\mathcal{C}$ consisting of all $\mathcal{C}$-spaces (P, u) with the following property: If $\mathcal{U}$ is an ultrafilter on P such that $\tilde{\mathcal{U}} = \{uB: B \in \mathcal{U}\}$ has an α-intersection property, then $\bigcap_{B \in \mathcal{U}} uB \neq \emptyset$. For any infinite cardinal α, $\mathcal{K}_2^{\alpha} = \mathcal{K}^{\alpha} \cap \mathcal{C}_2$ is a reflective subcategory of $\mathcal{C}$.

The conglomerate $\mathcal{R}$ of all reflective subcategories of the category $\mathcal{C}$ is partially ordered by the inclusion-relation and has the smallest element - the subcategory $\mathcal{A}_{-1}$ of $\mathcal{C}$ consisting of all singletons. The subcategory $\mathcal{A}_0$ of $\mathcal{C}$ consisting of all $\mathcal{C}$-spaces (P, u) for which card $P \leq 1$ is the smallest element in $\mathcal{R} - \{\mathcal{A}_{-1}\}$. Denote by $\mathcal{R}^*$ the collection $\mathcal{R} - \{\mathcal{A}_{-1}, \mathcal{A}_0\}$. Our next study of reflective subcategories in $\mathcal{C}$ is devoted to the minimal elements of $\mathcal{R}^*$.

2.6. Proposition. The category $\mathcal{A}_1$ of all anti-discrete spaces is a minimal element of $\mathcal{R}^*$.

Proof. Let $\mathcal{A} \in \mathcal{R}^*$ such that $\mathcal{A} \subset \mathcal{A}_1$. Let $(P, u) \in \operatorname{ob} \mathcal{A}$ for which card $P > 1$. Then there exists an extremal $\mathcal{C}$-monomorphism $r: A_2 \longrightarrow (P, u)$. Moreover, it is evident that r is a section and therefore $A_2 \in$ ob $\mathcal{A}$. Since A_2 is an extremal coseparator of $\mathcal{A}_1$ and each subspace of any $\mathcal{C}$-power A_2^I is a sect, it holds $\mathcal{A}_1 \subset \mathcal{A}$.

Next we show that the space C_2 has a reflective hul $\mathcal{A}_2$ in $\mathcal{C}$ which is a minimal element of $\mathcal{R}^*$. To show this, we shall use the following proposition:

2.7. Proposition [3]. Let $\mathcal{B}$ be a complete well-powered and co-well-powered category and $\mathcal{A}$ be a subcategory of $\mathcal{B}$. Let $\mathcal{E}(\mathcal{A})$ be an epireflective hull of $\mathcal{A}$ in $\mathcal{B}$. If the category $\mathcal{E}(\mathcal{A})$ is co-well-powered, then the epireflective hull of the category $\mathcal{A}$ in $\mathcal{E}(\mathcal{A})$ is the reflective hull of $\mathcal{A}$ in $\mathcal{B}$.

The category $\mathcal{F}_0$ is the epireflective hull of the space C_2 in $\mathcal{C}$. In fact, let $(P, u) \in$ ob $\mathcal{F}_0$ and $\mathcal{S}$ be the family of all closed subsets of P. For each $M \in \mathcal{S}$ define $\mathcal{C}$-morphism $f_M: (P, u) \longrightarrow C_2$ by $f_M[M] \subset \{0\}$ and $f_M[P - M] \subset \{1\}$. Let $m: (P, u) \longrightarrow C_2^{\mathcal{S}}$ be the $\mathcal{C}$-morphism defined by the family $\{f_M\}_{M \in \mathcal{S}}$. It is easy to show that m is an extremal $\mathcal{C}$-monomorphism and therefore $\mathcal{E}(C_2) = \mathcal{F}_0$. Since $\mathcal{F}_0$ is co-well-powered (see 1.18) the epireflective hull $\mathcal{A}_2$ of C_2 in $\mathcal{F}_0$ is the reflective hull of C_2 in $\mathcal{C}$.

2.8. Proposition. 1. If $(P, u) \in$ ob $\mathcal{A}_2$, $P \neq \emptyset$, then there exists $p \in P$ such that $u\{p\} = P$.

2. If $(P, u) \in$ ob $\mathcal{A}_2$ such that card $P \geq 2$, then (P, u) has a subspace isomorphic with C_2.

Proof. To prove 1, we need to show that for any $M \subset C_2^I$ such that $M^* = M$ there exists $x \in M$ for which $\{\bar{x}\} \supset M$. Since $C_2^I \in$ ob $\mathcal{F}_0$, M is closed in C_2^I. For each $i \in I$ put

$x_i = \max p_i[M]$. Let $x \in C_2^I$ such that $p_i(x) = x_i$ for each $i \in I$. Evidently, $\{\bar{x}\} = \bar{M}$. Thus, $\{\bar{x}\} \cap M = M$ and therefore $\overline{\{\bar{x}\} \cap M} = \bar{M} = \{\bar{x}\}$. Hence $x \in M^*$. Since $M^* = M$, x belongs to M.

2. Let $p \in P$ such that $u\{p\} = P$. Let $q \in P - \{p\}$. Since $(P, u) \in \operatorname{ob} \mathcal{S}_0$, it is evident that $\{p, q\} \cong C_2$.

<u>2.9. Theorem.</u> $\mathcal{A}_2$ is a minimal element of $\mathcal{R}^*$.

<u>Proof.</u> Let $\mathcal{A} \in \mathcal{R}^*$ and $\mathcal{A} \subset \mathcal{A}_2$. If $(P, u) \in \operatorname{ob} \mathcal{A} - \operatorname{ob} \mathcal{A}_0$, then there is an extremal monomorphism $m: C_2 \longrightarrow (P, u)$ which is evidently a section. Hence $C_2 \in \operatorname{ob} \mathcal{A}$ and therefore $\mathcal{A}_2 \subset \mathcal{A}$.

<u>2.10. Definition.</u> a) A $\mathcal{C}$-space is said to be rigid provided that $\hom_{\mathcal{C}}(X, X)$ contains only constant morphisms and identity morphism.

b) A $\mathcal{C}$-space X is said to be quasi-rigid provided that any non-constant $\mathcal{C}$-morphism $f: X \longrightarrow X$ is an isomorphism and for any two different morphisms $f, g \in \hom_{\mathcal{C}}(X, X)$ $\operatorname{card}\{x \in X: f(x) = g(x)\} \leq 1$.

<u>2.11. Theorem.</u> Let X be a quasi-rigid $\mathcal{C}_1$-object. Then the subcategory $\mathcal{A}_X$ of $\mathcal{C}$ consisting of $\emptyset$ and all $\mathcal{C}$-powers of the space X is a reflective subcategory of $\mathcal{C}$.

<u>Proof.</u> Let Y be a non-empty $\mathcal{C}$-object and $\tilde{C}(Y, X)$ be the set of all non-constant $\mathcal{C}$-morphisms $Y \longrightarrow X$. Let E be the equivalence-relation on $\tilde{C}(Y, X)$ defined as follows: $(f, g) \in E$ iff there exists an isomorphism $\varphi: X \longrightarrow X$ such that $f = \varphi \circ g$. Denote by C_Y the chosen representative set of the decomposition $\tilde{C}(Y, X)/E$. Then C_Y defines exactly one $\mathcal{C}$-morphism $r: Y \longrightarrow X^{C_Y}$ such that $p_f \circ r = f$ for each $f \in C_Y$. It can be proved that

(r, X^{C_Y}) is the $\mathcal{A}_X$-reflection of Y. To show it, we need to prove that for any $\mathcal{C}$-morphism $h: Y \longrightarrow X^A$ (X^A is an arbitrary non-empty $\mathcal{A}_X$-object) there exists exactly one $\mathcal{C}$-morphism $t: X^{C_Y} \longrightarrow X^A$ such that $h = t \circ r$.

1. Existence. Let $a \in A$ and $p_a: X^A \longrightarrow X$ be the corresponding projection. Let $p_a \circ h = f_a$ be non-constant. Then there exists an isomorphism $\varphi : X \longrightarrow X$ such that $\varphi \circ f_a \in C_Y$. Since (by the definition of r) $p_{\varphi \circ f_a} \circ r = \varphi \circ f_a$, $f_a = \varphi^{-1} \circ p_{\varphi \circ f_a} \circ r$. Let $a \in A$. If f_a is non-constant, then put $t_a = \varphi^{-1} \circ p_{\varphi \circ f_a}$ where φ is above defined isomorphism. If f_a is constant, then t_a is the constant $\mathcal{C}$-morphism $X^{C_Y} \longrightarrow X^A$ for which $t_a[X^{C_Y}] = f_a[Y]$. Now, let $t : X^{C_Y} \longrightarrow X^A$ be the unique $\mathcal{C}$-morphism defined by the family $\{t_a\}_{a \in A}$. Evidently, $t \circ r = h$.

2. Uniqueness. Let $t, t': X^{C_Y} \longrightarrow X^A$ be $\mathcal{C}$-morphisms for which $t \circ r = t' \circ r = h$. Put $p_a \circ t = m_a$, $p_a \circ t' = n_a$. Then $m_a \circ r = p_a \circ t \circ r = p_a \circ t' \circ r = n_a \circ r$. If m_a is non-constant, then by 1.7. there exists an isomorphism $\varphi : X \longrightarrow X$ such that $m_a = \varphi \circ p_f$ where $f \in C_Y$ so that $p_f = \varphi^{-1} \circ m_a$. Since f is non-constant, there exist some points x, y $\in$ Y such that $f(x) \neq f(y)$. Then $n_a(r(x)) = m_a \circ r(x) = \varphi \circ p_f \circ r(x) = \varphi \circ f(x) \neq \varphi \circ f(y) = \varphi \circ p_f \circ r(y) = m_a \circ r(y) = n_a(r(y))$; hence n_a is non-constant. Clearly, the converse is also true so that m_a is non-constant iff n_a is non-constant for each $a \in A$. Now, suppose m_a, n_a to be non-constant. Then $m_a = \varphi \circ p_f$, $n_a = \psi \circ p_g$ where $f, g \in C_Y$ and φ, ψ are suitable isomorphisms $X \longrightarrow X$. Since $\varphi \circ f = \varphi \circ p_f \circ r = m_a \circ r = n_a \circ r = \psi \circ p_g \circ r = \psi \circ g$, we have $f = (\varphi^{-1} \circ \psi) \circ g$. Then $(f, g) \in E$ and since each of f, g belongs to C_Y, $f = g$.
Hence $m_a = \varphi \circ p_f$, $n_a = \psi \circ p_f$. The equation $m_a | r[Y] = n_a | r[Y]$ implies $\varphi | p_f \circ r[Y] = \psi | p_f \circ r[Y]$. Since $p_f \circ r[Y] = f[Y]$

and card $f[Y] > 1$, we have $\varphi = \psi$ so that $m_a = n_a$. If m_a, n_a are constant, then $m_a \mid r[Y] = n_a \mid r[Y]$ implies $m_a = n_a$. Therefore for each $a \in A$ $p_a \circ t = m_a = n_a = p_a \circ t'$; hence $t = t'$.

R e m a r k . If a subcategory $\mathcal{a}_X$ of $\mathcal{C}$ consisting of all $\mathcal{C}$-powers of some $\mathcal{C}$-object X is reflective in $\mathcal{C}$, then it is a minimal element of $\mathcal{R} - \{\mathcal{a}_0\}$. Indeed, the diagonal morphism $\Delta : X \longrightarrow X^A$ is a section for any non-empty set A.

2.12. C o r o l l a r y . The subcategory $\mathcal{D}_2$ of $\mathcal{C}$ consisting of all $\mathcal{C}$-powers of D_2 is minimal in $\mathcal{R}^*$.

2.13. E x a m p l e s of homogeneous quasi-rigid $\mathcal{C}_1$-spaces.

a) $D_3 = (\{0, 1, 2\}, -)$ where $\overline{M} = M$ whenever card $M \leqq 1$ and $\overline{M} = \{0, 1, 2\}$ whenever card $M > 1$.

b) Let Z_p be the group of integers modulo p where $p \geqq 5$ and p is odd. Let $M \subset Z_p$. Then $\overline{M} = M$ whenever card $M \leqq 1$ and $\overline{M} = Z_p$ whenever card $M \geqq 3$. For each $k \in Z_p$ put $\overline{\{k, k+1\}} = \{k, k+1, k+2, k+3\}$. If $m, n \in Z_p$ such that $m \neq n+1$ and $n \neq m+1$, then $\overline{\{m, n\}} = \{m, n, \frac{1}{2}(m+n)\}$. (Since p is odd, any equation $x + x = k$ in Z_p has exactly one solution denoted by $\frac{1}{2}k$.) $(Z_p, -)$ satisfies the required conditions.

c) Let P be the set of all integers or real numbers. Let $M \subset P$. Put $\overline{M} = M$ if card $M \leqq 1$ and $\overline{M} = P$ if card $M \geqq 3$. For each $r \in P$ $\overline{\{r, r+1\}} = \{t \in P : t \leqq r+1\}$. If $r, s \in P$ such that $r < s$ and $s - r > 1$, then $\overline{\{r, s\}} = \{r-1, r, s\}$. The space P satisfies the required conditions.

2.14. C o r o l l a r y . If X is a rigid $\mathcal{C}_1$-space, then $\mathcal{a}_X$ is minimal in $\mathcal{R}^*$.

Next we construct an injective map of the class of all ordinals into the conglomerate of minimal elements of $\mathcal{R}^*$. Let ω_α be an initial ordinal. Define $\mathcal{C}$-topology on the set $W(\omega_\alpha + 1)$ as follows:

1. If $M \subset W(\omega_\alpha + 1)$ and card $M \leqq 1$, then $\overline{M} = M$.

2. If $M \subset W(\omega_\alpha)$ and card $M > 1$, then $\overline{M} = W(\eta + 1)$ where η is the smallest ordinal for which $M \subset W(\eta)$.

3. Let $0 < \xi < \omega_\alpha$. If $\xi = \zeta + 1$, then $\overline{\{\xi, \omega_\alpha\}} = \{\zeta, \xi, \omega_\alpha\}$. If ξ is a limit ordinal, then $\overline{\{\xi, \omega_\alpha\}} = (W(\xi + \omega_0) - W(\xi)) \cup \{\omega_\alpha\}$.

4. For all other $M \subset W(\omega_\alpha + 1)$ put $\overline{M} = W(\omega_\alpha + 1)$.

Denote by S_{ω_α} the just defined $\mathcal{C}$-space. Clearly, $S_{\omega_\alpha} \in$ ob $\mathcal{C}_1$.

<u>2.15. Theorem.</u> For any initial ordinal ω_α, S_{ω_α} is a rigid $\mathcal{C}_1$-space. If $\alpha \neq \beta$, then S_{ω_α} and S_{ω_β} are not isomorphic so that $a_{S_{\omega_\alpha}} \neq a_{S_{\omega_\beta}}$.

<u>Proof.</u> Evidently, by the definition of S_{ω_α} a subset M of $W(\omega_\alpha + 1)$ is closed iff $M \in \{\emptyset, W(\omega_\alpha + 1)\}$ or card $M = 1$. This implies that any non-constant $\mathcal{C}$-morphism $f: S_{\omega_\alpha} \longrightarrow S_{\omega_\alpha}$ is a monomorphism. Next we prove that f is necessarily $1_{S_{\omega_\alpha}}$. Clearly, $f[\overline{\{0, \omega_\alpha\}}] \subset \overline{\{f(0), f(\omega_\alpha)\}}$. If $\{f(0), f(\omega_\alpha)\} \neq \{0, \omega_\alpha\}$, then card $\overline{\{f(0), f(\omega_\alpha)\}} <$ $<$ card $f[\overline{\{0, \omega_\alpha\}}]$ = card $\overline{\{0, \omega_\alpha\}}$ = card $W(\omega_\alpha + 1)$. Therefore $\{f(0), f(\omega_\alpha)\} = \{0, \omega_\alpha\}$. It implies 1. $f(0) = 0$, $f(\omega_\alpha) = \omega_\alpha$ or 2. $f(0) = \omega_\alpha$, $f(\omega_\alpha) = 0$. The second case is impossible. In fact, if $f(\omega_\alpha) = 0$, then $f[\{1, 2, \omega_\alpha\}] = \{f(1), f(2), 0\}$. Clearly, $f(1), f(2) \notin \{0, \omega_\alpha\}$ (because f is a monomorphism). Then $\overline{\{f(1), f(2), 0\}} = W(\max\{f(1), f(2)\} + 2)$. Therefore card $\overline{\{1, 2, \omega_\alpha\}}$ = card $f[\overline{\{1, 2, \omega_\alpha\}}] >$

$>$ card $\{\overline{f(1), f(2), f(\omega_\alpha)}\}$ - a contradiction. Hence the case 1 is true.

Let $\omega_\alpha > \eta > 0$ and suppose that for each $\xi \in W(\eta)$ $f(\xi) = \xi$.

A. Let $f(\eta) < \eta$. Then $f(f(\eta)) = f(\eta)$ and this yields $f(\eta) = \eta$ - a contradiction.

B. Let $f(\eta) > \eta$. Clearly, $f(\eta) < \omega_\alpha$.

a) If $\eta = \xi + 1$, then $\xi < \eta$ so that $f[\overline{\{\eta, \omega_\alpha\}}] =$ $= f[\{\xi, \eta, \omega_\alpha\}] = \{\xi, f(\eta), \omega_\alpha\} \subset \overline{\{f(\eta), \omega_\alpha\}}$. If $f(\eta)$ is a limit ordinal, then $\overline{\{f(\eta), \omega_\alpha\}} =$ $= (W(f(\eta) + \omega_0) - W(f(\eta)) \cup \{\omega_\alpha\}$ so that $\xi \notin \overline{\{f(\eta), \omega_\alpha\}}$ what is impossible. Therefore $f(\eta)$ is not a limit ordinal and then $\overline{\{f(\eta), \omega_\alpha\}} = \{f(\eta) - 1, f(\eta), \omega_\alpha\}$. Because $f(\eta) \neq$ $\neq \xi < \eta$, we get $f(\eta) = \eta$ - a contradiction. Hence, if $\eta =$ $= \xi + 1$, then $f(\eta) = \eta$.

b) Finally, let η be a limit ordinal. Then $f(\eta)$ is also a limit ordinal. In fact, otherwise we have card $\{\overline{f(\eta), \omega_\alpha}\} =$ $= 3 <$ card $f[\overline{\{\eta, \omega_\alpha\}}] = \aleph_0$ what is impossible. Since $\eta < f(\eta)$, there exists an ordinal ϑ for which $\eta < \vartheta < f(\eta)$. The set $O_{f(\eta)} = W(\omega_\alpha + 1) - W(\vartheta)$ is a neighbourhood of $f(\eta)$ because $f(\eta) \notin \overline{W(\vartheta)} = W(\vartheta + 1)$. Clearly, $\eta < \xi$ for each $\xi \in O_{f(\eta)}$. Since f is continuous, there exists a neighbourhood O_η of η such that $f[O_\eta] \subset O_{f(\eta)}$ (see [5]). But by the definition of S_{ω_α} any neighbourhood of η intersects $W(\eta)$. Now, if $\xi \in O_\eta \cap W(\eta)$, then $f(\xi) = \xi < \eta < \vartheta$ so that $f(\xi) \notin O_{f(\eta)}$ - a contradiction. Hence, $f(\eta) = \eta$. So we have proved that $f/W(\omega_\alpha)$ is the identity. Because $f(\omega_\alpha) = \omega_\alpha$ (see above) it holds $f = 1_{S_{\omega_\alpha}}$. If $\alpha \neq \beta$, then card $W(\omega_\alpha + 1) \neq$ card $W(\omega_\beta + 1)$. Therefore S_{ω_α} and S_{ω_β} cannot be isomorphic.

2.16. Corollary. The map $\alpha \longmapsto \mathcal{A}_{S_{\omega_\alpha}}$ is an injective map of the class of all ordinals into the conglomerate of minimal elements in $\mathcal{R}^*$.

2.17. Examples. 1. Let $A_n = \{0, 1, \dots, n\}$, $n > 3$. Put $\overline{M} = M$ whenever $M \subset A_n$ and card $M \leq 1$, $\overline{\{0, 1\}} = \{0, 1, n\}$. If $\{k, t\} \neq \{0, 1\}$ and $k < t < n$, then $\overline{\{k, t\}} = \{0, \dots, t\}$. If $0 < k < n$, then $\overline{\{k, n\}} = \{k - 1, k, n\}$. For all other $M \subset A_n$ put $\overline{M} = A_n$. It can be easily verified that A_n with the just defined $\mathcal{C}$-topology is a rigid $\mathcal{C}_1$-space.

2. There exist rigid compact Hausdorff $\mathcal{T}$-spaces with more than one point (see [2]).

3. There exist rigid local connected subspaces of R^2 (with the usual topology) containing more than one point (see [2]).

The situation in the conglomerate $\mathcal{E}$ of all epireflective subcategories of $\mathcal{C}$ is simpler (analogous as in $\mathcal{T}$).

2.18. Theorem. The conglomerate $\mathcal{E} - \{\mathcal{A}_0\}$ has exactly two minimal elements - the category $\mathcal{A}_1$ (antidiscrete spaces) and the subcategory $\mathcal{D}$ of $\mathcal{C}$ which is the epireflective hull of the space D_2 in $\mathcal{C}$.

Proof. Let $\mathcal{A} \in \mathcal{E} - \{\mathcal{A}_0\}$ and (r, R) be the $\mathcal{A}$-reflection of the space D_2. Then R is isomorphic with A_2 and then $\mathcal{A}_1 \subset \mathcal{A}$ or R is isomorphic with D_2 and then $\mathcal{D} \subset \mathcal{A}$. The case R is isomorphic with C_2 is impossible and if card $R = 1$, then $\mathcal{A} = \mathcal{A}_0$.

The next step in our study is to show that in $\mathcal{R} - \{\mathcal{C}\}$ there is the greatest element. First we prove that $\mathcal{C}$ has an extremal coseparator.

Define a $\mathcal{C}$-topology on $\{0, 1, 2\}$ by $\overline{\emptyset} = \emptyset$, $\overline{\{0\}} = \{0, 1\}$

and $\overline{M} = \{0, 1, 2\}$ for all other $M \subset \{0, 1, 2\}$ and denote this space by C.

2.19. T h e o r e m . The space C is an extremal coseparator of the category $\mathscr{C}$.

P r o o f . Let $(P, u) \in$ ob $\mathscr{C}$. For each $M \in 2^P$ define $f_M: (P, u) \longrightarrow C$ by $f_M[M] \subset \{0\}$, $f_M[uM - M] \subset \{1\}$ and $f_M[P - uM] \subset \{2\}$. Let $m: (P, u) \longrightarrow C^{2^P}$ be the (unique) $\mathscr{C}$-morphism defined by the family $\{f_M\}_{M \in 2^P}$. Clearly, m is a $\mathscr{C}$-monomorphism. To show that m is extremal $\mathscr{C}$-monomorphism, we need to prove that $m[uM] = \overline{m[M]} \cap m[P]$ for each $M \in 2^P$. Evidently, $m[uM] \subset \overline{m[M]} \cap m[P] \subset \overset{-1}{p_M} [\{0, 1\}] \cap m[P]$ for each $M \in 2^P$. Now, let $y \in p_M^{-1}[\{0, 1\}] \cap m[P]$. Then $p_M(y) \in \in \{0, 1\}$ and there exists $x \in P$ such that $m(x) = y$. But since $p_M(y) = p_M(m(x)) = f_M(x) \in \{0, 1\}$, then x belongs to uM. Hence $y \in m[uM]$ so that $p_M^{-1}[\{0, 1\}] \cap m[P] \subset m[uM]$. Consequently, $m[uM] = \overline{m[M]} \cap m[P]$ for each $M \in 2^P$.

R e m a r k . Moreover $\mathscr{C}$ is the reflective hull of C (see 2.7).

2.20. P r o p o s i t i o n . Let $(P, u) = \prod_{a \in A} (P_a, u_a)$. Let K be a subspace of (P, u) isomorphic with C. Then there exists $a_0 \in A$ such that $K_{a_0} = p_{a_0}[K]$ is isomorphic with C.

P r o o f . Let $\{x, y, z\} = K$ be a subspace of (P, u) isomorphic with C. Let $u\{x\} \cap K = \{x, y\}$ and $u\{y\} \cap K = u\{z\} \cap K = K$. Then $u_a\{x_a\} \supset \{x_a, y_a\}$ and $u_a\{y_a\} \cap K_a = u_a\{z_a\} \cap K_a = K_a$ for each $a \in A$ where $K_a = p_a[K]$. If card $K_a \leq 2$ for some $a \in A$, then K_a is isomorphic with A_2 or card $K_a \leq 1$. Indeed, if $x_a = y_a \neq z_a$ and if we put $t_a = x_a = y_a$ then $u_a\{t_a\} \cap K_a = \{t_a, z_a\} = u_a\{z_a\} \cap K_a$. The other cases for card $K_a = 2$ can be shown analogously. Since

$z \notin u\{x\}$, there exists $a_0 \in A$ such that $z_{a_0} \notin u_{a_0}\{x_{a_0}\}$. But then card $K_{a_0} = 3$ and K_{a_0} is isomorphic with C.

2.21. Corollary. A $\mathcal{C}$-space (P, u) is an extremal coseparator of $\mathcal{C}$ iff there exists an extremal $\mathcal{C}$-monomorphism $m: C \longrightarrow (P, u)$.

2.22. Theorem. The subcategory $\mathcal{B}$ of $\mathcal{C}$ for which ob $\mathcal{B} = \{(P, u) \in$ ob $\mathcal{C}$: (P, u) is not an extremal coseparator of $\mathcal{C}\}$ is the greatest element of $\mathcal{R} - \{\mathcal{C}\}$.

Proof. $\mathcal{B}$ is closed under the formation of $\mathcal{C}$-products by 2.20. and it is evident that $\mathcal{B}$ is closed under subspaces. Hence, $\mathcal{B} \in \mathcal{R}$. Moreover $\mathcal{B} \in \mathcal{R} - \{\mathcal{C}\}$ because $\mathcal{B} \neq \mathcal{C}$. Let $\mathcal{A} \in \mathcal{R}$ such that $\mathcal{B} \not\supset \mathcal{A}$ and $(P, u) \in$ ob $\mathcal{A}$ - ob $\mathcal{B}$. Then there exists an extremal $\mathcal{C}$-monomorphism $m : C \longrightarrow (P, u)$ which is moreover evidently a section (the map $f: (P, u) \longrightarrow C$ defined by $f(m(0)) = 0$, $f[u\{m(0)\} - \{m(0)\}] = \{1\}$ and $f[P - u\{m(0)\}] =$ $= \{2\}$ is a left inverse $\mathcal{C}$-morphism of m). Therefore $C \in$ ob $\mathcal{A}$. But reflective hull of C in $\mathcal{C}$ is $\mathcal{C}$; hence $\mathcal{A} = \mathcal{C}$.

As we have shown (2.2.e) (the category $\mathcal{T}$ is not reflective in $\mathcal{C}$. It holds:

2.23. Proposition. The category $\mathcal{F}$ is the reflective hull of $\mathcal{T}$ in $\mathcal{C}$.

Proof. Because $\mathcal{a}_1 \subset \mathcal{T}$, any reflective subcategory containing $\mathcal{T}$ is bireflective. Therefore the reflective hull of $\mathcal{T}$ coincides with the epireflective one. Consider the space C_3 defined on the set $\{0, 1, 2\}$ by $\bar{\emptyset} = \emptyset$, $\overline{\{0\}} = \{0\}$ and $\bar{M} =$ $= \{0, 1, 2\}$ otherwise. Clearly $\mathcal{S}(C_3) = \mathcal{F}$ and $C_3 \in$ ob $\mathcal{T}$. (Let $(P, u) \in$ ob $\mathcal{F}$. For $M \in 2^P$ define $f_M: (P, u) \longrightarrow C_3$ by $f_M[M] \subset \{0\}$ and $f_M[P - M] \subset \{2\}$ if M is closed and $f_M[M] = \{1\}$

and $f_M[P - M] = \{2\}$ if $uM \neq M$. Then $m : (P, u) \longrightarrow C_3^{2^P}$ defined by the family $\{f_M\}_{M \in 2^P}$ is an extremal $\mathscr{C}$-monomorphism.) Because $C_3 \in \mathrm{ob}\, \mathscr{T}$, $\mathscr{F} = \mathscr{E}(C_3) \subset \mathscr{E}(\mathscr{T})$. But $\mathscr{T} \subset \mathscr{F}$ implies $\mathscr{E}(\mathscr{T}) \subset \mathscr{F}$. Hence, $\mathscr{E}(\mathscr{T}) = \mathscr{F}$.

3. Coreflective subcategories of the category $\mathscr{C}$

Recall that all subcategories are assumed to be both full and isomorphism-closed.

By [2] - 13.1.2. and 1.4. it holds:

3.1. Proposition. For a subcategory $\mathcal{A}$ of $\mathscr{C}$ for which $\mathcal{A} \neq \{\emptyset\}$ the following are equivalent:

a) $\mathcal{A}$ is coreflective

b) $\mathcal{A}$ is bicoreflective

c) $\mathcal{A}$ is closed under the formation of $\mathscr{C}$-coproducts and $\mathscr{C}$-extremal quotients.

This implies that the intersection of any collection of coreflective subcategories of $\mathscr{C}$ is a coreflective subcategory of $\mathscr{C}$ and therefore the conglomerate $\mathscr{L}$ of all coreflective subcategories of $\mathscr{C}$ together with the inclusion-relation is a complete (large) lattice. The smallest element in $(\mathscr{L}, \subset)$ is evidently $\mathscr{B}_0 = \{\emptyset\}$. Analogously as in $\mathscr{T}$ it can be shown that the category $\mathscr{B}_1$ of all discrete spaces is the smallest element in $\mathscr{L} - \{\mathscr{B}_0\}$ and the category $\mathscr{B}_2$ of all $\mathscr{C}$-coproducts of all antidiscrete spaces is the smallest element in $\mathscr{L} - \{\mathscr{B}_0, \mathscr{B}_1\}$.

Let $\mathcal{A}$ be a subcategory of $\mathscr{C}$ which is not necessarily full and isomorphism-closed. Then $\mathcal{C}(\mathcal{A})$ denotes the coreflective hull of $\mathcal{A}$ i.e. the intersection of all $\mathscr{B} \in \mathscr{L}$ such that $\mathcal{A} \subset \mathscr{B}$. Clearly, ob $\mathcal{C}(\mathcal{A})$ is the class of all $\mathscr{C}$-extremal quotients of all $\mathscr{C}$-coproducts of $\mathcal{A}$-objects.

3.2. **Examples.** a) The category $\mathfrak{A}$ of all A-spaces is a coreflective subcategory of $\mathcal{C}$.

b) Let ξ be an infinite cardinal. Denote by $\mathfrak{A}_\xi$ the subcategory of $\mathcal{C}$ for which $(P, u) \in \mathrm{ob}\, \mathfrak{A}_\xi$ iff for any family $\{B_a\}_{a\in A}$ of subsets of P such that $\mathrm{card}\, A \leqq \xi$ $u(\bigcup_{a\in A} B_a) = \bigcup_{a\in A} uB_a$. Then $\mathfrak{A}_\xi$ is coreflective in $\mathcal{C}$ and for any infinite cardinals ξ, ζ $\xi \neq \zeta$ implies $\mathfrak{A}_\xi \neq \mathfrak{A}_\zeta$. It can be easily verified that $\mathfrak{A}_\xi$ is closed under the formation of $\mathcal{C}$-coproducts and $\mathcal{C}$-extremal quotients.

c) The subcategory $\widetilde{\mathfrak{A}}$ of $\mathcal{C}$ for which $(P, u) \in \mathrm{ob}\, \widetilde{\mathfrak{A}}$ iff for any two closed subsets $M, N \subset P$ $M \cup N$ is closed is a coreflective subcategory of $\mathcal{C}$.

d) Let ξ be an infinite cardinal. Then the subcategory $\mathfrak{A}_\xi$ of $\mathcal{C}$ consisting of all $(P, u) \in \mathcal{C}$ for which the union of any family $\mathcal{S}$ of closed subsets such that $\mathrm{card}\, \mathcal{S} \leqq \xi$ is closed in (P, u) is coreflective in $\mathcal{C}$ and for $\xi \neq \zeta$ (infinite cardinals) $\mathfrak{A}_\xi \neq \mathfrak{A}_\zeta$.

Similarly as in $\mathcal{T}$ we can define:

3.3. **Definition.** Let $\mathcal{a}$ be a subcategory of $\mathcal{C}$. Then $\mathcal{a}_{gen}$ is the subcategory of $\mathcal{C}$ defined as follows: $(P, u) \in \mathrm{ob}\, \mathcal{a}_{gen}$ iff for each $M \in 2^P$ $uM = M \cup (\bigcup_{a\in A} u_a(M \cap Q_a))$ where $\{(Q_a, u_a)\}_{a\in A}$ is the family of all subspaces of (P, u) belonging to ob $\mathcal{a}$.

3.4. **Proposition.** If a subcategory $\mathcal{a}$ of $\mathcal{C}$ is epimorphism-closed i.e. $f: X \longrightarrow Y$ is an $\mathcal{C}$-epimorphism and $X \in \mathrm{ob}\, \mathcal{a}$ implies $Y \in \mathrm{ob}\, \mathcal{a}$, then $\mathcal{a}_{gen}$ is the coreflective hull of $\mathcal{a}$ in $\mathcal{C}$.

Proof. Let $(P, u) \in \mathrm{ob}\, \mathcal{C}$. Define $v: 2^P \longrightarrow 2^P$ putting $vM = \bigcup_{a \in A} u_a(M \cap Q_a)$ where $\{(Q_a, u_a)\}_{a \in A}$ is the family of all subspaces of (P, u) belonging to $\mathcal{A}$. It can be easily verified that $((P, v), i)$, where i is the identity map on P, is $\mathcal{A}_{gen}$-coreflection of (P, u). Thus, $\mathcal{A}_{gen}$ is coreflective in $\mathcal{C}$ so that $\mathcal{C}(\mathcal{A}) \subset \mathcal{A}_{gen}$. On the other hand, since $\mathcal{A}$ is epimorphism-closed, then every $\mathcal{A}_{gen}$-object is the direct limit of the family of all its subspaces belonging to $\mathcal{A}$ so that $\mathcal{A}_{gen} \subset \subset \mathcal{C}(\mathcal{A})$. Hence $\mathcal{A}_{gen} = \mathcal{C}(\mathcal{A})$.

3.5. E x a m p l e s. a) Let ξ be any cardinal. Let $\mathcal{B}^{\xi}$ be the subcategory of $\mathcal{C}$ for which $(P, u) \in$ ob $\mathcal{B}^{\xi}$ iff card $P \leqq \xi$. Then $\mathcal{B}^{\xi}$ is epimorphism-closed so that $\mathcal{B}^{\xi}_{gen}$ is coreflective. It will be shown later that for any cardinal ξ , $\mathcal{B}^{\xi}_{gen}$ has an extremal separator and for $\xi < \zeta$ $\mathcal{B}^{\xi}_{gen} \subsetneqq \mathcal{B}^{\zeta}_{gen}$.

b) The category $\mathcal{K}$ of compact $\mathcal{C}$-spaces (see 1.10) is epimorphism-closed so that $\mathcal{K}_{gen} \in \mathcal{L}$.

c) The category of all connected $\mathcal{C}$-spaces (a $\mathcal{C}$-space (P, u) is said to be connected provided that every $\mathcal{C}$-morphism $f: (P, u) \longrightarrow D_2$ is constant) and the category of all path-connected $\mathcal{C}$-spaces are epimorphism-closed.

d) For any cardinal ξ the subcategory $\mathcal{D}^{\xi}$ of $\mathcal{C}$ such that $(P, u) \in$ ob $\mathcal{D}^{\xi}$ iff there exist $M \subset P$ for which card $M \leqq \xi$ and $uM = P$ is epimorphism-closed.

3.6. D e f i n i t i o n . Let ξ be a cardinal and $\xi \geqq 2$. Let P be a set for which card $P = \xi$ and $p \in P$. Put $M = P - \{p\}$ and define $u: 2^P \longrightarrow 2^P$ as follows: $uN = N$ for each $N \in 2^P - \{M\}$, $uM = P$. Denote this space by S^{ξ} .

3.7. T h e o r e m . Let α be a cardinal and $\mathcal{S}^{\alpha} = \{S^{\xi} : \xi \geqq \alpha\}$. Then for any cardinal α $\mathcal{C}(\mathcal{S}^{\alpha}) = \mathcal{C}$.

P r o o f . Clearly, $\mathcal{B}_1 \subset \mathbf{C}(\mathcal{S}^{\alpha})$. If $2 \leqq \xi < \zeta$ are cardinals, then S^{ξ} is an extremal quotient of S^{ζ}. (Let x, y be the unique points in S^{ζ}, S^{ξ} respectively for which $\{x\}$, $\{y\}$ are not open in S^{ζ}, S^{ξ} respectively. Then any map $p: S^{\zeta} \longrightarrow S^{\xi}$ such that $p(x) = y$ and $p[S^{\zeta} - \{x\}] = S^{\xi} - \{y\}$ is an extremal $\mathcal{C}$-epimorphism.) Therefore $S^{\xi} \in \mathbf{C}(\mathcal{S}^{\alpha})$ for any cardinal $\xi \geqq 2$. Let $(P, u) \in \mathrm{ob}\,\mathcal{C}$, $A \subset P$ and $a_0 \in uA$. Denote by (A, a_0) the $\mathcal{C}$-space with the underlying set $A \cup \{a_0\}$ and $\mathcal{C}$-topology defined by $\overline{M} = M$ if $M \subset A \cup \{a_0\}$ and $M \neq A$

and $\overline{A} = A \cup \{a_0\}$. So we have $(A, a_0) \in \mathrm{ob}\,\mathcal{B}_1$ or (A, a_0) is isomorphic with $S^{\mathrm{card}\,A + 1}$ and therefore $(A, a_0) \in \mathbf{C}(\mathcal{S}^{\alpha})$. The set $\mathcal{S} = \{(A, a_0): A \subset P,\ a_0 \in uA\}$ is downward-directed by $(B, b_0) \leqq (A, a_0)$ iff $B \subset A$ and $b_0 \in B$ or $B = A$ and $b_0 = a_0$. Let $j^{(A, a_0)}: (A, a_0) \longrightarrow (P, u)$ be the $\mathcal{C}$-morphism defined by $j^{(A, a_0)}(x) = x$ for each $x \in A \cup \{a_0\}$. Then $(\{j^{(A, a_0)}\}_{(A, a_0) \in \mathcal{S}}, (P, u))$ is a direct limit of the diagram $D: \mathcal{S} \longrightarrow \mathcal{C}$ such that $D(A, a_0) = (A, a_0)$ and $D((B, b_0) = (A, a_0)) = (B, b_0) \hookrightarrow (A, a_0)$ (an embedding). Therefore $(P, u) \in \mathrm{ob}\,\mathbf{C}(\mathcal{S}^{\alpha})$. Hence, $\mathcal{C} \subset \mathbf{C}(\mathcal{S}^{\alpha})$.

3.8. P r o p o s i t i o n . For any cardinal $\xi \geqq 2$, S^{ξ} is an extremal separator of $\mathcal{B}^{\xi}_{gen}$ (see 3.5 a)) and for $\zeta > \xi \geqq 2$ $\mathcal{B}^{\xi}_{gen} \subsetneqq \mathcal{B}^{\zeta}_{gen}$.

P r o o f . $S^{\xi} \in \mathrm{ob}\,\mathcal{B}^{\xi}_{gen}$ implies $\mathbf{C}(S^{\xi}) \subset \mathcal{B}^{\xi}_{gen}$. Clearly, for any ζ for which $2 \leqq \zeta \leqq \xi$, $S^{\zeta} \in \mathrm{ob}\,\mathbf{C}(S^{\xi})$. Let $(P, u) \in \mathrm{ob}\,\mathcal{B}^{\xi}$. Then for each $A \subset P$ $\mathrm{card}\ uA \leqq \xi$. Therefore by the proof of the preceding theorem we have $\mathcal{B}^{\xi} \subset \mathbf{C}(S^{\xi})$. Hence $\mathcal{B}^{\xi}_{gen} \subset \mathbf{C}(S^{\xi})$. If $\eta > \xi$, then $S^{\eta} \notin \mathrm{ob}\,\mathcal{B}^{\xi}_{gen}$. In fact, every subspace Q of S^{η} for which $\mathrm{card}\ Q \leqq \xi$ is discrete. Let M be the unique subset of S^{η} for which $\overline{M} \neq M$. Then $\overline{M} = \bigcup_{a \in A} u_a(M \cap Q_a) = M$ where $\{(Q_a, u_a)\}_{a \in A}$ is the family of all subspaces of S^{η} belonging to ob $\mathcal{B}^{\xi}$. Hence $\mathcal{B}^{\xi}_{gen} \subsetneqq \mathcal{B}^{\eta}_{gen}$.

3.9. Proposition. Let $(\{m_a\}_{a\in A}, (P, u))$ be the $\mathcal{C}$-coproduct of $\{(P_a, u_a)\}_{a\in A}$ and $p: (P, u) \longrightarrow S^{\xi}$ be an extremal $\mathcal{C}$-epimorphism. Then there exists $a_0 \in A$ such that $p_{a_0} = p \circ m_{a_0} : (P_{a_0}, u_{a_0}) \longrightarrow S^{\xi}$ is an extremal $\mathcal{C}$-epimorphism.

Proof. Let y be the unique point of S^{ξ} for which $M = S^{\xi} - \{y\}$ is not closed. Then $\{y\}$ is not open in S^{ξ} and since p is an extremal $\mathcal{C}$-epimorphism, this implies $p^{-1}[y]$ is not open in (P, u). Therefore there exists $a_0 \in A$ such that $N = m^{-1}_{a_0}[p^{-1}[y]]$ is not open in (P_{a_0}, u_{a_0}). We show that $p_{a_0} =$ $= p \circ m_{a_0}$ is an extremal $\mathcal{C}$-epimorphism. If p_{a_0} is not an epimorphism, then $p_{a_0}[P_{a_0} - N] \neq M$. This implies that $B =$ $= p_{a_0}[P_{a_0} - N]$ is closed in S^{ξ} so that $p^{-1}[B]$ is closed in (P, u). Hence $P_{a_0} - N = m^{-1}[p^{-1}[B]]$ is closed in (P_{a_0}, u_{a_0}) what is impossible. Thus, p_{a_0} is an epimorphism. It is easy to verify that for each $N \subset S^{\xi}$ $N = p_{a_0}[u_{a_0} p^{-1}_{a_0}[N]]$ so that p_{a_0} is an extremal $\mathcal{C}$-epimorphism.

Proposition 3.9. implies:

3.10. Theorem. In $\mathcal{L} - \{\mathcal{C}\}$ there are no maximal elements.

Proof. Let $\mathcal{A} \in \mathcal{L} - \{\mathcal{C}\}$. Then there is a cardinal ξ such that $S^{\xi} \notin$ ob $\mathcal{A}$. Therefore $\mathcal{C}(\mathcal{A} \cup \{S^{\xi}\}) \supsetneqq \mathcal{A}$. But $\mathcal{C}(\mathcal{A} \cup \{S^{\xi}\}) \neq \mathcal{C}$ because for $\zeta > \xi$ $S^{\zeta} \notin$ ob $\mathcal{C}(\mathcal{A} \cup \{S^{\xi}\})$. In fact, if $S^{\zeta} \in$ ob $\mathcal{C}(\mathcal{A} \cup \{S^{\xi}\})$, then by 3.9. S^{ζ} is an $\mathcal{C}$-extremal quotient of some $\mathcal{A}$-object (P, u) (because S^{ζ} cannot be an $\mathcal{C}$-extremal quotient of S^{ξ}), i.e. there exists an extremal $\mathcal{C}$-epimorphism $p: (P, u) \longrightarrow S^{\zeta}$. But $\zeta > \xi$ implies that there exists an extremal $\mathcal{C}$-epimorphism $q: S^{\zeta} \longrightarrow S^{\xi}$.

Since in $\mathscr{C}$ extremal epimorphisms coincide with regular ones, $q \circ p\colon (P, u) \longrightarrow S^{\mathfrak{f}}$ is an extremal $\mathscr{C}$-epimorphism. Hence $S^{\mathfrak{f}}$ belongs to ob $\mathscr{A}$ what is a contradiction.

The category $\mathscr{T}$ of all topological spaces (i.e. AF-spaces) is not coreflective in $\mathscr{C}$. In fact, if (Q, t) is the space for which $Q = \{0, 1, 2, 3\}$, $t\{0\} = t\{1\} = t\{0, 1\} = \{0, 1\}$, $t\emptyset = \emptyset$ and $tM = Q$ for all other $M \subset Q$, then $(Q, t) \in \text{ob}\, \mathscr{T}$. Define the decomposition $R = \{x, y, z\}$ by $x = \{0\}$, $y = \{1, 2\}$, $z = \{3\}$ and let $p\colon Q \longrightarrow R$ be the corresponding projection. If u is the $\mathscr{C}$-topology on R for which $p\colon (Q, t) \longrightarrow (P, u)$ is an extremal $\mathscr{C}$-epimorphism, then $u(u\{x\}) = Q \neq \{x, y\} = u\{x\}$. Hence $\mathscr{T}$ is not closed under the formation of $\mathscr{C}$-extremal quotients.

<u>3.11. Theorem.</u> $C(\mathscr{T}) = \mathfrak{A}$ ($\mathfrak{A}$ - the category of all A-spaces).

<u>Proof.</u> Let (P, u) be an A-space, $M \subset P$. Define $u_M\colon 2^P \longrightarrow 2^P$ as follows: $u_M X = X \cup uM$ whenever $X \supset M$ and $u_M X = (u(X \cap M) - M) \cup X$ whenever $X \not\supset M$. Then (P, u_M) is a $\mathscr{T}$-space (i.e. AF-space) for which $u_M M = uM$ and $u_M \leqq u$. Clearly, (P, u_M) is a $\mathscr{C}$-space. Let $X \in 2^P$. If $X \supset M$, then $u_M(u_M(X)) = u_M(X \cup uM) = (X \cup uM) \cup uM = u_M X$. If $X \not\supset M$, then $((u(X \cap M) - M) \cup X) \cap M = X \cap M$. Therefore $u_M(u_M X) =$
$= u_M((u(X \cap M) - M) \cup X) = (u(((u(X \cap M) - M) \cup X) \cap M) - M) \cup$
$\cup\ ((uX \cap M) - M) \cup X) = (u(X \cap M) - M) \cup (u(X \cap M) - M) \cup X =$
$= u_M X$. Hence (P, u_M) is an F-space.
Next we show that (P, u_M) is also an A-space. Let $X, Y \in 2^P$.

1. $X \supset M$, $Y \supset M$, then evidently $u_M(X \cup Y) = u_M X \cup u_M Y$.
2. $X \not\supset M$, $Y \supset M$, then $X \cup Y \supset M$. $u_M X \cup u_M Y =$
$= ((u(X \cap M) - M) \cup X) \cup (Y \cup uM) = (X \cup Y) \cup uM = u_M(X \cup Y)$.

(Clearly $u(X \cap M) - M \subset uM$)

3. $X \not\supset M$, $Y \not\supset M$. Then we have two cases:

a) $X \cup Y \supset M$. Then $u_M X \cup u_M Y = ((u(X \cap M) - M) \cup X) \cup$ $\cup ((u(X \cap M) - M) \cup Y) = X \cup Y \cup (u(X \cap M) - M) \cup$ $\cup (u(Y \cap M) - M) = X \cup Y \cup (u((X \cup Y) \cap M) - M) =$ $= X \cup Y \cup (uM - M) = u_M(X \cup Y)$.

b) $X \cup Y \not\supset M$. Then $u_M(X \cup Y) = (u((X \cup Y) \cap M) - M) \cup$ $\cup (X \cup Y) = (u((X \cap M) \cup (Y \cap M)) - M) \cup X \cup Y =$ $= ((u(X \cap M) \cup u(Y \cap M)) - M) \cup X \cup Y = (u(X \cap M) - M) \cup$ $\cup X \cup (u(Y \cap M) - M) \cup Y = u_M X \cup u_M Y$. Hence, (P, u_M) is an A-space.

Clearly, $u_M M = M \cup uM = uM$. Let $X \in 2^P$. If $X \supset M$, then $u_M X = X \cup uM \subset uX$ because $uX \supset uM$. If $X \not\supset M$, then $u_M X =$ $= (u(X \cap M) - M) \cup X \subset uX \cup X = uX$. Hence $u_M \leqq u$.

Let $(\{m_M\}_{M \in 2^P}, \coprod_{M \in 2^P} (P, u_M))$ be a $\mathscr{C}$-coproduct. Put $(Q, t) = \coprod_{M \in 2^P} (P, u_M)$. Let $p: Q \longrightarrow P$ be the map for which $p(m_M(x)) = x$ for each $x \in P$ and $M \in 2^P$. Let v be the quotient $\mathscr{C}$-topology on P induced by p (i.e. $p: (Q, t) \longrightarrow (P, v)$ is an extremal $\mathscr{C}$-epimorphism). Then for each $M \in 2^P$, $vM =$ $= p[tp^{-1}[M]] = p[t(\bigcup_{X \in 2^P} m_X[M])] = p[\bigcup_{X \in 2^P} [u_X M]]$. Because for each $X \in 2^P$ $u_X M \subset uM = u_M M$, $p[\bigcup_{X \in 2^P} m_X[u_X M]] = uM$. Hence $v = u$ so that $(p, (P, u))$ is a $\mathscr{C}$-extremal quotient of a $\mathscr{C}$-coproduct of $\mathscr{T}$-objects. Therefore $(P, u) \in \text{ob}\ \mathfrak{C}(\mathscr{T})$. Thus, $\mathfrak{A} \subset \mathfrak{C}(\mathscr{T})$. Since $\mathscr{T} \subset \mathfrak{A}$, then $\mathfrak{C}(\mathscr{T}) \subset \mathfrak{A}$.

Finally we prove that $\mathscr{C}$ has no proper both reflective and coreflective subcategories.

3.12. Theorem. If $\mathscr{A}$ is a both reflective and coreflective subcategory of $\mathscr{C}$, then $\mathscr{A} = \mathscr{C}$.

P r o o f . $\mathcal{a}_{-1} \subset \mathcal{A}$ implies $\mathcal{A} \neq \{\emptyset\}$ so that $\mathcal{B}_1 \subset \mathcal{A}$. Denote by D_3 the three-point discrete space with underlying set $\{0, 1, 2\}$. Since $\mathcal{A}$ is reflective, $D_3 \sqcap D_3 \in$ $\in$ ob $\mathcal{A}$. Define the decomposition $R = \{x, y, z\}$ of $\{0, 1, 2\} \times$ $\times \{0, 1, 2\}$ by $x = \{(0, 1), (1, 0)\}$, $y = \{(0, 0), (1, 1),$ $(2, 2)\}$, $z = \{(2, 0), (2, 1), (0, 2), (1, 2)\}$ and let $p: \{0, 1, 2\} \times \{0, 1, 2\} \longrightarrow \{x, y, z\}$ be the corresponding projection. Then (R, v) where v is the quotient $\mathcal{C}$-topology induced by p (i.e. $p: D_3 \sqcap D_3 \longrightarrow (R, v)$ is an extremal $\mathcal{C}$--epimorphism) is isomorphic with C_3. Hence $C_3 \in$ ob $\mathcal{A}$ so that $\mathcal{A} = \mathcal{C}$.

R e m a r k s . Omitting (u_3) or (u_2), (u_3) or all (u_i) $i = 1, 2, 3$ in 1.1, we get more general spaces which will be called ${}^1\mathcal{C}$ -, ${}^2\mathcal{C}$ - and ${}^3\mathcal{C}$ -spaces respectively. Continuous maps of ${}^i\mathcal{C}$ -spaces $i = 1, 2, 3$ are defined analogously as in $\mathcal{C}$. Denote the corresponding categories by ${}^1\mathcal{C}$, ${}^2\mathcal{C}$ and ${}^3\mathcal{C}$ respectively and put ${}^0\mathcal{C} = \mathcal{C}$. Then ${}^i\mathcal{C}$ is full subcategory of ${}^j\mathcal{C}$ for $0 \leqq i \leqq j \leqq 3$ and moreover ${}^i\mathcal{C}$ is both reflective and coreflective in ${}^j\mathcal{C}$ for $0 \leqq i \leqq j \leqq 3$. (It is easy to see that Theorem 1.5. holds for each ${}^i\mathcal{C}$ $i = 0, 1, 2, 3$).

R E F E R E N C E S

[1] ČECH E., Topological Spaces. Academia, Prague 1966

[2] HERRLICH H., Topologische Reflexionen und Coreflexionen Springer-Verlag Berlin-Heidelberg-New York 1968

[3] HERRLICH H., - Strecker G. E., Category Theory. Allyn and Bacon Inc., Boston 1973

[4] SEKANINA M., Systems of topologies on a given set (Russian) Czech. Math. Journal, 15, 90, 1965, s. 9 - 29

[5] Topological Papers of E. ČECH.Academia, Prague 1968

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REFLEKTÍVNE A KOREFLEKTÍVNE PODKATEGÓRIE JEDNEJ KATEGÓRIE ZOVŠEOBECNENÝCH TOPOLOGICKÝCH PRIESTOROV

JURAJ ČINČURA, BRATISLAVA

S ú h r n

Topológia u topologického priestoru (P, u) v Čechovom zmysle z roku 1937 spĺňa nasledujúce axiomy:

1. $u\emptyset = \emptyset$,

2. $M \subset P \Longrightarrow M \subset uM$,

3. $M_1 \subset M_2 \subset P \Longrightarrow uM_1 \subset uM_2$.

Zobrazenie $f: (P, u) \longrightarrow (Q, v)$ sa nazýva spojité, ak $M \subset P \Longrightarrow f[uM] \subset vf[M]$. Práca sa zaoberá štúdiom reflektívnych a koreflektívnych podkategórií kategórie $\mathscr{C}$ topologických priestorov v uvedenom zmysle a spojitých zobrazení.

Р Е З Ю М Е

РЕФЛЕКТИВНЫЕ И КОРЕФЛЕКТИВНЫЕ ПОДКАТЕГОРИИ ОДНОЙ КАТЕГОРИИ ОБОБЩЕННЫХ ТОПОЛОГИЧЕСКИХ ПРОСТРАНСТВ

ЮРАЙ ЧИНЧУРА, БРАТИСЛАВА

Топология и топологического пространства (P, u) в смысле Чеха из 1937 г. исполняет следующие аксиомы: 1. $u\emptyset = \emptyset$, 2. $M \subset P \Longrightarrow M \subset uM$, 3. $M_1 \subset M_2 \subset P \Longrightarrow uM_1 \subset uM_2$. Отображение $f : (P, u) \longrightarrow (Q, v)$ называется непрерывным, если $M \subset P \Longrightarrow f\,[uM] \subset vf\,[M]$. В работе исследуются рефлективные и коре-

флективные подкатегории категории $\mathscr{C}$ топологических пространств в приведенном выше смысле и непрерывных отображений.

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ON THE UNIFIED MEASURE AND INTEGRATION THEORY

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There is an analogy between the measure theory and the integration theory, that can be discovered by the help of an ordering. Denote by L the family of all Lebesgue integrable functions (say e.g. on the interval $\langle a, b \rangle$) and for $f \in L$ put $J(f) = \int_a^b f(x)\,dx$. Then it holds:

1. If $f, g \in L$, $f \leq g$, then $J(f) \leq J(g)$
2. $J(f) + J(g) = J(\max(f, g)) + J(\min(f, g))$ for all $f, g \in L$.
3. If $f_n \in L$, $f_n \leq f_{n+1}$ $(n = 1, 2, \ldots)$, $f(x) = \lim_{n\to\infty} f_n(x)$ for all x and the sequence $\{J(f_n)\}_{n=1}^{\infty}$ is bounded, then $f \in L$ and

$$J(f) = J(\lim_{n\to\infty} f_n) = \lim_{n\to\infty} J(f_n)$$

Every measure has some analogous properties. Let J be a finite non-negative measure defined on a σ-ring. Then it holds:

1. If $A, B \in S$, $A \subset B$, then $J(A) = J(B)$
2. $J(A) + J(B) = J(A \cup B) + J(A \cap B)$ for all $A, B \in S$.
3. If $A_n \in S$, $A_n \subset A_{n+1}$ $(n = 1, 2, \ldots)$, $A = \bigcup_{n=1}^{\infty} A_n$ and the sequence $\{J(A_n)\}_{n=1}^{\infty}$ is bounded, then $A \in A$ and

$$J(A) = \lim_{n\to\infty} J(A_n)$$

It seems to be natural to study a common generalization of the both theories. Let H be a lattice, S its sublattice, $J: S \longrightarrow R$. The conditions mentioned above can be formulated as follows:

1. If $x, y \in S$, $x \leqq y$, then $J(x) \leqq J(y)$.

2. $J(x) + J(y) = J(x \vee y) + J(x \wedge y)$ for all $x, y \in S$. A function J satisfying the condition is called a valuation.

3. If $x_n \in S$, $x_n \leqq x_{n+1}$ $(n = 1, 2, \ldots)$, $x = \sup_n x_n \in H$ and the sequence $\{J(x_n)\}_{n=1}^{\infty}$ is bounded, then $x \in S$ and

$$J(x) = \lim_{n\to\infty} J(x_n)$$

In this article we want to give a review of some investigations in this direction. Here we shall present general results. If we consider a set of real-valued functions as a special case (ordered as usually, i.e. $f \leqq g$ iff $f(x) \leqq g(x)$ for every x) then we get results of the integration theory. If we consider a set of sets (ordered by the inclusion, i.e. $A \leqq B$ iff $A \subset B$) then we get results of the measure theory.

Of course, we can obtain not only the two special classical cases but also some new results as well as in the integration theory as in the measure theory, e.g. the theory of integral on lattice ordered groups or the measure theory on some types of lattices.

1. Integrals on lattices

Definition 1. Let H be a lattice, A be a sublattice of H. We say that a mapping $J: A \longrightarrow R$ (R is the set of all real numbers) is an integral, if the following three conditions are satisfied:

1. J is non-decreasing, i.e. for every $x, y \in A$ the relation $x \leqq y$ implies $J(x) \leqq J(y)$.

2. J is a valuation, i.e. $J(x) + J(y) = J(x \vee y) + J(x \wedge y)$ for every $x, y \in A$.

3. If $x_n \nearrow x$ (or $x_n \searrow x$)[1], $x_n \in A$ $(n = 1, 2, \ldots)$ and $x \in A$, then $J(x) = \lim_{n\to\infty} J(x_n)$

<u>Examples.</u>

1. Arbitrary measure J on a ring A of subsets of a set X. Of course, by a measure we mean a non-negative σ-additive real-valued function.

2. Let A be the family of all integrable functions, $J(f) = \int f\, d\mu$. Another possibility: A would be the set of all simple integrable functions.

3. Let E be a locally compact topological space, $A = K(E)$ be the set of all continuous functions with compact support. Let $J\colon K(E) \longrightarrow R$ be a positive Radon measure (see [4]). It is known ([4] , chap. IV, § 1, Th. 1) that J satisfies the above conditions 1 - 3.

4. Every additive capacity ([9] , chap. III, § 2) is an example of an integral on a family A of sets closed under finite unions and finite intersections.

5. The following example is a generalization of the second and the third. Let A be an Abelian lattice ordered group and $J\colon A \to R$ be a linear, positive and continuous functional (i.e. $x_n \nearrow x$ $J(x_n) \nearrow J(x)$). The second property holds, since J is linear and $x + y = x \vee y + x \wedge y$ for every $x, y \in A$.

6. The following example is mentioned in [26] . Let A be a family of non-negative real-valued functions defined on a set X, satisfying the following conditions: $0 \in A$; $f_n \in A$ $(n = 1, 2, \ldots)$ $\sup f_n \in A$, $\inf f_n \in A$, $f_1 + f_2 \in A$, $\alpha f_1 \in A$ $(\alpha > 0)$, $f_1 - \min(f_1, f_2) \in A$. Finally, a function $J\colon A \to R$ is studied, where $J(\alpha f) = \alpha J(f)$ for $\alpha > 0$, $J(f + g) = J(f) + J(g)$ and

---

[1] We write $x_n \nearrow x$, if $x_n \leq x_{n+1}$ $(n = 1, 2, \ldots)$ and $x = \bigvee_n x_n$.

such that $f_n \nearrow f$, $f \in A$ implies $J(f) = \sup J(f_n)$. Then J is an integral on A. It is sufficient to prove that $f \geqq 0 \Longrightarrow$ $\Longrightarrow J(f) \geqq 0$. But $\frac{n}{n+1} f \nearrow f$, hence

$$J(f) = \sup_n \frac{n}{n+1} J(f)$$

If $J(f)<0$, then $J(f) = \sup \frac{n}{n+1} J(f) \leqq \frac{1}{2} J(f)$ what is impossible.

2. Full integral

The notion of full integral was introduced by Alfsen ([2]).

<u>Definition 2.</u> An integral J defined on the sublattice A of a lattice H is called a full integral if it has the following property:

If x_n $x \in H$ or x_n $x \in H$, $x_n \in A$ $(n = 1, 2, \ldots)$ and $\{J(x_n)\}_{n=1}^{\infty}$ is bounded, then $x \in A$.

Of course, H need not be σ-complete or relatively - complete. $x_n \nearrow x$ means that $\{x_n\}_{n=1}^{\infty}$ is non-decreasing and it has the least upper bound, namely x. It is possible that there are sequences (bounded) which have not the least upper bound.

First we present here two results concerning the full integrals. The first is concerned with the completness of the corresponding pseudometric space and the second with an approximation theorem. In the first of these theorems we need the following notion. A lattice H is called to be relatively σ-complete, if every bounded sequence has the least upper bound and the greatest lower bound. A relatively σ-complete lattice is called to be σ-continuous if the following implications are satisfied:

$$x_n \nearrow x,\ y_n \nearrow y \qquad x_n \wedge y_n \nearrow x \wedge y$$

$$x_n \searrow x,\ y_n \searrow y \qquad x_n \vee y_n \searrow x \vee y$$

3. Completness of L_1

<u>Theorem 1.</u> ([2] , Theorem 3, p. 425). Let H be a σ-continuous lattice, A be its relatively σ-complete sublattice. Let $J: A \longrightarrow R$ be a full integral. Put $\rho(x, y) = J(x \vee y) - J(x \wedge y)$ for all $x, y \in A$. Then (A, ρ) is a complete pseudometric space.

<u>Corollary 1.</u> (Riesz - Fischer theorem). Let A be the family of all integrable functions, $J(f) = \int f \, d\mu$. Put $\rho(f, g) = \int |f - g| \, d\mu$. Then (A, ρ) is a complete pseudometric space.

<u>Corollary 2.</u> Let A be the δ-ring of all sets of finite measure which is defined on a σ-ring. Put $\rho(E, F) = \mu(E \vartriangle F)$. Then (A, ρ) is a complete pseudometric space.

In some sense there is a better variant of Theorem 1(see [15] , Theorem 4). The condition 2 can be changed by a weaker one, but the algebraic structure of H in [15] is more complicated. We shall discuss this situation in the next text (Theorem 5).

4. Approximation theorem

<u>Theorem 2.</u> ([17] , Theorem 2.1). Let $A = H$ be a relatively σ-complete and distributive lattice, $J:: A \longrightarrow R$ be a full integral. Let L be a sublattice of the lattice A and M be the set of all elements that can be approximated by elements of L, i.e.

$$M = \{a \in A;\ \forall \varepsilon > 0 \ \exists \ b \in L,\ J(a \vee b) - J(a \wedge b) < \varepsilon\}$$

Then the set M is monotone, i.e. $x_n \in M$, $x_n \nearrow x \in A$ (or $x_n \searrow x \in A$) implies $x \in M$.

C o r o l l a r y 1. Let J be a finite measure on a σ-ring A, L be a ring, $L \subset A$. Then $M \supset S(L)$, i.e. to every $E \in S(L)$ and every $\varepsilon > 0$ there is $F \in L$ such that $J(E \triangle F) < \varepsilon$.

C o r o l l a r y 2. Let A be the family of all integrable functions, L be the set of all simple integrable functions. Then to every Baire integrable function f and every $\varepsilon > 0$ there is a simple integrable function g such that $\int |f - g| \, d\mu < \varepsilon$.

In a similar way there can be modified also some further classical theorems, e.g. the Lebesgue bounded convergence theorem, Fatou lemma (see [2]), the completion of an integral (see [17]) etc. The central role here plays the assumption that J is a full integral, i.e. the Beppo Levi requirement. Hence the question of constructing of a full integral is very important.

5. C o n s t r u c t i o n o f a f u l i n t e g r a l

Throughout of this paper we shall assume that H is a relatively σ-complete, σ- continuous lattice and A is a sublattice of H. Further there is given a function $J_0 : A \longrightarrow R$ which is an integral. We want to construct a full integral $J : S \longrightarrow R$ such that J is an extension of J_0 (i.e. $S \supset A$ and $J_0(a) = J(a)$ for every $a \in A$).

This problem can be solved in various ways, by various constructions. Of course, also various properties of H or J_0 resp. are assumed.

Almost all constructions have a common part: first we usually define

$$A^+ = \{ b \in H;\ \exists\, a_n \in A,\ a_n \nearrow b \}$$

$$J_1^+(b) = \lim_{n \to \infty} J(a_n), \quad b \in A^+$$

(Analogically can be defined A^-, J_1^-.). For all $x \in H$ one can define an upper integral

$$J^+(x) = \inf \{ J_1^+(b) ; b \in A^+, b \geqq x \}$$

or an lower integral

$$J^-(x) = \sup \{ J_1^-(c) ; c \in A^-, c \leqq x \}$$

Finally

$$S = \{ x; J^+(x) = J^-(x) \in R \}$$

$$J(x) = J^+(x) = J^-(x), \ x \in S$$

It is interesting that J^+ has the following property: If $x_n \nearrow x$, x_n, $x \in H$, then $J^+(x_n) \nearrow J^+(x)$ (see [2] Prop. 3.1, [12] theorem 3.1, [13] theorem 1). Of course we are not able to prove that J is a full integral on S.

<u>Theorem 3.</u> ([2] , theorem 5, p. 429). Let J_0 be a decreasing valuation on A satisfying the following condition:

(A) $y_n \in A$, $z_n \in A$ $(n = 1, 2, \ldots)$, $y_n \nearrow y$, $z_n \searrow z$, $y \geqq z$

implies $\sup J_0(y_n) \geqq \inf J_0(z_n)$

Then J is a full integral on S.

The additional condition (A) is weaker than the original condition $x_n \nearrow x \in A \Longrightarrow J_0(x_n) \nearrow J_0(x)$. We can omit (A) if we assume that there is a richer algebraic structure on H. The main problem is the following: how to cross from increasing sequences to decreasing sequences ?

Assume that there are defined two binary operations + and -. on H (The concept was introduced in [12] , recently a similar theory was studied in [5] .) In our special cases the operations +, - can be interpreted as follows: 1. If H is a lattice of functions, then +, - are usual sum and difference. More general, +, - are the group operetions in an Abelian -group. 2. If H is a lattice of sets, then $+$, $-$ are the set theoretic union and difference. More generally, in a complemented lattice, $a + b = a \vee b$, $a - b = a \wedge b'$, where $b \wedge b' = 0$, $b \vee b' = 1$.

In the paper [13] the following conditions are assumed. (These conditions are weaker than the original ones from [12] .)

1. $x + y = y + x$ for all $x, y \in H$.

2. If $x, y, z \in H$ are arbitrary elements and $x \leqq y$ then $x + z \leqq y + z$, $x - z \leqq y - z$, $z - x \geqq z - y$.

3. If $x, y \in H$ are arbitrary elements and $x \leqq y$ then $y = x + (y - x)$.

4. If $x_n, y_n \in H$ are arbitrary elements $(n = 1, 2, \ldots)$ and $x_n \nearrow x$, $y_n \nearrow y$, then $x_n + y_n \nearrow x + y$.

5. If $x_n \in H$ are arbitrary elements $(n = 1, 2, \ldots)$ and $x_n \nearrow x$, then $x_n - y \nearrow x - y$.

6. If $x_n, y \in H$ are arbitrary elements $(n = 1, 2, \ldots)$ and $x_n \searrow x$, then $y - x_n \nearrow y - x$.

Further it is assumed that A is closed under $+$ and $-$ and to every $x \in H$ there are a_n, b_n A $(n = 1, 2, \ldots)$ such that $\bigwedge_{n=1}^{\infty} a_n \leqq x \leqq \bigvee_{n=1}^{\infty} b_n$.

Finally, J_0 satisfies the following additional conditions (of course, J_0 is an integral):

a) If $x, y \in A$ and $x \leqq y$, then $J_0(y) = J_0(x) + J_0(y - x)$.

b) If $x, y \in A$, then $J_0(x + y) = J_0(x) + J_0(y)$.

<u>Theorem 4.</u> ([13] theorem 6, p. 465). Under the previous assumptions the function J is a full integral on S.

By a similar way other constructions can be modified also. E. Futáš in [7] extends J_0 first on $A_\sigma = A^+$ resp. $A_\delta = A^-$

and then once more on $A_{\delta\sigma} \cup A_{\delta\sigma}$ and the last one is a convenient extension.

M. Šabo in [23] extends J_0 by the transfinite induction. He puts

$$A_1 = \{ x \in H;\ \exists\, a_n \in A,\ a_n \to x \},\quad J_1(x) = \lim_{n \to \infty} J_0(a_n)$$

Then he similarly constructs $A_2 \supset A_1$ and J_2 extending J_1, $A_3 \supset A_2$ etc.

For a limit ordinal he defines

$$A_\beta = \bigcup_{\alpha < \beta} A_\alpha ,\quad J_\beta (x) = J_\alpha (x) \quad (\alpha < \beta)$$

From the upper integral J^+ one can construct the set S by of the pseudometric

$$\varrho (x, y) = J^+((x \wedge y)) + (y - (x \wedge y))).$$

The set S is the topological closure of the set A with respect the topology induced by the pseudometric. This idea was realised by the author in [22] .

Also the known Carathéodory method can be adapted. In the case

$$S = \{x;\ \forall\ y \in H,\ J^+(y) = J^+(y \wedge x) + J^+(y - (x \wedge y))$$

See [21] ; similar results are contained in [5] , some special cases have been studied in [10] and [25] .

6. Other common formulations

We have mentioned the Riesz - Fischer theorem (Theorem 1) and the approximation theorem (Theorem 2). Now we first show that the Riesz - Fischer theorem can be improved (the valuation condition can be dropped) if a binary operation on H is defined.

T h e o r e m 5. Let H be a -complete lattice, A its sublattice. Let a binary operation Δ on H be given satisfying the following three conditions:

a) $\triangle$ is symmetric

b) $x \triangle y = (x \triangle z) \vee (z \triangle y)$ for all x, y, z.

c) $(x \wedge z) \triangle (x \wedge y \wedge z) \leqq x \triangle y$ for all x, y, z.

Further a real-valued function $J : A \longrightarrow R$ is given satisfying the following conditions:

1. $J(x \triangle x) = 0$ for all $x \in A$
2. If $x \leqq y$, $x, y \in A$, then $J(x) \leqq J(y)$, $J(y) \leqq J(x) + J(x \triangle y)$.
3. $J(x \vee y) \leqq J(x) + J(y)$ for all $x, y \in A$.
4. If $x_n \nearrow x (x_n \searrow x)$ and $\{J(x_n)\}_{n=1}^{\infty}$ is bounded, then

$x \in A$ and

$$J(x) = \lim_{n \to \infty} J(x_n)$$

Under the assumptions the couple (A, ϱ) where $\varrho(x, y) = J(x \triangle y)$ is a complete pseudometric space.

The theorem follows from the main result of [15] (theorem 1, p. 38). Since the corollary is not explicitely presented in [15] we here prove it. Put

$$R_i = \left\{ (x, y);\ x, y \in A,\ J(x \triangle y) < \frac{1}{2^i} \right\}$$

Then the following properties are satisfied:

1. R_i is a reflexive (see 1) and symmetric (see a)).
2. If $(x_i, x_{i+1}) \in R_i$ $(i = n, \ldots, n + r)$, then $(x_n, x_{n+r}) \in R_{n-1}$. (This follows from b) and 3).
3. If $(x, y) \in R_n$, then $(z \wedge x, z \wedge x \wedge y) \in R_n$ (see c) and the condition 2).
4. $R_{n+1} \subset R_n$
5. If $x_n \nearrow x$ or $x_n \searrow x$, $x_n \in A$ and $(x_n, x_{n+1}) \in R_n$ for $n > N$, then $x \in A$ and to any m there is n_0 such that $(x, x_n) \in R_m$ for any $n \geqq n_0$.

To prove the fifth property assume that e.g. $x_n \nearrow x$ and $J(x_n \triangle x_{n+1}) < \frac{1}{2^n}$. The assumption 2 implies that $\{J(x_n)\}_{n=1}^{\infty}$ is bounded, hence $x \in A$ and $J(x) = \lim_{n\to\infty} J(x_n)$ by 4. But then

$$J(x) = J(x_n) + J(x \triangle x_n)$$

and therefore $\lim_{n\to\infty} J(x \triangle x_n) = 0$, hence 5 is proved.

Now the completeness of (A, ϱ) follows from Theorem 1 in [15].

Let us mention the meaning of $\triangle$ in the classical examples. In the measure theory, $\triangle$ is the symmetric difference, in the integration theory $f \triangle g = |f - g|$.

As the second result contained in this section we present the decomposition theorem including the Jordan decomposition theorem for Daniell integrals. The theorem was proved in [24] and we present it here in a little special form.

Theorem 6. ([24], p. 370). Let A be a distributive lattice with the least element 0 and two binary operations $+$, $-$ such that the following conditions are satisfied:

1. $x - x = 0$ for all $x \in A$.
2. If $x \leq y \leq z$, then $y - x \leq z - x \leq z$
3. If $x = y$, $x, y \in A$, then $y = x + (y - x)$, $x + z \leq y + z$.
4. If $x_n \nearrow x (x_n \searrow x)$, then $x_n \wedge y \nearrow x \wedge y$, $y - x_n \searrow y - x$ $(x_n \vee y \nearrow x \vee y$, $y - x_n \nearrow y - x)$.

Further, let $J: A \longrightarrow R$ satisfy the following conditions:

1. $J(0) = 0$
2. $J(x) + J(y) = J(x \vee y) + J(x \wedge y)$ for all $x, y \in A$
3. If $x \leq v$, $y \leq u - v$, then $J(x + y) = J(x) + J(y)$
4. If $x_n \nearrow x$, or $x_n \searrow x$ and $x_n, x \in A$, then $J(x_n) \longrightarrow J(x)$.

Put

$J^+(x) = \sup\{J(y);\ y \leq x\}$, $J^-(x) = -\inf\{J(y);\ y \leq x\}$

Then J^+, J^- are integrals on lattices, $J^+(0) = J^-(0) = 0$ and $J(x) = J^+(x) - J^-(x)$ for all $x \in A$.

7. Quasilinear structure

Just mentioned algebraic systems enable us to formulate various problems concerning measure and integration theory. Let us call the system a quasilinear structure. Of course, in the problems there occurs a relatively large number of conditions. This situation is not very convenient because we cannot work with a standard algebraic system (as e.g. lattice ordered groups or Boolean algebras resp. present).

Recently S. Brehmer has published ([5]) a concept of so-called C-lattices which includes both the classical cases and presents a nice closed theory. It seems to us that all our conditions are satisfied in Brehmer's C-lattices, at least in the case that H has the least element 0. Of course, S. Brehmer works with an operation $\setminus$, which is in the function lattice case defined by the formula $f \setminus g = f - \min(f, g)$.

Another general algebraic system has been used in [2] by L. Berg. The system is very general, even it includes the Brehmer concept, but it is not so adequate to the discussed problems.

Finally we present a purely algebraic result of the author from [20] . It generalises the well-known assertions that in lattice ordered groups and in Boolean algebras the infinite distributive law holds.

<u>T h e o r e m 7.</u> Let S be a relatively σ-complete lattice with a binary operation $-$, satisfying the following conditions:

1. $a - \bigvee_n a_n = \bigwedge_n (a - a_n)$
2. If $x \geqq y$ and $x - y \leqq u - u$ for some u, then $x = y$
3. If $x \leqq y$, then $(z \wedge y) - (z \wedge x) \leqq y - x$ for every z.

Then S is an infinitely distributive lattice, i.e. the identity

$$z \wedge (\bigvee_n x_n) = \bigwedge_n (z \wedge z_n)$$

holds

8. Subadditive case

As corollaries we have obtained some known results at most. Now we present an example, where by generalizating results holding for subadditive measures we get new results for subadditive integrals. Of course, quasilinear structure is here a little complicated, we need a larger number of properties and they are less elegant:

1. $+$ is associative and commutative
2. There is $0 \in H$ such that $x - x = 0$ for every $x \in H$
3. If $a \leqq x \leqq b$, $c \leqq y \leqq d$, then

   $a - d \leqq x - y \leqq b - c$, $a + c \leqq x + y \leqq b + d$

   $(b - c) - (a - d) \leqq (b - a) + (d - c)$

   $(b + d) - (a + c) \leqq (b - a) + (d - c)$

   $(b - a \leqq (b - x) + (x - a)$

4. If $a \geqq c$, $b \geqq d$, then $(a \vee b) - (c \vee d) \leqq (a - c) + (b - d)$

   $(a \wedge b) - (c \wedge d) \leqq (a - c) + (b - d)$

5. If $a_n \nearrow a$, $b_n \nearrow b$, then $a_n + b_n \nearrow a + b$, $a_n \wedge b_n \nearrow a \wedge b$
6. If $a_n \searrow a$, $b_n \searrow b$, then $a_n + b_n \searrow a + b$, $a_n \vee b_n \searrow a \vee b$
7. If $a_n \nearrow a$, $b_n \searrow b$, then $a_n - b_n \nearrow a - b$, $a_n - b_n \searrow b - a$.

Properties of J_0:

1. $J_0(a + b) \leq J_0(a) + J_0(b)$ for every a, b
2. $J_0(b) \leq J_0(b - a) + J_0(a)$ for every a, b
3. If $a \leq b$, then $J_0(a) \leq J_0(b)$
4. If $a_n \searrow 0$, then $J_0(a_n) \searrow 0$
5. If $a_n \leq a_{n+1}$, $\{J_0(a_n)\}_{n=1}^{\infty}$ is bounded, then

$\lim_{n\to\infty} J_0(a_{n+1} - a_n) = 0.$

We shall not present explicitely the corresponding results (see [18]). We shall be only interested in the condition 5. The condition is known from the theory of subadditive measures (see [1] , [6]), although in a stronger form: A subadditive measure μ is called to be exhausting (in russian if E_n disjoint $\Longrightarrow \lim_{n\to\infty} \mu(E_n) = 0$.

We see that the general method is very economical. Simultaneously we get an extension theorem for subadditive measure, an extension theorem for subadditive integrals and even an integration theory with respect to a subadditive measure. The integral of a simple function f can be defined by

$$\int f \, d\mu := \inf\left\{ \sum_{i=1}^{n} \alpha_i \, \mu(E_i); \; f \geq \sum_{i=1}^{n} \alpha_i \, \lambda_{E_i} \right\}$$

or /another possibility)

$$\int f \, d\mu := \sup\left\{ \sum_{i=1}^{n} \alpha_i \, \mu(E_i); \; f \geq \sum_{i=1}^{n} \alpha_i \, \lambda_{E_i}, \; E_i \text{ disjoint} \right\}$$

9. Measures on lattices

The following definition is in accordance with the studied conception as well as with the classical notion of a measure as a set-function.

D e f i n i t i o n 3. Let A be a lattice with the least element 0. A mapping $\mu : A \longrightarrow \langle 0, \infty \rangle$ is called a measure if the following properties are satisfied:

1. $\mu(0) = 0$

2. μ is a valuation, i.e. $\mu(x) + \mu(y) = \mu(x \vee y) + \mu(x \wedge y)$ for every $x, y \in A$.

3. If $x_n \nearrow x$, $x_n \in A$ $(n = 1, 2, \ldots)$, $x \in A$, then $\mu(x_n) \longrightarrow \mu(x)$.

Of course, μ will be an integral on a lattice (i.e. also the property dual to 3 is satisfied), only if the domain A fulfils some further conditions (A is something like a ring of sets). There are at least three concepts in this connection:

1. H is an orthocomplemented lattice, i.e. H has the least element 0, the greatest element 1 and there is a mapping $x \longmapsto x^{\perp}$ such that

α) $x \text{---} x^{\perp}$ is bijective,

β) $x \leqq y \quad x^{\perp} \geqq y^{\perp}$

γ) $x^{\perp\perp} = x$

δ) $x \wedge x^{\perp} = 0$

ε) $x \vee x^{\perp} = 1$.

Let A be such a sublattice of H that $a, b \in A$ implies $a \wedge b^{\perp} \in A$.

T h e o r e m 8. ([14] , theorem 4, p. 48). Let H be a σ-complete, modular orthocomplemented lattice, A be its sublattice closed under the operation $(a, b) \quad a \wedge b^{\perp}$. Let $\mu : A \longrightarrow \langle 0, \infty \rangle$, $\mu(0) = 0$. Then μ is a measure iff $\mu(\bigvee_n a_n) = \sum_n \mu(a_n)$ for every disjoint sequence $\{a_n\}_{n=1}^{\infty}$ of A, i.e. such a sequence that $(\bigvee_{i \in \alpha} a_i) \wedge (\bigvee_{j \in \beta} a_j) = 0$

for every disjoint set ,

Now it would be easy to prove the measure extension theorem (see [14] , theorem 3, p. 47) in this theory.

2. H is a modular, complemented lattice and A such its sublattice that $a \wedge b' \in A$ for every $a, b \in A$ and every complement b' of b. Also in this case an analogy of theorem 8 holds ([14] , theorem 4, p. 48). The validity of the measure extension theorem is now not evident:

<u>Theorem 9.</u> ([8] , theorem 2, p. 159, theorem 3 p. 160). Let H be a relatively complemented, σ-continuous, σ-complete lattice with the least element. Let $A \subset H$ be a lattice ring (i.e. such a sublattice of H that $a - b = \{x;\ x \wedge a \wedge b = 0,\ x \vee (a \wedge b) = a\} \subset A$ for every $a, b \in A$). Let μ be a σ-finite measure on A. Then there is exactly one measure $\bar{\mu}$ on $S(A)$, that is an extension of μ. The measure $\bar{\mu}$ is σ-finite.

3. H is a logic, i.e. an orthocomplemented lattice satisfying the following two properties:

η) If $a_n \in H$ $(n = 1, 2, \ldots)$, $a_n \leqq a_n^{\perp}$ $(n \neq m)$ then there are $\bigvee_{n=1}^{\infty} a_n$, $\bigwedge_{n=1}^{\infty} a_n$.

ξ) To every $a_1, a_2 \in H$ such that $a_1 \leqq a_2$ there is $b \in H$ such that $b \leqq a_1^{\perp}$ and $b \vee a_1 = a_2$.

Measures on sublogics of H are studied (see [27]), but the definition of a measure on a logic does not agree with Definition 3.

<u>Definition 4.</u> A measure on a logic A is such a non-negative function $\mu: A \longrightarrow R$ that $\mu(0) = 0$ and

$$\mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} \mu(a_n)$$

whenever $\{a_n\}_{n=1}^{\infty}$ is a sequence of orthogonal elements (i.e. $a_n \leqq a_m^{\perp}$ for $n \neq m$).

If a, b are orthogonal (i.e. $a \leqq b^{\perp}$) then $a \wedge b = b^{\perp} \wedge b = 0$, hence a, b are disjoint. Even if $\{a_n\}_{n=1}^{\infty}$ is a sequence of orthogonal elements, then $\{a_n\}_{n=1}^{\infty}$ is disjoint (see Theorem 8). Indeed, if α, β are disjoint sets of indices, then $a_j \leqq a_i^{\perp}$ for all $j \in \beta$, $i \in \alpha$, hence

$$\bigvee_{j\in\beta} a_j \leqq \bigwedge_{i\in\alpha} a_i^{\perp} = \left(\bigvee_{i\in\alpha} a_i\right)^{\perp}$$

$$\left(\bigvee_{j\in\beta} a_j\right) \wedge \left(\bigvee_{i\in\alpha} a_i\right) = 0$$

We have seen (Theorem 8) that

$$\mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} (a_n)$$

if μ is a measure (in the sense of Definition 3) on a orthocomplemented modular latticė. Hence every measure on a logic in the sense of Definition 3 is a measure in the sense of Definition 4. The convergense is not true. A measure on a logic need not be a valuation and need not be subadditive (see [19] , example 2.1). No extension theorem is known.

10. Negative answers

We shall present here three examples when the Carathéodory method cannot be used.

<u>Example.</u> ([19] , example 3.2). Let H be the set of all linear subspaces of R^2, a consists of the following six sets: $\{0\}$, $\{(x_1, x_2); x_1 = 0\}$, $\{(x_1, x_2); x_2 = 0\}$, $\{(x_1, x_2); x_1 = x_2\}$,

$\{(x_1, x_2); x_2 = -x_1\}$, R_2. Further, let $\mu(0) = 0$, $\mu(R^2) = 2$ and $\mu(L) = 1$ otherwise. Then μ is a measure which is a valuation. No linear subspace is measurable in the Carathéodory sense, since if a, b are linear and not orthogonal, then $a \wedge b = 0$, $a \wedge b^{\perp} = 0$, hence

$$\mu(a) = 1 \neq 0 = \mu(0) + \mu(0) = \mu(a \wedge b) + \mu(a \wedge b^{\perp})$$

T h e o r e m 10. ([16] , prop. 6, p. 244). Let H be a modular, orthocomplemented lattice, $\mu: H \longrightarrow R$, $M_1 = \{ b; \mu(a) = \mu(a \wedge b) + \mu(a \wedge b^{\perp})$ for any $a\}$. Let M_1 be a sublattice and $a, b \in M_1$ imply $a \wedge b'^{\perp} \in M_1$. Let μ be increasing and additive on M_1 (i.e. $a \leqq b^{\perp} \Longrightarrow \mu(a \vee b) = \mu(a) + \mu(b)$). Then M_1 is a distributive lattice.

T h e o r e m 11. ([16] prop. 7, p. 244). Let H be a modular, complemented lattice, $\mu: H \longrightarrow R$, $M_3 = \{ b; \mu(a) = \mu(a \wedge b) + \mu(a \wedge b')$ for all a and all complements b' of $b\}$. Let M_3 be a sublattice of S and $b \in M_3$ imply $b' \in M_3$ for all complements b' of b. Let μ be additive and increasing on M_3. Then M_3 is a distributive lattice.

R E F E R E N C E S

[1] ALEKSIUK V.N. - BEZNOSIKOV F.D.: Extension of continuous outer measures on Boolean algebra. Izv. VUZ, 4, 119, 1972, p. 3 - 9

[2] ALFSEN E.M.: Order preserving maps and integration processes. Math. Ann., 149, 1963, p. 419 - 461

[3] BERG L.: Analysis in geordneten kommutativen Halbgruppen mit Nullelement. Nova Acta Leopoldina, 43, No 219, 1975, p. 5 - 28

[4] BOURBAKI N.: Éléments de mathématique VI. Integration Paris.

[5] BREHMER S.: Verbandtheoretische Charakterisierung des Mass- und Integralbegriffs von Carathéodory. Potsdam. Forsch., B, No 3, 1974, s. 88 - 91

[6] DREWNOWSKI L.: Topological rings of sets, continuous set functions, integration I, II. Bull. Acad. Pol. Sci., 20, 1972, p. 269 - 286

[7] FUTÁŠ E.: Extension of continuous functionals. Mat. čas., 21, 1971, p. 191 - 198

[8] GYORFFY L. - RIEČAN B.: On the extension of measures in relatively complemented lattices. Mat. čas., 23, 1973, p. 158 - 163

[9] MEYER P.A.: Probability and potentials. London 1966

[10] PUGLISI M.A.: Seminorme di Beppo Levi ed integrali di Daniell sopra uno spazio die Riesz astratto. Ricerche di Matematica, 13, 1969, p. 181 - 214

[11] RIEČAN B.: Sur une extension continue des certaines fonctions monotones. Spisy Přír. fak. Univ. J. E. Purkyně, 457, 1964, p. 481 - 483

[12] RIEČAN B.: On a continuous extension of monotone functionals of some types. Mat.-fyz. čas., 15, 1965, p. 116 - 125

[13] RIEČAN B.: On an extension of operators with values in linear semiordered spaces. Čas. pěst. mat., 93, 1968, p. 459 - 471

[14] RIEČAN B.: On the extension of a measure on lattices, Mat. čas., 19, 1969, p. 44 - 49

[15] RIEČAN, B.: On L_1-completness theorem. Acta fac.rer. nat. Univ. Comen., Mathem., 27, 1972, p. 37 - 43

[16] RIEČAN B.: A note on the extension of measures on lattices. Mat. čas., 20, 1970, p. 239 - 244

[17] RIEČAN B.: Regularity and approximation theorems for measures and integrals, Mat. čas., 24, 1974, p.209 - 224

[18] RIEČAN B.: An extension of Daniell integration scheme.Mat. čas., 25, 1975, p. 211 - 219

[19] RIEČAN B.: Abstract completion of a Lebesgue measure from a Borel measure. Mat. čas., 25, 1975, p. 49 - 58

[20] RIEČAN B.: On infinite distributive law. Mat. obzory, 7, 1974, p. 53 - 56

[21] RIEČAN B.: On the Caratheodory method of the extension of measures and integrals, Mat. čas., to appear

[23] ŠABO M., On an extension of finite functionals by the transfinite induction. Math. Slov., 26, 1976, p. 193 - 199

[24] ŠIPOŠ J., A decomposition of a functional as a difference of two positive functionals. Mat. čas. 23, 1973, p. 364 - 373

[25] ŠIPOŠ J., Integrals on lattice ordered groups. to appear.

[26] TOPSOE F., Topology and measure.Springer lecture notes in Math., 133, 1970

[27] VARADARAJAN, Geometry of quantum theory. Princeton 1968

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O ZJEDNOTENEJ TEÓRII MIERY A INTEGRÁLU

BELOSLAV RIEČAN, BRATISLAVA

S ú h r n

Predložená práca obsahuje prehľad výsledkov teórie, ktorá je zovšeobecnením teórie miery a teórie integrálu. V tejto teórii sa študujú reálne funkcie na nejakej čiastočne usporiadanej množine S. Ak za S vezmeme vhodný systém množín, dostávame výsledky z teórie miery, v prípade, že S je vhodne zvolený systém funkcií, dostávame výsledky z teórie integrovania.

Р Е З Ю М Е

ОБ ОБЪЕДИНЕННЕЙ ТЕОРИИ МЕРЫ И ИНТЕГРАЛА

БЕЛОСЛАВ РИЕЧАН, БРАТИСЛАВА

Настоящая работа цодержит обзор результатов теории являющейся обобщением теории меры и теории интеграда. В этой теории изучаются действительные функции определенные на некотором частично упорядо-

ченном множестве S . Если в качестве S возмем подходящее семейство множеств, то мы приходим к результатом из теории меры, в случае, когда S - подходящее семейство функций, мы получаем результаты из теории интегрирования.

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