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ALFA, VYDAVATEĽSTVO TECHNICKEJ A EKONOMICKEJ LITERATÚI
BRATISLAVA

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О ПОСТРОЕНИИ МЕРЫ ИЗ ОБЪЕМА

ЙОЗЕФ КАЛАС, Братислава

В настоящей работе обобщается метод предложенный И. Добраковым. И. Добраков строит так называемую полуаддитивную меру определенную на системе всех борелевских множеств, из некоторой функции множества, определенной на системе всех компактных множеств.

Наш метод похож методу Т. Неубрунна публикованному в статьи [2]. Мы тоже выходим из пары $(\mathscr{U}, \mathscr{B})$, где $\mathscr{U}$ и $\mathscr{B}$ какие-то системы подмножеств пространства X и из какой-то функции множества λ определенной на системе $\mathscr{U}$. Полуаддитивная мера строится на σ-кольце P_2, которое содержит σ-кольцо порожденное системой $\mathscr{B}$.

Если мы более того предположим, что функция множества λ аддитивная, из нашей работы следуют все результаты уже упомянутой работы Т. Неубрунна.

Пусть $(\mathscr{U}, \mathscr{B})$ пара систем подмножеств пространства X выполняющая следующие условия:

1. $\emptyset \in \mathscr{U} \cap \mathscr{B}$

2. Если множества A_1 и A_2 принадлежат системе $\mathscr{U}$, то тоже множество $A_1 \cup A_2$ принадлежит системе $\mathscr{U}$.

3. Если множество $A \in \mathscr{U}$ и множества B_1, B_2 принадлежат системе $\mathscr{B}$ так, что $A \subset B_1 \cup B_2$ то существуют множества A_1, A_2 принадлежащее системе $\mathscr{U}$ так, что $A_1 \subset B_1$, $A_2 \subset B_2$ и $A = A_1 \cup A_2$.

Пусть на системе $\mathscr{U}$ определенная функция множества λ, которая удовлетворяет следующим условиям:

1'. Функция множества λ конечна и $\lambda(\emptyset) = 0$.

2'. Если множества A_1, A_2 принадлежат системе $\mathscr{U}$ и если $A_1 \subset A_2$, то $\lambda(A_1) \leqq \lambda(A_2)$.

3'. Если множества A_1 и A_2 принадлежат системе $\mathscr{U}$, то $\lambda(A_1 \cup A_2) \leqq \leqq \lambda(A_1) + \lambda(A_2)$.

Из свойств 1' и 2' следует, что функция λ неотрицательна.

Теперь мы определим на системе $\mathscr{B}$ функцию множества $\bar{\lambda}$ следующим образом:

$$\bar{\lambda}(B)=\sup\{\lambda(C): C\subset B,\ C\in\mathscr{U}\}$$

для любого множества $B\in\mathscr{B}$.

Лемма 1. Функция множества $\bar{\lambda}$ обладает следующими свойствами:

а) $\bar{\lambda}(\emptyset)=0$,

б) $\bar{\lambda}$ монотонна функция множества,

в) Если B_1, B_2 принадлежат системе $\mathscr{B}$ и $B_1\cup B_2$ принадлежит системе $\mathscr{B}$, то $\bar{\lambda}(B_1\cup B_2)\leqq\bar{\lambda}(B_1)+\bar{\lambda}(B_2)$.

Доказательство. Свойство а) следует из определения функции $\bar{\lambda}$. Докажем б). Пусть A, $B\in\mathscr{B}$ и $A\subset B$. Если $C\in\mathscr{U}$ и $C\subset A$, то $\lambda(C)\leqq\bar{\lambda}(B)$. Отсюда следует, что $\sup\{\lambda(C): C\subset A,\ C\in\mathscr{U}\}\leqq\bar{\lambda}(B)$, значит $\bar{\lambda}(A)\leqq\bar{\lambda}(B)$. Теперь докажем в). Пусть $B_1\in\mathscr{B}$, $B_2\in\mathscr{B}$ и пусть тоже $B_1\cup B_2\in\mathscr{B}$. Пусть множество C из $\mathscr{U}$ заключенное в $B_1\cup B_2$. В силу условия 3 существуют множества C_1, $C_2\in\mathscr{U}$ так, что $C_1\subset B_1$, $C_2\subset B_2$ и $C=C_1\cup C_2$. Отсюда следует, что $\lambda(C)\leqq\lambda(C_1)+\lambda(C_2)\leqq\bar{\lambda}(B_1)+\bar{\lambda}(B_2)$. Тогда тоже $\bar{\lambda}(B_1\cup B_2)=\sup\{\lambda(C): C\subset B_1\cup B_2,\ C\in\mathscr{U}\}\leqq\bar{\lambda}(B_1)+\bar{\lambda}(B_2)$.

Пусть далее выполнены условия:

4. Если $B_1\in\mathscr{B}$, $B_2\in\mathscr{B}$, то тоже $B_1\cup B_2\in\mathscr{B}$.

5. Если $B\in\mathscr{B}$, $C\in\mathscr{U}$, то тоже $B-C\in\mathscr{B}$.

6. Если $A\in\mathscr{U}$, то существуют множества $B\in\mathscr{B}$, $C\in\mathscr{U}$ так, что $A\subset B\subset\subset C$.

4'. Если A любое множество принадлежащее системе $\mathscr{U}$, то

$$\lambda(A)=\inf\{\lambda(C): A\subset B\subset C,\ B\in\mathscr{B},\ C\in\mathscr{U}\}$$

Lemma 2. Функция множества $\bar{\lambda}$ конечно-полуаддитивна.

Доказательство. Доказать этот факт можно методом индукции из леммы 1.

Лемма 3. Если $B\in\mathscr{B}$, $C\in\mathscr{U}$ и если $C\subset B$, то $\bar{\lambda}(B)\leqq\bar{\lambda}(B-C)+\lambda(C)$.

Доказательство. Из условия 4' следует, что существуют множества $E\in\mathscr{B}$, $A\in\mathscr{U}$ так, что $C\subset E\subset A$ и $\lambda(C)+\varepsilon>\lambda(A)$. Если $M\in\mathscr{U}$, $M\subset E$, то $\lambda(M)\leqq\lambda(A)$. Из определения функции $\bar{\lambda}$ следует, что $\bar{\lambda}(E)\leqq\lambda(A)$. Наконец получим, что $\bar{\lambda}(B)\leqq\bar{\lambda}((B-C)\cup E)\leqq\bar{\lambda}(B-C)+\bar{\lambda}(E)$, $\bar{\lambda}(B)\leqq\bar{\lambda}(B-C)+\lambda(C)+\varepsilon$, $\varepsilon>0$.

Отсюда следует, что $\bar{\lambda}(B)\leqq\bar{\lambda}(B-C)+\lambda(C)$.

Пусть далее выполнены следующие условия:

7. Если

$$\{B_n\}_{n=1}^{\infty}$$

произвольная последовательность множеств принадлежащих системе $\mathscr{B}$, то

$$\bigcup_{n=1}^{\infty} B_n \in \mathscr{B}.$$

8. Если

$$C \in \mathscr{U}, \quad \text{и} \quad C \subset \bigcup_{n=1}^{\infty} B_n,$$

где $B_n \in \mathscr{B}$, то существует такое число M, что

$$C \subset \bigcup_{n=1}^{M} B_n.$$

9. Если $C \in \mathscr{U}$, $B \in \mathscr{B}$, то $C - B \in \mathscr{U}$.

5′. Если

$$\{C_n\}_{n=1}^{\infty}$$

любая последовательность взаимо непересекающихся множеств из системы $\mathscr{U}$, то к любому положительному числу ε существует натуральное число M такое, что для любого натурального числа p можно писать неравенство

$$\lambda\left(\bigcup_{n=M+1}^{M+P} C_n\right) < \varepsilon.$$

Обозначим теперь через P_2 систему всех подмножеств A пространства X, для которых $\inf\{\bar{\lambda}(B - C): C \subset A \subset B,\ C \in \mathscr{U},\ B \in \mathscr{B}\} = 0$.

Теорема 1. Система P_2 представляет собой σ-кольцо.

Доказательство. Мы покажем сначала, что если $A_1 \in P_2$, $A_2 \in P_2$, то тоже $A_1 - A_2 \in P_2$. Пусть $A_1 \in P_2$, $A_2 \in P_2$. Ввиду определения системы P_2 к любому положительному числу $\varepsilon/2$ существуют множества $C_1 \in \mathscr{U}$, $C_2 \in \mathscr{U}$, $B_1 \in \mathscr{B}$, $B_2 \in \mathscr{B}$, $C_1 \subset A_1 \subset B_1$, $C_2 \subset A_2 \subset B_2$ такие, что $\bar{\lambda}(B_1 - C_1) < \varepsilon/2$, $\bar{\lambda}(B_2 - C_2) < \varepsilon/2$. Очевидно $C_1 - B_2 \subset A_1 - A_2 \subset B_1 - C_2$. Далее $\bar{\lambda}((B_1 - C_2) - (C_1 - B_2)) \leqq \bar{\lambda}((B_1 - C_1) \cup (B_2 - C_2) \leqq \bar{\lambda}(B_1 - C_1) + \bar{\lambda}(B_2 - C_2) < \varepsilon$, значит $A_1 - A_2 \in P_2$. Из отношения $C_1 \cup C_2 \subset A_1 \cup A_2 \subset B_1 \cup B_2$ и из того, что $(B_1 \cup B_2) - (C_1 \cup C_2) \subset (B_1 - C_1) \cup (B_2 - C_2)$, следует, что $A_1 \cup A_2 \in P_2$.

Для того, чтобы P_2 было σ-кольцо, достаточно показать, что для любой последовательности

$$\{A_n\}_{n=1}^{\infty}$$

взаимо непересекающихся множеств принадлежащих системе P_2, тоже

$$\bigcup_{n=1}^{\infty} A_n \in P_2.$$

Пусть $A_n \in P_2$, $n = 1, 2, \ldots$, $A_i \cap A_j = \emptyset$. К любому положительному числу ε и к каждому множеству A_n существуют множества $C_n \in \mathscr{U}$, $B_n \in \mathscr{B}$, $C_n \subset A_n \subset B_n$ так, что

$$\bar{\lambda}(B_n - C_n) < \frac{\varepsilon}{2^{n+2}}.$$

Из условия 5′ следует, что существует натуральное число M такое, что

$$\lambda\left(\bigcup_{n=M+1}^{M+P} C_n\right) < \frac{\varepsilon}{4},$$

где p любое натуральное число. Из леммы 3 следует, что

$$\bar{\lambda}\left(\bigcup_{n=M+1}^{M+P} B_n\right) \leqq \bar{\lambda}\left(\bigcup_{n=M+1}^{M+P} B_n - \bigcup_{n=M+1}^{M+P} C_n\right) + \lambda\left(\bigcup_{n=M+1}^{M+P} C_n\right) \leqq$$

$$\leqq \bar{\lambda}\left(\bigcup_{n=M+1}^{M+P} (B_n - C_n)\right) + \lambda\left(\bigcup_{n=M+1}^{M+P} C_n\right) \leqq \sum_{n=M+1}^{M+P} \bar{\lambda}(B_n - C_n) + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}.$$

Пусть $E \in \mathscr{U}$,

$$E \subset \left(\bigcup_{n=M+1}^{\infty} B_n - \bigcup_{n=1}^{M} C_n\right) \subset \bigcup_{n=M+1}^{\infty} B_n.$$

Из условия 8 вытекает, что существует такое натуральное число p, что

$$E \subset \bigcup_{n=M+1}^{M+P} B_n.$$

Это значит, что

$$\lambda(E) \leqq \bar{\lambda}\left(\bigcup_{n=M+1}^{M+P} B_n\right) < \frac{\varepsilon}{2},$$

затем

$$\bar{\lambda}\left(\bigcup_{n=M+1}^{\infty} B_n - \bigcup_{n=1}^{M} C_n\right) \leqq \frac{\varepsilon}{2},$$

$$\bar{\lambda}\left(\bigcup_{n=1}^{\infty} B_n - \bigcup_{n=1}^{M} C_n\right) \leqq \bar{\lambda}\left(\bigcup_{n=1}^{M} B_n - \bigcup_{n=1}^{M} C_n\right) + \bar{\lambda}\left(\bigcup_{n=M+1}^{\infty} B_n - \bigcup_{n=1}^{M} C_n\right) \leqq$$

$$\leqq \bar{\lambda}\left(\bigcup_{n=1}^{M} (B_n - C_n)\right) + \frac{\varepsilon}{2} \leqq \sum_{n=1}^{M} \bar{\lambda}(B_n - C_n) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Потому, что

$$\bigcup_{n=1}^{M} C_n \subset \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} B_n,$$

справедливо $\bigcup_{n=1}^{\infty} A_n \in P_2$.

Лемма 4. Если множество $B \in \mathscr{B}$, то $\inf\{\bar{\lambda}(B-C): C \subset B,\ C \in \mathscr{U}\}=0$.

Доказательство. Допустим противоположное, значит, существует множество $B \in \mathscr{B}$, для которого выполняется:

$$\inf\{\bar{\lambda}(B-C): C \subset B,\ C \in \mathscr{U}\}=\varepsilon>0.$$

Тогда для любого множества $C \in \mathscr{U}$ и $C \subset B$ справедливо неравенство $\bar{\lambda}(B-C) \geqq \varepsilon$. Если мы положим $C_1=\emptyset$, то мы получим, что $\bar{\lambda}(B) \geqq \varepsilon$. Из определения функции $\bar{\lambda}$ следует, что существует множество $C_2 \in \mathscr{U}$, $C_2 \subset B$ такое, что $\bar{\lambda}(C_2)>\varepsilon/2$; очевидно $\bar{\lambda}(B-C_2) \geqq \varepsilon$. Так как $\bar{\lambda}(B-C_2) \geqq \varepsilon$, то существует множество $C_3 \subset B-C_2$ для которого $\lambda(C_3)>\varepsilon/2$; опять $\bar{\lambda}(B-C_2 \cup C_3) \geqq \varepsilon$. Пусть $C_1, \ldots, C_{n-1}$ взаимо непересекающиеся множества такие, что $C_i \subset B$, $C_i \in \mathscr{U}$ $(i=1, \ldots, n-1)$ и

$$\bar{\lambda}\left(B-\bigcup_{i=1}^{n-1} C_i\right) \geqq \varepsilon.$$

Отсюда опять следует, что существует множество $C_n \in \mathscr{U}$,

$$C_n \subset B-\bigcup_{i=1}^{n-1} C_i \quad \text{так, что} \quad \lambda(C_n)>\frac{\varepsilon}{2}.$$

Мы получили таким образом последовательность

$$\{C_n\}_{n=1}^{\infty}$$

взаимо непересекающихся множеств принадлежащих системе $\mathscr{U}$ таких, что $\lambda(C_n)>\varepsilon/2$ для любого положительного целого числа n, но этот факт в противоположении с условием 5'.

Из предшествующей леммы вытекает, что система $\mathscr{B}$ содержится в σ-кольце P_2.

Лемма 5. Функция множества $\bar{\lambda}$ счетно-полуаддитивна.

Доказательство. Мы уже знаем, что $\bar{\lambda}$ конечно-полуаддитивна. Пусть теперь

$$\{B_n\}_{n=1}^{\infty}$$

любая последовательность множеств принадлежащих системе $\mathscr{B}$ и пусть множество C заключенное в

$$\bigcup_{n=1}^{\infty} B_n.$$

В силу условия 8 существует положительное целое число M так, что

$$C \subset \bigcup_{n=1}^{M} B_n.$$

Потом

$$\lambda(C) \leqq \bar{\lambda}\left(\bigcup_{n=1}^{M} B_n\right) \leqq \sum_{n=1}^{M} \bar{\lambda}(B_n) \leqq \sum_{n=1}^{\infty} \bar{\lambda}(B_n)$$

Отсюда вытекает, что

$$\bar{\lambda}\left(\bigcup_{n=1}^{\infty} B_n\right) = \sup \{\lambda(C) : C \subset \bigcup_{n=1}^{\infty} B_n,\ C \in \mathscr{U}\} \leqq \sum_{n=1}^{\infty} \bar{\lambda}(B_n)$$

Определим теперь на σ-кольце P_2 функцию множества μ следующим образом:

$$\mu(A) = \sup \{\lambda(C) : C \subset A,\ C \in \mathscr{U}\}.$$

Теорема 2. Функция множества μ обладает следующими свойствами:

а) $\mu(\emptyset) = 0$, $\mu(B) = \bar{\lambda}(B)$ для любого множества $B \in \mathscr{B}$;

б) μ монотонна,

в) $\mu(A) = \inf \{\bar{\lambda}(B) : A \subset B,\ B \in B\}$;

г) μ счетно-полуаддитивна;

д) Если $A \subset B$, $B \in P_2$ и $\mu(B) = 0$, то тоже $A \in P_2$.

Доказательство. Свойство а) следует из определения функции μ. Докажем б). Пусть A_1, $A_2 \in P_2$, $A_1 \subset A_2$. Если $C \in \mathscr{U}$, $C \subset A_1$, то $\lambda(C) \leqq \mu(A_2)$. Из этого неравенства вытекает, что $\mu(A_1) = \sup \{\lambda(C) : C \subset A_1,\ C \in \mathscr{U}\} \leqq \mu(A_2)$. Теперь докажем в):

Если $A \in P_2$, $B \in \mathscr{B}$ и $A \subset B$, то $\mu(A) \leqq \mu(B) = \bar{\lambda}(B)$. Потому что $A \in P_2$, существуют множества $C \in \mathscr{U}$, $E \in \mathscr{B}$ так, что $C \subset A \subset E$ и $\bar{\lambda}(E - C) < \varepsilon$. Из леммы 3 вытекает, что $\bar{\lambda}(E) \leqq \bar{\lambda}(E - C) + \lambda(C) < \lambda(C) + \varepsilon \leqq \mu(A) + \varepsilon$, значит $\mu(A) = \inf \{\bar{\lambda}(B) : A \subset B,\ B \in \mathscr{B}\}$. Для доказательства свойства г) мы воспользуем предшествующее свойство функции μ и того, что функция $\bar{\lambda}$ счетно-полуаддитивна.

Пусть $\{A_n\}_{n=1}^{\infty}$ — некоторая последовательность множеств принадлежащих σ-кольцу P_2. Для любого $\varepsilon > 0$ и для любого $n = 1, 2, \dots$ существуют такие множества B_n из $\mathscr{B}$, что

$$A_n \subset B_n \text{ и } \bar{\lambda}(B_n) < \mu(A_n) + \frac{\varepsilon}{2^n}.$$

Отсюда следует, что

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leqq \bar{\lambda}\left(\bigcup_{n=1}^{\infty} B_n\right) \leqq \sum_{n=1}^{\infty} \bar{\lambda}(B_n) < \sum_{n=1}^{\infty} \mu(A_n) + \varepsilon$$

Так как ε произвольно, то из этих неравенств следует, что функция μ счетно-полуаддитивна. Наконец докажем свойство д):

Пусть множество A содержится в таком множестве $E \in P_2$, что $\mu(E) = 0$.

Из определения числа $\mu(E)$ следует, что если некоторое множество $C \in \mathscr{U}$ содержится в множестве E, то $\lambda(C)=0$. Ввиду того, что $E \in P_2$ существуют множества $C \in \mathscr{U}$, $B \in \mathscr{B}$, $C \subset E \subset B$ так, что $\bar{\lambda}(B-C)<\varepsilon$, значит $\varepsilon > \bar{\lambda}(B-C) \geqq \bar{\lambda}(B)-\lambda(C) = \bar{\lambda}(B)$. Так как $\emptyset \subset A \subset B$ и $\bar{\lambda}(B-\emptyset) = \bar{\lambda}(B)<\varepsilon$, мы видим, что множество $A \in P_2$.

Пусть далее выполнены условия:

10. Если $B_1 \in \mathscr{B}$, $B_2 \in \mathscr{B}$, то тоже $B_1 \cap B_2 \in \mathscr{B}$.

11. Если $C_1 \in \mathscr{U}$, $C_2 \in \mathscr{U}$, то тоже $C_1 \cap C_2 \in \mathscr{U}$.

6′. Если $\{C_n\}_{n=1}^{\infty}$ — убывающая последовательность множества из $\mathscr{U}$ такая, что

$$\bigcap_{n=1}^{\infty} C_n = \emptyset, \text{ то } \lim_{n\to\infty} \lambda(C_n)=0.$$

Лемма 6. Функция множества μ непрерывна сверху на пустом множестве.

Доказательство. Пусть $\{A_n\}_{n=1}^{\infty}$ убывающая последовательность множеств принадлежащих σ-кольцу P_2 и пусть

$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Ввиду определения σ-кольца P_2 существуют множества $C_n \in \mathscr{U}$, $B_n \in \mathscr{B}$ так, что $C_n \subset A_n \subset B_n$ и

$$\bar{\lambda}(B_n - C_n) < \frac{\varepsilon}{2^{n+2}}.$$

Положим

$$T_n = \bigcap_{i=1}^{n} C_i,\ E_n = \bigcap_{i=1}^{n} B_i, \qquad n = 1, 2, \ldots$$

Последовательность $\{T_n\}_{n=1}^{\infty}$ представляет собой убывающую последовательность множеств из системы $\mathscr{U}$, причем

$$\bigcap_{n=1}^{\infty} T_n = \emptyset.$$

Из условия 6′ следует, что

$$\lim_{n\to\infty} \lambda(T_n)=0.$$

Для каждого n справедливо, что $A_n \subset E_n$, $T_n \subset E_n$, значит

$$\mu(A_n) \leqq \bar{\lambda}(E_n) \leqq \bar{\lambda}(E_n - T_n) + \lambda(T_n) = \bar{\lambda}\Big(E_n - \bigcap_{i=1}^{n} C_i\Big) + \lambda(T_n) \leqq$$

$$\leqq \sum_{i=1}^{n} \lambda(E_i - C_i) + \lambda(T_n) \leqq \sum_{i=1}^{n} \bar{\lambda}(B_i - C_i) + \lambda(T_n) \leqq \frac{\varepsilon}{2} + \lambda(T_n).$$

Из этих неравенств следует, что

$$\lim_{n\to\infty} \mu(A_n) = 0.$$

Значит, если пара $(\mathcal{U}, \mathcal{B})$ и функция множества λ удовлетворяют всем высказанным условиям, можно на σ-кольце P_2 сконструировать *монотонную, счетно-полуаддитивную, непрерывную сверху на пустом множестве и обращающуюся в нуль на пустом множестве функцию* μ.

До сих пор мы не предполагали, что функция множества λ является на системе $\mathcal{U}$ *аддитивной*. Пусть далее функция λ аддитивна.

Теорема 3. Функция множества μ представляет собой меру на σ-кольце P_2.

Доказательство. Мы докажем, что μ конечна и аддитивна. Для того, чтобы μ была конечна достаточно ввиду теоремы 2 доказать, что функция $\bar{\lambda}$ конечна. Пусть $B \in \mathcal{B}$. Ввиду леммы 4 справедливо, что существует такое множество $C \in \mathcal{U}$, $C \subset B$, для которого $\bar{\lambda}(B - C) < \varepsilon$. Ввиду леммы 3 справедливо, что $\bar{\lambda}(B) \leqq \bar{\lambda}(B - C) + \lambda(C) < \varepsilon + \lambda(C)$. Из этого и из конечности функции λ следует, что функция $\bar{\lambda}$ конечна. Мы докажем теперь что μ аддитивна на σ-кольце P_2.

Пусть $A_1 \in P_2$, $A_2 \in P_2$ и пусть $A_1 \cap A_2 = \emptyset$. В силу определения μ и $\varepsilon > 0$ существуют множества C_1, C_2 из $\mathcal{U}$ так, что $C_1 \subset A_1$, $C_2 \subset A_2$, и

$$\lambda(C_1) \geqq \mu(A_1) - \frac{\varepsilon}{2}, \ \lambda(C_2) \geqq \mu(A_2) - \frac{\varepsilon}{2}.$$

Затем $\mu(A_1 \cup A_2) \geqq \lambda(C_1 \cup C_2) = \lambda(C_1) + \lambda(C_2) \geqq \mu(A_1) + \mu(A_2) - \varepsilon$. Число ε любое положительное, потому $\mu(A_1 \cup A_2) \geqq \mu(A_1) + \mu(A_2)$. Обратное неравенство следует из полуаддитивности функции μ. Уже знаем, что μ неотрицательна и непрерывна сверху на пустом множестве, значит μ представляет собой меру на σ-кольце P_2.

Лемма 7. Если $B \in \mathcal{B}$, $C \in \mathcal{U}$, и $C \subset B$, то $\bar{\lambda}(B) \geqq \bar{\lambda}(B - C) + \lambda(C)$.

Доказательство. Пусть $B \in \mathcal{B}$, $C \in \mathcal{U}$ и $C \subset B$, пусть далее $C_1 \subset B - C$, $C_1 \in \mathcal{U}$. Тогда $C \cup C_1 \subset B$, $C \cap C_1 = \emptyset$. Отсюда следует, что $\lambda(C) + \lambda(C_1) = \lambda(C \cup C_1) \leqq \bar{\lambda}(B)$, значит $\lambda(C) + \sup\{\lambda(C_1) : C_1 \subset B - C, \ C_1 \in \mathcal{U}\} \leqq \bar{\lambda}(B)$, значит $\lambda(C) + \bar{\lambda}(B - C) \leqq \bar{\lambda}(B)$.

Лемма 8. Система $\mathcal{U}$ содержится в σ-кольце P_2.

Доказательство. Мы должны показать, что для любого множества C из системы $\mathcal{U}$ справедливо равенство $\inf\{\bar{\lambda}(B - C) : C \subset B, \ B \in \mathcal{B}\} = 0$.

Пусть ε любое положительное число. Ввиду условия 4′ существуют множества $B \in \mathscr{B}$, $E \in \mathscr{U}$ так, что

$$C \subset B \subset E \text{ и } \lambda(C) + \frac{\varepsilon}{2} > \lambda(E) \geqq \bar{\lambda}(B)$$

$$\text{и } \bar{\lambda}(B - C) \leqq \bar{\lambda}(B) - \lambda(C) < \lambda(C) + \frac{\varepsilon}{2} - \lambda(C) = \frac{\varepsilon}{2}.$$

Мы обозначим через P_1 наследственное σ-кольцо порожденное классом $\mathscr{U} \cup \mathscr{B}$.

Лемма 9. Если $E \in P_1$, то существует такое множество B из $\mathscr{B}$, что $E \subset B$.

Доказательство. Можно легко убедится, что система $\mathscr{K} = \{E$: существует $B \in \mathscr{B}$, $E \subset B\}$ представляет наследственное σ-кольцо. Ввиду условия 6 $K \supset \supset \mathscr{U} \cup \mathscr{B}$. Отсюда следует что $\mathscr{K} \supset P_1$.

Мы определим на σ-кольце P_1 функцию множества μ^* следующим образом: $\mu^*(E) = \inf \{\bar{\lambda}(B) : E \subset B,\ B \in \mathscr{B}\}$.

Теорема 4. Функция μ^* является на наследственном σ-кольце P_1 внешней мерой.

Доказательство. То, что $\mu^*(A) \geqq 0$ и $\mu^*(\emptyset) = 0$ следует из определения функции μ^*. Отсюда тоже следует, что функция μ^* монотонна. Мы покажем, что μ^* счетно-полуаддитивна. Пусть $\{E_n\}_{n=1}^{\infty}$ любая последовательность множеств принадлежащих σ-кольцу P_1. К произвольному числу

$$\frac{\varepsilon}{2^n} > 0$$

существует $B_n \in \mathscr{B}$, так, что

$$E_n \subset B_n \quad \text{и} \quad \mu^*(E_n) + \frac{\varepsilon}{2^n} \geqq \bar{\lambda}(B_n).$$

Очевидно

$$\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} B_n.$$

Отсюда следует что

$$\mu^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leqq \bar{\lambda}\Big(\bigcup_{n=1}^{\infty} B_n\Big) \leqq \sum_{n=1}^{\infty} \bar{\lambda}(B_n) \leqq$$

$$\leqq \sum_{n=1}^{\infty} \Big(\mu^*(E_n) + \frac{\varepsilon}{2^n}\Big) = \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon$$

Так как ε произвольное то из этих неравенств следует, что μ^* счетно-полуаддитивна.

Лемма 10. Если $B \in \mathcal{B}$ то $\mu^*(B) = \bar{\lambda}(B)$.

Доказательство. Пусть $B \in \mathcal{B}$ Из отношения $B \subset B$ и из определения функции μ^* следует что $\mu^*(B) \leqq \bar{\lambda}(B)$.

Пусть E любое множество из $\mathcal{B}$, содержащее множество B. Тогда $\bar{\lambda}(B) \leqq \bar{\lambda}(E)$, значит

$$\mu^*(B) = \inf \{\bar{\lambda}(E) : B \subset E, E \in \mathcal{B}\} \geqq \bar{\lambda}(B)$$

Из этих неравенств, следует для любого множества B из $\mathcal{B}$ равенство $\mu^*(B) = \bar{\lambda}(B)$.

Обозначим через P_1^* систему всех μ^*-измеримых множеств. В следующих теоремах мы докажем, что $\mathcal{U} \subset P_1^*$ и $\mathcal{B} \subset P_1^*$.

Теорема 5. Любое множество принадлежащее системе $\mathcal{U}$ μ^*-измеримо.

Доказательство. Пусть $B \in \mathcal{B}$ и $A \in \mathcal{U}$. Мы покажем, что $\mu^*(B) \geqq \mu^*(B \cap A) + \mu^*(B \cap A')$. Пусть $E_1, E_2 \in \mathcal{U}$ $E_1 \subset B \cap A'$ $E_2 \subset B \cap E_1'$ Тогда

$$\mu^*(B) = \bar{\lambda}(B) \geqq \lambda(E_1 \cup E_2) = \lambda(E_1) + \lambda(E_2)$$

$$\mu^*(B) \geqq \lambda(E_1) + \sup \{\lambda(E_2) : E_2 \subset B \cap E_1'\} =$$
$$= \lambda(E_1) + \bar{\lambda}(B \cap E_1') = \lambda(E_1) + \mu^*(B \cap E_1')$$

Из того что $B \cap E_1' \supset B \cap A$ следует $\mu^*(B) \geqq \lambda(E_1) + \mu^*(B \cap A)$ значит

$$\mu^*(B) \geqq \sup \{\lambda(E_1) : E_1 \subset B \cap A'\} + \mu^*(B \cap A)$$

$$\mu^*(B) \geqq \bar{\lambda}(B \cap A') + \mu^*(B \cap A) = \mu^*(B \cap A) + \mu^*(B \cap A')$$

Пусть теперь E слюбое множество принадлежащее P_1 и пусть $B \in \mathcal{B}$ такое, что $B \supset E$. Тогда справедливо $\bar{\lambda}(B) = \mu^*(B) \geqq \mu^*(B \cap A) + \mu^*(B \cap A') \geqq \mu^*(E \cap A) + \mu^*(E \cap A')$, значит

$$\mu^*(E) = \inf \{\bar{\lambda}(B) : E \subset B, B \in \mathcal{B}\} \geqq \mu^*(E \cap A) + \mu^*(E \cap A')$$

Обратное неравенство следует из полуаддитивности внешней меры. Мы показали, что $\mathcal{U} \subset P_1^*$.

Лемма 11. Для любого множества $B \in \mathcal{B}$ и для любого положительного числа ε существует такое множество $A \in \mathcal{U}$, $A \subset B$, что $\mu^*(B - A) < \varepsilon$.

Доказательство. Ввиду леммы 4 и леммы 10 справедливо

$$\inf \{\mu^*(B - A) : A \subset B, A \in \mathcal{U}\} = \inf \{\bar{\lambda}(B - A) : A \subset B, A \in \mathcal{U}\} = 0$$

и из этого равенства уже следует утверждение леммы.

Лемма 12. Пусть E такое множество из наследственного σ-кольца P_1, что $E \subset B$, $B \in \mathcal{B}$. Тогда для любого числа $\varepsilon > 0$, существует множество A из $\mathcal{U}$ $A \subset B$ так, что $\mu^*(E) < \mu^*(E \cap A) + \varepsilon$.

Доказательство. Из предыдущей леммы следует, что существует множество $A \in \mathscr{U}$, $A \subset B$ так, что $\mu^*(B-A)<\varepsilon$. Но тогда $E = (E \cap A) \cup (E \cap (B-A))$

$$\mu^*(E) \leqq \mu^*(E \cap A) + \mu^*(E \cap (B-A)) \leqq$$
$$\leqq \mu^*(E \cap A) + \mu^*(B-A) < \mu^*(E \cap A) + \varepsilon$$

Лемма 13. Пусть $B \in \mathscr{B}$, E и C такие множества из P_1, что $E \subset B$, $C \subset B'$. Тогда $\mu^*(E \cup C) = \mu^*(E) + \mu^*(C)$.

Доказательство. Из предыдущей леммы следует, что существует такое множество A из $\mathscr{U}$, что $A \subset B$ и $\mu^*(E) < \mu^*(E \cap A) + \varepsilon$. Мы уже знаем, что каждое множество A из $\mathscr{U}$ μ^*-измеримо. Тогда справедливо

$$\mu^*((E \cap A) \cup C) = \mu^*(((E \cap A) \cup C) \cap A) +$$
$$+ \mu^*(((E \cap A) \cup C) \cap A') = \mu^*(E \cap A) + \mu^*(C)$$

$$\mu^*(E) + \mu^*(C) < \mu^*(E \cap A) + \varepsilon + \mu^*(C) =$$
$$= \mu^*((E \cap A) \cup C) + \varepsilon \leqq \mu^*(E \cup C) + \varepsilon$$

Следовательно $\mu^*(E) + \mu^*(C) \leqq \mu^*(E \cup C)$. Противоположное неравенство следует из полуаддитивности внешней меры.

Теорема 6. Любое множество принадлежащее системе $\mathscr{B}$ μ^*-измеримо.

Доказательство. Пусть A любое множество принадлежащее наследственному σ-кольцу P_1 и пусть B любое множество из системы $\mathscr{B}$. Множество $A = (A \cap B) \cup (A \cap B')$, затем из предыдущей леммы вытекает, что

$$\mu^*(A) = \mu^*((A \cap B) \cup (A \cap B') = \mu^*(A \cap B) + \mu^*(A \cap B'),$$

значит, $\mathscr{B} \subset P\dagger$.

Обозначим через P σ-кольцо порожденное классом $\mathscr{U} \cup \mathscr{B}$. Из теоремы 5 и 6 вытекает, что $P \subset P\dagger$. Тогда функция множества $\bar{\mu}$ определенная на P равенством $\bar{\mu}(E) = \mu^*(E)$ для любого E из P представляет собой меру на σ-кольце P.

Из леммы 4 и 8 вытекает, что $P \subset P_2$. Если $E \in P$, то $\mu(E) = \bar{\mu}(E)$.

Лемма 14. Если множество $E \in P_1$, то $\mu^*(E) = \inf\{\bar{\mu}(A) : E \subset A, A \in P\}$.

Доказательство. $\mu^*(E) = \inf\{\bar{\lambda}(B) : E \subset B, B \in \mathscr{B}\} \geqq \inf\{\bar{\mu}(C) : E \subset C, C \in P\} \geqq \mu^*(E)$.

Лемма 15. К любому множеству E из P_1 существует множество $C \in P$ такое, что $E \subset C$ и $\mu^*(E) = \bar{\mu}(C)$.

Доказательство. Пусть $E \in P_1$. Мы сначала покажем, что $\mu^*(E) < \infty$. Ввиду леммы 9 существует такое множество B из $\mathscr{B}$, что $E \subset B$. Тогда из монотонности функции μ^*, леммы 10 и из того, что $\bar{\lambda}$ конечна функция множества, следует, что $\mu^*(E) \leqq \mu^*(B) = \bar{\lambda}(B)$.

Ввиду предыдущей леммы к любому натуральному числу ε существует множество $C_n \in P$ такое, что $E \subset C_n$ и $\bar{\mu}(C_n) \leqq \mu^*(E) + 1/n$. Положим

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Тогда $E \subset C \in P$ и

$$\mu^*(E) \leqq \bar{\mu}(C) \leqq \bar{\mu}(C_n) \leqq \mu^*(E) + \frac{1}{n}$$

Ввиду того, что это неравенство справедливо для любого числа n, то $\mu^*(E) = \bar{\mu}(C)$.

Лемма 16. Если $A \in P_1$ и $\mu^*(A) = 0$, то $A \in P_2$.

Доказательство. Пусть $A \in P_1$. Тогда существует такое множество $C \in P \subset P_2$, что $A \subset C$ и далее $\mu(C) = \bar{\mu}(C) = \mu^*(A) = 0$. Из теоремы 2 следует, что $A \in P_2$.

Теорема 7. σ-кольцо всех μ^*-измеримых множеств содержится в σ-кольце P_2.

Доказательство. Пусть E μ^*-измеримое множество. Ввиду леммы 15 существует такое множество $C \in P$, что $E \subset C$ и $\mu^*(E) = \bar{\mu}(C)$. Затем $\mu^*(C - E) = \mu^*(C - E) = \mu^*(C) - \mu^*(E) = 0$. По предыдущей лемме $C - E \in P_2$, значит $E = C - (C - E) \in P_2$.

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SÚHRN

O KONŠTRUKCII MIERY Z OBJEMU

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Nech v priestore X sú dané dva systémy podmnožín $\mathcal{U}$, $\mathcal{B}$. Predpokladajme, že na $\mathcal{U}$ je definovaná množinová funkcia λ. Nech $\mathcal{U}$, $\mathcal{B}$ a λ spĺňajú určité podmienky. Vyjdúc z množinovej funkcie λ, na σ-okruhu P_2, ktorý je v práci definovaný, sa zostrojuje subaditívna miera μ. Ak množinová funkcia λ je aditívna, potom množinová funkcia μ je miera na P_2.

V poslednej časti práce sa uvažuje dedičný σ-okruh generovaný systémom $\mathcal{U} \cup \mathcal{B}$. Na tomto σ-okruhu definujeme množinovú funkciu μ^*, ktorá má všetky vlastnosti vonkajšej miery. Ak P_1^* je σ-okruhom všetkých μ^* merateľných množín a P je σ-okruh generovaný systémom $\mathcal{U} \cup \mathcal{B}$, tak je ukázané, že $P \subset P_1^* \subset P_2$.

SUMMARY

ABOUT CONSTRUCTION MEASURE FROM THE CONTENT

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Let in the space X, two systems $\mathcal{U}$ and $\mathcal{B}$ of subsets of X be given. Suppose a set function λ defined on the system $\mathcal{U}$. Let $\mathcal{U}$, $\mathcal{B}$ and λ satisfy certain conditions. Starting from the set function λ, a subadditive measure μ is constructed on the σ-ring P_2, which is defined in the paper. If the original set function λ is additive, then the set function μ is the measure on the σ-ring P_2.

In the last part of the paper the hereditary σ-ring generated by the system $\mathcal{U} \cup \mathcal{B}$ is considered. We define the set function μ^* on this σ-ring. The function μ^* has all properties of the outer measure. Let P_1^* be the σ-ring μ^* measurable sets and P be the σ-ring generated by the system $\mathcal{U} \cup \mathcal{B}$. It is shown, that $P \subset P_1^* \subset P_2$.

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O ROZKLADE KOMPLETNÉHO HYPERGRAFU NA FAKTORY S DANÝMI PRIEMERMI

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Úvod

Táto práca hovorí o rozklade kompletného hypergrafu na m faktorov s danými priemermi. Úloha rozkladu kompletného grafu na faktory s danými priemermi bola riešená v článku [1], z ktorého vychádza aj táto práca.

V prvej časti sa zaoberáme existenciou rozkladu kompletného hypergrafu a je istým zovšeobecnením niektorých výsledkov z [1]. V druhej časti riešime problém rozkladu kompletného hypergrafu na faktory s priemerom 1. V tretej časti riešime problém rozkladu pre $n = 2, 3, 4$.

1. Všeobecný prípad

Na začiatku uvedieme niektoré pojmy, s ktorými budeme neskôr pracovať (pozri tiež [2]).

Definícia. Nech X je konečná množina a M systém jej podmnožín. Usporiadanú dvojicu (X, M) budeme nazývať *hypergrafom.* Prvky množiny X nazveme *vrcholmi* a prvky množiny M *hranami* hypergrafu. Hovoríme, že vrcholy x a y sú susedné, ak $x \neq y$ a existuje hrana h hypergrafu taká, že $\{x, y\} \subset h$.

Postupnosť vrcholov a hrán v tvare

$$(x_0, h_1, x_1, h_2, \ldots, x_{n-1}, h_n, x_n)$$

nazveme *sledom,* ak každé dva vrcholy x_i, x_{i+1} sú susedné a platí inklúzia $\{x_i, x_{i+1}\} \subset h_i$. Pod dĺžkou sledu budeme rozumieť počet hrán v slede. Cestou nazveme sled, ktorého vrcholy aj hrany sú navzájom rôzne. Vzdialenosťou $d(x, y)$ vrcholov x a y budeme rozumieť dĺžku najkratšej cesty, ktorá spája vrcholy x a y. Ak takáto cesta neexistuje, kladieme $d(x, y) = \infty$.

Nech N je podmnožinou hranovej množiny hypergrafu (X, M); potom hypergraf (X, N) nazveme faktorom hypergrafu (X, M). Pod priemerom hypergrafu (X, M) rozumieme maximum vzdialeností vrcholov množiny X. Ak toto maximum neexistuje, kladieme priemer rovnajúci sa symbolu ∞. Priemer hypergrafu H budeme označovať symbolom $d(H)$.

Hypergraf $H=(X, M)$ sa nazýva kompletný, ak $M=2^X-\{\emptyset\}$. Kompletný hypergraf s n vrcholmi budeme v ďalšom označovať K_n.

Systém faktorov $\{H_i\}$, $i=1, 2, \ldots, m$, tvorí rozklad hypergrafu K_n, ak každá hrana z K_n patrí do hranovej množiny práve jedného z uvedených faktorov. Pretože hypergraf K_n obsahuje aj hranu, ktorá obsahuje všetky vrcholy, existuje v rozklade $\{H_i\}$ vždy aspoň jeden faktor s priemerom 1.

Veta 1. Nech K_n je kompletný hypergraf a $\{H_i\}$, $i=1, 2, \ldots, m$, je jeho rozklad na faktory s priemermi $d_1, d_2, \ldots, d_m$, pričom d_i sú prirodzené čísla alebo symboly ∞. Pre každé N väčšie alebo rovnajúce sa n možno rozložiť hypergraf K_N na m faktorov s priemermi $d_1, d_2, \ldots, d_m$.

Dôkaz. Faktory hypergrafu K_n možno podľa priemerov rozdeliť do troch skupín:

a) faktory s priemerom 1:

b) faktory s konečným priemerom väčším ako 1;

c) faktory s priemerom ∞.

Majme kompletný hypergraf K_N s N vrcholmi. Nech hypergraf K_n s n vrcholmi možno rozložiť na faktory H_i s priemermi d_i, $i=1, 2, \ldots, n$. Nech hypergraf K_n je podhypergrafom hypergrafu K_N. Vyberme ľubovoľne, ale pevne, vrchol x_0 hypergrafu K_n. Pomocou faktora H_i vytvoríme faktor G_i hypergrafu K_N s rovnakým priemerom ako H_i.

a) Ak $d(H_i)=1$, faktor G_i bude obsahovať tieto hrany: hrany, ktoré obsahoval faktor H_i a všetky hrany v tvare:

$$\{x_{k_1}, \ldots, x_{k_p}, x_{n+1}, x_{n+2}, \ldots, x_N\}$$

kde vrcholy $x_{n+1}, \ldots, x_N$ nie sú vrcholmi hypergrafu K_n a hrana $\{x_{k_1}, \ldots, x_{k_p}, x_0\}$ je hranou faktora H_i. Každé dva vrcholy faktora G_i sú spojené hranou, teda jeho priemer je 1. Treba si uvedomiť, že faktory H_i tvoria rozklad K_n, a teda každá z hrán je najviac v jednom z faktorov. Z toho dôvodu aj faktory G_i obsahujú rôzne hrany.

b) Ak faktor H_i má konečný priemer $d>1$, faktor G_i bude obsahovať hrany faktora H_i a všetky hrany typu $(x_{k_1}, \ldots, x_{k_p}, x_{n+1}, \ldots, x_N\}$, pričom $\{x_{k_1}, \ldots, x_{k_p}, x_0\}$ je hrana hypergrafu, H_i a x_0 je vybraný pevný vrchol. Ukážeme, že faktory H_i a G_i majú rovnaké priemery. Nech vzdialenosť vrcholov x a y vo faktore H_i je d. Vzdialenosť tých istých vrcholov vo faktore G_i nebude väčšia ako d, pretože G_i obsahuje všetky vrcholy faktora H_i. Nech by existovala vo faktore G_i cesta, ktorá spája vrcholy x a y dĺžky menšej ako d. Táto cesta obsahuje vrcholy a hrany hypergrafu K_N. Nahraďme tieto vrcholy a hrany prvkami z K_n, a to tak, že hranu $\{x_{k_1}, \ldots, x_{k_p}, x_{n+1}, \ldots, x_N\}$ nahradíme hranou $\{x_{k_1}, \ldots, x_{k_p}, x_0\}$ a vrcholy $x_{n+1}, \ldots, x_N$ vrcholom x_0. Takýmto spôsobom dostaneme spor s predpokladom, pretože sled vytvorený vo faktore H_i obsahuje cestu spájajúcu vrcholy x a y, ktorej dĺžka je menšia ako d_i. Ak by niektorý z vrcholov x a y ležal v K_N, nie však v K_n, potom by ich vzdialenosť bola 1 a teda nemá vplyv na priemer faktora G_i.

c) Ak faktor H_i má priemer ∞, potom faktor G_i bude obsahovať rovnaké hrany ako H_i, zrejme jeho priemer je opäť ∞.

Vyšetrením bodov a), b), c) sú vyčerpané všetky faktory rozkladu. Aby faktory G_i tvorili rozklad, musíme zabezpečiť, aby každá hrana z K_N patrila práve do jedného faktora G_i. Dosiahneme to tak, že všetky hrany, ktoré neboli medzi hranami niektorého z faktorov G_i, priradíme k niektorému faktoru s priemerom 1. Tým sme vetu dokázali.

Z vety 1 vyplýva, že ak vieme rozložiť kompletný hypergraf s n vrcholmi na faktory s danými priemermi, potom môžeme rovnako rozložiť aj kompletný hypergraf s väčším počtom vrcholov. Zostáva otázkou, či možno rozložiť hypergraf K_n na faktory s priemermi $d_1, d_2, \ldots, d_m$. Symbol $F(d_1, \ldots, d_m)$ bude označovať minimálne číslo n také, že hypergraf K_n možno rozložiť na faktory s priemermi $d_1, d_2, \ldots, d_m$. Ak takéto číslo neexistuje, kladieme $F(d_1, \ldots, d_m) = \infty$.

Veta 2. Nech $d_1, \ldots, d_m$, sú prirodzené čísla alebo symboly ∞, pričom existuje i také, že $d_i = 1$. Potom $F(d_1, \ldots, d_m)$ je konečné číslo a platí nerovnosť

$$F(d_1, \ldots, d_m) \leqq d_1 + \ldots + d_m = m$$

pričom ak d_j je symbol ∞, tak v súčte namiesto d_j dosadíme 1.

Dôkaz. Dôkaz urobíme konštrukciou rozkladu hypergrafu K_n, kde $n = d_1 + \ldots + d_m + m$. Rozdeľme vrcholy hypergrafu do m skupín, že v i-tej skupine bude $d_i + 1$ vrcholov. Zoberme do úvahy opäť tri prípady.

a) Nech $d_i = 1$. V i-tej skupine sú dva vrcholy x a y. Hranovú množinu faktora H_i bude tvoriť hrana $\{x, y\}$ a hrany, v ktorých chýba jedine vrchol x alebo y. Zrejme H_i má priemer 1.

b) Nech d_i je prirodzené číslo väčšie ako 1. Skupina vrcholov, ktorá prislúcha k d_i, obsahuje $d_i + 1$ vrcholov. Označme tieto vrcholy $x_1, x_2, \ldots, x_{d_i+1}$. Hranovú množinu faktora H_i potom budú tvoriť hrany $\{x_i, x_{i+1}\}$, $1 \leqq i \leqq d_i$, a hrana, ktorá obsahuje vrchol x_2 spolu s vrcholmi rôznymi od $x_1, \ldots, x_{d_i+1}$. Platí, že vzdialenosť x_1, x_{d_i+1} je d_i a tiež, že vzdialenosť ľubovoľných vrcholov u, v je menšia alebo sa rovná d_i. Ak vrcholy u, v nepatria do i-tej skupiny, potom ich vzdialenosť je 1. Inak je ich vzdialenosť celé číslo z intervalu $\langle 0, d_i \rangle$.

c) Nech $d_i = \infty$. V i-tej množine sú dva vrcholy x a y. Hranová množina faktora H_i bude obsahovať jedinú hranu, ktorá obsahuje všetky vrcholy hypergrafu okrem vrcholov x a y. Priemer faktora H_i bude zrejme ∞.

Hranové množiny faktorov sú zrejme disjunktné. Aby faktory H_i tvorili rozklad, treba aby každá hrana hypergrafu patrila do niektorého faktora. Všetky hrany, ktoré nepatrili do niektorého z faktorov H_i, pridáme k niektorému z faktorov s priemerom 1. Faktory H_i s priemermi $d_1, \ldots, d_m$ tvoria rozklad K_n; teda vetu sme dokázali.

Pri dôkazoch viet 1 a 2 mali faktory s priemerom 1 dôležitú úlohu. Pri grafoch sa takéto faktory nevyskytujú, pretože jediným faktorom kompletného grafu s priemerom 1 je samotný graf.

2. Rozklad K_n na faktory s priemerom 1

Aby faktor hypergrafu mal priemer 1, je nevyhnutné a stačí, aby vzdialenosť $d(x, y)$ bola 1 pre každú dvojicu (x, y) rôznych vrcholov. To znamená, že každé dva vrcholy hypergrafu sú spojené hranou.

Definícia. Hovoríme, že medzi vrcholmi x, y hypergrafu H je susednosť k, ak v H existuje práve k rôznych hrán incidentných s obidvoma vrcholmi x, y. Počtom susedností faktora rozumieme súčet susedností cez všetky dvojice vrcholov hypergrafu.

Poznámka. Susednosti môžeme sčítať rovnako aj cez hrany hypergrafu. Výsledok, ktorý dostaneme, je rovnaký.

Lema 1. Počet susedností v kompletnom hypergrafe s n vrcholmi je

$$W = \binom{n}{0}\binom{n}{2} + \ldots + \binom{n}{i}\binom{n-i}{2} + \ldots + \binom{n}{n-2}\binom{2}{2}$$

Dôkaz. Susednosti budeme sčítavať cez hrany hypergrafu. Hrana s k vrcholmi prispieva k celkovému súčtu susedností číslom $\binom{k}{2}$. V kompletnom hypergrafe K_n je práve $\binom{n}{n-k}$ takýchto hrán. Sčítaním cez všetky hrany hypergrafu dostaneme dokazovaný vzťah

Lema 2. Nech priemer faktora spĺňa rovnosť $d(H) = 1$; potom počet jeho susedností je aspoň $\binom{n}{2}$.

Dôkaz. Každé dva vrcholy faktora sú spojené aspoň jednou hranou. Každá dvojica vrcholov prispieva k celkovému súčtu susedností aspoň jednou susednosťou. Všetkých dvojíc vrcholov je $\binom{n}{2}$, teda susedností je aspoň $\binom{n}{2}$.

Veta 3. Nech faktory H_i, $i = 1, 2, \ldots, m$, $d(H_i) = 1$, tvoria rozklad kompletného hypergrafu s n vrcholmi. Potom platí: $m \leqq 2^{n-2}$.

Dôkaz. Celkový počet susedností hypergrafu K_n je

$$W_n = \sum_{i=0}^{n-2} \binom{n}{i}\binom{n-i}{2}$$

Každý faktor obsahuje aspoň $\binom{n}{2}$ susedností. Ak použijeme vzťah medzi binomickými koeficientmi

$$\binom{n}{i}\binom{n-i}{2}=\binom{n}{2}\binom{n-2}{i}$$

postupne dostávame

$$m \geqq \frac{W_n}{\binom{n}{2}}=\frac{\sum_i \binom{n}{i}\binom{n-i}{2}}{\binom{n}{2}}=\frac{\sum_i \binom{n}{2}\binom{n-2}{i}}{\binom{n}{2}}=$$

$$=\sum_i \binom{n-2}{i}=2^{n-2}$$

Tým sme dokázali tvrdenie vety.

Poznámka. Pre $n \geqq 4$ neexistuje rozklad hypergrafu K_n na faktory s priemerom 1 tak, že $m = 2^{n-2}$. Nech by totiž takýto rozklad existoval. Každý z faktorov H_i, $i = 1, \ldots, m$, obsahuje práve $\binom{n}{2}$ susedností. Vyšetrujeme hrany s $n-1$ vrcholmi. Faktor, ktorý obsahuje takúto hranu, obsahuje okrem nej už len dvojvrcholové hrany. Aby priemer faktora bol 1, faktor musí obsahovať $n-1$ dvojvrcholových hrán. Celý hypergraf K_n obsahuje $\binom{n}{2}$ takýchto hrán; lenže už v n faktoroch s $n-1$ dvojvrcholovými hranami leží až $n(n-1)$ týchto hrán, čo je spor.

Kvôli ilustrácii uvedieme rozklad K_4 na faktory s priemerom 1. Nech K_4 má vrcholy x_1, x_2, x_3, x_4. Potom faktory majú tvar:

$H_1 = \{x_1, x_2, x_3, x_4\}$
$H_2 = \{x_1, x_3, x_4\},\ \{x_1, x_2, x_3\},\ \{x_2, x_4\}$
$H_3 = \{x_1, x_2, x_4\},\ \{x_3, x_2, x_4\},\ \{x_1, x_3\}$

Ostatné hrany $\{x_1, x_2\}$, $\{x_1, x_4\}$, $\{x_2, x_3\}$, $\{x_3, x_4\}$, $\{x_1\}$, $\{x_2\}$, $\{x_3\}$, $\{x_4\}$ môžeme pridať k ľubovoľnému z faktorov.

3. Prípad m = 2, 3, 4

V prvej časti, vo vete 2, sme dokázali existenciu rozkladu hypergrafu K_n na faktory s priemermi $d_1, \ldots, d_m$. V tejto časti určíme hodnoty $F(d_1, \ldots, d_m)$ pre $m = 2, 3, 4$, ak všetky d_i sú prirodzené čísla.

Poznámka. Ak $d_1, \ldots, d_m$ sú prirodzené čísla, existuje $d_i = 1$, tak $F(d_1, \ldots, d_m)$ je konečné číslo. V ďalšom budeme priamo predpokladať, že $d_1 = 1$ a čísla d_i sú usporiadané podľa veľkosti.

Pre hodnotu $F(d_1, \ldots, d_m)$ platí nerovnosť $F(d_1, \ldots, d_m) \geqq d_m + 1$. Ak faktor má priemer d_m, tak obsahuje cestu dĺžky d_m, a teda aspoň $d_m + 1$ rôznych vrcholov.

Vyšetrujme hodnoty $F(d_1, \ldots, d_m)$ pre jednotlivé m.

a) Nech $m = 2$. Podľa poznámky $d_1 = 1$. Ak $d_2 = 1$, tak zrejme $F(1, 1) = 3$. Ak $d_2 > 1$, tak $F(1, d_2) = d_2 + 1$. Stačí ak ukážeme, že $F(1, d_2) \leqq d_2 + 1$. Nech faktor H_2 hypergrafu K_{d_2+1} obsahuje len dvojvrcholové hrany také, že tvoria cestu dĺžky d_2 medzi dvoma vrcholmi hypergrafu. Faktor H_2 má priemer d_2, faktor H_1, ktorý obsahuje všetky ostatné hrany, má priemer 1. Našli sme rozklad K_{d_2+1} na faktory s priemermi 1 a d_2, teda nerovnosť platí.

b) Nech $m = 3$. Hľadáme hodnotu $F(1, d_2, d_3)$. Ak platí nerovnosť $d_3 \geqq 3$, tak $F(1, d_2, d_3) = d_3 + 1$. Ukážeme, že platí $F(1, d_2, d_3) = d_3 + 1$. Vytvoríme rozklad hypergrafu K_{d_3+1} na faktory s priemermi $1, d_2, d_3$. Podľa známeho výsledku z teórie grafov možno rozložiť kompletný graf s $2n$ vrcholmi na hamiltonovské cesty (t. j. cesty dĺžky $2n - 1$). Každý kompletný graf s aspoň štyrmi vrcholmi obsahuje dve hranovo disjunktné hamiltonovské cesty. Ak faktory H_2, H_3 obsahujú len hrany týchto ciest, tak platí $d(H_2) = d(H_3) = d_3$. Označme $(x_1, h_1, x_2, \ldots, x_{d_3}, h_{d_3}, x_{d_3+1})$ hamiltonovskú cestu. Ak k nej pridáme hranu z $(k + 2)$ vrcholov $\{x_1, x_2, \ldots, x_{k+2}\}$, pričom $k + 2 = d_3 + 1$, priemer faktora sa skráti na dĺžku $d_3 - k$. Uvedeným spôsobom môžeme priemer faktora H_2 skrátiť na ľubovoľnú dĺžku d_2, $1 \leqq d_2 \leqq d_3$.

Zistíme hodnoty $F(1, d_2, d_3)$, ak $d_3 \leqq 3$. Platí $F(1, 1, d) \geqq 4$ pre konečné d. Ak by $F(1, 1, d) = 3$, potom by d nevyhnutne muselo byť symbolom ∞. Faktory H_1 a H_2 spolu obsahujú 6 susedností, čo je celkový počet susedností v K_3. Faktor H_3 by bol potom totálne nesúvislý. Konštrukciou možno ukázať, že:

$$F(1, 1, 1) = 4$$
$$F(1, 1, 2) = 4$$
$$F(1, 2, 2) = 4$$

c) Nech $m = 4$. Ak jeden z faktorov má priemer $d \geqq 5$, tak $F(1, d_2, d_3, d_4) \leqq$ $\leqq d_4 + 1$. Ukážeme, že $F(1, d_2, d_3, d_4) \leqq d_4 + 1$ pre $d_4 \geqq 5$. Kompletný graf s n vrcholmi počnúc $n = 6$, obsahuje aspoň 3 hranovo disjunktné hamiltonovské cesty. Ak $d_2 = d_3 = d_4$, tak faktory H_2, H_3, H_4 budú obsahovať hrany týchto hamiltonovských ciest. Platí $F(1, d_4, d_4, d_4) = d_4 + 1$. Ak platí $d_2 = d_3 = d_4 - 1$ a hamiltonovské cesty sú v tvare $(x_1, h_1, \ldots, x_{d_4}, h_{d_4}, x_{d_4} + 1)$ a $(y_1, u_1, \ldots, y_{d_4}, u_{d_4}, y_{d_4} + 1)$, tak faktory H_2' a H_3', ktoré vzniknú z faktorov H_2 a H_3 pridaním hrán $\{x_1, x_2, x_3\}$, resp. $\{y_1, y_2, y_3\}$, budú mať priemery $d_2 = d_3 = d_4 - 1$. Ukázali sme, že $F(1, d_4 - 1, d_4 - 1, d_4) = d_4 + 1$. Ak $d_2 < d_4 - 1$, tak $d_2 = d_4 - k$ a faktor H_2' dostaneme z faktora H_2 pridaním hrany $\{x_1, x_2, \ldots, x_{k+2}\}$. Faktor H_3' s priemerom d_3 dostaneme z faktora H_3 pridaním vhodného počtu trojvrcholových hrán. Nech napr. $d_3 = d_4 - p$. Potom k hranám faktora H_3 pridáme všetky hrany typu $\{y_q, y_r, y_s\}$; $q, r, s \in \langle 1, p \rangle$. Rozklad hypergrafu K_{d_4+1} na faktory s priemermi $1, d_2, d_3, d_4 \leqq 5$ možno vždy zostrojiť. Uvedieme bez dôkazu hodnoty $F(1, d_2, d_3, d_4)$ pre $d_4 < 5$:

$F(1, 1, 1, 1)=5,\quad F(1, 1, 2, 3)=4$
$F(1, 1, 1, 2)=4,\quad F(1, 2, 2, 3)=4$
$F(1, 1, 2, 2)=4,\quad F(1, 1, 3, 3)=4$
$F(1, 2, 2, 2)=4,\quad F(1, 2, 3, 3)=4$
$F(1, 1, 1, 3)=4,\quad F(1, 3, 3, 3)=5$

$F(1, 1, 1, 4)=5,\quad F(1, 3, 3, 4)=5$
$F(1, 1, 2, 4)=5,\quad F(1, 1, 4, 4)=5$
$F(1, 2, 2, 4)=5,\quad F(1, 2, 4, 4)=5$
$F(1, 1, 3, 4)=5,\quad F(1, 3, 4, 4)=5$
$F(1, 2, 3, 4)=5,\quad F(1, 4, 4, 4)=6$

Výsledky poslednej časti môžeme zhrnúť do nasledujúcej vety.

Veta 4. Nech $d_2, \ldots, d_i$, $i=2, 3, 4$, sú prirodzené čísla; potom existuje rozklad kompletného hypergrafu na faktory s uvedenými priemermi a platí $F(1, d_2, \ldots, d_i) = d_i + 1$ pre $d_i \geqq i+1$. Pre menšie hodnoty d_i sú $F(d_1, \ldots, d_i)$ uvedené v texte.

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РЕЗЮМЕ

О РАЗБИЕНИИ ПОЛНОГО ГИПЕРГРАФА НА ФАКТОРЫ С ДАННЫМИ ДИАМЕТРАМИ

Л. Ниепел, Братислава

В этой работе изучается проблема разбиения полного гиперграфа на факторы с наперед заданными диаметрами.

В первой части показано существование разбиения и его «наследственное» свойство. Во второй части изучается проблема разбиения на факторы с диаметром 1. В третей части комплетно показан случай разбиения на 2, 3, 4 фактора.

SUMMARY

ON DECOMPOSITION OF COMPLETE HYPERGRAPHS INTO FACTORS WITH GIVEN DIAMETERS

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In this paper there is solved the problem of decomposition of complete hypergraphs into factors with given diameters.

In the first part there is shown the existence of the factorisation and its "hereditary" property. In the second part the problem of decomposition of complete hypergraphs into factors with diameter 1 is solved. In the third part there is shown the case of decomposition of complete hypergrafph into 2, 3, 4 factors.

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NOTE TO THE EXPONENT OF CONVERGENCE

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In the monograph [2] the notion of the exponent of convergence $\lambda(x)$ of a non-decreasing sequence $x = \{x_t\}_{t=1}^{\infty}$ of positive reals is introduced by the formula $\lambda(x) = \inf\left\{\sigma>0: \sum_{t=1}^{\infty} x_t^{-\sigma} < +\infty\right\}$. Professor T. Šalát suggested to investigate the exponent of convergence as a real function defined on the set s^+ of all real non-decreasing sequences $\{x_t\}_{t=1}^{\infty}$ with the property $x_1 \geqq \gamma (0<\gamma)$ endowed with the Fréchet metric $\varrho(x, y) = \sum_{t=1}^{\infty} 2^{-t} \frac{|x_t - y_t|}{1+|x_t - y_t|}$ $(x = \{x_t\}_{t=1}^{\infty}, y = \{y_t\}_{t=1}^{\infty})$. In this paper Borel classification and some other properties of sets $\{x \in s^+ : \lambda(x) < a\}$ and $\{x \in s^+ : \lambda(x) > a\}$ will be given. Also we will prove some properties of the exponents of convergence of two dimensional sequences.

It is well known, that the set X of all real sequences endowed with the Fréchet metric is a complete metric space. The convergence in this space is the pointwise convergence.

Lemma 1. The set s^+ is a perfect subset of (X, ϱ) and hence the space (s^+, ϱ) is complete.

Proof. If $x(i) = \{x(i)_t\}_{t=1}^{\infty}$, $x(i) \in s^+$ $(i = 1, 2, \ldots)$ and $\lim_{i\to\infty} x(i) = x$, then $x \in s^+$. This follows immediately from the pointwise convergence in the sense of Fréchet metric. Hence the set s^+ is closed. If $x \in s^+$ then also $x(i) \in s^+$, where $x(i)_t = x_t$ for $t = 1, 2, \ldots, i$ and $x(i)_t = x_t + 1$ for $t = i+1, i+2, \ldots$ $(i = 1, 2, \ldots)$. Obviously $\lim_{i\to\infty} x(i) = x$ and consequently the set s^+ is dense in itself.

Theorem 1. The function $\lambda : s^+ \to \langle 0, \infty\rangle$ is discontinuous in every point $x \in s^+$.

Proof. Let $x \in s^+$. Let $y = \{t^{\alpha}\}_{t=1}^{\infty}$, $\alpha > 0$ and $1/\alpha \neq \lambda(x)$. Then $\lambda(y) = \limsup_{t\to\infty} \log t / \log t^{\alpha} = 1/\alpha$ (see [2], p. 41). We construct for each $i = 1, 2, \ldots$ an element

$x(i) \in s$ as follows: $x(i)_t = x_t$ for $t = 1, 2, \ldots, i$. If $(i+1)^\alpha \geqq x_i$ then $x(i)_t = t^\alpha$ for $t = i+1, i+2, \ldots$. If $(i+1)^\alpha < x_i$ then $x(i)_t = x_i$ for $t = i+1, i+2, \ldots, j-1$, where j is the least natural number with the property $j^\alpha \geqq x_i$ and for $t = j, j+1, \ldots$ we put $x(i)_t = t^\alpha$. The elements of this sequence $\{x(i)\}_{i=1}^{\infty}$ obviously belong to s^+ and $\lambda(x(i)) = 1/\alpha$ holds for each $i = 1, 2, \ldots$. Since $\lim_{i\to\infty} x(i) = x$ and $\lim_{i\to\infty} \lambda(x(i)) = 1/\alpha \neq \lambda(x)$ the function λ is discontinuous in x.

Corollary 1. The function $\lambda: s^+ \to \langle 0, \infty\rangle$ does not belong to the first Baire class.

Proof. The statement of Corollary 1 is an immediate consequence of Lemma 1, Theorem 1 and a well known Baire theorem (see [3], p. 182).

Theorem 2. The sets $A^a = \{x \in s^+ : \lambda(x)\} < a\}$ and $A_a = \{x \in s^+ : \lambda(x) > a\}$ belong to the third additive Borel class for each $a \in (-\infty, +\infty)$.

Proof. We consider the set A^a. Obviously, if $a \leqq 0$ then $A^a = \emptyset$ and the statement of Theorem 2 is true. Let $a > 0$. Then $A^a = \{x : \lambda(x) < a\} = \left\{x : \exists_{\sigma, 0<\sigma<a} \sum_{t=1}^{\infty} x_t^{-\sigma} < +\infty\right\} = \left\{x : \exists_{k=k_0}^{\infty} \sum_{t=1}^{\infty} x_t^{-(a-1/k)} < +\infty\right\} = \bigcup_{k=k_0}^{\infty} \left\{x : \sum_{t=1}^{\infty} x_t^{-(a-1/k)} < +\infty\right\}$, where k_0 is the least positive integer number with the property $\sigma = a - 1/k > 0$. We put $M_k = \left\{x : \sum_{t=1}^{\infty} x_t^{-\sigma} < +\infty\right\}$, where $\sigma = a - 1/k$ $(k = k_0, k_0 + 1, \ldots)$. Then $M_k = \{x : \forall_{p=1}^{\infty} \exists_{q=1}^{\infty} \forall_{m=1}^{\infty} \forall_{n=1}^{\infty} x_{q+m}^{-\sigma} + \ldots + x_{q+m+n}^{-\sigma} \leqq 1/p\} = \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \{x : x_{q+m}^{-\sigma} + \ldots + x_{q+m+n}^{-\sigma} \leqq 1/p\}$. We put $M_{kpqmn} = \{x : x_{q+m}^{-\sigma} + \ldots + x_{q+m+n}^{-\sigma} \leqq 1/p\}$. Let $x(i) \in M_{kpqmn}$ $(i = 1, 2, \ldots)$ and $\lim_{i\to\infty} x(i) = x$. Then $\lim_{i\to\infty} x(i)_t^{-\sigma} = x_t^{-\sigma}$ for each $t = q+m, q+m+1, \ldots, q+m+n$ whence $x \in M_{kpqmn}$. Consequently each of sets M_{kpqmn} is closed and the set $\{x : \lambda(x) < a\} = \bigcup_{k=k_0}^{\infty} \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} M_{kpqmn}$ belongs to the third additive Borel class.

We investigate the set A_a. If $a < 0$, then obviously $A_a = s^+$ and the statement of Theorem 2 is true. Let $a \geqq 0$. Then $A_a = \{x : \lambda(x) > a\} = \left\{x : \exists_{k=1}^{\infty} \sum_{t=1}^{\infty} x_t^{-(a+1/k)} = +\infty\right\} = \bigcup_{k=1}^{\infty} \left\{x : \sum_{t=1}^{\infty} x_t^{-(a+1/k)} = +\infty\right\}$. We put $N_k = \left\{x : \sum_{t=1}^{\infty} x_t^{-\sigma} = +\infty\right\}$, where $\sigma = a + 1/k$ $(k = 1, 2, \ldots)$. Then $N_k = \left\{x : \forall_{p=1}^{\infty} \exists_{q=1}^{\infty} \forall_{m=1}^{\infty} \sum_{t=1}^{q+m} x_t^{-\sigma} \geqq p\right\} = \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{m=1}^{\infty} \left\{x : \sum_{t=1}^{q+m} x_t^{-\sigma} \geqq p\right\}$. It is easy to see that each of sets $N_{kpqm} = \left\{x : \sum_{t=1}^{q+m} x_t^{-\sigma} \geqq p\right\}$ is closed. Hence the set $\{x : \lambda(x) > a\} = \bigcup_{k=1}^{\infty} \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{m=1}^{\infty} N_{kpqm}$ belongs to the third additive Borel class.

Theorem 3. Every of sets $A^a = \{x \in s^+ : \lambda(x) < a\}$, $a \in (-\infty, +\infty)$, is of the first category in s^+.

Proof. It follows from the proof of Theorem 2 that $A^a = \bigcup_{k=k_0}^{\infty} \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} M_{kpqmn} = \bigcup_{k=k_0}^{\infty} \bigcap_{p=1}^{\infty} M_{kp}$, where $M_{kp} = \{x : \exists_{q=1}^{\infty} \forall_{m=1}^{\infty} \forall_{n=1}^{\infty} x_{q+m}^{-\sigma} + \dots + x_{q+m+n}^{-\sigma} \leqq \leqq 1/p\}$. We show that each of sets M_{kp} is of the first category in s^+. It is sufficient to show that M_{kp} is an F_σ-set and its complement is dense in s^+ (see [3], p. 88). Let $z = \{z_t\}_{t=1}^{\infty} \in s^+$. Let $\varepsilon > 0$ and let u be the smallest positive integer for which $\sum_{t=u+1}^{\infty} 2^{-t} < \varepsilon$. We put $y = \{t^\alpha\}_{t=1}^{\infty}$, where $\alpha = 1/\sigma$ and define the sequence $x = \{x_t\}_{t=1}^{\infty} \in s^+$ as follows: $x_t = z_t$ for $t = 1, 2, \dots, u$. If $x_u \leqq (u+1)^\alpha$, then $x_t = t^\alpha$ for $t = u+1, u+2, \dots$. If $x_u > (u+1)^\alpha$, then we put $x_t = x_u$ for $t = u+1, u+2, \dots, v-1$, where v is the smallest positive integer with the property $v^\alpha \geqq x_u$, and $x_t = t^\alpha$ for $t = v, v+1, \dots$. Obviously, $\varrho(x, z) < \varepsilon$ holds for the sequence x. There is t_0 such that $x_t = t^\alpha$ holds for each $t = t_0, t_0 + 1, \dots$. Hence for every q there are m and n such that $x_{q+m}^{-\sigma} + \dots + x_{q+m+n}^{-\sigma} = (q+m)^{-\alpha\sigma} + \dots + (q+m+n)^{-\alpha\sigma} = \sum_{t=q+m}^{q+m+n} t^{-1} > p^{-1}$. Consequently the complement of M_{kp} is dense. Each of sets M_{kpqmn} is closed and hence $M_{kp} = \bigcup_{q=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} M_{kpqmn}$ is an F_σ-set. Since every of sets M_{kp} is of the first category in s^+ the statement of Theorem 3 immediately follows from the expression $A^a = \bigcup_{k=1}^{\infty} \bigcap_{p=1}^{\infty} M_{kp}$.

Remark 1. Using the method from the proof of Theorem 3 it is possible to see that the set $s_\infty^+ = \{x \in s^+ : \lim_{t\to\infty} x_t = +\infty\}$ is dense in s^+.

Theorem 4. The set $\{x \in s^+ : \lambda(x) = +\infty\}$ is residual in s^+.

Proof. According to Theorem 3 the set $\{x : \lambda(x) < +\infty\} = \bigcup_{n=1}^{\infty} \{x : \lambda(x) < n\}$ is of the first category in s^+. According to Lemma 1 the space s^+ is complete, hence the statement of Theorem 4 is fulfilled.

Lemma 2. The set $s_\infty^+ = \{x \in s^+ : \lim_{t\to\infty} x_t = +\infty\}$ is residual in s^+.

Proof. Since each of sequences $x = \{x_t\}_{t=1}^{\infty} \in s^+$ is increasing there exists always $\lim_{t\to\infty} x_t$. Let $B_n = \{x \in s^+ : \lim_{t\to\infty} x_t \leqq n\}$ for $n = 1, 2, \dots$. It follows from the definition of B_n and from the fact that every sequence in s^+ is increasing, that for every $n = 1, 2, \dots$ the set B_n is closed. Hence $\bigcup_{n=1}^{\infty} B_n$ is an F_σ-set. According to Remark 1 $s_\infty^+ = s^+ - \bigcup_{n=1}^{\infty} B_n$ is dense in s^+. Consequently $\bigcup_{n=1}^{\infty} B_n$ is of the first category in s^+ and the statement of Lemma 2 follows.

There is a natural question whether the statements of Theorem 2 and Theorem 3 are true if we replace the space s^+ by its subspace s^+_∞. The answer is positive in the case of Theorem 2. This is a consequence of a well known fact, that a set A in a metric space $Y(\subset Z)$ belongs to the additive Borel class α if and only if there is a set A_0 of the additive Borel class α in Z such that $A = A_0 \cap Y$. The answer to the question in the case of Theorem 3 is also positive.

Theorem 5. Each of sets $A^a = \{x \in s^+_\infty : \lambda(x) < a\}$, $a \in (-\infty, +\infty)$ is of the first category in s^+_∞.

Proof. It is sufficient to show that any set B nowhere dense in s^+ is also nowhere dense in s^+_∞. Let $K(x, \varepsilon) = \{y : \varrho(x, y) < \varepsilon\}$ be an open sphere, $x \in s^+_\infty$. There exists a sphere $K(y, \delta) \subset K(x, \varepsilon)$ such that $K(y, \delta) \cap B = \emptyset$. Since s^+_∞ is dense in s^+ there is $z \in s^+_\infty$ and $\eta > 0$ such that $K(z, \eta) \subset K(y, \delta)$ and hence $K(z, \eta) \cap B = \emptyset$. The set B is nowhere dense s^+_∞.

Further a formula for the calculation of the exponent of convergence of the two dimensional sequence is given. This formula is analogous to that derived in [2] for the one dimensional case.

Let us give some definitions which will be used in the next text. Let N be the set of natural numbers. A finite real function defined on $N \times N$ is said to be a two dimensional sequence and is denoted by $S = \{s_{mn}\}_{m,n=1}^{\infty}$. The set $N \times N$ will be considered with the product partial order $\leqq$, i. e. $(m, n) \leqq (p, q)$ if and only if $m \leqq p$ and $n \leqq q$. If some of the inequalities $m \leqq p$ and $n \leqq q$ hold in the strict sense we write $(m, n) < (p, q)$. A two dimensional sequence $S = \{s_{mn}\}_{m,n=1}^{\infty}$ is said to be increasing (decreasing) if $s_{mn} \leqq s_{pq}$ $(s_{mn} \geqq s_{pq})$ holds whenever $(m, n) \leqq (p, q)$. We will say that $\limsup\limits_{m,n\to\infty} s_{mn} = s \in E_1$ if for every $\varepsilon > 0$ there exists $(m_0, n_0) \in N \times N$ such that $s_{mn} < s + \varepsilon$ holds for each $(m, n) \geqq (m_0, n_0)$ and simultaneously for every $\varepsilon > 0$ and each $(m_0, n_0) \in N \times N$ there exists $(m, n) \geqq (m_0, n_0)$ such that $s_{mn} > s - \varepsilon$ holds. We will say that $\limsup\limits_{m,n\to\infty} s_{mn} = +\infty$ if for every $K > 0$ and each $(m_0, n_0) \in$ $\in N \times N$ there exists $(m, n) \geqq (m_0, n_0)$ such that $s_{mn} > K$ and $\limsup\limits_{m,n\to\infty} s_{mn} = -\infty$ if for each $K < 0$ there exists $(m_0, n_0) \in N \times N$ such that $s_{mn} < K$ holds whenever $(m, n) \geqq (m_0, n_0)$. The notion $\liminf\limits_{m,n\to\infty} s_{mn}$ will be reasoned analogously to the notion $\limsup\limits_{m,n\to\infty} s_{mn}$. We sill say $\lim\limits_{m,n\to\infty} s_{mn} = s$ if $\limsup\limits_{m,n\to\infty} s_{mn} = \liminf\limits_{m,n\to\infty} s_{mn} = s$.

The two dimensional series corresponding to a two dimensional sequence $\{a_{mn}\}_{m,n=1}^{\infty}$ is the two dimensional sequence $\{s_{mn}\}_{m,n=1}^{\infty}$ of its partial sums $s_{mn} = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} a_{ij}$. It will be signed by $\sum\limits_{i,j=1}^{\infty} a_{ij}$. We shall say that the two dimensional series is convergent if there is a finite limit $\lim\limits_{m,n\to\infty} s_{mn} = s$. This fact will be signed by

$\sum_{i,j=1}^{\infty} a_{ij}=s$. Further we shall use some properties of series with non-negative terms, i.e. $a_{ij} \geqq 0$ for $i, j = 1, 2, \dots$ (see [1], p. 377). If the series $\sum_{i,j=1}^{\infty} a_{ij}$ is convergent then $\lim_{i,j\to\infty} a_{ij} = 0$ (this statement holds also without assumption $a_{ij} \geqq 0$). Let $\sum_{i,j=1}^{\infty} a_{ij}$ be a series with non-negative terms. Then there exists the limit $\lim_{m,n\to\infty} s_{mn} = \sup_{m,n=1,2,\dots} s_{mn}$. If π is any permutation of the set $N \times N$ then $\sum_{i,j=1}^{\infty} a_{ij} = \sum_{i,j=1}^{\infty} a_{\pi(i,j)}$.

Let us give some lemmas which will be used in the next text.

Lemma 3. Let $\sum_{i,j=1}^{\infty} b_{ij}$ be a convergent series with non-negative terms and let $a_{ij} \geqq 0$. Let $a_{ij} \leqq b_{ij}$ hold for $(i, j) \in N \times N$ with the exception at most a finite number of pairs of indices. Then the series $\sum_{i,j=1}^{\infty} a_{ij}$ is convergent.

Proof. Let $a_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$, $t_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij}$ and let us suppose that $0 \leqq a_{ij} \leqq b_{ij}$ holds for each pair $(i, j) \in N \times N$. Then $s_{mn} \leqq t_{mn}$ holds for every pair $(m, n) \in N \times N$, consequently $\lim_{m,n\to\infty} s_{mn} = \sup_{m,n} s_{mn} = \sup_{m,n} t_{mn} = \lim_{m,n\to\infty} t_{mn} < +\infty$. If inequalities $a_{ij} \leqq b_{ij}$ do not hold for a finite number of pairs of indices, it is clear that $\sup_{m,n} s_{mn} (= \lim_{m,n\to\infty} s_{mn})$ is again a finite number.

Lemma 4. Let $\sum_{i,j=1}^{\infty} a_{ij}$ be a convergent series with non-negative terms and let the two dimensional sequence $\{a_{mn}\}_{m,n=1}^{\infty}$ be decreasing. Then $\lim_{m,n\to\infty} mna_{mn} = 0$.

Proof. Let $s_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$ and $\sum_{i,j=1}^{\infty} a_{ij} = s$. Since for each $\varepsilon > 0$ there exists (m_0, n_0) such that $0 \leqq s - s_{mn} < \varepsilon$ holds whenever $(m, n) \geqq (m_0, n_0)$, for $(m, n) > (u, v) \geqq (m_0, n_0)$ $0 \leqq s_{mn} - s_{uv} < |s_{mn} - s| + |s - s_{uv}| < 2\varepsilon$ is valid. The developed inequality and the decrease of $\{a_{mn}\}_{m,n=1}^{\infty}$ imply $(mn - m_0 n_0)a_{mn} \leqq s_{mn} - s_{m_0 n_0} < 2\varepsilon$ for $(m, n) > (m_0, n_0)$. Hence $mna_{mn} < 2\varepsilon + m_0 n_0 a_{mn}$ and from the fact $\lim_{m,n\to\infty} a_{mn} = 0$ the statement of Lemma 4 follows.

Let $R = \{r_{mn}\}_{m,n=1}^{\infty}$, $r_{mn} > 1$, be a two dimensional sequence. If $\sigma < \tau$ then, according to Lemma 3, the convergence of $\sum_{m,n=1}^{\infty} r_{mn}^{-\sigma}$ implies the convergence of the series $\sum_{m,n=1}^{\infty} r_{mn}^{-\tau}$. This fact enables us to define the exponent of convergence $\lambda(R)$ of the sequence R by the following way:

$$\lambda(R)=\inf\left\{\sigma>0: \sum_{m,n=1}^{\infty} r_{mn}^{-\sigma}<+\infty\right\}$$

In the next text it is shown that the exponent of convergence equals (under certain assumptions) to the number

$$s(R)=\limsup_{m,n\to\infty} \log mn/\log r_{mn}$$

The main purpose of the next text will be to give conditions under which the formula $\lambda(R)=s(R)$ can be used.

Theorem 6. Let $R=\{r_{mn}\}_{m,n=1}^{\infty}$ be an increasing sequence, $r_{mn}>1$. Then $s(R)\leqq\lambda(R)$.

Proof. Obviously it is sufficient to prove the inequality $s(R)\leqq\lambda(R)$ (further $s<\lambda$) only if $\lambda<+\infty$. Let ε be any positive number. Then $\sum_{m,n=1}^{\infty} r_{mn}^{-(\lambda+\varepsilon)}<+\infty$ and from Lema 4 it follows the existence of (m_0, n_0) such that $mnr_{mn}^{-(\lambda+\varepsilon)}<1$ holds for every $(m, n) \geqq (m_0, n_0)$. From the last inequality $\log mn/\log r_{mn} < \lambda+\varepsilon$ follows. Hence $s\leqq\lambda$.

The fact, that the sequence R in Theorem 6 is increasing is essential. This shows the following example.

Example 1. Let $R=\{r_{mn}\}_{m,n=1}^{\infty}$, $r_{mn}=mn$, be a two dimensional sequence. From the definition of the exponent of convergence it follows $\lambda(R)=1$. Since $r_{mn}\geqq 1$, the sequence $R'=\{r'_{mn}\}_{m,n=1}^{\infty}$, $r_{mn}=r_{\pi(m,n)}$ (π is any permutation of the set $N\times N$) has the same exponent of converganc as R, i.e. $\lambda(R')=\lambda(R)=1$. Let π' be the permutation of the set $N\times N$ which commutes the terms on the places (k, k^2) and (k^2, k^2) for each $k=1, 2, \dots,$ and does not influence the others. Obviously the sequence R' obtained from R by the application of the permutation π' is not increasing. Since $r'_{k^2k^2}=k^3$ we have $\limsup_{m,n\to\infty} \log mn/\log r'_{mn}=4/3$, consequently the assumption of the increase in Theorem 6 is essential.

Theorem 7. Let $R=\{r_{mn}\}_{m,n=1}^{\infty}$ be an increasing two dimensional sequence, $r_{mn}>1$, and

$$\mu(R)=\sup\{\sigma>0: \text{there exists } (m_0, n_0)\in N\times N \text{ such that}$$

$$\sum_{m=m_0,\,n=n_0}^{\infty} r_{mn}^{-\sigma}<+\infty \text{ and } \sum_{m,n=1}^{\infty} r_{mn}^{-\sigma}=+\infty\}$$

Let $\lambda(R)$ and $s(R)$ have the introduced meaning. Then

a) $\mu(R)\leqq s(R)$ implies $\lambda(R)=s(R)$;
b) $\mu(R)>s(R)$ implies $\lambda(R)>s(R)$.

Proof. a) Let us prove, that the inequality $\mu(R)\leqq s(R)$ (further $\mu\leqq s$)

implies the inequality $\lambda(R) \leqq s(R)$ $(\lambda \leqq s)$. The statement a) follows from the last inequality and Theorem 6. It is sufficient to prove the inequality for $s < +\infty$. In this case for each $\varepsilon > 0$ there exists (m_0, n_0) such that $\log mn/\log r_{mn} < s + \varepsilon$ holds for every $(m, n) \geqq (m_0, n_0)$. The last inequality implies successively these inequalities: $mn < r_{mn}^{s+\varepsilon}$, $(mn)^{(s+2\varepsilon)/(s+\varepsilon)} < r_{mn}^{s+2\varepsilon}$ and $(mn)^{-(s+2\varepsilon)/(s+\varepsilon)} > r^{-(s+2\varepsilon)}$. Since $(s+2\varepsilon)/$ $/(s+\varepsilon) > 1$ from the last inequality and Lemma 3 it follows that the series $\sum_{m=m_0, n=n_0}^{\infty} r_{mn}^{-(s+2\varepsilon)}$ is convergent. The assumption $\mu \leqq s$ implies $\mu < s + 2\varepsilon$ and the convergence of $\sum_{m,n=1}^{\infty} r_{mn}^{-(s+2\varepsilon)}$ is a consequence of the definition of μ. Hence $\lambda \leqq s + 2\varepsilon$. Since the last inequality holds for any $\varepsilon > 0$ we have $\lambda = s$.

b) First we will prove the inequality $\lambda \geqq \mu$ by contradiction. Let $\lambda < \mu$. Then there exists σ, $\sigma > \lambda$ such that $\sigma \leqq \mu$ and $\sum_{m,n=1}^{\infty} r_{mn}^{-\sigma} = +\infty$ (it follows from the definition of μ). Since $\sigma > \lambda$, the definition of the exponent of convergence implies $\sum_{m,n=1}^{\infty} r_{mn}^{-\sigma} < +\infty$. This is a contradiction. Hence it holds the inequality $\lambda \geqq \mu$ and with respect to the assumption $\mu > s$ the statement b) follows.

Remark 2. The condition $r_{mn} > 1$ for $(m, n) \geqq (1, 1)$, which is introduced in the definition of the exponent of convergence and in the assumption of Theorem 6 and Theorem 7, it is possible to replace by the condition: $r_{mn} > 0$ for $(m, n) \geqq$ $\geqq (1, 1)$ and with the exception at most of the finite number of pairs of indices $r_{mn} > 1$. It follows from the given proofs. If $0 < r_{mn} \leqq 1$ holds for infinite number pairs of indices then obviously $\lambda = +\infty$.

It is possible to generalize introduced definitions for n-dimensional sequence $(n \in N)$. It follows from the used methods in the proofs of Theorem 6 and Theorem 7 that these theorems hold in n-dimensional case. Let us remark that in one dimensional case the inequality $\mu \leqq s$ is always fulfilled since $\{\sigma > 0$: there exists $m_0 \in N$ such that $\sum_{m=m_0}^{\infty} r_m^{-\sigma} < +\infty$ and $\sum_{m=1}^{\infty} r_m^{-\sigma} = +\infty\} = \emptyset$ and $\mu = \sup \emptyset = -\infty$. Hence the case a) of Theorem 7 is possible only. In two dimensional case it is possible also the inequality $\mu > s$. This shows the following example.

Example 2. Let $r_{m1} = r_{1m} = m$ for each $m = 1, 2, \ldots$ and $r_{mn} = (mn)^2$ for each $(m, n) \geqq (2, 2)$. From the definition of the exponent of convergence it follows for this increasing sequence that $\lambda = 1$. It is obvious that $s = \limsup_{m,n\to\infty} \log mn/\log r_{mn} =$ $= 1/2$, i.e. $\lambda > s$. According to Theorem 7a) the case $\mu \leqq s$ is impossible, hence $\mu > s$.

Theorem 7 gives the necessary and sufficient condition under which it is possible, for the calculation of the exponent of convergence of the increasing two dimensional sequence $R = \{r_{mn}\}_{m,n=1}^{\infty}$, $r_{mn} > 1$, to use the formula

$$\lambda(R) = \limsup_{m,n\to\infty} \log mn/\log r_{mn}$$

To verify this condition in a concrete example it is relatively laborious. In the next text we will give a sufficient condition under which it is possible to use the above mentioned formula. This condition will be more suitable for the practical calculation.

Definition. We shall say that the series $\sum_{m,n=1}^{\infty} a_{mn}$, $a_{mn} > 0$, is row-wise perfect if for each m and m_0 there exists m', $m' \geqq m_0$, such that $\limsup_{n\to\infty} a_{mn}/a_{m'n} <$ $< +\infty$. The series $\sum_{m,n=1}^{\infty} a_{mn}$, $a_{mn} > 0$, is column-wise perfect if for each n and n_0 there exists n', $n' \geqq n_0$, such that $\limsup_{m\to\infty} a_{mn}/a_{mn'} < +\infty$. A row-wise and column- -wise perfect series will be called a perfect series.

Lemma 5. Let $\sum_{m,n=1}^{\infty} a_{mn}$ be a perfect series with positive terms and let there exist $(m_0, n_0) \in N \times N$ such that the series $\sum_{m=m_0, n=n_0}^{\infty} a_{mn}^{\sigma}$ converges ($\sigma > 0$). Then the series $\sum_{m,n=1}^{\infty} a_{mn}^{\sigma}$ converges.

Proof. For each $m \geqq m_0$ the convergence of the series $\sum_{n=n_0}^{\infty} a_{mn}^{\sigma}$ implies the convergence of the series $\sum_{n=1}^{\infty} a_{mn}^{\sigma}$. It follows from row-wise perfectnees of $\sum_{m,n=1}^{\infty} a_{mn}$ that for each m, $1 \leqq m < m_0$ there exists $m' \geqq m_0$ such that $a_{mn}^{\sigma} \leqq K a_{m'n}^{\sigma}$ for any $n = 1, 2, \ldots$, where the constant K does not depend on n. Hence for any $m = 1, 2, \ldots, m_0 - 1$ the series $\sum_{n=1}^{\infty} a_{mn}^{\sigma}$ converges. Analogously using the column-vise perfectnees of the series $\sum_{m,n=1}^{\infty} a_{mn}$ and the convergence of $\sum_{m=1}^{\infty} a_{mn}^{\sigma}$, $n \geqq n_0$, it is possible to check the convergence of each of series $\sum_{m=1}^{\infty} a_{mn}^{\sigma}$ for any $n = 1, 2, \ldots, n_0 - 1$. If we put $\sum_{m=m_0, n=n_0}^{\infty} a_{mn}^{\sigma} = s$, $\sum_{n=1}^{\infty} a_{mn}^{\sigma} = M_m$ for $m = 1, 2, \ldots, m_0 - 1$ and $\sum_{m=1}^{\infty} a_{mn}^{\sigma} = N_n$ for $n = 1, 2, \ldots, n_0 - 1$, then obviously for each partial sum s_{mn} of the series $\sum_{m,n=1}^{\infty} a_{mn}^{\sigma}$ the inequality $s_{mn} \leqq s + \sum_{m=1}^{m_0-1} M_m + \sum_{n=1}^{n_0-1} N_n$ follows. Hence the series $\sum_{m,n=1}^{\infty} a_{mn}^{\sigma}$ converges.

Theorem 8. Let $\{r_{mn}\}_{m,n=1}^{\infty}$ be an increasing two dimensional sequence, $r_{mn} > 1$, and the series $\sum_{m,n=1}^{\infty} r_{mn}^{-1}$ be perfect.

Then

$$\lambda = \limsup_{m,n\to\infty} \log mn/\log r_{mn}$$

Proof. If we put $a_{mn} = r_{mn}^{-1}$ then the perfectnees and Lemma 5 imply that the set $\{\sigma > 0$: there exists $(m_0, n_0) \in N \times N$ such that $\sum_{m=m_0, n=n_0}^{\infty} r_{mn}^{-\sigma} < +\infty$ and $\sum_{m,n=1}^{\infty} r_{mn}^{-\sigma} = +\infty\} = \emptyset$. Since $\mu = -\infty \leqq s$ Theorem 7a) implies the statement of Theorem 8.

Remark 3. Although the assumptions of Theorem 8 are stronger then those of Theorem 7 (as shown by the following Example 3), they are essential.

The two dimensional sequence R' constructed in Example 1 is not increasing, but the series $\sum_{m,n=1}^{\infty} r'^{-1}_{mn}$ is perfect, because R' differs from R in any row or any column at most in two terms and the series $\sum_{m,n=1}^{\infty} r_{mn}^{-1}$ is perfect. Hence the assumption of the increase in Theorem 8 is essential.

The two dimensional sequence constructed in Example 2 is increasing, but the series $\sum_{m,n=1}^{\infty} r_{mn}^{-1}$ is not perfect, because $\limsup_{n\to\infty} r_{1n}^{-1}/r_{mn}^{-1} = \limsup_{n\to\infty} (mn)^2/n + \infty$ holds for each $m > 1$. Hence the assumption of the perfectnees of the series $\sum_{m,n=1}^{\infty} r_{mn}^{-1}$ in Theorem 8 is essential.

Example 3. Let $R = \{r_{mn}\}_{m,n=1}$, $r_{mn} = 2^{mn}$. Theorem 8 is not acceptable for the calculation of the exponent of convergence, because the series $\sum_{m,n=1}^{\infty} 2^{-mn}$ is not perfect. Really, $\limsup_{n\to\infty} r_{mn}^{-1}/r_{m'n}^{-1} = \limsup_{n\to\infty} 2^{(m'-m)n} = +\infty$ holds whenever $m' > m$. Since $\{\sigma > 0$: there exists $(m_0, n_0) \in N \times N$ such that $\sum_{m=m_0, n=n_0}^{\infty} 2^{-\sigma mn} < +\infty$ and $\sum_{m,n=1}^{\infty} 2^{-\sigma mn} = +\infty\} = \emptyset$ according to Theorem 2a) it is possible to use the derived formula. Hence $\lambda(R) = \limsup_{m,n\to\infty} \log mn/\log 2^{mn} = 0$.

Corollary 2. Let z be any positive number and $R_z = \{m^z + n^z\}_{m,n=1}^{\infty}$. Then $\lambda(R_z) = 2/z$.

Proof. It is easy to verify that the assumptions of Theorem 8 for the two dimensional sequence R_z are fulfilled. Hence for the calculation of the exponent of convergence $\lambda(R_z)$ it is possible to use the formula $\lambda(R_z) = \limsup_{m,n\to\infty} \log mn/$ $/\log (m^z + n^z)$.

The inequalities $0 \leqq (m^{z/2} - n^{z/2})^2 = m^z - 2(mn)^{z/2} + n^z$ imply $2(mn)^{z/2} \leqq$ $\leqq m^z + n^z$, $\log mn/\log (m^z + n^z) \leqq (2/z)\ (1 - \log 2/\log (m^z + n^z))$, i.e. $\log mn/\log (m^z + n^z) < 2/z$. Hence $\lambda(R_z) \leqq 2/z$. Since $\lim_{m\to\infty} \log m^2/\log 2m^z =$

$= 2/z$ for every $\varepsilon > 0$ and (m_0, n_0) there exists $(m, m) \geqq (m_0, n_0)$ such that $\log m^2/\log 2m^z > 2/z - \varepsilon$. Hence $\lambda(R_z) = 2/z$.

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SÚHRN

POZNÁMKA O EXPONENTE KONVERGENCIE

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V článku sa vyšetruje exponent konvergencie $\lambda(x)$ (zavedený v [2]) ako reálna funkcia definovaná na množine s^+ všetkých reálnych neklesajúcich postupností $x = \{x_i\}_{i=1}^{\infty}$, $x_1 \geqq \gamma > 0$, s Fréchetovou metrikou. Dokazuje sa, že λ je funkcia tretej Bairovej triedy a každá z množín $\{x : \lambda(x) < a\}$, $a \in (-\infty, +\infty)$, je prvej kategórie v s^+. Taktiež sa zavádza pojem exponentu konvergencie $\lambda(R)$ dvojnej postupnosti $R = \{r_{mn}\}_{m,n=1}^{\infty}$. Ukazuje sa, že za istých predpokladov je možné k jeho výpočtu použiť formulu

$$\lambda(R) = \limsup_{m,n\to\infty} \log mn/\log r_{mn}$$

РЕЗЮМЕ

ЗАМЕТКА О ПОКАЗАТЕЛЕ СХОДИМОСТИ

П. Костырко, Братислава

В статье рассматривается показатель сходимости $\lambda(x)$ (введеный в [2]) как действительная функция определенная на пространстве s^+ всех вещественных неубывающих последовательностей $x = \{x_i\}_{i=1}^{\infty}$, $x_1 \geqq \gamma > 0$, с метрикой Фреше. Доказано, что λ является функцией третьего класса Бера и что множества $\{x : \lambda(x) < a\}$, $a \in (-\infty, +\infty)$, первой категории в s^+. Тоже вводится понятие показателя сходимости $\lambda(R)$ для двойной последовательности $R = \{r_{mn}\}_{m,n=1}^{\infty}$. Показано, что при некоторых условиях возможно для его вычисления пользоваться формулой

$$\lambda(R) = \limsup_{m,n\to\infty} \log mn/\log r_{mn}$$

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ON AN APPLICATION OF THE TYCHONOFF FIXED POINT THEOREM TO A CAUCHY NON-LINEAR PROBLEM

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I. Introduction and notations

In this paper the existence of a solution of the Cauchy non-linear problem

$$\frac{\partial^m u(x,t)}{\partial t^m} + a(x,t)\frac{\partial^n u(x,t)}{\partial x^n} = B\left(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}, \dots, \frac{\partial^{i+j} u}{\partial^i \partial x^j}, \dots, \frac{\partial^{m+k-1} u}{\partial t^{m-1} \partial x^k}\right) \tag{1}$$

$m \geqq n$; $i = 0, 1, \dots, m-1$; $j = 0, 1, \dots, k$; k is an arbitrary non-negative integer

$$\left[\frac{\partial^i u(x,t)}{\partial t^i}\right]_{t=0} = u_i(x) \quad i = 0, 1, \dots, m-1, \quad \frac{\partial^0 u}{\partial t^0} \equiv u \tag{2}$$

in the compact domain

$$E_0 = \{(x,t) \mid x_1 \leqq x \leqq x_2,\ 0 \leqq t \leqq d,\ x_1 < x_2,\ d > 0\} \tag{3}$$

is proved.

To this aim we use Tychonoff's fixed point theorem in the locally convex topological spaces.

First let us introduce some notations. Let R^n mean the n-dimentional Euclidean space; $R^1 \equiv R$, $R^+ = \{x \in R \mid x \geqq 0\}$, $R^- = \{x \in R \mid x \leqq 0\}$, $I^+ = \{1, 2, 3, \dots\}$ $I_0^+ = \{0, 1, 2, \dots\}$,
$C(E, N) = \{f(x_1, x_2, \dots, x_n) \mid f$ is a continuous function of n variables $E \in R^n \to \to N \subset R\}$,

$$G_x^+(\alpha, \varrho, M) = \left\{f(x,t) \in C(E_0, R^+) \,\middle|\, \frac{\partial^s f}{\partial x^s} \in C(E_0, R^+) \right. \tag{4}$$

$$\left.\text{for all } s \in I_0^+ \text{ and } 0 \leqq \frac{\partial^s f}{\partial x^s} \leqq M\frac{(s!)^\alpha}{\varrho^s},\ M \neq 0,\ \varrho \neq 0,\ \alpha \neq 0 \in R^+\right\},$$

$$G_x^-(\alpha, \varrho, M) = \left\{f(x,t) \in C(E_0, R^-) \,\middle|\, \frac{\partial^s f}{\partial x^s} \in C(E_0, R^-) \right. \tag{5}$$

for all $s \in I_0^+$ and $0 \leqq -\frac{\partial^s f}{\partial x^s} \leqq M \frac{(s!)^\alpha}{\varrho^s}, M \neq 0, \varrho \neq 0, \alpha \neq 0 \in R^+\Big\}$,

where $\frac{\partial^0 f}{\partial x^0} \equiv f$.

Further let us denote $u_{i,j} = \frac{\partial^{i+j} u(x, t)}{\partial t^i \, \partial x^j}$, $i, j \in I_0^+$,

$$B_{x^{\lambda_0}, u_{00}^{\lambda_1}, u_{10}^{\lambda_2}, u_{01}^{\lambda_2}, \ldots, u_{m-1,k}^{\lambda_\varkappa}} =$$

$$= \frac{\partial^{\lambda_0+\lambda_1+\ldots+\lambda_\varkappa} B(x, t, u_{00}, u_{10}, u_{01}, \ldots, u_{m-1,k})}{\partial x^{\lambda 0} \, \partial u_{00}^{\lambda_1} \, \partial u_{10}^{\lambda_2} \, \partial u_{01}^{\lambda_3}, \ldots, \partial u_{m-1,k}^{\lambda \varkappa}} \tag{6}$$

where $\varkappa = m(k+1)$, $\lambda_i \in I_0^+$.

Let be $\frac{\partial}{\partial x} \{B(x, t, u_{00}, \ldots, u_{m-1,k})\} = \frac{\partial}{\partial x} \Big\{B(x, t, u(x, t), \ldots, \frac{\partial^{m-1+k} u(x, t)}{\partial t^{m-1} \, \partial x^k}\Big\}$,

hence $\frac{\partial}{\partial x} \{\ \}$ denotes the derivative of the composite function. If E_0 is defined by (3), then

$$H(E_0, R) = \{f \in C(E_0, R) \,\Big|\, \frac{\partial^{i+j} f}{\partial t^i \, \partial x^j} \in C(E_0, R)$$
$$\text{for } i = 0, 1, \ldots, m;\, j = 0, 1, 2, \ldots\} \tag{7}$$

The sum of functions and the multiplication by a real number is defined in the set H in the natural way. We introduce the topology in this linear space with the help of the following countable system of semi-norms: for $\varphi \in H$ we set

$$p_s(\varphi) = \max_{\substack{i = 0, 1, -., m \\ j = 0, 1, \ldots, s}} \left\{\max_{E_0} \left|\frac{\partial^{i+j} \varphi}{\partial t^i \, \partial x^j}\right|\right\}, \qquad s \in I_0^+ \tag{8}$$

This system of semi-norms is a non-decreasing one and fulfils the axiom of separation, because $p_0(\varphi_0) > 0$ for $\varphi_0 \neq 0$. The set

$$U(\varphi_0, s, \varepsilon) = \{\varphi \in H \,|\, p_s(\varphi - \varphi_0) < \varepsilon;\, s \in I_0^+, \varepsilon > 0\} \tag{9}$$

defines a neighbourhood of the point φ_0 in H. According to the well-known theorems (e.g. [4]) $H(E_0, R)$ is a locally convex linear topological space and moreover (e.g. [6]) it is metrizable.

II. Existence of a solution

For the locally convex, complete linear spaces the following Tychonoff's theorem [1], [2] is true.

Theorem 1. Let H be a complete locally convex linear topological space and $N \subset H$ a closed convex set. Let T be a continuous operator on N and $T(N) \subset N$. If $\overline{T(N)}$ is compact, then there exists a fixed point of T in N.

We shall show that the space H defined by (8) is complete. The convergence in this space means the uniform convergence of the functions together with all their partial derivatives with respect to t to the order m and with respect to x to an arbitrary order. In fact the convergence $\{\varphi_n\}_{n=1}^{\infty} \to \varphi$ means that for an arbitrary $U(\varphi, s, \varepsilon)$ there exists such $r_0 \in I^+$, that for $r > r_0$ is $\varphi_r \in U(\varphi, s, \varepsilon)$ or $p_s(\varphi - \varphi_r) < \varepsilon$. Considering the metrizability of H the sequential completeness means the completeness and hence each Cauchy sequence in H converges to an element from H.

We remark that there exists a one-to-one correspondence between the solution of the problem (1), (2) and the fixed point of the integro-differential operator

$$Tu = \sum_{i=0}^{m-1} \frac{t^i}{i!} u_i(x) + \int_0^t \frac{(t-\tau)^{m-1}}{(m-1)!} [B(x, \tau, u_{00}, \dots, u_{m-1,k}) - a(x, \tau) u_{0_n}] \, d\tau \tag{10}$$

Let us suppose

(H_1): $a(x, t) \in G_x^-(\alpha, \varrho, M)$ in E_0, $u_i(x) \in G_x^+(\alpha, \varrho, M)$ for $x_1 \leqq x \leqq x_2$ and $i = 0, 1, \dots, m-1$;

(H_2): $B(x, t, u_{00}, u_{10}, u_{01}, \dots, u_{m-1,k}) \in C(E_0 x R^{+\varkappa}, R^+)$,

$$B_{x^{\lambda_0}, u_{00}^{\lambda_1}, u_{10}^{\lambda_2}, \dots u_{m-1 \cdot k}^{\lambda_\varkappa}} \in C(E_0 x R^{+\varkappa}, R^+)$$

where $\varkappa = m(k+1)$ $\lambda_i \in I_0^+$, $i = 0, 1, \dots, \varkappa$;

(H_3): Let there exist the functions $b_1(x, t)$, $b_2(x, t)$ with the properties:

$$b_i(x, t) \in G_x^+(\alpha, \varrho, M) \text{ in } E_0 \text{ for } i = 1, 2, \quad 0 < \alpha < \frac{m}{n} \tag{a}$$

If we denote

$$N = \left\{ u(x, t) \in H(E_0, R^+) \,\middle|\, 0 \leqq \frac{\partial^{i+j} u}{\partial t^i \, \partial x^j} \leqq \frac{\partial^{i+j} u_0}{\partial t^i \partial x^j}, \begin{array}{l} i = 0, 1, \dots, m \\ j = 0, 1, \dots, \end{array} \right\} \tag{11}$$

where $u_0(x, t)$ is the solution of the equation

$$\frac{\partial^m u}{\partial t^m} + a(x, t) \frac{\partial^n u}{\partial x^n} - b_2(x, t) u = b_1(x, t) \tag{12}$$

with the initial conditions (2), then let for each $s \in I_0^+$, $u \in N$ in E_0

$$\frac{\partial^s}{\partial x^s} \{B(x, t, u_{00}, \dots, u_{m-1,k})\} \leqq \frac{\partial^s b_1(x, t)}{\partial x^s} + \frac{\partial^s b_2(x, t) u(x, t)}{\partial x^s} \tag{b}$$

hold.

We remark that by the theorems 1 till 3 [3] $u_0(x, t)$ exists, is unique in the class $G_x^+(\alpha, \varrho e^{-1-\alpha}, \Omega)$, where α, ϱ, Ω are given constants.

Now we state our main result:

Theorem 2. If the assumptions (H_1), (H_2), (H_3) are satisfied, then there exists a solution of the problem (1), (2) in E_0 for which $0 \leqq \frac{\partial^s u}{\partial x^s} = \frac{\partial^s u_0}{\partial x^s}$ holds, for an arbitrary $s \in I_0^+$, where $u_0(x, t)$ is the solution of the linear problem (12), (2).

Before proving this theorem we shall prove the following lemmas.

Lemma 1. Let $M \subset H(E_0, R)$ be a set of functions, which are together with their partial derivatives up to the order $\frac{\partial^{m+s}}{\partial t^m \partial x^s}$ (m fixed, s arbitrary) equicontinuous. Let M be a bounded set. Then each sequence from M contains a subsequence which converges uniformly together with the sequences of its derivatives up to the order $\frac{\partial^{m+s}}{\partial t^m \partial x^s}$ to an element of the space H.

Proof. Let us take an arbitrary sequence $\{\overset{0}{\varphi}_\lambda\}_{\lambda=1}^\infty$, $\overset{0}{\varphi}_\lambda \in M$ for $\lambda = 1, 2, \ldots$. Since the boundedness in H means a uniform boundedness of the functions $\overset{0}{\varphi}_\lambda$ together with all their derivatives up to the order $\frac{\partial^{m+s}}{\partial t^m \partial x^s}$ it follows from the Ascoli-Arzela theorem, that there exists a sequence $\{\overset{1}{\varphi}_\lambda\}_{\lambda=1}^\infty$ chosen from $\{\overset{0}{\varphi}_\lambda\}_{\lambda=1}^\infty$ which converges uniformly in E_0 to $\varphi \in C(E_0, R)$.

Let us suppose that we have defined for $i = 0, 1, \ldots, m-1$ a system of sequences $\{\overset{i}{\varphi}_\lambda\}_{\lambda=1}^\infty$ with the properties:

$$\{\overset{i+1}{\varphi}_\lambda\}_{\lambda=1}^\infty \text{ is a subsequence of } \{\overset{i}{\varphi}_\lambda\}_{\lambda=1}^\infty \tag{a}$$

$$\left\{\frac{\partial^i \overset{i+1}{\varphi}_\lambda}{\partial t^i}\right\}_{\lambda=1}^\infty \text{ converges uniformly in } E_0 \text{ to } \frac{\partial^i \varphi}{\partial t^i} \in C(E_0, R) \tag{b}$$

Let us show that from $\{\overset{m}{\varphi}_\lambda\}_{\lambda=1}^\infty$ we can choose $\{\overset{m+1}{\varphi}_\lambda\}_{\lambda=1}^\infty$ with the property (b). Consider $\left\{\frac{\partial^m \overset{m}{\varphi}_\lambda}{\partial t^m}\right\}_{\lambda=1}^\infty$. In this sequence there exists a subsequence $\{\overset{m+1}{\varphi}_\lambda\}_{\lambda=1}^\infty$ which is uniformly convergent in E_0 to a function, say, $f(x, t)$. We shall show that $\frac{\partial^m \varphi}{\partial t^m} = f(x, t)$. In fact for $0 \leqq t_0 < t \leqq d$ is

$$\Phi_{\lambda, m} \equiv \int_{t_0}^{t} \frac{\partial^m \overset{m+1}{\varphi}_\lambda}{\partial \tau^m} d\tau = \frac{\partial^{m-1} \overset{m+1}{\varphi}_\lambda(x, t)}{\partial t^{m-1}} - \frac{\partial^{m-1} \overset{m+1}{\varphi}_\lambda(x, t_0)}{\partial t^{m-1}} \rightrightarrows$$

$$\rightrightarrows \frac{\partial^{m-1} \varphi(x, t)}{\partial t^{m-1}} - \frac{\partial^{m-1} \varphi(x, t_0)}{\partial t^{m-1}} \equiv \Phi .$$

Since $\frac{\partial \Phi_{\lambda, m}}{\partial t} = \frac{\partial^m \overset{m+1}{\varphi}_\lambda}{\partial t^m} \rightrightarrows \frac{\partial \Phi}{\partial t}$ and $\frac{\partial \Phi}{\partial t} = \frac{\partial^m \varphi}{\partial t^m}$, it follows from the uniqueness of the limit $f=\dfrac{\partial^m\varphi}{\partial t^m}\in C(E_0, R)$. Hence the system $\{\varphi_\lambda^i\}_{\lambda=1}^\infty$ has the properties (a), (b) for $i=m$ too.

We construct from the sequence $\{\varphi_\lambda^{m+1}\}_{\lambda=1}^\infty$ a system of sequences $\{\varphi_\lambda^{m+1,s}\}_{\lambda=1}^\infty$ with the properties:

$$\{\varphi_\lambda^{m+1,s}\}_{\lambda=1}^\infty \text{ is a subsequence of } \{\varphi_\lambda^{m+1,s-1}\}_{\lambda=1}^\infty \tag{a'}$$

$$\left\{\frac{\partial^s\varphi_\lambda^{m+1,s}}{\partial x^s}\right\}_{\lambda=1}^\infty \text{ converges uniformly in } E_0 \text{ to } \frac{\partial^s\varphi}{\partial x^s}\in C(E_0, R) \tag{b'}$$

for $s=1, 2, \ldots$, if $\varphi_\lambda^{m+1,0}\equiv\varphi_\lambda^{m+1}$. The existence of such a system (with the properties (a'), (b')) can be proved analogically to the first part of the proof of the Lemma 1 with the help of the mathematical induction. The sequence constructed by the diagonal process $\{\varphi_s^{m+1,s}\}_{s=1}^\infty$ uniformly converges in E_0 together with the partial derivatives up to the order mentioned in Lemma 1 to the function $\varphi(x, t)\in H$. Lemma 1 is proved.

Lemma 2. If $u(x, t)\in H(E_0, R)$ and $B(x, t, u_{00}, \ldots, u_{m-1,k})$ fulfils the assumptions (H_2), then for an arbitrary $s\in I_0^+$, $\varkappa=m(k+1)$, $\lambda_0, \lambda_1, \ldots, \lambda_\varkappa\in I_0^+$

$$\frac{\partial^s}{\partial x^s}\{B(x, t, u_{00}, \ldots, u_{m-1,k})\}=$$

$$=\sum_{\lambda_0+\lambda_1+\ldots+\lambda_\varkappa\leqq s} B_{x^{\lambda_0}, u_{00}^{\lambda_1}, \ldots, u_{m-1,k}^{\lambda_\varkappa}}(x, t, u_{00}, \ldots, u_{m-1,k})P_{\lambda_0,\lambda_1,\ldots,\lambda_\varkappa}(u_{01}, \ldots, u_{m-1,k+s}) \tag{13}$$

holds, where $P_{\lambda_0,\lambda_1,\ldots,\lambda_\varkappa}(u_0, \ldots, u_{m-1,k+s})$ is a polynomial in $\varkappa+m(s-1)$ variables u_{ij}, $i=0, 1, \ldots, m-1$, $j=1, 2, \ldots, k+s$ of the degree at most s.

The **proof of the Lemma 2** will be given by the mathematical induction. For $s=1$ we obtain

$$\frac{\partial}{\partial x}\{B(x, t, u_{00}, \ldots, u_{m-1,k})\}=$$

$$=B_x+B_{u_{00}}\cdot u_{01}+B_{u_{10}}\cdot u_{11}+\ldots+B_{u_{m-1,k}}\cdot u_{m-1,k+1}$$

Therefore for $s=1$ (13) is true. Let us suppose its validity for $s\geqq 1$ and we shall prove it for $s+1$:

$$\frac{\partial^{s+1}}{\partial x^{s+1}}\{B(x, t, u_{00}, \ldots, u_{m-1,k})\}=$$

$$=\sum_{\lambda_0+\lambda_1+\ldots+\lambda_\varkappa\leqq s}\Big[P_{\lambda_0,\lambda_1,\ldots,\lambda_\varkappa}(u_{01}, \ldots, u_{m-1,k+s})\cdot\frac{\partial}{\partial x}\{B_{x^{\lambda_0}, u_{00}^{\lambda_1}, \ldots, u_{m-1,k}^{\lambda_\varkappa}}\}+B_{x^{\lambda_0}, u_{00}^{\lambda_1}, \ldots, u_{m-1,k}^{\lambda_\varkappa}}\cdot$$

$$\cdot\frac{\partial}{\partial x}\{P_{\lambda_0,\lambda_1,\ldots,\lambda_\varkappa}(u_{01}, \ldots, u_{m-1,k+s})\}\Big]=\sum_{\lambda_0+\lambda_1+\ldots+\lambda_\varkappa\leqq s}[(B_{x^{\lambda_0+1}, u_{00}^{\lambda_1}, \ldots, u_{m-1,k}^{\lambda_\varkappa}}+B_{x^{\lambda_0}, u_{00}^{\lambda_1+1}, \ldots, u_{m-1,k}^{\lambda_\varkappa}}$$

$$\cdot u_{01} + \ldots + B_{x^{\lambda_0}, u_{00}^{\lambda_1}, \ldots, u_{m-1,k}^{\lambda_\varkappa}} \cdot u_{m-1,k+1}) \cdot P_{\lambda_0, \lambda_1, \ldots, \lambda_\varkappa}(u_{01}, \ldots, u_{m-1,k+s}) +$$

$$+ B_{x^{\lambda_0}, u_{00}^{\lambda_1}, \ldots, u_{m-1,k}^{\lambda_\varkappa}} \cdot \tilde{P}_{\lambda_0, \ldots, \lambda_\varkappa}(u_{01}, \ldots, u_{m-1,k+s+1})]$$

where $P_{\lambda_0, \lambda_1, \ldots, \lambda_\varkappa}(u_{01}, \ldots, u_{m-1,k+s+1})$ is a polynomial of the same degree as $P_{\lambda_0, \lambda_1, \ldots, \lambda_\varkappa}(u_{01}, \ldots, u_{m-1,k+s})$ but in $\varkappa + ms$ variables u_{ij} $i = 0, 1, \ldots, m-1$, $j = 1, 2, \ldots, k+s+1$. From the last equation it follows $\frac{\partial^{s+1}}{\partial x^{s+1}} \{B(x, t, u_{00}, \ldots, u_{m-1,k})\} =$

$$= \sum_{\mu_0 + \mu_1 + \ldots + \mu_\varkappa \leqq s+1} B_{x^{\mu_0}, u_{00}^{\mu_1}, \ldots, u_{m-1,k}^{\mu_0}} \cdot Q_{\mu_0, \mu_1, \ldots, \mu_\varkappa}(u_{01}, \ldots, u_{m-1,k+s+1})$$

where $Q_{\mu_0, \mu_1, \ldots, \mu_\varkappa}(u_{01}, \ldots, u_{m-1,k+s+1})$ is a polynomial of a degree at most $s+1$ in $\varkappa + ms$ variables u_{ij}, $i = 0, 1, \ldots, m-1$, $j = 1, 2, \ldots, s+1$. Thus Lemma 2 is proved.

Proof of Theorem 2. Let N be the set defined in the assumption (H_3). Clearly it is closed and convex. We shall show that $T(N) \subset N$. Let $u \in N$. For $r = 0, 1, \ldots, m-1$ and an arbitrary $s \in I_0^+$ in making use of the assumption (H_3), b) we have

$$\frac{\partial^{r+s} Tu}{\partial t^r \, \partial x^s} = \sum_{i=0}^{m-r-1} \frac{t^i}{i!} u_{i+r}^{(s)}(x) +$$

$$+ \int_0^t \frac{(t-\tau)^{m-r-1}}{(m-r-1)!} \left[\frac{\partial^s}{\partial x^s} \{B(x, \tau, u_{00}, \ldots, u_{m-1,k})\} - \frac{\partial^s}{\partial x^s} (a(x, \tau) u_{0n}(x, \tau)) \right] d\tau \leqq$$

$$\leqq \sum_{i=0}^{m-r-1} \frac{t^i}{i!} u_{i+r}^{(s)}(x) + \int_0^t \frac{(t-\tau)^{m-r-1}}{(m-r-1)!} \frac{\partial^s b_1(x, \tau)}{\partial x^s} d\tau -$$

$$- \int_0^t \frac{(t-\tau)^{m-1-r}}{(m-1-r)!} \left[\frac{\partial^s}{\partial x^s} (a(x, \tau) u_{0n}(x, \tau)) - \right.$$

$$\left. - \frac{\partial^s}{\partial x^s} (b_2(x, \tau) u_{00}(x, \tau)) \right] d\tau \tag{14}$$

and

$$\frac{\partial^{m+s} Tu}{\partial t^m \, \partial x^s} = \frac{\partial^s}{\partial x^s} \{B(x, t, u_{00}, \ldots, u_{m-1,k})\} -$$

$$- \frac{\partial^s}{\partial x^s} (a(x, t) \cdot u_{0n}(x, t)) \leqq$$

$$\leqq \frac{\partial^s b_1(x, t)}{\partial x^s} + \frac{\partial^s}{\partial x^s} (b_2(x, t) u_{00}(x, t)) - \frac{\partial^s}{\partial x^s} (a(x, t) u_{0n}(x, t)) \tag{15}$$

From the assumptions (H_1) and (H_2) we can see the validity of the inequality

$\dfrac{\partial^{r+s}Tu}{\partial t^r\,\partial x^s}\geqq 0$ for $r=0, 1, \dots, m$ and arbitrary $s\in I_0^+$. The right sides of the inequalities (14) and (15) are equal to $\dfrac{\partial^{r+s}u_0(x,t)}{\partial t^r\,\partial x^s}$, $r=0, 1, \dots, m$, $s\in I_0^+$, because we have for $u_0(x,t)$ the relation (35) in the paper [3], i.e.

$$u_0(x,t)=\sum_{i=0}^{m-1}\frac{t^i}{i!}u_i(x)+\int_0^t\frac{(t-\tau)^{m-1}}{(m-1)!}b_1(x,\tau)\,d\tau-$$

$$-\int_0^t\frac{(t-\tau)^{m-1}}{(m-1)!}\left[a(x,\tau)\frac{\partial^n u_0(x,\tau)}{\partial x^n}-b_2(x,\tau)u_0(x,\tau)\right]d\tau \qquad (16)$$

Then the following inequalities

$$0\leqq\frac{\partial^{r+s}Tu}{\partial t^r\,\partial x^s}\leqq\frac{\partial^{r+s}u_0}{\partial t^r\,\partial x^s},\qquad r=0, 1, \dots, m, s\in I_0^+ \qquad (17)$$

hold and thus $T(N)\subset N$.

We shall show now the continuity of the operator T in $N\subset H(E_0, R^+)$. The continuity will be proved, if for any $\varepsilon>0$, any $p\in I^+$ and arbitrary function $\tilde{u}\in N$ there exists such $\delta>0$, $q\in I^+$ that for each $u\in U(\tilde{u}, q, \delta)$ $Tu\in U(T\tilde{u}, p, \varepsilon)$ is true. For $r=0, 1, \dots, m-1$ and for $s=0, 1, \dots, p$ we have

$$\left|\frac{\partial^{r+s}Tu}{\partial t^r\,\partial x^s}-\frac{\partial^{r+s}T\tilde{u}}{\partial t^r\,\partial x^s}\right|=$$

$$=\left|\int_0^t\frac{(t-\tau)^{m-r-1}}{(m-r-1)!}\left[\frac{\partial^s}{\partial x^s}\{B(x,\tau,u_{00},\dots,u_{m-1,k})-\right.\right.$$

$$-B(x,\tau,\tilde{u}_{00},\dots,\tilde{u}_{m-1,k})\}+\frac{\partial^s}{\partial x^s}(a(x,\tau)\tilde{u}_{0n}(x,\tau)-$$

$$\left.\left.-a(x,\tau)u_{0n}(x,\tau))\right]d\tau\right| \qquad (18)$$

For $r=m$, $s=0, 1, \dots, p$ we obtain

$$\left|\frac{\partial^{m+s}Tu}{\partial t^m\,\partial x^s}-\frac{\partial^{m+s}T\tilde{u}}{\partial t^m\,\partial x^s}\right|=$$

$$=\left|\frac{\partial^s}{\partial x^s}\{B(x,t,u_{00},\dots,u_{m-1,k})-B(x,t,\tilde{u}_{00},\dots,\tilde{u}_{m-1,k})\}+\right.$$

$$\left.+\frac{\partial^s}{\partial x^s}(a(x,t)\tilde{u}_{0n}(x,t)-a(x,t)u_{0n}(x,t))\right| \qquad (19)$$

By Lemma 2 it is true that

$$\frac{\partial^s}{\partial x^s}\{B(x,t,u_{00},\dots,u_{m-1,k})-B(x,t,\tilde{u}_{00},\dots,\tilde{u}_{m-1,k})\}=$$

$$=\sum_{\lambda_0+\lambda_1+\dots+\lambda_\varkappa\leq s}[(B_{x^{\lambda_0},u_{00}^{\lambda_1},\dots,u_{m-1,k}^{\lambda_\varkappa}}(x,t,u_{00},\dots,u_{m-1,k})-$$

$$-B_{x^{\lambda_0},u_{00}^{\lambda_1},\dots,u_{m-1,k}^{\lambda_\varkappa}}(x,t,\tilde{u}_{00},\dots,\tilde{u}_{m-1,k}))\cdot P_{\lambda_0,\lambda_1,\dots,\lambda_\varkappa}(u_{01},\dots,u_{m-1,k+s})+$$

$$+B_{x^{\lambda_0},u_{00}^{\lambda_1},\dots,u_{m-1,k}^{\lambda_\varkappa}}(x,t,\tilde{u}_{00},\dots,\tilde{u}_{m-1,k})\cdot(P_{\lambda_0,\lambda_1,\dots,\lambda_\varkappa}(u_{01},\dots,u_{m-1,k+s})-$$

$$-P_{\lambda_0,\lambda_1,\dots,\lambda_\varkappa}(\tilde{u}_{01},\dots,\tilde{u}_{m-1,k+s}))], \tag{20}$$

where $P_{\lambda_0,\lambda_1,\dots,\lambda_\varkappa}(u_{01},\dots,u_{m-1,k+s})$ is a polynomial with the properties mentioned in Lemma 2. With regard to the assumption (H_2) and to the from there following boundedness of the functions $B_{x^{\lambda_0},u_{00}^{\lambda_1},\dots,u_{m-1,k}^{\lambda_\varkappa}}$ for $(x,t)\in E_0$, $u\in N$ and considering the boundedness of the polynomial $P_{\lambda_0,\lambda_1,\dots,\lambda_\varkappa}$ and taking in account the validity of (20) for $u\in N$ it holds, that to arbitrary $\frac{\varepsilon}{2l}>0$ $\left(\text{where } l=\max_{i=0,1,\dots,m}\left\{\frac{d^i}{i!}\right\}\right)$ and arbitrary $p\in I^+$ there exists such a $\delta_1>0$ that for all $u\in N$ which satisfy the inequalities

$$\left|\frac{\partial^{i+\lambda}u}{\partial t^i\,\partial x^\lambda}-\frac{\partial^{i+\lambda}\tilde{u}}{\partial t^i\,\partial x^\lambda}\right|<\delta_1,\qquad i=0,1,\dots,m;\ \lambda=0,1,\dots,p+k$$

it holds

$$\left|\frac{\partial^s}{\partial x^s}\{B(x,t,u_{00},\dots,u_{m-1,k})-B(x,t,\tilde{u}_{00},\dots,\tilde{u}_{m-1,k})\}\right|<\frac{\varepsilon}{2l}$$

is true for $s=0,1,\dots,p$. Taking into consideration the assumption (H_1) to any $\frac{\varepsilon}{2l}>0$ there exists such $\delta_2>0$, that for all $u\in N$ fulfilling the conditions $\left|\frac{\partial^\lambda u}{\partial x^\lambda}-\frac{\partial^\lambda\tilde{u}}{\partial x^\lambda}\right|<\delta_2$ for $\lambda=0,1,\dots,p+n$ the inequalities

$$\left|\frac{\partial^s}{\partial x^s}(a(x,t)\tilde{u}_{0n}(x,t)-a(x,t)u_{0n}(x,t))\right|<\frac{\varepsilon}{2l},\qquad s=0,1,\dots,p$$

hold. If we put $\delta=\min\{\delta_1,\delta_2\}$, $q=\max\{p+n,p+k\}$, then we see that for $u\in U(\tilde{u},q,\delta)$ considering (18) for $r=0,1,\dots,m-1$, $s=0,1,\dots,p$

$$\left|\frac{\partial^{r+s}Tu}{\partial t^r\,\partial x^s}-\frac{\partial^{r+s}T\tilde{u}}{\partial t^r\,\partial x^s}\right|\leq\frac{\varepsilon}{l}\int_0^t\frac{(t-\tau)^{m-r-1}}{(m-r-1)!}\,d\tau\leq\frac{\varepsilon}{l}\,\frac{d^{m-r}}{(m-r)!}\leq\varepsilon \tag{21}$$

In a similar way using the equality (19) we obtain the validity of the last inequality also for $r = m$. Thus the continuity of T on N is proved.

In order to end the proof of Theorem 2 it is sufficient to prove the compactness of the set $\overline{T(N)}$. It will be proved on the basis of Lemma 1 when $T(N)$ is bounded and equicontinuous together with all its partial derivatives up to the order $\frac{\partial^{m+s}}{\partial t^m \partial x^s}$ (see e.g. [7]). The boundedness of $T(N)$ follows immediately from the relation (17) because the function $u_0(x, t)$ is on the compact set E_0 continuous with all its partial derivatives $\frac{\partial^{r+s} u_0}{\partial t^r \partial x^s}$, $r = 0, 1, \dots, m$, $s \in I_0^+$. Therefore there exists a constant $K(s)$ such that $p_s(u_0) \leqq K(s)$. For the proof of the equicontinuity of the set $\{Tu\}_{u \in N}$ we use the fact that if $u \in N$ then $p_s(u) < K(s)$ for $s \in I_0^+$ therefore $\frac{\partial^{i+j} u}{\partial t^i \partial x^j}$ are for $i = 0, 1, \dots, m$, $j = 0, 1, \dots, s$, uniformly bounded and as it follows from the Lagrange mean value theorem for $i = 0, 1, \dots, m-1$, $j = 0, 1, \dots, s-1$, they are equicontinuous. Since $T(N) \subset N$, it suffices to prove the equicontinuity of the set $\left\{\frac{\partial^{m+s} Tu}{\partial t^m \partial x^s}\right\}_{u \in N}$. If $u \in N$, $p \in I_0^+$, (ξ_1, τ_1), $(\xi_2, \tau_2) \in E_0$, then

$$\left|\frac{\partial^{m+p} Tu(\xi_1, \tau_1)}{\partial t^m \partial x^p} - \frac{\partial^{m+p} Tu(\xi_2, \tau_2)}{\partial t^m \partial x^p}\right| =$$

$$= \left|\frac{\partial^p}{\partial x^p}\left\{B(x, t, u_{00}, \dots, u_{m-1,k}) - a(x, t) u_{0n}\right\}_{\substack{x=\xi_1 \\ t=\tau_1}} - \right.$$

$$\left. - \frac{\partial^p}{\partial x^p}\left\{B(x, t, u_{00}, \dots, u_{m-1,k}) - a(x, t) u_{0n}\right\}_{\substack{x=\xi_2 \\ t=\tau_2}}\right| \tag{22}$$

If we apply the statement of Lemma 2 to the last equality we obtain

$$\left|\frac{\partial^{m+p} Tu(\xi_1, \tau_1)}{\partial t^m \partial x^p} - \frac{\partial^{m+p} Tu(\xi_2, \tau_2)}{\partial t^m \partial x^p}\right| \leqq$$

$$\leqq \sum_{\lambda_0+\lambda_1+\dots+\lambda_\varkappa \leqq p} \left|\left[B_{x^{\lambda_0}, u_{00}^{\lambda_1}, \dots, u_{m-1,k}^{\lambda_\varkappa}}\right]_{\substack{x=\xi_1 \\ t=\tau_1}}\right| \cdot$$

$$\cdot \left|P_{\lambda_0, \lambda_1, \dots, \lambda_\varkappa}\left(u_{01}, \dots, u_{m-1,k+p}\right)_{\substack{x=\xi_1 \\ t=\tau_1}} - \right.$$

$$\left. - P_{\lambda_0, \lambda_1, \dots, \lambda_\varkappa}\left(u_{01}, \dots, u_{m-1,k+p}\right)_{\substack{x=\xi_2 \\ t=\tau_2}}\right| +$$

$$+ \sum_{\lambda_0+\lambda_1+\dots+\lambda_\varkappa \leqq p} \left|\left[B_{x^{\lambda_0}, u_{00}^{\lambda_1}, \dots, u_{m-1,k}^{\lambda_\varkappa}}\right]_{\substack{x=\xi_1 \\ t=\tau_1}} - \right.$$

$$-\left[B_{x^{\lambda_0}, u_{00}^{\lambda_1}, \ldots, u_{m-1,k}^{\lambda_\varkappa}}\right]_{\substack{x=\xi_2\\t=\tau_2}}\right| \cdot \left|P_{\lambda_0, \lambda_1, \ldots, \lambda_\varkappa}(u_{01}, \ldots, u_{m-1,k+p})_{\substack{x=\xi_2\\t=\tau_2}}\right| +$$

$$+\sum_{i=0}^{p}\binom{p}{i}\left|\left[\frac{\partial^i a}{\partial x^i}\cdot u_{0,p+n-i}\right]_{\substack{x=\xi_2\\t=\tau_2}} - \left[\frac{\partial^i a}{\partial x^i}\cdot u_{0,p+n-i}\right]_{\substack{x=\xi_1\\t=\tau_1}}\right|, \tag{23}$$

where $B_{x^{\lambda_0}, u_{00}^{\lambda_1}, \ldots, u_{m-1,k}^{\lambda_\varkappa}}$ is given by (6) and $P_{\lambda_0, \lambda_1, \ldots \lambda_\varkappa}$ is a polynomial with the properties mentioned in Lemma 2. The function B ind $(x^{\lambda_0}, u_{00}^{\lambda_1}, \ldots, u_{m-1,k}^{\lambda_\varkappa})$ is for $u \in N$ uniformly continuous in E_0 by (H_2), its components u_{ij}, $i = 0, 1, \ldots, m-1$, $j = 0, 1, \ldots, k$ are continuous functions, $P_{\lambda_0, \lambda_1, \ldots, \lambda_\varkappa}$ is a polynomial in the variables u_{ij}, $i = 0, 1, \ldots, m-1$, $j = 1, 2, \ldots, k+p$ and finally $\frac{\partial^i a}{\partial x^i}$ is for $i = 0, 1, \ldots$ in E_0 uniformly continuous following (H_1). With regard to these facts to an arbitrary $\varepsilon > 0$ there exists such $\delta > 0$, that for all (ξ_1, τ_1), $(\xi_2, \tau_2) \in E_0$ for which $|\xi_1 - \xi_2| < \delta$, $|\tau_1 - \tau_2| < \delta$ the inequality

$$\left|\frac{\partial^{m+p} Tu(\xi_1, \tau_1)}{\partial t^m\, \partial x^p} - \frac{\partial^{m+p} Tu(\xi_2, \tau_2)}{\partial t^m\, \partial x^p}\right| < \varepsilon \tag{24}$$

for any $p \in I_0^+$ holds. Hence the compactness of $T(N)$ is proved and with respect to Theorem 1 the validity of Theorem 2 is proved.

Remark 1. The preceding theory can be illustrated by the following simple example. Assume that $a(x, t) \equiv 0$ in E_0, for each $i = 0, 1, \ldots, m-1$, $u_i(x) = u_i =$ $=$ const. > 0, $x_1 \leqq x \leqq x_2$, and $b_i = b_i(t) > 0$, $i = 1, 2$, are continuous functions for $t \in \langle 0, d\rangle$. Let $v = v(t)$ be the solution of the problem $\frac{d^m v}{dt^m} - b_2(t)\, v = b_1(t)$, $v^{(i)}(0) = u_i$, and c be a constant such that $0 < c \leqq \frac{\min b_2(t)}{\max v(t)}$, where min and max are taken on $\langle 0, d\rangle$. Finally suppose that $B(x, t, u, \ldots) = cu^2$ on $E_0 \times R^{+\varkappa}$. Then the conditions (H_1), (H_2) and (H_3) (a) are satisfied. The assumption (H_3) (b) can be verified to be true in the following way. By using the method of mathematical induction we can obtain the equalities

$$\frac{\partial^s}{\partial x^s}[cu^2(x, t)] = c\sum_{j=0}^{s}\binom{s}{j}\frac{\partial^j u(x, t)}{\partial x^j}\cdot\frac{\partial^{s-j} u(x, t)}{\partial x^{s-j}}$$

for all $s \in I_0^+$, $(x, t) \in E_0$ and $u(x, t) \in C_\infty(E_0)$. Thus the assumption (H_3) (b) in this case has the form

$$c\sum_{i=0}^{s}\binom{s}{i}\frac{\partial^i u}{\partial x^i}\cdot\frac{\partial^{s-i} u}{\partial x^{s-i}} \leqq \frac{\partial^s b_1}{\partial x^s} + \sum_{i=0}^{s}\binom{s}{i}\frac{\partial^i b_2}{\partial x^i}\frac{\partial^{s-i} u}{\partial x^{s-i}}$$

for each $u \in N$ (see (11), where instead of u_0, v has to be put). Since $\frac{\partial^i u}{\partial x^i} \equiv 0$ in E_0

for $i \geqq 1$, $u \in N$, the last inequality is trivially satisfied for $s \geqq 1$. When $s = 0$, we have $0 \leqq cu^2 \leqq cuv \leqq uc \max v \leqq u \min b_2 \leqq b_2 u$.

Remark 2. Now consider $B = b_2^*(x, t)u + b_1^*(x, t)$, $b_i^* \in G_x^+(\alpha, \varrho, M)$ for $i = 1, 2$. Suppose $\frac{\partial^s b_i^*}{\partial x^s} \leqq \frac{\partial^s b_i}{\partial x^s}$, $i = 1, 2$, $s \in I_0^+$. Then the assumptions (H_2) and (H_3) are satisfied. Assuming also (H_1) we get from Theorem 2 that for the solutions $u_0(x, t)$ and $u(x, t)$ of the problems (12), (2) and (2), $\frac{\partial^m u}{\partial t^m} + a \frac{\partial^n u}{\partial x^n} - b_2^* u = b_1^*$ respectively, the inequalities $0 \leqq \frac{\partial^s u}{\partial x^s} \leqq \frac{\partial^s u_0}{\partial x^s}$, $s \in I_0^+$, are true. This is a comparison result between two solutions of linear initial problems.

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SÚHRN

O APLIKÁCII TICHONOVOVEJ VETY O PEVNOM BODE NA CAUCHYHO NELINEÁRNY PROBLÉM

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V práci sa dokazuje existencia riešenia Cauchyho nelineárneho problému

$$\frac{\partial^m u(x,t)}{\partial t^m} + a(x,t)\frac{\partial^n u(x,t)}{\partial x^n} = $$

$$= B\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \ldots, \frac{\partial^{i+j} u}{\partial t^i \partial x^j}, \ldots, \frac{\partial^{m+k+1} u}{\partial t^{m-1} \partial x^k}\right) \tag{1}$$

$m \geqq n$, $i = 0, 1, \ldots, m-1$; $j = 0, 1, \ldots, k$ (k je ľubovoľné nezáporné celé číslo)

$$\left[\frac{\partial^i u(x,t)}{\partial t^i}\right]_{t=0} = u_i(x), \qquad i = 0, 1, \ldots, m-1 \tag{2}$$

v kompaktnom obore $x_1 \leqq x \leqq x_2$, $0 \leqq t \leqq d$. K tomuto cieľu sa využíva Tichonovova veta o pevnom bode v lokálne konvexnom lineárnom topologickom priestore.

РЕЗЮМЕ

ПРИМЕНЕНИЕ ТЕОРЕМЫ ТИХОНОВА О НЕПОДВИЖНОЙ ТОЧКЕ НА НЕЛИНЕЙНУЮ ЗАДАЧУ КОШИ

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В работе доказывается существование решения нелинейной задачи Коши

$$\frac{\partial^m u(x,t)}{\partial t^m} + a(x,t)\frac{\partial^n u(x,t)}{\partial x^n} = B\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \ldots, \frac{\partial^{i+j} u}{\partial t^i \partial x^j}, \ldots, \frac{\partial^{m+k-1} u}{\partial t^{m-1} \partial x^k}\right) \tag{1}$$

$m \geqq n$, $i = 0, 1, \ldots, m-1$; $j = 0, 1, \ldots, k$ (k-произвольное неотрицательное целое число)

$$\left[\frac{\partial^i u(x,t)}{\partial t^i}\right]_{t=0} = u_i(x), \qquad i = 0, 1, \ldots, m-1 \tag{2}$$

на компактном множестве $x_1 \leqq x \leqq x_2$, $0 \leqq t \leqq d$. При доказателстве существования решения задачи (1), (2) променяем теорему Тихонова о неподвижной точке в локально выпуклом линейном топологическом пространстве.

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AN A PRIORI BOUND OF THE ROUND-OFF ERROR IN THE INTEGRATION BY MULTISTEP DIFFERENCE METHOD FOR THE DIFFERENTIAL EQUATION $y^{(s)}=f(x, y)$

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Introduction

The purpose of this paper is to generalize the assertion about the a priori estimation for round-off error by the multistep difference methods for the solution of ordinary differential equations. The subject under discussion is the generalization of the considerations from [4] on the pages 242, 262, 312 and 318 with respect to the order of the differential equation and at the same time it is the proof of a part of Theorem 3.11 and an estimation of the constant K_2 from [2]. The proof is made under the assumption that the order k of the difference formula is greater or at least equal to the order s of the differential equation.

Let us consider the differential equation

$$y^{(s)}=f(x, y) \quad \text{for} \quad s \geqq 1 \tag{1}$$

The general k-th order difference formula for its solution may be written in the form

$$\sum_{\mu=0}^{k} \alpha_\mu y_{n+\mu}=h^s \sum_{\mu=0}^{k} \beta_\mu f(x_{n+\mu}, y_{n+\mu}), \qquad n=0, 1, \ldots, N-k \tag{2}$$

where

$$h=(b-a)/N, \quad x_n=a+nh \tag{3}$$

N is an arbitrary positive integer, k is a fixed integer and α_μ and β_μ ($\mu=0, 1, \ldots, k$) denote real constants which do not depend on n. We shall always assume that $\alpha_k \neq 0$, $|\alpha_0|+|\beta_0|>0$ and in addition, that $k \geqq s$.

Definition 1. The formula (2) will be said to be stable in the sense of Dahlquist, if all roots ζ_i of the characteristic polynomial

$$\varrho(\zeta)=\sum_{\mu=0}^{k} \alpha_\mu \zeta^\mu \tag{4}$$

are such that $|\zeta_i| \leqq 1$ and the multiplicity of the roots for which $|\zeta_i| = 1$ is at most s.

Then the following two lemmas are true.

Lemma 1. Let the polynomial $\varrho(\zeta) = \alpha_k\zeta^k + \alpha_{k-1}\zeta^{k-1} + \ldots + \alpha_0$ satisfy the condition of stability and let the coefficients γ_l $(l = 0, 1, 2, \ldots)$ be defined by

$$\frac{1}{\alpha_k + \alpha_{k-1}\zeta + \ldots + \alpha_0\zeta^k} = \gamma_0 + \gamma_1\zeta + \gamma_2\zeta^2 + \ldots, \; |\zeta| < 1 \tag{5}$$

Then there exist constants $\Gamma_0, \Gamma_1, \ldots, \Gamma_{s-1}$ such that

$$|\gamma_l| \leqq \sum_{i=0}^{s-1} \binom{l+i-1}{i} \Gamma_i, \qquad l = 0, 1, 2, \ldots \tag{6}$$

Proof. We proceed by induction with using Lemma 5.5 and Lemma 6.2 from [4]. This two lemmas prove the validity of Lemma 1 for $s = 1$ and $s = 2$. Let the lemma be true for $s - 1$ and let $\alpha_k + \alpha_{k-1}\zeta + \ldots + \alpha_0\zeta^k = \hat{\varrho}(\zeta) = \zeta^k\varrho(\zeta^{-1})$. Since $\varrho(\zeta)$ has no roots outside $|\zeta| \leqq 1$ and since the roots on $|\zeta| = 1$ are of multiplicity at most s, the polynomial $\hat{\varrho}(\zeta)$ has no roots in $|\zeta| < 1$, and its roots on $|\zeta| = 1$ are at most s-tuple. If the roots of $\varrho(\zeta)$ on $|\zeta| = 1$ with multiplicity s are denoted by ζ_s, $\zeta_{2s}, \ldots, \zeta_{cs}$, then for a suitable choise of the constants $A_1, A_2, \ldots, A_c$ the function

$$f(\zeta) = \frac{1}{\hat{\varrho}(\zeta)} - \sum_{\mu=1}^{c} \frac{A_\mu}{(\zeta - \zeta_{\mu s}^{-1})^s} \tag{7}$$

is holomorphic in $|\zeta| < 1$ and has at most a finite number of poles with multiplicity $(s - 1)$ on $|\zeta| = 1$. By the induction step we have that the coefficients of the Taylor expansion of $f(\zeta)$ in $\zeta = 0$ are bounded. In view of

$$\frac{1}{(\zeta - \zeta_\mu^{-1})^s} = \frac{(-\zeta_\mu)^s}{(1 - \zeta_\mu\zeta)^s} = (-\zeta_\mu)^s \sum_{i=0}^{\infty} \binom{s-1+i}{s-1} (\zeta_\mu\zeta)^i \tag{8}$$

the Taylor coefficients at $\zeta = 0$ of each term on the right of (7) satisfy an inequality of the form (6). It follows that the same is true for the function

$$\frac{1}{\hat{\varrho}(\zeta)} = f(\zeta) + \sum_{\mu=1}^{c} \frac{A_\mu}{(\zeta - \zeta_{\mu s}^{-1})^s} \tag{9}$$

Further we shall use the identity

$$\alpha_k\gamma_l + \alpha_{k-1}\gamma_{l-1} + \ldots + \alpha_0\gamma_{l-k} = \begin{cases} 1 & l = 0 \\ 0 & l > 0 \end{cases} \tag{10}$$

where it is assumed that $\gamma_l = 0$ for $l < 0$. This identity is proved by multiplying both sides of (5) by $\alpha_k + \alpha_{k-1}\zeta + \ldots + \alpha_0\zeta^k$ and comparing coefficients in the resulting expansions in the same powers of ζ. The following lemma concerns the growth of solutions of the nonhomogeneous linear difference equation

$$\sum_{\mu=0}^{k} \alpha_\mu z_{m+\mu} = h^s \sum_{\mu=0}^{k} \beta_{\mu,m} z_{m+\mu} + \lambda_m \tag{11}$$

Lemma 2. Let the polynomial $\varrho(\zeta) = \alpha_k \zeta^k + \ldots + \alpha_0$ satisfy the condition of stability, let B^*, β and Λ be nonnegative constants such that

$$|\beta_{k,m}| + |\beta_{k-1,m}| + \ldots + |\beta_{0,m}| \leqq B^*,\ |\beta_{k,m}| \leqq \beta,\ |\lambda_m| \leqq \Lambda,\ 0 \leqq m \leqq N \tag{12}$$

and let $0 < h^s < |\alpha_k| \beta^{-1}$.

Then every solution of (11) for which

$$|z_\mu| \leqq Z, \qquad \mu = 0, 1, \ldots, k-1 \tag{13}$$

satisfies

$$|z_n| \leqq K^* e^{nh^s L^*}, \qquad 0 \leqq n \leqq N \tag{14}$$

Here

$$L^* = \frac{B^* \sum_{i=0}^{s-1} \binom{N-k+i-1}{i} \Gamma_i}{1 - h^s |\alpha_k^{-1}| \beta}$$

$$K^* = \frac{\Lambda \sum_{i=0}^{s-1} \binom{N-k+i}{i+1} \Gamma_i + kAZ \sum_{i=0}^{s-1} \binom{N-k+i-1}{i} \Gamma_i}{1 - h^s |\alpha_k^{-1}| \beta} \tag{15}$$

where $A = |\alpha_k| + |\alpha_{k-1}| + \ldots + |\alpha_0|$.

The proof is analogous to the proofs of Lemma 5.6 and Lemma 6.3 from [4]. For $l = 0, 1, \ldots, n-k$ multiply the equation (11) corresponding to $m = n - k - l$ by γ_l and add the resulting equations. If the sum is called S_n we obtain by summing on the left

$$S_n = (\alpha_k z_n + \alpha_{k-1} z_{n-1} + \ldots + \alpha_0 z_{n-k})\gamma_0 +$$
$$+ (\alpha_k z_{n-1} + \alpha_{k-1} z_{n-2} + \ldots + \alpha_0 z_{n-k-1})\gamma_1 +$$
$$+ \ldots + (\alpha_k z_k + \alpha_{k-1} z_{k-1} + \ldots + \alpha_0 z_0)\gamma_{n-k}$$

Rearranging we get

$$S_n = \alpha_k \gamma_0 z_n + (\alpha_k \gamma_1 + \alpha_{k-1} \gamma_0) z_{n-1} + \ldots$$
$$\ldots + (\alpha_k \gamma_{n-k} + \ldots + \alpha_0 \gamma_{n-2k}) z_k + (\alpha_{k-1} \gamma_{n-k} + \ldots$$
$$\ldots + \alpha_0 \gamma_{n-2k+1}) z_{k-1} + \ldots + \alpha_0 \gamma_{n-k} z_0$$

By (10) this reduces to

$$S_n = z_n + (\alpha_{k-1}\gamma_{n-k} + \ldots + \alpha_0\gamma_{n-2k+1})z_{k-1} + \ldots + \alpha_0\gamma_{n-k}z_0 \tag{16}$$

Summing on the right, we find

$$\begin{aligned} S_n = h^s\{&\beta_{k,n-k}\gamma_0 z_n + (\beta_{k-1,n-k}\gamma_0 + \beta_{k,n-k-1}\gamma_1)z_{n-1} + \ldots + \\ &+ (\beta_{0,n-k}\gamma_0 + \ldots + \beta_{k,n-2k}\gamma_k)z_{n-k} + \ldots + \beta_{0,0}\gamma_{n-k}z_0\} + \\ &+ \lambda_{n-k}\gamma_0 - \lambda_{n-k-1}\gamma_1 + \ldots + \lambda_0\gamma_{n-k} \end{aligned} \tag{17}$$

Equating (16) and (17) we find, using (12) and (13)

$$\begin{aligned} |z_n| \leqq h^s\beta|\alpha_k^{-1}|\,|z_n| + h^s B^* \sum_{i=0}^{s-1}\binom{N-k+i-1}{i}\Gamma_i \sum_{m=0}^{n-1} z_m + \\ + A\sum_{i=0}^{s-1}\binom{N-k+i}{i+1}\Gamma_i + kAZ\sum_{i=0}^{s-1}\binom{N-k+i-1}{i}\Gamma_i \end{aligned}$$

Solving the resulting inequality for $|z_n|$, there follows

$$|z_n| \leqq h^s L^* \sum_{m=0}^{n-1} |z_m| + K^* \tag{18}$$

where L^* and K^* are given by (15). We now proceed by induction. Since $A\sum_{i=0}^{s-1}\binom{N-k+i-1}{i}\Gamma_i > 1$ and hence $K^* \geqq Z$, the estimate

$$|z_m| \leqq K^*(1 + h^s L^*)^m \tag{19}$$

is true for $m = 0, 1, \ldots, k-1$. Assuming its truth for $m = 0, 1, \ldots, n-1$ and using it on the right of (18), we obtain

$$|z_n| \leqq h^s L^* K^* \frac{(1 + h^s L^*)^n - 1}{h^s L^*} + K^* = K^*(1 + h^s L^*)^n$$

Relation (19) is thus established for $m = n$ and this holds generally for $m = 0, 1, \ldots, N$. The statement of the lemma now follows by using $1 + h^s L^* \leqq e^{h^s L^*}$.

Instead of satisfying the difference equation (2) exactly, the quantities $\tilde{y}_n$ actually calculated satisfy an equation which we write in the form

$$\sum_{\mu=0}^{k} \alpha_\mu \tilde{y}_{n+\mu} = h^s \sum_{\mu=0}^{k} \beta_\mu f(x_{n+\mu}, \tilde{y}_{n+\mu}) + \varepsilon_{n+k}, \qquad n = 0, 1, 2, \ldots \tag{20}$$

The quantities ε_n will be called the local round-off errors. The problem to be studied in this paper is the influence of these local errors on the accumulated round-off error $r_n = \tilde{y}_n - y_n$. We shall derive an a priori estimate for r_n under the sole assumption that $|\varepsilon_{n+k}| \leqq \varepsilon$ $(n = 0, 1, 2, \ldots)$ where ε is a constant independent of n.

Subtracting from (20) the corresponding equation (2) and writing

$$g_m = r_m^{-1}[f(x_m, \tilde{y}_m) - f(x_m, y_m)], \quad \text{if} \quad r_m \neq 0$$
$$g_m = 0, \qquad \text{if} \quad r_m = 0$$

so that in any case $|g_m| \leqq L$, L is the Lipschitz constant, we obtain

$$\sum_{\mu=0}^{k} \alpha_\mu r_{n+\mu} = h^s \sum_{\mu=0}^{k} \beta_\mu g_{n+\mu} r_{n+\mu} + \varepsilon_{n+k} \tag{21}$$

Applying Lemma 2 to this relation with $z_m = r_m$, $\Lambda = \varepsilon$, $N = (b-a)/h$, $Z = 0$ (since $r_0 = r_1 = \ldots = r_{k-1} = 0$), $B^* = LB = L \sum_{\mu=0}^{k} |\beta_\mu|$, $\beta = L|\beta_k|$ under the assumption that $h^s < L^{-1}|\beta_k^{-1}\alpha_k|$ we get

$$|r_n| \leqq \varepsilon h^{-s}(x_n - a)^s \Gamma^* \exp\left[(x_n - a)^s \Gamma^* B^*\right] \tag{22}$$

where

$$\Gamma^* = \frac{\max\left[\max_i \left(\frac{\Gamma_{i-1}}{\binom{s}{i} i!}\right), \max_j \left(\frac{\Gamma j}{\binom{s-1}{j} j!}\right)\right]}{1 - h^s L |\alpha_k^{-1}\beta_k|}$$

Therefore the following theorem is true.

Theorem 1. Let the function $f(x, y)$ be defined, continuous and satisfy the Lipschitz condition in y with the constant L in the region $a \leqq x \leqq b$, $|y| < \infty$. Let $s \geqq 1$, $k \geqq s$ be integers and let be given the differential equation (1) with the difference formula (2) stable in the sense of Dahlquist under the conditions (3), where $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_0| > 0$. Let the calculated results satisfy (20), where for the local round-off errors $|\varepsilon_n| \leqq \varepsilon$, $n = 0, 1, \ldots, N$ is true and let $h^s < L^{-1}|\beta_k^{-1}\alpha_k|$. Then for the accumulated round-off error (22) is valid.

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SÚHRN

ODHAD ZAOKRÚHĽOVACEJ NEPRESNOSTI DIFERENČNEJ VIACKROKOVEJ METÓDY PRE DIFERENCIÁLNU ROVNICU $y^{(s)}=f(x,y)$

A. Huťa jun., Bratislava

V článku sa vyšetruje vplyv lokálnych zaokrúhľovacích nepresností na celkovú zaokrúhľovaciu nepresnosť pre diferenciálnu rovnicu $y^{(s)}=f(x,y)$.

Dokazuje sa nasledovná veta:

Veta 1. Nech funkcia $f(x,y)$ je definovaná, spojitá a spĺňa Lipschitzovu podmienku vzhľadom na y s konštantou L v obore $a \leqq x \leqq b$, $|y|<\infty$. Nech $s \geqq 1$, $k \geqq s$ sú celé čísla a nech je daná diferenciálna rovnica (1) s diferenčnou schémou (2) stabilnou v zmysle Dahlquista za predpokladov (3), kde $\alpha_k \neq 0$, $|\alpha_0|+|\beta_0|>0$. Nech skutočné výsledky spĺňajú (20), kde pre lokálne zaokrúhľovacie nepresnosti platí $|\varepsilon_n| \leqq \varepsilon$ $(n=0,1,\ldots,N)$ a nech $h^s < L^{-1}|\beta_k^{-1}\alpha_k|$. Potom pre výsledné zaokrúhľovacie nepresnosti platí (22).

РЕЗЮМЕ

ОЦЕНКА ПОГРЕШНОСТИ ОКРУГЛЕНИЯ РАЗНОСТНОГО МНОГОШАГОВОГО МЕТОДА ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $y^{(s)}=f(x,y)$

А. Хутя мл., Братислава

В этой статье исследуется влияние локальных погрешностей округления на общую погрешность округления для дифференциального уравнения $y^{(s)}=f(x,y)$.

Доказана следующая теорема:

Теорема 1. Пусть функция $f(x,y)$ определена, непрерывна и удовлетворяет условию Липшица по y с постоянной L в области $a \leqq x \leqq b$, $|y|<\infty$. Пусть $s \geqq 1$, $k \geqq s$ целые числа и пусть задано дифференциальное уравнение (1) с разностной формулой (2), устойчивой в смысле Дальквиста при условиях (3), где $\alpha_k \neq 0$, $|\alpha_0| + |\beta_0|>0$. Пусть, кроме того, фактические результаты удовлетворяют условию (20), где для локальных погрешностей округления имеет место отношение $|\varepsilon_n| \leqq \varepsilon$ $(n=0,1,\ldots,N)$ причем $h^s < L^{-1}|\beta_k^{-1}\alpha_k|$. Тогда для конечной погрешности округления имеет место отношение (22).

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MODELS OF THE KLEIN BOTTLE

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1. Introduction

The Klein bottle Kb is a topological factor space I^2/R where R is an equivalence relation given as follows: if $(x, y) \neq (x', y')$ are two points of the unite square I^2, then

$$(x, y)R(x', y') \Leftrightarrow \left\langle \begin{array}{l} \{x, x'\} = \{0, 1\} \text{ and } y = y' \\ \{y, y'\} = \{0, 1\} \text{ and } x + x' = 1 \end{array} \right.$$

It is well known that Kb may be embedded neither into the 3-dimensional euclidean space E^3 nor into the 3-dimensional real projective space RP^3. Roughly spoken: the Klein bottle cannot be constructed as a surface in "our" space. However, there is a way in which to model Kb in "our" space, namely using lines instead of points.

The aim of this article is to prove the following three assertions.

Theorem. Let in RP^3 be given a quadric S^2 of the signature (3, 1). Let M be a line in RP^3 which does not tangent S^2. The space K of all tangents U to S^2 intersecting the line M (regarded as a topological subspace of the Grassmannian $G_1(RP^3)$) is homeomorphic to the Klein bottle Kb.

Corollary 1. The subspace K_1 of $G_1(E^3)$ which consists of all tangents U to the unite sphere $S^2 \subset E^3$ intersecting or parallel to the x^3-axis $(x^1 = x^2 = 0)$ is homeomorphic to Kb.

Corollary 2. The subspace K_2 of $G_1(E^3)$ which consists of all tangents U to the unite sphere $S^2 \subset E^3$ parallel to a given plane $E^2 \subset E^3$ is homeomorphic to Kb.

2. Preliminaries

In any real projective space RP^n a coordinate system

$$\sigma: R^{n+1} - \{0\} \to RP^n$$

is assumed to be given.

We recall (see [1] pg 356) the Plücker map

$$\pi: G_1(RP^3) \to RP^5, \quad U \mapsto \sigma(u^0, \ldots, u^5)$$

which goes from the set $G_1(RP^3)$ of all lines in RP^3 into RP^5 and is defined as follows: choose arbitrary two distinct points $A = \sigma(a^0, a^1, a^2, a^3)$, $B = \sigma(b^0, b^1, b^2, b^3)$ of a line $U \in G_1(RP^3)$. Denote $[i, j] = a^i b^j - a^j b^i$, $i, j \in \{0, 1, 2, 3\}$; then

$$\begin{aligned} u^0 &= [0, 1], \quad u^1 = [0, 2], \quad u^2 = [0, 3] \\ u^3 &= [2, 3], \quad u^4 = [3, 1], \quad u^5 = [1, 2] \end{aligned}$$

This map is well defined, injective, and

$$G_1(RP^3) = Q = \{\sigma(u^0, \ldots, u^5);\ u^0u^3 + u^1u^4 + u^2u^5 = 0\} \tag{1}$$

is the regular hyperquadric of the signature (3, 3) in RP^5. From now on, $G_1(RP^3)$ is a topological space with a topology induced by π-hence $G_1(RP^3)$ is homeomorphic to Q.

Let $Y = \sigma(y^0, y^1, y^2, y^3) \in RP^3$ be a point, $U = \pi^{-1} \circ \sigma(u^i)$, $V = \pi^{-1} \circ \sigma(v^i) \in G_1(RP^3)$ two lines and

$$S^2 = \{\sigma(x^0, x^1, x^2, x^3);\ (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 0\} \tag{2}$$

a "unit sphere" in RP^3. Then by [1] (3.185) and (3.192) it is

$$U \cap V \neq 0 \Leftrightarrow u^0v^3 + u^1v^4 + u^2v^5 + u^3v^0 + u^4v^1 + u^5v^2 = 0 \tag{3}$$

and

$$Y \in U \Leftrightarrow \begin{array}{l} y^0u^5 - y^1u^1 + y^2u^0 = 0 \\ y^0u^4 + y^1u^2 \qquad\quad - y^3u^0 = 0 \\ y^0u^3 \qquad\quad - y^2u^2 + y^3u^1 = 0 \\ \qquad y^1u^3 + y^2u^4 + y^3u^5 = 0 \end{array} \tag{4}$$

Lemma. A line U is tangent to the quadric S^2 if and only if

$$(u^0)^2 + (u^1)^2 + (u^2)^2 = (u^3)^2 + (u^4)^2 + (u^5)^2 \tag{5}$$

Proof. Let $Y \in U \cap S^2$ and suppose $u^0 \neq 0$. Then y^2 and y^3 may be expressed as functions of y^0 and y^1 using first two of the equations (4). After substituing these into the equation (2) the quadratic equation with the discriminant

$$D = 4u^0[(u^0)^2 + (u^1)^2 + (u^2)^2 - (u^3)^2 - (u^4)^2 - (u^5)^2]$$

is obtained. Obviously U is tangent to S^2 if and only if $D = 0$ i.e. (5). It is obvious how to finish the proof in the case $u^0 = 0$: take any other $u^i \neq 0$ instead of u^0 and corresponding two of the equations (4).

3. Proof of the Theorem

Two *cases* are to be destinguished: 1. $M \cap S^2$ consists of two points $C_1 \neq C_2$; 2. $M \cap S^2$ is empty. The space K will be denoted by K_1 in the first and by K_2 in the second case.

1. Let $M \cap S^2 = \{C_1, C_2\}$. There exists such a coordinate system in RP^3 that S^2 is given by (2) and $C_{1,2} = \sigma(\pm 1, 0, 0, 1)$. Thus $M = C_1C_2 = \pi^{-1} \circ \sigma(0, 0, 1, 0, 0, 0)$ and according to (3) lines $U = \pi^{-1} \circ \sigma(u^i)$ and M intersect (or are equal) if and only if $u^5 = 0$. Therefore from (1) and (5) it follows

$$K_1 = \{\pi^{-1} \circ \sigma(u^i);\ u^5 = 0,\ u^0u^3 + u^1u^4 = 0,$$
$$(u^0)^2 + (u^1)^2 + (u^2)^2 = (u^3)^2 + (u^4)^2\}$$

Since $(u^3)^2 + (u^4)^2 \neq 0$, it may be assumed (normalized) that

$$(u^0)^2 + (u^1)^2 + (u^2)^2 = 1,\ (u^3)^2 + (u^4)^2 = 1 \tag{6}$$

Hence there exists $x \in I$ such that

$$u^3 = \varepsilon \sin \pi y,\ u^4 = \varepsilon \cos \pi y,\ \varepsilon = \pm 1 \tag{7}$$

Moreover, this y is uniquely defined by (7) except the case of $u^3 = 0$. The relation $u^0u^3 + u^1u^4 = 0$ together with (7) yields

$$u^0 = \varepsilon z \cos \pi y,\ u^1 = -\varepsilon z \sin \pi y \tag{8}$$

and using (6) it is $z^2 + (u^2)^2 = 1$. Hence there exists $x \in I$ for which

$$z = \cos 2\pi x,\ u^2 = \sin 2\pi x \tag{9}$$

Thus each point $\pi^{-1} \circ \sigma(u^i) \in K_1$ may be expressed in the form

$$\begin{aligned} u^0 &= \cos 2\pi x \cos \pi y, & u^3 &= \sin \pi y \\ u^1 &= -\cos 2\pi x \sin \pi y, & u^4 &= \cos \pi y \\ u^2 &= -\sin 2\pi x, & u^5 &= 0 \end{aligned}$$

These relations define a surjective continuous map

$$\Phi : I^2 \to K_1,\ (x, y) \mapsto (u^i(x, y))$$

and K_1 is homeomorphic to the factor space I^2/R_1 where R_1 is the equivalence relation given by

$$(x, y)R_1(x', y') \Leftrightarrow \Phi(x, y) = \Phi(x, y) = \Phi(x', y')$$

The proof of case 1. will be finished by showing $R = R_1$. It is not difficult to check the inclusion $R \subset R_1$. To prove $R_1 \subset R$, let us take two different points (x, y) and (x', y') for which

$$u^i(x, y) = tu^i(x', y') \quad \text{for} \quad i = 0, \ldots, 5$$

Since $(u^3)^2 + (u^4)^2 = 1$, it is $t = +1$ or -1. If $t = +1$, then $i = 3{,}4$ yields $y = y'$ and $i = 0, 1, 2$ yields $\{x, x'4 = \{0, 1\}$. If $t = -1$, then $i = 3, 4$ yields $\{y, y'\} = \{0, 1\}$ and $i = 0, 1, 2$ yields $x + x' = 1$. The proof of case 1 is finished.

2. Let $M \cap S^2 = \emptyset$. There exists such a coordinate system σ in RP^3 that S^2 is given by (2) and $M = \{\sigma(0, x^1, x^2, 0)\}$. Thus $M = \pi^{-1} \circ \sigma(0, 0, 0, 0, 0, 1)$ and similarly as in the case I) it is

$$K_2 = \{\pi^{-1} \circ \sigma(u^i);\ u^2 = 0,\ u^0 u^3 + u^1 u^4 = 0,$$
$$(u^0)^2 + (u^1)^2 = (u^3)^2 + (u^4)^2 + (u^5)^2\}$$

Further, K_2 is homeomorphic to K_1 for there exists an involutive homeomorphism

$$\varphi: RP^5 \to RP^5,\ \sigma(u^0, u^1, u^2, u^3, u^4, u^5) \mapsto \sigma(u^3, u^4, u^5, u^0, u^1, u^2)$$

with invariant subspace Q which mutually interchanges K_1 and K_2. The proof of the Theorem is finished. Both Corollaries come immediately from an inclusion map $E^3 \to RP^3$, $(x^1, x^2, x^3) \to \sigma(1, x^1, x^2, x^3)$ and its extension to Grassmannians.

4. Example

At the end of the paper we turn our attention to the omitted case "M is tangent to S^2". An instructive example of a non-Serre fibration will be obtained.

Let $S^2 \equiv x^2 + y^2 + z^2 = 1$ be a unit sphere in E^3 and M its tangent at the north pole $N = (0, 0, 1)$ given by $y = 0$, $z = 1$. Let K_0 be a topological subspace of $E^1 \times G_1(E^3)$ which consists of all couples (V, U) where $V \in M$ is a point and U a tangent to S^2 passing through V. Then there is a fibre map $p: K_0 \to M$, $(V, U) \mapsto V$ which is not a Serre fibration, although $p^{-1}(V)$ is homeomorphic to S^1 for each $V \in M$. To prove the last assertion let us regard the commutative diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{\ \mu\ } & K_0 \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ S^1 \times I & \xrightarrow{\ \lambda\ } & M \end{array}$$

where i: $e^{i\alpha} \mapsto (e^{i\alpha}, 0)$ is an inclusion map,

$$\mu:\ e^{i\alpha} \mapsto \left(N, \left\{N + s\left(\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}, 0\right), s \in R\right\}\right)$$
$$\lambda:\ (e^{i\alpha}, t) \mapsto (t, 0, 1).$$

By showing that there is no lift $\Lambda: S^1 \times I \to K_0$ preserving the commutativity of the diagram the non-CHP (covering homotopy property) for (K_0, p, M), and hence the assertion, will be proved.

Suppose by contradiction that there exists a lift Λ such that $p \circ \Lambda = \lambda$. Then Λ may be viewed as a homotopy in $G_1(E^3)$ which takes an uncontractible circle $\Lambda_0(S^1) = \mu(S^1)$ into a contractible one $\Lambda_1(S^1)$; here $\Lambda_1(S^1)$ is a "cone" which consists of all lines

$$\left\{\left(1 + \frac{s}{2}(1 + \cos\alpha),\ \frac{s}{\sqrt{2}}\sin\alpha,\ 1 + \frac{s}{2}(1 - \cos\alpha)\right);\ s \in R\right\}$$

where $\alpha \in [0, 2\pi]$.

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SÚHRN

MODELY KLEINOVEJ FĽAŠE

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Topologický podpriestor priestoru $Gr_1(RP^3)$, ktorý sa skladá zo všetkých dotyčníc kvadriky signatúry (3, 1), ktoré pretínajú priamku nedotýkajúcu sa tejto kvadriky — to je model Kleinovej fľaše.

РЕЗЮМЕ

МОДЕЛИ БУТЫЛКИ КЛЕЙНА

М. Гейны, Братислава

Топологическое подпространство пространства $Gr_1(RP^3)$, состоящее из всех кассательных квадрики сигнатуры (3, 1) пересекающих прямую не кассающуюся этой квадрики — это модель бутылки Клейна.

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REMARK ON MATCHINGS IN REGULAR GRAPHS

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In this paper, all notions not defined here will be used in the sense of [4]. Given a graph $G=(V, E)$, V and E denote its point set $V(G)$ and line set $E(G)$, respectively. For a set $M \subseteq V(G)$, $G(M)$ denotes the induced subgraph of G with point set M and $h_G(M)$ denotes the number of odd components of $G(V-M)$. (By an odd component we mean that with an odd number of points.)

A matching of a graph G can be defined as a subgraph H of G in which each point has degree 1. We refer to the difference $|V(G)|-|V(H)|$ as the deficiency of the matching. Thus a 1-factor of G is a matching with deficiency zero.

The paper deals with matchings in regular graphs whose line-connectivity differs from the degree no more than by unity. There are several results on this subject. We mention here only those from among them, which will be used in the sequel.

Theorem A. ([2]). Let G be an $(r-1)$-line-connected regular multigraph of degree $r>0$ with an even number of points. Then G has a 1-factor containing an arbitrarily prescribed line.

Theorem A appears as a consequence of the following assertion.

Theorem B ([5]). Let G be an $(r-1)$-line-connected regular multigraph of degree $r>0$ with an even number of points. Then G has a 1-factor not containing $r-1$ arbitrarily prescribed lines.

The following two results are due to Grant, Holton and Little (see [3]).

Theorem C. Let G be an $(r-1)$-point-connected regular graph of degree $r>0$. If d is a non-negative integer and the number of points of G has the parity of d, then G has a matching with deficiency d containing one arbitrarily prescribed line.

Theorem D. Let G be an r-point-connected regular graph of degree $r>1$ with an odd number of points. Then for any $v \in V(G)$, $G-v$ has a 1-factor.

In this paper we have generalized the abovementioned four results. Our first theorem is based on the following assertion which immediately follows from a result due to Berge (see [1] or [2]).

Lemma 1. Let G be a multigraph, and let d be a non-negative integer. Then in order that G shall have no matching of deficiency d it is necessary and sufficient that one of the following three conditions shall hold:

1. $d > |V(G)|$,
2. $d \not\equiv |V(G)| \pmod 2$,
3. There is a subset S of $V(G)$ such that $d + |S| < h_G(S)$.

The next theorem is a generalization of Theorem B and the proof of it given below is an extension of that of Theorem B from [5].

Theorem 1. Let $G = (V, E)$ be an $(r-1)$-line-connected regular multigraph of degree $r > 0$ with p points. Let d and x be integers where $0 \leqq d \leqq p$, $d \equiv p \pmod 2$, and $x \leqq r + rd/2 - 1$. If $E' \subseteq E$ with $|E'| = x$, then the multigraph $G' = (V, E - E')$ has a matching with deficiency d.

Proof. Assume that the multigraph G' has no matching with deficiency d. Then by Lemma 1 we have the case (3), i.e. there is a set $S \subseteq V$ such that $G'(V-S)$ has n odd components (say) $G'_1, G'_2, \ldots, G'_n$, where

$$n = h_{G'}(S) > d + |S| \tag{1}$$

As $n - |S| \equiv n + |S| \equiv p \equiv d \pmod 2$, (1) implies

$$n - |S| \geqq d + 2 \tag{2}$$

Denote $V(G'_i)$ by V_i for $i = 1, 2, \ldots, n$. The number of all lines of G incoming to V_i from S, or from $V - S - V_i$ will be denoted by s_i, or by t_i, respectively. Denoting by d_i the sum of degrees of points in $G(V_i)$, we have

$$0 \equiv d_i = r|V_i| - (s_i + t_i) \leqq r|V_i| - (r-1) = r(|V_i| - 1) + 1 \equiv 1 \pmod 2$$

It follows that $r - 1 < s_i + t_i$, or

$$r \leqq s_i + t_i \qquad (i = 1, 2, \ldots, n) \tag{3}$$

From S exactly $\sum_{i=1}^{n} s_i$ lines income to $\bigcup_{i=1}^{n} V_i$. As G is regular of degree r,

$$\sum_{i=1}^{n} s_i \leqq r|S| \tag{4}$$

Further, the condition $|E'| = x$ gives

$$\sum_{i=1}^{n} t_i \leqq 2x \tag{5}$$

Using (3), (4), and (5), we obtain $rn \leqq r|S| + 2x$, or

$$r(n - |S|) \leqq 2x \tag{6}$$

From (2) and (6) we have $r(d+2) \leqq 2x$ which contradicts the assumptions of Theorem 1. This completes the proof.

Corollary. Let r, p, d be integers, where $r \geqq 1$, $0 \leqq d \leqq p$ and $d \equiv p \pmod 2$. Let G be an $(r-1)$-line-connected regular multigraph of degree r with p points. a) If p is even, then G has a matching with deficiency d not containing $d/2$ arbitrarily prescribed points and $r-1$ arbitrarily prescribed lines. b) If p is odd, then G has a matching with deficiency d not containing $(d+1)/2$ arbitrarily prescribed points and $r/2-1$ arbitrarily prescribed lines.

The proof is obvious. Note that Corollary a) or b) involves Theorem B or D, respectively (put $d=0$ or $d=1$, respectively).

The following two constructions enable us to make from a given regular multigraph a new one with further useful properties.

Let G be a multigraph and let $e_1, e_2, \ldots, e_n$ be n lines of $G (n \geqq 1)$. Let $e_i = u_i v_i$ for $i = 1, 2, \ldots, n$. 1. Provided that $w \notin V(G)$ we put $S^+(G; e_1, \ldots, e_n) = (V(G) \cup \{w\}, (E(G) - \{e_1, \ldots, e_n\}) \cup \{u_1 w, wv_1, \ldots, u_n w, wv_n\})$. 2. Provided that $x_1, \ldots, x_n, y_1, \ldots, y_{n-2} \notin V(G)$, we put $S^{++}(G; e_1, \ldots, e_n) = (V(G) \cup \{x_1, \ldots, x_n, y_1, \ldots, y_{n-2}\}, (E(G) - \{e_1, \ldots, e_n\}) \cup \{u_1 x_1, x_1 v_1, \ldots, u_n x_n, x_n v_n\} \cup \{x_i y_j \mid i = 1, \ldots, n; j = 1, \ldots, n-2\})$.

Lemma 2. Let G be an $(r-1)$-line-connected regular multigraph of even degree $r \geqq 2$ and let $e_1, \ldots, e_{r/2}$ be $r/2$ arbitrary lines of G. Then the multigraph $S^+ = S^+(G; e_1, \ldots, e_{r/2})$ is $(r-1)$-line-connected and regular of degree r.

Proof. Clearly S^+ is regular of degree r. Assume that there is a set Q of $r-2$ lines which disconnects S^+. Let S_1^+ and S_2^+ be two components of $S^+ - Q$. If S_1^+ or S_2^+ consists only of the point w, we have a contradiction because the degree of w is greater than $r-2$. If boths S_1^+ and S_2^+ contain points of $V(G)$, we have a contradiction again because in the construction of S^+ every line e_i is substituted by the path $u_i w v_i$ and G is $(r-1)$-line-connected.

Lemma 3. Let G be an $(r-1)$-line-connected regular multigraph of degree $r \geqq 2$ and let $e_1, \ldots, e_r$ be r arbitrary lines of G. Then the multigraph $S^{++} = S^{++}(G; e_1, \ldots, e_r)$ is $(r-1)$-line-connected and regular of degree r.

Proof. The regularity is immediate. Assume that there is a set Q of $r-2$ lines which disconnects S^{++}. If there are two components of $S^{++} - Q$ each containing at least one point of G, we have a contradiction because every line e_i is substituted by the path $u_i x_i v_i$ and G is $(r-1)$-line-connected. Therefore the set $V(G)$ lies in a component S_1^{++} of $S^{++} - Q$. For any two points y_i, y_j there are r line-disjoint $y_i - y_j$ paths (namely $y_i x_k y_j$; $k = 1, \ldots, r$), so that all the points $y_1, \ldots, y_{r-2}$ lie in the same component of $S^{++} - Q$. For any two points x_i, x_j there are at least $r-1$ line-disjoint paths joining them (e.g. the paths $x_i y_k x_j$, $k = 1, \ldots, r-2$ and a path $x_i u_i - u_j x_j$ where $u_i - u_j$ is a path containing no line $x_s y_t$ the existence of which follows from the connectedness of G). For any two points x_i, y_j there are the following $r-1$ $x_i - y_j$ paths: $x_i y_j$, $x_i y_k x_k y_j$ ($k = 1, \ldots, r-2$ and $k \neq j$), $x_i u_i - u_f x_f y_j$

where $u_i - u_f$ is a path containing no line $x_s y_t$ and $f \in \{r-1, r\} - \{i\}$. Thus $S^{++} - Q$ has only two components: S_1^{++} containing only all the points of G and S_2^{++} containing the other points of S^{++}. However, there are $2r$ lines going from $V(G)$ to $V(S^{++}) - V(G)$. We have a contradiction and the lemma is proved.

Theorem 2. Let r, d, p be integers with $r \geqq 1, 0 \leqq d \leqq p, d \equiv p \pmod 2$. Let G be an $(r-1)$-line-connected regular multigraph of degree r. Then G has a matching with deficiency d containing one arbitrarily prescribed line and not containing other $rd/2$ arbitrarily prescribed lines of G and having all the non-matched points in the set of all points incident with the $rd/2$ lines.

Proof. If $r=1$, the assertion is trivial. Therefore let $r \geqq 2$. If $d=0$ we have Theorem A, so we can suppose that $d \geqq 1$. Consider $1 + rd/2$ arbitrary lines of G (say) $e_0, e_1, \ldots, e_{rd/2}$. If d is odd then r is even and we can construct a multigraph $G_0 = S^+(S^{++}(\ldots S^{++}(S^{++}(G; e_1, \ldots, e_r); e_{r+1}, \ldots, e_{2r}) \ldots; e_{(d-3)r/2+1}, \ldots, e_{(d-1)r/2});$ $e_{(d-1)r/2+1}, \ldots, e_{dr/2})$. If d is even we can construct a multigraph $G_e = S^{++}(\ldots S^{++}(S^{++}(G; e_1, \ldots, e_r); e_{r+1}, \ldots, e_{2r}) \ldots; e_{(d-2)r/2+1}, \ldots, e_{dr/2})$. According to Lemmas 2 and 3 both G_0 and G_e are $(r-1)$-line-connected regular multigraphs of degree r, so that by Theorem A they have 1-factors containing the prescribed line e_0. From the definition of S^{++} we see that each of points $y_1^{(i)}, \ldots, y_{r-2}^{(i)}$ is matched with a point of the set $X^{(i)} = \{x_1^{(i)}, \ldots, x_r^{(i)}\}$ (i corresponds to the i-th r-tuple of prescribed lines $(e_{(i-1)r+1}, \ldots, e_{ir})$). Therefore, there are exactly two points of $X^{(i)}$ which are matched with points of G. From the construction of S^+ we see that the point w is matched with a point of G. In any case the points of G matched with points not in G are incident with some of the lines $e_1, \ldots, e_{rd/2}$. Thus, being restricted to G, every 1-factor of G_0 (G_e respectively) gives a desirable matching of G with deficiency d containing e_0 and not containing $e_1, \ldots, e_{rd/2}$. Q. E. D.

Note that Theorem 2 involves Theorems A and C as special cases.

Remark. The preceding proof can be easily modified to deduce Theorem 1 from Theorem B.

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SÚHRN

POZNÁMKA O PÁRENIACH V PRAVIDELNÝCH GRAFOCH

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Dokazuje sa nasledovná veta: Nech $G=(V, E)$ je pravidelný graf stupňa $r>0$ s p vrcholmi a hranovou súvislosťou aspoň $r-1$. Nech d a x sú celé čísla, kde $0 \leqq d \leqq p$, $d \equiv p \pmod 2$ a $x \leqq r + rd/2 - 1$. Ak $E' \subseteq E$ a $|E'| = x$, potom graf $G' = (V, E - E')$ má párenie neobsahujúce práve d vrcholov. Tiež sa dokazujú niektoré ďalšie tvrdenia tohto typu. Tieto výsledky sú zovšeobecnením niektorých doterajších výsledkov.

РЕЗЮМЕ

ЗАМЕЧАНИЕ О ПАРОСОЧЕТАНИЯХ В ОДНОРОДНЫХ ГРАФАХ

Я. Плесник, Братислава

Доказывается следующая тоерема: Пусть дан p-вершинный однородный $(r-1)$-реберно-связный граф $G=(V, E)$ степени $r>0$. Пусть d и x такие целые числа что $0 \leqq d \leqq p$, $d \equiv p$ (mod 2) и $x \leqq r + rd/2 - 1$. Если $E' \subseteq E$ и $|E'| = x$, тогда граф $G' = (V, E - E')$ обладает паросочетанием которое не содержит точно d вершин. Тоже доказываются некоторые дальнейшие утверждения этого типа. Эти результаты являются обобщением некоторых прежних результатов.

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BEMERKUNGEN ZU EINER RANDWERTAUFGABE IN DREI PUNKTEN FÜR DIE DIFFERENTIALGLEICHUNG DRITTER ORDNUNG

JOZEF VENCKO, Bratislava

1. Erwägen wir die Differentialgleichung dritter Ordnung

$$[r(x)y']'' + [\lambda q(x) + \mu p(x)]y = 0 \tag{1}$$

wo $r(x)>0$, $q(x)>0$ stetige Funktionen von $x \in \langle a, c\rangle$ sind und $p(x)$ ist eine definierte und stetige Funktion von $x \in \langle b, c\rangle$, $a<b<c$, wobei $p(x)=0$ für $x \in \langle a, b\rangle$ und $p(x)>0$ für $x \in (b, c)$ ist $\lambda\mu \in (0, \infty)$ sind Parametern.

In dieser Bemerkung zeigen wir, dass es möglich ist die Parametern λ, μ so zu wählen, daß die entsprechende Lösung $y(x)$ der Gleichung (1) die Randbedingungen

$$y(a) = y'(a) = y(b) = y(c) = 0 \tag{2}$$

erfüllt. Ein ähnliches Problem wurde in der Arbeit [1] für die Differentialgleichung

$$y''' + q(x, \lambda, \mu)y = 0$$

gelöst.

2. Es gilt daher den Satz:

Hilfsatz 1. Es seien I, J Intervalle der realen Veränderlichen, $r(x)>0$ eine stetige Funktion auf I, $q(x, \lambda) \geqq 0$ eine stetige Funktion für $(x, \lambda) \in I \times J$. Weiter sei $y(x,\lambda)$ eine nichttriviale Lösung der Differentialgleichung

$$[r(x)y']'' + q(x, \lambda)y = 0 \tag{3}$$

mit der Eigenschaft $y(a, \lambda)=0$, $a \in I$. Wenn wir mit $\varphi(\lambda) \in I$ die Nullstelle der Funktion $y(x, \lambda)$ bezeichnen, welche rechts von a liegt, dann ist $\varphi(\lambda)$ stetig für einen solchen Parameter λ für welchen $\varphi(\lambda) \in I$ gilt.

Beweis. Wir bemerken, dass die Lösung $y(x, \lambda)$ keine doppelte Nullstelle in einem Punkte $x_1 \in I$ haben kann, welcher rechts von Punkte a liegt. Tatsächlich, wenn $y(x_1, \lambda) = y'(x_1, \lambda) = 0$, dann erhalten wir in der Integralidentität

$$ry'[ry']' - \int_{x_0}^{x} \{[ry']'^2 - rqyy'\}\, dt = \{ry'[ry']'\}_{x=x_0}$$

welche für die Lösung der Differentialgleichung (3) gilt (wir erhalten diese durch multiplizieren der Gleichung (3) mit dem Produkt ry' und durch Integration von x_0 bis x, x_0, $x \in I$) wenn wir $x = x_1$ wählen und x_0 ist die erste links von x_1 liegende Nullstelle von y', erhalten wir die Gleichheit $\int_{x_0}^{x_1} \{[ry']'^2 - rqyy']\} dt = 0$ was im Hinblick auf das konstante Zeichen der Funktion unter dem Integral zu einem Widerspruch führt. Es gilt also $y(\varphi(\lambda), \lambda) = 0$, $y'(\varphi(\lambda),(\lambda) \equiv \frac{\partial}{\partial x} y(x, \lambda)\Big|_{x=\varphi(\lambda)} \neq 0$. Da $y(x, \lambda)$ eine stetige Function der Veränderlichen x, λ in jedem geschlossenen zweidimensionalen Rechteck ist, welches ein Teil von $I \times J$ ist folgt aus dem Satz über implizite Funktionen die Stetigkeit $\varphi(\lambda)$.

Hilfssatz 1 ist in den Arbeiten [1], [2] für die Fälle $r \equiv 1$, eventuell $q(x, \lambda) = \lambda q(x)$ angeführt.

In der Arbeit [2] finden wir den Beweis für den folgenden Oszillationssatz.

Oszillationssatz. $y(x, \lambda)$ sei die Lösung der Differentialgleichung

$$[r(x)y']'' + \lambda q(x)y = 0 \tag{4}$$

mit der Eigenschaft $y(a, \lambda) = 0$. Es seien $-\infty < a < b < \infty$, $r(x) > 0$, $q(x) \geqq 0$ stetige Funktionen von $x \in \langle a, \infty)$, $\lambda > 0$. Dann existieren zu der natürlichen Zahl γ solche $\lambda_\gamma > 0$, dass für $\lambda > \lambda_\gamma$ die Funktion $y(x, \lambda)$ im Intervall (a, b) wenigstens γ Nullstellen hat.

Bemerkung. Ganz ähnlich wie der Beweis des Oszillationssatzes in der Arbeit [2] kann auch der folgende Satz beweisen werden.

Hilfssatz 2. Die Koeffizienten der Differentialgleichung (3) mögen folgende Bedingungen erfühlen: $r(x) > 0$, $q(x, \lambda) \geqq 0$ sind stetige Funktionen von $x \in \langle a, \infty)$ und $\lambda \in (\Lambda_1, \Lambda_2)$, $\lim_{\lambda \to \Lambda_2} q(x, \lambda) = 0$ sei gleichmässig für alle $x \in \langle a, \infty)$ und es sei $a < b < \infty$. Wenn $y(x, \lambda)$ die Lösung der Differentialgleichung (3) mit der Eigenschaft $y(a, \lambda) = 0$ ist, dann existiert zu einer beliebigen natürlichen Zahl γ ein solches $\lambda_\gamma \in (\Lambda_1, \Lambda_2)$ dass $y(x, \lambda)$ für $\lambda > \lambda_\gamma$ in (a, b) wenigstens γ Nullstellen hat.

3. Die Koeffizienten der Differentialgleichung (1) sollen die in Abschnitt 1 angeführten Bedingungen erfüllen. Erweitern wir ihre Definition auf den Intervall $\langle a, \infty)$ auf folgende Art:

$$r(x) = r(c) \quad \text{für} \quad x \geqq c$$
$$q(x) = q(c) \quad \text{für} \quad x \geqq c$$

$$p(x)=\begin{cases}0 & \text{für } x\in\langle a,b\rangle\\ p(c) & \text{für } x\geqq c\end{cases}$$

Beweisen wir folgenden Satz.

Satz 1. Es seien alle oben angeführten Voraussetzungen erfüllt. Dann existiert eine solche natürliche Zahl N und solche Folgen von Parametern $\{\lambda_{N+p}\}_{p=0}^{\infty}$, $\{\mu_{N+p}\}_{p=0}^{\infty}$ für welche die Folge der Lösungen der Differentialgleichung (1) $\{y_{N+p}\}_{p=0}^{\infty}$ derart existiert, dass $y_{N+y} = y(x,\lambda_{N+p},\mu_{N+p})$ für $p=0, 1, \dots$ die Randbedingungen (2) erfüllen und im Intervall (a,b) gerade $N+p$ Nullstellen haben.

Beweis. Es sei $y(x,\lambda,\mu)$ die Lösung der Gleichung (1) mit der Eigenschaft $y(a,\lambda,\mu) = y_x'(a,\lambda,\mu) = 0$, $[ry_x']_x'(a,\lambda,\mu)\neq 0$. Aus dem Oszillationssatz folgt, dass bei genügend grossem λ im Intervall (a,b) weinigstens eine Nullstelle der Funktion $y(x,\lambda,\mu)$ liegt. N bezeichne die Anzahl der Nullstellen in diesem Intervall. Es existiert also ein solches $\bar{\lambda}$, μ, N dass

$$x_N(\bar{\lambda},\mu)<b\leqq x_{N+1}(\bar{\lambda},\mu)$$

gilt, wo x_N die N-te Nullstelle der Lösung $y(x,\lambda,\mu)$ in (a,b) ist. Weiter impliziert der Oszillationssatz die Existenz eines solchen $\lambda^*>\bar{\lambda}$ dass $x_{N+1}(\lambda^*,\mu)<b$. Aus dem Hilfssatz 1 folgt dann die Existenz eines solchen $\lambda_N\in(\bar{\lambda},\lambda^*)$ dass $y(b,\lambda_N,\mu)=0$ und $y(x,\lambda_N,\mu)$ hat im Intervall (a,b) gerade N Nullstellen. Fixieren wir jetzt λ_N und bezeichnen wir mit v die Anzahl der Nullstellen der Lösung $y(x,\lambda_N,\mu)$ im Intervall (b,c) für irgendeinen Parameter $\bar{\mu}$. Also wenn $\xi_v(\lambda_N,\bar{\mu})$ die v-te Nullstelle der Lösung $y(x,\lambda_N,\bar{\mu})$ in (b,c) ist, gilt

$$\xi_v(\lambda_N,\bar{\mu})<c\leqq\xi_{v+1}(\lambda_N,\bar{\mu})$$

Der Oszillationssatz verbürgt uns die Existenz eines solchen $\mu^*>\bar{\mu}$, dass $\xi_{v+1}(\lambda_N,\mu^*)<c$ ist. Aus der stetigen Abhängigkeit der Nullstellen folgt die Existenz eines solchen $\mu_N\in(\bar{\mu},\mu^*)$ dass $y(c,\lambda_N,\mu_n)=0$ ist. Damit haben wir bewiesen, dass $y_N=y(x,\lambda_N,\mu_n)$ die Lösung der Randwertaufgabe (1), (2) ist und im Intervall (a,b) gerade N Nullstellen und im Intervall (a,b) γ Nullstellen hat.

Wenn wir auf diese Art fortfahren, beweisen wir vollständig die Behauptung dess Satzes 1.

Erwägen wir jetzt die Differentialgleichung

$$[r(x)y']''+q(x,\lambda\mu)y=0 \tag{5}$$

Dan gilt folgender Satz:

Satz 2. Es seien $a<b<c$ reale Zahlen. Es sei $r(x)>0$ eine stetige Funktion von $x\in\langle a,c\rangle$ und es sei $q(x,\lambda,\mu) = q_\lambda(x,\lambda) + q_\mu(x,\mu)$ wo $q_\lambda(x,\lambda)$ eine stetige Funktion von $x\in\langle a,c\rangle$ und $\lambda\in(\Lambda_1,\Lambda_2)$ ist, weiter

$$q_\mu(x,\mu)=\begin{cases} t(x) & \text{für } x\in\langle a,b\rangle \\ s(x,\mu) & \text{für } x\in\langle b,c\rangle,\ \mu\in\langle M_1,M_2\rangle \end{cases}$$

$s(b,\mu)=t(b)$. Es sei $q_\mu(x,\mu)$ eine stetige Funktion von $x\in\langle a,c\rangle$, und $\mu\in(M_1,M_2)$. Weiter sei $q(x,\lambda,\mu)\geqq 0$ für alle $x\in\langle a,c\rangle$, $\lambda\in(\Lambda_1,\Lambda_2)$, $\mu\in(M_1,M_2)$ und $\lim\limits_{\lambda\to\Lambda_2} q_\lambda(x,\lambda)=\infty$ gelte gleichmässig für alle $x\in\langle a,c\rangle$ und endlich gelte $\lim\limits_{\mu\to M_2} q_\mu(x,\mu)=\infty$ gleichmässig für alle $x\in\langle\beta,c\rangle$ wo $b<\beta<c$ ist. Dann existiert eine solche Zahl N und Folgen der Werte von Parametern $\{\lambda_{N+p}\}_{p=0}^{\infty}$, $\{\mu_{N+p}\}_{p=0}^{\infty}$ zu velchen eine Folge von Funktionen $\{y_{N+p}\}_{p=0}^{\infty}$ existiert wo $y_{N+p}\equiv y(x,\lambda_{N+p},\mu_{N+p})$ die Lösung der Differentialgleichung (5) ist, welche die Bedingungen (2) erfüllen und im Intervall (a,b) gerade $N+p$ Nullstellen haben.

Der Beweis des Satzes 2 wird analogisch genau wie der Beweis des Satzes 1 durchgeführt nur wird anstatt des Oszillationssatzes Der Hilfssatz 2 verwendet.

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SÚHRN

POZNÁMKA K JEDNÉMU OKRAJOVÉMU PROBLÉMU V TROCH BODOCH PRE DIFERENCIÁLNU ROVNICU TRETIEHO RÁDU

J. Vencko, Bratislava

V poznámke sa hovorí o diferenciálnej rovnici $[r(x)y']'' + [\lambda q(x) + \mu p(x)]y=0$ (1). Ukážeme, že parametre λ, μ možno vybrať tak, aby riešenie $y(x)$ rovnice (1) spĺňalo okrajové podmienky $y(a) = y'(a) = y(b) = y(c) = 0$ pre $a<b<c$.

РЕЗЮМЕ

ПРИМЕЧАНИЕ К ОДНОЙ КРАЕВОЙ ЗАДАЧЕ В ТРЕХ ТОЧКАХ ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ТРЕТЬЕГО ПОРЯДКА

И. Венцко, Братислава

В этой задаче рассматривается дифференциальное уравнение $[r(x)y']'' + [\lambda q(x) + \mu p(x)]y = 0$ (1). Показано, что параметры λ, μ можно подобрать таким способом, чтобы решение $y(x)$ уравнения (1) удовлетворяло краевым условиям $y(a) = y'(a) = y(b) = y(c) = 0$ для $a < b < c$.

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REARRANGEMENTS OF SERIES AND A TOPOLOGICAL CHARACTERIZATION OF THE ABSOLUTE CONVERGENCE OF SERIES

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Let E denote the set of all permutations of the set of all positive integers. Define the function ϱ on $E \times E$ as follows. For $x = \{x_n\}_{n=1}^{\infty}$, $y = \{y_n\}_{n=1}^{\infty} \in E$, put

$$\varrho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

It has been proved [1] that ϱ is a metric and that the metric space (E, ϱ) is of the second category at each of its points. In [1], [4], [5] and [6], some properties of the rearrangements of series are studied from the point of view of the topology in E.

Let

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \tag{1}$$

be a series with complex (real) terms. To every permutation $x = \{x_n\}_{n=1}^{\infty} \in E$ there corresponds a rearrangement

$$\sum_{n=1}^{\infty} a(x_n), \qquad (a(x_n) = a_{x_n},\ n = 1, 2, \ldots)$$

of the series (1). The main result of [1] and [5] is the following Theorem:

Theorem. Let $\sum_{n=1}^{\infty} a_n$ be a non-absolutely convergent series with real terms. Then for all $x = \{x_n\}_{n=1}^{\infty} \in E$ with the exception of the points of a set of the first category in E, we have

$$\liminf_{n\to\infty} \sum_{k=1}^{n} a(x_k) = -\infty, \quad \limsup_{n\to\infty} \sum_{k=1}^{n} a(x_k) = +\infty$$

In the present paper we give a simple generalization of the said result (see

Theorem 1.1). Further we shall prove a "topological" criterion for absolute convergence of series with complex terms (Theorem 1.3) and also study some properties of certain real-valued functions defined on E.

Definitions and notations

1. Let $\sum_{n=1}^{\infty} a_n$ be a series with real terms. Put

$$N_+ = \{n, a_n \geqq 0\}, \quad N_- = \{n, a_n < 0\}$$

A series $\sum_{n=1}^{\infty} a_n$ with real terms is said to be of the type (α) or (β), if $\sum_{n \in N_+} a_n = +\infty$ and $\sum_{n \in N_-} |a_n| < +\infty$, or $\sum_{n \in N_+} a_n < +\infty$ and $\sum_{n \in N_-} a_n = -\infty$, respectively.

A series $\sum_{n=1}^{\infty} a_n$ with real terms is said to be of the type (γ), if $\sum_{n \in N_+} a_n = \sum_{n \in N_-} |a_n| = +\infty$ $\left(\text{we put } \sum_{n \in 0} a_n = 0\right)$.

2. a) Suppose that $\sum_{n=1}^{\infty} a_n$ is a series with real terms. Define a function $f = f\left(\sum_{n=1}^{\infty} a_n\right)$ on E as follows.

Let $x = \{x_n\}_{n=1}^{\infty} \in E$. If the series $\sum_{n=1}^{\infty} a(x_n)$ converges, then we put

$$f(x) = \frac{\sum_{n=1}^{\infty} a(x_n)}{1 + \left|\sum_{n=1}^{\infty} a(x_n)\right|}$$

If $\sum_{n=1}^{\infty} a(x_n) = +\infty$, or $\sum_{n=1}^{\infty} a(x_n) = -\infty$, then we put $f(x) = 1$, or $f(x) = -1$, respectively. For an oscillating series $\sum_{n=1}^{\infty} a(x_n)$ we put $f(x) = 0$.

b) Suppose that $\sum_{n=1}^{\infty} z_n$ is a series with complex terms and $x = \{x_n\}_{n=1}^{\infty} \in E$. Define the function $\varphi = \varphi\left(\sum_{n=1}^{\infty} z_n\right)$ as follows. If $\sum_{n=1}^{\infty} z(x_n)$ converges, put

$$\varphi(x) = \frac{\sum_{n=1}^{\infty} z(x_n)}{1 + \left|\sum_{n=1}^{\infty} z(x_n)\right|}$$

If $\sum_{n=1}^{\infty} z(x_n)$ does not converge, then $\varphi(x)=0$.

3. a) Let $\sum_{n=1}^{\infty} a_n$ be a series with reals terms. Denote by $K\left(\sum_{n=1}^{\infty} a_n\right)$ the set of all those $x=\{x_n\}_{n=1}^{\infty} \in E$ for which the series $\sum_{n=1}^{\infty} a(x_n)$ is convergent. Let $G_1\left(\sum_{n=1}^{\infty} a_n\right)$ and $G_2\left(\sum_{n=1}^{\infty} a_n\right)$ denote the set of all $x=\{x_n\}_{n=1}^{\infty} \in E$ with $\sum_{n=1}^{\infty} a(x_n)=+\infty$ and with $\sum_{n=1}^{\infty} a(x_n)=-\infty$, respectively. Define then

$$K^*\left(\sum_{n=1}^{\infty} a_n\right)=K\left(\sum_{n=1}^{\infty} a_n\right)\cup G_1\left(\sum_{n=1}^{\infty} a_n\right)\cup G_2\left(\sum_{n=1}^{\infty} a_n\right)$$

$$D\left(\sum_{n=1}^{\infty} a_n\right)=E-K\left(\sum_{n=1}^{\infty} a_n\right); \qquad D^*\left(\sum_{n=1}^{\infty} a_n\right)=E-K^*\left(\sum_{n=1}^{\infty} a_n\right)$$

b) Let $\sum_{n=1}^{\infty} z_n$ be a series with complex terms. Denote by $\tilde{K}\left(\sum_{n=1}^{\infty} z_n\right)$ the set of all $x=\{x_n\}_{n=1}^{\infty} \in E$ such that the series $\sum_{n=1}^{\infty} z(x_n)$ converges. Set $\tilde{D}\left(\sum_{n=1}^{\infty} z_n\right)=$ $=E-\tilde{K}\left(\sum_{n=1}^{\infty} z_n\right)$.

4. A point b is called an accumulation point of a sequence $\{x_n\}_{n=1}^{\infty}$ iff for each neighbourhood $0(b)$ of b there are infinitely many n with $x_n \in 0(b)$. The set of all accumulation points of the sequence $\{b_n\}_{n=1}^{\infty}$ will be denoted by $\{b_n\}_n'$.

5. If $x \in E$, $\delta>0$, then we put $S(x, \delta) = \{y \in E, \varrho(x, y)<\delta\}$.

6. The symbol $\langle -\infty, +\infty\rangle$ stands for the set $(-\infty, +\infty) \cup \{-\infty\} \cup \{+\infty\}$, and $\langle 0, +\infty\rangle$ denotes the set $\langle 0, +\infty) \cup \{+\infty\}$.

7. Let X be a metric space, let g be a real — valued function defined on X. We say that g has the strong Daboux property if for each $\varepsilon>0$, $c \in (-\infty, +\infty)$, and every two points $u, z \in \overline{S(x, \varepsilon)}$ ($\overline{M}$ denotes the closure of M) with $g(u)<c<g(z)$ there exists $y \in S(x, \varepsilon)$ with $g(y)=c$ (see [3]).

Suppose now that g is an arbitrary function defined on X. The function g is called strongly locally recurrent on X if for each $x \in X$ and $\varepsilon>0$ there exists an uncountable set (having the power of the continuum) of points $y \in S(x, \varepsilon)$ such that $g(y)=g(x)$ (see [8]).

1

The following theorem is essentially due to R. P. Agnew and H. M. Sengupta (cf. [1], [5]).

Theorem 1.1. Let

$$\sum_{n=1}^{\infty} a_n = a_1, + a_2 + \dots + a_n + \dots \tag{2}$$

be a series with real terms.

a) If the series (2) is of the type (α) ((β)) then for each $x = \{x_n\}_{n=1}^{\infty} \in E$ we have $\sum_{n=1}^{\infty} a(x_n) = +\infty \left(\sum_{n=1}^{\infty} a(x_n) = -\infty\right)$.

b) If (2) is of the type (γ), then the set of all those $x = \{x_n\}_{n=1}^{\infty} \in E$ for which

$$\liminf_{n\to\infty} \sum_{k=1}^{n} a(x_k) = -\infty, \qquad \limsup_{n\to\infty} \sum_{k=1}^{n} a(x_k) = +\infty \tag{A}$$

is a residual G_δ set.

c) If (2) is a series of the type (γ) and $a_n \to 0$ then for all $x = \{x_n\}_{n=1}^{\infty} \in E$, with the exception of points of a set of the first category in E, we have

$$\left\{\sum_{k=1}^{n} a(x_k)\right\}_n' = \left\langle -\infty, +\infty \right\rangle \tag{B}$$

Remark. The hypothesis in b) is evidently weaker than assuming non—absolute convergence of series (2). Consequently the assertion b) is a little stronger than the said result by R. P. Agnew and H. M. Sengupta. The assumption in c) is clearly fulfilled if (2) is a non-absolutely convergent series, and so c) refines the theorem by R. P. Agnew and H. M. Sengupta.

Proof of the Theorem. The assertion a) is evident. We are going to prove b). Let (2) be a series of the type (γ). Denote by $A(k)$ the set of all those $x = \{x_n\}_{n=1}^{\infty} \in E$ for which there can be chosen an $l = l(x)$ with $\sum_{j=1}^{l} a(x_j) > k$. We show at first that $A(k)$ is an open set in E. In fact, assuming that $x = \{x_n\}_{n=1}^{\infty} \in A(k)$, we show $S(x, 2^{-l-1}) \subset A(k)$, $l = l(x)$. Let $y = \{y_n\}_{n=1}^{\infty} \in S(x, 2^{-l-1})$. Then clearly $y_i = x_i$ $(i = 1, 2, \dots, l)$, hence $\sum_{i=1}^{l} a(y_i) > k$ and therefore $S(x, 2^{-l-1}) \subset A(k)$.

Put $A = \bigcap_{k=1}^{\infty} A(k)$. Then A is a G_δ set in E. Evidently A is the set of all such $x = \{x_n\}_{n=1}^{\infty} \in E$ that $\limsup_{n\to\infty} \sum_{i=1}^{n} a(x_i) = +\infty$. Now we show that A is dense in E. Let $z = \{z_n\}_{n=1}^{\infty} \in E$, $\delta > 0$. It is sufficient to prove that $A \cap S(z, \delta) \neq \emptyset$. Let m be a positive integer with $2^{-m}<\delta$. Define a sequence $x=\{x_n\}_{n=1}^{\infty}$ as follows

$$x_i = z_i \qquad (i=1, 2, \ldots, m) \tag{3}$$

Put $N_- - \{z_1, z_2, \ldots, z_m\} = \{l_1 < l_2 < \ldots\}$.

Choose a positive integer k_1 and pairwise distinct positive integers $x_{m+1}, \ldots, x_{m+k_1}$ which are all different from the numbers $x_1, x_2, \ldots, x_m$ in such a way that the following conditions are satisfied

1_1. there is none of the numbers l_i $(i=1, 2, \ldots)$ among the numbers $x_{m+1}, x_{m+2}, \ldots, x_{m+k_1}$;

2_1. the numbers $x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+k_1}$ include all positive integers less than l_1.

3_1. $\sum_{i=1}^{m+k_1} a(x_i) > \sum_{i=1}^{m} a(x_i) + 1$

In view of the hypothesis of the theorem, such numbers can be chosen. Put $x_{m+k_1+1} = l_1$. Choose a positive integer $k_2 > k_1 + 1$ and pairwise distinct positive integers

$$x_{m+k_1+2}, x_{m+k_1+3}, \ldots, x_{m+k_2}$$

that are all different from numbers $x_1, \ldots, x_{m+k_1+1}$, in such a manner that the following conditions are fulfilled:

1_2. none of the numbers l_i $(i=1, 2, \ldots)$ can be found among the numbers $x_{m+k_1+2}, x_{m+k_1+3}, \ldots, x_{m+k_2}$.

2_2. All the positive integers less than l_2 are included among the numbers $x_1, x_2, \ldots, x_{m+k_2}$.

3_2. $\sum_{i=m+k_1+1}^{m+k_2} a(x_i) > \sum_{i=1}^{m+k_1} a(x_i) + 1$

Put $x_{m+k_2+1} = l_2$. Choose a positive integer $k_3 > k_2 + 1$ and pairwise distinct numbers $x_{m+k_2+2}, \ldots, x_{m+k_3}$ that are different from numbers $x_1, \ldots, x_{m+k_2+1}$ and such that conditions analogous to 1_2, 2_2, 3_2 are fulfilled.

Thus a sequence $x=\{x_n\}_{n=1}^{\infty}$ can be constructed by induction. Clearly $x \in E$ and by 3_s $(s=1, 2, \ldots)$ we have also $x \in A$. Besides, in view of (3) we have $x \in S(z, \delta)$ and hence $x \in A \cap S(z, \delta)$. The last statement proves that A is a dense G_δ set in E. Therefore A is a residual set in E (see [2]).

Analogously it can be proved that the set B of all those $x=\{x_n\}_{n=1}^{\infty} \in E$ for which $\liminf_{n\to\infty} \sum_{i=1}^{\infty} a(x_i) = -\infty$ is a residual G_δ set in E. On the other hand, $A \cap B$ is the set of all $x=\{x_n\}_{n=1}^{\infty} \in E$ for which (A) holds.

Assertion c) is a consequence of part b) and the following theorem that can be found e.g. in [9]:

"Let X be a compact metric space, $\xi = \{x_n\}$ a sequence of ist points and $C(\xi)$ the set of all accumulation points of the sequence ξ. Let $\lim_{n\to\infty} \varrho(x_n, x_{n+1}) = 0$. Then $C(\xi)$ is a connected set."

In fact, suppose $\sum_{n=1}^{\infty} a_n$ be a series of the type (γ) and $a_n \to 0$. Then by part b) of our theorem there exists a residual G_δ set M in E such that for $x = \{x_n\}_{n=1}^{\infty} \in M$ we have

$$\liminf_{n\to\infty} s_n(x) = -\infty, \qquad \limsup_{n\to\infty} s_n(x) = +\infty$$

where $s_n(x) = \sum_{i=1}^{n} a(x_i)$ $(n = 1, 2, \dots)$.

We shall now verify the assumptions of the said result from [9]. Consider the sequence $\{s_n(x)\}_{n=1}^{\infty}$, $x \in M$. It is a sequence of real numbers and in view of the assumption $a_n \to 0$ it can be shown that $\lim_{n\to\infty} |s_n(x) - s_{n+1}(x)| = 0$.

It is still necessary to verify that $\langle -\infty, +\infty \rangle$ is a compact. This is done e. g. in [12].

So all hypotheses of Theorem 1 of [9] are satisfied, that means that $\{s_n(x)\}_n'$ is a connected set. By what has already been said we obtain

$$-\infty \in \{s_n(x)\}_n'; \quad \text{and also} \quad +\infty \in \{s_n(s)\}_n'$$

hence $\{s_n(x)\}_n' = \langle -\infty, +\infty \rangle$ which means that $\left\{\sum_{k=1}^{\infty} a(x_k)\right\}_n' = \langle -\infty, +\infty \rangle$ for all $x = \{x_n\}_{n=1}^{\infty} \in M$ and completes the proof of the theorem.

Theorem 1.2. Let $\sum_{n=1}^{\infty} z_n$ be a series with complex terms. Let $\sum_{n=1}^{\infty} |z_n| = +\infty$. Then $\bar{K}\left(\sum_{n=1}^{\infty} z_n\right)$ is a set of the first category in E.

Proof. Denote $z_n = a_n + ib_n$ $(n = 1, 2, \dots)$ where a_n, b_n are real numbers. The assertion of the theorem follows from Theorem 1.1 and the evident inclusion

$$\bar{K}\left(\sum_{n=1}^{\infty} z_n\right) \subset K\left(\sum_{n=1}^{\infty} a_n\right) \cap K\left(\sum_{n=1}^{\infty} b_n\right).$$

Theorem 1.3. A series $\sum_{n=1}^{\infty} z_n$ with complex terms converges absolutely if and only if there exists a set $H \subset E$ of the second category in E and such that for every $x = \{x_n\}_{n=1}^{\infty} \in H$ the series $\sum_{n=1}^{\infty} z(x_n)$ converges.

Proof. If $\sum_{n=1}^{\infty} |z_n| < +\infty$, it is sufficient to put $H = E$ and observe that E is of the second category in itself (see [1]). Conversely, let there exist such a set $H \subset E$ of the second category in E that for each $x = \{x_n\}_{n=1}^{\infty} \in H$ the series $\sum_{n=1}^{\infty} z(x_n)$ cenverges. Suppose that $\sum_{n=1}^{\infty} |z_n| = +\infty$. By Theorem 1.2 the set $\bar{K}\left(\sum_{n=1}^{\infty} z_n\right)$ is of the first category in E. Then it follows from the obvious inclusion $H \subset \bar{K}\left(\sum_{n=1}^{\infty} z_n\right)$ that H is also a set of the first category in E. This contradicts the assumptions of the theorem. Therefore $\sum_{n=1}^{\infty} |z_n|$ necessarily converges. The theorem is proved.

In what follows we shall investigate the topological structure of the sets $K\left(\sum_{n=1}^{\infty} a_n\right)$, $K^*\left(\sum_{n=1}^{\infty} a_n\right)$, $\left(\bar{K}\left(\sum_{n=1}^{\infty} z_n\right)\right)$. We shall show that for any series $\sum_{n=1}^{\infty} a_n$ $\left(\sum_{n=1}^{\infty} z_n\right)$ these sets belong to the second Borel class.

Theorem 1.4. Let $\sum_{n=1}^{\infty} a_n$ be a series with reals terms. Then each of the sets $K\left(\sum_{n=1}^{\infty} a_n\right)$, $K^*\left(\sum_{n=1}^{\infty} a_n\right)$ is an $F_{\sigma\delta}$ set.

Corollary. Each of the set $D\left(\sum_{n=1}^{\infty} a_n\right)$, $D^*\left(\sum_{n=1}^{\infty} a_n\right)$ is a $G_{\delta\sigma}$ set.

Proof of the theorem. By the Cauchy—Bolzano test for convergence of series we have

$$K\left(\sum_{n=1}^{\infty} a_n\right) = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} C(k, m, n) \tag{4}$$

where

$$C(k, m, n) = \left\{x = \{x_i\}_{i=1}^{\infty} \in E, \left|\sum_{i=m+1}^{m+n} a(x_i)\right| \leq \frac{1}{k}\right\}$$

Observe that the sequence $\{x^{(k)}\}_{k=1}^{\infty}(x^{(k)} = \{x_n^{(k)}\}_{n=1}^{\infty})$ of points of the space E converges to $x = \{x_n\}_{n=1}^{\infty}E$ if and only if for each $r = 1, 2, \dots$ we have $\lim_{k\to\infty} x_r^{(k)} = x_r$. This implies that for any positive integers k, m, n the set $C(k, m, n)$ is closed. Then by (4) we obtain that $K\left(\sum_{n=1}^{\infty} a_n\right)$ is an $E_{\sigma\delta}$ set.

Since $K^*\left(\sum_{n=1}^{\infty} a_n\right) = K\left(\sum_{n=1}^{\infty} a_n\right) \cup G_1\left(\sum_{n=1}^{\infty} a_n\right) \cup G_2\left(\sum_{n=1}^{\infty} a_n\right)$ it is sufficient to prove now that $G_1\left(\sum_{n=1}^{\infty} a_n\right)$, $G_2\left(\sum_{n=1}^{\infty} a_n\right)$ are $F_{\sigma\delta}$ sets in E. We are going to prove this for $G_1\left(\sum_{n=1}^{\infty} a_n\right)$. For $G_2\left(\sum_{n=1}^{\infty} a_n\right)$ the proof is analogous.

It follous directly from the definition of $G_1 = G_1\left(\sum_{n=1}^{\infty} a_n\right)$ that

$$G_1 = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{n=m}^{\infty} B(k, n)$$

where

$$B(k, n) = \left\{x = \{x_j\}_{j=1}^{\infty} \in E,\ \sum_{n=1}^{n} a(x_j) \geqq k\right\}$$

Clearly $B(k, n)$ is a closed set in E (for any k, n), and hence $G_1\left(\sum_{n=1}^{\infty} a_n\right)$ is an $F_{\sigma\delta}$ set in E.

Theorem 1.5. Let $\sum_{n=1}^{\infty} z_n$ be a series with complex terms. Then $\tilde{K}\left(\sum_{n=1}^{\infty} z_n\right)$ is an $F_{\sigma\delta}$ set in E.

Corollary. $\tilde{D}\left(\sum_{n=1}^{\infty} z_n\right)$ is a $G_{\delta\sigma}$ set in E.

Proof of the theorem. Consider the series $\sum_{n=1}^{\infty} z_n$ where $z_n = a_n + ib_n$ for $n = 1$, 2, ... We may write $\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} (a_n + ib_n)$. The last equality implies that

$$\tilde{K}\left(\sum_{n=1}^{\infty} z_n\right) = K\left(\sum_{n=1}^{\infty} a_n\right) \cap K\left(\sum_{n=1}^{\infty} b_n\right)$$

By Theorem 1.4 $K\left(\sum_{n=1}^{\infty} a_n\right)$ and $K\left(\sum_{n=1}^{\infty} b_n\right)$ are $F_{\sigma\delta}$ sets, therefore $\tilde{K}\left(\sum_{n=1}^{\infty} z_n\right)$ is also an $F_{\sigma\delta}$ set in E.

2.

In the sequel we shall study some properties of functions $f\left(\sum_{n=1}^{\infty} a_n\right)$ and $\varphi\left(\sum_{n=1}^{\infty} z_n\right)$.

Let $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$ be a series of the type (γ) and let $a_n \to 0$. Although the definition of $f\left(\sum_{n=1}^{\infty} a_n\right)$ depends on the series $\sum_{n=1}^{\infty} a_n$, we shall prove that all the function $f\left(\sum_{n=1}^{\infty} a_n\right)$ for all the series of the type (γ) and with $a_n \to 0$ have some interesting common properties.

Theorem 2.1. Let $\sum_{n=1}^{\infty} a_n$ be a series of the type (γ) and let $a_n \to 0$. Then the function $f = f\left(\sum_{n=1}^{\infty} a_n\right)$ has the following properties.

1. f is strongly locally recurrent at each point of E.
2. $f(E) = \langle -1, 1 \rangle$; moreover for each $x \in E$ and any $\varepsilon > 0$ we have $f(S(x, \varepsilon)) = \langle -1, 1 \rangle$.
3. f possesses the strong Darboux property.
4. f belongs to the third Borel class.

Remark. It follows from 2 that $f = f\left(\sum_{n=1}^{\infty} a_n\right)$ is discontinuous at each point $x \in E$. Thus f may serve as an example of a function which is discontinuous at every point and, at the same time, has the strong Darboux property.

Proof of the Theorem. Proof of 1. Let $x = \{x_n\}_{n=1}^{\infty} \in E$, $\delta > 0$. Put $f(x) = \frac{\eta}{1 + |\eta|}$, where $\eta = \sum_{n=1}^{\infty} a(x_n)$ if the series $\sum_{n=1}^{\infty} a_n$ converges, $\eta = 0$ if that series oscillates, and $\eta = +\infty(-\infty)$ if $\sum_{n=1}^{\infty} a_n = +\infty \left(\sum_{n=1}^{\infty} a_n = -\infty\right)$. (In the latter case we put $f(x) = 1$ and $f(x) = -1$ respectively.) Choose a positive integer N so that $2^{-N} < \delta$. Put $y_i = x_i$ $(i = 1, 2, \ldots, N)$. The series

$$a(x_{N+1}) + a(x_{N+2}) + \ldots + a(x_{N-k}) + \ldots$$

is of the type (γ) and $a(x_i) \to 0$ (with $i \to \infty$).

Therefore there exists such an uncountable set (whose power is that of the continuum) of permutations $\{k_i\}_i^{\infty} = N+1$ of the set $\{N+1, N+2, \ldots, N+k, \ldots\}$ that $\sum_{i=N+1}^{\infty} a(x_{k_i}) = \eta - \sum_{i=1}^{N} a(x_i)$ (see [7]). Putting $y_i = x_{k_i}$ $(i = N+1, N+2, \ldots)$ we obtain an uncountable set, again of the power of the continuum, of points $y = \{y_n\}_{n=1}^{\infty} \in S(x, \delta)$ wiht $f(y) = f(x)$.

Proof of part 2 is similar to that of 1.

Assertion 3 follows directly from 2.

We are going to prove assertion 4. In fact we have to show that, for every real a, each of the sets

$$M^a = \{x \in E, f(x) < a\}, \quad M_a = \{x \in E, f(x) > a\}$$

is an $F_{\sigma\delta\sigma}$ set. We carry out the proof for M^a.

Suppose $a > 1$ or $a \leqq -1$. Then the assertion for M^a is obvious. Let therefore $-1 < a \leqq 1$. Consider the following two cases.

a) $0 < a \leqq 1$

b) $-1 < a \leqq 0$

In case a) we get

$$M^a = D^*\left(\sum_{n=1}^{\infty} a_n\right) \cup R^a \tag{5}$$

where $R^a = \left\{x \in K^*\left(\sum_{n=1}^{\infty} a_n\right),\ f(x) < a\right\}$. Put

$$F(k, n) = \left\{x = \{x_i\}_{i=1}^{\infty} \in K^*\left(\sum_{n=1}^{\infty} a_n\right),\ \frac{\sum_{j=1}^{n} a(x_j)}{1 + \left|\sum_{j=1}^{n} a(x_j)\right|} < a - \frac{1}{k}\right\}$$

then

$$R^a = \bigcup_{k=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{n=p}^{\infty} F(k, n) \tag{6}$$

Since $F(k, n)$ is a closed set in $K^*\left(\sum_{n=1}^{\infty} a_n\right)$, it follows that R^a is an F_σ set in $K^*\left(\sum_{n=1}^{\infty} a_n\right)$. However $K^*\left(\sum_{n=1}^{\infty} a_n\right)$ is an $F_{\sigma\delta}$ set in E and hence R^a is an $F_{\sigma\delta}$ set in E. By Corollary of Theorem 1.4, $D^*\left(\sum_{n=1}^{\infty} a_n\right)$ is a $G_{\delta\sigma}$ set in E. Applying (5) we obtain that M^a is an $F_{\sigma\delta\sigma}$ set in E.

In case b) we have $M^a = \left\{x \in K^*\left(\sum_{n=1}^{\infty} a_n\right),\ f(x) < a\right\}$ and a similar reasoning yields that M^a is an $F_{\sigma\delta\sigma}$ set in E.

In a similar way in can be proved that M_a is an $F_{\sigma\delta\sigma}$ set in E. Thus the Theorem is proved.

Theorem 2.2. Let $\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} (a_n + ib_n)$ be a series of with complex terms, where a_n, b_n $(n = 1, 2, \ldots)$ are real numbers. Then we have:

A. If $\sum_{n=1}^{\infty} |z_n| < +\infty$, then $\varphi(E)$ contains a single point.

B. If one of the series $\sum_{n=1}^{\infty} a_n$ or $\sum_{n=1}^{\infty} b_n$ is of the type (α) or the type (β), then $\varphi(E) \equiv \{0\}$.

C. If both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are of the type (γ) and $z_n \to 0$, then

1. φ is strongly locally recurrent at each point of E.

2. If $\sum_{n=1}^{\infty} z_n$ is a nonabsolutely convergent series, then the function φ maps the set E into

2′. a part of a conic section, lying in the unit circle,
2″. the interior of the unit circle, i. e. $\varphi(E)=\{z=a+bi;\ |z|<1\}$.
3. The function φ belongs to the third Borel class in E.

Proof. A. Since $\sum_{n=1}^{\infty}|z_n|<+\infty$, all the series $\sum_{n=1}^{\infty}z(x_n)$, for all $x=\{x_n\}_{n=1}^{\infty}\in E$, converge to the same value $z=a+bi$. Hence $\varphi(x)=\frac{z}{1+|z|}$ for all $x\in E$.

B. Let the series $\sum_{n=1}^{\infty}a_n$ be of the type (α). Then by part a) of Theorem 1.1 we have $\sum_{n=1}^{\infty}a(x_n)=+\infty$ for each $x=\{x_n\}_{n=1}^{\infty}\in E$. This however happens if and only if none of the series $\sum_{n=1}^{\infty}z(x_n)$, for $x\in E$, converges, i. e. for all $x\in E$ we have $\varphi(x)=0$. The situation is quite analogous if $\sum_{n=1}^{\infty}a_n$ is a series of the type (β) and if $\sum_{n=1}^{\infty}b_n$ is of the type (α) or of the type (β).

C. 1. Let $x^0=\{x_n^0\}_{n=1}^{\infty}$ be an arbitrary point in E. Since $\sum_{n=1}^{\infty}a_n$ is a series of the type (γ) and $a_n\to 0$ with $n\to\infty$, it follows that $\sum_{n=1}^{\infty}a(x_n^0)$ is also a series of the type (γ) and $a(x_n^0)\to 0$ for $n\to\infty$. Hence there is a subsequence $\{a(x_{n_i}^0)\}_{i=1}^{\infty}$ of the sequence $\{a(x_n^0)\}_{n=1}^{\infty}$ such that for each $i=1, 2, \ldots$ we have

$$|a(x_{n_i}^0)|\leqq\frac{1}{2^i} \tag{7}$$

Since we have also assumed that $\sum_{n=1}^{\infty}b_n$ is a series of the type (γ) and $b_n\to 0$ with $n\to\infty$, we infer that $\sum_{n=1}^{\infty}b(x_n^0)$ is a series of the type (γ) and $b(x_n^0)\to 0$ for $n\to\infty$, and hende $b(x_{n_i}^0)\to 0$. Therefore there exists a subsequence $\{b(x^0\ \mathrm{ind}\ (n_{i_k}))\}_{k=1}^{\infty}$ of $\{b(x_{n_i}^0)\}_{i=1}^{\infty}$ such that for all $k=1, 2, \ldots$ we have

$$|b(x^0\ \mathrm{ind}\ (n_{i_k}))|\leqq\frac{1}{2^k} \tag{8}$$

The sequence $\{a(x^0\ \mathrm{ind}\ (n_{i_k}))\}_{k=1}^{\infty}$ is a subsequence of $\{(x_{n_i}^0)\}_{i=1}^{\infty}$, and hence, in view of (7), we have

$$|a(x^0\ \mathrm{ind}\ (n_{i_k}))|\leqq\frac{1}{2^k} \tag{9}$$

The inequalities (8) and (9) prove that $\sum_{k=1}^{\infty} a(x^0 \text{ ind } (n_{i_k}))$ and $\sum_{k=1}^{\infty} b(x^0 \text{ ind } (n_{i_k}))$ are absolutely convergent series.

Let $\varepsilon > 0$ and choose a positive integer N with

$$\frac{1}{2^n} > \varepsilon \tag{10}$$

Let k^0 be the least positive integer such that

$$n_{i_k 0} > N + 1 \tag{11}$$

Denote by W the set of all those $\overline{x^0} \in E$ that can be obtained from x^0 by some permutation of the coordinates x^0 ind (n_{i_k}) for $k \geqq k^0$.

Being a set of all the permutations of an infinite subset of all positive integers, W has the power of the continuum.

To complete the proof it remains to show that for every $\overline{x^0} \in W$ the following statements are true:

a) $\varrho(x^0, \overline{X^0}) < \varepsilon$

b) 1. $\sum_{n=1}^{\infty} a(\overline{x_n^0}) = \sum_{n=1}^{\infty} a(x_n^0)$ if $\sum_{n=1}^{\infty} a(x_n^0)$ converges; if $\sum_{n=1}^{\infty} a(x_n^0)$ does not converge, neither does the series $\sum_{n=1}^{\infty} a(\overline{x_n^0})$.

2. $\sum_{n=1}^{\infty} b(\overline{x_n^0}) = \sum_{n=1}^{\infty} b(x_n^0)$ if $\sum_{n=1}^{\infty} b(x_n^0)$ converges; and if $\sum_{n=1}^{\infty} b(x_n^0)$ does not converge, neither does $\sum_{n=1}^{\infty} a(\overline{x_n^0})$.

a) Consider any $\overline{x^0} \in W$. Then by the definition of the metric ϱ and by (10) (11) we have

$$\varrho(x^0, \overline{x^0}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n^0 - \overline{x_n^0}|}{1 + |x_n^0 - \overline{x_n^0}|} =$$

$$= \sum_{n=n_{i_k 0}}^{\infty} \frac{1}{2n} \frac{|x_n^0 - \overline{x_n^0}|}{1 + |x_n^0 - \overline{x_n^0}|} < \sum_{n=n_{i_k 0}}^{\infty} \frac{1}{2^n} = \frac{1}{2^{n_{i_k 0}} - 1} < \frac{1}{2^n} < \varepsilon$$

b) 1. Let $\sum_{n=1}^{\infty} a(x_n^0) = l$. Then

$$\sum_{n=1}^{\infty} a(x_n^0) = \left[\sum_{n=1}^{\infty} a(x_n^0) - \sum_{k=k^0}^{\infty} a(x^0 \text{ ind } (n_{i_k}))\right] + \sum_{k=k^0}^{\infty} a(x^0 \text{ ind } (n_{i_k}))$$

This in view of (9) implies

$$\sum_{n=1}^{\infty} a(\overline{x_n^0}) = \left[\sum_{n=1}^{\infty} a(\overline{x_n^0}) - \sum_{k=k^0}^{\infty} a(x^0 \text{ ind } (n_{i_k}))\right] + \sum_{k=k^0}^{\infty} a(x^0 \text{ ind } (n_{i_k})) =$$

$$\left[\sum_{n=1}^{\infty} a(x_n^0) - \sum_{k=k^0}^{\infty} a(x^0 \text{ ind } (n_{i_k}))\right] + \sum_{k=k^0}^{\infty} a(x^0 \text{ ind } (n_{i_k})) = \sum_{n=1}^{\infty} a(x_n^0)$$

If $\sum_{n=1}^{\infty} a(x_n^0)$ does not converge, then clearly its convergence character will not be changed by any permutation of $a(x^0 \text{ ind } (n_{i_k}))$ entries $(k \geqq k^0)$ of the sequence $\{a(x_n^0)\}_{n=1}^{\infty}$.

The assertion b) 2 can by proved analogously as b) 1. C. 2. By the convergence domain of a series we shall mean the set of all the numbers that can be obtained as the sum of a convergent rearrangements of the series. By the well-known Steinitz theorem (see. e. g. [10]), the convergence domain of a conditionally convergent series with complex terms may turn out to be one of the following sets:

2′. The straight line ζ consisting of all those points z of the complex plane for which $\mathcal{R}(z \cdot e^{i\psi}) = \mathcal{R}(s \cdot e^{i\psi})$, where s is any finite sum of some rearrangement of $\sum_{n=1}^{\infty} z_n$ and $\mathcal{R}(u)$ denotes the real part of a complex number u. This is the case if there exists exactly one angle $\psi (0 \leqq \psi < \pi)$ with the property that the real part $\sum_{n=1}^{\infty} \mathcal{R}(z_n \cdot e^{i\psi})$ of the series obtained from the given one by turning its terms by converges absolutely.

2″. The whole complex plane in the case that no angle ψ has the property described in 2′.

Consequently, in case 2′, we have $\varphi(E) = \{\mathcal{H}(z)\colon \mathcal{H}(z) = \dfrac{z}{1+|z|};$ where z runs through the convergence domain, i. e. the straight line $\xi\}$. Consider the mapping $\mathcal{H}$: $\mathcal{H}(z) = \dfrac{z}{1+|z|}$ where $z = x + iy$. It can easily be seen that for all z we have $|\mathcal{H}(z)| < 1$.

To get a better idea of the set $\varphi(E)$ we have to realise what is the image of ζ in the above mapping $\mathcal{H}$.

Let in R^3 with the coordinate system (x, y, t) be given the conical surface $x^2 + y^2 = t^2$, let Φ be the part of the conical surface given by $t \geqq 0$; let $p = (0, 0, 1)$ be a point. Identify the plane $t = 0$ with the plane C (C — the set of all complex numbers) in the usual way $R^2 \to C$; $(x, y) \mapsto x + iy = z$ (see *Fig. 1*).

The mapping $\mathcal{H}$ may be viewed upon as the superposition $\beta \circ \alpha$ where

$$\alpha\colon \quad C \to \Phi,\ z \to (wx, wy, (1-w)) \text{ where } w = \frac{2-|z|}{2(z\bar{z}-1)}$$

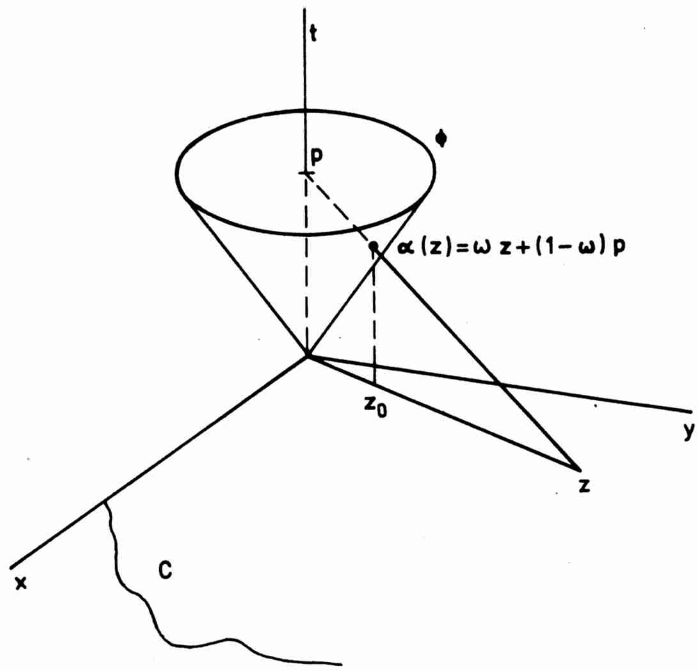

Fig.1.

and

$$\beta:\quad \Phi\rightarrow C,\ (x,y,t)\mapsto(x,y)$$

Geometrically, β is the orthogonal projection into the plane $t=0$. It is not difficult to verify that $\alpha(z)$ is the common point of the abscissa pz and the surface Φ.

It is now evident that $\mathcal{H}(\zeta)=\beta\circ\alpha(\zeta)$. Since, however, $\alpha(\zeta)$ is a conic section (the section of Φ by the plane $p\zeta$), $\beta(\alpha(\zeta))$ is again a conic section or a part of a conic section.

The following discussion can be deduced from the properties of Φ.

1. If ζ has no common point with the unit circle in the complex plane $t=0$, then $\varphi(E)$ is a part of an ellipse.
2. If ζ is a tangent of the unit circle, then $\varphi(E)$ is a part of a parabola.
3. If ζ cuts the unit circle, then $\varphi(E)$ is a part of one branch of a hyperbola.

In the case 2" it is easy to see that the mapping

$$\varphi(x)=\frac{\sum_{n=1}^{\infty} z(x_n)}{1+\left|\sum_{n=1}^{\infty} z(x_n)\right|},$$

defined under 2b) in the „Definitions and Notations" section of the present paper, maps the set of all complex numbers onto the set of all those complex numbers u for which $|u|<1$.

C. 3. It is sufficient to prove that

$$Q_1=\varphi^{-1}\{z=a+bi;\, a<A,\, b<B;\, A, B-\text{real}\}$$

and

$$Q_2=\varphi^{-1}\{z=a+bi;\, a>A,\, b>B;\, A, B-\text{real}\}$$

are $F_{\sigma\delta\sigma}$ sets. ($\varphi^{-1}(M)$ means the inverse image of M, not the inverse function).

If $A\leqq -1$ and $B\leqq -1$ or alternately $A>1$ and $B>1$, the assertion is evident. Suppose therefore that the contrary is true. We prove the assertion for Q_1. Put

$$\varphi(x)=\frac{\sum_{n=1}^{\infty} z(x_n)}{1+\left|\sum_{n=1}^{\infty} z(x_n)\right|}=\frac{\sum_{n=1}^{\infty} a(x_n)}{1+\left|\sum_{n=1}^{\infty} z(x_n)\right|}+\mathrm{i}\,\frac{\sum_{n=1}^{\infty} b(x_n)}{1+\left|\sum_{n=1}^{\infty} z(x_n)\right|}=$$

$$=\varphi_1(x)+\mathrm{i}\varphi_2(x)$$

Now we have

$$Q_1=\{\varphi_1^{-1}\{x;\, x\text{-real},\, x<A\}\}\cap\{\varphi_2^{-1}\{x;\, x\text{-real},\, x<B\}\}$$

Denote $Q_1=Q_1^1\cap Q_1^2$.

As we have already observed, it is sufficient to take the values A, B from the interval $(-1, 1\rangle$. Consider the following two cases

1. $0<A\leqq 1$
   $0<B\leqq 1$
2. $-1<A\leqq 0$
   $-1<B\leqq 0$

In the case 1, we get

$$Q_1^1=D\Big(\sum_{n=1}^{\infty} z_n\Big)\cup S_1^A \tag{12}$$

where

$$S_1^A=\Big\{x\in\tilde{K}\Big(\sum_{n=1}^{\infty} z_n\Big);\ \varphi_1(x)<A\Big\}$$

Put

$$F_1(k,n)=\left\{x=\{x_j\}_{j=1}^{\infty}\in\tilde{K}\Big(\sum_{n=1}^{\infty} z_n\Big);\ \frac{\sum_{n=1}^{\infty} a(x_n)}{1+\left|\sum_{n=1}^{\infty} z(x_n)\right|}<A-\frac{1}{k}\right\}$$

then

$$S_1^A = \bigcup_{k=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{n=p}^{\infty} F_1(k, n)$$

Since $F_1(k, n)$ is closed in $\tilde{K}\left(\sum_{n=1}^{\infty} z_n\right)$ the set S_1^A is of the type F_σ in $K\left(\sum_{n=1}^{\infty} z_n\right)$. On the other hand, $\tilde{K}\left(\sum_{n=1}^{\infty} z_n\right)$ is an $F_{\sigma\delta}$ set in E, hence S_1^A is also an $F_{\sigma\delta}$ set in E. $\tilde{D}\left(\sum_{n=1}^{\infty} z_n\right)$ is a $G_{\delta\sigma}$ set in E (see Corollary to Theorem 1.5). From what has been said we infer by (12) that Q_1^1 is an $F_{\sigma\delta\sigma}$ set in E.

In an analogous way we could prove that Q_1^2 is also an $F_{\sigma\delta\sigma}$ set in E. Then $Q_1 = Q_1^1 \cap Q_1^2$, being an intersection of two $F_{\sigma\delta\sigma}$ sets, is an $F_{\sigma\delta\sigma}$ set in E.

The proof that Q_2 is an $F_{\sigma\delta\sigma}$ set in E runs in an analogous way.

The proof of the theorem is complete.

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SÚHRN

ÚPRAVA RADOV A TOPOLOGICKÉ KRITÉRIÁ ABSOLÚTNEJ KONVERGENCIE RADOV

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V prvej časti tejto práce je jednoduché zovšeobecnenie hlavného výsledku prác [1], [5] autorov R. P. Agnewa a H. M. Sengupta. Ďalej sa tu študujú niektoré topologické kritériá absolútnej konvergencie radu s komplexnými členmi. V druhej časti sa hovorí o niektorých vlastnostiach istých reálnych funkcií, definovaných na priestore E všetkých permutácií množiny všetkých prirodzených čísiel. V závere práce sa opisujú niektoré spoločné vlastnosti týchto funkcií.

РЕЗЮМЕ

ОФОРМЛЕНИЕ РЯДА И ТОПОЛОГИЧЕСКИЕ ПРИЗНАКИ АБСОЛЮТНОЙ СХОДИМОСТИ РЯДА

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В первой части этой работы приведено простое обобщение главного результат работ [1], [5] авторов Р. П. Агнева и Г. М. Сенгупта. Дальше здесь изучаются некоторые топологические признаки абсолютной сходимости комплексного ряда. Во второй части работы изучаются свойства некоторых действительных функций, определенных на пространстве E всех перестановок множества всех целых положительных чисел. Наконец обращается внимание на некоторые общие свойства этих функций.

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ON SMALL SYSTEMS

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The problems studied in the present paper are of part of the field of measure theory in which the notion of measure is replaced by systems of sets resembling the families $\mathcal{M}_n = \left\{E \in \mathcal{S}: \mu(E) < \frac{1}{n}\right\}$ where μ is a measure on the σ-ring $\mathcal{S}$. This type of questions is discussed in [6], [7] and other papers.

In [1], J. Lloyd constructs in a natural way the system of null sets (analogous to $\{E \in \mathcal{S}: \mu(E) = 0\}$ see e. g. [4], [5]). He also gives a construction of small systems $\mathcal{N}_n^*$ similar to the Caratheodory construction of outer measure. In section 1 of the present paper a generalization of Lloyd's constructions is given. An outer measure v which is an aproximation of systems $\mathcal{N}_n^*$ is constructed in Section 3. In section 2 it is shown that one of the theorems in [1] is false and corrected theorem is proved. This correction has been obtained in a more detailed study of nonatomic measures on a ring.

In the sequel we shall use the following notation. N denotes the set of positive integers. $N^{-1} = \left\{\frac{1}{n}: n \in N\right\}$. If ε is a family of subsets of X, then $\mathcal{H}(\varepsilon)$ will stand for the hereditary σ-ring generated by ε, and $\sigma(\varepsilon)$ will stand for the σ-ring generated by ε. If $A \subset X$ then $A \mid \varepsilon$ will denote the family $\{E \in \varepsilon: E \subset A\}$.

All notations of measure theory not defined here are used in the same sense as in [2].

1.

Throughout the paper X is an abstract set and $\mathcal{S}$ is a σ-ring of subsets of X. We shall denote by T a subset of $(0; \infty)$ having 0 as its limit point.

Definition. Let X be an abstract set, $\mathcal{S}$ a σ-ring of subsets of X, and $\{\mathcal{M}_n\}_{n=1}^{\infty}$ a sequence of subfamilies of $\mathcal{S}$, such that

(A) for each $n \in N$, $\mathcal{M}_n \neq \emptyset$;

(B) for each $n \in N$, there exists a sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers such that $E_i \in \mathcal{M}_{k_i}$ $(i=1, 2, \ldots)$ implies $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}_n$;

(C) for each $n \in N$, if $E \in \mathcal{M}_n$ and $F \in \mathcal{S}$, then $E \cap F \in \mathcal{M}_n$.

A sequence $\{\mathcal{M}_n\}_{n=1}^{\infty}$ satisfying all the above properties will be called a small system on $\mathcal{S}$.

In what follows let $\{\mathcal{N}_t\}_{t \in T}$ be a family of nonempty subfamilies of a ring $\mathcal{R}$.

Definition. Given $E \in \mathcal{H}(\mathcal{R})$, we say the class of sets $\{E_i\}_{i \in I}$, $I \neq \emptyset$, $I \subset N$, is an n-cover for E, provided that $E_i \in \mathcal{N}_{t_i}$, for some $t_i \in T$ $(i \in I)$, $\bigcup_{i \in I} E_i \supset E$ and $\sum_{i \in I} t_i \leqq \frac{1}{n}$, where $n \in N$.

Definition. For each $n \in N$, we define $\mathcal{N}_n^* = \{E \in \mathcal{H}(\mathcal{R}) : E$ has an n-cover$\}$.

Remark 1. If we choose in particular $T = N^{-1}$ then the above construction of $\{\mathcal{N}_n^*\}_{n=1}^{\infty}$ coincides with the construction of $\{\mathcal{N}_n^*\}_{n=1}^{\infty}$ in [1]. Therefore our construction generalizes that of Lloyd.

To avoid confusion let us point out that the systems $\mathcal{N}_n$ in [1] are understood differently than our systems $\mathcal{N}_t$. In [1] $\mathcal{N}_n$ corresponds to the set $\left\{E \in \mathcal{R} : \mu(E) < \frac{1}{n}\right\}$ while in our paper $\mathcal{N}_t$ corresponds to $\{E \in \mathcal{R} : \mu(E) < t\}$.

Remark 2. Suppose $E_i \in \mathcal{N}_{k_i}^*$, where $i \in I \subset N$ and $\sum_{i \in I} \frac{1}{k_i} \leqq \frac{1}{n}$, then $\bigcup_{i \in I} E_i \in \mathcal{N}_n^*$.

Theorem 1. The family $\{\mathcal{N}_n^*\}_{n=1}^{\infty}$ constructed above is a small system in $\mathcal{H}(\varepsilon)$ having the following properties:

a) for each $n \in N$, $\emptyset \in \mathcal{N}_n^*$;

b) for each $n \in N$, $\mathcal{N}_{n+1}^* \subset \mathcal{N}_n^*$;

c) for each $n \in N$, if $E \in \mathcal{N}_n^*$ and $F \subset E$, then $F \in \mathcal{N}_n^*$.

Proof. Since 0 is a limit point of T, there is $t_0 \in T$ such that $t_0 \leqq \frac{1}{n}$. Since $\mathcal{N}_{t_0} \neq \emptyset$ there exists an n-cover for the empty set; a) is proved.

Since every $(n+1)$-cover for E is at the same time an n-cover for E, b) is obvious.

Property c) is also evident, since every n-cover for E is an n-cover for each of its subsets.

Axiom (A) for small systems follows from a).

To verify (B) choose for each $n \in N$ a sequence $\{k_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} \frac{1}{k_i} \leqq \frac{1}{n}$ (for example $k_i = n2^i$, $i = 1, 2, \ldots$). If $E_i \in \mathcal{N}_{k_i}^*$, by Remark 2 we have $\bigcup_{i=1}^{\infty} E_i \in \mathcal{N}_n^*$.

(C) is an obvious consequence of c).

Definition. We will call $\{\mathcal{N}_n^*\}_{n=1}^{\infty}$ the small system induced by $\{\mathcal{N}_t\}_{t\in T}$.

2.

Let μ be a measure defined on a ring $\mathcal{R}$. Throughout this section $\mathcal{N}_t$ will denote the system $\mathcal{N}_t = \{E \in \mathcal{R}: \mu(E) < t\}$. We are going to study the following question: Under what conditions on T or $(\mathcal{R}, \mu)$ is the equality $\mathcal{N}_n^* = \left\{A \in \mathcal{H}(\mathcal{R}): \mu^*(A) < \frac{1}{n}\right\}$ true, where μ^* is the outer measure induced by μ.

Definition. If μ is a measure on a ring $\mathcal{R}$, a set $E \in \mathcal{R}$ of positive measure is called an atom if, given $F \in \mathcal{R}$ such that $F \subset E$, then either $\mu(F) = 0$ or $\mu(E - F) = 0$.

A. Zaanen states the following lemma in [3].

Lemma 1. Let $\mathcal{R}$ be a σ-ring, and μ a measure on $\mathcal{R}$. If $E \in \mathcal{R}$ is of finite positive measure and E does not contain any atoms, then for real number c, such that $0 < c < \mu(E)$, there exists a subset F of E such that $F \in \mathcal{R}$ and $\mu(F) = c$.

Remark 3. J. Lloyd quotes this lemma in 1 incorrectly. Instead of the hypothesis that $\mathcal{R}$ is a σ-ring he assumes only that $\mathcal{R}$ is a ring. As shown by the following example Lemma 1 is no more valid if $\mathcal{R}$ is assumed to be a ring only.

Example 1. Let $\mathcal{R}_0$ be a ring of subsets of the real line consisting of all those A that can be written $A = \bigcup_{i=1}^{n} [a_i, b_i)$, where a_i, b_i are rational numbers and $[a_i, b_i)$ are pairwise disjoint. Define a measure μ by $\mu(A) = \sum_{i=1}^{n} (b_i - a_i)$. Evidently μ is a measure, being a restriction of the Lesbegue measure to $\mathcal{R}_0$. Since μ attains only rational values the conclusion of Lemma 1 is false.

Using the above mentioned incorrectly quoted lemma J. Lloyd "proved" the following proposition [1].

"Let μ be a measure on a ring $\mathcal{R}$ and $\{\mathcal{N}_n^*\}_{n=1}^{\infty}$ be a small system induced by $\{\mathcal{N}_t\}_{t\in T}$, where $T = N^{-1}$. If μ has no atoms, then $\mathcal{N}_n^* = \left\{E \in \mathcal{H}(\mathcal{R}): \mu^*(E) < \frac{1}{n}\right\}$, for each $n \in N$".

The following example shows that this proposition is actually false.

Example 2. Let R be the set of reals and $\mathcal{R}$ the ring of those subsets of R that can be represented in the form $A = \bigcup_{i=1}^{n} [a_i, b_i)$ where $a_i, b_i \in R$ and the intervals $[a_i, b_i)$ are pairwise disjoint. Put $\mu(A) = \sum_{i=1}^{n} [g(b_i) - g(a_i)]$ where g is the function defined by: $g(x) = 0$ for $x \leqq 0{,}35$ and $g(x) = x$ for $x \in (0{,}35; 0{,}45)$ and $g(x) = 0{,}45$ for $x \geqq 0{,}45$. Then $\mu^*(R) < \frac{1}{2}$ whenever $R \notin \mathcal{N}_2^*$.

We shall now correct Theorem 10 in [1], by assuming that "μ^* has no atoms on $\sigma(\mathscr{R})$" instead of the original asumption that "μ has no atoms on $\mathscr{R}$". To prove the corrected theorem we shall use the following lemma.

Lemma 2. Let μ be a measure on a ring $\mathscr{R}$ and μ^* the Caratheodory extension of μ to $\sigma(\mathscr{R})$. Then μ^* has no atom of finite measure if and only if the following condition holds:

(D) For each $A \in \mathscr{R}$ with $0<\mu(A)<\infty$ the set $\{\mu(E): E \in A \mid \mathscr{R}\}$ is dense in the interval $[0; \mu(A)]$

Proof. Assumed that (D) is satisfied and let $B \in \sigma(\mathscr{R})$ $0<\mu^*(B)<\infty$. Put $\varepsilon = \frac{1}{3}\mu^*(B)$. There exists a sequence $\{E_n\}_{n=1}^{\infty}$ of sets in $\mathscr{R}$ such that $B \subset \bigcup_{i=1}^{\infty} E_i$ and $\sum_{i=1}^{\infty} \mu(E_i) < \mu^*(B) + \varepsilon$. We may and do assume that E_n are pairwise disjoint since if they were not, we could consider $F_n = E_n - \bigcup_{i=1}^{n-1} E_i$ instead of E_n for $n \geqq 2$. Then there exists n_0 such that

$$\mu\left(\bigcup_{i=1}^{n_0} E_i\right) = \sum_{i=1}^{n_0} \mu(E_i) > \mu^*(B) - \varepsilon = 2\varepsilon$$

By (D) there exists $F \in \left[\left(\bigcup_{i=1}^{n_0} E_i\right) \middle| \mathscr{R}\right]$ with $\varepsilon < \mu(F) < 2\varepsilon$. Since $\mu^*\left(\bigcup_{i=1}^{\infty} E_i - B\right) < \varepsilon$ and $F \subset \bigcup_{i=1}^{\infty} E_i$ we obtain

$$\mu^*(F) = \mu^*\left[\left(\bigcup_{i=1}^{\infty} E_i - B\right) \cap F\right] + \mu^*(B \cap F) < \varepsilon + \mu^*(B \cap F)$$

and hence $\mu^*(B \cap F) > \mu^*(F) - \varepsilon > 0$. On the other hand $\mu^*(B \cap F) < 2\varepsilon < \mu^*(B)$ and thus we have proved that B is not an atom.

Conversely. Let μ^* has no atom having finite μ^*-measure. Let $A \in R$, $0<\mu(A)<\infty$ and $0<c<\mu(A)$. It is sufficient to prove that for any $\varepsilon>0$ there is $F \in A \mid \mathscr{R}$ such that $\mu(F) \in (c-\varepsilon; c+\varepsilon)$. By Lemma 1 there exists $B \in A \mid \sigma(\mathscr{R})$ with $\mu^*(B) = c$. Hence there is a sequence of sets B_n in $A \mid \mathscr{R}$ such that $B \subset \bigcup_{i=1}^{\infty} B_i$ and $\sum_{i=1}^{\infty} \mu(B_i) < c + \varepsilon$. We may and do assume that B_n are pairwise disjoint. There must exist n_0 such that $\mu\left(\bigcup_{i=1}^{n_0} B_i\right) = \sum_{i=1}^{n_0} \mu(B_i) > c - \varepsilon$. It is now sufficient to put $F = \bigcup_{i=1}^{n_0} B_i$.

Lemma 3. For each $n \in N$ we have $\mathscr{N}_n^* \subset \left\{A \in \mathscr{H}(\mathscr{R}) : \mu^*(A) < \frac{1}{n}\right\}$

Proof. Let $E \in \mathcal{N}_n^*$. Then there is $\{E_k\}_{k\in K}$ where K is countable nonempty subset of T with $E_k \in \mathcal{N}_{t_k}$ for all $t_k (k \in K)$, while $\bigcup_{k\in K} E_k \supset E$ and $\sum_{k\in K} t_k \leqq \frac{1}{n}$. Then for every $k \in K$ we have $\mu(E_k) < t_k$ hence $\mu^*\left(\bigcup_{k\in K} E_k\right) < \frac{1}{n}$. It follows $\mu^*(E) < \frac{1}{n}$ which proves the lemma.

Theorem 2. Let μ be a measure on $\mathcal{R}$ such that its extension μ^* to $\sigma(\mathcal{R})$ has no atoms. Then for every $n \in N$ we have

$$\mathcal{N}_n^* = \left\{E \in \mathcal{H}(\mathcal{R}) : \mu^*(E) < \frac{1}{n}\right\}$$

Proof. In view of Lemma 3 it is sufficient to show that $\mathcal{N}_n^* \supset \left\{E \in \mathcal{H}(\mathcal{R}) : \mu^*(E) < \frac{1}{n}\right\}$. For a given $n \in N$ assume that $\mu^*(E) < \frac{1}{n}$. Then there exists a sequence $\{E_i\}_{i=1}^{\infty}$ such that $E_i \in \mathcal{R}$ $(i = 1, 2, \ldots)$ $E \subset \bigcup_{i=1}^{\infty} E_i$ and $\sum_{i=1}^{\infty} \mu(E_i) < \frac{1}{n}$. Let ε be given by

$$\varepsilon = \frac{1}{n} - \sum_{i=1}^{\infty} \mu(E_i) \tag{1}$$

Since 0 is a limit point for T, for each $i \in N$ there is $t_i \in T$ with $0 < t_i < \frac{\varepsilon}{2^i}$. We can then choose, for each $i \in N$, a positive integer p_i such that

$$(p_i - 1) \cdot t_i \leqq \mu(E_i) < p_i \cdot t_i \leqq \mu(E_i) + \frac{\varepsilon}{2^i} \tag{2}$$

Define δ_i for each $i \in N$ by

$$\delta_i = p_i \cdot t_i - \mu(E_i) \tag{3}$$

In view of Lemma 2 for each $i \in N$ we can then construct by induction the collection $\{E_i^k\}_{k=1}^{p_i-1}$ of pairwise disjoint subsets of E_i such that

$$t_i - \frac{\delta_i}{p_i} \leqq \mu(E_i^k) < t_i \qquad (k = 1, 2, \ldots, p_i - 1) \tag{4}$$

The set $E_i^{p_i} = E_i - \bigcup_{k=1}^{p_i-1} E_i^k$ then satisfies

$$\mu(E_i^{p_i}) = \mu(E_i) - \sum_{k=1}^{p_i-1} \mu(E_i^k) \tag{5}$$

and combining (5) with (4) for $k=1, 2, \ldots, p_i-1$ we obtain

$$\mu(E_i)-(p_i-1)\cdot t_i<\mu(E_i^{p_i})\leqq\mu(E_i)-(p_i-1)\cdot\left(t_i-\frac{\delta_i}{p_i}\right)$$

Aplying (2) we get $\mu(E_i^{p_i})>0$ and by (3) it follows:

$$\mu(E_i^{p_i})\leqq\mu(E_i)-p_i\cdot t_i+\delta_i+t_i-\frac{\delta_i}{p_i}=t_i-\frac{\delta_i}{p_i}$$

and hence $E_i^{p_i}\in\mathcal{N}_{t_i}$, $i\in N$.

The countable family $\{E_i^k\}_{k=1}^{p_i}{}_{i=1}^{\infty}$ covers E and moreover by (2) and (1) we have:

$$\sum_{i=1}^{\infty}p_i\cdot t_i\leqq\sum_{i=1}^{\infty}\left[\mu(E_i)+\frac{\varepsilon}{2^i}\right]=\varepsilon+\sum_{i=1}^{\infty}\mu(E_i)=\frac{1}{n}$$

Thus we have that $\{E_i^k\}_{k=1}^{p_i}{}_{i=1}^{\infty}$ is an n-cover for E. And therefore $E\in\mathcal{N}_n^*$ for every set $E\in\mathcal{H}(\mathcal{R})$ with $\mu^*(E)<\frac{1}{n}$.

Theorem 3 is another result concerning the question: when do we have $\mathcal{N}_n^*=\left\{E\in\mathcal{H}(\mathcal{R}):\mu^*(E)<\frac{1}{n}\right\}$?

Theorem 3. If $T^-\supset[0, 1]$ then $\mathcal{N}_n^*=\left\{F\in\mathcal{H}(\mathcal{R}):\mu^*(F)<\frac{1}{n}\right\}$ for each $n\in N$.

Proof. In view of Lemma 3 it is sufficient to prove that $\mathcal{N}_n^*\supset\supset\left\{F\in\mathcal{H}(\mathcal{R}):\mu^*(F)<\frac{1}{n}\right\}$.

Let $\mu^*(E)<\frac{1}{n}$. Then there is a sequence $\{E_k\}_{k=1}^{\infty}$ of sets in the ring $\mathcal{R}$ such that $E\subset\bigcup_{k=1}^{\infty}E_k$ and $\sum_{k=1}^{\infty}\mu(E_k)<\frac{1}{n}$. Put $\delta=\frac{1}{n}-\sum_{k=1}^{\infty}\mu(E_k)$. Since $T\cap[0, 1]$ is dense in $[0, 1]$ for each $k\in N$ there is $t_k\in T$ with $\mu(E_k)<t_k\leqq\mu(E_k)+\frac{\delta}{2^k}$. Then $E_k\in\mathcal{N}_{t_k}$ for each $k\in N$ and

$$\sum_{k=1}^{\infty}t_k\leqq\sum_{k=1}^{\infty}\left[\mu(E_k)+\frac{\delta}{2^k}\right]=\delta+\sum_{k=1}^{\infty}\mu(E_k)=\frac{1}{n}$$

which proves that $E\in\mathcal{N}_n^*$.

3.

In this section let $\mathcal{R}$ be a ring and let $\mathcal{H}(\mathcal{R})$ be the hereditary σ-ring generated by $\mathcal{R}$.

Suppose that φ is a set function defined on $\mathcal{R}$. Define set function ν on $\mathcal{H}(\mathcal{R})$ by:

$$\nu(A)=\inf\left\{\sum_{k=1}^{\infty}\varphi(E_k): A\subset\bigcup_{k=1}^{\infty}E_k,\ E_n\in\mathcal{R},\ n=1,2,\ldots\right\},\quad A\in\mathcal{H}(\mathcal{R}) \tag{6}$$

The following lemma states: in order that ν be an outer measure it is sufficient that φ satisfy the following conditions

I. φ is nonnegative,

II. $\varphi(\emptyset)=0$.

Lemma 4. if φ satisfies I. and II. then the function ν defined od $\mathcal{H}(\mathcal{R})$ by:

$$\nu(A)=\inf\left\{\sum_{k=1}^{\infty}\varphi(E_k): A\subset\bigcup_{k=1}^{\infty}E_k,\ E_n\in\mathcal{R},\ n=1,2,\ldots\right\},$$

is an outer measure.

Proof is analogous to that of Theorem 5.1. in [2].

In a similar way as in section 1 assume that $T\subset(0,\infty)$ has 0 as its limit point. Consider a family $\{\mathcal{N}_t\}_{t\in T}$ of subfamilies of $\mathcal{R}$ such that for each $t\in T$ we have $\emptyset\in\mathcal{N}_t$.

Define φ on $\mathcal{R}$ by $\varphi(E)=\inf\{t\in T: E\in\mathcal{N}_t\}$. (Hereafter φ will always denote this function determined by the family $\{\mathcal{N}_t\}_{t\in T}$).

In terms of φ we can define the function ν on $\mathcal{H}(\mathcal{R})$ by (6). Since the fixed function φ satisfies I. and II., it follows from Lemma 3 that ν is an outer measure.

Theorem 4. For every $n\in N$ we have:

$$\left\{E\in\mathcal{H}(\mathcal{R}):\nu(E)<\frac{1}{n}\right\}\subset\mathcal{N}_n^*\subset\left\{E\in\mathcal{H}(\mathcal{R}):\nu(E)\leqq\frac{1}{n}\right\}$$

Proof. Assume $\nu(E)<\frac{1}{n}$. From definition of ν it is clear there are sets $E_k\in\mathcal{R}(k\in N)$ with $E\subset\bigcup_{k=1}^{\infty}E_k$ and $\sum_{k=1}^{\infty}\varphi(E_k)<\frac{1}{n}$. Put $\delta=\frac{1}{n}-\sum_{k=1}^{\infty}\varphi(E_k)$. It follows from definition of φ that for each $k\in N$ there exist $t_k\in T$ with $\varphi(E_k)\leqq t_k<\varphi(E_k)+\frac{\delta}{2^k}$ and $E_k\in\mathcal{N}_{t_k}$. On the other hand $\sum_{k=1}^{\infty}t_k<\sum_{k=1}^{\infty}\left[\varphi(E_k)+\frac{\delta}{2^k}\right]=\delta+\sum_{k=1}^{\infty}\varphi(E_k)=\frac{1}{n}$ which proves that E has an n-cover, and the inclusion $\left\{E\in\mathcal{H}(\mathcal{R}):\nu(E)<\frac{1}{n}\right\}\subset\mathcal{N}_n^*$ is proved.

Now let $E\in\mathcal{N}_n^*$. Then there exists a sequence $\{E_k\}_{k=1}^{\infty}$ such that $E\subset\bigcup_{k=1}^{\infty}E_k$, $E_k\in\mathcal{N}_{t_k}$ and $\sum_{k=1}^{\infty}t_k\leqq\frac{1}{n}$. By the definition of φ we have $\varphi(E_k)\leqq t_k$ for each $k\in N$ and so $\nu(E)\leqq\sum_{k=1}^{\infty}\varphi(E_k)\leqq\sum_{k=1}^{\infty}t_k\leqq\frac{1}{n}$. It follows that $\nu(E)\leqq\frac{1}{n}$.

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SÚHRN

O MALÝCH SYSTÉMOCH

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Nech $\mathscr{R}$ je okruh, $\sigma(\mathscr{R})$, nech je σ-okruh generovaný $\mathscr{R}$. V časti 1 uvádzame zovšeobecnenie Lloydovej konštrukcie malého systému $\{\mathscr{N}_n\}_{n=1}^{\infty}$ definovaného na $\sigma(\mathscr{R})$, ktorý je odvodený zo systému $\{\mathscr{N}_n\}_{n=1}^{\infty}$ neprázdnych podmnožín. Nech μ je miera na $\mathscr{R}$ a μ^* je vonkajšia miera na dedičnom σ-okruhu $\mathscr{H}(\mathscr{R})$ generovanom $\mathscr{R}$. Hlavným výsledkom práce je veta 2, ktorá tvrdí: Ak $\mathscr{N}_n = \left\{E \in \mathscr{R} : \mu(E) < \frac{1}{n}\right\}$, $n = 1, 2, \ldots$ a μ^* nemá atom na $\sigma(\mathscr{R})$, tak pre všetky n platí rovnosť $\mathscr{N}_n^* = \left\{E \in \mathscr{H}(\mathscr{R}) : \mu^*(E) < \frac{1}{n}\right\}$. Tento výsledok opravuje nesprávne tvrdenie J. Lloyda. Veta 2 sa dosahuje pomocou lemy 2, ktorá tvrdí, že μ^* nemá na $\sigma(\mathscr{R})$ atom konečnej miery $\Leftrightarrow$ ak pre všetky $A \in \mathscr{R}$; $0 < \mu(A) < \infty$ je množina $\{\mu(E) : E \in A \mid \mathscr{R}\}$ hustá v intervale $[0, \mu(A)]$. V časti 3 konštruujeme vonkajšiu mieru ν aproximujúcu malý systém $\{\mathscr{N}_n^*\}_{n=1}^{\infty}$.

РЕЗЮМЕ

О МАЛЫХ СИСТЕМАХ

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Пусть $\mathcal{R}$-кольцо, и $\sigma(\mathcal{R})$ обозначает σ-кольцо порожденное кольцом $\mathcal{R}$. В части 1 дано обобщение конструкции Й. Лойда, малой сыстемы $\{\mathcal{N}_n^*\}_{n=1}^{\infty}$ определенной на $\sigma(\mathcal{R})$, и полученной из системы $\{\mathcal{N}_n\}_{n=1}^{\infty}$ непустых подмножеств $\mathcal{R}$. Пусть μ-мера определенная на $\mathcal{R}$ и μ^*-внешняя мера на наследственном σ-кольце $\mathcal{H}(\mathcal{R})$, порожденном $\mathcal{R}$. Основным результатом работы является теорема 2: Если

$$\mathcal{N}_n = \left\{E \in \mathcal{R} : \mu(E) < \frac{1}{n}\right\}, n = 1, 2, \ldots$$

и μ^* не содержит атом на $\sigma(\mathcal{R})$, то для всех имеет место

$$\mathcal{N}_n^* = \left\{E \in \mathcal{H}(\mathcal{R}) : \mu^*(E) < \frac{1}{n}\right\}.$$

Этот результат исправляет ошибочное утверждение Й. Лойда. Упоминавшийся резултат получается с помощью Леммы 2, утверждающей, что μ^* не имеет на $\sigma(\mathcal{R})$ атом конечной меры $\Leftrightarrow$ если для всех $A \in \mathcal{R}$, для которых $0 < \mu(A) < +\infty$ множество $\{\mu(E) : E \in A \mid \mathcal{R}\}$-плотно в промежутке $[0, \mu(A)]$, части 3 работы построена внешняя мера ν приближающая малую систему $\{\mathcal{N}_n^*\}_{n=1}^{\infty}$.

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ASYMPTOTICAL PROPERTIES OF SOLUTIONS OF DELAYED LINEAR DIFFERENTIAL EQUATIONS

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In a series of papers (see e. g. [5] or [6] among others) J. A. Rjabov has proved that the solutions of a delayed differential equation

$$x'(t) = a(t) + b(t)x(t - \tau(t)) \tag{1}$$

generated by arbitrary initial functions can be asymptotically approximated (for $t \to \infty$) by so called complete or bilateral solutions, i. e. solutions of (1) for all $t \in R$, provided that $\|a\|$, $\|b\|$, and $\|\tau\|$ are sufficiently small (see remark after Theorem 4 below). Recently [7] he generalizes his results also to some weakly non-linear equations. For complete references the reader is referred to the review paper of R. Driver [2] where also some improvements of Ryabov's results can be found. Some new conditions for existence of complete solutions in more general cases can be found in [8].

In the present paper there are given some generalizations and improvements of above quoted results, for linear equations, in section 3 below. This results are obtained by using some modification of the Ważewski's topological principle which is developed in section 2, for the case of more general non-linear delayed differential equations.

In the sequel the set of reals, the set of non-negative reals, and the n-dimensional Euclidean space will be denoted by R, R^+, and R_n, respectively. The norm $\|.\|$ in R_n is defined by $\|(x_1, \ldots, x_n)\| = \max_i |x_i|$. For x, $y \in R_n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ the inequality $x > y$ (resp. $x \geqq y$) means that $x_i > y_i$ (resp. $x_i \geqq y_i$) for each i.

Preliminary constructions

Consider the delayed differential equation

$$x'(t) = f(t, x(t - \tau_1(t)), \ldots, x(t - \tau_m(t))) \tag{2}$$

where $\tau_i: R \to R^+$ are continuous functions, $f: R_{mn+1} \to R_n$ is continuous and satisfies a Lipschitz condition with respect to all variables except for first, on each set of the form $I \times R_{mn}$ where $I \subset R$ is a bounded interval, i. e. for each such I there is some positive constant N such that $\|f(t, y_1, \dots, y_{mn}) - f(t, z_1, \dots, z_{mn})\| \leqq \leqq N \max_i |y_i - z_i|$ whenever $t \in I$. These assumptions guarantee that for each continuous function $\varphi_0: (-\infty, a] \to R_n$ there is exactly one function $\varphi: (-\infty, b) \to R_n$ such that φ is a solution of (2) for $t \in (a, b)$, and $\varphi(t) = \varphi_0(t)$ for $t \leqq a$. Note that in this case $\varphi'(a)$ may not exist.

We begin with the following

Lemma 1. Let φ be a continuous function $(-\infty, k] \to R_n$ which is a solution of (2) for $t \in (s, k)$, with initial function $\varphi_0(t) = \varphi(t)$ for $t \in (-\infty, s]$. Let $\varepsilon > 0$. Then there is some $\delta > 0$ with the following property: If ψ_0 is a continuous function $(-\infty, s + h] \to R_n$ such that $\|\varphi_0(\xi) - \psi_0(\xi + h)\| < \delta$ for each $\xi \leqq s$ and if $|h| < \delta$ then there exists a function ψ which is a solution of (2) in $(s + h, k + h]$, with ψ_0 as initial function and such that $\|\varphi(\xi) - \psi(\xi + h)\| < \varepsilon$ for each $\xi \in [s, k]$.

Proof. Let $I = [s - 1, k + 1]$, $u = \min\{\xi - \tau_i(\xi);\ \xi \in I,\ i = 1, 2, \dots, m\}$. Let J be a closed bounded interval in R_n such that $\varphi(\xi) \in J$ for each $\xi \in [u, k]$. Let ω_φ be the modulus of continuity of φ in $[u, k]$, ω_f the modulus of continuity of f in $I \times J^m$ and ω_i the modulus of continuity of τ_i in I. For $\xi \geqq 0$ put $\omega_\tau(\xi) = \max_i \omega_i(\xi)$. Let N be Lipschitz constant of f in $I \times R_{mn}$. Let $\delta < 1$ be a positive number and assume that $|h| < \delta$ and that $\sup_{\xi \leqq s} \|\varphi(\xi) - \psi_0(\xi + h)\| < \delta$. Let $(s + h, a + h)$ be the largest interval in which there exists a solution ψ of (2) with ψ_0 as initial function. Put $b = \min\{a, k\}$ and let $t \in [s, b)$, Then we have the following estimate:

$$\max_{[s,t]} \|\varphi(\xi) - \psi(\xi + h)\| \leqq \|\varphi(s) - \psi_0(s + h)\| +$$

$$+ \int_s^t \|f(\xi, \varphi(\xi - \tau_1(\xi)), \dots, \varphi(\xi - \tau_m(\xi))) -$$

$$- f(\xi + h, \varphi(\xi - \tau_1(\xi)), \dots, \varphi(\xi - \tau_m(\xi)))\| \, d\xi +$$

$$+ \int_s^t \|f(\xi + h, \varphi(\xi - \tau_1(\xi)), \dots, \varphi(\xi - \tau_m(\xi))) -$$

$$- f(\xi + h, \varphi(\xi - \tau_1(\xi + h)), \dots, \varphi(\xi - \tau_m(\xi + h)))\| \, d\xi +$$

$$+ \max_i N \int_s^t \|\varphi(\xi - \tau_i(\xi + h)) - \psi(\xi + h - \tau_i(\xi + h))\| \, d\xi \leqq$$

$$\leqq \delta + (\omega_f(\delta) + \omega_f(\omega_\varphi(\omega_\tau(\delta))))\,(k-s) + N\int_s^t (\delta + \max_{[s,t]} \|\varphi(\xi) - \psi(\xi+h)\|)\,d\xi.$$

Using the well-known Gronwall lemma (see e. g. [3]) we get

$$\sup_{[s,b)} \|\varphi(\xi) - \psi(\xi+h)\| \leqq$$

$$\leqq (\delta + (k-s)\,(\omega_f(\delta) + \omega_f(\omega_\varphi(\omega_\tau(\delta))) + N\delta)) \cdot e^{N(k-s)}.$$

The right-hand side of the inequality tends to zero whenever $\delta \to 0$. This means that ψ is bounded in the interval $[s+h, b+h)$ hence ψ can be continuously extended to the interval $[s+h, b+h]$. To finish the proof it suffices to note that $b = k$ since in the case $b < k$, ψ is a solution of (2) in $(s+h, a+h]$ which is impossible. Thus the Lemma is proved.

In the sequel we shall use the following notation: Let C_n^* be the metric space of all uniformly continuous functions $(-\infty, 0] \to R_n$ with the metric $\mu(f, g) = \min\{\sup_{\xi \leqq 0} \|f(\xi) - g(\xi)\|, 1\}$. Let $L_n = R \times C_n^*$ be the metric space with metric $\varrho((r, f), (s, g)) = |r - s| + \mu(f, g)$. If φ is a uniformly continuous function $I \to R_n$ where I is a below unbounded subinterval of R then for each $t \in I$, $\varphi_t \in C_n^*$ is defined by $\varphi_t(u) = \varphi(t+u)$ for each $u \leqq 0$.

Let φ^0 be a uniformly continuous function $(-\infty, t] \to R_n$ and let φ be the solution of (2) with φ^0 as initial function defined on the largest possible interval $[t, b)$ ($\varphi(\xi) = \varphi^0(\xi)$ for $\xi \leqq t$). Then the set $\{(\xi, \varphi_\xi);\ \xi \in [t, b)\}$ is called the integral curve of (2) going from the point $P = (t, \varphi_t^0)$ and it will be denoted by $\Omega(P)$.

Let T be an open subset of the space L_n and let $P \in T$. If $\Omega(P) \subset T$ then $\Omega(P)$ is called an asymptotical curve. Otherwise there is the first point $Q(P) \in L_n$ on the curve $\Omega(P)$ which does not belong to T. Let $G \subset T$ be the set of all those points from which do not go asymptotical curves and let $S = \{Q(P), P \in G\}$. Define a subset S^* of S as follows: If $P \in G$ and $Q(P) = (s, \psi_s)$ then $Q(P) \in S^*$ if and only if there is a positive δ such that, for $t \in (s, s+\delta)$ we have $(t, \psi_t) \notin \overline{T}$. Here $\overline{A}$ denotes, as usually, the closure of A.

Finally we recall that a set A is a retract of $A \cup B$ in the sense of Borsuk whenever there exists a continuous mapping (a retraction) g from $A \cup B$ onto A such that $g(t) = t$ for each $t \in A$.

Now we are able to prove the following theorem which is a generalization of Theorem 9.8.1 from [1].

Theorem 1. Let $S = S^*$ and let Φ be a mapping from $G \cup S$ to S such that $\Phi(P) = Q(P)$ for $P \in G$ and $\Phi(P) = P$ for $P \in S$. Then Φ is a retraction from $G \cup S$ onto S.

Proof. It suffices to show that Φ is continuous. Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of points from $G\cup S$ converging to some $P_0\in G\cup S$. We show that $\Phi(P_n)\to\Phi(P_0)$. Let $P_n=(t(n), \varphi^n_{t(n)})$, $Q_n=\Phi(P_n) = (s(n), \varphi^n_{s(n)})$, where φ^n is a solution of (9) in the interval $(t, (n), c(n))$, $n=1, 2, \ldots$.

First we assume that $P_0\in G$. Since $G\subset T$, T is open and $T\cap S=\emptyset$, it suffices to consider the case $P_n\in G$ for all n. Let $\varepsilon>0$. Let $A_0=(x(0), \varphi^0_{x(0)})$, $B_0=(y(0), \varphi^0_{y(0)})$ be two points from $\Omega(P_0)$ such that the curve P_0A_0 lies in G, $B_0\notin \overline{T}$ and the diameter of the curve A_0B_0 is less then ε. Clearly $x(0)<s(0)<(0)$. Let $h(n)= =t(n)-t(0)$. By Lemma 1, $c(n)>y(0)+h(n)$, for all sufficiently large n. For each such n put $A_n=(x(0)+h(n), \varphi^n_{x(0)}+h(n))$, $B_n=(y(0)+h(n), \varphi^n_y(0)+ +h(n))$. Let $\varepsilon_1>0$ be the distance of B_0 from the set T and $\varepsilon_2>0$ the distance of the curve P_0A_0 from the set $L_n\setminus T(P_0A_0$ is a compact). Let $\varepsilon_3 = \min(\varepsilon, \varepsilon_1, \varepsilon_2)$. By Lemma 1 there is some integer N such that $\varrho((\xi, \varphi^0_\xi), (\xi+h(n), \varphi^n_{\xi+h(n)}))<\varepsilon_3$ for all $n>N$ and $\xi\in[t(0), y(0)]$. But from this follows that for each $n>N$ the curve P_nA_n lies in G and $B_n\notin\overline{T}$, hence Q_n lies between A_n and B_n. Now it is easy to see that $\varrho(Q_0, Q_n) < \varepsilon+\varepsilon_3 \leqq 2\varepsilon$, for each $n>N$.

If $P_0\in S$, then we may assume without loss of generality that $P_n\in G$, for $n=1$, $2, \ldots$. In this case put $A_n=P_n$, for each n. Now the proof is similar, as in the preceeding case. We assume that the diameter of the curve P_0B_0 is less then $\varepsilon>0$. For all sufficiently large n, Q_n lies between P_n and B_n, and again we have $\varrho(Q_0, Q_n) < 2\varepsilon$, for $n>N$. This finishes the proof of Theorem 1.

Theorem 2. Let $S=S^*$ and $Z\subset T\cup S$. If $Z\cap S$ is a retract of S, but not a retract of Z, then there is some $P\in Z$ such that $\Omega(P)\subset T$.

Proof of the theorem is based on Theorem 1 and can be found e. g. in [1].

Now we are able to prove the main result of this section:

Theorem 3. Let the above quoted assumption on f and τ_i, $i=1, \ldots, m$, be satisfied. Assume that $\psi>\varphi$ are continuous functions $R\to R_n$, ω a uniformly continuous function $R\to R_n$ and $\varepsilon\in R_n$ a positive constant such that

$$\psi(t)-\varepsilon>\omega(t)>\varphi(t)+\varepsilon \tag{3}$$

for all $t\leqq t^*$ where t^* is a constant, and

$$\max\{\overline{D}^+\psi(t), \overline{D}^-\psi(t)\}<f(t, y_1, \ldots, y_m)<\min\{\underline{D}^+\varphi(t), \underline{D}^-\varphi(t)\} \tag{4}$$

for $y_i\in[\varphi(t-\tau_i(t)), \psi(t-\tau_i(t))]$, $i=1, 2, \ldots, m$, $t\in R$ (the symbols $\overline{D}^+$, $\overline{D}^-$, $\underline{D}^+$, $\underline{D}^-$ denotes, as usually, the Dini derivatives). Then there exists a function $\overline{x}: R\to R_n$, which is, for each t, a solution of (2) and such that $\varphi\leqq x\leqq\psi$.

In the proof of Theorem 3 we shall use the following two lemmas:

Lemma 2. Let the assumptions of Theorem 3 be satisfied. Let $b\in R$ be such that, for each $i=1, 2, \ldots, m$, either $\tau_i(b)>0$ or $\tau_i(t)=0$ for each t in a certain neighbourhood of b. Then there exists a continuous function $x^b: R\to R_n$, which is a solution of (2) in the interval (b, ∞) and such that $\varphi\leqq x^b\leqq\psi$.

Proof. We apply Theorem 2. Let $T=\{(t, y)\in L_n;\ \psi_t - c > y > \varphi_t + c$, for a suitable positive constant $c \in R_n\}$. T is an open subset of the space L_n and since $(t^*, \omega_{t^*}) \in T$, T is non-empty.

The set Z is defined as follows: Choose $\eta > 0$ such that for each i we have either $t-\tau_i(t) < b-\eta$ for all $t\in[b-\eta, b]$, or $\tau_i(t)=0$ for all t from a certain open neighbourhood of the interval $[b-\eta, b]$. From the continuity of functions τ_i it follows that there is such a $d < b-\eta$, that $\max\{t-\tau_i(t);\ t\in[b-\eta, b],\ \tau_i(t)>0,\ i=1, 2, \ldots, m\} < d$. To each $u \in [\varphi(b), \psi(b)]$ assigne a function $y^u\colon (-\infty, b] \to R_n$ defined by

$$y^u(t) = \omega(t) \quad \text{for} \quad t \leqq d$$

$$y^u(t) \;=\; u + \int_b^t f(t, y^u(t-\tau_1(t)),\ \ldots,\ y^u(t-\tau_m(t)))\, dt \ \text{ for } t\in[b-\eta, b].$$

The assumptions on f, η guarantee that for $u \neq v$ and for $t \in [b-\eta, b]$ there is $y^u(t) \neq y^v(t)$, that $y^u(t)$ depends continuously on u in the interval $[b-\eta, b]$, and from (4) we get

$$\varphi(t) < y^u(t) < \psi(t) \tag{5}$$

for $t\in[b-\eta, b)$. Now let y^u be continuous in the interval $[d, b-\eta]$ and such that for each $t\in[d, b-\eta]$ (5) holds, and let for each u, v

$$\max_{[d, b-\eta]} \|y^u(t) - y^v(t)\| = \|y^u(b-\eta) - y^v(b-\eta)\|.$$

We have

$$S = \{(t, \varphi_t);\ t\in R\} \cup \{(t, \psi_t);\ t \in R\}$$

Put

$$Z = \{(b, (y^u)b);\ u\in[\varphi(b), \psi(b)]\}$$

and

$$Z_1 = \{(b, (y^u)_b),\ u \text{ is from the boundary of } [\varphi(b), \psi(b)]\}$$

It is easy to verify that the mapping $v\colon (b, (y^u)_b) \mapsto u$ is a homeomorphism from Z onto the closed interval $[\varphi(b), \psi(b)]$ in R_n such that $v(Z\cap S) = v(Z_1)$ is the boundary of the interval $[\varphi(b), \psi(b)]$. But the boundary of any closed bounded non-degenerate interval in R_n cannot be a retract of this interval (see e. g. [4]), so $Z \cap S$ is not a retract of Z.

It is easy to see that $Z\cap S$ is a retract of S. From (4) it follows that $S = S^*$. Hence all assumptions from Theorem 2 are satisfied and hence there is a point $P = (b, (y^s)_b \in Z$ such that $\Omega(P) \subset T$.

Let x be a solution of (2) in the interval (b, ∞) such that $x(t) = y^s(t)$ for $t \leqq b$. Clearly $\varphi(t) \leqq x(t) \leqq \psi(t)$ for each $t \in R$ and the proof of Lemma 2 is finished.

Lemma 3. Let f, $\tau_i \geqq 0$ be continuous. Let $\varphi \leqq \psi$ be continuous functions $R \to R_n$. Assume that $\{t_k\}_{k=1}^{\infty}$ is a decreasing sequence of real numbers which tends to $-\infty$. Let for each natural k there is a function z^k: $R \to R_n$, which is a solution of (2) in the interval (t_k, ∞) and such that $\varphi \leqq z^k \leqq \psi$. Then there is some $z: R \to R_n$, which is everywhere a solution of (2) and $\varphi \leqq x \leqq \psi$.

Proof. Since φ and ψ are continuous the functions $\{z^k\}_{k=1}^{\infty}$ are almost uniformly bounded (i. e. uniformly bounded on each subcompact of R) and since their derivatives $(z^k)'(t) = f(t, z^k(t-\tau_1(t)), \ldots, z^k(t-\tau_m(t)))$ are almost uniformly bounded the functions $\{z^k\}_{k=1}^{\infty}$ are almost equicontinuous. So there is a subsequence $\{z^{k(n)}\}_{n=1}^{\infty}$ which converges almost uniformly to some function z. It is not difficult to verify that z is a solution of (2) with the desired properties, q. e. d.

Proof of Theorem 3. We shall define a sequence $\{t_k\}_{k=1}^{\infty}$ as follows: Put for $i = 1, 2, \ldots, m$, $A_i = \{t;\ \tau_i(t) > 0\}$. If all sets A_i are below bounded by some constant c let $\{t_k\}_{k=1}^{\infty}$ be an arbitrary sequence of numbers $< c$ decreasing to $-\infty$. If some A_i is below unbounded let B be a set of the form $B = A_{i(1)} \cap \ldots \cap A_{i(s)}$ which is below unbounded and such that $B \cap A_j$ is below bounded by a constant d for each $j \neq i(1), \ldots, i(s)$. Such a set clearly exists and is open. In this case let $\{t_k\}_{k=1}^{\infty}$ be a sequence of points from the set $B \cap (-\infty, d)$ decreasing to $-\infty$. It is easy to verify that for each i and each k either $\tau_i(t_k) > 0$, or $\tau_i(t) = 0$ for each t from a certain neighbourhood of t_k. By Lemma 2 there is a sequence $\{x^k\}_{k=1}^{\infty}$ of functions $R \to R_n$ such that $\varphi \leqq x^k \leqq \psi$ and x^k is a solution of (2) in the interval (t_k, ∞), for each k. By Lemma 3 there is a solution x of (2) in the whole R such that $\varphi \leqq x \leqq \psi$. Theorem 3 is proved.

3. Main results.

In this section we apply Theorem 3 to the delayed linear differential equation

$$x'(t) = A_1(t)x(t-\tau_1(t)) + \ldots + A_m(t)x(t-\tau_m(t)) \tag{6}$$

where A_i are continuous square matrices $R \to R_n \times R_n$, τ_i: $R \to R^+$ are continuous and x: $R \to R_n$. Define a function τ: $R \to R^+$ by $\tau(t) = \max_i (\tau_i(t)$. The norm of matrix (a_{ij}) is defined by $\|(a_{ij})\| = \max_{i,j} |a_{ij}|$. Now we can state the following.

Theorem 4. Let for each $t \in R$

$$\sum_{i=1}^{m} \int_{t-\tau(t)}^{t} \|A_i(\xi)\| \, d\xi < 1/ne \tag{7}$$

Let x: $R \to R_n$ be a solution of (6) for $t > b$. Assume that τ is bounded. Then there exists a complete solution y of (6) and a positive r such that for each $t > b$,

$$\|x(t) - y(t))\| < r . \exp\left(ne\int_t^b \|A_1(\xi)\| + \ldots + \|A_m(\xi)\| \, d\xi\right). \tag{8}$$

In other words, y asymptotically approximates x, for $t \to +\infty$.

Remark. As a consequence of Theorem 4 we have the following result concerning the differential equation

$$x'(t) = a(t)x(t - \tau(t)) \tag{9}$$

where $a: R \to R$ is continuous and bounded: Let $L = \sup_t |a(t)|$, $T = \sup_t \tau(t)$, and let $LT < 1/e$. Then each solution of (9) can be asymptotically approximated by a complete solution. The recent results in the case of bounded functions a, τ are more restrictive (see e. g. [2]). There is assumed that, moreover, the complete solutions of (9) are bounded for $t > b$. Or the boundedness is not required but there is assumed that LT$<c$ where $c < 1/e$ is a certain, precisely defined constant.

Proof of Theorem 4. Define the function $\lambda: R \to R$ by $\lambda(t) = \exp\left(ne\int_t^b (\|A_1(\xi)\| + \ldots + \|A_m(\xi)\|)\,d\xi\right)$. Choose $d > b$ such that $t - \tau(t) > b$ for $t > d$. Let $c = \max_{[b,d]} \int_{t-\tau(t)}^t \|A_1(\xi)\| + \ldots + \|A_m(\xi)\|)\,d\xi$. Clearly $c < 1/ne$. Let r be a positive constant. Let $I \in R_n$ be the vector whose all coordinates are 1. Define functions ψ and φ as follows:

$$\psi(t) = r\lambda(t) \cdot I + x(t) \quad \text{for } t \geqq b$$
$$\psi(t) = (rI + x(b))\lambda(t) \quad \text{for } t < b$$

Assume that r is sufficiently large such that $rI + x(b) > 0$ (coordinatewise inequality). Similarly

$$\varphi(t) = -r\lambda(t) \cdot I + x(t) \quad \text{for } t \geqq b$$
$$\varphi(t) = -\psi(t) + 2x(b) \quad \text{for } t < b$$

We show that for sufficiently large $r = r(b)$ the functions ψ, φ satisfy the conditions (4) of Theorem 3. Choose

$$z_i \in [\varphi(t - \tau_i(t)), \psi(t - \tau_i(t))], \qquad i = 1, \ldots, m \tag{10}$$

First assume that $t - \tau(t) \geqq b$. Then we have $A_1(t)z_1 + \ldots + A_m(t)z_m = A_1(t)(u_1 + x(t - \tau_1(t))) + \ldots + A_m(t)(u_m + x(t - \tau_m(t))) \leqq -n\|A_1(t)\| r \lambda(t - \tau_1(t))I - \ldots - n\|A_m(t)\| r\lambda(t - \tau_m(t))I + x'(t) > -n(\|A_1(t)\| + \ldots + \|A_m(t)\|)r\lambda(t)eI + x'(t) = \psi'(t)$ (the last inequality follows from the fact that $\lambda(t - \tau(t)) < \lambda(t) \cdot e$). Similarly the inequality $A_1(t)z_1 + \ldots + A_m(t)z_m < \varphi'(t)$ can be verified. The inequalities (4) are satisfied also for $t < b$, the proof is similar as in the preceeding case. It remains to consider the case $t \geqq b$ and $t - \tau(t) < b$. Let $M = \max_{[b,d]} \|x(t - \tau(t))\|$. From (10) we have $z_i = u_i + x(t - \tau_i(t))$ for $i = 1, \ldots, m$, where $u_i \in [-(rI + x(b))\lambda(t - \tau_i(t)) - MI, (rI + x(b))\lambda(t - \tau_i(t)) + MI]$.

We have $A_1(t)z_1 + \dots + A_m(t)z_m = A_1(t)u_1 + \dots + A_m(t)u_m + x'(t) >$ $> -n(\|A_1(t)\| + \dots + \|A_m(t)\|)\,(rI + x(b))\lambda(t)e^{nec} + MI) + x'(t)$. It is easily checked that, for sufficiently large r the right-hand side of the last inequality is greater then $\psi'(t)$ (resp. greater then max $(\psi'_+(b), \psi'_-(b))$ when $t=b$). Similarly one can prove that for sufficiently large r, $A_1(t)z_1 + \dots + A_m(t)z_m$ is less then $\varphi'(t)$ (resp. less then min $(\varphi'_+(b), \varphi'_-(b))$ when $t=b$). Thus for suitable $r>0$ the functions ψ, φ satisfy the conditions (4). Hence by Theorem 3 there exists a complete solution $y: R \to R_n$ of (6) such that $\varphi < y < \psi$. From this and from the inequality $\varphi(t) < x(t) < \psi(t)$, for $t > b$, we obtain (8), q. e. d.

The following theorem gives information on the asymptotical dimension of the space of all solutions of (6).

Theorem 5. Let the assumptions of Theorem 4 be satisfied and let $\vartheta \in (0, 1)$ be a constant such that

$$\sum_{i=1}^{m} \int_{t-\tau(t)}^{t} \|A_i(\xi)\| \,d\xi < \vartheta/ne \qquad \text{for all} \quad t \in R$$

Then there exists an n-dimensional linear subspace U of the space of all complete solutions of (6) such that each solution of (6) generated by arbitrary initial function can be asymptotically (for $t \to +\infty$) approximated by some member of U. In other words, the space of all solutions of (6) is asymptotically n-dimensional.

Proof. Denote by U_{as} the class of all complete solutions x of (6) such that $\|x(t) \leqq s \cdot \exp\left(ne\left|\int_t^a \left\|\sum_{i=1}^{m} A_i(\xi)\right\| d\xi\right|\right) = \alpha(a, s, t)$ for each $t \in R$. Let $U_a = \bigcup_s U_{as}$. We show that through each point $(u, w) \in R \times R_n$ goes exactly one member of U_a.

It is easy to verify that for each $x \in U_{as}$ and each t, we have $\|x'(t)\| < \partial\alpha(a, s, t)/\partial t$. From this we have, for each $x \in U_{as}$:

$$\text{If } \|x(t_0)\| = \alpha(a, s, t_0) \text{ for some } t_0, \text{ then } t_0 = a\,. \tag{11}$$

Now let $y, z \in U_{as}$ such that $y(u) = z(u)$ for some $u \in R$. Let $v = \inf\{s\,;\, y, z \in U_{us}\}$. From (11) it follows that $y, z \in U_{uv}$ and that $\|y(u)\| = \|z(u)\| = v$. But the assumptions of Theorem 5 guarantee that through each point $(u, w) \in R \times R_n$ goes exactly one complete solution of (6) which belongs to $U_{u\|w\|}$, cf. [8]. Hence $y = z$.

To finish the proof it suffices to note that for each $a, b \in R$, $U_a = U_b = U$, U is n-dimensional linear space, and that if some complete solution x of (6) satisfies the inequality $\varphi < x < \psi$, where φ, ψ are the functions involved in the proof of Theorem 4, then $x \in U$.

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SÚHRN

ASYMPTOTICKÉ VLASTNOSTI RIEŠENÍ LINEÁRNYCH DIFERENCIÁLNYCH ROVNÍC S ONESKORENÝM ARGUMENTOM

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V práci je dokázaná táto veta: Nech v rovnici

$$x'(t) = A_1(t)x(t + \tau_1(t)) + \ldots + A_m x(t - \tau_m(t)) \tag{1}$$

sú matice $A_i(t): R \to R_n \times R_n$ a aj funkcie $\tau_i: R \to R^+$ spojité a nech

$$\sum_{i=1}^{m} \int_{t-\tau(t)}^{t} \|A_i(s)\| \, ds < 1/ne$$

Nech $x: R \to R_n$ je riešenie rovnice (1) pre všetky $t \geqq t_0$ a nech $\tau(t) = \max \tau_i(t)$ je ohraničená funkcia. Potom existuje úplné riešenie $y(t)$ rovnice (1) (definované pre všetky t) a kladné r také, že

$$\|x(t) - y(t)\| < r \cdot \exp\left(ne \int_t^{t_0} \|A_1(s)\| + \ldots + \|A_m(s)\| \, ds\right)$$

pre všetky $t > t_0$.

Pri dôkaze vety sme použili Ważevského topologický princíp.

РЕЗЮМЕ

АСИМПТОТИЧЕСКИЕ СВОЙСТВА РЕШЕНИЙ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ЗАПАДЫВАЮЩИМСЯ АРГУМЕНТОМ

К. Смиталова, Братислава

В статье доказывается следующая теорема: Пусть в уравнении

$$x'(t) = A_1(t)x(t-\tau_1(t)) + \ldots + A_m x(t-\tau_m(t)) \quad (1)$$

матрицы $A_i(t): R \to R_n \times R_n$ и функции $\tau_i: R \to R^+$ непрерывные и

$$\sum_{i=1}^{m} \int_{t-\tau(t)}^{t} \|A_i(s)\| \, ds < 1/ne$$

Предположим, что $x: R \to R_n$ является решением (1) для всех $t \geqq t_0$ и что $\tau(t) = \max_i \tau_i(t)$ ограниченная функция. Тогда существует двустороннее решение $y(t)$ уравнения (1) и положительное r так, что

$$\|x(t) - y(t)\| < r \cdot \exp\left(ne \int_t^{t_0} \|A_1(s)\| + \ldots + \|A_m(s)\| \, ds\right)$$

для всех $t > t_0$.

Теорема доказывается при помощи топологического принципа Важевского.

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D-СТРУКТУРА

ПЕТЕР ЦВИК, Братислава

1. Введение

Одным из важнейших понятий аффинной геометрии является понятие середина пары точек. Алгебраический подход к изучению этого понятия показывает Ян Гатяль в работах [1] и [2]. Середина пары точек это бинарное соответствие

$$\sigma: A \times A \to A$$

на которое накладывают дальнейшие требования. По Гатялю это именно:

1. (A, σ) является квазигруппой, которая
2. идемпотентна: $\forall a \in A: \sigma(a, a) = a$
3. медиальна: $\forall a, b, c, d \in A: \sigma(\sigma(a, b)\sigma(c, d)) = \sigma(\sigma(a, c)\sigma(b, d))$
4. коммутативна: $\forall a, b \in A: \sigma(a, b) = \sigma(b, a)$

Подражая этому построению «почти аффинного» пространства, изучает Ян Дуплак (см. [3] и [4]) поворот на прямой угол в Эвклидовой плоскости. Определяет ROT-структуру как квазигруппу $(A'; \circ)$ для которой справедлива только одна аксиома:

$$x \circ (x \circ y) = z \circ ((x \circ z) \circ y)$$

В этой статье решаем вопросы алгебраического подхода к изучению симметрии отностительно прямой на множество всех прямых пучка Эвклидовой плоскости. Наша точка зрения та же самая как в упомянутых работах, хотя в течении изучения проблемы мы не пользовались результатами Бахмана (см. [5]).

2. Определение

Алгебраизация геометрического понятия M начинается исканием тех надлежащих свойств понятия M, которые принимаем за определяющие. В выборе тех или других свойств отражается, конечно, точка зрения автора.

После тщательного пересмотра различных моделей приняли мы следующее

Определение 1. Группоид (D, σ) обладающий четырьмя свойствами (будем их называть аксиомами)

1. идемпотентность: $\forall a \in D: \sigma(a, a) = a$
2. медиальность: $\forall a, b, c, d \in D: \sigma(\sigma(a, b), \sigma(c, d)) = \sigma(\sigma(a, c), \sigma(b, d))$
3. эластичность: $\forall a, b \in D: \sigma(\sigma(a, b), b) = a$
4. для любых $a, b \in D$ уравнение $\sigma(a, x) = b$ имеет точно два решения —

будем называть D-структурой.

В дальнейшем будем $\sigma(a, b)$ писать просто ab, $ab \cdot c$ значит $\sigma(\sigma(a, b), c)$ и т. д.

Не тяжело проверить следующую основную модель D-структуры:

Модель 1. Пусть D-множество L всех прямых Эвклидовой плоскости проходящих через определенную точку 0. (Пучок с вершиной 0). Для $a, b \in L$ определяем $\sigma(a, b) = ab$ как прямую симметричную с прямой a относительно прямой b.

3. Независимость аксиом

Покажем что аксиоматическая система состоящая из аксиом 1—4 является независимой. Это выходит из существования моделей 2—5. Все эти модели построенны как произведение $(K, .) = (D, .) \times (E, *)$ любой D-структуры $(D, .)$ и надлежащим образом определенного группоида $(E, *)$

Модель 2. Пусть $(E, *)$ эластичная, медиальная, но неидемпотентная квазигруппа; напр. $(Z_2, +)$ Тогда для $(K, *)$ выполнены все аксиомы за исключением аксиомы 1.

Модель 3. Пусть $(E, \times)$ идемпотентная, эластичная но немедиальная квазигруппа; например $E = Z^* = Z - \{0\}$ и

$$a * b = \begin{cases} a & \text{если } a = b \\ b - a & \text{если } a \neq b \end{cases}$$

Тогда для K, выполнены все аксиомы за исключением аксиомы 2.

Модель 4. Пусть $(E, *)$ является A-структурой (см. [1]); например $E = R$ и $a * b = \frac{1}{2}(a + b)$. Тогда для $(K, .)$ выполнены все аксиомы за исключением аксиомы 3.

Модель 5. Пусть $(E, *) = (D, .)$ является D-структурой. Тогда для $(K, \circ) = (D, .) \times (D, .)$ выполняются первые три аксиомы, но последняя уже не выполняется, так как уравнение $a \cdot x = b$ повидимому допускает точно четыре решения.

4. Модели

Покажем несколько дальнейших моделей D-структуры.

Модель 6. На действительной прямой R определяем операцию

$$\varrho: R \times R \to R, \ (x, y) \to 2y - x$$

Заметим, что

$$\varrho(., a): R \to R, \ x \to x, a$$

является симметрией относительно точки a. Не сложно проверить, что (R, ϱ) является идемпотентной, медиальной, эластической квазигруппой (между прочим: ее мы также могли брать как пример вместо модели 5.) Так как подмножество Z всех целых чисел замкнуто относительно операции ϱ то существует фактор-структура $(R/Z, \varrho_1)$. Обозначая $\bar{y}$ образ числа y при канонической проекции

$$p: R \to R/Z$$

получаем:

$$\varrho_1: R/Z \times R/Z \to R/Z, \ (\bar{x}, \bar{y}) \to \overline{2y - x}$$

Операция ϱ_1 сохраняет первые три свойства операции ϱ, но не сохраняет свойство квазигруппы. На самом деле уравнение $\varrho_1(\bar{a}, \bar{x}) = \bar{b}$ омеет два решения $\overline{\frac{1}{2}(a+b)}, \overline{\frac{1}{2}(a+b+1)}$. Итак $(R/Z, \varrho_1)$ является D-структурой; обозначим ее как модель 6.

Модель 7 получим геометризацией модели 6. Пусть $S^1 \leqq \{z \in C: |z| = 1\}$ единечная окружность в комплексной плоскости. При помощи биективного отображениа

$$\varphi: R/Z \to S^1, \bar{x} \to e^{2\pi ix} \ \forall x \in R$$

можно окружность отождествить с множеством R/Z. Тогда операция

$$\varrho_2: S^1 \times S^1 \to S^1, (e^{i\alpha}, e^{i\beta}) \to e^{i(2\beta - \alpha)}$$

является φ-образом операции ϱ_1 и (S^1, ϱ_2) есть дальнейшим примером D-структуры; ее будем называть модель 7.

Заметим, что $\varrho_2(e^{i\alpha}, e^{i\beta})$ есть точка симметрична с точкой $e^{i\alpha}$ относительно прямой соединяющей начало 0 с точкой $e^{i\beta}$. Если в модели 7 множество S^1 точек $e^{i\alpha}$ заменить множеством K всех полупрямых $0e^{i\alpha}$ и естественным образом заменить операцию ϱ_2 на операцию, которую обозначим ϱ_3, то получаем модель 7′: (K, ϱ_3).

Примечание: Непосредственно из построения моделей 6, 7, 7′ вытекает, что они изоморфны. Более того, даже модель 1 также изоморфен с этими тремя. Отображение

$$\psi: R/Z \to L, \ \bar{x} \to 0e^{\pi i x}$$

отображающее класс $\bar{x}$ на прямую, соединяющую начало координат 0 с точкой $e^{\pi i x}$ является биективным и

$$\varrho_1(\bar{x}, \bar{y}) = \sigma(\psi(\bar{x}), \psi(\bar{y})) \ \ \forall \bar{x}, \bar{y} \in R/Z$$

Итак, хотя были построены уже четыре модели D-структуры, на самом деле это только один пример. Возникает вопрос, существует ли модель D-структуры не изоморфной модели 1. Ответ есть положительный. В следующем примере мы построим даже большой класс не изоморфных D-структур.

Модель 8. Пусть $(E, *)$ иденпотентная, медиальная, эластичная квазигруппа. Тогда ее произведение $(K, \circ) = (D, \circ) \times (E, *)$ с любой D-структурой $(D, \circ)$ является опять D-структурой. Проверить этот факт оставим читателю.

В качестве конкретного примера такого рода возьмем

Модель 9. Пусть E множество всех точек Эвклидовой плоскости, которую ради простоты отождествляем с плоскостью C всех комплексных чисел. Операция $*$ на $E \equiv C$ определяется через

$$x * y = 2y - x$$

т. е. отображение $E \to E$, $x \mapsto x * a$ является симметрией плоскости относительно «точки a».

Потому что $(E, *)$ является иденпотентной, медиальной и эластичной квазигруппой и (S^1, ϱ_2) D-структурой (см. модель 7), то $(K, \circ) = (S^1, \varrho_2) \times (E, *)$ является также D-структурой. Если теперь элементу $(e^{i\alpha}, a) \in S^1 \times E$ дать более наглядный вид полупрямой $\{a + te^{i\alpha}; t \in R_+\}$ с началом в точке a и направлением вектора $e^{i\alpha}$, то получаем геометрическую модель 9, элементами которой служат все полупрямые Эвклидовой плоскости.

Заметим, что аналогичным образом может быть построена D-структура на множестве всех полупрямых плоскости Лобачевского.

Последняя модель нашего обзора получается в результате следующего вопроса: Пусть $(L, \circ)$ является модель 1 D-структуры. Найти самую D-подструктуру $(M, \circ)$ D-структуры $(L, \circ)$

Посредством отображения ψ (или более точно ψ^{-1}) сводится эта задача к задаче отыскания самой малой D-подструктуры (H, τ) D-struktury $(R/Z, \varrho_1)$. Не теряя общности предположим $\bar{0} \in H$. Из аксиомы 4 вытекает, что потом $\frac{\bar{1}}{2} \in H$. Новое применение этой аксиомы влечет $\frac{\bar{1}}{4} \in H$, $\frac{\bar{3}}{4} \in H$ и т. д. Ясно, что множество H состоит из всех классов $\overline{a2^{-n}}$ диадических дробей. В любом таком классе сусцествует точно одно число из интервала $[0, 1)$ которое можно писать в виде 0 или

$$0, a_1a_2 \ldots a_n 1 = a_1 2^{-1} + a_2 2^{-2} + \ldots + a_n 2^{-n} + 2^{-(n+1)} \qquad \text{(A)}$$

где $a_i = 0$ или 1 для всех $i = 1, 2, \ldots, n$. Этот факт показывает следующую конструкцию искомой модели.

Модель 10. Пусть множество H состоит из всех диадических чисел интервала $[0, 1)$, которые будем писать в виде $0, a_1 \ldots a_n 1$ (ноль пишем просто 0) как показанно в (A) На множестве H определяем операцию $\tau\colon H \times H \to H$ через

$$\begin{aligned}
&\tau(0, a_1 \ldots a_n 1\,;\, 0, b_1 \ldots b_m 1) = 0,\ a_1'a_2' \ldots a_n'1 + 0,\ b_2b_3 \ldots b_m 1 - \varepsilon\\
&\tau(0\,;\, 0, b_1 \ldots b_m 1) \qquad\quad\ = 0,\ b_2 \ldots b_m 1\\
&\tau(0, a_1 \ldots a_n 1\,;\, 0) \qquad\quad\ = 0,\ a_1' \ldots a_n'1\\
&\tau(0\,;\, 0{,}1) \qquad\qquad\qquad\ = 0
\end{aligned}$$

где штрихом обозначается инволюция на $\{0, 1\}$ — t. e. $0' = 1,\ 1' = 0$;

$$\begin{aligned}
\varepsilon = 0 \quad &\text{если} \quad 0, a_1'a_2' \ldots a_n'1 + 0, b_2 \ldots b_m 1 < 1\\
\varepsilon = 1 \quad &\text{если} \quad 0, a_1'a_2' \ldots a_n'1 + 0, b_2 \ldots b_m 1 \geqq 1
\end{aligned}$$

и операции выпольняются над полем Z_2.

Например $\tau(0{,}0011\,;\, 0{,}01011001) = 0{,}1101 + 0{,}1011001 - \varepsilon$ и так как $0{,}1101 + 0{,}1011001 = 1{,}1000001 > 1$, то $\varepsilon = 1$ и $\tau(0{,}0011\,;\, 0{,}01011001) = 0{,}1000001$.

На самом деле это можно наглядно писать также так:

$$0{,}0011 \equiv \overline{\frac{3}{16}}; \quad 0{,}01011001 \equiv \overline{\frac{89}{256}}$$

$$0{,}1101 \equiv -\overline{\left(\frac{3}{16}\right)} = \overline{\left(1 - \frac{3}{16}\right)} = \overline{\left(\frac{13}{16}\right)};\ 0{,}1011001 \equiv \overline{\left(2\,\frac{89}{256}\right)} = \overline{\left(\frac{89}{128}\right)}$$

Проверить, что (H, τ) является D-структурой, оставляем читателю.

5. Теоремы

Для D-структуры $(D, \circ)$ покажем несколько основных утверждений. Чтобы сократить язык доказательств, будем через i обозначать условие применяя иденпотентность получаем; подобно для медиальности пишем m, для эластичности пишем e и для аксиомы (4) пишем d.

Теорема 1. Для любых $a, b, c, d, \in D$ выпольняется

1. $ac = bc \Rightarrow a = b$ (1)
2. $ab = c \Rightarrow a = cb$ (2)
3. $ab = b \Rightarrow a = b$ (3)
4. $a \cdot bc = ab \cdot ac, \; ab \cdot c = ac \cdot bc$ (4)
5. $ab \cdot a = a \cdot ba$ (5)
6. $ab \cdot ba = a(a \cdot ba)$ (6)
7. $(a \cdot bd)c = (a \cdot cd)b$ (7)

Доказательство. Утверждения доказываются постепенно.

1. Пусть $ac = bc$. Тогда $a \overset{e}{=} ac \cdot c = bc \cdot c \overset{e}{=} b$ итак (1) справедливо.
2. Пусть $ab = c$. Тогда $a \overset{e}{=} ab \cdot b = cb$ доказано (2).
3. Пусть $ab = b$. Тогда из два вытекает $a = bb \overset{i}{=} b$ доказано (3).
4. $a \cdot bc \overset{i}{=} aa \cdot bc \overset{m}{=} ab \cdot ac$ и $ab \cdot c \overset{i}{=} ab \cdot cc \overset{m}{=} ac \cdot bc$ доказано (4).
5. Если в (4) вставить $a = c$, то получаем (5).
6. $ab \cdot ba \overset{4}{=} (ab \cdot b)(ab \cdot a) \overset{e}{=} a \cdot (ab \cdot a) \overset{5}{=} a \cdot (a \cdot ba)$
7. $pc \cdot bd = pc \cdot bd \overset{2}{\Rightarrow} ((pc \cdot bd)bd)c = p$ подобно $((pb \cdot cd)cd)b = p$ пусть

$a = (pc \cdot bd) = (pb \cdot cd)$, вытекает $(a \cdot bd)c = (a \cdot cd)b$.

Определение 2. Уравнение $ax = a$ имеет точно два решения; одним является a и второе обозначим через a'. Отображение

$$v: D \to D, \; a \to a'$$

будем называть перпендикулярностью. Так как в модели 1 прямая a' перпендикулярная к прямой a. Итак

$$aa' = a \quad \text{и} \quad a \neq a' \tag{8}$$

Теорема 2. если $ab = ac$, то либо $b = c$, либо $b' = c$

Доказательство. Пусть $ab = ac$ и $b \neq c$. Обозначим $ab = d$, тогда $ac = d$. Так как по d уравнение $ax = d$ имеет точно два решения, то $ax = d$ влечет $x = b$, либо $x = c$. Теперь имеем

$$d \overset{i}{=} dd = ab \cdot ac \overset{4}{=} a \cdot bc$$

и по сказанному $bc = b$ либо $bc = c$. Вторая возможность влечет (применяя (3)) $b = c$, что противоречит предположению; и так $bc = b$, откуда по определению $c = b'$.

Следствие. Если c одно из двух решений уравнения $ax=b$, то c' второе из этих решений.

Теорема 3. Перпендикулярность есть инволютивностью, т. е.

$$v^2=1 \text{ и } (a')'=a \text{ для всех } a\in D$$

Доказательство. Для любого $a\in D$ выпольняется $a\cdot a'a \overset{5}{=} aa'\cdot a \overset{8}{=} a\cdot a \overset{i}{=}$ $\overset{i}{=} a$. Из определения 2 вытекает, что $a'a=a$ либо $a'a=a'$. Первое из соотношений противоречит (3) и (8). Получаем:

$$a'a=a' \tag{9}$$

Теперь уравнение $a'x=a$ имеет 2 решения, но a, a', $(a')'$ являются решениями и свер того $a\neq a'\neq(a')'$. Это возможно при условии $a=(a')'$, что потребовалось доказать.

Теорема 4. Отображение v является афтоморфизмом D-структуры.

Доказательство. Нам требуется доказать, что $(ab)'=a'b'$ для любых a, $b\in D$. Обозначая $ab=c$ получаем; уравнение $cx=c$ имеет точно два решения: $x=c$ и $x=c'$.

Не сложно проверить, что помимо c и c' также $a'b'$ является решением. Так как $c\neq c'$, то $a'b'=c'$ или $a'b'=c$. Последнее равенство приводит к противоречию $a'b' = ab = ab' \overset{1}{\Rightarrow} a' = a$. Итак $a'b' = (ab)'$

Определение 3. Элементы a, a' будем называть решениями первой степени уравнения $ax=a$.

Как $b_1, \ldots, b_k$ решения $(k-1)$-ой степени уравнеия $ax=a$, тогда решения ξ, ξ' всех уравнений $b_i\cdot\xi=b_j$ будем называть решениями k-той степени уравнения $ax=a$.

Лема. Пусть L множество решений k--той степени то card $L=2^k$.

Доказательство. *Математическая индукция.*

1. На основании определения 3.
2. Для обозначения 2^{n-1} решений удобно использовать модель. 10. Здесь выражены как числа $0, a_1 \ldots a_{n-1}$ где a_i равно 0 или 1. Наглядно, что решения уравнения $0, a_1 \ldots a_{n-1}\cdot x = 0, b_1 \ldots b_{n-1}$ имеют вид $0, 1c_1 \ldots c_{n-1}$; $0, 0c_1 \ldots c_{n-1}$. Но это уже уравнения решения k-той степени.

Существуют четыре типа уравнений:

I. $0, a_1 \ldots a_{n-2}\,0\cdot x=0, b_1 \ldots b_{n-2}\,0$
II. $0, a_1 \ldots a_{n-2}\,1\cdot x=0, b_1 \ldots b_{n-2}\,1$
III. $0, a_1 \ldots a_{n-2}\,0\cdot x=0, b_1 \ldots b_{n-2}\,1$
IV. $0, a_1 \ldots a_{n-2}\,1\cdot x=0, b_1 \ldots b_{n-2}\,0$

Уравнения I. и II. типа не допускают новых решений. У уравнений III. и IV. типа все решения новые. Это значит, что 2^{n-1} новых решений и 2^{n-1} решений $(k-1)$--ой степени. Из того вытекает, что card $L = 2^k$.

Теорема 5. Не существует конечная D-структура.

Доказательство ясно из предыдущей лемы.

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SÚHRN

D-ŠTRUKTÚRA

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D-štruktúrou nazývame idempotentný, mediálny a elastický grupoid, v ktorom rovnica $a \cdot x = b$ má vždy práve dve riešenia. Táto univerzálna algebra je zovšeobecnením zväzku priamok, v ktorom $a \cdot b$ je priamka súmerná s priamkou a vzhľadom na priamku b. Dokázali sme bezospornosť a nezávislosť axiomatického systému, ukázali sme šesť rôznych modelov. Päť dokázaných viet predstavuje základnú informáciu o tejto algebre.

RESUME

D-STRUCTURE

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Under D-structure we mean an idempotent, medial, and elastic grupoid in which any equation $a \cdot x = b$ has exactly two different solutions. This universal algebra is a generalization of a set of all straight lines passing through a given point 0 where $a \cdot b$ is the line symmetric to a line a whith respect to b. Consistency and idependence of axiomatic system is proved and six different models are described. Five theorems give basic information about this algebra.

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CHAIN IN S_1

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Let K be a field of all complex numbers and let S_1 be a onedimensional projective space over K.

Let 1. P_1, P_2, P_3 be three mutually different points of the space S_1, 2. $P=(x_\nu, y_\nu)$, $\nu=1, 2, 3$, in a coordinate system Σ of the space S_1, 3. $P=(x, y)$ be any point of the space S_1 (with regard to Σ).

A complex number defined by the expression

$$\delta=\delta(P, P_1, P_2, P_3)=\frac{\begin{vmatrix} x & y \\ x_2 & y_2 \end{vmatrix}\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} x & y \\ x_3 & y_3 \end{vmatrix}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}} \tag{1}$$

is called the cross ratio of the points P, P_1, P_2, P_3 taken in the given order, if the denominator in (1) is not zero. If the denominator is zero and numerator differs from zero we say that the cross ratio is infinite and we write $\delta=\infty$.

Remark. From 1. it is obvious that a case when both numerator and denominator are zero will not occur.

Definition 1. Let P_1, P_2, P_3 be three mutually different points of S_1. The set of all points $P \in S_1$ such that the cross ratio $\delta(P, P_1, P_2, P_3)$ is a real number or equals infinite is called a chain determined by points P_1, P_2, P_3 (in the diven order). We denote $\mathcal{R}$ (P_1, P_2, P_3) or $\mathcal{R}$ (P_ν).

Properties. 1. The cross ratio is invariant with respect to exchange of a coordinate system in S_1. If Σ is such a coordinate system in which $P_1=(1, 1)$, $P_2=(0, 1)$, $P_3=(1, 0)$ then $\delta(P, P_1, P_2, P_3)=\frac{x}{y}$, hence $\mathcal{R}(P_1, P_2, P_3)$ is the set of all $P=(x, y)$ for which $\frac{x}{y}$ is real or infinite.

2. Obviously holds

$$P_1, P_2, P_3 \in (P_1, P_2, P_3)$$

3. Besides:

a) $\mathcal{R}(P_1, P_2, P_3) = \mathcal{R}(P_1, P_3, P_2)$

b) $\mathcal{R}(P_1, P_2, P_3) = \mathcal{R}(P_2, P_1, P_3)$

To prove this it suffices to note:

a) $\delta(P, P_1, P_3, P_2) = \frac{y}{x}$ for $x \neq 0$ and $\delta = \infty$ for $x = 0$;

b) $\delta(P, P_2, P_1, P_3) = \frac{-x+y}{y}$ for $x \neq 0$ and $\delta = \infty$ for $x = 0$.

Result. $\mathcal{R}(P_1, P_2, P_3) = \mathcal{R}(P_{\nu_1}, P_{\nu_2}, P_{\nu_3})$, where (ν_1, ν_2, ν_3) is an arbitrary permutation of numbers 1, 2, 3.

4. Let Q_1, Q_2, Q_3 be three mutually different points of the chain $\mathcal{R}(P_1, P_2, P_3)$. Then $\mathcal{R}(Q_1, Q_2, Q_3) = \mathcal{R}(P_1, P_2, P_3)$.

Proof. Denote $Q_\nu = (x_\nu, y_\nu)$, $\nu = 1, 2, 3$ in Σ. Let us distinguish two cases:

1. For just one of the points Q_ν there is $y_\nu = 0$,
2. $y_\nu \neq 0$, $\nu = 1, 2, 3$.

Case 1. With regard to Property 3 we shall suppose $y_3 = 0$, hence $y_1 \neq 0$, $y_2 \neq 0$. From the assumption on Q_ν it follows $\frac{x_1}{y_1} = r_1$; $\frac{x_2}{y_2} = r_2$, where r_1, r_2 are reals and $r_1 \neq r_2$. Let $Q \in \mathcal{R}(Q_1, Q_2, Q_3)$ and $Q = (x, y)$. Then the cross ratio $\delta(Q, Q_1, Q_2, Q_3) = \frac{x - r_2 y}{y(r_1 - r_2)}$ is real if $y \neq 0$, and is equal to ∞ if $y = 0$. In both cases $Q \in \mathcal{R}(P_1, P_2, P_3)$. Conversely, let $P \in \mathcal{R}(P_1, P_2, P_3)$ and $P = (x, y)$. Then $\delta(P, P_1, P_2, P_3) = \frac{x}{y}$ is real if $y \neq 0$, and $\delta = \infty$ if $y = 0$. In both cases $P \in \mathcal{R}(Q_1, Q_2, Q_3)$.

Case 2. From assumptions on Q_ν it follows that $\frac{x_\nu}{y_\nu} = r_\nu$, $\nu = 1, 2, 3$ is real and r_ν are mutually different. Let $Q \in \mathcal{R}(Q_1, Q_2, Q_3)$ and $Q = (x, y)$. Then the cross ratio

$$\delta(Q, Q_1, Q_2, Q_3) = \frac{x - r_2 y}{x - r_3 y} \cdot \frac{r_1 - r_3}{r_1 - r_2}$$

is real if $x - r_3 y \neq 0$ and $\delta = \infty$ if $x - r_3 y = 0$. Here we distinguish two cases: a) $y = 0$, *b*) $y \neq 0$. In the case a) $Q = P_3$, so that $Q \in \mathcal{R}(P_1, P_2, P_3)$. In the case b) if $x - r_3 y = 0$, then $\frac{x}{y}$ is real, hance $Q \in \mathcal{R}(P_1, P_2, P_3)$; if $x - r_3 y \neq 0$, then $\frac{x - r_2 y}{x - r_3 y}$ is real. Therefore $\frac{\frac{x}{y} - r_2}{\frac{x}{y} - r_3}$ is a real number. This number differs from 1 (since $r_2 = r_3$

yields $Q_2 = Q_3$), therefore $\frac{x}{y}$ is real. Hence $Q \in \mathcal{R}(P_1, P_2, P_3)$. Let $P \in \mathcal{R}(P_1, P_2, P_3)$, and $P = (x, y)$. If $y = 0$, then

$$\delta(P, Q_1, Q_2, Q_3) = \frac{r_1 - r_3}{r_1 - r_2} \text{ is real, thus } P \in \mathcal{R}(Q_1, Q_2, Q_3).$$

If $y \neq 0$, then $\frac{x}{y}$ is real. Besides $\delta(P, Q_1, Q_2, Q_3) = \infty$ if $x - r_3 y = 0$, thus $P \in \mathcal{R}(Q_1, Q_2, Q_3)$ and if $x - r_3 y \neq 0$

$$\delta(P, Q_1, Q_2, Q_3) = \frac{x - r_2 y}{x - r_3 y} \cdot \frac{r_1 - r_3}{r_1 - r_2} = \frac{\frac{x}{y} - r_2}{\frac{x}{y} - r_3} \cdot \frac{r_1 - r_3}{r_1 - r_2}$$

consequently δ is real and $P \in \mathcal{R}(Q_1, Q_2, Q_3)$.

Definition 2. The subset $\mathcal{R} \subset S_1$ is said to be a chain if there are three mutually different points $P_1, P_2, P_3 \in S_1$ such that $\mathcal{R} = \mathcal{R}(P_1, P_2, P_3)$.

Definition 3 *(Cauchy's representation of point).* Let $\mathcal{L} = \mathcal{R} \times \mathcal{R} \cup \{\infty\}$. Elements of $\mathcal{L}$ are called points. The point ∞ is called an ideal element of $\mathcal{L}$. Let P be any point of S_1, and let $P = (x, y)$ in a fixed coordinate system Σ of the space S_1. In the case $y \neq 0$ a point $[a, b] \in \mathcal{L}$, where $a + bi = \frac{x}{y}$ $(a \in \mathcal{R},\ b \in \mathcal{R})$ is called Cauchy's representation of P. We denote it P^0 and we write $P^0 = [a, b]$. In the case $y = 0$ we take an ideal element of $\mathcal{L}$ for Cauchy's representation of P. Then we write $P^0 = \infty$.

Remark 1. The set $\mathcal{L}$ is called the Cauchy plane. Point ∞ is called the ideal point of the Cauchy plane, other points are called its points or non-ideal points.

Definition 4. Let P_1^0, P_2^0, P_3^0 be three points of $\mathcal{L}$ and

a) at least one of them is the ideal point
b) let $P_\nu^0 = [a_\nu, b_\nu]$ for $\nu = 1, 2, 3$ and

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0.$$

Then the points P_1^0, P_1^0, P_3^0 are said to be collinear.

Corollaries. If two of points P_ν^0 $(\nu = 1, 2, 3)$ coincide, then points P_1^0, P_2^0, P_3^0 are collinear.

Remark 2. Let P_1^0, P_2^0, be two non-ideal points of $\mathcal{L}$ which don't coincide. In the next let $\mathcal{H}(P_1^0, P_2^0)$ denote the subset of $\mathcal{L}$ defined as follows: $P^0 \in \mathcal{H}$ if and only if points P^0, P_1^0, P_2^0 are collinear.

It follows immediately: $\{P_1^0, P_2^0\} \subset \mathcal{K}(P_1^0, P_2^0)$.

Remark 3. Let Q_1^0, Q_2^0 be two non-ideal points of the set $\mathcal{K}(P_1^0, P_2^0)$. Then $\mathcal{K}(P_1^0, P_2^0) = \mathcal{K}(Q_1^0, Q_2^0)$.

Proof. Let $P_1^0 = [a_1, b_1]$, $P_2^0 = [a_2, b_2]$, $Q_1^0 = [c_1, d_1]$, $Q_2^0 = [c_2, d_2]$. The assumption gives:

$$\begin{vmatrix} c_1 & d_1 & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} c_2 & d_2 & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$$

Hence

$$(c_1, d_1, 1) = \alpha(a_1, b_1, 1) + \beta(a_2, b_2, 1)$$
$$(c_2, d_2, 1) = \gamma(a_1, b_1, 1) + \delta(a_2, b_2, 1)$$

where $\alpha, \beta, \gamma, \delta \in \mathcal{K}$. Since $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0$ (in the other case $\gamma = \lambda\alpha$, $\delta = \lambda\beta$, $\lambda \in \mathcal{K}$ implies $Q_1^0 = Q_2^0$) it holds

$$(a_1, b_1, 1) = \alpha'(c_1, d_1, 1) + \beta'(c_2, d_2, 1)$$
$$(a_2, b_2, 1) = \gamma'(c_1, d_1, 1) + \delta'(c_2, d_2, 1)$$

Let now $X^0 \in \mathcal{K}(P_1^0, P_2^0)$; if $X^0 = \infty$ then $X^0 \in \mathcal{K}(Q_1^0, Q_2^0)$; if $X^0 \neq \infty$ and $X^0 = [x, y]$ then

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$$

Hence

$$\begin{vmatrix} x & y & 1 \\ c_1 & d_1 & 1 \\ c_2 & d_2 & 1 \end{vmatrix} = 0$$

so $X^0 \in \mathcal{K}(Q_1^0, Q_2^0)$. Similarly, $X^0 \in \mathcal{K}(Q_1^0, Q_2^0)$ gives $X^0 \in \mathcal{K}(P_1^0, P_2^0)$.

Definition 5. Let $\mathcal{M}$ be a subset of $\mathcal{L}$ defined as follows: There exist two non-ideal points P_1^0, $P_2^0 (P_1^0 \neq P_2^0)$ of $\mathcal{L}$ such, that $\mathcal{M} = \mathcal{K}(P_1^0, P_2^0)$. Then $\mathcal{M}$ is called a straight line of the Cauchy plane $\mathcal{L}$, more precisely — a straight line of $\mathcal{L}$ defined by points P_1^0, P_2^0.

Remark 4. From previous it follows that a straight line $\mathcal{M}$ is defined by any pair of its different non-ideal points.

Remark 5. From the definition 5 follows immediately that the ideal point $P^0 = \infty$ is a point of each straight line in the Cauchy plane.

Definition 6. Let $P_1, P_2, P_3 \in \mathcal{L}$ be three non collinear points, $P_\nu^0 = [a_\nu, b_\nu]$. The set $\mathcal{M}$ of all points $P^0 = [a, b]$ $(P \neq \infty)$ for which

$$\begin{vmatrix} a^2+b^2 & a & b & 1 \\ a_1^2+b_1^2 & a_1 & b & 1 \\ a_2^2+b_2^2 & a_2 & b_2 & 1 \\ a_3^2+b_3^2 & a_3 & b_3 & 1 \end{vmatrix} = 0$$

Then $\mathcal{M}$ is called a circle of the Cauchy plane.

Remark 6. Since points P_1^0, P_2^0, P_3^0 are non-collinear, they all are not ideal and it holds

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} \neq 0$$

Remark 7. From definition it is clear that P_1^0, P_2^0, $P_3^0 \in \mathcal{M}$.

Remark 8. None circle in Cauchy plane contains the ideal point.

Theorem 1. Let a chain R contain a point $U = (1, 0)$ (in a coordinate system Σ of the space S_1). Let P_1, P_2, P_3 be three points of the chain R. Then Cauchy's representations P_1^0, P_2^0, P_3^0 are collinear.

Proof. If at least one of the points P_v, $v = 1, 2, 3$ coincide with U, the proposition is clear by the Definition 4. If some of points P_v coincide, we get the same result.

Suppose $P_v \neq U$, $v = 1, 2, 3$ and no two of points P_v coincide; denote $P_v = (x_v, y_v)$ for $v = 1, 2, 3$ in the coordinate system Σ. Let $\frac{x_v}{y_v} = a_v + ib_v$ for $v = 1, 2, 3$ where a_v, b_v are real. From definition of a chain it follows that there exist three points Q_1, Q_2, Q_3 such that no two of them coincide and $\mathcal{R} = \mathcal{R}(Q_1, Q_2, Q_3)$. By property 4 $\mathcal{R} = \mathcal{R}(P_1, P_2, P_3)$ take place. Since $U \in \mathcal{R}$, $\delta(U, P_1, P_2, P_3)$ is a real number or ∞. But

$$\delta(U, P_1, P_2, P_3) = \frac{\begin{vmatrix} 1 & 0 \\ x_2 & y_2 \end{vmatrix} \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ x_3 & y_3 \end{vmatrix} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}} = \frac{y_2}{y_3} \cdot \frac{\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}}$$

As denominator differs from zero, δ is real. Thus

$$\delta(U, P_1, P_2, P_3) = \frac{\frac{x_1}{y_1} - \frac{x_3}{y_3}}{\frac{x_1}{y_1} - \frac{x_2}{y_2}} = \frac{(a_1 + ib_1) - (a_3 + ib_3)}{(a_1 + ib_1) - (a_2 + ib_2)} =$$

$$= \frac{[(a_1 - a_3)(a_1 - a_2) + (b_2 - b_3)(b_1 - b_3)]}{(a_1 - a_2)^2 + (b_1 - b_2)^2} +$$

$$+ \frac{i[(a_1 - a_2)(b_2 - b_3) - (a_1 - a_3)(b_1 - b_2)]}{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

is a real number. Therefore

$$(a_1-a_2)(b_1-b_3)-(a_1-a_3)(b_1-b_2)=\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}=0$$

which means that P_1^0, P_2^0, P_3^0 are collinear.

Theorem 2. Let $\mathcal{R}=\mathcal{R}(P_1, P_2, P_3)$ be a chain. If the Cauchy representations (in a given coordinate system Σ) P_1^0, P_2^0, P_3^0 of P_1, P_2, P_3 are collinear, then $\mathcal{R}$ contains the point $U=(1, 0)$.

Proof. From property 4 it is clear that $\mathcal{R}=\mathcal{R}(P_1, P_2, P_3)$. We distinguish two cases:

a) One of the points P_1^0, P_2^0, P_3^0 is the ideal point of the Cauchy plane. Then the assumption of the theorem is true.

b) None of points P_1^0, P_2^0, P_3^0 is the ideal point of the Cauchy plane. Denote $P_\nu^0=[a_\nu, b_\nu]$, for $\nu=1, 2, 3$ and count $\delta(U, P_1, P_2, P_3)$. It holds (see proof of Theorem 1):

$$\delta(U, P_1, P_2, P_3)=\frac{\dfrac{x_1}{y_1}-\dfrac{x_3}{y_3}}{\dfrac{x_1}{y_1}-\dfrac{x_2}{y_2}}=\frac{(a_1+ib_1)-(a_3+ib_3)}{(a_1+ib_1)-(a_2+ib_2)}=$$

$$=\frac{(a_1-a_3)(a_1-a_2)+(b_1-b_3)(b_1-b_2)}{(a_1-a_2)^2+(b_1-b_2)^2}$$

Therefore δ is a real number and hence $U\in\mathcal{R}$.

Consequence. If $\mathcal{R}(P_1, P_2, P_3)=\mathcal{R}(Q_1, Q_2, Q_3)$ then the collinearity of P_1^0, P_2^0, P_3^0 imply the collinearity of Q_1^0, Q_2^0, Q_3^0.

Definition 7. The set of all Cauchy's representations of points of a chain $\mathcal{R}$ is called the Cauchy's representation of a chain $\mathcal{R}$. We denote it $\mathcal{R}^0$.

Theorem 3. Cauchy's representation $\mathcal{R}^0$ of a chain $\mathcal{R}$ is a straight line if and only if $\mathcal{R}$ and the point $U=(1, 0)$ are incident (with respect to given coordinate system Σ).

Proof. Let $\mathcal{R}$ be a chain containing the point $U=(1, 0)$. The there exist three different points P_1, P_2, P_3 such that $\mathcal{R}=\mathcal{R}(P_1, P_2, P_3)$. Let us assume $P_1\neq U$, $P_2\neq U$. Let X be any point of $\mathcal{R}$. From Theorem 1 it follows: X^0, P_1^0, P_2^0 are collinear, so that the point X^0 and a straight line defined by points P_1^0, P_2^0 are incident. Conversely, if a point Y^0 and a straight line defined by P_1^0, P_2^0 are incident then Y^0, P_1^0, P_2^0 are collinear. The cases $Y^0=\infty$, or $Y^0=P_1^0$, or $Y^0=P_2^0$ are obvious. In the rest of cases $\delta(U, Y, P_1, P_2)$ is a real number (see Proof of Theorem 2). Therefore $Y\in\mathcal{R}$. Let $\mathcal{R}^0$ be a straight line of $\mathcal{L}$, then $\mathcal{R}^0$ and the ideal point of the Cauchy plane are incident. Thus $\mathcal{R}$ and the point $U=(1, 0)$ are incident.

Theorem 4. Let $\mathcal{M}$ be a subset of S_1 such that its Cauchy's representation $\mathcal{M}^0$ is a straight line of $\mathcal{L}$ (in a coordinate system Σ of the space S_1). Then $\mathcal{M}$ is a chain (by Theorem 3 $\mathcal{M}$ and the point $U=(1, 0)$ in Σ are incident).

Proof. Since M^0 is a straight line of $\mathcal{L}$ there exist two non-ideal points P_1^0, $P_2^0 \neq P_1^0$ of $\mathcal{L}$ such that $\mathcal{M}^0 = \mathcal{K}(P_1^0, P_2^0)$. Denote $P_1^0 = [a_1, b_1]$, $P_2^0 = [a_2, b_2]$. Let $P_3^0 = \infty$ (the ideal point of $\mathcal{L}$). Then $P_3^0 \in \mathcal{M}$, so that $\mathcal{M}$ is incident with each of points P_1, P_2, $P_3=(1, 0)$, while no two of this points coincide. We are going to prove: $\mathcal{M} = \mathcal{R}(P_1, P_2, P_3)$. Denote $P_1=(x_1\ y_1)$, $P_2=(x_2, y_2)$. Let $P=(x, y)$ be any point of $\mathcal{M}$. Then $P^0 \in \mathcal{M}^0$ and we have:

$$\delta(P, P_1, P_2, P_3) = \frac{y_1}{y} \cdot \frac{y_2 x - y x_2}{x_1 y_2 - y_1 x_2} \quad \text{for} \quad y \neq 0$$

and

$$\delta(P, P_1, P_2, P_3) = \infty \quad \text{for} \quad y = 0$$

In other case, clearly $P \in \mathcal{R}(P_1, P_2, P_3)$. In the first case

$$\delta(P, P_1, P_2, P_3) = \frac{\frac{x}{y} - \frac{x_2}{y_2}}{\frac{x_1}{y_1} - \frac{x_2}{y_2}} = \frac{(a+b\mathrm{i}) - (a_2 + b_2\mathrm{i})}{(a_1 + b_1\mathrm{i}) - (a_2 + b_2\mathrm{i})} =$$

$$= \frac{(a-a_2)(a_1-a_2) + (b-b_2)(b_1-b_2) + \mathrm{i}[(a_1-a_2)(b-b_2) - (a-a_2)(b_1-b_2)]}{(a_1-a_2)^2 + (b_1-b_2)^2}.$$

Since $P^0 \in \mathcal{M} = \mathcal{K}(P_1^0, P_2^0)$ it holds

$$0 = \begin{vmatrix} a & b & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = (a_1 - a_2)(b_2 - b) - (b_1 - b_2)(a_2 - a)$$

Thus $\delta(P, P_1, P_2, P_3)$ is a real number. Therefore $P \in \mathcal{R}(P_1, P_2, P_3)$. Let now $P=(x, y)$ be a point of S_1 such that $P \in \mathcal{R}(P_1, P_2, P_3)$. Then

$$\delta(P, P_1, P_2, P_3) = \frac{y_1}{y} \cdot \frac{x y_2 - y x_2}{x_1 y_2 - y_1 x_2} \quad \text{for} \quad y \neq 0$$

and

$$\delta(P, P_1, P_2, P_3) = \infty \quad \text{for} \quad y = 0$$

In the second case P^0 is the ideal point of $\mathcal{L}$ so that $P^0 \in \mathcal{M}^0$ and $P \in \mathcal{M}$. In the first case we have

$$\delta(P, P_1, P_2, P_3) = \frac{(a-a_2)(a_1-a_2)+(b-b_2)(b_1-b_2)}{(a_1-a_2)^2+(b_1-b_2)^2} +$$

$$+\frac{i[(a_1-a_2)(b-b_2)-(b_1-b_2)(a-a_2)]}{(a_1-a_2)^2+(b_1-b_2)^2}$$

As δ is a real number its imaginary part equals zero, so:

$$\begin{vmatrix} a & b & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$$

Therefore $P^0 \in \mathcal{K}(P_1^0, P_2^0) = \mathcal{M}^0$ thus $P \in \mathcal{M}$.

Theorem 5. Let a chain $\mathcal{R}$ and a point $U=(1, 0)$ are not incident (in relation to a coordinate system Σ). Then its Cauchy's representation $\mathcal{R}^0$ is a circle (in relation to the coordinate system Σ).

Proof. Since $\mathcal{R}$ is a chain there exist three points P_1, P_2, P_3 such, that no two of them coincide and $\mathcal{R} = \mathcal{R}(P_1, P_2, P_3)$. As P_1, P_2, $P_3 \in \mathcal{R}$, points P_1^0, P_2^0, P_3^0 are incident with $\mathcal{R}^0$ and they all are non-ideal and not collinear; otherwise a chain $\mathcal{R}$ should be incident with a point $U=(1, 0)$. In the next we shall use the notation Δ

$$\Delta = \begin{vmatrix} a^2+b^2 & a & b & 1 \\ a_1^2+b_1^2 & a_1 & b_1 & 1 \\ a_2^2+b_2^2 & a_2 & b_2 & 1 \\ a_3^2+b_3^2 & a_3 & b_3 & 1 \end{vmatrix}$$

Let $P^0 \in \mathcal{R}^0$, then $P \in \mathcal{R}$ and $\delta(P, P_1, P_2, P_3)$ is a real number or ∞. In the case $P = P_3 : P^0 = P_3^0$ and $a = a_3$, $b = b_3$, therefore $\Delta = 0$. In the case $P \neq P_3$ the denominator of the expresion

$$\delta(P, P_1, P_2, P_3) = \frac{\begin{vmatrix} x & y \\ x_2 & y_2 \end{vmatrix} \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} x & y \\ x_3 & y_3 \end{vmatrix} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}}$$

is not zero and $\delta(P, P_1, P_2, P_3)$ is real. But

$$\delta(P, P_1, P_2, P_3) = \frac{\begin{vmatrix} \frac{x}{y} & 1 & \frac{x_1}{y_1} & 1 \\ \frac{x_2}{y_2} & 1 & \frac{x_3}{y_3} & 1 \end{vmatrix}}{\begin{vmatrix} \frac{x}{y} & 1 & \frac{x_1}{y_1} & 1 \\ \frac{x_3}{y_3} & 1 & \frac{x_2}{y_2} & 1 \end{vmatrix}} =$$

$$=\frac{\begin{vmatrix} a+bi & 1 \\ a_2+b_2i & 1 \end{vmatrix} \begin{vmatrix} a_1+b_1i & 1 \\ a_3+b_3i & 1 \end{vmatrix}}{\begin{vmatrix} a+bi & 1 \\ a_3+b_3i & 1 \end{vmatrix} \begin{vmatrix} a_1+b_1i & 1 \\ a_2+b_2i & 1 \end{vmatrix}}=$$

$$=\frac{[(a-a_2)(a_1-a_3)-(b-b_2)(b_1-b_3)]+i[(a_1-a_3)(b-b_2)+(a-a_2)(b_1-b_3)]}{[(a-a_3)(a_1-a_2)-(b-b_3)(b_1-b_2)]+i[(a_1-a_2)(b-b_3)+(a-a_3)(b_1-b_2)]}=$$

$$=\frac{A+iB}{C+iD}=\frac{(AC+BD)+i\left(\begin{vmatrix} A & B \\ C & D \end{vmatrix}\right)}{C^2+D^2}$$

Hence

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix}=0$$

but

$$-\begin{vmatrix} A & B \\ C & D \end{vmatrix}=\begin{vmatrix} a^2+b^2 & a & b & 1 \\ a_1^2+b_1^2 & a_1 & b_1 & 1 \\ a_2^2+b_2^2 & a_2 & b_2 & 1 \\ a_3^2+b_3^2 & a_3 & b_3 & 1 \end{vmatrix}$$

The proof of the last statement is lenghtly but elementary. Therefore $\Delta=0$.

Let now P^0 ba a non-ideal point of $\mathscr{L}$ and let for $P^0=[a, b]$, $P_v^0=[a_v, b_v]$, $v=1, 2, 3$, holds $\Delta=0$. Denote $P=(x, y)$. If $P=P_3$, then $P\in\mathscr{R}(P_1, P_2, P_3)$. If $P\neq P_3$, then

$$\delta(P, P_1, P_2, P_3)=\frac{\begin{vmatrix} x & y \\ x_2 & y_2 \end{vmatrix} \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} x & y \\ x_3 & y_3 \end{vmatrix} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}}$$

with the denominator not being zero, so that

$$\delta(P, P_1, P_2, P_3)=\frac{(AC+BD)+i\left(-\begin{vmatrix} A & B \\ C & D \end{vmatrix}\right)}{C^2+D^2}$$

by the first part of the proof.

But $0=\Delta=-\begin{vmatrix} A & B \\ C & D \end{vmatrix}$, therefore $\delta(P, P_1, P_2, P_3)$ is a real number, i. e. $P\in\mathscr{R}(P_1, P_2, P_3)$ and $P^0\in\mathscr{R}^0$.

Theorem 6. Let $\mathscr{M}$ be a circle of the Cauchy plane $\mathscr{L}$. Then there exist a chain $\mathscr{R}\subset S_1$ such that $\mathscr{R}^0=\mathscr{M}$.

Proof. As $\mathcal{M}$ is a circle of $\mathcal{L}$ there exist three non-collinear points P_1^0, P_2^0, P_3^0 of $\mathcal{L}$ such, that $P^0 \in \mathcal{L} \wedge P^0 \in \mathcal{M} \Leftrightarrow$ a) P^0 is non-ideal point, b) for $P^0 = [a, b]$, $P_\nu^0 = [a_\nu, b_\nu]$, $\nu = 1, 2, 3$ it holds $\Delta = 0$. Consider the subset $\mathcal{R}(P_1, P_2, P_3) \subset \subset S_1 \cdot \mathcal{R}(P_1, P_2, P_3)$ is a chain of S_1; denote it $\mathcal{R}$. Let $P^0 \in \mathcal{R}^0$; if $P^0 = P_3^0$ we have $P^0 \in \mathcal{M}$; if $P^0 \neq P_3^0$ we have $P \in \mathcal{R} = \mathcal{R}(P_1, P_2, P_3)$, thus $\delta(P, P_1, P_2, P_3)$ is a real number. But

$$\delta(P, P_1, P_2, P_3) = \frac{(AC + BD) + \mathrm{i}\left(-\begin{vmatrix} A & B \\ C & D \end{vmatrix}\right)}{C^2 + D^2}$$

Therefore $-\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0$, i. e. $\Delta = 0$, and $P^0 \in \mathcal{M}$.

Let now $P^0 \in \mathcal{M}$; then P^0 is not the ideal point and $\Delta = 0$. If $P^0 = P_3^0$, then $P = P_3$, $P \in \mathcal{R}(P_1, P_2, P_3) = \mathcal{R}$; therefore $P^0 \in \mathcal{R}^0$. If $P^0 \neq P_3^0$, then $P \neq P_3$, $P \in \mathcal{R}$ $(P_1, P_2, P_3) = \mathcal{R}$; therefore $P^0 \in \mathcal{R}^0$. Hence $\delta(P, P_1, P_2, P_3)$ is a real number, so $P \in \mathcal{R}(P_1, P_2, P_3) = \mathcal{R}$ and $P^0 \in \mathcal{R}^0$.

Theorem 7. Let $\mathcal{R}^0$ be a Cauchy's representations of a chain $\mathcal{R} \subset S_1$. Then $\mathcal{R}^0$ is a straight line if $U = (1, 0) \in \mathcal{R}^0$ and $\mathcal{R}^0$ is a circle if $U = (1, 0) \in \mathcal{R}$.

Proof. Follows from theorems 3, 4, 5, 6. Let γ be a set of all chains of the space S_1.

Let E_3^* be a three dimensional Euclidean space completed by the ideal elements and let $G(0, X, Y, Z)$ be its rectangular coordinate system with an origin 0. Let $\varkappa$ be a unit sphere of E_3^* with the centre in 0. Symbols int $\varkappa$ and ext $\varkappa$ denote the sets of all interior and exterior points of $\varkappa$ respectively. Now we are going to describe an important mapping $\varphi: \gamma \to \text{ext } \varkappa$.

Construction. Consider a plane (XY) of E_3^*. Let all its ideal points coincide; we have a model of the Cauchy plane. Denote it $[XY]$. Let Σ is a coordinate system in S_1. Let $P = (x, y)$ be any point of S_1, and let $P^0 = [a, b]$ be the Cauchy's representation of P in the case $y \neq 0$. By the same symbol we denote a point of $[XY]: P^0 = (a, b, 0)$ with respect to $G(0, X, Y, Z)$. The stereographic projection of the point P^0 from $S = (0, 0, 1)$ on $\varkappa$ will be called the Riemann's representation of P, and denoted P^r. In the case $y = 0$ let the Riemann's representation of P be S. It is well-known that just defined representation of S_1 is a bijective transformation of S_1 into $\varkappa$ (clearly depending on Σ). Let now $\mathcal{R} \in \gamma$; we distinguish two cases:

1. $\mathcal{R}$ and a point $U = (1, 0)$ are incident. Then the Cauchy's representation of $\mathcal{R}^0$ in $[XY]$ is a straight line and therefore the Riemann's representation of the chain $\mathcal{R}$ is a circle of a plane ϱ (ϱ and S are incident). The pole R of this plane with respect to $\varkappa$ is clearly a point of ext $\varkappa$. We cane take a point R for the representation $\varphi(\mathcal{R})$ of the chain $\mathcal{R}$.

2. $\mathcal{R}$ and a point $U = (1, 0)$ are not incident. Then the Cauchy's representation of $\mathcal{R}^0$ in $[XY]$ is a circle. Therefore the Riemann's representation of the chain

$\mathcal{R}$ is a circle in a plane ϱ (ϱ is not incident with S). Its pole R with respect to $\varkappa$ is clearly a point of ext $\varkappa$. As in 1. take a point R for the representation $\varphi(\mathcal{R})$ of the chain $\mathcal{R}$.

Just defined representation of the set γ (1, 2) is clearly a bijective transformation of γ on ext $\varkappa$. Denote it φ, i. e. $\varphi(\mathcal{R})=R$.

Theorem 8. Representation φ defined above has the following property: Let p be a straight line of the space E_3^*. Then the set of all chains which are the representation of the points of $\{p \cap \text{ext}\, \varkappa\}$ in φ^{-1} is a set of coaxal circles in $[XY]$. The set is

a) elliptic — if p intersect $\varkappa$ in two distinct points;

b) hyperbolic — if p does not intersect $\varkappa$;

c) parabolic — if p is a tangent to $\varkappa$.

Conversely. Let us consider a set of coaxal circles in $[XY]$. Then their representation in φ are points of $\{p \cap \text{ext}\, \varkappa\}$, where p is a straight line which a) intersect $\varkappa$ in two non-incident points (elliptic case), b) does not intersect $\varkappa$ (hyperbolic case), c) is a tangent to $\varkappa$ (parabolic case).

Proof. *Case 1.* The straight line p intersects $\varkappa$ in two points Q_1, $Q_2 \neq Q_1$. Then its conjugate polar line p' in E_3^* does not intersect $\varkappa$. Let one of points Q_1, Q_2 be the point $S=(0, 0, 1)$. In such a case the set of circles in which polar planes of points of $\{p \cap \text{ext}\, \varkappa\}$ intersect $\varkappa$ is projected from S into a set of concetric circles of $[XY]$ whose common centre is the point of intersection of p and the plane $[XY]$ ([3]).

Let none of points Q_1, Q_2 coincide with S. In such a case polar planes of points of $\{p \cap \text{ext}\, \varkappa\}$ intersect $\varkappa$ in circles, whose planes are incident with the conjugate polar p'. This circles are projected from S into an elliptic set of coaxal circles of $[XY]$ with the radical axis in the projection of the straight line p' from the point S ([3]). If the intersection circle and the point S are incident then it is projected from S into a straight line (the radical axis).

Case 2. The straight line p does not intersect $\varkappa$. In this case its conjugate polar line intersect $\varkappa$ in two points Q_1, $Q_2 \neq Q_1$. Let one of points Q_1, Q_2 be the point S (i. e. p is incident with a tangent plane of the point S to $\varkappa$). In this case polar planes of all points of the straight line p are incident with S, and so are the intersection circles. Therefore they are projected from S into pencil straight lines with the centre in the point of intersection $Q_1Q_2 \cap [XY]$.

Let none of points Q_1, Q_2 coincide with S. In this case polar planes of points of the straight line p are not incident with the point S. Therefore the intersection circles are projected from S into the hyperbolic set of coaxal circles of $[XY]$ with the limiting points in the projections of points Q_1, Q_2 from S into $[XY]$.

Case 3. The straight line p is a tangent to $\varkappa$; denote its tangent point by T.In this case the conjugate polar line p' is incident with a tangent plane of T. Moreover, p' and T are incident and p' is perpendicular to p.

Let T and S coincide. In this case polar planes of points $P \neq T$ of the straight line p are incident with the straight line p' and intersect $\varkappa$ in circles, which being incident with S an projected from S into the parallel straight lines of $[XY]$ (as p' is parallel to $[XY]$).

Let now $T \neq S$; in this case polar planes of points $P \neq T$ of the straight line p intersect $\varkappa$ in circles with a common tangent p' in T. Therefore they are projected from S into the set of circles of $[XY]$ with a common tangent t in projection of the straight line from S.If the intersection circle is incident with S (in only one case), it is projected from S into the straight line t.

The converse assumption follows from well-known properties of the stereographic projection:

1. The stereographic projection of a straight line is a circle on $\varkappa$ passing through the centre S of the projection. The stereographic projection of a circle is a circle on $\varkappa$ which does not pass through the centre S.

2. The stereographic projection of the set of coaxal circles of $[XY]$ is the set of circles of an axial pencil. The axis of the pencil intersects $\varkappa$ in two points Q_1, $Q_2 \neq Q_1$ (in the case of the hyperbolic set) or does not intersect $\varkappa$, or is tangent to $\varkappa$ (in the case of the elliptic or the parabolic set).

3. The stereographic projection of a pencil of straight lines of $[XY]$ with the centre in a non-ideal point O, or the stereographic projection of a pencil of parallel straight lines of $[XY]$ is a set of circles of an axial pencil, which are incident with S. The axis of the pencil is SO in the first case; in the second case it is incident with S and parallel to $[XY]$, i. e. it is incident with a tangent plane of S to $\varkappa$.

From above it follows that poles of planes of an axial pencil (which are incident with the intersection circles with respect to $\varkappa$) are incident with a conjugate polar line to the axis of the pencil and have the property from the theorem.

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SÚHRN

REŤAZEC V S_1

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Práca študuje reťazce v jednorozmernom projektívnom priestore nad telesom komplexných čísiel. Študuje obraz reťazca v cauchyovskej rovine, ktorá je modelom komplexnej projektívnej priamky. V závere práce sa na základe predchádzajúcej metódy a výsledkov zavádza zobrazenie množiny všetkých reťazcov na guľovú plochu, ktorá je riemanovským modelom komplexnej projektívnej priamky a konštruuje sa zobrazenie množiny všetkých reťazcov na množinu vonkajších bodov vzhľadom na guľovú plochu v rozšírenom euklidovskom trojrozmernom priestore.

РЕЗЮМЕ

ЦЕПЬ В S_1

И. Тренчански, Братислава

В статье изучаются цепи в одномерном проективном пространстве над полом комплексных чисел. Изучается образ цепи в плоскости Коши, которая является моделью комплексной проективной прямой. В заключении этой статьи приводится на основании предшествующего метода и результатов отображение множества всех цепей на модель комплексной проективной прямой Римана и строится отображение множества всех цепей на множество внешних точек по отношению к сферической поверхности в расширенном пространстве Евклида.

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ON A MODIFICATION OF THE METHOD OF MAJORANTS

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The Cauchy problem for partial differential equations (in short p. d. e.) in the class of analytic functions has been solved by means of the method of undetermined coefficients and by the method of majorants. While the former gives a formal power series solution, the latter consists of finding a majorant power series for that formal solution. The majorant power series is a solution of a majorant p. d. e. satisfying majorant initial conditions. Thus the method of majorants can be roughly stated as the following principle: In the class of analytic functions the solution of the majorant Cauchy problem majorizes the solution of the given problem.

We shall show that suitably changing the notion of the majorant the same is true in a wider class C^∞ of functions. The Cauchy problem has been very intensively studied in that class, especially in function spaces of Gevrey type. A survey of the methods and results can be found in the paper [5]. Different formulations of that principle will be given especially for linear p. d. e. Some elements of the idea of modification of the mentioned method are contained in the paper [3] (Theorems 3 and 4) and in the thesis [4] (Theorem 3.1).

Notation and Preliminaries

Let Z be the set of all natural numbers, R^n the n-dimensional euclidean space, $R^1=R$, $R^+ = \{x\in R: x\geqq 0\}$.

Let $N, m, m_1+1, \dots, m_N+1\in Z$ and $t_0\in R$. Let $M = m(m_1+1)\dots(m_N+1)$.

Let $\alpha=(\alpha_0, \alpha_1, \dots, \alpha_N)$ be an $(N+1)$-dimensional multiindex, $|\alpha| = \sum_{i=0}^{N}\alpha_i$ its length. α determines the N-dimensional multiindex $(\alpha_1, \dots, \alpha_N)$ which will be denoted as α^0.

Let

$$A=\{(\alpha_0, \alpha_1, \dots, \alpha_N);\ 0\leqq\alpha_0\leqq m-1,\\ 0\leqq\alpha_1<\infty, \dots, 0\leqq\alpha_N<\infty\}$$

$$A^m = \{(\alpha_0, \alpha_1, \ldots, \alpha_N);\ 0 \leqq \alpha_0 \leqq m,$$
$$0 \leqq \alpha_1 < \infty, \ldots, 0 \leqq \alpha_N < \infty\}$$

$$A^0 = \{\alpha^0 : \alpha \in A\}$$

$$A_{m_1, m_2, \ldots, m_N} = \{(\alpha_0, \alpha_1, \ldots, \alpha_N);\ 0 \leqq \alpha_0 \leqq m-1,$$
$$0 \leqq \alpha_1 \leqq m_1, \ldots, 0 \leqq \alpha_N \leqq m_N\}$$

The set $A_{m_1, m_2, \ldots, m_N}$ (as well as A, A^m, A^0) can be well ordered, e. g. by the rule: for $\alpha, \beta \in A_{m_1, m_2, \ldots, m_N}$ $\alpha < \beta$ if and only if either $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$, $\alpha_0 > \beta_0$, or $|\alpha| = |\beta|$, $\alpha_0 = \beta_0$, $\ldots$, $\alpha_{j-1} = \beta_{j-1}$, $\alpha_j > \beta_j$ for a $j \in \{1, 2, \ldots, N\}$. Then $A_{m_1, m_2, \ldots, m_N}$ can be written in the form of a sequence $\alpha_1, \alpha_2, \ldots, \alpha_M$.

Let $G \in R^N$ be a region of points $x = (x_1, \ldots, x_N)$ such that there exists a nondecreasing sequence of bounded regions $\{G_q\}_{q=1}^{\infty}(\bar{G}_q \subset G_{q+1}$ for each $q \in Z)$ with the property $G = \bigcup_{q=1}^{\infty} \bar{G}_q$, $\bar{G}_q$ is the closure of G_q for each $q \in Z$.

Let $\langle t_0, \infty) \times G = H$, $\langle t_0, t_0 + p \rangle \times \bar{G}_q = H_{p,q}$ $(p, q \in Z)$, $\langle t_0, t_0 + r \rangle \times G = H_r (r \in R, r > 0)$.

Let $L(L^m)$ be the linear space of all real functions defined on H with the property

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial t^{\alpha_0} \partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \in C(H) \text{ for all } \alpha \in A \tag{1}$$

(for all $\alpha \in A^m$).

Hence $L^m \subset L$.

Let $u_i(x) : G \to R$, $i = 0, 1, \ldots, m-1$, be such that

$$D^{\alpha^0} u_i \in C(G) \quad \text{for all} \quad \alpha^0 \in A^0 \quad (\text{i. e. } u_i \in C^\infty(G)) \tag{2}$$

Let $B(t, x, y_1, \ldots, y_M) : H \times R^M \to R$. Consider a $(1+N+M)$-dimensional multiindex $\beta = (\beta_0, \beta_1, \ldots, \beta_N, \beta_{N+1}, \ldots, \beta_{N+M})$ where all $\beta_j \geqq 0$ are integers, $j = 0, 1, \ldots, N+M$. Then $\beta^0 = (\beta_1, \ldots, \beta_N, \beta_{N+1}, \ldots, \beta_{N+M})$ is the associated with β, $(N+M)$-dimensional multiindex.

Denote

$$D^{\beta^0} B = \frac{\partial^{|\beta^0|} B}{\partial x_1^{\beta_1} \ldots \partial x_N^{\beta_N} \partial y_1^{\beta_{N+1}} \ldots \partial y_M^{\beta_{N+M}}}$$

$(\beta_j \geqq 0$ are integers, $j = 1, \ldots, N+M)$.

Let the function B possess the following property

$$D^{\beta^0} B \in C(H \times R^M) \text{ for each } (N+M)\text{-dimensional multiindex} \tag{3}$$

Specially, $B \in C(H \times R^M)$.

The class of all functions $B: H \times R^M \to R$ satisfying the assumption (3) will be denoted by $\mathcal{B}$.

Formulation of the Problem

To find a solution $u \in L^m$ of the initial value problem

$$\frac{\partial^m u}{\partial t^m} = B(t, x, D^{\alpha_1}u, \ldots, D^{\alpha_M}u), \ (t, x) \in H \tag{4}$$

$$\frac{\partial^i u(t_0, x)}{\partial t^i} = u_i(x), \ x \in G, \ i = 0, \ldots, m-1 \tag{5}$$

Remarks 1. The p. d. e. (4) is in the variable t of the order m, in the variable x_i of the order m_i, $i = 1, \ldots, N$.

2. Neither the equation (4), nor the initial conditions (5) (under weaker conditions on u_i, $i = 0, \ldots, m-1$) require from the solution u to have the partial derivatives with respect to x_i, $i = 1, \ldots, N$, of all orders. This demand has been made only for the sake of simplifying the future considerations.

Lemma 1. Let $u \in L$ (i. e. u satisfies the condition (1)) and let B satisfy (3). Then the composite function $B_1 = B \circ u$ defined in H by the relation

$$B_1(t, x) = B[t, x, D^{\alpha_1}u(t, x), \ldots, D^{\alpha_M}u(t, x)]$$

shows the following property:

For each $\alpha^0 \in A^0$

$$D^{\alpha^0}B_1 = \sum_{|\beta^0| \leqq |\alpha^0|} D^{\beta^0}B[t, x, D^{\alpha_1}u(t, x), \ldots, D^{\alpha_M}u(t, x)] \cdot$$

$$\cdot P_{\beta^0}(D^{\alpha_a}u(t, x), \ldots, D^{\alpha_c}u(t, x)) \tag{6}$$

where P_{β^0} are polynomials with nonnegative coefficients which are independent on B and u in the variables $D^{\alpha_a}u, \ldots, D^{\alpha_c}u$ such that the components α_i, $i = 0, 1, \ldots, N$, of the multiindices $\alpha_a, \ldots, \alpha_c$ satisfy the inequalities

$$0 \leqq \alpha_0 \leqq m-1, \ 0 \leqq \alpha_i \leqq m_i + |\alpha^0|, \qquad i = 1, \ldots, N \tag{7}$$

P_{β^0} are polynomials of at most $m \prod_{i=1}^{N} (m_i + |\alpha^0| + 1)$ variables and of the degree at most $|\alpha^0|$.

Proof. By mathematical induction in the length of multiindex $|\alpha^0|$. For $|\alpha^0| = 0$, $D^{\alpha^0}B_1 = B \cdot 1$ (for short, the argument of B is not written). When $|\alpha^0| = 1$, we have

$$\frac{\partial B_1}{\partial x_i}=\frac{\partial B}{\partial x_i}+\frac{\partial B}{\partial u}\frac{\partial u}{\partial x_i}+\frac{\partial B}{\partial\frac{\partial u}{\partial t}}\frac{\partial^2 u}{\partial t\,\partial x_i}+\frac{\partial B}{\partial\frac{\partial u}{\partial x_1}}\cdot\frac{\partial^2 u}{\partial x_1\,\partial x_i}+\ldots+$$

$$+\frac{\partial B}{\partial\frac{\partial u}{\partial x_N}}\frac{\partial^2 u}{\partial x_N\,\partial x_i}+\ldots+\frac{\partial B}{\partial\frac{\partial^{m-1}u}{\partial t^{m-1}}}\frac{\partial^m u}{\partial t^{m-1}\,\partial x_i}+\ldots+$$

$$+\frac{\partial B}{\partial\frac{\partial^{m_1}u}{\partial x_1^{m_1}}}\frac{\partial^{m_1+1}u}{\partial x_1^{m_1}x_i}+\ldots+\frac{\partial B}{\partial\frac{\partial^{m_N}u}{\partial x_N^{m_N}}}\frac{\partial^{m_N+1}u}{\partial x_N^{m_N}\,\partial x_i} \tag{8}$$

We see that the lemma is true for $|\alpha^0|=1$. Let us suppose that it is true for all multiindices of the length smaller or equal to k. Each multiindex of the length $k+1$ determines a derivative of the form $\frac{\partial}{\partial x_i}D^{\alpha^0}B_1$, where $|\alpha^0|=k$. If the operator $\frac{\partial}{\partial x_i}$ is applied to the equality (6) which is supposed to be true by the induction assumption, on the right hand side we get a sum of expressions which contain either $\frac{\partial}{\partial x_i}D^{\beta^0}B$, or $\frac{\partial}{\partial x_i}P_{\beta^0}$. The calculation of $\frac{\partial}{\partial x_i}D^{\beta^0}B$ gives a similar sum to (8). With respect to the fact that the derivative of a composite function of many variables is a linear combination with nonnegative coefficients of some derivatives, further that the derivative of a polynomial with nonnegative coefficients is again a polynomial with nonnegative coefficients we get that the application of the operator $\frac{\partial}{\partial x_i}$ to (6) yields an expression on the right hand side which is similar to the right hand side of (6) with the only difference that D^{β^0} can be of the order at most $k+1$, the degree of the polynomial P_{β^0} will increase by 1 and in all cases at most by 1 $\left(\text{due to } \frac{\partial}{\partial x_i}D^{\beta^0}B\right)$ and the components of multiindices $\alpha_a, \ldots, \alpha_e$ can increase by at most 1 (the increase comes from the members containing $\frac{\partial}{\partial x_i}P_{\beta^0}$). Thus the estimations (7) as well as the assertion about the degree of P_{β^0} are true. The coefficients of the polynomials P_{β^0} for $|\alpha^0|=1$ depend neither on B nor on u. From the induction assumption this statement remains valid for the multiindices with the length $k+1$. Thus the proof of the lemma is complete.

Continuity of the composite function $B\circ u$ implies

Lemma 2. The function $u\in L^m$ is a solution of the initial problem (4), (5) if and only if $u\in L$ is a solution of the integrodifferential equation

$$u(t,x)=\sum_{i=0}^{m-1}u_i(x)\frac{(t-t_0)^i}{i!}+\int_{t_0}^{t}\frac{(t-s)^{m-1}}{(m-1)!}B[s,x,D^{\alpha_1}u(s,x),\ldots D^{\alpha_M}u(s,x)]\,ds \tag{9}$$

This lemma leads us to consider the operator T defined in L by

$$Tu(t,x)=\sum_{i=0}^{m-1} u_i(x)\frac{(t-t_0)^i}{i!}+\int_{t_0}^{t}\frac{(t-s)^{m-1}}{(m-1)!}\times$$
$$\times B[s,x,D^{\alpha_1}u(s,x),\ldots,D^{\alpha_M}u(s,x)]\,ds \tag{10}$$

By Lemma 2, each fixed point of that operator is a solution of the problem (4), (5). With help of the same lemma we get that for each $\alpha\in A$ there exists and belongs to $C(H)$

$$D^{\alpha}Tu(t,x)=\sum_{i=\alpha_0}^{m-1} D^{\alpha^0}u_i(x)\frac{(t-t_0)^{i-\alpha_0}}{(i-\alpha_0)!}+$$
$$+\int_{t_0}^{t}\frac{(t-s)^{m-1-\alpha_0}}{(m-1-\alpha_0)!}\cdot D^{\alpha^0}B[s,x,D^{\alpha_1}u(s,x),\ldots,D^{\alpha_M}u(s,x)]\,ds \tag{11}$$

and

$$\frac{\partial^m Tu(t,x)}{\partial t^m}=B[t,x,D^{\alpha_1}u(t,x),\ldots,D^{\alpha_M}u(t,x)]\in C(H)$$

Hence

$$T\colon L\to L^m \tag{12}$$

In order to be able to continue in further considerations, we shall introduce in L the topology of a locally convex linear topological space by means of the system of seminorms

$$v(u;p,q,r)=\max_{\substack{|\alpha|\leq r\\ \alpha\in A}}\{\max_{(t,x)\in H_{p,q}}|D^{\alpha}u(t,x)|\}$$

for each $p,q,r\in Z$ (13)

which satisfies Hausdorff's axiom of separation. As usual, under the $\varepsilon(>0),p,q,r$ neighbourhood $U(u_0;\varepsilon,p,q,r)$ of the point u_0 the set $\{u\in L\colon v(u-u_0;p,q,r)<<\varepsilon\}$ will be understood. From the countability of the system (13) the metrizability of the space L follows. Convergence of a sequence $\{u_n\}_{n=1}^{\infty}$ in this space means the uniform convergence of each sequence $\{D^{\alpha}u_n\}_{n=1}^{\infty}$ for all $\alpha\in A$ on each $H_{p,q}$. The Cauchy sequence is recognized by the fact that each sequence of admissible derivatives is uniformly Cauchy sequence on each $H_{p,q}$ and hence, uniformly convergent. Therefore the considered space L (denoted by the same sign as the one without topology) is complete. On the basis of the example in [6] (p. 32) L is a quasi-normed space and hence, a Fréchet space. A set $S\subset L$ of functions u is bounded, when on each $H_{p,q}$ the set of $D^{\alpha}u$ for each $\alpha\in A$ is uniformly bounded.

Lemma 3. When the set $S\subset L^m$ is bounded in L and the set

$$\left\{D^{\alpha^0}\frac{\partial^m u}{\partial t^m}: u\in S\right\} \tag{14}$$

is uniformly bounded for each $\alpha^0\in A^0$ on each $H_{p,q}(p, q\in Z)$, then S is relatively compact.

Proof. Since L is metrizable, it suffices to prove that each sequence $\{u_n\}_{n=1}^{\infty}$, $u_n\in S$, contains a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ which is convergent and hence for each $\alpha\in A$ $\{D^\alpha u_{n_k}\}_{k=1}^{\infty}$ is uniformly convergent on each $H_{p,q}$. From the boundedness of S the uniform boundedness of each sequence $\{D^\alpha u_n\}_{n=1}^{\infty}$ on each $H_{p,q}$ for all $\alpha\in A$ follows. This, as well as the uniform boundedness of the set (14), implies the equicontinuity of $\{D^\alpha u_n\}_{n=1}^{\infty}$ on each $H_{p,q}$. On the basis of the Ascoli-Arzelà Theorem each sequence $\{D^\alpha u_n\}_{n=1}^{\infty}$ on each $H_{p,q}$ $(\alpha\in A,\ p,\ q\in Z)$ contains a subsequence which is uniformly convergent. There are countable many sequences $\{D^\alpha u_n\}$ and countable many sets $H_{p,q}$ (the set $A\times Z^2$ is countable), therefore applying the diagonal process of choice we can get a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ with the desired properties.

Lemma 4. The operator T defined by (10), is continuous and compact in L.

Proof. Continuity of T. Let $u_0\in L$ be arbitrary, but fixed and $Tu_0=v_0$. Let an arbitrary neighbourhood $U(v_0;\varepsilon,p,q,r)$ of the point v_0 be given. From (11) for each $u\in L$, $|\alpha|\leqq r$ it follows

$$D^\alpha Tu(t,x)-D^\alpha Tu_0(t,x)=$$

$$=\int_{t_0}^{t}\frac{(t-s)^{m-1-\alpha_0}}{(m-1-\alpha_0)!}\{D^{\alpha^0}(B\circ u)(s,x)-D^{\alpha^0}(B\circ u_0)(s,x)\}\,ds$$

By means of (6) we get that the expression in brackets can be written in the form

$$\sum_{|\beta^0|\leqq|\alpha^0|}\{D^{\beta^0}B[s,x,D^{\alpha_1}u(s,x),\dots,D^{\alpha_M}u(s,x)]\cdot$$

$$\cdot(P_{\beta^0}[D^{\alpha_a}u(s,x),\dots,D^{\alpha_c}u(s,x)]-P_{\beta^0}[D^{\alpha_a}u_0(s,x),\dots,D^{\alpha_c}u_0(s,x)])+$$

$$+(D^{\beta^0}B[s,x,D^{\alpha_1}u(s,x),\dots,D^{\alpha_M}u(s,x)]-$$

$$-D^{\beta^0}B[s,x,D^{\alpha_1}u_0(s,x),\dots,D^{\alpha_M}u_0(s,x)])\cdot P_{\beta^0}[D^{\alpha_a}u_0(s,x),\dots,D^{\alpha_c}u_0(s,x)]\} \tag{15}$$

Since $|\alpha^0|\leqq|\alpha|\leqq r$, by Lemma 1, the derivatives $D^{\beta^0}B$ in (15) occur at most of the order r and the derivatives $D^{\alpha_a}u,\dots,D^{\alpha_c}u$ at most of the order $r_1=m-1+\sum_{i=1}^{N}m_i+Nr$. From the continuity and boundedness of these functions as well as of the functions P_{β^0} on compact subsets of their domain it follows that there exists such a $\delta_1>0$ that for all $u\in U(u_0;\delta_1,p,q,r_1)$ the relation $Tu\in U(v_0;\varepsilon,p,q,r)$ is true.

Compactness of T. Let an arbitrary $H_{p,q}$ and an arbitrary $\alpha \in A$, $|\alpha| = r$, be given. When S is a bounded set, then the set of all $D^{\alpha}u$, $u \in S$, up to the order r_1 (with the above meaning) is uniformly bounded on $H_{p,q}$. Then from the continuity of the functions $D^{\beta^0}B$, on the basis of Lemma 1, it follows that also the functions $D^{\alpha^0}(B \circ u)$ for $u \in S$ are uniformly bounded on $H_{p,q}$, and hence TS is bounded in L, too. $D^{\alpha^0}\frac{\partial^m u}{\partial t^m} = D^{\alpha^0}(B \circ u)$ are by the above said uniformly bounded on $H_{p,q}$. Taking this and (12) into account, by Lemma 3 we get the relative compactness of TS.

With respect to the last Lemma the Tichonov fixed point theorem can be used for proving the existence of a fixed point of the operator T if a closed, convex and bounded set $S \subset L$ can be found such that T maps it into itself. This set may be determined by a modification of the method of majorants and by partial ordering of the space L.

Modification of the method of majorants

We shall say that $u_2 \in L$ majorizes $u_1 \in L$ if $|D^{\alpha}u_1(t, x)| \leqq D^{\alpha}u_2(t, x)$ for all $\alpha \in A$ and all $(t, x) \in H$.

In symbols,

$$u_1 \ll u_2$$

Similarly, if B_1, B_2 are two functions from the class $\mathscr{B}$, then $B_1 \ll B_2$ when and only when $|D^{\beta^0}B_1(t, x, y_1, \ldots, y_M)| \leqq D^{\beta^0}B_2(t, x, y_1, \ldots, y_M)$ for all admissible $(N+M)$ — dimensional multiindices and all points from $H \times R^M$.

If v_1, $v_2 \colon G \to R$ satisfy the conditions (2), then $v_1 \ll v_2$ if and only if $|D^{\alpha^0}v_1| \leqq D^{\alpha^0}v_2$ for each $\alpha^0 \in A^0$.

The most important properties of the relation $\ll$.

Let $u_j \in L$, $j = 1, 2, 3, 4$, $B_1, B_2 \in \mathscr{B}$. Then the following statements are true:

1. If $u_1 \ll u_2$, $u_2 \ll u_1$, then $u_1 = u_2$.
2. If $u_1 \ll u_2$, $u_2 \ll u_3$, then $u_1 \ll u_3$.
3. $u_1 \ll u_1$ if and only if $0 \ll u_1$.
4. If $u_1 \ll u_2$, $u_3 \ll u_4$, then $u_1 + u_3 \ll u_2 + u_4$.
5. If $u_1 \ll u_2$, $u_3 \ll u_4$, then $u_1 u_3 \ll u_2 u_4$.
6. If $u_1 \ll u_2$, $B_1 \ll B_2$, then $B_1 \circ u_1 \ll B_2 \circ u_2$.

The proof of the properties 1—5 is simple and will be omitted. The sixth property is based on Lemma 1 and its generalization if instead of α^0, α is considered and in its proof the nonnegativity of the coefficients of polynomials P_{β^0} as well as the nonnegativity and monotonicity of the derivatives $D^{\beta^0}B_2$ are used. Similar properties are shown by the majorant power series, see [2].

With respect to the property 3, the relation $\ll$ does not mean a partial ordering of the space L. However if we restrict ourselves to $L^+=\{u\in L: 0\ll u\}$, then this relation defines a partial ordering on L^+. This can be extended to L by introducing the relation $<$.

Let $u_1, u_2\in L$. We shall say that u_1 is a predecessor of u_2, in denotation $u_1<u_2$ if $D^\alpha u_1\leqq D^\alpha u_2$ for all $\alpha\in A$.

Similarly, if B_1, B_2 are two functions from the class $\mathcal{B}$, then $B_1<B_2$ if and only if $D^{\beta^0}B_1\leqq D^{\beta^0}B_2$ for all admissible $(N+M)$ — dimensional multiindices β^0.

Finally, if $v_1, v_2: G\to R$ satisfy the assumption (2), then $v_1<v_2$ when and only when $D^{\alpha^0}v_1\leqq D^{\alpha^0}v_2$ for each $\alpha^0\in A^0$.

From the properties of the relation $<$ we shall consider the following ones.

Let $u_j\in L$, $j=1, 2, 3, 4$, $B\in\mathcal{B}$. Then the following statements are true:

1. If $u_1<u_2$, $u_2<u_1$, then $u_1=u_2$.
2. If $u_1<u_2$, $u_2<u_3$, then $u_1<u_3$.
3. $u_1<u_1$ for each $u_1\in L$.
4. If $u_1<u_2$, $u_3<u_4$, then $u_1+u_3<u_2+u_4$.
5. If $u_1<u_2$, $0<u_3$, then $u_1u_3<u_2u_3$.
6. If $0<u_1<u_2$, $0<B$, then $B\circ u_1<B\circ u_2$.
7. $0<u_1$ if and only if $0\ll u_1$. Similarly, $0<B$ if and only if $0\ll B$.
8. If $0<u_1$, $0<u_2$, then $u_1<u_2$ if and only if $u_1\ll u_2$.

The proof of the property 6. follows from the sixth property of the relation $\ll$ if we take into consideration that from the assumption $0<u_1<u_2$, $0<B$ the statements $u_1\ll u_2$, $B\ll B$ follow.

From Lemma 2 and (10), (11) the following assertion follows which will be useful in the subsequent considerations and which will be given as

Lemma 5. Let the functions u_i, $i=0, \dots, m-1$, satisfy (2) and let $B\in\mathcal{B}$. Further, let

$$0\ll B,\ 0\ll u_i, \qquad i=0, \dots, m-1 \tag{16}$$

Then each solution u of the Cauchy problem (4), (5) satisfies $0\ll u$.

Generalized comparison test

By means of the foregoing lemmas the following theorem will be proved.

Theorem 1. Let the functions B, $B_1\in\mathcal{B}$ and let u_i, u_i^1, $i=0, \dots, m-1$, satisfy the assumption (2). Let

$$B\ll B_1,\ u_i\ll u_i^1, \qquad i=0, \dots, m-1 \tag{17}$$

Let there exist a solution $u^\dagger$ of the Cauchy problem

$$\frac{\partial^m u}{\partial t^m}=B_1(t,x,D^{\alpha_1}u,\ldots,D^{\alpha_M}u)\quad (t,x)\in H \tag{4_1}$$

$$\frac{\partial^i u(t_0,x)}{\partial t^i}=u_i^1(x),\ x\in G,\qquad i=0,\ldots,m-1 \tag{5_1}$$

Then there exists a solution u of the problem (4), (5) for which $u\ll u^\dagger$.

Proof. Let $S=\{u\in L: u\ll u^\dagger\}$. From Lemma 5 it follows that $u^\dagger\in S$ and hence $S\neq\emptyset$. Further it is true that S is a convex, closed and bounded subset of L. We shall show that $TS\subset S$. In fact, let $u\in S$ and $\alpha\in A$. On the basis of (11), of the assumptions of the theorem and of the property 6. of the relation $\ll$ we have

$$|D^\alpha Tu(t,x)|\leqq\sum_{i=\alpha_0}^{m-1}|D^{\alpha^0}u_i(x)|\frac{(t-t_0)^{i-\alpha_0}}{(i-\alpha_0)!}+$$

$$+\int_{t_0}^{t}\frac{(t-s)^{m-1-\alpha_0}}{(m-1-\alpha_0)!}\cdot|D^{\alpha^0}(B\circ u)(s,x)|\,\mathrm{d}s\leqq$$

$$\leqq\sum_{i=\alpha_0}^{m-1}D^{\alpha^0}u_i^1(x)\frac{(t-t_0)^{i-\alpha_0}}{(i-\alpha_0)!}+\int_{t_0}^{t}\frac{(t-s)^{m-1-\alpha_0}}{(m-1-\alpha_0)!}\cdot$$

$$\cdot D^{\alpha^0}(B_1\circ u^\dagger)(s,x)\,\mathrm{d}s=D^\alpha Tu^\dagger(t,x)=D^\alpha u^\dagger(t,x)\text{ for each }(t,x)\in H$$

Lemma 4 guarantees the fulfilment of all remaining assumptions of the Tichonov theorem, by which there exists in S at least one fixed point u of the operator T. By Lemma 2, u is a solution of the Cauchy problem (4), (5). As an element of S, u satisfies $u\ll u^\dagger$.

Since the functions B_1, u_i^1, $i=0,\ldots,m-1$, satisfy with (17) also (16), the problem arises what can be said about the existence of a solution of the Cauchy problem (4), (5) under the assumptions of Lemma 5. Except the trivial case when $B=0$ and all $u_i=0$, $i=0,\ldots,m-1$, for such functions B and u_i, $i=0,\ldots,m-1$, and two constants k_1, $k_2\in R^+$ the relation $k_1B\ll k_2B$ as well as $k_1u_i\ll k_2u_i$, $i=0,\ldots,m-1$, are true if and only if $k_1\leqq k_2$. With respect to this, from Theorem 1 the following corollary follows.

Corollary. Let the functions B and u_i, $i=0,\ldots,m-1$ satisfy the assumptions of Lemma 5. Then there exists a unique number λ_0, $0\leqq\lambda_0\leqq\infty$ such that for each $\lambda\geqq0$ satisfying $\lambda<\lambda_0$ there exists at least one solution of the Cauchy problem

$$\frac{\partial^m u}{\partial t^m}=\lambda B(t,x,D^{\alpha_1}u,\ldots,D^{\alpha_M}u)\qquad (t,x)\in H \tag{4_λ}$$

$$\frac{\partial^i u(t_0,x)}{\partial t^i}=\lambda u_i(x),\ x\in G,\qquad i=0,\ldots,m-1 \tag{5_λ}$$

and for λ, $\lambda>\lambda_0$, the problem (4_λ), (5_λ) has no solution.

Examples. 1. The solution u of the problem $\frac{du}{dt}=\lambda e^{u}$, $u(0)=\lambda$, for $\lambda>0$ is the function $u(t)=\log\frac{1}{e^{-\lambda}-\lambda t}$ which is not defined for all t, $0\leqq t<\infty$. Thus in the case of the functions $B=e^{u}$, $u_0=1$, $\lambda_0=0$.

2. Clearly for the problem $\frac{du}{dt}=u$, $u(0)=1$, $\lambda_0=\infty$.

3. The problem $\frac{du}{dt}=\lambda\frac{2}{\pi}\frac{1}{1+t^2}u^2$, $u(0)=\lambda$ has the unique solution $u(t)=$ $=\lambda\pi/(\pi-2\lambda^2 \operatorname{arc\,tg} t)$ which exists in $\langle 0,\infty)$ only for $-1\leqq\lambda\leqq 1$. Thus $\lambda_0=1$.

In the next theorem the relation $<$ will be used.

Theorem 2. Let the functions u_i, $i=0,\ldots,m-1$, satisfy (2) and let $B\in\mathscr{B}$. Then a necessary and sufficient condition that there exist a solution u of the initial problem (4), (5) is that there exist two functions u_1^*, $v_1^*\in L^m$ with the following properties:

1. $u_1^*<v_1^*$ (18)
2. For all $u_1, u_2\in S=\{u\in L: u_1^*<u<v_1^*\}$ and each $\alpha^0\in A^0$ the implication

$$u_1<u_2 \Rightarrow D^{\alpha^0}(B\circ u_1)\leqq D^{\alpha^0}(B\circ u_2) \text{ is true;} \tag{19}$$

3.
$$D^{\alpha^0}\frac{\partial^i u_1^*(t_0,x)}{\partial t^i}\leqq D^{\alpha^0}u_i(x)\leqq D^{\alpha^0}\frac{\partial^i v_1^*(t_0,x)}{\partial t^i}$$
$$\text{for each } \alpha^0\in A^0, x\in G \text{ and } i=0,\ldots,m-1 \tag{20}$$

4.
$$D^{\alpha^0}\frac{\partial^m u_1^*}{\partial t^m}\leqq D^{\alpha^0}(B\circ u_1^*);\; D^{\alpha^0}(B\circ v_1^*)\leqq$$
$$\leqq D^{\alpha^0}\frac{\partial^m v_1^*}{\partial t^m} \quad \text{for all} \quad \alpha^0\in A^0 \tag{21}$$

If such functions u_1^*, v_1^* exist, then the sequences $\{u_n^*\}_{n=1}^{\infty}$, $\{v_n^*\}_{n=1}^{\infty}$ defined by the relations

$$u_{n+1}^*=Tu_n^*,\; v_{n+1}^*=Tv_n^* \qquad (n\in Z) \tag{22}$$

where the operator T is defined by (10), show these properties:

1. $u_1^*<u_2^*<\ldots<u_n^*<\ldots<v_n^*<\ldots<v_2^*<v_1^*$
2. There exist $\lim_{n\to\infty}u_n^*=u^*$, $\lim_{n\to\infty}v_n^*=v^*$ in L;
3. $u_n^*<u^*<v^*<v_n^*$ for each $n\in Z$;
4. u^*, v^* are solutions of the problem (4), (5).

Proof. Necessary condition. Let there exist a solution u of the problem (4), (5). Then the functions $u_1^*=v_1^*=u$ satisfy (18) and, on the basis of (4), (5), the equalities in (20), (21). S is reduced to one point u and (19) is true for this element.

Sufficient condition. With respect to (10) and (11) from the assumption (19) it follows that for any two elements $u_1, u_2 \in S$ such that $u_1 < u_2$ the inequality $Tu_1 < Tu_2$ is true. Hence T is isotone in S. As $u^\dagger$ can be written in the form

$$u^\dagger(t,x) = \sum_{i=0}^{m-1} \frac{\partial^i u^\dagger(t_0,x)}{\partial t^i} \frac{(t-t_0)^i}{i!} + \\ + \int_{t_0}^{t} \frac{(t-s)^{m-1}}{(m-1)!} \frac{\partial^m u^\dagger(s,x)}{\partial t^m} \, ds \tag{23}$$

by comparing (23) with (10) and (11), on the basis of (20) and (21), we come to $D^\alpha u^\dagger \leqq D^\alpha T u^\dagger \leqq D^\alpha T v^\dagger \leqq D^\alpha v^\dagger$ for each $\alpha \in A$. Thus $u^\dagger < T u^\dagger < T v^\dagger < v^\dagger$. With respect to the monotony of T, this implies that the sequences $\{u_n^*\}_{n=1}^{\infty}$, $\{v_n^*\}_{n=1}^{\infty}$ defined by (22) show the property 1. Thus for each $\alpha \in A$ the sequence $\{D^\alpha u_n^*\}_{n=1}^{\infty}$ is nondecreasing and bounded from above by $D^\alpha v_m^*$ for an arbitrary $m \in Z$. Hence at each point $(t,x) \in H$ there exists

$$\lim_{n\to\infty} D^\alpha u_n^*(t,x) = u_\alpha(t,x) \leqq D^\alpha v_m^*(t,x) \qquad (\alpha \in A, m \in Z) \tag{24}$$

The sequence $\{u_n^*\}_{n=1}^{\infty}$ is bounded in L. By the compactness of T there exists a subsequence of the sequence $\{u_n^*\}_{n=1}^{\infty}$, say $\{u_{n_k}^*\}_{k=1}^{\infty}$ which is convergent in the space L, and hence for each $\alpha \in A$ the sequence $\{D^\alpha u_{n_k}^*\}_{k=1}^{\infty}$ converges on each $H_{p,q}$ uniformly and the limit function is u_α (on the basis of the uniqueness of the limit). Therefore u_α are continuous on H and, by Dini's theorem, the convergence of each sequence $\{D^\alpha u_n^*\}_{n=1}^{\infty}$ on each $H_{p,q}$ is uniform. Therefore there is a function $u^* \in L$ such that $u_\alpha = D^\alpha u^*$ and $\lim_{n\to\infty} u_n^* = u^*$ in L. The inequalities (24) give

$$u^* < v_m^* \quad \text{for each} \quad m \in Z \tag{25}$$

In a similar way we get $\lim_{n\to\infty} v_n^* = v^*$, whereby from (25) $u^* < v^*$ follows. This together with the inequalities $u_n^* < u^*$, $v^* < v_n^*$ shows that the properties 2. and 3. of the sequence $\{u_n^*\}_{n=1}^{\infty}$, $\{v_n^*\}_{n=1}^{\infty}$ are true.

From (10) and (22) we get

$$u_{n+1}^*(t,x) = \sum_{i=0}^{m-1} u_i(x) \frac{(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{m-1}}{(m-1)!} \cdot (B \circ u_n^*)(s,x) \, ds$$

where by the limit process, using the Lebesgue Dominated Convergence Theorem and taking into consideration the continuity of B we obtain that u^* is a solution of the equation (9) as well as of the initial problem (4), (5). The same is true about v^*.

If in addition to the assumptions of Theorem 2 $0 < B$, then, on the basis of the property 6. of the relation $<$, the operator T is isotone on the positive cone

$L^+ = \{u \in L: 0 < u\}$. If moreover $0 < u_i$, $i = 0, \ldots, m-1$, then $0 < T0$ and $u† = 0$ can be put in Theorem 2. In this case the following corollary is true.

Corollary. If in addition to the assumptions of Theorem 2 $0 < B$, $0 < u_i$, $i = 0, \ldots, m-1$, are true, then a necessary and sufficient condition for the existence of a solution to the Cauchy problem (4), (5) is that there exist a function $v† \in L^m$ showing the following properties:

1. $0 < v†$

2. $D^{\alpha^0} u_i(x) \leqq D^{\alpha^0} \dfrac{\partial^i v†(t_0, x)}{\partial t^i}$ for each $\alpha^0 \in A^0$, every $x \in G$ and each $i = 0, \ldots,$ $m-1$.

3. $D^{\alpha^0}(B \circ v†) \leqq D^{\alpha^0} \dfrac{\partial^m v†}{\partial t^m}$ for all $\alpha^0 \in A^0$.

If such a function exists, then the sequences $\{u_n^*\}_{n=1}^{\infty}$, $u† = 0$, $\{v_n^*\}_{n=1}^{\infty}$ defined by the relations (22), possess the properties 1—4 from Theorem 2.

Suitably modified Theorem 2 applies to ordinary differential equations. Let us consider the initial — value problem

$$\frac{d^m u}{dt^m} = f\left(t, u, \frac{du}{dt}, \ldots, \frac{d^{m-1}u}{dt^{m-1}}\right), \qquad t_0 \leqq t < \infty \tag{4_0}$$

$$\frac{d^i u(t_0)}{dt^i} = u_0^{(i)}, \qquad i = 0, \ldots, m-1 \tag{5_0}$$

under standart assumption $f(t, u_1, \ldots, u_m) \in C(<t_0, \infty) \times R^m)$; $u_0^{(i)} \in R$, $i = 0, \ldots, \ldots, m-1$. When by $L(L^m)$ the locally convex topological vector space $C_{m-1}(<t_0, \infty))$ $(C_m(<t_0, \infty)))$ provided by the system of seminorms

$$\nu_0(u; p) = \max_{i=0, \ldots, m-1} \{\max_{t_0 \leqq t \leqq t_0 + p} |u^{(i)}(t)|\}$$

$$(\nu_0(u; p) = \max_{i=0, 1, \ldots, m} \{\max_{t_0 \leqq t \leqq t_0 + p} |u^{(i)}(t)|\}) \tag{13_0}$$

is understood, Lemma 2 and Lemma 4 remain valid; $u_1 < u_2$ will now mean that $\dfrac{d^i u_1}{dt^i} \leqq \dfrac{d^i u_2}{dt^i}$, $i = 0, \ldots, m-1$. Theorem 2 will change in the following way.

Theorem 2′. In order that there exist a solution u of the initial value problem (4_0), (5_0), it is necessary and sufficient that there exist two functions $u†$, $v† \in C_m(<t_0, \infty)$ with the properties:

1. $u† < v†$

2. For all $u_1, u_2 \in S = \{u \in C_{m-1}(<t_0, \infty): u† < u < v†\}$ the following implication holds:

$$u_1 < u_2 \Rightarrow f\left[t, u_1(t), \ldots, \frac{d^{m-1}u_1(t)}{dt^{m-1}}\right] \leqq$$

$$\leqq f\left[t, u_2(t), \ldots, \frac{d^{m-1}u_2(t)}{dt^{m-1}}\right] \quad (t \in t_0, \infty));$$

3. $\frac{d^i u_1^*(t_0)}{dt^i} \leqq u_0^{(i)} \leqq \frac{d^i v_1^*(t_0)}{dt^i}$ for all $i = 0, \ldots, m-1$

4. $\frac{d^m u_1^*(t)}{dt^m} \leqq f\left[t, u_1^*(t), \ldots, \frac{d^{m-1}u_1^*(t)}{dt^{m-1}}\right]$

$$\frac{d^m v_1^*(t)}{dt^m} \geqq f\left[t, v_1^*(t), \ldots, \frac{d^{m-1}v_1^*(t)}{dt^{m-1}}\right]$$

for all $t \in \langle t_0, \infty)$.

If such functions exist, then the sequences $\{u_n^*\}_{n=1}^{\infty}$, $\{v_n^*\}_{n=1}^{\infty}$ defined by the relations (22) where the operator T is defined by $(Tu)(t) = \sum_{i=0}^{m-1} u_0^{(i)} \frac{(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{m-1}}{(m-1)!} \cdot f\left[s, u(s), \ldots, \frac{d^{m-1}u(s)}{dt^{m-1}}\right] ds$, possess the properties 1—4 from Theorem 2 with the only difference that u^* and v^* are solutions of the problem (4_0), (5_0).

The proof goes on similarly like the proof of Theorem 2.

In a similar way one could modify Theorem 1 and Corollary of Theorem 2 for the problem (4_0), (5_0).

Consider now the problem (4), (5). For the sake of simplicity of notation let us rearrange the set $A_{m1, \ldots, m_N}$ in such a way that the first m terms $D^{\alpha_1}u, D^{\alpha_2}u, \ldots, \ldots, D^{\alpha_m}u$ mean $u, \frac{\partial u}{\partial t}, \ldots, \frac{\partial^{m-1}u}{\partial t^{m-1}}$ (as to the other members of the considered set, in this case they can be ordered in an arbitrary manner). Let us suppose that the function $B \in \mathscr{B}$ does not depend explicitly on x, i. e. $B = B(t, y_1, \ldots, y_M)$ and that the initial functions $u_i(x) = \text{const.} = u_0^{(i)}$, $i = 0, \ldots, m-1$. Then each solution of the initial-value problem

$$\frac{d^m u}{dt^m} = B\left(t, u, \frac{du}{dt}, \ldots, \frac{d^{m-1}u}{dt^{m-1}}, 0, \ldots, 0\right), \qquad t_0 \leqq t < \infty$$

$$\frac{d^i u(t_0)}{dt^i} = u_0^{(i)}, \qquad i = 0, \ldots, m-1$$

is a solution of the problem (4), (5). From Theorem 2′ the following corollary follows.

Corollary. Let the function $B \in \mathscr{B}$ do not depend explicitly on x and let the initial functions $u_i(x) = \text{const.} = u_0^{(i)}$, $i = 0, \ldots, m-1$. Let

$$\frac{\partial B(t, y_1, \ldots, y_m, 0, \ldots, 0)}{\partial y_i} \geqq 0$$

for all $i = 1, \ldots, m$ and at all points $(t, y_1, \ldots, y_m) \in \langle t_0, \infty) \times R^m$. Let further, exist on $\langle t_0, \infty)$ two solutions $u^{\dagger}(t), v^{\dagger}(t) \in C_m(\langle t_0, \infty))$ of the differential inequalities

$$\frac{d^m u^{\dagger}}{dt^m} \leqq B\left(t, u^{\dagger}, \frac{du^{\dagger}}{dt}, \ldots, \frac{d^{m-1}u^{\dagger}}{dt^{m-1}}, 0, \ldots, 0\right)$$

and

$$\frac{d^m v^{\dagger}}{dt^m} \geqq B\left(t, v^{\dagger}, \frac{dv^{\dagger}}{dt}, \ldots, \frac{d^{m-1}v^{\dagger}}{dt^{m-1}}, 0, \ldots, 0\right)$$

respectively, with the properties

$$u^{\dagger} < v^{\dagger}, \frac{d^i u^{\dagger}(t_0)}{dt^i} \leqq u_0^{(i)} \leqq \frac{d^i v^{\dagger}(t_0)}{dt^i}, \qquad i = 0, \ldots, m-1$$

Then there exists a solution of the initial — value problem (4), (5).

Let us consider the problem (4), (5) in the special case when the differential equation (4) is a linear and hence of the form

$$\frac{\partial^m u}{\partial t^m} = \sum_{\gamma \in A \text{ ind } (m_1, \ldots, m_N)} a_\gamma(t, x) D^\gamma u + f(t, x) \tag{4_1}$$

Condition (3) means the assumption

$$D^{\alpha^0} a_\gamma \in C(H),\ D^{\alpha^0} f \in C(H) \text{ for each } \alpha^0 \in A^0 \text{ and each } \gamma \in A_{m_1, \ldots, m_N} \tag{3_1}$$

In accordance with the definition of the relation $B_1 \ll B_2$,

$$\sum_{\gamma \in A \text{ ind } (m_1, \ldots, m_N)} a_\gamma^1(t, x) D^\gamma u + f^1(t, x) \ll \sum_{\gamma \in A \text{ ind } (m_1, \ldots, m_N)} a_\gamma(t, x) \cdot D^\gamma u + f(t, x)$$

if

$$|D^{\alpha^0} a_\gamma^1(t, x)| \leqq D^{\alpha^0} a_\gamma(t, x)$$

$$|D^{\alpha^0} f^1(t, x)| \leqq D^{\alpha^0} f(t, x)$$

everywhere in H for each $\alpha^0 \in A^0$.

Hence Lemma 5 remains to be true for (4_1) if instead of (16)

$$D^{\alpha^0} u_i(x) \geqq 0,\ i = 0, \ldots, m-1,\ D^{\alpha^0} a_\gamma(t, x) \geqq 0,\ D^{\alpha^0} f(t, x) \geqq 0 \tag{16_1}$$

at each point $(t, x) \in H$ and for each $\alpha^0 \in A^0$ is valid.

Besides the mentioned results the following statement for the differential equation (4 $\cdot$ $\cdot$ is true.

Theorem 3. Suppose that besides the standart assumptions (2), (3_1) $D^{\alpha^0}a_\gamma \geqq 0$ for each $\alpha^0 \in A^0$ and each $\gamma \in A_{m_1, \dots, m_N}$ is true and that there is a number $\lambda > 1$ and a solution w in H of the differential equation

$$\frac{\partial^m u}{\partial t^m} = \lambda \sum_{\gamma \in A \text{ ind } (m_1, \dots, m_N)} a_\gamma(t, x) D^\gamma u$$

with the property

$$D^\alpha w(t, x) > 0 \text{ for each } \alpha \in A \text{ and each } (t, x) \in H \tag{26}$$

Then a necessary and sufficient condition for the existence of a solution to the problem (4_1), (5) is that there exist such a function $u^* \in L$ that for each $p, q \in Z$ there exist a constant $M_{p,q}$ for which

$$|D^\alpha(Tu^* - u^*)(t, x)| \leqq M_{p,q} D^\alpha w(t, x) \; ((t, x) \in H_{p,q}, \alpha \in A) \tag{27}$$

Here T is the operator associated with the problem (4_1), (5) and hence of the form

$$Tu(t, x) = \sum_{i=0}^{m-1} u_i(x) \frac{(t-t_0)^i}{i!} + \int_{t_0}^t \frac{(t-s)^{m-1}}{(m-1)!} f(s, x) \, ds +$$
$$+ \sum_{\gamma \in A \text{ ind } (m_1, \dots, m_N)} \int_{t_0}^t \frac{(t-s)^{m-1}}{(m-1)!} a_\gamma(s, x) D^\gamma u(s, x) \, ds \; (u \in L, (t, x) \in H)) \tag{10_1}$$

If the equation (4_1) is reduced to the form

$$\frac{\partial^m u}{\partial t^m} = a(t, x) u + f(t, x) \tag{4_2}$$

then there exists a unique solution of the problem (4_2), (5).

Proof. Necessary condition is clear. For u^* we take a solution of the problem (4_1), (5) and then (27) is fulfilled for all $M_{p,q} \geqq 0$.

Sufficient condition. L can be provided, besides by the system (13), also by the system of seminorms μ introduced by

$$\mu(u; p, q, r) = \max_{\substack{|\alpha| \leqq r \\ \alpha \in A}} \left\{ \max_{(t, x) \in H_{p,q}} \frac{|D^\alpha u(t, x)|}{D^\alpha w(t, x)} \right\} \tag{28}$$

for each $p, q, r \in Z$.

Let

$$\min_{\substack{|\alpha| \leqq r \\ \alpha \in A}} \{ \min_{(t, x) \in H_{p,q}} D^\alpha w(t, x) \} = k_{p,q,r}$$

$$\max_{\substack{|\alpha| \leqq r \\ \alpha \in A}} \{ \max_{(t, x) \in H_{p,q}} D^\alpha w(t, x) \} = K_{p,q,r}$$

Then, by comparing (28) with (13), we get

$$\frac{\nu(u;p,q,r)}{K_{p,q,r}} \leqq \mu(u;p,q,r) \leqq$$

$$\leqq \frac{\nu(u;p,q,r)}{k_{p,q,r}} \quad (u \in L, p, q, r \in Z)$$

Hence both systems of seminorms define the same topology in L.

Let $p, q, r \in Z$ be given. For $(t,x) \in H_{p,q}$, $\alpha \in A$ with $|\alpha| \leqq r$, $u, v \in L$ we have

$$\frac{D^{\alpha}[Tu(t,x) - Tv(t,x)]}{D^{\alpha}w(t,x)} = \frac{1}{D^{\alpha}w(t,x)} \int_{t_0}^{t} \frac{(t-s)^{m-1-\alpha_0}}{(m-1-\alpha_0)!} \cdot$$

$$\cdot \sum_{\gamma \in A \text{ ind } (m_1, \ldots, m_N)} D^{\alpha^0}(a_\gamma(s,x) \cdot D^{\gamma}(u(s,x) - v(s,x)))\, ds \tag{30}$$

On the basis of the Leibniz rule for the derivative of product ([1], p. 50)

$$D^{\alpha^0}(a_\gamma(s,x) D^{\gamma}(u(s,x) - v(s,x))) =$$

$$\sum_{\delta + \varrho = \alpha^0} \frac{\alpha^0!}{\delta!\,\varrho!} D^{\delta} a_\gamma(s,x) \cdot D^{\varrho+\gamma}(u(s,x) - v(s,x))$$

With respect to (28) we have

$$|D^{\varrho+\gamma}(u(s,x) - v(s,x))| \leqq \mu(u - v; p, q, |\varrho| + |\gamma|) D^{\varrho+\gamma} w(s,x)$$

and hence

$$|D^{\alpha^0}(a_\gamma(s,x) D^{\gamma}(u(s,x) - v(s,x)))| \leqq$$

$$\leqq \sum_{\sigma + \varrho = \alpha^0} \frac{\alpha^0!}{\delta!\,\varrho!} \cdot D^{\delta} a_\gamma(s,x) D^{\varrho+\gamma} w(s,x) \cdot$$

$$\cdot \mu(u - v; p, q, r + m - 1 + m_1 + \ldots + m_N) =$$

$$= D^{\alpha^0}(a_\gamma(s,x) D^{\gamma} w(s,x)) \mu(u - v; p, q, r + K) \tag{31}$$

where $K = m - 1 + m_1 + \ldots + m_N$.

From (30), using (31), we come to the inequality

$$\frac{|D^{\alpha}[Tu(t,x) - Tv(t,x)]|}{D^{\alpha}w(t,x)} \leqq \frac{1}{D^{\alpha}w(t,x)} D^{\alpha} \int_{t_0}^{t} \frac{(t-s)^{m-1}}{(m-1)!} \cdot$$

$$\cdot \sum_{\gamma \in A \text{ ind } (m_1, \ldots, m_N)} a_\gamma(s,x) D^{\gamma} w(s,x)\, ds \cdot \mu(u - v; p, q, r + K) \tag{32}$$

The function w satisfies initial conditions (5) with some functions $u_i(x)$, say $u_i(x) = \tilde{u}_i(x)$, $i = 0, \ldots, m-1$. Then Lemma 2 gives

$$w(t,x)=\sum_{i=0}^{m-1}\tilde{u}_i(x)\frac{(t-t_0)^i}{i!}+$$

$$+\lambda\int_{t_0}^{t}\frac{(t-s)^{m-1}}{(m-1)!}\cdot\sum_{\gamma\in A_{m_1,\ldots,m_N}}a_\gamma(s,x)D^\gamma w(s,x)\,\mathrm{d}s$$

and further, by (11)

$$D^\alpha w(t,x)=\sum_{i=\alpha_0}^{m-1}D^{\alpha^0}\tilde{u}_i(x)\frac{(t-t_0)^{i-\alpha_0}}{(i-\alpha_0)!}+$$

$$+\lambda D^\alpha\int_{t_0}^{t}\frac{(t-s)^{m-1}}{(m-1)!}\cdot\sum_{\gamma\in A\ \mathrm{ind}\ (m_1,\ldots,m_N)}a_\gamma(s,x)D^\gamma w(s,x)\,\mathrm{d}s$$

Therefore

$$\frac{1}{D^\alpha w(t,x)}D^\alpha\int_{t_0}^{t}\frac{(t-s)^{m-1}}{(m-1)!}\sum_{\gamma\in A\ \mathrm{ind}\ (m_1,\ldots,m_N)}a_\gamma(s,x)D^\gamma w(s,x)\,\mathrm{d}s\leqq\frac{1}{\lambda} \tag{33}$$

because the assumption (26) guarantees that all $D^{\alpha^0}\tilde{u}_i(x)>0$. Combining (32) with (33) we finally obtain

$$\mu(Tu-Tv\,;p,q,r)\leqq\frac{1}{\lambda}\mu(u-v\,;p,q,r+K) \tag{34}$$

The operator T, given by (10_1), maps L into itself, therefore there exists the sequence $\{u_n^*\}_{n=0}^{\infty}$ where $u_0^*=u^*$ and $u_n^*=Tu_{n-1}^*$, $n=1,2,\ldots$. By (34), the sequence shows the following property. For $n,k\in Z$ $\mu(u_{n+k}^*-u_n^*;\,p,q,r)\leqq\leqq\sum_{l=1}^{k}\mu(u_{n+l}^*-u_{n+l-1}^*;\,p,q,r)\leqq\sum_{l=1}^{k}\frac{1}{\lambda^{n+l-1}}\mu(u_1^*-u_0^*;\,p,q,r+(n+l-1)K)$. The assumption (27) means that $\mu(u_1^*-u_0^*;\,p,q,r)\leqq M_{p,q}$, and thus

$$\mu(u_{n+k}^*-u_n^*;p,q,r)\leqq M_{p,q}\frac{1}{\lambda^n}\frac{1-\lambda^{-k}}{1-\lambda^{-1}} \tag{35}$$

This, with respect to (29), implies that each sequence $\{D^\alpha u_n^*\}_{n=0}^{\infty}$ for $\alpha\in A$ is uniformly Cauchy on each $H_{p,q}$ and hence $\{u_n^*\}_{n=0}^{\infty}$ is in the space $\{L,\{v\}\}$ a Cauchy sequence. Because this space is complete, there exists $\lim_{n\to\infty}u_n^*=u^{**}$ and by the continuity of T, u^{**} is a fixed point of T.

If the equation (4_1) is of the form (4_2), (31) turns into

$$|D^{\alpha^0}(a(s,x)\,(u(s,x)-v(s,x)))|\leqq$$

$$\leqq\sum_{\delta+\varrho=\alpha^0}\frac{\alpha^0!}{\delta!\varrho!}D^\delta a(s,x)\cdot D^\varrho w(s,x)\mu(u-v\,;p,q,r)$$

and hence in (34) $K=0$ can be put. Then (35) may be written as

$$\mu(u^*_{n+k}-u^*_n; p, q, r) \leqq \frac{1}{\lambda^n} \frac{1-\lambda^{-k}}{1-\lambda^{-1}} \mu(u^*_1-u^*_0; p, q, r)$$

From that the existence of the solution u^{**} of the problem (4_2), (5) follows, similarly as in the general case. Its uniqueness is assured by (34) for $K=0$.

Corollary. Let the assumptions of Theorem 3 be satisfied and let, further, f and u_i, $i = 0, \ldots, m-1$, be such that the function F defined in H by $F(t, x) = \sum_{i=0}^{m-1} u_i(x) \cdot \frac{(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{m-1}}{(m-1)!} \cdot f(s, x)\, ds$ shows the following property. For each p, $q \in Z$ there is a constant $M_{p,q}>0$ such that

$$|D^\alpha F(t, x)| \leqq M_{p,q} D^\alpha w(t, x) \ ((t, x) \in H_{p,q}, \alpha \in A).$$

Then there exists at least one solution of the problem (4_1), (5).

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SÚHRN

O MODIFIKÁCII METÓDY MAJORÁNT

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V práci sa rozširuje použitie metódy majoránt na riešenie Cauchyho úlohy z triedy analytických funkcií na triedu nekonečne diferencovateľných funkcií. Na tejto triede sa definuje čiastočné usporiadanie a platí: Riešenie majorantného Cauchyho problému majorizuje riešenie daného problému.

РЕЗЮМЕ

О МОДИФИКАЦИИ МЕТОДА МАЖОРАНТ

В. Шеда, Братислава

В работе расширяется применение метода мажорант для решения задачи Коши из класса аналитических функций на класс бесконечно дифференцируемых функций. В этом классе определяется частичное упорядочение и имеет место: Решение мажорантной задачи Коши мажорирует решение данной проблемы.

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AN A PRIORI BOUND OF THE DISCRETIZATION ERROR IN THE INTEGRATION BY MULTISTEP DIFFERENCE METHOD FOR THE DIFFERENTIAL EQUATION $y^{(s)}=f(x, y)$

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Introduction

The purpose of this paper is to generalize the assertion about the a priori estimation for discretization error by the multistep difference methods for the solution of ordinary differential equations. The subject under discussion is the generalization of the considerations from [4] outhe fages 221, 247, 298 and 314 with respect to the order of the differential equation and at the same time it's the proof of a part of Theorem 3.11 and an estimation of the constants K_1 and K_3 from [2]. (The estimation of the constant K_2 cf. [6]).

Let us consider the differential equation

$$y^{(s)}=f(x, y) \quad \text{for} \quad s \geqq 1 \tag{1}$$

supposing that the function $f(x, y)$ is defined, continuous and satisfies the Lipschitz condition in y with the constant L in the region $a \leqq x \leqq b$, $|y| < \infty$.

The general k-th order difference formula for its solution may be written in the form

$$\sum_{\mu=0}^{k} \alpha_\mu y_{n+\mu} = h^s \sum_{\mu=0}^{k} \beta_\mu f(x_{n+\mu}, y_{n+\mu}), \qquad n=0, 1, \ldots, N-k \tag{2}$$

where

$$h=(b-a)/N, \quad x_n=a+nh \tag{3}$$

N is an arbitrary positive integer, k is a fixed integer and α_μ and β_μ ($\mu=0, 1, \ldots, k$) denote real constants which do not depend on n. We shall always assume that $\alpha_k \neq 0$, $|\alpha_0|+|\beta_0|>0$ and in addition that $k \geqq s$.

Definition 1. The formula (2) will be said to be stable in the sense of Dahlquist, if all roots ζ_i of the characteristic polynomial

$$\varrho(\zeta)=\sum_{\mu=0}^{k}\alpha_\mu\zeta^\mu \tag{4}$$

are such that $|\zeta_i|\leqq 1$ and the multiplicity of the roots for which $|\zeta_i|=1$ is at most s.

Then the following two lemmas are true.

Lemma 1. Let the polynomial (4) satisfy the condition of stability and let the coefficients γ_l $(l=0, 1, 2, \ldots)$ be defined by

$$\frac{1}{\alpha_k+\alpha_{k-1}\zeta+\ldots+\alpha_0\zeta^k}=\gamma_0+\gamma_1\zeta+\gamma_2\zeta^2\ldots,\ |\zeta|<1$$

Then there exist constants $\Gamma_0, \Gamma_1, \ldots, \Gamma_{s-1}$ such that

$$|\gamma_l|\leqq\sum_{i=0}^{s-1}\binom{l+i-1}{i}\Gamma_i, \qquad l=0, 1, 2, \ldots$$

Lemma 2. Let the polynomial (4) satisfy the condition of stability, let B^*, β and Λ be nonnegative constants such that

$$\sum_{\mu=0}^{k}|\beta_{\mu,m}|\leqq B^*,\ |\beta_{k,m}|\leqq\beta,\ |\lambda_m|\leqq\Lambda,\ 0\leqq m\leqq N$$

and let $0<h^s<|\alpha_k|\beta^{-1}$. Then every solution of the equation

$$\sum_{\mu=0}^{k}\alpha_\mu z_{m+\mu}=h^s\sum_{\mu=0}^{k}\beta_{\mu.m}z_{m+\mu}+\lambda_m$$

for which

$$|z_\mu|\leqq Z, \qquad \mu=0, 1, \ldots, k-1$$

satisfies

$$|z_n|\leqq K^*e^{nh^sL^*}, \qquad 0\leqq n\leqq N$$

Here

$$L^*=\frac{B^*\sum_{i=0}^{s-1}\binom{N-k+i-1}{i}\Gamma_i}{1-h^s|\alpha_k^{-1}|\beta}$$

$$K^*=\frac{\Lambda\sum_{i=0}^{s-1}\binom{N-k+i}{i+1}\Gamma_i+kAZ\sum_{i=0}^{s-1}\binom{N-k+i-1}{i}\Gamma_i}{1-h^s|\alpha_k^{-1}|\beta}$$

where $A=\sum_{\mu=0}^{k}|\alpha_\mu|$.

The proof of Lemma 1 and Lemma 2 cf. [6].

In the next we associate with (2) the difference operator

$$L[y(x);h]=\sum_{\mu=0}^{k}\alpha_\mu y(x+\mu h)-h^s\sum_{\mu=0}^{k}\beta_\mu y^{(s)}(x+\mu h) \tag{5}$$

We shall apply the operator L only to functions which have continuous derivatives of sufficiently high order.

Definition 2. The given difference operator of the form (5) is said to be of order $p\geqq 0$, if it satisfies the following conditions

$$\sum_{\mu=0}^{k}\mu^t\alpha_\mu=0 \quad \text{for } t=0,1,2,\ldots,s-1$$

$$\frac{1}{t!}\sum_{\mu=0}^{k}\mu^t\alpha_\mu=\frac{1}{(t-s)!}\sum_{\mu=0}^{k}\mu^{t-s}\beta_\mu \quad \text{for } t=s,s+1,\ldots,s+p-1 \tag{6}$$

As a special case of this definition cf. Definition 3.1. in [2]. Examples of operators for which $s\geqq 3$ cf. [3].

Remark. From the practical point of view the difference equation (2) is completely equivalent to the equation

$$\sum_{\mu=0}^{k}\alpha_\mu y_{n+t+\mu}=h^s\sum_{\mu=0}^{k}\beta_\mu f(x_{n+t+\mu},y_{n+t+\mu})$$

where t is any fixed (positive or negative) integer. Proceeding as above, we may associate with the relation stated above the difference operator

$$L_t[y(x);h]=\sum_{\mu=0}^{k}\alpha_\mu y(x+(t+\mu)h)-h^s\sum_{\mu=0}^{k}\beta_\mu y^{(s)}(x+(t+\mu)h)$$

Clearly $L_t[y(x);h]=L[y(x+th);h]$ is true. We thus find that if $L[y(x);h]$ is of order p, then $L_t[y(x);h]$ is of order p too and the order p does not depend on t.

If (5) is an operator of order p, we shall assume that the exact solution $y(x)$ has a continuous derivative of order $p+s$ for $x\in[a,b]$ and set

$$Y=\max_{x\in[a,b]}|y^{(p+s)}(x)| \tag{7}$$

It will be shown that for sufficiently accurate starting values the accumulated discretization error is of the order of h^p.

Lemma 3. Let $L[y(x);h]$ be a difference operator of order $p>0$. There exists a constant $G>0$, depending only on L such that

$$|L[y(x);h]|\leqq h^{p+s}GY, \qquad a\leqq x,\ x+kh\leqq b \tag{8}$$

for all functions $y(x)$ having a continuous derivative of order $p+s$ in $[a,b]$.

The proof is analogous to the proof of Lemma 5.7 in [4]. An expression for G can be found by using Taylor's theorem with the integral form of the remainder. If

$y(x)$ has a continuous derivative of order q in (a, b), and if x and $\bar{x}$ are in (a, b), then we have

$$y(\bar{x}) = y(x) + (\bar{x} - x)y'(x) + \ldots + \frac{(\bar{x} - x)^{q-1}}{(q-1)!} y^{(q-1)}(x) +$$

$$+ \frac{1}{(q-1)!} \int_x^{\bar{x}} (\bar{x} - t)^{q-1} y^{(q)}(t)\, dt$$

Applying this for $q = p + s$ with $\bar{x} = x + \mu h$ ($\mu = 1, 2, \ldots, k$) to both $y(x)$ and $y^{(s)}(x)$ we obtain after putting $t = x + hw$

$$y(x + \mu h) = \ldots + h^{p+s} \int_0^{\mu} \frac{(\mu - w)^{p+s-1}}{(p+s-1)!} y^{(p+s)}(x + hw)\, dw$$

$$y^{(s)}(x + \mu h) = \ldots + h^{p} \int_0^{\mu} \frac{(\mu - w)^{p-1}}{(p-1)!} y^{(p+s)}(x + hw)\, dw$$

where the dots represent terms involving derivatives of order $< p + s$. If the operator $L[y(x); h]$ has order p, and if each term on the right of (5) is replaced by one of the above expressions, then the terms represented by dots cancel owing to the definition of p, and what remains may be written in the form

$$L[y(x); h] = h^{p+s} \int_0^k G(w) y^{(p+s)}(x + hw)\, dw \qquad (9)$$

where the function $G(w)$ is called the kernel of operator $L[y(x); h]$ and it is represented in each of the intervals $[\mu, \mu + 1)$ ($\mu = 0, 1, \ldots, k - 2$) and in the interval $[k - 1, k]$ as follows; for $w \in [k - 1, k]$

$$G(w) = \alpha_k \frac{(k - w)^{p+s-1}}{(p+s-1)!} - \beta_k \frac{(k - w)^{p-1}}{(p-1)!}$$

and for $w \in [\mu, \mu + 1)$ ($\mu = 0, 1, 2, \ldots, k - 2$)

$$G(w) = \sum_{i=\mu+1}^{k} \left(\alpha_i \frac{(i - w)^{p+s-1}}{(p+s-1)!} - \beta_i \frac{(i - w)^{p-1}}{(p-1)!} \right)$$

It now easily follows that

$$|L[y(x); h]| \leqq h^{p+s} \int_0^k |G(w)|\, |y^{(p+s)}(x + wh)|\, dw$$

and that (8) is true with

$$G = \int_0^k |G(w)|\, dw$$

Let y_n $(n=0, 1, 2, \ldots)$ be now the exact solution of the difference equation (2), defined by the starting values $y_\mu = \eta_\mu(h)$, $\mu = 0, 1, \ldots, k-1$. As to the starting errors, we assume that

$$|y_\mu - y(x_\mu)| \leqq h^{s-1}\delta, \qquad \mu = 0, 1, \ldots, k-1 \tag{10}$$

We now subtract $\dot{L}[y(x_m); h]$ from the corresponding relation

$$\sum_{\mu=0}^{k} \alpha_\mu y_{m+\mu} - h^s \sum_{\mu=0}^{k} \beta_\mu f(x_{m+\mu}, y_{m+\mu}) = 0$$

satisfied by the values y_m. Writing $d_m = y_m - y(x_m)$, $m = 0, 1, \ldots$ and setting

$$\begin{aligned} g_m &= [f(x_m, y_m) - f(x_m, y(x_m))]d_m^{-1}, && \text{if } \; d_m \neq 0 \\ g_m &= 0, && \text{if } \; d_m = 0 \end{aligned}$$

so that in any case $|g_m| \leqq L$, where L is the Lipschitz constant, we get

$$\sum_{\mu=0}^{k} \alpha_\mu d_{m+\mu} = h^s \sum_{\mu=0}^{k} \beta_\mu g_{m+\mu} d_{m+\mu} + R_m$$

where $|R_m| \leqq GYh^{p+s}$.

Applying Lemma 2 to this equation with $z_m = d_m$, $Z = h^{s-1}\delta$, $N = (x_n - a)/h$, $A = GYh^{p+s}$, $\beta_{\mu,m} = g_{m+\mu}\beta_\mu$, $\beta = L|\beta_k|$, $B^* = LB = L\sum_{\mu=0}^{k} |\beta_\mu|$ under the assumption that $h^s < L^{-1}|\beta_k^{-1}\alpha_k|$ we get for $a \leqq x_n \leqq b$

$$|d_n| \leqq \Gamma^*((x_n - a)^{s-1}kA\delta + (x_n - a)^s GYh^p) \exp((x_n - a)^s \Gamma^* B^*) \tag{11}$$

where

$$\Gamma^* = \frac{\max\left[\max\limits_i \left(\dfrac{\Gamma_{i-1}}{\binom{s}{i} i!}\right), \max\limits_j \left(\dfrac{\Gamma j}{\binom{s-1}{j} j!}\right)\right]}{1 - h^s L |\alpha_k^{-1}\beta_k|}$$

As in the first and second order case (cf. [4]), the estimate (11) exhibits very clearly the influence of the various sources of error on the accumulated error. The term involving $kA\delta$ represents the influence of the starting error and the term involving GY represents the error resulting from local discretization.

Remark. This form of the discretization error estimate is analogous to the form of estimate from [4]. In the case that we assumed about the starting errors $Z = \delta = \max\limits_{\mu=0, 1, \ldots, k-1} |y_\mu - y(x_\mu)|$, we should get the error estimate in the form analogous to the form from [2], namely

$$|d_n| \leqq \Gamma^*((x_n - a)^{s-1}kA\delta h^{1-s} + (x_n - a)^s GYh^p) \exp((x_n - a)^s \Gamma^* B^*)$$

Therefore the following theorem is true.

Theorem 1. Let the function $f(x, y)$ be defined, continuous and satisfy the Lipschitz condition in y with the constant L in the region $a \leqq x \leqq b$, $|y| < \infty$. Let $s \geqq 1$, $k \geqq s$ be integers and let be given the differential equation (1) with the difference formula (2) stable in the sense of Dahlquist under the conditions (3), where $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_0| > 0$. Let the associated difference operator (5) have order p, let the sequence y_n be the exact solution of (2) where for the starting errors the relation (10) is true and let $h^s < L^{-1}|\beta_k^{-1}\alpha_k|$. Then for the accumulated discretization error (11) is valid.

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SÚHRN

APRIÓRNY ODHAD DISKRETIZAČNEJ NEPRESNOSTI DIFERENČNEJ VIACKROKOVEJ METÓDY PRE DIFERENCIÁLNU ROVNICU $y^{(s)}=f(x, y)$

A. Huťa jun., Bratislava

V článku sa vyšetruje vplyv začiatočných a lokálnych diskretizačných nepresností na celkovú diskretizačnú nepresnosť pre diferenciálnu rovnicu $y^{(s)}=f(x, y)$.

Dokazuje sa nasledovná veta.

Veta 1. Nech funkcia $f(x, y)$ je definovaná, spojitá a spĺňa Lipschitzovu podmienku vzhľadom na y s konštantou L v obore $a \leqq x \leqq b$, $|y|<\infty$. Nech $s \geqq 1$, $k \geqq s$ sú celé čísla a nech je daná diferenciálna rovnica (1) s diferenčnou schémou (2) stabilnou v zmysle Dahlquista za predpokladov (3), kde $\alpha_k \neq 0$, $|\alpha_0|+|\beta_0|>0$, a nech pridružený operátor (5) je rádu p. Nech $\{y_n\}$ je exaktným riešením (2), kde začiatočné nepresnosti spĺňajú (10) a nech $h^s<L^{-1}|\beta_k^{-1}\alpha_k|$. Potom pre výsledné diskretizačné nepresnosti platí (11).

РЕЗЮМЕ

ОЦЕНКА ПОГРЕШНОСТИ ДИСКРЕТИЗАЦИИ РАЗНОСТНОГО МНОГОШАГОВОГО МЕТОДА ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $y^{(s)}=f(x, y)$

А. Хутя мл., Братислава

В этой статье исследуется влияние начальных и локальных погрешностей дискретизации на общую погрешность дискретизации для дифференциального уравнения $y^{(s)}=f(x, y)$.

Доказана следующая теорема.

Теорема 1. Пусть функция $f(x, y)$ определена, непрерывна и удовлетворяет условию Липшица по у с постоянной L в области $a \leqq x \leqq b$, $|y|<\infty$. Пусть $s \geqq 1$, $k \geqq s$ целые числа и пусть задано дифференциальное уравнение (1) с разностной формулой (2) устойчивой в смысле Дальквиста при условиях (3), где $\alpha_k \neq 0$, $|\alpha_0|$ + $|\beta_0|>0$, и пусть сопряженный оператор (5) порядка p. Пусть $\{y_n\}$ является точным решением (2), где начальные погрешности удовлетворяют условию (10) причем $h^s<L^{-1}|\beta_k^{-1}\alpha_k|$. Тогда для конечной погрешности дискретизации имеет место отношение (11).

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ON MULTIPLICATIVELY k-THIN SETS

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In this paper we shall consider a special case of the following problem: Let us have the function $f\ N^k \to N$ (N is the set of natural numbers, $k \in N$, $k > 1$. The set $M \subset N$ is called f-thin provided $f(x_1, x_2, \ldots, x_k) \notin M$ for every k-tuple $(x_1, x_2, \ldots, x_k) \in M^k$ such that $x_i \neq x_j$ for $i \neq j$. We shall determine the properties of the function $f^*: N \to N$, where $f^*(n)$ is the greatest natural number such that the set $\{n, n+1, \ldots, f^*(n)\}$ can be decomposed into two disjoint f-thin subsets.

The additive k-thin sets (i. e. f-thin sets, where $f(x_1, x_2, x_3, \ldots, x_k) = x_1 + x_2 + x_3 + \ldots + x_k$) and the multiplicative k-thin sets (i. e. f-thin sets (i. e. f-thin sets, where $f(x_1, x_2, \ldots, x_k) = x_1 \cdot x_2 \cdot \ldots \cdot x_k$) were considered in the paper [1]. We shall continue the investigation of the multiplicative k-thin sets. To simplify the procedure let $a_k(x_1, x_2, \ldots, x_k) = x_1 + x_2 + \ldots + x_k$ and $m_k(x_1, x_2, \ldots, x_k) = x_1 \cdot x_2 \cdot \ldots \cdot x_k$ ($k = 2, 3, \ldots$). We shall give a lower estimation for $m^*(n)$ and asymptotic formulas for the functions m_2^* and m_3^*.

The method used in the paper [1] for the determination of lower estimation for $a_k^*(n)$ can be used for the determination of lower estimation for $m_k^*(n)$.

Theorem 1. For each $k > 1$ there exists a polynomial $p_k(n)$ of degree $k^2 + k - 1$,

$$p\text{-}k(n) = n^{k^2+k-1} + \frac{1}{2}(k-1)(k^2+k-2) \, . \, n^{k^2+k-2} + \ldots$$

such that $m_k^*(n) \geqq p_k(n)$ ($n = 1, 2, \ldots$).

Proof. Let

$$\alpha = n(n+1) \ldots (n+k-1) = n^k + \frac{1}{2} k(k-1)n^{k-1} + \ldots$$

$$\beta = \alpha(\alpha+1) \ldots (\alpha+k-1) = n^{k^2} + \frac{1}{2} k^2(k-1)n^{k^2-1} + \ldots$$

$$\gamma = n(n+1) \ldots (n+k-2) \cdot \beta = n^{k^2+k-1} + \frac{1}{2}(k-1)(k^2+k-2)n^{k^2+k-2} + \ldots$$

Let us consider the following decomposition of the set $\{n, n+1, \ldots, \gamma - 1\}$ into two subsets A and B:

$$A = \{n, n+1, \ldots, \alpha - 1\} \cup \{\beta, \beta+1, \ldots, \gamma - 1\}$$
$$B = \{\alpha, \alpha+1, \ldots, \beta - 1\}$$

It is easy to check that the sets A and B are multiplicatively k-thin. Hence $m_k^*(n) \geqq \gamma - 1 = p_k(n)$.

Corollary 1. Let $k > 1$. Then

$$\liminf_{n\to\infty} \left(\frac{m_k^*(n)}{n^{k^2+k-2}} - n\right) \geqq \frac{1}{2}(k-1)(k^2+k-2)$$

Theorem 2. Let the function $m_2^*(n)$ have the introduced meaning. Then

$$\liminf_{n\to\infty} \left(\frac{m_2^*(n)}{n^4} - n\right) \geqq 2 \text{ and } \limsup_{n\to\infty} \left(\frac{m_2^*(n)}{n^4} - n\right) \leqq 4$$

Proof. The first part of Theorem 2 follows immediately from Corollary 1. The second inequality $m_2^*(n) < n^2(n+1)^2(n+2)$ which has been proved in [1].

Theorem 3. Let the function $m_3^*(n)$ have the introduced meaning. Then

$$\liminf_{n\to\infty} \left(\frac{m_3^*(n)}{n^{10}} - n\right) \geqq 10 \text{ and } \limsup_{n\to\infty} \left(\frac{m_3^*(n)}{n^{10}} - n\right) \leqq 13$$

Proof. The first part of Theorem 3 follows from Corollary 1. Now we prove the second part of Theorem 3. Fix some n and assume that there is a decomposition $\{n, n+1, \ldots, m_3^*(n)\} = A \cup B$ into two disjoint multiplicatively 3-thin sets A and B.

We shall investigate the following three possible cases for the numbers n, $n+1$, $n+2$, $n+3$ and $n+4$:

A. all the five elements belong to one of the sets A and B;

B. four elements belong to one set and one element belongs to the other set;

C. three numbers belong to one set and two to the other set.

We shall use the symbols a, b, c, d, e for the investigated elements.

A. $\{a, b, c, d\} \subset A$. It is easy to derive $\{abc, abd, acd, bcd\} \subset B$, further $a^3b^2c^2d^2 \in A$, as well as

$$a^4b^3c^2d^2 \in B \tag{1}$$

Since $\{a, d\} \subset A$ and also $ad(a^2b^2c^2d) = a^3b^2c^2d^2 \in A$ we have

$$a^2b^2c^2d \in B \tag{2}$$

We show, that $ab \in A$ cannot be held. If $ab \in A$, then $a \cdot b(ab) = a^2b^2 \in B$. Similarly, $(ab) \cdot c \cdot d \in B$ and then $a^2b^2(abcd)(acd) = a^4b^3c^2d^2 \in A$, which contradicts (1). Hence $ab \in B$. Similarly, $ac \in B$.

Finally we have $a^2b^2c^2d = (ab)(ac)(bcd) \in A$, which contradicts (2). Hence

$$m_3^*(n) < a^4b^3c^2d^2$$

B. $(a, b, c, d\} \subset A$, $e \in B$. It is easy to derive that $\{abc, abd, acd, bcd\} \subset B$, $a^2b^2cde \in A$

$$\{a^3b^3cde, a^3b^2c^2de\} \subset B \tag{3}$$

and $a^3b^2c^2d^2 \in A$. It follows from the last relation, $\{a, d\} \subset A$ and from the expression $a^3b^2c^2d^2 = a \cdot d \cdot (a^2b^2c^2d)$ that necessarily

$$a^2b^2c^2d \in B \tag{4}$$

If $ab \in A$, then $\{a^2b^2, abcd\} \subset B$ and $a^3b^3cde \in A$, which contradicts (3). Hence $ab \in B$.

If $ac \in A$, then $\{a^2bc, abcd\} \subset B$, and $a^3b^2c^2de \in A$, which contradicts (3). Hence $ac \in B$.

At the end we have $(ab)\ (ac)\ (bcd) = a^2b^2c^2d \in A$, which contradicts (4). Hence $m_3^*(n)$ is smaller than the greatest from the numbers a^3b^3cde, $a^3b^2c^2de$, $a^3b^2c^2d^2$.

C. $\{a, b, c\} \subset A$, $\{d, e\} \subset B$. Clearly $abc \in B$, $abcde \in A$. From this and from the equalities $ac(cde) = b \cdot c \cdot (ade) = a \cdot c \cdot (bde)$ it follows $\{cde, bde, ade\} \subset B$. Hence $abcd^3e^3 \in A$. Further we have $a^2bc^2de = abcde(a)\ (c) \in B$ and $ab^2c^2de = abcde(b)\ (c) \in B$. From $\{abc, cde, e\} \subset B$ we have

$$abc^2de^2 \in A \tag{5}$$

There is $a \cdot b \cdot (cd^3e^3) \in A$, $\{a, b\} \subset A$ and then $cd^3e^3 \in B$. From the equality $cd^3e^3 = cde(d)\ (de^2) \in B$ and from $\{d, cde\} \subset B$ we have $de^2 \in A$. Since $abcde \in A$, we

$$abc^2d^2e^3 \in B \tag{6}$$

If $de \in A$, then $bcde \in B$, $acde \in B$ and then $abc^2d^2e^3 = (bcde)\ (acde)\ (e) \in A$, which contradicts (6). Hence $de \in B$. Since $\{a^2bc^2de, abc\} \subset B$ and $a^2bc^2de = (abc)\ (de)\ (ac)$ we have $ac \in A$. Similarly $\{ab^2c^2de, abc\} \subset B$ implies $bc \in A$. Then $abc^2de^2 = (ac)\ (bc)\ (de^2) \in B$, which contradicts (5). Hence $m_3^*(n)$ is smaller than the greatest from the numbers $abcd^3e^3$ and $abc^2d^2e^3$.

It follows from the proseeded considerations, that we must take the bound from the case A (the polynomial of the 11-th degree in n). If we substitute $a = n$, $b = n + 1$, $c = n + 2$, $d = n + 3$, then $m_3^*(n) < n^4(n + 1)^3(n + 2)^2(n + 3)^2 = n^{11} + 13n^{10} + \dots$. The second part of Theorem 3 follows from the last inequality.

Corollary 2. Let the functions $m_2^*(n)$, $m_3^*(n)$ have the introduced meaning. Then

$$\frac{m_2^*(n)}{n^4} = n + \Omega(1) \quad \text{and} \quad \frac{m_3^*(n)}{n^{10}} = n + \Omega(1)$$

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SÚHRN

O MULTIPLIKATÍVNYCH k-RIEDKYCH MNOŽINÁCH

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Nech $m_k^*(n)$ je také najväčšie prirodzené číslo, že množina $\{n, n+1, \ldots, m_k^*(n)\}$ sa dá rozložiť na dve multiplikatívne k-riedke množiny. V práci je uvedený dolný odhad pre $m_k^*(n)$ a asymptotické formuly pre $m_2^*(n)$ a $m_3^*(n)$.

РЕЗЮМЕ

О МУЛЬТИПЛИКАТИВНЫХ k-РЕДКИХ МНОЖЕСТВАХ

Е. Нилатиова, Братислава

Пусть $m_k^*(n)$ самое больше натуральное число, для которого множество $\{n, n+1, \ldots, m_k^*(n)\}$ можно разложить на два мультипликативно k-редких множества. В работе установлена нижная оценка для $m_k^*(n)$ и ассимптотические формулы для $m_2^*(n)$ и $m_3^*(n)$.

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ON THE KOLMOGOROFF THEOREM AND THE INFINITE PRODUCT OF PROBABILITY SPACES

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There is a well-known correspondence between the Kolmogoroff theorem and the infinite product of probability spaces. Let T be an arbitrary index set and for $t \in T$ let X_t be a non-empty set and $\mathscr{S}_t$ be an algebra of subsets of X_t. Put $X = \times_{t \in T} X_t$. Let Γ be the set of all finite subsets of T. For $\alpha \in \Gamma$ we put

$$X_\alpha = \times_{t \in \alpha} X_t$$

and

$$\pi_\alpha : X \to X_\alpha, \quad \pi_\alpha((x_t)_{t \in T}) = (x_{t_1}, \ldots, x_{t_k})$$

where $\alpha = \{t_1, \ldots, t_k\}$. For $\alpha, \beta \in \Gamma$, $\alpha \supset \beta$ we denote by $\pi_{\alpha,\beta}$ the projection from X_α to X_β. Let $\mathscr{S}_\alpha$ be the algebra generated by the projections of $\mathscr{S}_{t_1}, \ldots, \mathscr{S}_{t_k}$, i. e. the algebra generated by the family

$$\pi_{\alpha,t_1}^{-1}(\mathscr{S}_{t_1}) \cup \pi_{\alpha,t_2}^{-1}(\mathscr{S}_{t_2}) \cup \ldots \cup \pi_{\alpha,t_k}^{-1}(\mathscr{S}_{t_k})$$

Let $\mathscr{S}$ be the family of all sets of the form $\pi_\alpha^{-1}(E)$ where $E \in \mathscr{S}_\alpha$ and $\alpha \in \Gamma$. The Kolmogoroff theorem starts with a compatible family of probability measures

$$\mu_\alpha : \mathscr{S}_\alpha \to R$$

i. e.

$$\mu_\alpha(\pi_{\alpha,\beta}^{-1}(E)) = \mu_\beta(E)$$

for every $E \in \mathscr{S}_\beta$ and every $\alpha, \beta \in \Gamma$ such that $\alpha \supset \beta$. The condition permits to define

$$\mu : \mathscr{S} \to R$$

by the formula

$$\mu(\pi_\alpha^{-1}(E)) = \mu_\alpha(E), \; E \in \mathscr{S}_\alpha, \; \alpha \in \Gamma$$

The Kolmogoroff theorem states that μ is (under certain assumptions) a probability measure. (An original form of the theorem is included in [3], chap. III, sec. 4, some further results are contained in [4], theorem T 31 of chap. III and in [5] theorem III. 3.)

A compatible family of measures can be obtained from an arbitrary family $\{\mu_t\}_{t\in T}$ of probability measures by the formulas

$$\mu_\alpha(E_1 \times E_2 \times \ldots \times E_k) = \mu_{t_1}(E_1)\mu_{t_2}(E_2) \ldots \mu_{t_k}(E_k)$$

As before, $\alpha = \{t_1, \ldots, t_k\}$, $E_i \in \mathcal{S}_{t_i}$ $(i = 1, \ldots, k)$. The function μ induced by the compatible family is called the product measure. The theorem on the product measure states that the function μ is a measure. (A first result concerning this matter is probably [2]; see also [1] theorem 2, §38 and [5] proposition V.1.2.)

But the theorem on the product of measures does not follow from the Kolmogoroff theorem, since the Kolmogoroff theorem contains some assumptions of a topological nature, while the product measure theorem is known in a purely abstract form. Only in the most important but special case where $X_t = R\,(t \in T)$, the Kolmogoroff theorem implies the product measure theorem.

The aim of the article is to find a common generalization of the both theorems.

We have also one reason for constructing such a generalization. Usual proofs of the two theorems contain something similar. In the Kolmogoroff theorem (we shall call it shortly "compact case") the sets from $\mathcal{S}$ are approximated by compact sets and then the following property of the family $\mathcal{K}$ of all compact non-empty sets is used:

$$\text{If } E_n \in \mathcal{K},\ E_n \supset E_{n+1}\ (n = 1, 2, \ldots), \text{ then } \bigcap_{n=1}^{\infty} E_n \in \mathcal{K}.$$

In the product measure theorem (we shall call it shortly "product case") first the implication $\mu \times \nu(E) \geqq \varepsilon \Rightarrow \mu(\{x;\ \nu(E^x) \geqq \varepsilon/2\}) \geqq \varepsilon/2$ is used and then the following property of the family $\mathcal{F}$ of all sets of a measure larger than $\varepsilon > 0$:

$$\text{If } E_n \in \mathcal{F},\ E_n \supset E_{n+1}\ (n = 1, 2, \ldots), \text{ then } \bigcap_{n=1}^{\infty} E_n \in \mathcal{F}.$$

This is the idea of our common formulation. We shall say that a family of sets satisfying the mentioned property is an ε-family.

Definition. Let $\mathcal{F}$ be a non-empty family of subsets of a non-empty set Y. We shall say that $\mathcal{F}$ is an epsilon-family, if the following property is satisfied:

$$\text{If } E_n \in \mathcal{F},\ E_n \supset E_{n+1}\ (n = 1, 2, \ldots), \text{ then } \bigcap_{n=1}^{\infty} E_n \in \mathcal{F}.$$

Of course, we shall work with a collection $\mathcal{S}_\alpha^k$ of ε-families $(k \in N)$ and also with another collection $\mathscr{B}_\alpha^{k,j}$ $(k, j \in N)$ of families of subsets of X_α. In the special cases the families have the following meaning:

compact case	product case
$\mathscr{S}_\alpha^k = \{E \in \mathscr{S}_\alpha ; E = \pi_{\alpha,\beta}^{-1}(D),$ $D \neq \emptyset$, D compact, $D \in \mathscr{S}_\beta,$ $\alpha \supset \beta\} \cup \{X_\alpha\}$	$\mathscr{S}_\alpha^k = \left\{E \in \mathscr{S}_\alpha ; \mu_\alpha(E) \geqq \frac{1}{2^k}\right\}$
$\mathscr{B}_\alpha^{k,j} = \left\{E \in \mathscr{S}_\alpha ; \mu_\alpha(E) < < \frac{1}{2^k} + \ldots + \frac{1}{2^{k+j}}\right\}$	$\mathscr{B}_\alpha^{k,j} = \mathscr{S}_\alpha$

Now we shall list the axioms. In the formulations we shall use the following notation: a set $B \in \mathscr{S}_\alpha$ belongs to $\mathscr{D}_{\alpha,i}^{k+1}$ (where $i \in \alpha$), if $\pi_{\alpha,i}(B) \in \mathscr{S}_i^{k+1}$ and

$$\{y ; (x, y) \in B\} \in \mathscr{S}_{\alpha-\{i\}}^{k+1}$$

for every $x \in \pi_{\alpha,i}(B)$. By other words, a set B of $\mathscr{S}_\alpha$, $B = \bigcup_{j=1}^{n} (B_j \times C_j)$, B_j pairwise disjoint, belongs to $\mathscr{D}_{\alpha,i}^{k+1}$,

if

$$\pi_{\alpha,i}(B) = \bigcup_{j=1}^{n} B_j \in \mathscr{S}_i^{k+1}$$

and

$$C_j \in \mathscr{S}_{\alpha-\{i\}}^{k+1}$$

for every $j \in \{1, \ldots, n\}$.

Theorem. Let the following assumptions be satisfied:

1. $\mathscr{S}_\alpha^k$ is an ε-family for every $\alpha \in \Gamma$, $k \in N$.

2. To any $E \in \mathscr{S}_\alpha$ such that $\mu_\alpha(E) \geqq \frac{1}{2^k}$ there is $G \subset E$ such that $G \in \mathscr{D}_{\alpha,i}^{k+1}$ and $E - G \in \mathscr{B}_\alpha^{k,0}$.

3. To any $E, F \in \mathscr{S}_\alpha$ such that $F \subset E$, $\mu_\alpha(F) \geqq \frac{1}{2^k}$ and to any $G \in \mathscr{D}_{\alpha,i}^{k+1}$ such that $E - G \in \mathscr{B}_\alpha^{k,j}$ there is $H \in \mathscr{D}_{\alpha,i}^{k+1}$ such that $H \subset G \cap F$, $F - H \in \mathscr{B}_\alpha^{k,j+1}$.

4. If $D \in \mathscr{B}_\alpha^{k,j}$ and $\beta \supset \alpha$ then $\pi_{\beta\alpha}^{-1}(D) \in \mathscr{B}_\beta^{k,j}$.

5. If $D \in \mathscr{S}_\beta^k$, $\gamma \supset \beta \supset \alpha$, then $\pi_{\sigma\beta}^{-1}(D) \in \mathscr{S}_\gamma^k$, $\pi_{\beta\alpha}(D) \in \mathscr{S}_\alpha^k$.

Then the function μ is σ-additive on $\mathscr{S}$.

Proof. It sufficies to prove that μ is upper continuous in $\emptyset$, i. e. $A_n \supset A_{n+1}$,

$A_n \in \mathcal{S}$ $(n=1, 2, \ldots)$, $\bigcap_{n=1}^{\infty} A_n = \emptyset$ implies $\lim_{n\to\infty} \mu(A_n) = 0$. We shall prove it indirectly. Let there be $\varepsilon > 0$ such that $\mu(A_n) \geqq \varepsilon$ $(n=1, 2, \ldots)$. Choose k such that $\frac{1}{2^k} < \varepsilon$. Since $A_n \in \mathcal{S}$, there are $\alpha_n \in \Gamma$ and $A'_n \in \mathcal{S}_{\alpha_n}$ such that

$$A_n = \overset{-1}{\pi}_{\alpha_n}(A'_n), \quad A'_n \in \mathcal{S}_{\alpha_n}$$

We can assume that $\alpha_1 \subset \alpha_2 \subset \alpha_3 \subset \ldots$, since otherwise we should take $\alpha'_n = \bigcup_{i=1}^{n} \alpha_i$ instead of α_n and $A''_n = \overset{-1}{\pi}_{\alpha_n' \alpha_n}(A'_n)$ instead of A'_n.

Put $\bigcup_{n=1}^{\infty} \alpha_n = \{i_1, i_2, \ldots\}$ and let $i_1 \in \alpha_1$. In the condition 2 we put $\alpha = \alpha_1$, $i = i_1$, $E = A'_1$. Further, denote $B_1 = G$, hence

$$B_1 \subset A'_1, B_1 \in \mathcal{D}^{k+1}_{\alpha_1, i_1}, \quad A'_1 - B_1 \in \mathcal{B}^{k,0}_{\alpha_1}$$

Now we put in 3

$$E = \overset{-1}{\pi}_{\alpha_2\alpha_1}(A'_1), \quad F = A'_2, \quad G = \overset{-1}{\pi}_{\alpha_2\alpha_1}(B_1)$$

By 4 we have

$$E - G = \overset{-1}{\pi}_{\alpha_2\alpha_1}(A'_1 - B_1) \in \mathcal{B}^{k,0}_{\alpha_2}$$

since $A'_1 - B_1 \in \mathcal{B}^{k,0}_{\alpha_1}$. Now by 5 we have

$$G = \overset{-1}{\pi}_{\alpha_2\alpha_1}(B_1) \in \mathcal{D}^{k+1}_{\alpha_2, \alpha_1}$$

hence there exist

$$B_2 = H \in \mathcal{D}^{k+1}_{\alpha_2, i_1}, \quad B_2 \subset \overset{-1}{\pi}_{\alpha_2\alpha_1}(B_1)$$

such that

$$B_2 \subset A'_2, \quad A'_2 - B_2 \in \mathcal{B}^{k,1}_{\alpha_2}$$

Repeating the process infinitely many times we obtain a sequence $\{B_n\}_{n=1}^{\infty}$ such that

$$B_n \subset A'_n, \quad B_n \subset \overset{-1}{\pi}_{\alpha_n\alpha_{n-1}}(B_{n-1}), \quad B_n \in \mathcal{D}^{k+1}_{\alpha_n, i_1}$$

Put

$$C_n = \pi_{\alpha_n, i_1}(B_n)$$

By the definition of $\mathcal{D}^{k+1}_{\alpha_n, i_1}$ we have

$$C_n \in \mathcal{S}^{k+1}_{i_1}$$

Further

$$C_{n+1}=\pi_{\alpha_{n+1},i_1}(B_n)\subset\pi_{\alpha_{n+1},i_1}(\pi^{-1}_{\alpha_{n+1}\alpha_n}(B_n))=\pi_{\alpha_n,i_1}(B_n)=C_n$$

Therefore by 1 we have $\bigcap_{n=1}^{\infty} C_n\neq\emptyset$, hence there is

$$x^0_{i_1}\in\pi_{\alpha_n,i_1}(B_n)\qquad(n=1,2,\ldots)$$

Since $x^0_{i_1}\in\pi_{\alpha_n,i_1}(B_n)$ and $B_n\in\mathscr{D}^{k+1}_{\alpha_n,i_1}$, we obtain

$$\{y\,;\,(x^0_{i_1},y)\in B_n\}\mathscr{S}^{k+1}_{\alpha_n-\{i_1\}}\qquad(n=1,2,\ldots)$$

i. e. to any n there is y_n such that $(x^0_{i_1},y_n)\in B_n$. Put

$$D_n=\{y\,;\,(x^0_{i_1},y)\in B_n\}$$

Of course, $D_n\in\mathscr{S}^{k+1}_{\alpha_n-i_1}$. Now take sufficiently large n (such that $i_2\in\alpha_n$) and put

$$E_n=\pi_{\alpha_n-i_1,i_2}(D)$$

Then $E_n\in\mathscr{S}^{k+1}_{i_2}$ by 5, $E_n\supset E_{n+1}$, hence $\bigcap_{n=1}^{\infty}E_n\neq\emptyset$ by 1. Hence there is $x^0_{i_2}$ such that

$$x^0_{i_2}\in E_n\qquad(n \text{ such that } i_2\in\alpha_n)$$

Therefore there is such z_n that

$$(x^0_{i_2},z_n)\in D_n$$

or there is y_n such that

$$(x^0_{i_1},x^0_{i_2},y_n)\in B_n$$

etc. So we get (for such n that $\{i_1,\ldots,i_k\}\subset\alpha_n$) there is y_n for which

$$(x^0_{i_1},\ldots,x^0_{i_k},y_n)\in B_n\subset A'_n$$

If we denote $\beta=\{i_1,\ldots,i_k\}$ then

$$(x^0_{i_1},\ldots,x^0_{i_k})\in\pi_{\alpha_n,\beta}(A'_n)=\pi_\beta(\pi^{-1}_{\alpha_n}(A'_n))=\pi_\beta(A_n)$$

for sufficiently large n. But $A_{n+1}\subset A_n$ $(n=1,2,\ldots)$ hence

$$(x^0_{i_1},\ldots,x^0_{i_k})\in\pi_\beta(A_n)\quad\text{for every } n$$

Now define $x^0\in X$ as follows:

$$x^0(i_t)=x^0_{i_t},\qquad t=1,2,\ldots$$

$x^0(i)=$ arbitrary element of X_i otherwise.

We want to prove that $x^0 \in \bigcap_{n=1}^{\infty} A_n$. Take any n and k such that $\alpha_n \subset \{i_1, \ldots, \ldots, i_k\} = \beta$. Then

$$\pi_\beta(x^0) = (x^0_{i_1}, \ldots, x^0_{i_k}) \in \pi_\beta(A_n)$$

Hence there exists $y \in A_n$ such that $y_{i_1} = x^0_{i_1}, \ldots, y_{i_k} = x^0_{i_k}$. Denote $\alpha_n = \{u_1, \ldots, \ldots, u_t\} \subset \beta$. Then also

$$y_{u_1} = x^0_{u_1}, \ldots, y_{u_t} = x^0_{u_t}$$

hence

$$\pi_{\alpha_n}(x^0) = (x^0_{u_1}, \ldots, x^0_{u_t}) \in \pi_{\alpha_n}(A_n) = A'_n$$

It means $x^0 \in \pi^{-1}_{\alpha_n}(A'_n) = A_n$. Of course, n was arbitrary.

Corollary 1. Let $X_t (t \in T)$ be topological spaces and $\mathscr{C}_\alpha \subset \mathscr{S}_\alpha$ be families of compact sets such that

$$\mu_\alpha(E) = \sup \{\mu_\alpha(C) ; C \subset E, C \in \mathscr{C}_\alpha\}$$

for every $E \in \mathscr{S}_\alpha$. Then μ is σ-additive on $\mathscr{S}$.

Proof. As before we put

$$\mathscr{S}^k_\alpha = \{E \in \mathscr{S}_\alpha ; E = \pi^{-1}_{\alpha\beta}(D), D \neq \emptyset, D \text{ compact}, D \in \mathscr{S}_\beta, \alpha \supset \beta\}$$

for every k, and

$$\mathscr{B}^{k,j}_\alpha = \left\{E \in \mathscr{S}_\alpha ; \mu_\alpha(E) < \frac{1}{2^k} + \frac{1}{2^{k+1}} + \ldots + \frac{1}{2^{k+j}}\right\}$$

We prove that the assumptions 1—5 of Theorem are satisfied.

1. Let $E_n \in \mathscr{S}^k_\alpha$, $E_n \supset E_{n+1}$. Since α is a finite set, there is $\beta \subset \alpha$ such that $E_n = \pi^{-1}_{\alpha\beta}(D_n)$ for infinitely many n. Denote the set of indices by ϱ. Then

$$\bigcap_{n=1}^{\infty} E_n = \bigcap_{n \in \varrho} E_n = \bigcap_{n \in \varrho} \pi^{-1}_{\alpha\beta}(D_n) = \pi^{-1}_{\alpha\beta}\Big(\bigcap_{n \in \varrho} D_n\Big)$$

But the set $\bigcap_{n \in \varrho} D_n$ is compact and non-empty, hence $\bigcap_{n=1}^{\infty} E_n \in \mathscr{S}^k_\alpha$.

2. By the assumption of Corollary there is $G \in \mathscr{S}^k_\alpha$ such that

$$\mu_\alpha(E - G) < \frac{1}{2^k}$$

i. e.

$$E - G \in \mathscr{B}^{k,0}_\alpha$$

Further, the set $\pi_{\alpha,i}(G)$ is compact, i. e. $\pi_{\alpha,i}(G) \in \mathscr{S}^{k+1}_i$. Similarly, the set $\{y ;$

$(x, y) \in G\}$ is compact and also non-empty (for $x \in \pi_{\alpha,i}(G)$). Hence $G \in \mathscr{D}_{\alpha,i}^{k+1}$.

3. The proof is similar to the proof of the preceeding case.

4. and 5. Obvious.

Corollary 2. Let $(X_t, \mathscr{S}_t, \mu_t)$ be a collection of probability measures. For $E \in \mathscr{S}_\alpha$, $E = \bigcup_{i=1}^{n} (E_1^i \times E_2^i \times \ldots \times E_k^i)$, where $\alpha = \{t_1, \ldots, t_k\}$, $E_j^i \in \mathscr{S}_{t_j}$ and $E_1^i \times \ldots \times E_k^i$ are pariwise disjoint we put

$$\mu_\alpha(E) = \sum_{i=1}^{n} \mu_{t_1}(E_1^i)\mu_{t_2}(E_2^i) \ldots \mu_{t_k}(E_k^i)$$

Then the corresponding function μ on $\mathscr{S}$ is σ-additive.

Proof. Put $\mathscr{S}_\alpha^k = \left\{E \in \mathscr{S}_\alpha \,;\, \mu_\alpha(E) \geqq \frac{1}{2^k}\right\}$, $\mathscr{B}_\alpha^{k,i} = \mathscr{S}_\alpha$. Then the first assumption is satisfied evidently. Similarly 4 and 5 are obvious. We prove 2, the proof of 3 is similar.

Let $E \in \mathscr{S}_\alpha$, $\mu_\alpha(E) \geqq \frac{1}{2^k}$. Let $i \in \alpha$, $E = \bigcup_{j=1}^{n} (B_j \times C_j)$, B_j pairwise disjoint. Denote

$$\varrho = \left\{j \,;\, \mu_{\alpha-i}(C_j) \geqq \frac{1}{2^{k+1}}\right\}$$

and

$$G = \bigcup_{j \in \varrho} (B_j \times C_j) \subset E$$

Then

$$\pi_{\alpha,i}(G) = \bigcup_{j \in \varrho} B_j$$

and for every $x \in \pi_{\alpha,i}(G)$ we have

$$\{y \,;\, (x, y) \in G\} \in \mathscr{C}_{\alpha-i}^{k+1}$$

The proof will be finished if we show that

$$\mu_i\Big(\bigcup_{j \in \varrho} B_j\Big) \geqq \frac{1}{2^{k+1}}$$

But

$$\frac{1}{2^k} \leqq \mu_\alpha(G) = \sum_{j=1}^{n} \mu_i(B_j)\mu_{\alpha-i}(C_j) \leqq$$

$$\leqq \sum_{j \in \varrho} \mu_i(B_j) \cdot 1 + \sum_{j \notin \varrho} \mu_i(B_j) \cdot \frac{1}{2^{k+1}} =$$

$$= \mu_i\Big(\bigcup_{j \in Q} B_j\Big) + \frac{1}{2^{k+1}} \mu_i\Big(\bigcup_{j \notin Q} B_j\Big) \leqq$$

$$\leqq \mu_i\Big(\bigcup_{j \in Q} B_j\Big) + \frac{1}{2^{k+1}}.$$

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SÚHRN

O KOLMOGOROVEJ VETE A NEKONEČNOM SÚČINE PRAVDEPODOBNOSTNÝCH PRIESTOROV

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V článku je dokázaná jedna všeobecná veta, ktorej bezprostrednými dôsledkami sú Kolmogorova veta a veta o súčine ľubovoľného systému pravdepodobnostných priestorov. Ako je známe, tieto vety sú vo všeobecnosti nezávislé.

РЕЗЮМЕ

О ТЕОРЕМЕ КОЛМОГОРОВА И БЕЗКОНЕЧНОМ ПРОИЗВЕДЕНИИ ВЕРОЯТНОСТНЫХ ПРОСТРАНСТВ

Б. Риечан, Братислава

Приводится одна теоретико-множественная теорема, непосредственными следствиями которой являются с одной стороны теорема Колмогорова и с другой стороны теорема о произведении произвольного семейства вероятностных пространств. Как известно в общем случае эти теоремы взаимно независимы.

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TRANSFINITE CONVERGENCE AND LOCALLY SEPARABLE METRIC SPACES

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The papers [6], [7] deal with the limits of transfinite sequences of various types of functions $f: X \to Y$, where X, Y are supposed to be metric spaces. The separability of X guarantees that the pointwise limit of a transfinite sequence of so called quasticontinuous, cliquish and simply continuous functions is quasicontinuous, cliquish and simply continuous respectively. The same is true for some other types of functions.

When the separability of X is substituted by the locally separability, then for some of the theorems many of the above results follow immediately (the theorems on quasicontinuous and cliquish functions). The others need some modifications. We shall give such modifications. On the other hand a question whether the locally separability is essential to preserve some of the types of functions under the transfinite convergence, leads to a characterization of locally separable spaces. To give such a characterization is the other question discussed in this note.

A function $f: X \to Y$ (X, Y are topological spaces) is said to be simply continuous provided that for any open $G \subset Y$, $f^{-1}(G) = U \cup Z$, where U is open and Z nowhere dense in X.

The transfinite (pointwise) convergence (see [4]) of a sequence $\{f_\xi\}_{\xi<\Omega}$ (Ω means the first uncountable ordinal) of functions defined on X with values in Y to a function f means that $\lim_{\xi<\Omega} f_\xi(x) = f(x)$ for any $x \in X$, i. e. for $x \in X$ and any $\varepsilon > 0$ there exists $\lambda < \Omega$ such that $\xi > \lambda$ implies $\varrho(f_\xi(x), f(x)) < \varepsilon$ where ϱ is the metric in Y.

Theorem 1. Let $\{f_\xi\}$ ($\xi < \Omega$) be a transfinite sequence of simply continuous functions defined on a locally separable metric space X with the values in a metric space Y. Suppose that $\{f_\xi\}_{\xi<\Omega}$ converges to f. Then f is simply continuous.

Proof. For the case when X is separable we proved the theorem in [6]. Now we present the mentioned modification. According to a theorem of Sierpiński ([5]) $X = \bigcup_{t \in T} G_t$ where G_t are open in X, separable subspaces of X, and $G_{t'} \cap G_{t''} = \emptyset$ for $t' \neq t''$, t', $t'' \in T$.

Hence for any open $G \subset Y$

$$f^{-1}(G) = \bigcup_{t \in T} f^{-1}(G) \cap G_t \tag{1}$$

Put $f_t = f|G_t$, $f_\xi^t = f_\xi|G_t$. For any $x \in G_t$, $f_t(x) = \lim_{\xi<\Omega} f_\xi^t(x)$. If G is any open set in Y, then

$$(f_\xi^t)^{-1}(G) = f_\xi^{-1}(G) = f_\xi^{-1}(G) \cap G_t = (U_\xi \cup Z_\xi) \cap G_t =$$

$$= (U_\xi \cap G_t) \cup (Z_\xi \cap G_t)$$

where U_ξ is open and Z_ξ nowhere dense in X. Hence $U_\xi \cap G_t$ is open in G_t and $Z_\xi \cap G_t$ is nowhere dense in G_t. Thus each of the functions f_t is simply continuous on the subspace G_t. Since G_t is separable for any $t \in T$, we have (see [6] Theorem 1) that f_t is simply continuous on G_t. From (1) follows

$$f^{-1}(G) = \bigcup_{t \in T} f^{-1}(G) \cap G_t = \bigcup_{t \in T} f_t^{-1}(G) =$$

$$= \bigcup_{t \in T} (U_t \cup Z_t) = (\bigcup_{t \in T} U_t) \cup (\bigcup_{t \in T} Z_t)$$

Since U_t is open in G_t and Z_t nowhere dense in G_t, we have that $\bigcup_{t \in T} U_t$ is open in X and $\bigcup_{t \in T} Z_t$ is nowhere dense in X because of the fact that $G_t (t \in T)$ are open and pairwise disjoint. Thus f is simply continuous on X.

A function $f: X \to Y$ is said to be quasicontinuous at the point $x_0 \in X$ if for any neighbourhood $W(x_0)$ of the point x_0 and any neighbourhood $V(f(x_0))$ of $f(x_0)$ there exists an open set

$$U \subset W(x_0), \quad U \neq \emptyset, \quad \text{such that} \quad f(U) \subset V(f(x_0))$$

It is said to be cliquish at $x_0 \in X$ (X is a topological space, Y a metric space with the metric ϱ) if for any $\varepsilon > 0$ and any neighbourhood $W(x_0)$ there exists a non-empty open set $\bigcup \subset W(x_0)$ such that $\varrho(f(x), f(x')) < \varepsilon$ for any two points x', $x'' \in \bigcup$. The function f is said to be quasicontinuous or cliquish on X if it is quasicontinuous or cliquish respectively in any $x \in X$.

The transfinite convergence of quasicontinuous and cliquish functions on separable metric spaces preserves the quasicontinuity and cliquishness respectively ([7] Theorems 1 and 2). The generalizations of these theorems for locally separable metric spaces is obvious because of the fact, that any point has separable neighbourhood and because of the fact, that causicontinuity and cliquishness on X is defined by means of the quasicontinuity and cliquishness at any point $x \in X$, so the problem may be reduced to a separable space. But the following theorem is of some interest.

Theorem 2. A metric space X is locally separable if the following holds for any subspace $Y \subset X$: if $\{f_\xi\}_{\xi<\Omega}$ is a transfinite sequence of real quasicontinuous functions on Y such that $f = \lim_{\xi<\Omega} f_\xi$, then f is quasicontinuous on Y.

Proof. The sufficiency was discussed above, so the proof of this part may be omitted. Let us prove the necessity. Suppose X not to be locally separable. Then there exists an isolated set $M \subset X$ such that M has a point of condensation y, i. e. $U(y) \cap M$ is uncountable for any neighbourhood $U(y)$ of y. (see [8]). Let $I_n = \left\{x : x \in M,\ \varrho(x, y) < \frac{1}{n}\right\}$, $n = 1, 2, \ldots$ (ϱ is the metric in X).

Each of the sets J_n is uncountable. Construct a transfinite sequence $\{x_\xi\}_{\xi<\Omega}$ as follows. Let $x_1 \in M$ be any point. Suppose that the sequence $\{x_\zeta\}_{\zeta<\xi}$ is already constructed for $\xi < \Omega$, $\zeta < \xi$, $\xi > 1$. If ξ is not a limit ordinal then there exists such a limit number ξ_0 that $\xi = \xi_0 + n$ where n is positive integer. Choose $x_\xi \in J_n$ such that $x_\xi \neq x_\zeta$ for $\zeta < \xi$, If ξ is a limit ordinal number, we choose as x_ξ any point belonging to M such that $x_\xi \neq x_\zeta$ for $\zeta < \xi$. Hence we have a sequence $\{x_\xi\}_{\xi<\Omega}$ such that for any $\mu < \Omega$ and any positive integer n there exists $\xi > \mu$ with $x_\xi \in J_n$. Denote by Z the set of the values of $\{x_\xi\}_{\xi<\Omega}$ and put $Y = Z \cup \{y\}$. Then Y considered as a subspace X is not locally separable. Define on Y $\{f_\xi\}_{\xi<\Omega}$ as follows:

$$f_\xi(x) = \begin{cases} 0 & \text{if} \quad x = x_\eta,\ \eta < \xi \\ 1 & \text{if} \quad x = x_\eta,\ \eta \geqq \xi \\ 1 & \text{if} \quad x = y \end{cases}$$

Each of the functions $f_\xi (\xi < \Omega)$ is quasicontinuous on Y. In fact, let $\xi_0 < \Omega$. Quasicontinuity at the points of Z is evident since the points are isolated. If suffices to prove the quasicontinuity in y. Choose $\varepsilon > 0$ and any neighbourhood U of the point y. Let n be such that the sphere with the center y and radius $\frac{1}{n}$ is contained in U. There exists $x_\eta \in J_n$ such that $\eta > \xi_0$. The one-point set $\{x_\eta\} \subset U$ is open and $|f_{\xi_0}(x_\eta) - f_{\xi_0}(y)| = 0$. The quasicontinuity at y is proved. Morever $\lim_{\xi<\Omega} f_\xi(x) = 0$ for $x \in Z$ and $\lim_{\xi<\Omega} f_\xi(y) = 1$. But the function

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in Z \\ 1 & \text{if} \quad x = y \end{cases}$$

is not quasicontinuous on Y because it is not quasicontinuous at y.

Such a characterization of locally separable metric spaces as was given in

Theorem 2 is not possible by means of cliquish functions. On the contrary the following is true.

Theorem 3. There exists a metric space which is not locally separable and such that on each of its subspaces the pointwise convergence of a sequence $\{f_\xi\}_{\xi<\Omega}$ of real cliquish functions implies the cliquishness of the limit function.

Proof. The metric space (Y, ϱ) from the proof of the preceeding theorem may serve as an example. Any of the real functions defined on Y is cliquish. In fact if f is defined on Y then the cliquishness at any point different from y is evident, because it is isolated. At the point y cliquishness follows from the fact that in any neighbourhood of U of the point y there is an isolated point x and hence an open set $\{x\} \subset U$. The situation on any subspace of Y is the same.

Note 1. The same is the situation when the cliquish functions in the preceeding Theorem are substituted by pseudocontinuous (see [1, 2] for the definition) or simply continuous functions. For the proof it suffices to note that any f defined on Y is pseudocontinuous as well as simply continuous.

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SÚHRN

TRANSFINITNÁ KONVERGENCIA A LOKÁLNE SEPARABILNÉ METRICKÉ PRIESTORY

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V práci sú charaktèrizované lokálne separabilné metrické priestory pomocou transfinitnej konvergencie. Dokazuje sa veta, ktorá tvrdí, že priestor X je lokálne separabilný vtedy a len vtedy, ak na každom jeho podpriestore limita každej transfinitnej postupnosti kvázispojitých funkcií je kvázispojitá funkcia.

РЕЗЮМЕ

ТРАНСФИНИТНАЯ СХОДИМОСТЬ И ЛОКАЛЬНО СЕПАРАБЕЛЬНЫЕ МЕТРИЧЕСКИЕ ПРОСТРАНСТВА

А. Нойбруннова, Братислава

В работе охарактеризованы локально сепарабельные метрические пространства с помощью трансфинитной сходимости. Доказывается теорема, в которой утверждается, что пространство X есть локально сепарабельное тогда и только тогда, когда в каждом его подпространстве предел каждой трансфинитной последовательности квазинепрерывных функций является квазинепрерывной функцией.

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K PROBLÉMU POROVNANIA DYNAMICKEJ A STATICKEJ NÁSOBNOSTI V TEÓRII IDEÁLOV

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Nech A je d-rozmerný lokálny noetherovský okruh (komutatívny s jednotkou) s maximálnym ideálom $\mathcal{M}$. Nech $\mathcal{G} \subset A$ je parametrický ideál, t. j. $\mathcal{M}$-primárny ideál, ktorý je vytvorený systémom parametrov ([7]). Pre dostatočne veľké n je $l(A/\mathcal{G}^{n+1})$ (dĺžka A-modulu $A/\mathcal{G}^{n+1}$) polynóm s premennou n a s celočíselnými koeficientami stupňa d, nazývaný Hilbertov—Samuelov polynóm; zapíšeme ho v tvare

$$l(A/\mathcal{G}^{n+1}) = e_0(\mathcal{G}, A)\binom{n+d}{d} - e_1(\mathcal{G}, A)\binom{n+d-1}{d-1} + \ldots + (-1)^d e_d(\mathcal{G}, A)$$

([7] alebo [4]).

Koeficient $e_0(\mathcal{G}, A) > 0$ sa nazýva násobnosťou $\mathcal{G}$ v A (dynamická násobnosť v zmysle Serra). W. Gröbner v [2] definuje násobnosť $\mathcal{G}$ v A ako $l(A/\mathcal{G})$ (statická násobnosť). Tieto dve násobnosti sa vo všeobecnosti líšia a ich vzťah je daný nerovnosťou ([7])

$$D = : l(A/\mathcal{G}) - e_0(\mathcal{G}, A) \geqq 0$$

Rovnosťou $D = 0$ sú charakterizované Cohenove—Macaulayho okruhy ([7]). Existuje obsiahlejšia trieda okruhov, v ktorých D je invariant nezávislý od $\mathcal{G}$. Tieto okruhy sú študované v [6] a nazývajú sa B-okruhy (v [6] I-okruhy). Podotýkam, že $l(A/\mathcal{G}^{n+1}) < \infty$ $[l(A/\mathcal{G}) < \infty)$, pretože $A/\mathcal{G}^{n+1}$ $[A/\mathcal{G}]$ je noetherovský a artinovský okruh.

Nech $l(A/\mathcal{G}) = \mu$. Potom existuje reťazec $\mathcal{M}$-primárnych ideálov v A v tvare

$$\mathcal{G} = : \mathcal{G}_1 \subsetneqq \mathcal{G}_2 \subsetneqq \ldots \subsetneqq \mathcal{G}_{\mu-1} \subsetneqq \mathcal{G}_\mu = : \mathcal{M}$$

Je zrejmé, že $\{l(A/\mathcal{G}_i)\}_{i=1}^{\mu} = \{1, 2, \ldots, \mu\}$, a teda medzi ideálmi $\mathcal{G}_i$ pre všetky $i = 1, \ldots, \mu$ existuje práve jeden $\mathcal{G}_s$ s vlastnosťou

$$l(A/\mathcal{G}_s) = e_0(\mathcal{G}, A)$$

Konštrukcia takého $\mathcal{G}_s$ je obsahom tejto práce.

Pre ľubovoľný ideál $\mathcal{A} \subset A$ označme $\mathrm{Ass}(\mathcal{A})$ množinu všetkých prvoideálov $\mathcal{P}$, ktoré patria k neskrátitelnému primárnemu rozkladu ideálu $\mathcal{A}$. (V ďalšom budeme predpokladať, že každý primárny rozklad je už neskrátitelný). Ďalej označme:

$$\mathrm{Assh}(\mathcal{A}) =: \{\mathcal{P} \in \mathrm{Ass}(\mathcal{A})\,;\, \dim(\mathcal{P}) = \dim(\mathcal{A})\}$$

Nech $\mathcal{G} = (a_1, \ldots, a_d)$ je parametrický ideál v A a $(0) = \bigcap_{i=1}^{s_0} \mathcal{G}_i^0$ je primárny rozklad nulového ideálu. Konštruujme nasledujúce ideály $*$-procesom. Vyberme z primárneho rozkladu nulového ideálu tie komponenty $\mathcal{G}_i^0$, pre ktoré

$$\dim(\mathcal{G}_i^0) = \dim((0))$$

Nech sú to $\mathcal{G}_1^0, \ldots, \mathcal{G}_{t_0}^0$, $t_0 \leqq s_0$. Pretože takéto komponenty sú izolovanými komponentmi primárneho rozkladu ideálu (0), je ich výber jednoznačný. Nech

$$\left(a_1, \bigcap_{j=1}^{t_0} \mathcal{G}_j^0\right) = \bigcap_{i=1}^{s_1} \mathcal{G}_i^1$$

je primárny rozklad ideálu $\left(a_1, \bigcap_{j=1}^{t_0} \mathcal{G}_j^0\right)$. Vyberme z neho opäť všetky tie komponenty $\mathcal{G}_i^1$, pre ktoré

$$\dim(\mathcal{G}_i^1) = \dim\left(\left(a_1, \bigcap_{j=1}^{t_0} \mathcal{G}_j^0\right)\right)$$

Nech sú to $\mathcal{G}_1^1, \ldots, \mathcal{G}_{t_1}^1$, $t_1 \leqq s_1$. Takýmto postupom skonštruujeme ideály

$$\left(a_2, \bigcap_{j=1}^{t_1} \mathcal{G}_j^1\right), \left(a_3, \bigcap_{j=1}^{t_2} \mathcal{G}_j^2\right), \ldots, \left(a_d, \bigcap_{j=1}^{t_{d-1}} \mathcal{G}_j^{d-1}\right) =: \tilde{\mathcal{G}}$$

Lema 1. A. $(a_1, \ldots, a_s) \subseteqq \left(a_s, \bigcap_{j=1}^{t_{s-1}} \mathcal{G}_j^{s-1}\right)$ pre všetky $s = 1, \ldots, d$.

B. Ideál $\tilde{\mathcal{G}}$ je $\mathcal{M}$-primárny.

Dôkaz. Vyplýva z konštrukcie.

Lema 2. $(0):(a_i^n) \subseteqq \bigcap_{j=1}^{t_0} \mathcal{G}_j^0$ pre všetky $i = 1, \ldots, d$ a pre všetky prirodzené čísla n.

Dôkaz. Nech $x \in (0):(a_i^n)$. Potom $x \cdot a_i^n = 0$; to znamená, že $a_i^n \in (0):(x)$. Pretože (a_i) je $(d-1)$-rozmerný ideál (7), platí $a_i^n \notin \mathcal{P}_j^0 = \sqrt{\mathcal{G}_j^0}$ pre všetky $j = 1, \ldots, t_0$. To znamená, že $(0):(x) \not\subset \mathcal{P}_j^0$ pre všetky $j = 1, \ldots, t_0$. Posledné je však ekvivalentné s tým, že $x \in \mathcal{G}_j^0$ pre všetky $j = 1, \ldots, t_0$ ([7]), teda

$$x \in \bigcap_{j=1}^{t_0} \mathscr{G}_j^0$$

čo sme mali dokázať.

W. Vogel vyslovil domnienku, že $l(A/\bar{\mathscr{G}}) = e_0(\mathscr{G}, A)$. Ukážeme, že za istých okolností a predpokladov o ideále $\mathscr{G}$ platí táto domnienka ako veta. Vychádzajúc z [5] definujeme induktívne vzhľadom na rozmer okruhu A pre všetky $i \in N$ nasledujúci ideál

$$M_A^i(a_1, \ldots, a_d)$$

Pred $d = 0$ nech $M_A^i(0) = (0)$.

Pred $d \geqq s > 0$ nech

$$M_A^i(a_1, \ldots, a_s) = (a_s) + M_A^i(a_1, \ldots, a_{s-1}) : (a_s^i)$$

Lema 3. Pre všetky $i \in N$ platí:

A. $$M_A^i(a_1, \ldots, a_s) \subseteqq M_A^i(a_1, \ldots, a_{s+1})$$

pre všetky $s = 1, \ldots, d-1$.

B. $$(a_1, \ldots, a_d) \subseteqq M_A^i(a_1, \ldots, a_d)$$

C. $$M_A^i(a_1, \ldots, a_d) \subseteqq M_A^{i+1}(a_1, \ldots, a_d)$$

D. $$M_A^i(a_1, \ldots, a_d)/M = M_{\bar{A}}^i(\bar{a}_2, \ldots, \bar{a}_d),$$

pričom

$$M = : M_A^i(a_1), \bar{A} = A/M, \bar{a}_j = ((a_j) + M)/M$$

pre všetky $j = 2, \ldots, d$.

Dôkaz. Tvrdenia A, B, C sú triviálne. Tvrdenie D dokážeme indukciou vzhľadom na d. Pre $d = 1$ je rovnosť zrejmá. Nech teraz

$$M_A^i(a_1, \ldots, a_{d-1})/M = M_{\bar{A}}^i(\bar{a}_2, \ldots, \bar{a}_{d-1})$$

Potom

$$M_{\bar{A}}^i(\bar{a}_2, \ldots, \bar{a}_d) = :(a_d) + M_{\bar{A}}^i(\bar{a}_2, \ldots, \bar{a}_{d-1}) : (\bar{a}_d^i) =$$
$$= ((a_d) + M)/M + M_A^i(a_1, \ldots, a_{d-1})/M : ((a_d^i) + M)/M =$$
$$= ((a_d) + M)/M + (M_A^i(a_1, \ldots, a_{d-1}) : (a_d^i))/M =$$
$$= ((a_d) + M_A^i(a_1, \ldots, a_{d-1}) : (a_d^i))/M = M_A^i(a_1, \ldots, a_d)/M$$

Dôkaz sme týmto ukončili.

Vychádzajúc z C lemy 3 definujeme

$$M_A(a_1, \ldots, a_d) = : \bigcup_{i=1}^{\infty} M_A^i(a_1, \ldots, a_d)$$

Pretože A je noetherovský, existuje také prirodzené číslo n, že

$$M_A(a_1, \ldots, a_d) = M_A^n(a_1, \ldots, a_d)$$

Definícia. Nech R je okruh s jednotkou. Hovoríme, že prvoideál okruhu R má výšku k (označujeme $h(\mathcal{P}) = k$), ak existuje reťazec $\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \ldots \ldots \subsetneq \mathcal{P}_{k-1} \subsetneq \mathcal{P}_k = : \mathcal{P}$ prvoideálov okruhu R a neexistuje taký reťazec, ktorý pozostáva z viac ako $k+1$ členov.

Nech $\mathcal{G} = (a_1, \ldots, a_d)$ je parametrický ideál v d-rozmernom lokálnom noetherovskom okruhu A (komutatívnom s jednotkou).

Podmienka W. Nech pre všetky $i = 1, \ldots, d$ $a_i \notin \mathcal{P}$ pre všetky izolované ideály $\mathcal{P} \in \mathrm{Ass}((a_1, \ldots, a_{i-1}))$ s $\dim(\mathcal{P}) = d - i$ (pre $i = 1$ definujeme $(a_1, \ldots, a_{i-1}) = = : (0)$).

Veta 1. Nech parametrický ideál $\mathcal{G} = (a_1, \ldots, a_d)$ spĺňa podmienku W. Potom pre všetky $i = 1, \ldots, d$ a pre všetky prvoideály $\mathcal{P} \in \mathrm{Assh}((a_1, \ldots, a_i))$ platí $h(\mathcal{P}) = i$.

Dôkaz. Vetu dokážeme indukciou vzhľadom na i. Pre $i = 1$ nech $\mathcal{P} \in \mathrm{Assh}((a_1))$ a nech $h(\mathcal{P}) = 0$. Potom $\mathcal{P}$ je $(d-1)$-rozmerný izolovaný prvoideál asociovaný s (0) a $a_1 \in \mathcal{P}$. To je však v spore s podmienkou W, teda $h(\mathcal{P}) = 1$ ([7]). Nech teraz pre každé $\mathcal{P} \in \mathrm{Assh}((a_1, \ldots, a_{i-1}))$ platí $h(\mathcal{P}) = i - 1$. Nech ďalej $\mathcal{P}_0 \in \mathrm{Assh}((a_1, \ldots, a_i))$. Potom $\mathcal{P}_0$ je $(d-i)$-rozmerný prvoideál; ktorý obsahuje $(a_1, \ldots, a_{i-1})$. To znamená, že $\mathcal{P}_0$ obsahuje izolovaný prvoideál

$$\mathcal{P}_1 \in \mathrm{Ass}((a_1, \ldots, a_{i-1})), \text{ pre ktorý } \dim(\mathcal{P}_1) \geqq d - i\,.$$

Keby $\dim(\mathcal{P}_1) = d - i$, potom by $\mathcal{P}_0 = \mathcal{P}_1$ a $a_i \in \mathcal{P}_1$, čo je spor s podmienkou W. Je teda $\dim((\mathcal{P}_1) = d - i + 1$, *t. j.* $\mathcal{P}_1 \in \mathrm{Assh}((a_1 \ldots, a_{i-1}))$ a $\mathcal{P}_1 \subsetneq \mathcal{P}_0$. Pretože $h(\mathcal{P}_1) = i - 1$, platí $h(\mathcal{P}_0) = i$, čo bolo treba dokázať.

Veta 2. Nech A je d-rozmerný lokálny neotherovský okruh (komutatívny s jednotkou) s maximálnym ideálom $\mathcal{M}$. Nech $\mathcal{G} = (a_1, \ldots, a_d)$ je parametrický ideál v A, ktorý spĺňa podmienku W. Potom

$$\tilde{\mathcal{G}} = M_A(a_1, \ldots, a_d)$$

Dôkaz. Vetu dokážeme indukciou vzhľadom na d. Pre $d = 0$ je tvrdenie triviálne. Vychádzame z dohovoru, že prázdnou množinou je generovaný nulový ideál. Pre $d = 1$ je $\dim(A) = 1$, $\mathcal{G} = (a_1)$ a $a_1 \notin \mathcal{P}$ pre všetky $\mathcal{P} \in \mathrm{Assh}((0))$. Ak a_1 nie je deliteľ nuly v A (t. j. $\mathcal{M} \notin \mathrm{Assh}((0))$), potom $(0):(a_1^i) = (0)$ pre všetky prirodzené čísla i ([7]) a nulový ideál je nezmiešaný. To znamená, že pre všetky $n \in N$

$$\tilde{\mathcal{G}} = : \left(a_1, \bigcap_{i=1}^{t_0} \mathcal{G}_i^0\right) = (a_1, (0)) = (a_1) + (0) : (a_1^n) = M_A(a_1)$$

Nech a_1 je deliteľ nuly v A, $(0) = \mathcal{G}_1 \cap \ldots \cap \mathcal{G}_s \cap \mathcal{G}_{s+1}$ je primárny rozklad nulového ideálu, pričom $\sqrt{\mathcal{G}_i}=\mathcal{P}_i$ pre všetky $i=1, \ldots, s$ a $\sqrt{\mathcal{G}_{s+1}}=\mathcal{M}$. Potom pre dostatočne veľké n platí

$$(0):(a_1^n)=\bigcap_{i=1}^{s+1}\mathcal{G}_i:(a_1^n)=\bigcap_{i=1}^{s}\mathcal{G}_i$$

To znamená, že pre dostatočne veľké n je

$$\mathcal{G}=:\left(a_1, \bigcap_{i=1}^{s}\mathcal{G}_i\right)=(a_1)+(0):(a_1^n)=M_A(a_1)$$

K dôkazu vety 2 pre $d>1$ potrebujeme nasledujúce pomocné vety.

Pomocná veta 1. Za predpokladov vety 2 nech

$$(0)={}^d\mathcal{G}_1^0\cap\ldots\cap{}^d\mathcal{G}_s^0\cap{}^{d-1}\mathcal{G}_1^0\cap\ldots\cap{}^0\mathcal{G}_1^0$$

$$(a_1)={}^{d-1}\mathcal{G}_1^1\cap\ldots\cap{}^{d-1}\mathcal{G}_t^1\cap{}^{d-2}\mathcal{G}_1^1\cap\ldots\cap{}^0\mathcal{G}_1^1 .$$

sú príslušné primárne rozklady a $\sqrt{{}^i\mathcal{G}_k^j}={}^i\mathcal{P}_k^j$ (index i vyjadruje rozmer príslušného primárneho ideálu, a teda i radikálu). Potom pre dostatočne veľké n

A. $\dim\left(\left(a_1, \bigcap_{j=1}^{s}{}^d\mathcal{G}_j^0\right)\right)=\dim((a_1)+(0):(a_1^n))=d-1.$

B. Ak $\mathcal{R}_i={}^{d-1}\mathcal{G}_i^1+\bigcap_{j=1}^{s}{}^d\mathcal{G}_j^0$ pre všetky $i=1, \ldots, t$, potom $\sqrt{\mathcal{R}_i}={}^{d-1}\mathcal{P}_i^1$, a teda primárny rozklad ideálu $\mathcal{R}_i$ obsahuje jediný izolovaný komponent, a to ${}^{d-1}\mathcal{P}_i^1$ — primárny ideál ${}^{d-1}\mathcal{G}_i$.

C. Ak

$$\mathcal{A}=:\left(a_1, \bigcap_{j=1}^{s}{}^d\mathcal{G}_j^0\right)={}^{d-1}\mathcal{G}_1^*\cap\ldots\cap{}^{d-1}\mathcal{G}_m^*\cap{}^{d-2}\mathcal{G}_1^*\cap\ldots\cap{}^0\mathcal{G}_1^*$$

$$\mathcal{B}=:(a_1)+(0):(a_1^n)={}^{d-1}\mathcal{G}_1^{**}\cap\ldots\cap{}^{d-1}\mathcal{G}_n^{**}\cap{}^{d-2}\mathcal{G}_1^{**}\cap\ldots\cap{}^0\mathcal{G}_1^{**}$$

sú príslušné primárne rozklady, potom $t=n=m$ a pri vhodnom usporiadaní

$${}^{d-1}\mathcal{G}_i^{**}={}^{d-1}\mathcal{G}_i^*={}^{d-1}\mathcal{G}_i \quad \text{pre všetky } i=1, \ldots, t$$

Dôkaz pomocnej vety 1. A. $\dim(a_1)=d-1$([7]). Ďalej $\dim(\mathcal{A})\leqq\dim((a_1))=$ $=d-1$, $\dim(\mathcal{B})\leqq\dim((a_1))=d-1$, lebo $(a_1)\subseteqq\mathcal{A}$, $(a_1)\subseteqq\mathcal{B}$. Pretože existuje $(d-1)$-rozmerný prvoideál, napr. ${}^{d-1}\mathcal{P}_1^1$, ktorý obsahuje ideál $\mathcal{A}$ (ideál $\mathcal{B}$), platí $\dim(\mathcal{A})=\dim(\mathcal{B})=d-1$.

B. Tvrdenie vyplýva z vlastností radikálu ideálu.

C. Pretože $\{{}^{d-1}\mathcal{P}_i^1\}_{i=1}^t$ je množina všetkých $(d-1)$-rozmerných prvoideálov, ktoré obsahujú $(d-1)$-rozmerný ideál $\mathcal{A}$, platí:

$$\mathrm{Assh}(\mathcal{A})=\{{}^{d-1}\mathcal{P}_i^1\}_{i=1}^t$$

Špeciálne $t = m$ a $\sqrt{{}^{d-1}\mathcal{G}_1^*} = {}^{d-1}\mathcal{P}_i^1$ pre všetky $i = 1, \ldots, t$. Teraz si všimnime, že primárny rozklad ideálu $(0):(a_1^n)$ pre dostatočne veľké n dostaneme z primárneho rozkladu nulového ideálu vynechaním všetkých p-primárnych komponentov pre $a_1 \in \mathcal{P}$. Z toho okamžite vyplýva, že

$$\mathrm{Assh}(\mathcal{B}) = \{{}^{d-1}\mathcal{P}_i^1\}_{i=1}^t$$

$$t = n \text{ a } \sqrt{{}^{d-1}\mathcal{G}_1^{**}} = {}^{d-1}\mathcal{P}_i^1 \quad \text{pre všetky } i = 1, \ldots, t.$$

Pretože $\mathcal{B} \subseteq \mathcal{A} \subseteq \mathcal{R}_i$ pre všetky $i = 1, \ldots, t$ (lema 2), platí

$${}^{d-1}\mathcal{G}_1^{**} \subseteq {}^{d-1}\mathcal{G}_1^* \subseteq {}^{d-1}\mathcal{G}_i$$

pre všetky $i = 1, \ldots, t$ ([7]).

Pretože $(a_1) \subseteq \mathcal{B}$, je ${}^{d-1}\mathcal{G}_i^1 \subseteq {}^{d-1}\mathcal{G}_1^{**}$ pre všetky $i = 1, \ldots, t$.

Nech teraz pre ľubovolné i, $1 \leqq i \leqq t$, $x \notin {}^{d-1}\mathcal{G}_1^{**}$. Potom $\mathcal{B}:(x) \subseteq {}^{d-1}\mathcal{P}_i^1$ ([7]). To znamená, že

$$((0):(a_1^n)):(x) \subseteq {}^{d-1}\mathcal{P}_i^1$$

Pretože primárny rozklad ideálu $(0):(a_1^n)$ pre dostatočne veľké n neobsahuje $\mathcal{P}$-primárny komponent pre $\mathcal{P} \in \mathrm{Ass}((a_1))$, vyplýva z posledného, že

$$\bigcap_{j=1}^{s} {}^d\mathcal{G}_j^0 : (x) \subseteq {}^{d-1}\mathcal{P}_i^1, \quad \text{teda} \quad x \notin \bigcap_{j=1}^{s} {}^d\mathcal{G}_j^0$$

Tým sme ukázali, že

$$\mathcal{R}_i \subseteq {}^{d-1}\mathcal{G}_1^{**}$$

pre všetky $i = 1, \ldots t$. Použijúc na posledné poznámku za vetou 8 ([7]), dostávame

$${}^{d-1}\mathcal{G}_i \subseteq {}^{d-1}\mathcal{G}_1^{**}$$

pre všetky $i = 1, \ldots, t$. Teda

$${}^{d-1}\mathcal{G}_1^{**} \subseteq {}^{d-1}\mathcal{G}_1^* \subseteq {}^{d-1}\mathcal{G}_i \subseteq {}^{d-1}\mathcal{G}_1^{**}$$

pre všetky $i = 1, \ldots, t$; čo znamená, že všade platí rovnosť; a to bolo treba dokázať.

Pomocná veta 2. Za predpokladov vety 2 nech $\bar{A} = A/\mathcal{B}$, $\bar{a}_i = ((a_i) + \mathcal{B})/\mathcal{B}$ pre všetky $i = 2, \ldots, d$. Potom

A. $\bar{A}$ je $(d-1)$-rozmerný lokálny neotherovský okruh s maximálnym ideálom $\bar{\mathcal{M}} = \mathcal{M}/\mathcal{B}$.

B. $\bar{\mathcal{G}} = (\bar{a}_2, \ldots, \bar{a}_d)$ je parametrický ideál v $A\bar{A}$.

C. Ideál $\bar{\mathcal{G}}$ spĺňa podmienku W.

Dôkaz pomocnej vety 2. A. $\dim(\bar{A}) = \dim(A/\mathcal{B}) = :\dim(\mathcal{B}) = d-1$.

B. Pretože $\mathcal{G} = (a_1, \ldots, a_d)$ je $\mathcal{M}$-primárny ideál v A a $\mathcal{G}/\mathcal{B} = \bar{\mathcal{G}} = (\bar{a}_2, \ldots,$

..., $\bar{a}_d$), je $\bar{\mathcal{G}}$ $\bar{\mathcal{M}}$-primárny ideál v $\bar{A}$. Pretože má $d-1$ generátorov, je parametrickým ideálom v $\bar{A}$.

C. Pretože pre každé $\mathcal{P} \in \mathrm{Assh}((a_1, \ldots, a_{i-1}))$ sa $h(\mathcal{P}) = i-1$ (veta 1) a $(a_1, \ldots, a_{i-1}) + \mathcal{B} = (a_1, \ldots, a_{i-1}) + (0):(a_1^n)$ pre dostatočne veľké n a $i>1$, platí

$$(0):(a_1^n) \subseteq \mathcal{P}$$

pre všetky $\mathcal{P} \in \mathrm{Assh}((a_1, \ldots, a_{i-1}))$, $i>1$ a n dostatočne veľké. To znamená, že pre $i>1$ sa

$$\mathrm{Assh}((a_1, \ldots, a_{i-1}) + \mathcal{B}) = \mathrm{Assh}((a_1, \ldots, a_{i-1}))$$

Teraz je zrejmé, že každý izolovaný $(d-i)$-rozmerný prvoideál asociovaný s $(a_1, \ldots, a_{i-1}) + \mathcal{B}$ pre $i>1$ je izolovaným prvoideálom asociovaným s $(a_1, \ldots, a_{i-1})$. Keby teraz $\bar{a}_i \in \bar{\mathcal{P}}$ pre $\bar{\mathcal{P}} \in \mathrm{Ass}((\bar{a}_2, \ldots, \bar{a}_{i-1}))$ izolovaný s $\dim(\mathcal{P}) = d-i$, potom by $a_i \in \mathcal{P}$ pre $\mathcal{P} \in \mathrm{Ass}((a_1, \ldots, a_{i-1}) + \mathcal{B})$ izolovaný s $\dim(\mathcal{P}) = d-i$, teda by $a_i \in \mathcal{P}$ pre $\mathcal{P} \in \mathrm{Ass}((a_1, \ldots, a_{i-1}))$ izolovaný s $\dim(\mathcal{P}) = d-i$, čo je v spore s podmienkou W. Tým sme dôkaz pomocnej vety 2 skončili.

Pokračujeme v dôkaze vety 2. Vychádzajúc z pomocnej vety 2 skonštruujme v okruhu $\bar{A}$ $*$-procesom ideály

$$\left(\bar{a}_2, \bigcap_{j=1}^{k_1} \bar{\mathcal{G}}_j^1\right), \left(\bar{a}_3, \bigcap_{j=1}^{k_2} \bar{\mathcal{G}}_j^2\right), \ldots, \left(\bar{a}_d, \bigcap_{j=1}^{k_{d-1}} \bar{\mathcal{G}}_j^{d-1}\right)$$

Pretože parametrický ideál $\bar{\mathcal{G}} = (\bar{a}_2, \ldots, \bar{a}_d)$ v okruhu $\bar{A}$ spĺňa podmienky vety 2 (pomocná veta 2), na základe indukčného predpokladu platí:

$$\left(\bar{a}_d, \bigcap_{j=1}^{k_{d-1}} \bar{\mathcal{G}}_j^{d-1}\right) = M_{\bar{A}}(\bar{a}_2, \ldots, \bar{a}_d)$$

Pretože $(d-1)$-rozmerné komponenty primárneho rozkladu ideálov $\mathcal{A}$ a $\mathcal{B}$ sa zhodujú (pomocná veta 1) a $\mathcal{B} \subseteq \mathcal{A}$, je

$$\bigcap_{j=1}^{k_1} \bar{\mathcal{G}}_j^1 = \bigcap_{j=1}^{t_1} \bar{\mathcal{G}}_j^1$$

To znamená, že

$$\left(a_2, \bigcap_{j=1}^{t_1} \mathcal{G}_j^1\right)/\mathcal{B} = \left(\bar{a}_2, \bigcap_{j=1}^{t_1} \bar{\mathcal{G}}_j^1\right) = \left(\bar{a}_2, \bigcap_{j=1}^{k_1} \bar{\mathcal{G}}_j^1\right)$$

Teraz je zrejmé, že ideály v A konštruované $*$-procesom z $(a_1, \ldots, a_d)$ a ideály v $\bar{A}$ konštruované tým istým procesom z $(\bar{a}_2, \ldots, \bar{a}_d)$ si $(1-1)$ — značne zodpovedajú v kanonickom homomorfizme z A do $\bar{A}$, t. j.

$$\left(a_i, \bigcap_{j=1}^{t_{i-1}} \mathcal{G}_j^{i-1}\right)/\mathcal{B} = \left(\bar{a}_i, \bigcap_{j=1}^{k_{i-1}} \bar{\mathcal{G}}_j^{i-1}\right)$$

pre všetky $i = 2, \ldots, d$. Špeciálne teda platí:

$$\mathscr{G}/\mathscr{B} = \left(a_d, \bigcap_{j=1}^{t_{d-1}} \mathscr{G}_j^{d-1}\right)/\mathscr{B} = \left(\bar{a}_d, \bigcap_{j=1}^{k_{d-1}} \bar{\mathscr{G}}_j^{d-1}\right) =$$

$$= M_{\bar{A}}(\bar{a}_2, \ldots, \bar{a}_d)$$

z toho na základe D lemmy 3

$$\bar{\mathscr{G}}/\mathscr{B} = M_A(a_1, \ldots, a_d)/\mathscr{B}$$

Pretože $\mathscr{B} \subseteq \mathscr{A} \subseteq \bar{\mathscr{G}}$ a $\mathscr{B} \subseteq M_A(a_1, \ldots, a_d)$ (lema 3) vyplýva z $(1-1)$-značnej korešpondencie medzi ideálmi v $\bar{A}$ a ideálmi v A obsahujúcimi $\mathscr{B}$, že

$$\bar{\mathscr{G}} = M_A(a_1, \ldots, a_d)$$

Dôkaz vety 2 sme ukončili.

Veta 3. Za predpokladov vety 2 platí:

$$e_0(\mathscr{G}, A) = l(A/\bar{\mathscr{G}})$$

Dôkaz. Indukciou vzhľadom na d dokážeme, že

$$e_0(\mathscr{G}, A) = l(A/M_A(a_1, \ldots, a_d)$$

Pre $d = 0$ je $e_0(\emptyset, A) = l(A) = l(A/(0))$ ([7]). Pre ľubovoľný parametrický ideál $(a_1, \ldots, a_d)$ v A platí

$$e_0((a_1, \ldots, a_d), A) = e_0((\bar{a}_2, \ldots, \bar{a}_d)\,\bar{A})$$

(tento výsledok vyplýva z [3]).

Na základe indukčného predpokladu a D lemy 3 dostaneme

$$e_0((a_1, \ldots, a_d), A) = e_0((\bar{a}_2, \ldots, \bar{a}_d), \bar{A}) =$$

$$= l(\bar{A}/M_{\bar{A}}(\bar{a}_2, \ldots, \bar{a}_d)) = l((A/\mathscr{B})/(M_A(a_1, \ldots, a_d)/\mathscr{B})) =$$

$$= l(A/\mathscr{B}) - l(M_A(a_1, \ldots, a_d)/\mathscr{B}) = l(A/M_A(a_1, \ldots, a_d))$$

Týmto sme vetu dokázali.

Poznámka. Z algebraicko-geometrického hľadiska poukazuje práca na jednu z ciest zisťovania násobnosti $i(C, V \cdot W)$ komponenty C prieseku projektívnych variet V a W. P. Samuel totiž dokázal, že ak $\dim(C) = \dim(V \cap W)$, potom $i(C, V \cdot W) = e_0(\mathscr{G}, A)$ pre určitý parametrický ideál $\mathscr{G}$ v určitom lokálnom noetherovskom okruhu A. Ak teda ideál $\mathscr{G}$ spĺňa podmienku W (to je triviálne splnené v prípade, že A je B-okruh, resp. regulárny lokálny okruh, resp. Cohenov—Macaulayho okruh), potom $i(C, V \cdot W) = l(A/\bar{\mathscr{G}})$.

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РЕЗЮМЕ

К ПРОБЛЕМЕ СРАВНЕНИЯ ДИНАМИЧЕСКОЙ И СТАТИЧЕСКОЙ КРАТНОСТЕЙ В ТЕОРИИ ИДЕАЛОВ

З. Бодя, Братислава

В работе обсуждена проблема существования и построения такого $\mathcal{M}$-примарного идеала $\mathcal{G}$ в локальном нётеровом кольце A (коммутативном с единицей) с максимальным идеалом $\mathcal{M}$, для которого статическая кратность (длина A-модуля $A/\mathcal{G}$) и динамическая кратность первоначально данного параметрического идеала $\mathcal{G}$ (старший член многочлена Гильберт—Самюэля $l(A/\mathcal{G}^{n+1})_{n\geq 0}$) равняются. Вводится т. н. $*$-процесс и показывается, что при его употреблении на параметрический идеал определённого класса параметрических идеалов кольца A получается $\mathcal{M}$-примарный идеал требованного свойства. В заключительном примечании показывается на геометрическое употребление этого результата.

RESUMMÉ

ZUM PROBLEM DES VERGLEICHES DER DYNAMISCHEN UND STATISCHEN MULTIPLIZITÄT IN DER IDEALTHEORIE

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In der Arbeit betrachtet man das Problem der Existenz und des Konstruieren solches $\mathcal{M}$-primären Ideals $\mathcal{G}$ in lokalem noetherschen Ring A (kommutativem mit Einselement) mit maximalen Ideal $\mathcal{M}$, für

welches statische Multiplizität (die Länge von A-Modul $A/\tilde{\mathscr{G}}$) und dynamische Multiplizität des ursprünglich gegebenen parametrischen Ideals $\mathscr{G}$ (die Leitkoefizient von Hilbert—Samuel Polynom $l(A/\mathscr{G}^{n+1})_{n\gg 0}$ gleich sind. Man definiert sog. $*$-Prozess und zeigt, dass man durch die Anwendung dieses Prozesses auf parametrisches Ideal von gewisser Klasse parametrischer Ideale des Ringes A ein $\mathscr{M}$-primäres Ideal erhält, das geforderte Eingeschaft hat. In Abschlussbemerkung weist man auf die geometrische Anwendungen hin.

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POZNÁMKA O JEDNOROZMERNEJ MIERE V DVOJROZMERNOM PRIESTORE

FRANTIŠEK VINŠ, Trnava

1. Úvod

Nech $\mathscr{A}$ je systém podmnožín metrického priestoru X, d reálna funkcia s oborom $\mathscr{A}$, r kladné číslo. Potom na systéme 2^X definujeme:

$$H[\mathscr{A}, d, r](\emptyset) = 0, \; H[\mathscr{A}, d](\emptyset) = 0 \text{ a pre } E \neq \emptyset$$

$$H[\mathscr{A}, d, r](E) = \inf \left\{ \sum_{i=1}^{\infty} d(E_i); \; E \subset \bigcup_{i=1}^{\infty} E_i, \; d(E_i) < r, \; E_i \in \mathscr{A} \right\}$$

$$H[\mathscr{A}, d](E) = \sup \{ H[\mathscr{A}, d, r](E); \; r > 0 \}$$

Systém množín E_i $(i = 1, 2, \ldots)$ nazveme r-pokrytím množiny E, ak $E \subset \bigcup_{i=1}^{\infty} E_i$ a $d(E_i) < r$ pre každé i. Funkciu $H[\mathscr{A}, d]$ pre ľubovoľné $\mathscr{A}$, d nazývame mierou Hausdorffovho typu. Možno dokázať, že $H[\mathscr{A}, d]$ je vonkajšou mierou, ak d je nezáporná funkcia.

V nasledujúcej práci sa obmedzíme na dvojrozmerný euklidovský priestor a pod $d(E)$ budeme rozumieť priemer množiny E, t. j. $d(E) = \sup \{\varrho(X, Y); X, Y \in E\}$, kde $\varrho(X, Y)$ je veľkosť úsečky XY. Pre stručnosť namiesto $H[\mathscr{A}, d, r]$, resp. $H[\mathscr{A}, d]$ budeme písať $H[\mathscr{A}, r]$, resp. $H[\mathscr{A}]$.

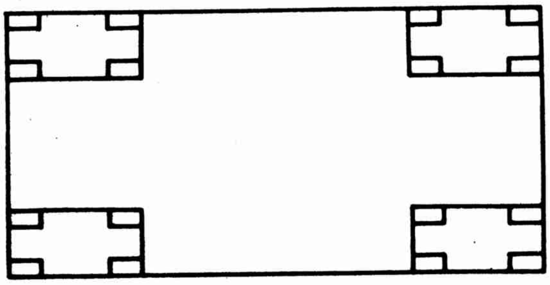
V práci [2] F. Hausdorff o. i. študoval funkciu $H[\mathscr{S}]$, kde $\mathscr{S}$ je systém všetkých kruhov. Už predtým zaviedol C. Carathédory jednorozmernú mieru $H[\mathscr{C}]$ pomocou systému $\mathscr{C}$ všetkých konvexných množín v rovine. Zrejme $H[\mathscr{C}] \leqq H[\mathscr{S}]$. Besicovitch v práci [1] dokázal existenciu takej množiny $\mathscr{A}$, že $H[\mathscr{C}](\mathscr{A}) < H[\mathscr{S}](\mathscr{A})$.

V práci [3] zostrojil B. Riečan takú množinu B, že $H[\mathscr{S}](B) < H[\mathscr{R}](B)$, pričom $\mathscr{R}$ je systém všetkých obdĺžnikov v rovine. V tejto práci zostrojíme takú množinu S, že $H[\mathscr{R}](S) < H[\mathscr{Q}](S)$, kde $\mathscr{R}$ je systém všetkých obdĺžnikov v rovine, ktorých rozmery sú v pomere $2:1$ a $\mathscr{Q}$ je systém všetkých štvorcov v rovine.

(Poznamenajme, že pre „rozumné“ množiny D platí $H[\mathscr{C}](D) = H[\mathscr{S}](D) = H[\mathfrak{R}](D) = H[\mathfrak{Q}](D)$.)

2. Konštrukcia

Zvoľme si obdĺžnik s rozmermi 2 a 1, ktorý nazveme obdĺžnikom rádu 1. Nech n je ľubovoľné prirodzené číslo. Rohom obdĺžnika rádu n nazveme jeho obraz v hociktorej rovnoľahlosti so stredom v jeho vrchole a koeficientom $\frac{1}{4}$. Rohy obdĺžnika rádu n nazveme obdĺžnikmi rádu $n+1$. (Na *obr. 1* sú zobrazené všetky obdĺžniky rádu $n = 1, 2, 3$). Vrcholy obdĺžnikov rádu n nazveme vrcholmi rádu n. Zjednotenie všetkých obdĺžnikov rádu n označme M_n a množinu všetkých vrcholov rádu n označme V_n.



Obr. 1.

Tvrdenie 1. $\bigcap_{n=1}^{\infty} M_n \supset \bigcup_{n=1}^{\infty} V_n$, ale $\bigcap_{n=1}^{\infty} M_n \neq \bigcup_{n=1}^{\infty} V_n$.

Dôkaz inklúzie je zrejmý, pretože pre každé n platí $V_n \subset M_n$, $M_n \supset M_{n+1}$ a tiež $V_n \subset V_{n+1}$. Obrátená inklúzia však neplatí. Predovšetkým $\operatorname{card}\left(\bigcup_{n=1}^{\infty} V_n\right) = \operatorname{card}(N)$. Ďalej uvážme, že pre Cantorovo diskontinuum C platí $\operatorname{card}(C) = \operatorname{card}(\langle 0, 1\rangle)$ takže $\operatorname{card}\left(\bigcap_{n=1}^{\infty} M_n\right) = \operatorname{card}(\langle 0, 1\rangle \times \langle 0, 1\rangle)$.

Tvrdenie 2. S je kompaktná množina.

Dôkaz. Množiny M_n sú uzavreté a ohraničené, teda kompaktné. Preto je aj S, ako prienik kompaktných množín, kompaktná množina.

Lema 1. Nech a, b $(a > b)$ sú rozmery obdĺžnika, m strana minimálneho štvorca, ktorý pokrýva daný obdĺžnik.

Potom platí:

a) $$m = \frac{a+b}{\sqrt{2}}, \quad \text{ak} \quad \frac{b}{a} < \text{tg}\,\frac{\pi}{8}$$

b) $$m = a, \quad \text{ak} \quad \frac{b}{a} \geqq \text{tg}\,\frac{\pi}{8}$$

Dôkaz. a) Danému obdĺžniku možno opísať štvorec, ktorého strany zvierajú so stranami obdĺžnika uhol veľkosti $\frac{\pi}{4}$. Pre veľkosť strany štvorca m platí:

$$m = a\cos\frac{\pi}{4} + b\sin\frac{\pi}{4} = \frac{a+b}{\sqrt{2}}$$

To, že ide o minimálny štvorec, dokážeme nepriamo. Nech m' $\left(m' < \frac{a+b}{\sqrt{2}}\right)$ je strana minimálneho štvorca, ktorý pokrýva daný obdĺžnik, pričom strana a zviera s jednou stranou štvorca uhol $\alpha < \frac{\pi}{4}$. Potom platí

$$m' \geqq a\cos\alpha + b\sin\alpha$$

a teda tým skôr

$$\frac{a+b}{\sqrt{2}} > a\cos\alpha + b\sin\alpha \tag{1}$$

Nech φ je ostrý uhol, pre ktorý je $\text{tg}\,\varphi = \frac{b}{a}$. Vydelením nerovnosti (1) výrazom $\sqrt{a^2+b^2}$ a úpravou dostaneme

$$\cos\left(\frac{\pi}{4} - \varphi\right) > \cos(\alpha - \varphi) \tag{2}$$

Z (2) dostaneme: $\alpha > \frac{\pi}{4}$, alebo $\alpha > 2\varphi - \frac{\pi}{4}$, čo je spor s predpokladmi.

b) Danému obdĺžniku možno opísať štvorec, ktorého strana $m = a$. To, že ide o minimálny štvorec, dokážeme opäť nepriamo. Nech $m''(m'' < a)$ je strana minimálneho štvorca, ktorý pokrýva daný obdĺžnik a β nech je ostrý uhol, ktorý zviera strana a s jednou stranou štvorca. Potom platí:

$$a > a\cos\beta + b\sin\beta \tag{3}$$

Vydelením (3) výrazom $\sqrt{a^2+b^2}$ a úpravou dostaneme

$$\cos\varphi > \cos(\beta - \varphi), \quad \text{t. j.} \quad \beta > 2\varphi, \quad \text{alebo} \quad \beta < 0$$

čo je spor s predpokladmi.

Lema 2. Nech $A_k (4 \leqq k \leqq 4^n)$ je k-prvková podmnožina množiny V_n. Potom pre priemer d_k minimálneho obdĺžnika (t. j. obdĺžnika s minimálnym priemerom) s rozmermi $2:1$, ktorý pokrýva množinu A_k, platí:

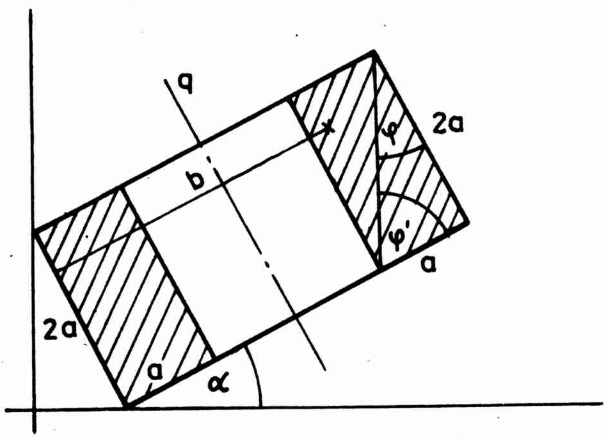
$$d_k \geqq \frac{k\sqrt{5}}{4n} \tag{4}$$

Dôkaz. Stačí, ak (4) dokážeme pre také množiny A_k, ktoré možno pokryť minimálnymi obdĺžnikmi požadovaných vlastností vzhľadom na pevne zvolené k. Také A_k nazveme d-minimálne.

Z konštrukcie množiny S vyplýva, že rohové obdĺžniky rádu $m \leqq n$ sú minimálne obdĺžniky, ktoré pokrývajú d-minimálne množiny s $p = 4^{n-m+1}$ prvkami. Priemer týchto obdĺžnikov je $d_p = \frac{\sqrt{5}}{4^{m-1}}$, čo možno vyjadriť $d_p = \frac{4^{n-m+1}\sqrt{5}}{4^n}$, t. j. pre $k = 4^{n-m+1}$ tvrdenie (4) platí.

Podobne z konštrukcie množiny S vyplýva, že dva najbližšie susedné rohové obdĺžniky rádu $m'(m' < n)$ sú pokrytím d-minimálnej množiny s $2 \cdot 4^{n-m+1}$ prvkami. Minimálny obdĺžnik, ktorý pokrýva takúto množinu, má priemer dvakrát väčší ako d_p. Teda (4) platí pre $k = 2^l$ $(2 \leqq l \leqq 2n)$.

Ak pre k platí $2^l < k < 2^{l+1}$, potom minimálna množina s k prvkami obsahuje d-minimálnu množinu s 2^l prvkami. Nech napr. 2^l vrcholov patrí jedinému rohovému obdĺžniku, ktorý je pre túto množinu minimálny. Ľahko dokážeme, že jeho dlhšia strana bude kratšou stranou minimálneho obdĺžnika, ktorý pokrýva d-minimálnu množinu s $2^l + 1$ bodmi. (Teda už v tomto prípade sa priemer minimálneho obdĺžnika zväčší dvakrát. Tým bude (4) dokázané pre každé k.) Dôkaz urobíme nepriamo. Nech $2a$, a sú strany rohového obdĺžnika, ktorý pokrýva d-minimálnu množinu s 2^l bodmi. Ďalší bod, ktorý patrí najbližšiemu



susednému rohovému obdĺžniku, nech má vzdialenosť b od strany $2a$ — pozri *obr. 2.* Zrejme $b \geqq 3a$. Keby týchto $2^l + 1$ bodov bolo možné pokryť obdĺžnikom, ktorého kratšia strana je menšia ako $2a$, potom by pre priemety do tejto strany platilo:

$$a \sin \alpha + 2a \cos \alpha < 2a, \; 3a \sin \alpha < 2a \qquad (5)$$

alebo

$$a \cos \alpha + 2a \sin \alpha < 2a, \; 3a \cos \alpha < 2a \qquad (6)$$

Ak φ je ostrý uhol, pre ktorý platí $\operatorname{tg} \varphi = \frac{a}{2a}$, tak zrejme $\varphi > \frac{\pi}{8}$. Úpravou nerovníc (2) a (3) dostaneme:

$$\cos(\alpha - \varphi) < \cos \varphi, \; \sin \alpha < \frac{2}{3} \qquad (7)$$

$$\sin(\alpha + \varphi) < \cos \varphi, \; \cos \alpha < \frac{2}{3} \qquad (8)$$

t. j. $\alpha > 2\varphi$ a $\sin \alpha < \frac{2}{3}$, alebo $\alpha < \frac{\pi}{2} - 2\varphi$ a $\cos \alpha < \frac{2}{3}$, čo nemôže platiť, pretože $2\varphi > \frac{\pi}{4}$.

Lema 3. Nech $A_k (4 \leqq k \leqq 4^n)$ je ľubovoľná množina k vrcholov n-tého rádu, S_k nech je minimálny štvorec, t. j. štvorec s minimálnou dĺžkou strany, ktorý pokrýva A_k. Potom platí

$$d(S_k) \geqq \frac{2k\sqrt{2}}{4^n} \qquad (9)$$

Dôkaz. Stačí zobrať do úvahy také množiny A_k, ktoré možno pokryť najmenším štvorcom pri pevne zvolenom k. Také množiny A_k nazveme s-minimálne. Dôkaz rozdelíme na niekoľko častí:

I. Ak $k = 2^p$, kde $p = 2, 4, \ldots, 2n$, tak z konštrukcie množiny S vyplýva, že s-minimálne množiny sú zahrnuté v rohových obdĺžnikoch rádu $r = n - \frac{p}{2} + 1$. Podľa lemy 1 príslušný minimálny štvorec S_k má stranu rovnajúcu sa dlhšej strane rohového obdĺžnika, t. j.

$$d(S_k) = \frac{2\sqrt{2}}{4^{n-p/2}} = \frac{2 \cdot 2^p\sqrt{2}}{4^n} = \frac{2k\sqrt{2}}{4^n}$$

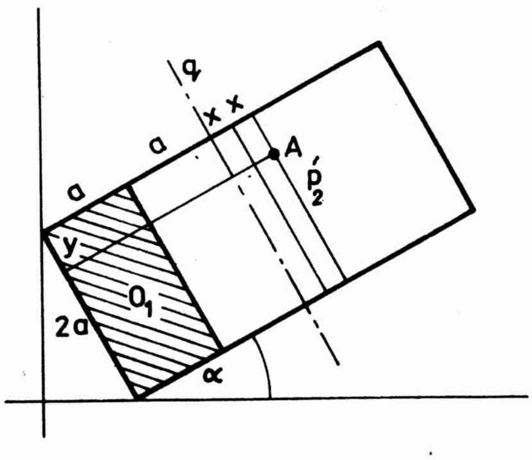
čo sme mali dokázať.

II. Podobne ako v časti I dokážeme (9) pre $k = 2^p$, kde $p = 3, 5, \ldots, 2n - 1$. V tomto prípade, ako to opäť vyplýva z konštrukcie množiny S, s-minimálne množiny budú zahrnuté v dvoch najbližších susedných rohových obdĺžnikoch rádu $r = n - \frac{p-1}{2} + 1$, ktoré sú zahrnuté v obdĺžniku s dvakrát väčšími rozmermi a jeho vrcholy sú prvkami A_k. Preto platí:

$$d(S_k) = 2 \cdot \frac{2\sqrt{2}}{4^{n-(p-1)/2}} = \frac{2 \cdot 2^p \sqrt{2}}{4^n} = \frac{2k\sqrt{2}}{4^n}$$

čo sme chceli dokázať.

III. Nech teraz $2^p < k < 2^{p+1}$ $(p = 2, 4, \ldots, 2n)$; s-minimálna množina A_k bude zahrnutá do dvoch najbližších susedných rohových obdĺžnikov rádu $r = n - \frac{p}{2} + 1$. Jeden z obdĺžnikov obsahuje 2^p bodov s-minimálnej množiny A_k, označme ho O_1, ďalší nech je O_2. Obdĺžniky O_1 a O_2 sú časti obdĺžnika s dvojnásobnými rozmermi, ktorý rozdelíme priečkou q na dva zhodné štvorce S_1 a S_2. Nech S_2 zahrňuje O_2. Urobme nasledovnú transformáciu: Vrcholy rádu n, ktoré obsahuje obdĺžnik O_2, sa nachádzajú na $2^{p/2}$ úsečkách $p_1, p_2, \ldots, p_m$ $(m = 2^{p/2})$ počítajúc od q a sú s q rovnobežné. Tieto úsečky posuňme tak, aby ich obrazy $p'_1, p'_2, \ldots, p'_m$ rozdeľovali S_2 na zhodné časti, pričom sa zachová ich poradie a $p'_m = p_m$. Zoberme do úvahy zobrazenie f množiny 2^{p+1} vrcholov rádu n, ktoré je identické na časti v O_1 a v časti z O_2 bodom na úsečkách p_k $(k = 1, 2, \ldots, m)$ priraďuje body na úsečkách p'_k získané spomínaným posunutím. Obraz množiny A_k v tomto zobrazení označme A'_k. Ľahko zistíme, že $d(A_k) \geqq d(A'_k)$. Pravda A'_k nemusí byť s-minimálna. Ak B_k bude s-minimálna množina obrazov vrcholov rádu



n v zobrazení f, potom bude tým skôr $d(A_k) \geqq d(B_k)$. Pre jednoduchosť zvoľme v tomto prípade stranu štvorca S_1 za jednotkovú. Potom vzdialenosť úsečiek p'_l a p'_{l+1} bude $x = 2^{-(p/2)}$.

a) Ak $k = 2^p + l \cdot 2^{p/2}$ $(l = 1, 2, \ldots, 2^{p/2})$, potom pre platnosť (9) stačí dokázať, že minimálny štvorec S_k príslušný množine B_k zahrňuje $O_1, p'_1 \ldots, p'_l$, t. j. že jeho strana je $1 + lx$. Skutočnosť, že S_k zahrňuje O_1, je zrejmá z konštrukcie množiny S. Dokážeme, že S_k neobsahuje ani jeden bod úsečky p'_{l+1}, čo v našom prípade pre $l = 1$ znamená (pozri *obr. 3*), že sústava nerovníc

$$y \sin \alpha + (1 + 2x) \cos \alpha \leqq 1 + x \tag{10}$$

$$(1 - y) \cos \alpha + (1 + 2x) \sin \alpha \leqq 1 + x \tag{11}$$

nemá riešenie pre $\alpha \in \left\langle 0, \frac{\pi}{2} \right\rangle$, $x \in \left(0, \frac{1}{2} \right\rangle$, $y \in \left\langle 0, \frac{1}{4} \right\rangle$. Označme β, resp. β' ostrý uhol, pre ktorý $\operatorname{tg} \beta = \frac{y}{1 + 2x}$, resp. $\operatorname{tg} \beta' = \frac{\sqrt{y^2 + (1 + 2x)^2 - (1 + x)^2}}{1 + x}$ a ostrý uhol γ, resp. γ' pre ktorý $\operatorname{tg} \gamma = \frac{1 - y}{1 + 2x}$, resp. $\operatorname{tg} \gamma' = \frac{\sqrt{(1 - x)^2 + (1 + 2x)^2 - (1 + x)^2}}{1 + x}$.
Potom sústava (10), (11) je ekvivalentná so sústavou

$$\cos(\alpha - \beta) \leqq \cos \beta', \tag{12}$$

$$\sin(\alpha + \gamma) \leqq \sin\left(\frac{\pi}{2} - \gamma'\right) \tag{13}$$

Preto by muselo platiť:

$$\sin(\beta + \beta') \leqq \sin\left(\frac{\pi}{2} - \gamma - \gamma'\right) \tag{14}$$

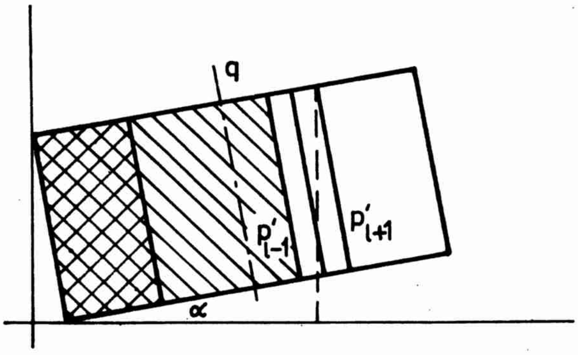
Zo vzťahu (14) dostaneme:

$$\frac{y(1 + x) + (1 + 2x)\sqrt{(1 + 2x)^2 + y^2 - (1 + x)^2}}{(1 + 2x)^2 + y^2} \leqq$$

$$\leqq \frac{(1 + 2x)(1 + x) - (1 - y)\sqrt{(1 + 2x)^2 + (1 - y)^2 - (1 + x)^2}}{(1 + 2x)^2 + (1 - y)^2} \tag{15}$$

Ak zoberieme do úvahy, že $(1 + 2x)^2 + y^2 - (1 + x)^2 > (2x + y)^2$ a $(1 + 2x)^2 + (1 - y)^2 - (1 + x)^2 > (1 - y)^2$, zo vzťahu (15) dostaneme

$$\frac{y(1 + x) + (1 + 2x)(2x + y)}{(1 + 2x)^2 + y^2} < \frac{(1 + 2x)(1 + x) - (1 - y)^2}{(1 + 2x)^2 + (1 - y)^2} \tag{16}$$



Ekvivalentnou úpravou (16) dostaneme:

$$x + 2x^2 + 4x^3 + 8x^4 + 2xy + 4x^2y + 6x^2y^2 + 12x^3y + 3xy^3 +$$
$$+ 2y + 2y^4 < 3xy^2 + 3y^2 \qquad (17)$$

Nerovnosť (17) neplatí pre žiadne $x \in \left(0, \frac{1}{2}\right\rangle$, $y \in \left\langle 0, \frac{1}{4}\right\rangle$ už aj preto, že $x > 3xy^2$ a $2y \geqq 3y^2$.

Ak $l > 1$, stačí dokázať (pozri *obr. 4*), že sústava nerovníc

$$\cos\alpha + (1 + (l-1)x)\sin\alpha \leqq 1 + lx \qquad (18)$$

$$\sin\alpha + (1 + (l-1)x)\cos\alpha \leqq 1 + lx \qquad (19)$$

$$(1 + (l+1)x)\cos\alpha \leqq 1 + lx \qquad (20)$$

nemá riešenie pre $\alpha \in \left\langle 0, \frac{\pi}{2}\right\rangle$, $x = \frac{1}{4}$, $l = 2, 3$; $x = \frac{1}{8}$, $l = 2, 3, \dots, 7$; $x = \frac{1}{16}$, $l = 2, 3, \dots, 15$ atď. Hodnotu $x = \frac{1}{2}$ nemusíme brať do úvahy, pretože v tom prípade pre l je prípustná hodnota len 1. Ak porovnáme nerovnice (18) a (19), zistíme, že stačí zobrať do úvahy sústavu nerovníc (19), (20) a $\alpha \in \left\langle 0, \frac{\pi}{4}\right\rangle$. Ľavá strana nerovnice (19) je rastúca, pri pevne zvolenom lx, práve vtedy, ak $\alpha \leqq \delta$, kde δ je ostrý uhol a $\operatorname{tg}\delta = \frac{1}{1 + (l-1)x}$.

Ľavá strana nerovnice (20) je všade klesajúca. Rovnosť obidvoch strán nastane práve vtedy, ak $\operatorname{tg}\alpha = 2x$. Pre takýto uhol α je ľavá strana nerovnice (19) ešte stále rastúca, ale nerovnica (20) nie je splnená. V intervale $\left(\delta, \frac{\pi}{4}\right\rangle$ je ľavá strana nerovnice (19) klesajúca, avšak ani pre $\alpha = \frac{\pi}{4}$ nerovnica (19) nie je splnená.

b) Teraz dokážeme, že pridaním ďalšieho bodu, t. j. ak $k = 2^p + l \cdot 2^{p/2} + 1$ ($l = 0, 1, \ldots, 2^{p/2} - 1$), sa strana príslušného minimálneho štvorca zväčší viac ako o $\frac{1}{2}x$.

Ak $l = 0$, stačí dokázať (pozri *obr. 3*), že sústava

$$y \sin \alpha + (1 + x) \cos \alpha \leqq 1 + \frac{1}{2}x \tag{21}$$

$$(1 - y) \cos \alpha + (1 + x) \sin \alpha \leqq 1 + \frac{1}{2}x \tag{22}$$

nemá riešenie. K tomu však stačí zobrať do úvahy, že sústava (10), (11) nemá riešenie a použiť substitúciu $2x = t$.

Ak $0 < l < 2^{p/2} - 1$, tak vzhľadom na prípad a) stačí dokázať, že nerovnica

$$y \sin \alpha + (1 + (l + 1)x) \cos \alpha \leqq 1 + \left(l + \frac{1}{2}\right)x \tag{23}$$

nemá riešenie, ak $\operatorname{tg} \alpha \leqq x$. To však vyplýva zo skutočnosti, že nemá riešenie nerovnica

$$\frac{1 + (l + 1)x}{\sqrt{1 + x^2}} < 1 + \left(l + \frac{1}{2}\right)x \tag{24}$$

Ekvivalentnou úpravou totiž dostaneme

$$1 + \left(l + \frac{3}{4}\right)x < x\left(1 + \left(l + \frac{1}{2}\right)x\right)^2$$

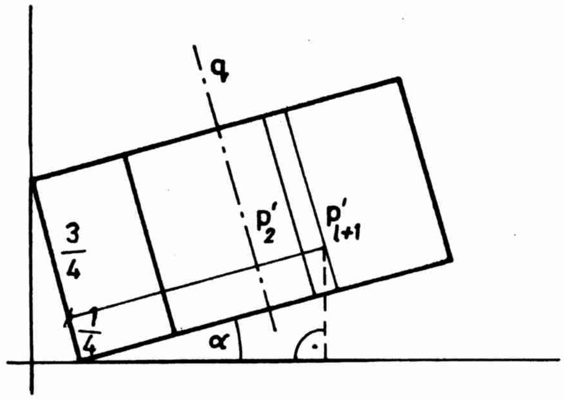
čo zrejme pre $l = 1$ neplatí. Pre $l > 1$ stačí odhad $\left(l + \frac{1}{2}\right)x < 1$, takže by mala mať riešenie nerovnica

$$1 + \left(2 + \frac{3}{4}\right)x < 4x \tag{25}$$

ktorá však pre $x \leqq \frac{1}{2}$ nemá riešenie.

c) Teraz dokážeme, že prírastok strany minimálneho štvorca pridaním $2^{p/2} + 1$ bodov sa zväčší o x. K tomu stačí dokázať, ak uvážime prípad III a) (pozri *obr. 5*), že nerovnica

$$\frac{3}{4} \sin \alpha + (1 + (l + 1)x) \cos \alpha < 1 + (l + 1)x \tag{26}$$



pre α splňujúce podmienku $\operatorname{tg}\alpha < x$ nemá riešenie.

Položme $\operatorname{tg}\varphi = \frac{3}{4(1+(l+1)x)}$. Potom z (26) dostaneme:

$$\cos(\alpha - \varphi) < \cos\varphi \tag{27}$$

Na základe (27) platí:

$$\alpha > 2\varphi \tag{28}$$

Ak uvážime, že $\operatorname{tg}\varphi > \frac{3}{8}$ a $\operatorname{tg}\alpha \leqq x$, tak ľahko zistíme, že (28) nemôže platiť.

IV. K úplnosti treba ešte dokázať, že (9) platí pre k, ktoré vyhovuje podmienke $2^p < k < 2 \cdot 2^p$, kde $p = 3, 5, \ldots, 2n-1$. Ak však zoberieme do úvahy okolnosti, ktoré sme uviedli v odseku II, dôkaz by bol analogický ako v odseku III, a preto ho neuvádzame.

Veta 1. Nech $\mathscr{R}$ je systém všetkých otvorených obdĺžnikov s rozmermi v pomere 2:1. Potom platí $H[\mathscr{R}](S) = \sqrt{5}$.

Dôkaz. Nech r je ľubovoľné kladné číslo, ε ľubovoľné kladné číslo menšie než $\frac{r}{2}$. Zoberme M_{n_0} tak, aby rohové obdĺžniky rádu n_0 mali priemer menší ako ε. Nech M'_{n_0} je systém všetkých otvorených obdĺžnikov, ktoré „vznikli" z obdĺžnikov rádu n_0 predĺžením uhlopriečok o $\frac{\varepsilon}{2 \cdot 4^{n_0 - 1}}$ na obidve strany. Zrejme M'_{n_0} je r-pokrytím množiny S. Pretože súčet priemerov všetkých obdĺžnikov rádu n_0 je $\sqrt{5}$ a je ich 4^{n_0-1}, platí:

$$H[\mathscr{R}, r](S) \leqq \sqrt{5} + \varepsilon \tag{29}$$

Pretože (29) platí pre každé $\varepsilon > 0$ a pre každé $r > 0$, platí aj

$$H[\mathcal{R}](S) \leqq \sqrt{5} \tag{30}$$

Teraz dokážeme obrátenú nerovnosť k nerovnosti (30). Z kompaktnosti množiny S vyplýva, že z ľubovoľného otvoreného r-pokrytia množiny S možno vybrať konečný podsystém, ktorý je tiež r-pokrytím množiny S. Počet obdĺžnikov tohto pokrytia nech je k. Označme ich R_i $(i = 1, 2, \ldots, k)$. Z konečnosti uvedeného pokrytia vyplýva existencia takého čísla n_1, že každý obdĺžnik R_i obsahuje aspoň jeden obdĺžnik rádu n_1. Pretože $V_{n_1} \subset \bigcup_{i=1}^{k} R_i$, podľa lemy 2 bude platiť

$$\sum_{i=1}^{k} \delta(R_i) \geqq \sum_{i=1}^{k} \frac{k_i\sqrt{5}}{4^{n}1} \tag{31}$$

kde k_i je počet vrcholov rádu n_1, ktoré obsahuje obdĺžnik R_i. Ak zoberieme do úvahy, že $\sum_{i=1}^{k} k_i \geqq 4^{n}1$, tak na základe (31) platí:

$$\sum_{i=1}^{k} \delta(R_i) \geqq \sqrt{5} \tag{32}$$

Pretože (32) platí pre ľubovoľné pokrytie, platí aj $H[\mathcal{R}](S) \geqq \sqrt{5}$, čím sme dôkaz vety ukončili.

Veta 2. Nech $\mathcal{Q}$ je systém všetkých otvorených štvorcov. Potom platí $H[\mathcal{Q}](S) = 2\sqrt{2}$.

Dôkaz. Nech r je ľubovoľné kladné číslo, ε ľubovoľné kladné číslo menšie než $\frac{r}{2}$. Rozdeľme každý obdĺžnik rádu n na dva zhodné štvorce, ktoré nazveme štvorcami rádu n. Systém všetkých takýchto štvorcov označme $\mathcal{Q}_n$. Nech n_2 je také prirodzené číslo, že štvorce systému $\mathcal{Q}_{n_2}$ majú priemer menší než ε. Systém $\mathcal{Q}'_{n_2}$ nech je systém všetkých otvorených štvorcov, ktoré „vzniknú" zo štvorcov rádu n_2 predĺžením ich uhlopriečok na obidve strany o $\frac{\varepsilon}{4^{n}2}$. Zrejme $\mathcal{Q}'_{n_2}$ je r-pokrytím množiny S. Pretože súčet priemerov všetkých štvorcov rádu n_2 je $2\sqrt{2}$ a je ich $2 \cdot 4^{n}2^{-1}$ platí:

$$H[\mathcal{Q}, r](S) \leqq 2\sqrt{2} + \varepsilon \tag{33}$$

Nerovnosť (33) platí pre ľubovoľné $r > 0$ a pre ľubovoľné $\varepsilon > 0$. Preto platí aj

$$H[\mathcal{Q}](S) \leqq 2\sqrt{2} \tag{34}$$

Dôkaz obrátenej nerovnosti k nerovnosti (34) je podobný dôkazu nerovnosti (31) v dôkaze predošlej vety, len namiesto lemy 2 treba použiť lemu 3.

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РЕЗЮМЕ

ЗАМЕТКА О ЛИНЕЙНОЙ МЕРЕ В ДВУХМЕРНОМ ПРОСТРАНСТВЕ

Ф. Винш, Трнава

Для любой системы $\mathcal{A}$ подмножества множества R^2, для любого $r>0$ и $E \subset R^2$, $E \neq 0$ положим

$$H[\mathcal{A}, r](E) = \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam} E_i ; \quad E \subset \bigcup_{i=1}^{\infty} E_i, \operatorname{diam} E_i < r, E_i \in \mathcal{A} \right\}$$

$$H[\mathcal{A}](E) = \sup \{ H[\mathcal{A}, r](E); r > 0 \}, H[\mathcal{A}](\emptyset) = 0$$

В статье построено такое множество $S \subset R^2$, что $H[\mathcal{R}](S) < H[\mathcal{Q}](S)$, где $\mathcal{R}$ система всех прямоугольников и $\mathcal{Q}$ система всех квадратов в R^2.

SUMMARY

A NOTE ON THE ONE-DIMENSIONAL MEASURE IN THE TWO-DIMENSIONAL SPACE

F. Vinš, Trnava

For any family $\mathcal{A}$ of subsets of R^2, $r>0$ and $E \subset R^2$, $E \neq \emptyset$ put

$$H[\mathcal{A}, r](E) = \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam} E_i ; E \subset \bigcup_{i=1}^{\infty} E_i, \operatorname{diam} E_i < r, E_i \in \mathcal{A} \right\}$$

$$H[\mathcal{A}](E) = \sup \{H[\mathcal{A}, r](E); r>0\}, H[\mathcal{A}](\emptyset) = 0$$

In the paper such a set $S \subset R^2$ is constructed, that $H[\mathcal{R}](S) < H[\mathcal{Q}](S)$, where $\mathcal{R}$ is the family of all rectangles and $\mathcal{Q}$ is the family of all squares in the plane.

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NUMERICAL ENTROPY

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In this paper there is a notion of numerical entropy introduced. This is an abstraction of some known types of entropy e. q. topological and algebraical one.

The following result is shown. The numerical entropy of Bernoulli shift over the σ-algebra generated by cylinders is equal to infinity.

1. Definitions and notations

Definition 1. 1. By the base of numerical entropy (BNE) we mean a couple (X, T) where X is a distributive lattice with a maximal element I, $T: X \to X$ is a lattice homomorphism, such that $T(I) = I$.

2. By BNE — morphism $(X_1, T_1) \to (X_2, T_2)$ we mean a lattice — — homomorphism $f: X_1 \to X_2$ with properties:

$$f(I_1) = I_2$$
$$f \circ T_1 = T_2 \circ f$$

Definition 2. Let (X, T) be BNE.

1. A finite set $Q = \{x_i, i = 1, \dots, n\} \subset X$ is called a covering of the maximal element I if $\bigvee_{i=1}^{n} x_i = I$. The set of all finite coverings we denote by $\mathcal{P}_0$.

2. A set $Q \subset X$ is called the covering of I if there exists $Q_0 \subset Q$ such that $Q_0 \in \mathcal{P}_0$. The set of all coverings we denote by $\mathcal{P}$.

3. Let $Q \in \mathcal{P}_0$. An element $x \in Q$ is called a proper element of Q if $\vee(Q - \{x\}) \neq I$.

4. A covering $Q \in \mathcal{P}_0$ is called minimal if every its element is proper.

Definition 3. We introduce the following structure on the set $\mathcal{P}$.

1. a partial ordering

$$Q < R \Leftrightarrow \forall x \in R \; \exists y \in Q: x \leqq y$$

R is a refinement of Q;

2. a binary operation

$$Q \stackrel{\vee}{-} R = \{x \wedge y : x \in Q, y \in R\}$$

3. a transformation

$$\bar{T}: \mathcal{P} \to \mathcal{P}, \ \bar{T}(Q) = \{T(x): x \in Q\}$$

4. for any $n \in Z^+$ we put

$$Q^n = \underset{j=0}{\overset{n-1}{\vee}} \bar{T}^j(Q)$$

2. Basic properties

Lemma 1. The previous structure has the following properties:
(1.1) $Q \stackrel{\vee}{-} R$ is a join in the poset $(\mathcal{P}, <)$
(1.2) $Q<R$, $Q'<R'$ implies

$$\bar{T}(Q)<\bar{T}(R)$$
$$Q \stackrel{\vee}{-} Q' < R \stackrel{\vee}{-} R'$$

and for any $n \in Z^+$

$$Q^n < R^n$$

(1.3) $\bar{T}(Q \stackrel{\vee}{-} R) = \bar{T}(Q) \stackrel{\vee}{-} \bar{T}(R)$

Definition 4. We define the following real functions on the set $\mathcal{P}$

$$N(Q) = \min \{\text{card } Q_0 : Q_0 \in \mathcal{P}_0, Q_0 \subset Q\}$$

$$H(Q) = \log N(Q)$$

$$h(T, Q) = \lim_{n \to \infty} \frac{1}{n} H(Q^n)$$

$$H(T) = \sup_{Q \in \mathcal{P}} h(T, Q)$$

Proposition. The introduced functions have the following properties.

(2.1) To any covering $Q \in \mathcal{P}$ there exists a minimal finite subcovering Q_0, such that $N(Q) = \text{card } Q_0$.

(2.2) A number of proper elements of a finite covering Q is not greater then $N(Q)$.

(2.3) $$H(Q \stackrel{\vee}{-} R) \leqq H(Q) + H(R)$$

and

$$H(\bar{T}(Q)) \leqq H(Q)$$

(2.4) $Q<R$ implies $H(Q)\leqq H(R)$

(2.5) $h(T,Q)\leqq H(Q)$

Proof. (2.1) From the definition of $N(Q)$ there exists a finite subcovering Q_0 such that $H(Q)=\text{card } Q_0$. If Q_0 were not minimal covering, we can choose a smaller subcovering.

(2.2) Let $N(Q)=\text{card } Q_0$, $Q_0\subset Q$ and $x_0\in Q_0$ is a proper element of Q. Then $\vee(Q-\{x_0\})\geqq\vee(Q_0)=I$ what is a contradiction.

(2.3) Let $N(Q)=\text{card } Q_0$, $N(R)=\text{card } R_0$, then $Q_0\dot{\vee}R_0\subset Q\dot{\vee}R$

$$N(Q\dot{\vee}R)\leq\text{card }(Q_0\dot{\vee}R_0)\leq\text{card } Q_0\cdot\text{card } R_0=$$

$$=N(Q)\cdot N(R)$$

(2.4) Let $N(R)=\text{card } R_0$, $R_0=\{y_1,\ldots,y_{N(R)}\}$; since for every $\forall i=1,\ldots,N(R)$, $y_i\leqq x_i\in Q$, hence $\bigvee_{i=1}^{N(R)}x_i=I$. Therefore $N(Q)\leqq N(R)$.

(2.5) $Q<R$ implies $Q^n<R^n$, hence $h(T,Q)\leqq h(T,R)$.

Lemma 2. Let f be an injective BNE — morphism $(X_1,T_1)\rightarrow(X_2,T_2)$. Then $h(T_1)\leqq h(T_2)$ (see [8]).

Corollary. 1. Isomorfic BNE have the same entropy.

2. Let Y be a T — invariant sub — lattice of X containing I and $U=T\mid Y$. Then $h(U)\leqq h(T)$.

Remark. Let B be a topological space, $\tau:B\rightarrow B$ be a continuous mapping. We define on the lattice $\mathcal{O}$ of all open sets a mapping $T:\mathcal{O}\rightarrow\mathcal{O}$, $T(U)=\tau^{-1}(U)$. The topological entropy of (B,τ) is defined as the numerical entropy of $(\mathcal{O},T)$.

3. Entropy of Bernoulli shift

Let

$$\varrho_r=\{0,1,\ldots,r-1\},\qquad r\geqq 2$$

$$\varrho_r^s=\prod_{i=0}^{s-1}Y_i,\qquad Yi=\varrho_r$$

$$Z_r=\prod_{i=1}^{\infty}Y_i,\qquad Y_i=\varrho_r$$

and τ_r be the left — side shift on Z_r.

We define a homomorphism of the algebra 2^{Z_r} by

$$T_r(A)=\tau_r^{-1}(A)$$

Let C_r be the system of elementary cylinders

$$E_{i,j}=\{x\in Z_r,\ x_i=j\},\qquad i\in Z^+,\ j\in\varrho_r$$

Let $\mathcal{A}_r \subset \mathcal{O}_r \subset \mathcal{S}_r$ be algebra, topology and σ-algebra resp. generated by C_r and $\mathcal{P}\text{-}a \subset \mathcal{P}_\mathcal{O} \subset \mathcal{P}_s$ be systems of coverings of Z_r by those elements. We have

$$T_r(E_{i,j}) = E_{i+1,j}$$

therefore the mentioned sublattices of 2^{Z_r} are T_r invariant. It is well known that $h_a(T_r) = h_\mathcal{O}(T_r) = \ln r$ (see [8]).

From the corollary of lemma 3 it yields that $h_s(T_r) \geqq \ln r$. We shall shaw that $h_s(T_r) = \infty$. For any $t \in Z^+$ we shall construct a covering $Q_t \in \mathcal{P}_s$ such that $h(T_r, Q_t) \geqq \ln t$. Q_t consists of $t+1$ elements $F_0, \ldots, F_t$ such that for every natural number s the canonical mapping

$g_s: \varrho^s_{t+1} \to Q^s_t$, $v \mapsto \bigcap_{j=0}^{s-1} T^j F_{v_j}$ has the following properties:

1. $g_s | \varrho^s_t$ is injective.
2. For any $v \in \varrho^s_t \subset \varrho^s_{t+1}$, $g_s(v)$ is a proper element of Q^s_t. By Proposition we have

$N(Q^s_t) \geqq \operatorname{card}\{x: x$ is a proper element of $\{Q^s_t\} \geqq t^s$, hence $h_s(T_r, Q_r) \geqq \ln t$.
Because of simplicity we shall write T instead of T_r and E_i instead of $E_{i,1}$.
For any $p \in \varrho_t$, $k \in Z^+$ we put

$$\sigma_k = t \cdot \sum_{j=1}^{k} j = 1/2t \cdot k \cdot (k+1) \tag{1}$$

$$m(k, p) = \sigma_k + (k+1) \cdot p$$

$$F_p = \bigcap_{k=0}^{\infty} E_{m(k,p)},\ 0 \leqq p \leqq t-1,\ F_t = Z_r - \bigcup_{p=0}^{t-1} F_p \tag{2}$$

We shall prove that g_s has the mentioned properties.
1. By the definition we have

$$g_s(v) = \bigcap_{j=0}^{s-1} T^j(F_{v_j}) = \bigcap_{j=0}^{s-1} T^j\left[\bigcap_{k=0}^{\infty} E_{m(k,v_j)}\right] = \bigcap_{k=0}^{\infty}\bigcap_{j=0}^{s-1} E_{m(k,v_j)} + j$$

Let $k < s < k' \in Z^+$ and $i, j \in \varrho_s$, then

$$\sigma_k \leqq m(k, v_j) + j \leqq \sigma_k + v_j(k+1) + j \leqq$$

$$\leqq \sigma_{s-1} + (t-1) \cdot s + (s-1) = \sigma_s - 1 < \sigma_s$$

Hence

$$\sigma_k \leqq m(k, v_j) + j < \sigma_s \leqq m(s, v_i) + i < \sigma_{k'} \tag{3}$$

Let $v \neq v' \in \varrho^s_t$. Then there exists $i \in \varrho_s$ such that $v_i \neq v'_i$

$$m(s, v_j) + j \neq m(s, v'_i) + i \quad \text{for any} \quad j \in \varrho^s_t \tag{4}$$

a) $v_j = v'_i$. Then $i \neq j$ and $m(s, v'_i) + i \neq m(s, v_j) + j$;
b) $v_j \neq v'_i$, suppose $v_j + 1 \leqq v'_i$

$$m(s, v_j) + j = \sigma_s + v_j(s+1) + j < \sigma_s + (v_j + 1) \cdot (s+1) \leqq$$

$$\leqq \sigma_s + v'_i(s+1) + i \leqq m(s, v_i) + i$$

For $n \in Z^+$ we put

$$z_v(n) = \begin{cases} 1, & \text{if } n < \sigma_s, \text{ or } n = m(s, v_j) + j, \ j \in \varrho_s, \text{ or } n > \sigma_{s+1} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

From (3) and (5) we get

$$z_v[m(k, v_j) + j] = 1 \quad \text{hence} \quad z_v \in \bigcap_{j=0}^{s-1} \bigcap_{k=0}^{\infty} E_{m(k, v_j)+j} = g_s(v)$$

From (4) and (5)

$$z_v[m(k, v'_i) + i] = 0 \quad \text{for some } i, \text{ hence} \quad z_v \notin E_{m(s, v_i')+i} \supset g_s(v')$$

We have

$$z_v \in g(v) - g(v') \tag{6}$$

2. From (6) we get that for every $v \in \varrho^s_t$

$$z_v \in g(v) - \bigcup \{v' : v' \in \varrho^s_t - \{v\}\} \tag{7}$$

Let $w \in \varrho^s_{t+1} - \varrho^s_t$. There exists i, $0 \leqq i \leqq s-1$, $w(i) = t$.
Therefore

$$g_s(w) \subset T^i(F_t) \subset T^i(F^c_{v_i}) \subset [g_s(v)]^c = \bigcup_{j=0}^{s-1} [T^j(F_{v_j})]^c$$

Hence

$$g_s(v) \cap g_s(w) = \emptyset \tag{8}$$

From (7) and (8) we see that 2 holds.

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SÚHRN

NUMERICKÁ ENTROPIA

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V práci sme zaviedli pojem numerickej entropie ako zovšeobecnenie niektorých známych druhov entropie (napr. topologickej a algebraickej).

Odvodené výsledky sa dajú použiť v známom type úloh o porovnávaní rôznych druhov entropie.

Dokázali sme, že numerická entropia Bernoulliho posunutia nad súčinovou σ-algebrou sa rovná nekonečnu.

РЕЗЮМЕ

НУМЕРИЧЕСКАЯ ЭНТРОПИЯ

М. Коморникова, Братислава

В статье вводится понятие нумерической энтропии как обобщения некоторых известных типов энтропии, например топологической и алгебраической. Полученные результаты можно применить к задачам о сравнении различных типов энтропии.

Самый важный результат — нумерическая энтропия сдвига Бернулли над σ-полем порожденным цилиндрами равна бесконечности.

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ON SYSTEMS OF ISOMORPHIC COPIES OF AN ALGEBRA IN ANOTHER ALGEBRA

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Let $\mathscr{A}$ be a class of algebras of the same signature and let $A, B \in \mathscr{A}$. A set M is said to be a maximal system of disjoint isomorphic copies (m. s. d. i. c.) of A in B if

1. the elements of M are pairwise disjoint subalgebras of B isomorphic to A and
2. there is no subalgebra C of B isomorphic to A and disjoint to all elements of M.

For every cardinality m let $k(A, B, m)$ denote the cardinality of the set of all m. s. d. i. c. of A in B which have the cardinality m. The notion of m. s. d. i. c. can be generalized to classes of models (instead of classes of algebras) in the obvious way; then we can use the denotation $k(A, B, m)$, too.

At the Miniconference on lattice theory in Szeged, August 1974, M. Sekanina stated the following problem:

Let A, B be lattices and let for every natural number n there be n pairwise disjoint isomorphic copies of A in B. Are there $\aleph_0$ pairwise disjoint isomorphic copies of A in B? Analogical problem for graphs was formerly considered by R. Halin [1]. V. Koubek announced a negative solution for modular lattices. A. Bandelt stated this problem in categorical way.

Theorems of this paper will answer the following problem for some classes of algebras and models.

Problem. Let $\mathscr{A}$ be as above, $M_1 = \{\aleph_0, 1, 2, 3, \dots\}$, $M_2 = \{\aleph_0, 0, 1, 2, 3, \dots\}$ and φ be a mapping of M_1 into M_2. Are there algebras $A, B \in \mathscr{A}$ such that for all $m \in M_1$ it holds $k(A, B, m) = \varphi(m)$?

For example, Theorem 5 will state that if $\mathscr{A}$ is the class of all countable lattices, then for every mapping φ of M_1 into M_2 such $A, B \in \mathscr{A}$ exist. This implies the negative answer to Sekanina's problem; it is sufficient to consider the function φ such that $\varphi(m) = 1$ for $m < \aleph_0$, $\varphi(\aleph_0) = 0$.

Notice that if for some algebras A, B it holds $k(A, B, 0) \neq 0$, then $k(A, B, 0) = 1$ and $k(A, B, m) = 0$ for all $m > 0$. That is the reason for excluding 0 from the domain of φ in the Problem. The possibility to include uncountable cardinalities into the range or the domain of φ is not considered in the present paper.

1. Solution for Groupoids

Theorem 1. For every mapping φ of M_1 into M_2 there are countable groupoids A_1, B_1 such that $k(A_1, B_1, m) = \varphi(m)$ for all $m \in M_1$.

In the proof of Theorem 1 we shall use the following geometrical lemma.

Lemma. Let φ be a mapping of M_1 into M_2. Then there is a set P of lines in the (Euclidean) plane and a set Q of points of the same plane such that:

1. no three different lines from P intersect in one point;
2. all points of intersection of lines from P belong to Q;
3. every point $X \in Q$ belongs to a line $p \in P$;
4. the distance of any two different points from Q is at least 1; the distance of any two different parallel lines from P is at least 1;
5. every half-line of every line $p \in P$ contains infinitely many points of Q;
6. for every $m \in M_1$ there are exactly $\varphi(m)$ m-elements maximal subsets of P consisting of mutually parallel lines.

Proof. There exists a (finite or infinite) sequence $(a_1, a_2, a_3, \ldots)$ of real numbers containing exactly $\varphi(m)$ numbers exactly m-times, for every $m \in M_1$. Choose for every i a line p_i parallel with the vector $(1, a_i)$ in such a way that its distances from all points of intersection of lines $p_1, \ldots, p_{i-1}$ and from all lines $p_j \parallel p_i$, $j < i$, are at least 1. Then let $P = \{p_1, p_2, p_3, \ldots\}$ and let Q contain all points of intersection of lines from P and some other points of these lines if necessary (to fulfil (5)). We omit the detailed proof.

Below we shall use the sets P, Q from Lemma. The points $X, Y \in Q$ will be called neighbour points if $X \neq Y$, $XY \in P$ and no point between X, Y belongs to Q. Similarly, integers x, y will be called neighbour integers if $|x - y| = 1$.

Proof of Theorem 1. If $\varphi(m) = 0$ for all $m \in M_1$, we can easily find groupoids A_1, B_1 such that A_1 has no isomorphic copy in B_1 (e. g. A_1 = the set of all integers, B_1 = the set of positive integers, $x \circ y = \max(x, y)$ in both cases). Hence further we may assume that φ does not vanish on the whole M_1. (This part of proof will not be repeated in the proofs of other theorems.) Let $A_1 = (Z, \circ)$, where Z is the set of all integers and

$$x \circ y = \begin{cases} 2y - x, & \text{if } |x - y| = 1 \\ x, & \text{if } |x - y| \neq 1 \end{cases}$$

for all $x, y \in Z$.

Let $B_1 = (Q, \circ)$, where Q is the set of points from Lemma and

$$X \circ Y = \begin{cases} \text{the neighbour } Z \neq X \text{ of the point } Y \text{ on the line } XY, \\ \text{if} \qquad\qquad X, Y \text{ are neighbour points,} \\ X \text{ otherwise.} \end{cases}$$

It is easy to see that for every $p \in P$ the subgroupoid $(p \cap Q, \circ)$ of B_1 is isomorphic to A_1. Conversely, every isomorphic copy of A_1 in B_1 is of this form. To show it, consider such an isomorphic copy $(D, \circ)$. Let X, Y be images of 1, 2. Then $X \circ Y \neq X$, hence X, Y are neighbour points. Since the numbers 1, 2 generate A_1, the points X, Y generate $(D, \circ)$. Hence $D \subseteq p \cap Q$ where $p = XY$. However, A_1 is not isomorphic with its proper subgroupoid, A_1 is isomorphic with $(p \cap Q, \circ)$ and hence D cannot be a proper subset of $p \cap Q$. Therefore $D = p \cap Q$.

We have found the one-to-one correspondence $p \mapsto p \cap Q$ between the lines of P and the isomorphic copies of A_1 in B_1. If p, q are parallel, then the subgroupoids $p \cap Q$, $q \cap Q$ of B_1 are obviously disjoint. If p, q are not parallel, then its point of intersection X belongs to Q, and hence $p \cap Q$, $q \cap Q$ are not disjoint. Hence m. s. d. i. c. of A_1 in B_1 correspond to maximal sets of mutually parallel lines in P. Therefore by (6) for every $m \in M_1$ we have $k(A_1, B_1, m) = \varphi(m)$, q. e. d.

2. Solution for semilattices, partially ordered sets and graphs

Theorem 2. For every mapping φ of M_1 into M_2 there are infinite countable semilattices A_2, B_2 such that for all $m \in M_1$ it holds $k(A_2, B_2, m) = \varphi(m)$.

Proof. We may assume that φ does not vanish on the whole set M_1. Let $A_1 = (Z, \circ)$, $B_1 = (Q, \circ)$ be the groupoids from the previous section. For the simplicity we shall write x instead of $\{x\}$ if $x \in Z$ or $x \in Q$.

Let $A_2 = Z \cup \{\{x, x+1\} \mid x \in Z\} \cup \{[x, x+1] \mid x \in Z\} \cup \{R\}$ where R denotes the set of real numbers and $[x, x+1] = \{y \in R \mid x \leqq y \leqq x+1\}$. The set A_2 partially ordered by set-theoretical inclusion is a $\vee$-semilattice; its diagram is on *fig. 1.*

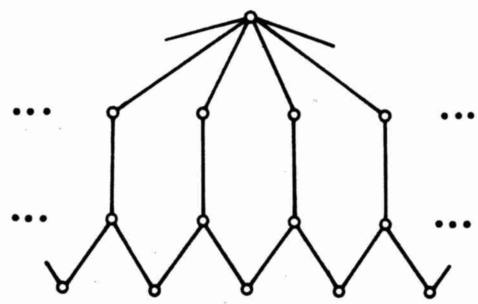

Fig. 1.

Now we shall construct the semilattice B_2. We may assume that no two points of intersection of lines from P are neighbour points. Let

$$B_2 = Q \cup \{X, Y\} \mid X, Y \text{ are neighbour points}\} \cup$$
$$\cup \{\overline{XY} \quad \mid X, Y \text{ are neighbour points}\} \cup P \cup \{\pi\}$$

where π is the whole Euclidean plane and $\overline{XY}$ is the segment with endpoits X, Y (in both cases considered as set of points). The set B_2 partially ordered by set-theoretical inclusion is a semilattice.

If $X, Y \in Q$ then

$$X \vee Y = \begin{cases} X & \text{if } X = Y, \\ \{X, Y\} & \text{if } X, Y \text{ are neighbour points,} \\ XY & \text{if } X \neq Y, XY \in P \text{ and } X, Y \text{ are not neighbour points,} \\ \pi & \text{if } X \neq Y \text{ and } XY \notin P. \end{cases}$$

If $a, b \in B_2$ and e. g. $a \in Q$, then $a \vee b$ exist because the elements greater than a form a finite chain.

For every $p \in P$ the set $\{x \in B_2 \mid x \leqq p\}$ is obviously a subsemilattice of B_2 isomorphic to A_2. Conversely, every isomorphic copy of A_2 in B_2 is of this form. To show that, consider arbitrary isomorphism ψ of A_2 into B_2. For $x \in Z$ obviously $\psi(x) \in Q$ because elements of Q are the only elements $y \in B_2$ such that $[y, \pi]$ is not a chain (and $[x, R]$ is not a chain in A_2). If x, y are neighbour integers (i. e. $|x - y| = 1$), then $A = \psi(x)$, $b = \psi(y)$ are neighbour points because the length of the interval $[A \vee B, \pi]$ is less than 2 in all other cases. Since no points of intersection of lines from P are neighbour points, at least one of the points $\psi(2)$, $\psi(3)$ is not such a point; assume e.g. $\psi(2)$ is not. The points $\psi(1)$, $\psi(2)$, $\psi(3)$ lie on a line $p \in P$ and it holds $\psi(R) = \psi(1) \vee \psi(2) \vee \psi(3) = p$. Hence ψ is an isomorphism of A_2 into $\{x \in B_2 \mid x \leqq p\}$. However, A_2 is not isomorphic to any its proper semilattice, hence ψ is an isomorphism of A_2 into $\{x \in B_2 \mid x \leqq p\}$.

To finish the proof, it is sufficient to consider that the one-to-one correspondence $p \mapsto \{x \in B_2 \mid x \leqq p\}$ between the lines from P and the isomorphic copies of A_2 in B_2 preserves disjointness in both directions.

Theorem 3. For every mapping φ of M_1 into M_2 there are countable partially ordered sets A_3, B_3 satisfying $k(A_3, B_3, m) = \varphi(m)$ for all $m \in M_1$.

Proof. We may assume that φ does not vanish on the whole M_1. It is sufficient to take $A_3 = A_2$, $B_3 = B - \{\pi\}$, and both sets partially ordered by set-theoretical inclusion. The proof is almost the same as the proof of Theorem 2, and we shall not repeat it. We only notice that the heights of A_3 and B_3 can be diminished by dropping out the segments.

Graphs of semilattices A_2, B_2 (and graphs of partially ordered sets A_3, B_3 as well) can be used to obtain a solution of our problem for the class of all graphs. However, we can prove a little stronger result.

Theorem 4. For every mapping φ from M_1 into M_2 there are infinite countable planar graphs A_4, B_4 such that for all $m \in M_1$ it holds $k(A_4, B_4, m) = \varphi(m)$.

Proof. We may assume that φ does not vanish on the whole set M_1. Let A_4 be the graph on the *fig. 2.* Its vertices are all integers and all rational numbers of the form $(8n \pm 1)/2$, where n is an integer; its edges join every pair neighbour integers and every pair x, y of its vertices such that $|x-y| = 1/2$. The graph A_4 is not isomorphic with any its proper subgraph.

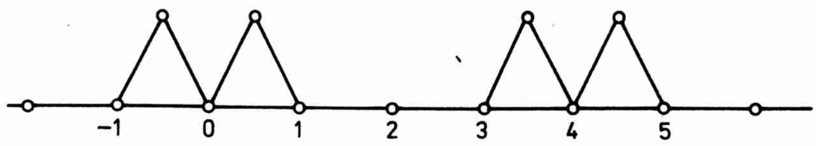

Fig. 2.

The graph B_4 will be illustrated in the plane in the usual way. Let P, Q be the sets from Lemma. We can arrange (by adding some new points to Q and by changing the unity of length, if necessary) that there exists a system $\{\alpha_p \mid p \in P\}$ such that:

a) every α_p is a one-to-one mapping of $p \cap Q$ onto the set Z of integers which preserves neighbourhood;

b) if X is the point of intersection of $p, q \in P$ then $\alpha_p(X)$, $\alpha_q(X)$ are even numbers and exactly one of them is divisible by 4.

Choose such a system $\{\alpha_p \mid p \in P\}$. An (unordered) pair $\{X, Y\}$ of neighbour points will be called a 4-pair if one of the numbers $\alpha_{XY}(X)$, $\alpha_{XY}(Y)$ is divisible by 4. To every 4-pair $\{X, Y\}$ a point M_{XY} can be chosen (imagine it lying near the midpoint of the segment $\overline{XY}$ but not on the line XY) in such a way that

c) if $\{X, Y\}$ is a 4-pair and the segment $\overline{XM_{XY}}$ has a common point U with a line $p \in P$ then $p = XY$ and $U = X$;

d) if $\{X, Y\}$ and $\{U, V\}$ are different 4-pairs and the segments $\overline{XM_{XY}}$, $\overline{UM_{UV}}$ have a common point W then $X = U = W$.

Now we can describe the graph B_4. Its vertices are all points from Q and all points M_{XY} for all 4-pairs $\{X, Y\}$. Its edges are (represented by) all segment $\overline{XY}$ where X, Y are neighbour points and all segments $\overline{XM_{XY}}$, $\overline{YM_{XY}}$ for all 4-pairs $\{X, Y\}$. By c) and d) B_4 is a planar graph.

For every $p \in P$ denote

$$\bar{p} = p \cap Q \cup \{M_{XY} \mid XY = p \text{ and } \{X, Y\} \text{ is a 4-pair}\}$$

and β_p the extension of the mapping α_p to the set $\bar{p}$ such that $\beta_p(M_{XY}) = (\alpha_p(X) + \alpha_p(Y))/2$. Then β_p is an isomorphism of the subgraph $\bar{p}$ of B_4 onto A_4. Conversely, it can be shown that every subgraph of B_4 isomorphic to A_4 is of the form $\bar{p}$ for some $p \in P$. The crucial point in the proof is the statement: If U, X, V are pairwise different, X is a point of intersection of lines $p, q \in P$, $\{U, X\}$ and $\{V, X\}$ are pairs of neighbour points then $UV \in P$ if and only if either both $\{U, X\}$, $\{X, V\}$ are 4-pairs or none of them is a 4-pair (see *fig. 3*).

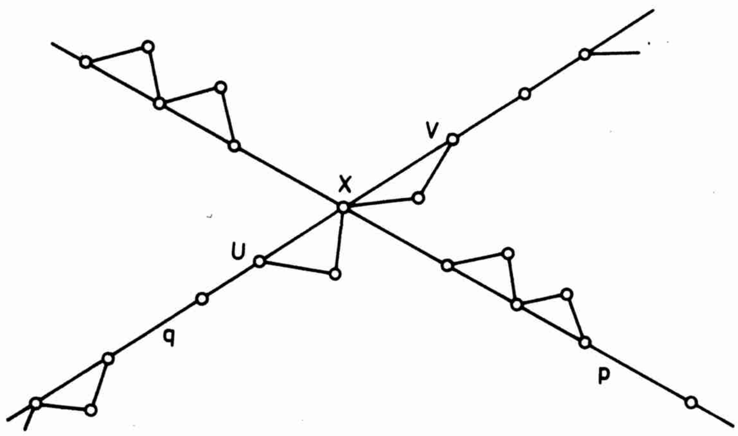

Fig. 3.

To finish the proof of Theorem 4 it is sufficient to consider that $\bar{p} \cap \bar{q} = \emptyset$ if and only if $p \cap q = \emptyset$ for all $p, q \in P$, and to use the point (6) of Lemma.

3. Solution for lattices

Theorem 5. Let φ be a mapping of M_1 into M_2. Then there are infinite countable lattices A_5, B_5 such that for all $m \in M_1$

$$k(A_5, B_5, m) = \varphi(m) \text{ holds}$$

Proof. It is sufficient to consider the case when does not vanish everywhere. Let $A_1 = (Z, \circ)$, $B_1 = (Q, \circ)$ are groupoids constructed in the proof of Theorem 1. We show how to construct the lattice B_5 from the groupoid B_1; A_5 is constructed from A_1 in the same way. Denote

$Q_2 = \{\{X, Y\} \subseteq Q \mid X, Y$ are neighbour points$\}$
$Q_3 = \{\{X, Y, Z\} \subseteq Q \mid X, Y, Z$ are pairwise different points of a line $p \in P\}$
$\bar{Q} = Q \cup Q_2 \cup Q_3$

For arbitrary $x, y, z \in Q$ we have

$$x \circ y = z \equiv ((\{x, y\} \notin Q_2 \wedge z = x) \vee (\{x, y\} \in Q_2 \wedge \{y, z\} \in Q_2 \wedge \wedge \{x, y, z\} \in Q_3))$$

Hence to define the operation $\circ$ it is sufficient to know the sets Q, Q_2, Q_3.

Let F be the free distributive lattice generated by the set of free generators Q. For simplicity of denotations let

$$x \vee y = \{x, y\} \quad \text{if} \quad \{x, y\} \in Q_2$$

and

$$x \vee y \vee z = \{x, y, z\} \quad \text{if} \quad \{x, y, z\} \in Q_3$$

Assign to all elements $a \in F$ pairwise disjoint lattices S_a, where $S_a = \{a\}$ if $a \notin \bar{Q}$ and S_a is an isomorphic copy of D displayed on *fig. 4* if $a \in \bar{Q}$; in this case let

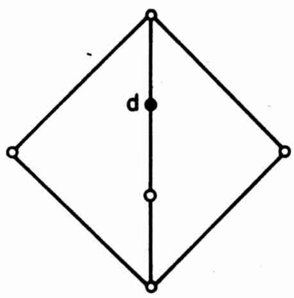

Fig. 4.

$a \in S_a$ be the image of $d \in D$ in the isomorphism of D onto S_a. Now let $S = \bigcup_{a \in F} S_a$ and for all $x, y \in S$

$$x \leqq y \equiv (\exists a \in F)\,(x, y \in S_a \wedge x \leqq y(S_a)) \vee$$
$$\vee (\exists a, b \in F)\,(x \in S_a \wedge y \in S_b \wedge a < b(F))$$

Then $B_5 = (S, \leqq)$ is a lattice.

We show that S_a, $a \in \bar{Q}$ are the only sublattices of B_5 which are isomorphic to D. Assume it is not true. Then there is an isomorphism ψ of D into B_5 and $a, b \in F$, $a \neq b$ such that $\psi(D) \cap S_a \neq \emptyset$, $\psi(D) \cap S_b \neq \emptyset$. The composition of ψ with the canonical homomorphism of B_5 onto F maps D onto at least two-element sublattice of F. This is a contradiction because D has no nontrivial distributive homomorphic images.

Hence it holds

$$x \in \bar{Q} \equiv \text{there is an isomorphism of } D \text{ into } B_5 \text{ which maps } d \text{ onto } x$$

Then it is easy to check that

$$x \in Q \equiv x \text{ is a minimal element of } \bar{Q} \text{ (with respect to the ordering in } B_5)$$
$$x \in Q_3 \equiv x \text{ is a maximal element of } \bar{Q}$$
$$x \in Q_2 \equiv x \in \bar{Q} \wedge x \notin Q \wedge x \notin Q_3.$$

Now let us construct the lattice A_5 from $A_1 = (Z, \circ)$ in the same way, and introduce the designations Z_2, Z_3, $\bar{Z}$ analogously to Q_2, Q_3, $\bar{Q}$. Let us investigate the isomorphic copies of A_5 into B_5.

For every $p \in P$ denote by $A(p)$ the sublattice of B_5 generated by $\bigcup_{a \in \bar{p}} S_a$ where

$$\bar{p} = \{y \in Q \mid x \in p\} \cup \{\{x, y\} \in Q_2 \mid x, y \in p\} \cup$$
$$\cup \{\{x, y, z\} \in Q_3 \mid x, y, z \in p\}$$

Every lattice $A(p)$, $p \in P$, is isomorphic with the lattice A_5; the isomorphism can be constructed from any injective mapping of Z onto $p \cap Q$ which preserves neighbourhood.

Now we show that, conversely, every isomorphic copy of A_5 in B_5 is some $A(p)$, $p \in P$. Let f be an isomorphism of A_5 into B_5. For $x \in \bar{Z}$ it holds $f(x) \in \bar{Q}$, hence f (more exactly: the restriction of f on the set Z) is an isomorphism of partially ordered set $\bar{Z}$ into partially ordered set $\bar{Q}$. Hence f maps Z into Q, Z_2 into Q_2, and Z_3 into Q_3. The elements $f(1)$, $f(2)$ are neighbour points in Q (because they have an upper bound in Q_2). Denote the line $f(1)f(2)$ by p. If $x \in Z - \{1, 2\}$ then $\{1, 2, x\} \in Z_3$ and hence $\{f(1), f(2), f(x)\} \in Q_3$; hence $f(x) \in p$. Therefore f maps Z into $p \cap Q$. Since every $f(x)$, $x \in Z$ has two neighbour points $f(x-1)$, $f(x+1)$, f is a mapping of Z onto $p \cap Q$. Now it is easy to check that f is an isomorphism of A_5 onto $A(p)$. Then $p \mapsto A(p)$ is a one-to-one correspondence between the lines from P and the isomorphic copies of A_5 in B_5. This correspondence also preserves disjointness in both directions: If $p, q \in P$ are parallel and different then $p \cap Q$, $q \cap Q$ are disjoint and hence the sublattices of F generated by $p \cap Q$, $q \cap Q$ are disjoint. Then $A(p)$, $A(q)$ are disjoint, too. If p, q are not parallel then their common point belongs to $A(p)$ and $A(q)$, hence $A(p)$, $A(q)$ are not disjoint. Now the condition (6) from Lemma implies $k(A_5, B_5, m) = \varphi(m)$ for all $m \in M_1$.

4. Algebras with unary operations

Theorem 6. Let φ be a mapping of M_1 into M_2. Then there are infinite countable algebras A_6, B_6 with two unary operations such that for every $m \in M_1$ it holds $k(A_6, B_6, m) = \varphi(m)$.

Proof. If φ vanishes on the whole M_1, then A_6, B_6 can be easily constructed. Assume the contrary. Let A_4, B_4 be (undirected) graphs with countable many vertices and edges, let $k(A_4, B_4, m) = \varphi(m)$ for all $m \in M_1$ and no different isomorphic copies of A_4 in B_4 have common edges. (Such graphs were constructed in the proof of Theorem 4.)

Let A_6' be the set of all vertices and all edges of the graph A_4 and let the unary operations α, β on the set A_6 be chosen in such a way that $\alpha(x) = \beta(x) = x$ if x is a vertex of A_4 and $\alpha(u)$, $\beta(u)$ are both vertices incident with u if u is an edge. (The choices of $\alpha(u)$, $\beta(u)$ can be done quite independently for different edges.)

Let B_6' be the set of all vertices and all edges of the graph B_4. For every isomorphic copy C of A_4 in B_4 choose one isomorphism f_c of A_4 into C. Now define the operations α, β on the set B_6' in such a way that

1. $\alpha(x) = \beta(x) = x$ if x is a vertex of B_4
2. $\alpha(u)$, $\beta(u)$ are both vertices incident with u, if u is an edge of B_4
3. $\alpha(f_c(u)) = f_c(\alpha(u))$, $\beta(f_c(u) = f_c(\beta(u))$, if u is an edge of some isomorphic copy C of A_4 in B_4.

Then there is a one-to-one correspondence between the set of all isomorphic copies of $A_6 = (A_6', \alpha, \beta)$ in $B_6 = (B_6', \alpha, \beta)$ and the set of all isomorphic copies of the graph A_4 in B_4; this correspondence preserves disjointness, and hence for every $m \in M_1$ we have $k(A_6, B_6, m) = k(A_4, B_4, m) = \varphi(m)$, q. e. d.

Remark. The analogon of Theorem 6 does not hold for algebras with one unary operation. E. g. for such algebras A_7, B_7 it is impossible $k(A_7, B_7, 1) = k(A_7, B_7, 2) = 1$

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SÚHRN

O SYSTÉMOCH DISJUNKTNÝCH IZOMORFNÝCH KÓPIÍ JEDNEJ ALGEBRY V DRUHEJ ALGEBRE

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Nech $M_1 = \{\aleph_0, 1, 2, 3, \ldots\}$, $M_2 = \{\aleph_0, 0, 1, 2, \ldots\}$ a nech A, B sú algebry rovnakej signatúry. Označme symbolom $k(A, B, m)$ počet takých maximálnych systémov disjunktných izomorfných kópií algebry A v algebre B, ktoré majú presne m prvkov. Dokazuje sa, že pre každé zobrazenie φ množiny M_1 do množiny M_2 existujú nekonečné spočítateľné zväzy A, B také, že pre všetky $m \in M_1$ platí $k(A, B, m) = \varphi(m)$. Analogické vety sa dokazujú tiež pre polozväzy, čiastočne usporiadané množiny, rovinné grafy a algebry s dvoma unárnymi operáciami.

РЕЗЮМЕ

О СИСТЕМАХ НЕПЕРЕСЕКАЮЩИХСЯ ИЗОМОРФНЫХ КОПИЙ ОДНОЙ АЛГЕБРЫ В ДРУГОЙ АЛГЕБРЕ

И. Корец, Братислава

Пусть $M_1 = \{\aleph_0, 1, 2, 3, \ldots\}$, $M_2 = \{\aleph_0, 0, 1, 2, \ldots\}$ и пусть A, B-алгебры одной сигнатуры. Обозначим через $k(A, B, m)$ число таких максимальных систем попарно непересекающихся изоморфных копий алгебры A в алгебре B, которые состоят из m элементов. Доказывается, что для всякого отображения φ множества M_1 в множество M_2 существуют бесконечные счетные решетки A, B, такие, что $k(A, B, m) = \varphi(n)$ для $m \in M_1$. Аналогичные теоремы доказываются также для полуструктур, частично упорядоченных множеств, плоских графов и алгебр с двумья одноместными операциями.

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GEOMETRICAL CONSTRUCTION OF COMPLEX HOPF FIBRATION OVER THE SPHERE S^4

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In this paper we present a geometrical construction of a spread in CP^3 which, regarded as a vector — boundle is isomorphic with the standard generator of the group $\tilde{K}_C(S^4)$.

As a special case we get a spread in RP^3 equivalent with the Hopf fibration, the generator of $\tilde{K}_R(S^2)$.

1. Notations

a) Let C or H be the field of the complex or quaternion numbers.

b) Let CP^n or HP^n be the n-dimensional projective space over the complex or quaternion numbers.

c) The space $G_K(CP^n)$ of all k-dimensional subspaces of CP^n cudomed with the natural topology, is homeomorphic to the Grassmannian $G_{k+1}(C^{n+1})$.

d) We consider the quaternions written in the form

$$q = c + d \cdot j, \ (c, d) \in C^2$$

with respect to a multiplicative rule

$$(c_1 + d_1 \cdot j) \cdot (c_2 + d_2 \cdot j) =$$
$$= (c_1 \cdot c_2 - d_1 \cdot \bar{d}_2) + (c_1 \cdot d_2 + d_1 \cdot \bar{c}_2) \cdot j$$

so that

α) $$j \cdot j = -1$$

β) $$\forall c \in C: c \cdot j = j \cdot \bar{c}$$

e) The mapping

$$\text{“}*\text{”}: H \to H, \ (c + d \cdot j) \mapsto (c - d \cdot j)$$

is an isomorphism of the algebra of quaternions.

$$(q_1^* \cdot q_2^* = (c_1 - d_1 \cdot j) + (c_2 - d_2 \cdot j) =$$

$$= (c_1 \cdot c_2 - d_1 \cdot \bar{d}_2) + (-c_1 \cdot d_2 - d_1 \cdot \bar{c}_2)j = (q_1 \cdot q_2)^*)$$

f) The mapping

$$g : C^4 \to H^2,\ (c_1, c_2, c_3, c_4) \mapsto (c_1 + c_2 \cdot j,\ c_3 + c_4 \cdot j)$$

is an isomorphism of the vector spaces over C.

g) Topological spaces HP^1 is homeomorphic with the sphere S^4 and $CP^1 \approx S^2$ (see [3]).

2. Hopf fibration over S^4

Let

$$s : H^n \to HP^{n-1},\ (q_1, \ldots, q_n) \mapsto [q_1, \ldots, q_n]$$

be a homogeneous coordinate system.

Let

$$b = [q_1, q_2] = [c_1 + d_1 \cdot j,\ c_2 + d_2 \cdot j] \in HP^1$$

We have

$$s^{-1}(b) = \{(q \cdot q_1, q \cdot q_2) : q \in H\} \in G_1(H^2)$$

and

$$h(b) = g^{-1} \cdot s^{-1}(b) = \{c \cdot (c_1, d_1, c_2, d_2) +$$
$$+ d \cdot (-\bar{d}_1, \bar{c}_1, -\bar{d}_2, \bar{c}_2) : (c, d) \in C^2\} \tag{1}$$

(because of $(c + d \cdot j) \cdot (c_i + d_i \cdot j) = c \cdot (c_i - d_i \cdot j) + d \cdot (-d_i - \bar{c}_i \cdot j),\ i = 1, 2$)

Lemma. System $\{h(b) : b \in HP\}$ as a spread in $G_2(H^4) \approx CP^3$. The induced vector — bundle $h^*(\gamma_2^4)$ is a generator of the group

$$\tilde{K}_c(S^4) \approx Z \quad \text{(see [1], [2])}.$$

3. Geometrical construction

Let $CP^3 \sim I = s(C^4)$ be the subset of HP^3 generated by the embeding $C \subset H$, $c \mapsto c + 0 \cdot j$.

In HP^3 we consider points

$$A = [1, -j, 0, 0],\quad A^* = [1, j, 0, 0],\quad B = [0, 0, 1, -j],\quad B^* = [0, 0, 1, j]$$

and the embeding

$$v : HP^1 \to p \equiv AB,\quad [q_1, q_2] \mapsto q_1 A + q_2 B$$

From 1.e we get

$$p^* \equiv A^* B^* = \{X \in HP^3 : X^* = [x_1^*, \ldots, x_4^*] \in p\}.$$

Let now take a point

$$X=(q_1A+q_2B)=[c_1+d_1\cdot j,\ d_1-c_1\cdot j,\ c_2+d_2\cdot j,\ d_2-c_2\cdot j]\in p$$

and the projective line XX^*. It is easy to show that this line intersects the subset I in the complex projective line YZ where

$$Y=\frac{1}{2}(X+X^*) \quad \text{and} \quad Z=\frac{1}{2}j(X-X^*).$$

If we put

$$f(X)=XX^*\cap I$$

we get

$$f(X)=\{[c\cdot(c_1,c_2,d_1,d_2)+d\cdot(-\bar{d}_1,\bar{c}_1,-\bar{d}_2,\bar{c}_2)]:(c,d)\in C^2\} \qquad (2)$$

From (1) and (2) we get

$$s^{-1}\cdot f\cdot v(b)=h(b)$$

for any

$$b\in HP^1\approx S^4.$$

Corollary. System $S=\{f(X)\}_{X\in p}$ is the spread in $I\approx CP^3$, which regarded as a vector-bundle is a generator of $\tilde{K}_C(S^4)$.

Remark. If we consider an embeding

$$e: C\to H,\quad r_1+ir_2\mapsto r_1+j\cdot r_2$$

we get the subset $J\subset CP^3$. We have

$$J\cap P\approx CP^1\approx S^2,\ J\cap I\approx RP^1.$$

With the same technique as above we get that $S\cap J$ is the spread in RP^3 which regarded as a vector — bundle (infact it is the Hopf fibration) is a generator of $\tilde{K}_R(S^2)$.

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SÚHRN

GEOMETRICKÁ KONŠTRUKCIA KOMPLEXNEJ HOPFOVEJ FIBRÁCIE NAD SFÉROU S^4

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V práci sme uviedli efektívnu konštrukciu generátora grupy $K_c(S^4)$. Uvedený generátor je spread v priestore CP^3, ktorý dostaneme prienikom so systémom priečok XX^* mimobežiek p, p^* združených v HP^3 vzhľadom na involúciu $(c+d\cdot j)^* = c-d\cdot j$.

Analogická konštrukcia v $RP^3 \subset CP^3$ dáva generátor grupy $K_R(S^2)$.

РЕЗЮМЕ

ГЕОМЕТРИЧЕСКАЯ КОНСТРУКЦИЯ КОМПЛЕКСНОГО РАССЛОЕНИЯ ХОПФА НАД СФЕРОЙ C^4

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В статье дается эффективное построение порождающего элемента группы $K_c(S^4)$. Упомянутым элементом является точное слоение пространства CP^3, определенное как пересечение с множеством всех трансверслей XX^*, соединяющих скрещивающиеся прямые p и p^*, сопряженные в HP^3 относительно инволюции $(c+d\cdot j)^* = c-d\cdot j$.

Аналогичное построение в $RP^3 \subset CP^3$ дает порождающий элемент группы $K_R(S^2)$.

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