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A NOTE ON RELATIONS IN DISTRIBUTIVE LATTICES

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There are two objects for this note. In chapter I we consider translations, defined by G. Szász [6 and 7], and congruence relations on a distributive lattice. In chapter II we define and consider a $\cup$-semicongruence, which we shall call a strong $\cup$-semicongruence.

I. Translations on distributive lattice L

1. Translations and congruences on distributive lattice L. In his paper [7] G. Szász defined a translation $l(x)$ on a lattice:

Definition 1. A mapping l on a lattice L is called a translation if it satisfies the following identity:

$$l(x) \cup y = l(x \cup y), \quad (x, y \in L) \tag{1}$$

Further in the same paper Szász proved ([7] Satz 4):

Lemma 1. A lattice L is distributive if and only if every translation l on L is a $\cap$-endomorphism.

M. Kolibiar [4] constructed a $\cup$-semicongruence on a lattice L relating to translation on L. He presented the following properties (Satz 1):

Lemma 2. On a lattice L there is a one-to-one mapping between translations l and $\cup$-semicongruences Θ_l having the following property:

1. There is in L a dual ideal $I_\cap$ such that every congruence class mod Θ_l contains exactly one element of $I_\cap$.

The $\cup$-semicongruence Θ_l relating to translation l and the translation l_Θ relating to $\cup$-semicongruence Θ_l are characterized, respectively, by

2. $x\Theta_l y$ if and only if $l(x) = l(y)$ $(x, y \in L)$,

3. $l_\Theta(x) = x'' \in I$ for which $x\Theta_l x''$.

By lemmas 1 and 2 we immediately obtain:

Corollary. On a distributive lattice L there is a one-to-one mapping between translations l and congruences Θ_l having the property 1 of lemma 2.

The congruence Θ_l relating to translation l and the translation l_Θ relating to congruence Θ_l are characterized, respectively, by properties 2 and 3 of lemma 2.

Now we can prove a property of the congruences Θ_l:

Theorem 1. Let L be a distributive lattice. The congruences Θ_l constitute a sublattice of the congruence lattice $\Theta(L)$ of L.

Proof. Let l and u be arbitrary translations on L. G. Szász has proved ([8] Satz 3: (3))

$$l(x)\cup u(x)=l(u(x))=lu(x)=ul(x)$$

where $lu=ul=p$ is a translation on L. At first we prove that in $\Theta(L)$ $\Theta_l\cup\Theta_u=\Theta_p$, where $p=ul$, by showing that 1. $\Theta_l\cup\Theta_u \leqq \Theta_p$ and 2. $\Theta_p\leqq\Theta_l\cup\Theta_u$.

1. $x\Theta_l y \Rightarrow l(x)=l(y) \Rightarrow ul(x)=ul(y) \Rightarrow x\Theta_p y \Rightarrow \Theta_l\leqq\Theta_p$ and similarly $\Theta_u\leqq\Theta_p$, whence $\Theta_u\cup\Theta_l\leqq\Theta_p$.

2. Let $x\Theta_p y$. Then $l(u(x))=l(u(y))$, hence $u(x)\Theta_l u(y)$. Moreover $u(u(x))= = u(x)$ (see [8]), hence $x\Theta_u u(x)$, $y\Theta_u u(y)$ and we get $x\Theta_l\cup\Theta_u y$.

Let l and u be arbitrary translations on L and let us consider the mapping $l\cap u$. $(l(x)\cap u(x))\cup y = (l(x)\cup y)\cap(u(x)\cup y) = l(x\cup y)\cap u(x\cup y)$, whence follows that $l\cap u=p$ is a translation on L. To complete the proof we show that in $\Theta(L)$, $\Theta_l\cap\Theta_u=\Theta_p$, where $p=l\cap u$, by showing that 3. $\Theta_l\cap\Theta_u\leqq\Theta_p$ and 4. $\Theta_l\cap\Theta_u\geqq\Theta_p$.

3. $x(\Theta_l\cap\Theta_u)y \Rightarrow x\Theta_l y$ and $x\Theta_u y \Rightarrow l(x)=l(y)$ and $u(x)=u(y) \Rightarrow$ $\Rightarrow u(x)\cap l(x) = u(y)\cap l(y) \Rightarrow x\Theta_p y$. Hence $\Theta_l\cap\Theta_u\leqq\Theta_p$.

4. We make use of the identity $l(x)\cup u(y) = l(x\cup u(y)) = l(u(x\cup y))= = lu(x\cup y)$. Now let $x\Theta_p y$. Then $l(x)\cap u(x) = l(y)\cap u(y)$. It follows $l(x)= = l(x)\cup(l(y)\cap u(y)) = (l(x)\cup l(y))\cap(l(x)\cup u(y)) = (l(x)\cup l(y))\cap lu(x\cup y)= = (l(x)\cup l(y))\cap(l(y)\cup u(x)) = l(y)\cup(l(x)\cap u(x)) = l(y)$, hence $x\Theta_l y$. Similarly $x\Theta_u y$, therefore $x\Theta_l\cap\Theta_u y$.

2. Translations and congruences generated by an ideal on distributive lattice. A well-known construction for the congruence relation Θ_P by means of a prime ideal P of lattice L is as follows:

$$x\Theta_P y \text{ if and only if either } x, y\in P \text{ or } x, y\in L-P$$

Then there are at most two congruence classes mod Θ_P in L, which implies that Θ_P is a maximal congruence on L. Let us consider the maximal congruences on the distributive lattice L.

Theorem 2. Every maximal congruence Θ on a distributive lattice L is generated by a prime ideal P of L.

Proof. Any distributive lattice with more than two elements has a proper congruence relation (see [1] p. 193, thm 15, cor 1). Hence L/Θ is a two-element chain $\bar{0}<\bar{1}$. This follows the assertion.

We characterize the congruences generated on a distributive lattice L by an ideal and the translation of L relating to it. According to condition 1 of corollary of lemma 2 we must at first consider the congruence classes modulo congruences generated by an ideal I of L. Grätzer and Schmidt have proved ([2] thm 2: f) that for any ideal I of L and for any $x, y \in L$, $x \geqq y$, $x\Theta[I]y$ if and only if $x = y \cup d$, where $d \in I$.

Lemma 3. Let L be a distributive lattice satisfying the maximum condition. Then for an arbitrary ideal I of L the following holds true:

1. In every congruence class mod $\Theta[I]$ there is one and only one element $x \in L$: $x \geqq a$, where $I = (a]$.

2. $I = (a]$. Let $x \geqq a$. Then $\{x\}$ is a congruence class mod $\Theta[I]$ or x is congruent mod $\Theta[I]$ to an element $x_0 \ngeqq a$.

Proof. According to the maximum condition every ideal I of L is a principal ideal $(a]$, $a \in L$. We may then modify the condition for the congruence relation $\Theta[(a]]$ as follows: $x\Theta[(a]]y \Leftrightarrow x \cup a = y \cup a$.

Let T be a congruence class mod $\Theta[(a]]$. According to the maximum condition there is in T a maximum element t_0. For arbitrary $t_1 \in T$, $t_1 \neq t_0, t_1 \cup a = = t_0 \cup a$ and $t_1 < t_0$. If $t_0 \ngeqq a$, then $t_0 \cup a = (t_0 \cup a) \cup a$, hence $t_0 < t_0 \cup a \in T$ which contradicts the maximality of t_0. Hence $t_0 \geqq a$. If $t_2 \geqq a$ and $t_2 \in T$ then $t_2 = t_2 \cup a = = t_0 \cup a = t_0$ which completes the proof of 1. The proof of 1 immediately shows the validity of the assertion 2.

Now by lemma 3 we can construct the dual ideal $I_\cap$ in 1 of lemma 2.

Theorem 3. Let L be a distributive lattice satisfying the maximum condition. For every congruence $\Theta[I]$ on L there is in L a dual ideal $I_\cap$ such that $I_\cap$ contains one and only one element of every congruence class mod $\Theta[I]$.

Proof. As in the proof before $I = (a]$, $a \in L$. We define $I_\cap = \{x : x \geqq a\}$. $I_\cap$ contains, according to lemma 3, one and only one element of every congruence class mod $\Theta[(a]]$. Further, the definition of $I_\cap$ implies that $I_\cap$ is the dual ideal $[a)$, which completes the proof.

Corollary of lemma 2 (or Satz 4 of M. Kolibiar [4]) shows that there is in L a translation l relating to the dual ideal $[a)$ and thus generating the congruence relation $\Theta[(a]]$. Hence

Corollary. In a distributive lattice L satisfying the maximum condition every congruence relation of type $\Theta[I]$ is generated by a translation of L.

We may now characterize lattices L in which every congruence relation is generated by a translation.

Theorem 4. Let L be a distributive, relatively complemented lattice satisfying the maximum condition, $0 \in L$. Then every congruence relation in L is generated by a translation of L.

Proof. There is a one-to-one correspondence between ideals and congruence relations of L (Hashimoto, [2] lemma 3); hence every congruence relation is generated by an ideal of L. Further, every homomorphic image of L is weakly complemented (G. J. Areškin [2] footnote on p. 153), which according to Grätzer and Schmidt ([2] thm 4) implies that every congruence relation is the minimal congruence for a suitable ideal of L. But then corollary of theorem 3 readily conveys the validity of the assertion.

We define a dual translation d on the lattice L: $d(x) \cap y = d(x \cap y)$, $x, y \in L$. Owing to the duality, the dual theorems of translation l on L are valid for d. We can characterize the minimal congruence relation Θ_{ab} on the distributive lattice L generated by an interval $[a, b]$, $a, b \in L$ ([2] thm 1): $c\Theta_{ab}d \Leftrightarrow (b \cup d) \cap c = c$ and $(a \cup d) \cap c = d$, where $c \geqq d$, $c, d \in L$. The proofs of lemma 3 and theorem 3 imply that the congruence relations $\Theta[(b]]$ and $\Theta[[a)]$ are generated, on one hand, by ideals $(b]$ and $[a)$, respectively, and on the other hand by the translation l and the dual translation d, respectively.

Lemma 4. Let L be distributive lattice. Then $\Theta_{ab} = \Theta[(b]] \cap \Theta[[a)]$, $[a, b] \subset L$.

The assertion and the proof of lemma are included in 1.4, corollary by M. Kolibiar [3].

As a corollary we now immediately obtain:

Corollary. Let L be a distributive lattice satisfying the maximum and the minimum condition. Then every congruence relation Θ_{ab} on L generated by an interval $[a, b]$ L is a meet of two congruences generated by a translation and a dual translation of L.

II. On strong $\cup$-semicongruence on a distributive lattice

3. Strong $\cup$-semicongruence. In the following we define a $\cup$-semicongruence on a lattice in such manner that by the properties of the $\cup$-semicongruence we may see where the power of lattice congruence lies.

Lemma 5 gives an equivalent form for the definition of the congruence Θ on a $\cup$-semilattice L. The proof of lemma is obvious, and we shall omit it.

Lemma 5. A binary relation Θ on a $\cup$-semilattice L is a congruence relation on L if and only if

1. $x\Theta x$ for every $x \in L$,
2. $x\Theta y \Leftrightarrow x\Theta y \cup x$ and $y\Theta y \cup x$ $(x, y \in L)$,
3. if $x \geqq y$, $x \geqq z$ and $x\Theta y$, $x\Theta z$, then $y\Theta z$ $(x, y, z \in L)$,
4. if $x \geqq y \geqq z$ and $x\Theta y$, $y\Theta z$, then $x\Theta z$ $(x, y, z \in L)$,
5. if $x \geqq y$ and $x\Theta y$, then $x \cup t\Theta y \cup t$ for every $t \in L$ $(x, y \in L)$.

We shall now alter the condition 2 of lemma 5 so that the binary relation produced by the alteration takes into account the $\cap$-operation on the lattice. This new binary relation we call a strog $\cup$-semicongruence, abbreviated s-semicongruence.

Definition 2. A binary relation Θ on lattice L is a strong $\cup$-semicongruence if and only if Θ satisfies the conditions 1, 3, 4, 5 of lemma 5 and if

2. $x\Theta y \Leftrightarrow x\cup y\Theta x\cap y \ (x, y\in L)$.

By definition 2 we immediately see that every lattice congruence satisfies the conditions 1, 5 of the definition, whence every lattice congruence is an s-semicongruence, and further that every s-semicongruence is a $\cup$-semicongruence on a lattice.

At first we consider the conditions which the lattice L has to satisfy in order that a $\cup$-semicongruence might be an s-semicongruence and that an s-semicongruence might be a lattice congruence.

Let x, y be elements of a lattice L. We denote by $\Theta_{\cup xy}$ and Θ_{sxy} the least $\cup$-semicongruence and the s-semicongruence on L which collapse the elements x, y.

Theorem 5. Every $\cup$-semicongruence of a lattice L is an s-semicongruence if and only if L is a chain.

Proof. Let x, y be an arbitrary pair of elements in L, $x\Theta_{\cup xy}y$, and suppose that every $\cup$-semicongruence on L is an s-semicongruence. Then $x\cup y\Theta_{\cup xy}x\cap y$. But then x and y are comparable according to the properties of the $\cup$-semicongruence and to the minimality of $\Theta_{\cup xy}$.

If, conversely, L is a chain, then $x\Theta_{\cup xy}y$ implies $x\cap y\Theta_{\cup xy}x\cup y$ on the strength of the comparability of x and y, and thus every $\cup$-semicongruence is an s-semicongruence too.

Theorem 6. Let the lattice L be a chain. Then every s-semicongruence is a lattice congruence on L.

The proof is trivial.

The converse of theorem 6 is valid on a distributive lattice (theorem 7). For the proof of the converse we need:

Lemma 6. Let L be a distributive lattice and Θ_{sab} a binary relation on L generated by arbitrary elements a, $b\in L$, and defined:

$$x\Theta_{sab}y \Leftrightarrow \begin{cases} 1. \ a\cup b\cup x = a\cup b\cup y \text{ and} \\ 2. \ (a\cap b)\cup(x\cap y) = x\cap y \text{ or } 3. \ x=y \end{cases}$$

Then Θ_{sab} is the least s-semicongruence on L collapsing the elements a and b.

Proof. At first we show by definition 2 that Θ_{sab} is an s-semicongruence:

1. is trivial according to 3. We assume in the following that $x\neq y$.

2. $x\Theta_{sab}y \Leftrightarrow (a\cup b\cup y)\cap(a\cup b\cup x) = (a\cup b)\cup(x\cap y) = (a\cup b)\cup(x\cup y)$ and $(a\cap b)\cup(x\cap y) = x\cap y \Leftrightarrow x\cup y\Theta_{sab}x\cap y$.

3. $x\Theta_{sab}y$, $x\Theta_{sab}z$ and $x\geqq y$, $x\geqq z \Rightarrow a\cup b\cup x = a\cup b\cup y = a\cup b\cup z$ and $(a\cap b)\cup(x\cap y) = y, (a\cap b)\cup(x\cap z) = z \Rightarrow (a\cap b)\cup(y\cap z) = y\cap z \Rightarrow y\Theta_{sab}z$.

4. $x\Theta_{sab}y$, $y\Theta_{sab}z$ and $x \geqq y \geqq z \Rightarrow a\cup b\cup x = a\cup b\cup y = a\cup b\cup z$ and $(a\cap b)\cup y = y$, $(a\cap b)\cup z = z \Rightarrow (a\cap b) \cup (x\cap z) = z \Rightarrow x\Theta_{sab}z$.

5. $x\Theta_{sab}y \Rightarrow (a\cup b) \cup (x\cup t) = (a\cup b) \cup (y\cup t)$ and $(a\cap b) \cup ((x\cup t) \cap \cap(y\cup t)) \doteq ((a\cap b) \cup (x\cap y))\cup t = (x\cap y)\cup t = (x\cup t) \cap (y\cup t) \Rightarrow x\cup t\Theta_{sab}y\cup t$ for every $t \in L$.

Let $\Theta_{\min ab}$ be the least s-semicongruence generated by elements $a, b \in L$. We shall show that $\Theta_{sab} \leqq \Theta_{\min ab}$, whence $\Theta_{sab} = \Theta_{\min ab}$ according to the minimality of $\Theta_{\min ab}$.

If $x\Theta_{sab}y$, then by the definition of Θ_{sab} $(a\cap b)\cup(x\cap y) = x\cap y$, which implies that $(a\cap b) \cup (x\cup y) = x\cup y$. Further, $y\cup x\Theta_{sab}y\cap x$ implies that

$$a\cup b\cup x\cup y = (a\cup b)\cup(x\cap y) \tag{2}$$

Certainly $a\Theta_{\min ab}b$ and thence by 5, $(a\cap b) \cup (x\cap y)\Theta_{\min ab} (a\cup b) \cup (x\cap y) \Rightarrow \Rightarrow x\cap y\Theta_{\min ab}(a\cup b) \cap (x\cap y)$ and $(a\cap b) \cap (x\cup y)\Theta_{\min ab}(a\cup b) \cap (x\cup y)$. The transitivity and (2) together imply that $x\cap y\Theta_{\min ab}(a\cap b) \cup (x\cup y) \Rightarrow \Rightarrow x\cap y\Theta_{\min ab}x\cup y \Rightarrow \Theta_{\min ab} \geqq \Theta_{sab}$.

Now we can prove:

Theorem 7. Let L be a distributive lattice. Each s-semicongruence is a lattice congruence if and only if L is a chain.

Proof. According to theorem 6 it is sufficient to show necessity. We show that if $t \leqq y$ and $x \leqq y$, then t is comparable with x. Hence L is a chain.

Let $x \leqq y$ and $t \leqq y$; then $x\Theta_{sxy}y \Rightarrow x\cap t\Theta_{sxy}y\cap t$ since Θ_{sxy} is a lattice congruence. $y\cap t = t \Rightarrow t\Theta_{sxy}x\cap t$. On account of 3 and 2 of lemma 6 we have two possibilities: I, by 3, $t = x\cap t \Rightarrow x \geqq t$, and II. by 2, $(x\cap y) \cup (x\cap t\cap y) = x\cup(x\cap t) = x = x\cap t \Rightarrow x \leqq t$.

In the following we shall examine whether the set of all s-semicongruences of a lattice L forms a lattice $\Theta_s(L)$. We define the $\cap$-operation for $\Theta_s(L)$ in like manner as that on $\Theta(L)$, and then obviously $\Theta_{s1}\cap\Theta_{s2}$, where Θ_{s1}, Θ_{s2} are arbitrary s-semicongruences on L, is an s-semicongruence on L. For the $\cup$-operation we define a binary relation Φ and prove theorem 8.

Definition 3. Let L be a lattice. Φ is a binary relation on L corresponding to a set $\{\Theta_{sj}\}$, $j \in A$, of s-semicongruences on L such that Φ satisfies the following conditions: $x, y \in L$, $x\Phi y$ if and only if there is a finite number m of finite sequences on L

$$\begin{matrix} z_{01}, & z_{11}, & \dots, & z_{n_1 1} \\ z_{02}, & z_{12}, & \dots, & z_{n_2 2} \\ \vdots & \vdots & & \vdots \\ z_{0m}, & z_{1m}, & \dots, & z_{n_m m} \end{matrix} \tag{3}$$

such that

1. $z_{ir}\Theta_{sj}z_{i+1r}$ for every $i=0, 1, \ldots, n_r-1$ and $r=1, 2, \ldots, m$ and for a $j\in A$.

2. For every sequence of (3) there is an interval $I_r = [I_r^1, I_r^2] = [z_{0r}\cap\ldots\cap z_{n_r r}, z_{0r}\cup\ldots\cup z_{n_r r}]$. The intervals given by (3) can be divided into different classes. In every class $I_{r1}, I_{r2}, \ldots, I_{rk}$ there is for every two intervals I_{rp}, I_{rq} a sequence of intervals of this class, $I_{rp}, I_{rp+1}, \ldots, I_{rp+n}=I_{rq}$, where I_{rp+i} and I_{rp+i+1} $(i=0, \ldots, n-1)$ have at least one common element. Further for every class there is an interval $[I_{r1}^1\cap I_{r_2}^1\cap\ldots\cap I_{rk}^1, I_{r_1}^2\cup\ldots\cup I_{rk}^2]$ which intervals can be divided into classes as before. After finite steps there is only one class which gives an interval $I_0 = [z_{01}\cap z_{11}\cap z_{21}\cap\ldots\cap z_{n_1 1}\cap z_{02}\cap\ldots\cap z_{n_m m}, z_{01}\cup\ldots\cup z_{n_m m}]$.

3. $[x\cap y, x\cup y]\subset I_0$.

Theorem 8. Let L be a distributive lattice and Φ the binary relation on L defined by definition 3. Then Φ is an s-semicongruence on L and $\Phi=\bigcup_{j\in A}\Theta_{sj}$.

Proof. If Φ is an s-semicongruence, then immediately by definition 3 $\Phi=\bigcup_{j\in A}\Theta_{sj}$. So it has to be shown that Φ is an s-semicongruence. 1. is obvious and the condition 3 of definition 3 implies 2. According to the property 5 of s-semicongruences we may to every term z_{i_r} of the sequences (3) add the part $\cup t$, where $t\in L$ is arbitrary. According to the distributivity, $t\cup(z_{01}\cap\ldots\cap z_{n_m m}) = (t\cup z_{01})\cap\ldots\cap(t\cup z_{n_m m})$, whence $t\cup(x\cap y)\Phi t\cup x\cup y$, which proves 5. Let $x\geqq y\geqq z$ and $x\Phi y$, $y\Phi z$, where the proof of 4 is obvious in respect of y. Similarly, the proof of 3 is in respect of x obvious.

Theorem 8 shows that $\Theta_s(L)$ is a lattice when L is a distributive lattice.

4. s-semicongruences on a distributive lattice. At first, we consider the pseudocomplement of an s-semicongruence Θ_s on a distributive lattice L. To this purpose we define the concept of the transpose.

Definition 4. Let L be a lattice and $[a, b]$ an interval of L, $t\in L$, t arbitrary. The transpose of the interval $[a, b]$ relating to t, written $[a, b]_t$, is the interval $[a\cup t, b\cup t]$ of L. A transpose is proper if $a\cup t\neq b\cup t$.

Theorem 9. Let Θ_s be an s-semicongruence on the lattice L and Θ_s^+ a binary relation on L satisfying the condition: $x\Theta_s^+y$ if and only if in every transpose of the interval $[x\cap y, x\cup y]$ every congruence class mod Θ_s consists of a single element. Θ_s^+ is an s-semicongruence on L and a pseudocomplement of Θ_s in $\Theta_s(L)$ if and only if L is a distributive lattice.

Proof. Let L be distributive. We show that Θ_s^+ is an s-semicongruence on L. 1 is obvious, and the definition of Θ_s^+ immediately implies 2 and 5.

4. Let $x\geqq y\geqq z$ and $x\Theta_s^+y, y\Theta_s^+z$, and suppose that $x\not\Theta_s^+z$. Then there is in L an element t and a transpose $[z, x]_t$ containing a proper interval $[a, b]$, such that $a\Theta_s b$. Denote by Θ_L a lattice congruence on L. In particular $a(\Theta_{Lxy}\cup\Theta_{Lyz})b$, which implies that there is a proper interval $[u, v]\subset[a, b]$ such that $u\Theta_{Lyz}v$ or

$u\Theta_{Lxy}v$. On account of analogy we shall only consider the first case. According to theorem 1 in [2], in distributive lattice then $(z\cup u)\cap v=u$ and $(y\cup u)\cap v=v$. If $z\cup u = y\cup u$, then according to the preceeding equations $u=v$ which is a contradiction. But then obviously there is a proper interval $[z, y]_u$ or $[y, x]_u$ collapsed by the s-semicongruence Θ_s, which is a contradiction.

3. Let $y\leqq x$, $z\leqq x$ and $x\Theta_s^+y$, $x\Theta_s^+z$ and suppose that $y\neq x$, $z\neq x$, and that y and z are noncomparable. According to 2 it is sufficient to prove that $y\cup z\Theta_s^+y$, $y\Theta_s^+y\cap z$ (or $y\cup z\Theta_s^+z$, $z\Theta_s^+z\cap y$, which has the same proof). $[y, y\cup z] \subset [y, x]$; hence $y\cup z\Theta_s^+y$. Let $[a, b]$ be a proper transpose of $[y\cap z, y]$, which implies that $a=t\cup(y\cap z) = (t\cup y) \cap (t\cup z)$ and $b=t\cup y$ for a given $t\in L$. Now $[a, b]$ has always a proper transpose contained in a proper transpose $[z, x]_t$. This is implied by the equations $a\cup(t\cup z) = ((t\cup y) \cap (t\cup z)) \cup (t\cup z) = t\cup z$ and $b\cup(t\cup z)= = (t\cup y) \cup (t\cup z) = t\cup z\cup y \Rightarrow [t\cup z, t\cup z\cup y] \subset [z, x]_t$. Further, if $t\cup z= = t\cup z\cup y$, then $t\cup y = (t\cup z) \cap (t\cup y) \Rightarrow a=b$, which is a contradiction. Hence $y\cap z\Theta_s^+y$.

Obviously Θ_s^+ is the pseudocomplement of Θ_s on $\Theta_s(L)$ when Θ_s^+ is s-semicongruence on L.

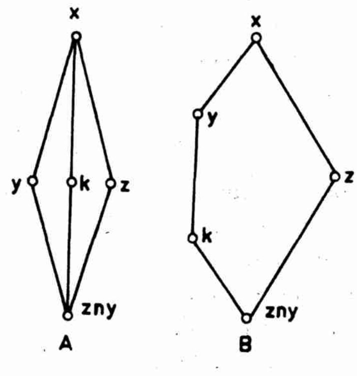

Fig. 1

The distributivity of L is necessary. If L is not distributive then it shall contain at least one of the sublattices of *figure 1*. In the sublattice A we consider the s-semicongruence Θ_{skx}. $y\Theta_{skx}^+x$ and $z\Theta_{skx}^+x$, but $y\Theta_{skx}^+z$ since $[k, x] = [y\cap z, y]_k$. In the sublattice B there is $y\Theta_{sky}^+x$ and $z\Theta_{sky}^+x$, but $y\Theta_{sky}^+z$ since $[k, y] \subset [y\cap z, y]_k$. This completes the proof.

In a distributive lattice L we can define a minimal s-semicongruence generated by an ideal I of L.

Theorem 10. Let L by a distributive lattice and I an ideal on L. $\Theta_s[I]$ is a binary relation on L satisfying the condition: $x\Theta_s[I]y$ if and only if $x \cup z = y \cup z$ for a given $z \in I$. Then $\Theta_s[I]$ is an s-semicongruence on L, and a minimal one generated by the ideal I.

Proof. It is well-known that the relation $\Theta_s[I]$ is a lattice congruence on L. It follows immediately that $\Theta_s[I]$ is an s-semicongruence.

Let Θ_s be an s-semicongruence on L with a class I. If $x\Theta_s[I]y$, then $x \cup t = y \cup t$, $t \in I$, whence $x = x \cap (y \cup t) = (x \cap y) \cup (x \cap t)$, $y = (y \cap x) \cup (y \cap t)$. But $x \cap t \Theta_s y \cap t$, since $x \cap t$, $y \cap t \in I$. Hence $x\Theta_s y$ and thus $\Theta_s[I] \leqq \Theta_s$, which implies the minimality of $\Theta_s[I]$.

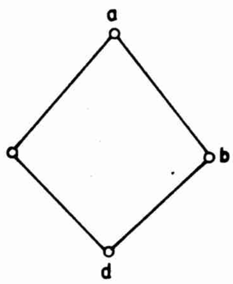

Fig. 2

Finally we consider some properties of a minimal s-semicongruence Θ_{sab} on a distributive lattice L. The distributive lattice L of *figure 2* implies that the lattice congruence Θ_{Lab} and the s-semicongruence Θ_{sab} are different, since $c\Theta_{Lab}d$ but $x \not\Theta_{sab} d$.

Theorem 11. Let L be a distributive lattice and $a<b$, $a, b \in L$. $[a, b]$ is a congruence class mod Θ_{sab}.

Proof. It is sufficient, owing to the transitivity of Θ_{sab}, to show that $x \notin [a, b]$, $x \in L$ imply $x \not\Theta_{sab} a$ or $x \not\Theta_{sab} b$. We have five different cases:

1. $x<a$. Then $(a \cap b) \cup (x \cap a) = a \cup x = a \neq a \cap x = x \Rightarrow x \not\Theta_{sab} a$.

2. $x>b$. Then $(a \cup b) \cup b = b \neq a \cup b \cup x = x \Rightarrow x \not\Theta_{sab} b$.

3. $x<b$, and x is noncomparable with a. Then $(a \cap b) \cup (x \cap a) = a \cup (x \cap a) = a \neq x \cap a \Rightarrow x \not\Theta_{sab} a$.

4. $x>a$, and x is noncomparable with b. Then $(a \cup b) \cup b = b \neq b \cup x \Rightarrow x \not\Theta_{sab} b$.

5. x is noncomparable with both a and b. The proof is similar as in the case 3 and 4.

Theorem 12. Let L be a distributive lattice. $\Theta_{sab} \cap \Theta_{scd} = 0$ for any $a>b \geqq c>d$, $a, b, c, d \in L$.

Proof. Let $a>b\geqq c>d$ and $x>y$ such that $x(\Theta_{sab}\cap\Theta_{scd})y$. Then $x\Theta_{sab}y$, $x\Theta_{scd}y$, whence $c\cup x = c\cup y$, $d\cup y=y$ and $b\cup y=y$, which imply that $x>y\geqq b\geqq$ $\geqq c>d$. But this is a contradiction to the equality $c\cup x=c\cup y$. Therefore $\Theta_{sab}\cap\Theta_{scd}=0$.

The opposites of theorems 11 and 12 are not valid. In order to show this, we consider the non-distributive sublattices A and B of figure 1. In the sublattice A we find by considering the minimal s-semicongruence $\Theta_{sy\cap z,y}$: $y\Theta_{sy\cap z,y}\, y\cap z \Rightarrow$ $\Rightarrow x\Theta_{sy\cap z,y}z$ and $x\Theta_{sy\cap z,y}k$. Then on account of the transitivity, $z\Theta_{sy\cap z,y}k \Rightarrow$ $\Rightarrow x\Theta_{sy\cap z,y}\,y\cap z$. Now e. g. $y\Theta_{sy\cap z,y}x$, whence $\Theta_{sy\cap z,y}\cap\Theta_{syx}\neq 0$. In the sublattice B we see, by considering the minimal s-semicongruences $\Theta_{sy\cap z,z}$, $\Theta_{sy\cap z,k}$ and $\Theta_{sy\cap z,y}$, that the assertions of theorems 11 and 12 remain valid, although B is not even modular. By the aid of the sublattice A we can now formulate two corollaries, the proofs of which are obvious.

Corollary 1. If L is a modular lattice and if for every pair $a, b\in L$, $a<b$, $[a, b]$ is a congruence class mod Θ_{sab}, then L is a distributive lattice.

Corollary 2. If L is a modular lattice and if $\Theta_{sab}\cap\Theta_{scd}=0$ for any $a>b\geqq$ $\geqq c>d$, $a, b, c, d\in L$, then L is a distributive lattice.

We demonstrate a property of an atom of the lattice $\Theta_s(L)$.

Theorem 13. In a distributive lattice L an interval $[a, b]$, $a\prec b$, $a, b\in L$, generates an atom of the lattice $\Theta_s(L)$ if and only if $[a, b]$ has no proper transpose.

Proof. Let $[a, b]$ be an interval with the property mentioned in the theorem. Then $a\Theta_{sab}b \Rightarrow \Theta_{sab}\neq 0$. On the other hand there is no other proper interval collapsed by Θ_{sab}, which implies that there cannot be any other minimal s-semicongruences Θ_s generated by the proper interval such that $\Theta_s<\Theta_{sab}$. Hence Θ_{sab} is an atom of $\Theta_s(L)$.

Let, on the contrary, Θ_{sab} be an atom of $\Theta_s(L)$ and suppose that $a\prec b$. If $[a, b]$ has a proper transpose $[c, d]$, then $c\Theta_{sab}d$, but by definition of Θ_{scd}, $a\Theta_{scd}b$, which is a contradiction.

I wish to give sincere thanks to Professor M. Kolibiar to whom I am obliged for numerous suggestions concerning corrections and improvements of this paper.

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SÚHRN

POZNÁMKA K RELÁCIÁM V DISTRIBUTÍVNYCH ZVÄZOCH

J. Nieminen, Heinola

Práca sa sústreďuje na dva problémy. V prvej kapitole sa autor zaoberá transláciami, ktoré zaviedol G. Szász v prácach [6], [7] a kongruenciami na distributívnych zväzoch. Druhá kapitola rozoberá špeciálne kongruencie vzhľadom na operáciu spojenia v distributívnych zväzoch.

РЕЗЮМЕ

ПРИМЕЧАНИЕ К ОТНОШЕНИЯМ В ДИСТРИБУТИВНЫХ РЕЩЕТКАХ

Ю. Ниминен, Гейнола

В работе рассматриваются два типа проблем. В первой главе изучаются трансляции, введенные Г. Сасом в [6, 7], и конгруэнции на дистрибутивных решетках. Во второй главе рассматривается специальная $\cup$-конгруэнция на дистрибутивной решетке.

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ASYMPTOTIC EQUIVALENCE OF VOLTERRA INTEGRAL EQUATIONS

ARLETE CERQUEIRA LIMA, Brasil

Introduction

Let us consider the perturbed system of Volterra equations

$$\mathbf{x}(t)=\mathbf{f}(t)+\int_0^t \mathbf{a}(t,s)\,[\mathbf{x}(s)+g(s,\mathbf{x}(s))]\,\mathrm{d}s\,,\qquad t\geqq 0 \tag{1}$$

where $\mathbf{x}$,$\mathbf{f}$, $\mathbf{g}$ are the component vectors and $\mathbf{a}$ is a given $n\times n$ matrix. Our aim will be to compare the solutions of (1) with those of the system of linear unperturbed equations

$$\mathbf{y}(t)=\mathbf{f}(t)+\int_0^t \mathbf{a}(t,s)\,y(s)\,\mathrm{d}s \tag{2}$$

obtained from (1) omitting the nonlinear term. The nature of this comparison may be stated as follows: to determine the conditions which should be imposed on $\mathbf{f}(t)$, $\mathbf{g}(t,\mathbf{x})$ and $\mathbf{a}(t,\mathbf{x})$ so that the solution $\mathbf{y}(t)$ of (2) corresponds to a solution $\mathbf{x}(t)$ of (1) such that

$$\lim_{t\to\infty}[\mathbf{x}(t)-\mathbf{y}(t)]=0 \tag{3}$$

and conversely, that is, given a solution $\mathbf{x}(t)$ of (1) there exists a solution $\mathbf{y}(t)$ of (2) such that (3) is true. In other words, the systems (1) and (2) should be asyptotically equivalent.

Together with (1) and (2) let us consider equations (4) and (5) below

$$\begin{cases}\mathbf{x}'(t)=\mathbf{F}(t)+\mathbf{A}(t)\mathbf{x}(t)+\int_0^t \mathbf{B}(t,s)\mathbf{x}(s)\,\mathrm{d}s+(\mathbf{G}\mathbf{x})(t)\\ \mathbf{x}(0)=\mathbf{x}_0\qquad (0\leqq t<\infty)\end{cases} \tag{4}$$

$$\begin{cases} \mathbf{y}'(t) = \mathbf{F}(t) + \mathbf{A}(t)\mathbf{y}(t) + \int_0^t \mathbf{B}(t, s)\, \mathbf{y}(s)\, ds \\ \mathbf{y}(0) = \mathbf{x}_0 \qquad (0 \leqq t < \infty) \end{cases} \tag{5}$$

where **F** and **G** are given vectors and **A** and **B** are given $n \times n$ matrices. In general the perturbations **G** are nonlinear functionals. Although it is possible to integrate (4) and (5) and then obtain (1) and (2), in many applications it is interesting to give criteria directly to (4) and (5).

J. A. Nohel, [1], [2], has studied the systems (1) and (2), (4) and (5), establishing asymptotic equivalence between continuous and bounded solutions of (1) and (2) [1, T.3] and of (4) and (5), [2, T.2].

We will prove that Nohel's theorems remain valid if we alter the assumptions, especially the ones about the functions $\mathbf{g}(t, \mathbf{x})$ and, $(\mathbf{G}\mathbf{x})\ (t)$.

Followig Nohel's approach, using the concept of associated resolvent to a given kernel, we have that the resolvent $r(t, s)$ is the solution of the equation

$$\mathbf{r}(t, s) = \mathbf{a}(t, s) + \int_s^t \mathbf{a}(t, u)\, \mathbf{r}(u, s)\, du\ , \qquad 0 \leqq s \leqq t < \infty \tag{6}$$

This solution exists (and is locally integrable on (t, s)) if the kernel $\mathbf{a}(t, s)$ is locally integrable in (t, s) for $0 \leqq s \leqq t < \infty$. For $s \geqq t \geqq 0$, we define $\mathbf{r}(t, s) = 0$.

Under these conditions the solution of the system (2) can be written as

$$\mathbf{y}(t) = \mathbf{f}(t) + \int_0^t \mathbf{r}(t, s)\, \mathbf{f}(s)\, ds\ , \tag{7}$$

provided we make some additional assumptions about $\mathbf{f}$.

Furthermore, it is possible to show that system (1) can be changed into an equivalent system

$$\mathbf{x}(t) = \mathbf{y}(t) + \int_0^t \mathbf{r}(t, s)\, \mathbf{g}(s, \mathbf{x}(s))\, ds \tag{8}$$

where $\mathbf{y}(t)$ is given by (7).

With this is mind we will first study the asymptotic equivalence between the solutions (7) and (8).

Inicially we will prove the following theorem

Theorem 1. Let us suppose that

a) $\mathbf{r}(t, s)$ is locally integrable on (t, s) for $0 \leqq s \leqq t < \infty$;

b) there is a constant $B > 0$ such that

$$\int_0^t |\mathbf{r}(t, s)|\, ds \leqq B\ , \qquad (t \geqq 0)$$

c) $$\lim_{h\to 0}\left[\int_t^{t+h}|\mathbf{r}(t+h,s)|\,ds+\int_0^t|\mathbf{r}(t+h,s)-\mathbf{r}(t,s)|\,ds\right]=0$$

for every $t>0$;

d) $$\lim_{t\to\infty}\int_0^T|\mathbf{r}(t,s)|\,ds=0\,,\qquad \text{for every } T>0$$

e) the function $\boldsymbol{g}(t,\boldsymbol{x})$ is continuous on (t,x) *for* $0\leqq t<\infty$ and $|x|<\infty$;

f) there is a function $F(t,u)$, continuous on (t,u) for $0\leqq t<\infty$ and $u\geqq 0$, non-decreasing with respect to u and such that,

$$|\boldsymbol{g}(t,x)|\leqq F(t,|\boldsymbol{x}|)\tag{9}$$

with

$$\lim_{t\to\infty}F(t,c)=0\,,\qquad \text{for every } c\geqq 0\tag{10}$$

and

$$\overline{\lim_{u\to\infty}}\frac{u}{F(t,u)}=\infty\,,\qquad \text{uniformly, for } t\geqq 0\tag{11}$$

or

$$\overline{\lim_{u\to\infty}}\frac{u}{F(t,u)}=c>0\,,\qquad \text{uniformly, for } t\geqq 0\tag{12}$$

with the additional hypothesis that $\dfrac{2B}{c-\gamma}<1$, for some γ, $0<\gamma<c$.

Let $\boldsymbol{y}(t)$ be a solution of (2), continuous and bounded on $[0,\infty)$. Then there is a solution $\mathbf{x}(t)$ of (1) which is continuous and bounded on $[0,\infty)$ and such that,

$$\lim_{t\to\infty}[\mathbf{x}(t)-\boldsymbol{y}(t)]=0\tag{13}$$

Conversely, if $\boldsymbol{x}(t)$ is a solution of (1), continuous and bounded on $[0,\infty)$, then there is a solution $\boldsymbol{y}(t)$ of (2) which is also continuous and bounded on $[0,\infty)$ and such that (13) holds.

Proof. Consider BC, the Banach space of all functions continuous and bounded on $[0,\infty)$, with norm $\|\psi\|=\sup_{t\geqq 0}|\psi(t)|$ and let $\boldsymbol{y}(t)\in BC$ be a solution of (2). Then there is a constant $K>0$ such that, $\|\boldsymbol{y}\|<K$. We will first prove that (1) has a solution $\boldsymbol{x}(t)$ belonging to BC.

Let us define on BC the operator $R\psi$ in the following manner:

$$(R\psi)(t) = \boldsymbol{y}(t) + \int_0^t r(t, s)\, \boldsymbol{g}(s, \psi(s))\, ds \tag{14}$$

Consider now

$$S_\varrho = \{|\psi(t) \in BC| \; \|\psi(t)\| \leqq 2\varrho\},$$

where ϱ will be determined afterwards.

We will prove that:

1.. There is $\varrho > 0$ $(\varrho > K)$ such that $RS_\varrho \subset S_\varrho$.

In fact, if $\psi \in S_\varrho$, using (14), (9), (10) and b), we have

$$\begin{aligned} |(R\psi)(t)| &\leqq |\boldsymbol{y}(t)| + \int_0^t |\mathbf{r}(t, s)|\, F(s, |\psi(s)|)\, ds \leqq \\ &\leqq K + \int_0^t |\mathbf{r}(t, s)|\, F(s, 2\varrho)\, ds \leqq K + \sup_{s \in [0, \infty)} F(s, 2\varrho) B \end{aligned} \tag{15}$$

Now, from (11), we have that for every B there is a sequence, $\{d_n\}_{n=1}^\infty$, with $\lim_{n\to\infty} d_n = \infty$, such that

$$2B < \frac{d_n}{F(t, d_n)}, \quad \text{for every } t \geqq 0 \text{ and } n = 1, 2, \ldots \tag{16}$$

Consider n such that $d_n > 2K$ and take $\varrho = \frac{d_n}{2}$.

From (16) it follows that

$$F(t, 2\varrho) = F(t, d_n) < \frac{d_n}{2B}$$

and therefore

$$\sup_{t \geqq 0} F(t, 2\varrho) \leqq \frac{d_n}{2B}$$

Then, from (15)

$$|(R\psi)(t)| \leqq \varrho + \frac{d_n}{2B} \cdot B = 2\varrho$$

and, consequently

$$\|R\psi\| \leqq 2\varrho$$

1. *We will show now that there is* $\varrho > 0$, $\varrho > k$, such that $RS_\varrho \subset S_\varrho$, using (12).

From (12) we have that for every arbitrary $\gamma > 0$, $\gamma < c$, there is a sequence

$\{u_n\}_{n=1}^{\infty}$, with $\lim\limits_{n\to\infty} u_n = \infty$, such that

$$c-\gamma<\frac{u_n}{F(t, u_n)}, \quad \text{for every } t\geqq 0 \text{ and } n = 1, 2, \dots .$$

Consider n such that $u_n > 2K$ and take $\varrho = \frac{u_n}{2}$.

Similarly, we find

$$F(t, 2\varrho) = F(t, u_n) < \frac{2\varrho}{c-\gamma} \quad \text{for every } t \geqq 0$$

therefore,

$$\sup_{t\in[0, \infty)} F(t, 2\varrho) \leqq \frac{2\varrho}{c-\gamma}$$

and

$$|(R\psi)(t)| \leqq \varrho + \frac{2\varrho}{c-\gamma} B = \varrho\left(1+\frac{2B}{c-\gamma}\right) < 2\varrho$$

2. The operator R is continuous on S_{ϱ}.

The proof of this statement, using either (11) or (12), is identical to the one given by Nohel [**1**, T. 1].

3. The functions in the set $RS\varrho$ are bounded (what is obvious from 1 or 1′) and equicontinuous on every closed interval $[0, d]$ of $[0, \infty)$.

In fact, supposing $t > t_0$, t_0, $t \in [0, d]$, we have

$$|(R\psi)(t) - (R\psi)(t_0)| \leqq |\boldsymbol{y}(t) - \boldsymbol{y}(t_0)| +$$

$$+ \sup_{0\leqq s\leqq t} |\boldsymbol{g}(s, \psi(s))| \left[\int_0^{t_0} |\mathbf{r}(t_0, s) - \mathbf{r}(t, s)|\, ds + \int_{t_0}^{t} |\mathbf{r}(t, s)|\, ds\right]$$

However, from (9) and (10), we find

$$\sup_{0\leqq s\leqq d} |\boldsymbol{g}(s, \psi(s))| \leqq \sup_{0\leqq s\leqq d} F(s, 2\varrho) < H, \text{ for some constant } H > 0 \tag{17}$$

On the other hand, given $\varepsilon > 0$, we choose $\delta_1 > 0$ such that

$$\|\boldsymbol{y}(t) - \boldsymbol{y}(t_0)\| \leqq \varepsilon\ 2 \quad \text{for } |t - t_0| \leqq \delta_1$$

We also choose $\delta_2 > 0$ (see c)) such that

$$\int_0^{t_0} |\mathbf{r}(t, s) - \mathbf{r}(t_0, s)|\, ds + \int_{t_0}^{t} |\mathbf{r}(t, s)|\, ds < \frac{\varepsilon}{2H},$$

for

$$|t-t_0|<\delta_2$$

Then, for

$$|t-t_0|\leqq\delta=\min(\delta_1,\delta_2)\,,$$

we find

$$|(R\psi)(t)-(R\psi)(t_0)|<\frac{\varepsilon}{2}+\frac{H\varepsilon}{2H}=\varepsilon$$

what proves the equicontinuity.

As we have shown, the proof of iii) is independent of the assumptions (11) and (12).

Under these conditions, from Schauder—Tychonoff's Theorem, R has a fixed point on S_ϱ, that is, there is a function $\tilde{\psi}\in S_\varrho$ such that

$$\tilde{\psi}(t)=\boldsymbol{y}(t)+\int_0^t \mathbf{r}(t,s)\,\boldsymbol{g}(s,\tilde{\psi}(s))\,\mathrm{d}s\,,\qquad (0\leqq t<\infty)$$

and from the equivalence between (1) and (8), we have that $\tilde{\psi}$ is a bounded solution of (1).

We shall prove now that

$$\lim_{t\to\infty}[\tilde{\psi}(t)-\boldsymbol{y}(t)]=0$$

Consider $\varepsilon>0$, arbitrarily given. Using (10) we can choose $T>0$ such that

$$F(t,2\varrho)\leqq\frac{\varepsilon}{2B}\,,\qquad (t\geqq T)\tag{18}$$

where B is the constant given in b) and considering also d), we can choose $T_1>T$ such that

$$\int_0^T|\mathbf{r}(t,s)|\,\mathrm{d}s<\frac{\varepsilon}{2M}\,,\qquad (t\geqq T_1)\tag{19}$$

where M is a positive constant, to be determined later.

Furthermore, from (8) and since $\|\tilde{\psi}\|\leqq 2\varrho$, we have, for $t\geqq T$,

$$\begin{aligned}|\tilde{\psi}(t)-\boldsymbol{y}(t)|&\leqq\int_0^T|\mathbf{r}(t,s)|\,F(s,|\tilde{\psi}(s)|)\,\mathrm{d}s+\int_T^t|\mathbf{r}(t,s)|\,F(s,|\tilde{\psi}(s)|)\,\mathrm{d}s\leqq\\&\leqq\sup_{0\leqq s\leqq T}F(s,2\varrho)\int_0^T|\mathbf{r}(t,s)|\,\mathrm{d}s+\sup_{T\leqq t<\infty}F(t,2\varrho)\int_0^t|\mathbf{r}(t,s)|\,\mathrm{d}s\,,\qquad t\geqq T\end{aligned}\tag{20}$$

Using (10) there is a constant M such that $\sup\limits_{0\leqq s\leqq T}F(s,2\varrho)<M$ and from (18)

$$\sup_{T \leqq t < \infty} F(t, 2\varrho) < \frac{\varepsilon}{2B}.$$

Then

$$|\tilde{\psi}(t) - \mathbf{y}(t)| \leqq M \frac{\varepsilon}{2M} + \frac{\varepsilon}{2B} B = \varepsilon , \qquad t \geqq T_1$$

Consequently

$$\lim_{t \to \infty} [\tilde{\psi}(t) - \mathbf{y}(t)] = 0$$

and the proof is independent of the assumptions (11) and (12).

To show the converse, let $\mathbf{u}(t)$ be a solution of (1) (that is, of the equivalent system (8)).

Consider, as a definition,

$$\mathbf{v}(t) = \mathbf{u}(t) - \int_0^t \mathbf{r}(t, s)\, \mathbf{g}(s, \mathbf{u}(s))\, ds \tag{21}$$

From this and using (9) and b), we have

$$|\mathbf{v}(t)| \leqq \|\mathbf{u}\| + H_1 B < \infty , \qquad 0 \leqq t < \infty$$

where $F(s, 2\varrho) < H_1$ for $s \in [0, \infty)$, since from (10) we conclude that $F(t, 2\varrho)$ is bounded on $[0, \infty)$. That is, v is bounded whichever hypothesis, (11) or (12), is considered.

We state that:

a) $$\lim_{t \to \infty} [\mathbf{v}(t) - \mathbf{u}(t)] = 0$$

To show this, consider an arbitrary $\varepsilon > 0$ and assume $\|\mathbf{u}\| < \varrho$. Defining T and T_1 as before (in (18) and (19), respectively), and from (21) and (9), for $t \geqq T$, we have

$$|\mathbf{v}(t) - \mathbf{u}(t)| \leqq \int_0^T |\mathbf{r}(t, s)|\, F(s, |u(s)|)\, ds + \int_T^t |\mathbf{r}(t, s)|\, F(s, |\mathbf{u}(s)|)\, ds$$

and from b), d), (18) and (19), for $t > T_1$

$$|\mathbf{v}(t) - \mathbf{u}(t)| \leqq M \frac{\varepsilon}{2M} + \frac{\varepsilon}{2B} B$$

in a similar way as it was done in (20); as a result, our statement is true independently of (11) and (12)

b) $\mathbf{v}(t)$ is a solution of (2).

The proof of this fact is also independent of (11) and (12) and is identical to the one given in [1, T. 3].

This ends the proof of the Theorem.

Note. For Nohel [1, T.3]

$$F(t, |\mathbf{x}|)=\begin{cases}\lambda(t)|\mathbf{x}|, & |\mathbf{x}|\geqq 1\\ \lambda(t), & |\mathbf{x}|<1\end{cases}$$

where

$$\lambda(t)\to 0 \quad \text{and} \quad \|\lambda B\|\leqq\frac{1}{2}$$

This is equivalent to the situation we presented when (12) holds, that is

$$\lim_{u\to\infty}\frac{u}{F(t,u)}=\lim_{\substack{u\to\infty\\ u\geqq 1}}\frac{u}{\lambda(t)u}=\frac{1}{\lambda(t)}\geqq\frac{1}{\sup\limits_{[0,\infty)}\lambda(t)}=\frac{1}{\|\lambda\|}$$

for every $t\in[0, \infty)$.

Theorem 2. Let $L_p(S)=\{\psi: \psi$ is measurable on $S=[0, \infty)$ and $\int_S |\psi(t)|^p\, dt<\infty\}$: if $\psi\in L_p(S)$ define $\|\psi\|_p = \left(\int_S |\psi(t)|^p\, dt\right)^{1/p}$.

Let us suppose that the resolvent $\mathbf{r}(t, s)$ satisfies the following conditions: There are constants $p>1$, and $B>0$ such that

$$\left(\int_0^t |\mathbf{r}(t, s)|^p\, ds\right)^{1/p}\leqq B, \qquad (0\leqq t<\infty) \tag{22}$$

$$\lim_{h\to 0}\left\{\left(\int_t^{t+h} |\mathbf{r}(t+h, s)|^p\, ds\right)^{1/p}+\left(\int_0^t |\mathbf{r}(t+h, s)-\mathbf{r}(t, s)|^p\, ds\right)^{1/p}\right\}=0, \qquad (0\leqq t<\infty) \tag{23}$$

and for every fixed $T>0$

$$\lim_{t\to\infty}\int_0^T |\mathbf{r}(t, s)|^p\, ds=0 \tag{24}$$

Consider $\mathbf{g}(t, \mathbf{x})$ maesurable with respect to t for every fixed $\mathbf{x}$ and continuous in $\mathbf{x}$ for every fixed t, $0\leqq t<\infty$, $|\mathbf{x}|<\infty$ and such that

$$|\mathbf{g}(t, \mathbf{x})|\leqq F(t, |\mathbf{x}|) \tag{25}$$

where $F(t, u)$ is a continuous function in (t, u), for $0\leqq t<\infty$ and $u\geqq 0$, non-decreasing with respect to u, with

$$F(t, k)\geqq 0 \text{ and } F(t, k)\in L_q(0, \infty), \text{for every } k\geqq 0, \tag{26}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Furthermore, suppose that

$$\overline{\lim_{u\to\infty}}\frac{u}{F(t,u)}=\infty\,,\quad \text{uniformly, for } t\geqq 0 \tag{27}$$

and let BC be the Banach space mentioned in the proof of Theorem 1.

Then, given a solution $\boldsymbol{y}\in BC$ of system (2), there is a solution $\boldsymbol{x}\in BC$ of system (1) such that

$$\lim_{t\to\infty}[\boldsymbol{x}(t)-\boldsymbol{y}(t)]=0$$

and, conversely, given a solution $\boldsymbol{u}\in BC$ of system (1) there is a solution $\boldsymbol{v}\in BC$ of system (2) such that

$$\lim_{t\to\infty}[\boldsymbol{u}(t)-\boldsymbol{v}(t)]=0$$

Proof. We will once again follow Nohel's proof [2, T. 1].

1. Assuming there is a solution $\boldsymbol{y}\in BC$ of (2), let us show that there is a solution $\boldsymbol{x}(t)$ of (1) on $0\leqq t<\infty$. We will use Volterra's equation, (8), which is equivalent to (1). From [3, Lemma 1.1] or from [4, T. 1.1], there is a solution $\boldsymbol{x}(t)$, locally continuous; from (23) $\boldsymbol{x}(t)$ is continuous. Consider a given $T>0$; let us prove that $\boldsymbol{x}(t)$ exists on $[0, T]$. In order to get this result, define a constant $k=k(T)>0$ by

$$k=\max\left\{\sup_{0\leqq t<\infty}|\boldsymbol{y}(t)|,\ \sup_{0\leqq t<T}\left|\int_0^T|\boldsymbol{r}(t,s)|^p\,ds\right|^{1/p}\right\} \tag{28}$$

Using (23) let us choose $h_0>0$ as small as necessary, so that

$$\left[\int_t^{t+h}|\boldsymbol{r}(t+h,s)|^p\,ds\right]^{1/p}\leqq\frac{1}{2k},\qquad (0\leqq h\leqq h_0,\quad 0\leqq t\leqq T) \tag{29}$$

Let us assume that the solution $\boldsymbol{x}(t)$ exists on $0\leqq t\leqq T_1\leqq T-h_0$. Consider

$$M=M(T_1)=\sup_{0\leqq t\leqq T_1}|\boldsymbol{x}(t)|$$

From (8) and using Hölder's inequality, we obtain

$$|\boldsymbol{x}(T_1)|\leqq|\boldsymbol{y}(T_1)|+\left(\int_0^{T_1}|\boldsymbol{r}(T_1,s)|^p\,ds\right)^{1/p}\left(\int_0^{T_1}F^q(s,|\boldsymbol{x}(s)|)\,ds\right)^{1/q}\leqq$$

$$\leqq k + k\left(\int_0^{T_1} F^q(s, M)\, ds\right)^{1/q}$$

Take

$$P_1 = \left(\int_0^{T_1} F^q(s, M) ds\right)^{1/q}$$

and let L be a number, to be determined later, such that

$$k + kP_1 < \frac{L}{2} \tag{30}$$

For h sufficiently small, $|\boldsymbol{x}(T_1+h)| < L$. We assert that this last inequality holds for every $0 \leqq h \leqq h_0$.

In fact if we assume that there exists a first $0 < h \leqq h_0$ such that $|x(T_1+h)| = L$, from (8), (25) and Hölder's inequality it follows that

$$L = |\boldsymbol{x}(T_1+h)| \leqq |\boldsymbol{y}(T_1+h)| + \int_0^{T_1} |\mathbf{r}(T_1+h, s)|\, F(s, |\boldsymbol{x}(s)|)\, ds +$$

$$+ \int_{T_1}^{T_1+h} |\mathbf{r}(T_1+h, s)|\, F(s, |\boldsymbol{x}(s)|)\, ds \leqq$$

$$\leqq k + k\left(\int_0^{T_1} F^q(s, M)\, ds\right)^{1/q} + \frac{1}{2k}\left(\int_{T_1}^{T_1+h} F^q(s, L)\, ds\right)^{1/q}$$

Taking

$$P_2 = \left(\int_{T_1}^{T_1+h} F^q(s, L)\, ds\right)^{1/q}$$

we have, using (30)

$$L = |\boldsymbol{x}(T_1+h)| \leqq k + kP_1 + \frac{1}{2k} P_2 < \frac{L}{2} + \frac{1}{2k} P_2$$

On the other hand, from (27) we have that for every $R > 0$ there is a sequence

$\{u_n\}_{n=1}^{\infty}$, with $\lim_{n\to\infty} u_n = \infty$, such that

$$R \leqq \frac{u_n}{F(s, u_n)}, \quad \text{for every } s \geqq 0 \text{ and } n = 1, 2, \ldots$$

Take the value of L as being equal to one of the u_n, as large as required.

Under these conditions,

$$F(s, L) \leqq \frac{L}{R}$$

and

$$P_2 = \left(\int_{T_1}^{T_1+h} F^q(s, L)\, \mathrm{d}s \right)^{1/q} \leqq \frac{L}{R} \left(\int_{T_1}^{T_1+h} \mathrm{d}s \right)^{1/q} = \frac{L}{R} h^{1/q}$$

Taking $R = \dfrac{h^{1/q}}{k}$ we have that

$$L = |\boldsymbol{x}(T_1 + h)| < \frac{L}{2} + \frac{1}{2k} \cdot Lk = L$$

which is a contradiction; therefore $\boldsymbol{x}(t)$ can be extended to $[0, T]$. Since $T > 0$ is arbitrarily chosen, it follows that the mentioned solution can be extended on $[0, \infty)$.

2. Let us show now that $\boldsymbol{x}(t) \in BC$. From (27) we conclude that there is an increasing sequence $\{c_n\}_{n=0}^{\infty}$, with $\lim_{n\to\infty} c_n = \infty$ and $c_0 > |\boldsymbol{x}(t_0)|$ such that the sequence $\left\{ \dfrac{F(t, c_n)}{c_n} \right\}$ is non-decreasing and

$$\lim_{n\to\infty} \frac{F(t, c_n)}{c_n} = 0, \quad \text{uniformly, for } t \geqq 0 \tag{31}$$

Let us assume that $\boldsymbol{x}(t)$ is not bounded. Then, since $|\boldsymbol{x}(t)|$ is continuous on $[0, \infty)$, we have that the equation $|\boldsymbol{x}(t)| = c_n$ has at last one solution for every $n = 1, 2, \ldots$. Let t_n be the smallest of them. This means that $|\boldsymbol{x}(t_n)| < c_n$ for $t \in [0, t_n)$ with $|\boldsymbol{x}(t_n)| = c_n$, or $\sup_{0 \leqq s \leqq c_n} |\boldsymbol{x}(s)| = c_n$. It is clear that $\{t_n\}_{n=1}^{\infty}$ is an increasing sequence and $\lim_{n\to\infty} t_n = \infty$.

Now, from (8) and Hölder's inequality, it follows that

$$|\boldsymbol{x}(t_n)| \leqq \|\boldsymbol{y}(t)\| + \left(\int_0^T |\mathbf{r}(t_n, s)|^p \,\mathrm{d}s \right)^{1/p} \left(\int_0^T F^q(s, c_n)\, \mathrm{d}s \right)^{1/q} +$$

$$+ \left(\int_T^{t_n} |\mathbf{r}(t_n, s)|^p \,\mathrm{d}s \right)^{1/p} \left(\int_T^{t_n} F^q(s, c_n)\, \mathrm{d}s \right)^{1/q}$$

for n as large as $t_n > T$.

From (31) we have that for $n_0 > 0$ there is $n_1 \geqq n_0$ such that

$$\frac{F(s, c_n)}{c_n} \leqq \frac{F(s, c_{n_0})}{c_{n_0}} \tag{32}$$

for every $n \geqq n_1$ and $s \geqq 0$.

We choose T so that

$$\left(\int_T^\infty \left(\frac{F(s, c_{n_0})}{c_{n_0}}\right)^q \mathrm{d}s\right)^{1/q} \leqq \frac{1}{2B} \tag{33}$$

Then, using (32), (33) and (22), we have that for n as large as $n \geqq n_1$ and $t_n > T$

$$c_n \leqq \|\boldsymbol{y}(t)\| +$$

$$+\left(\int_0^T |\mathbf{r}(t_n, s)|^p \mathrm{d}s\right)^{1/p} \cdot c_n \left(\int_0^T \left[\frac{F(s, c_{n_0})}{c_{n_0}}\right]^q \mathrm{d}s\right)^{1/q} + Bc_n \frac{1}{2B}$$

or

$$c_n \left\{\frac{1}{2} - \left(\int_0^T |\mathbf{r}(t_n, s)|^p \mathrm{d}s\right)^{1/p} \left(\int_0^T \left(\frac{F(s, c_{n_0})}{c_{n_0}}\right)^q \mathrm{d}s\right)^{1/q}\right\} \leqq \|\boldsymbol{y}(t)\|$$

Now, from (24), it follows that the expression in brackets will tend to 1/2 which implies a finite limit for the sequence $\{c_n\}_{n=1}^\infty$ and this is a contradiction. Therefore $\boldsymbol{x}(t)$ is bounded.

3. We will now show that $\lim_{t\to\infty}[\boldsymbol{x}(t) - \boldsymbol{y}(t)] = 0$, when $\boldsymbol{y}(t) \in BC$ is a solution of (2) and $\boldsymbol{x}(t) \in BC$ is a solution of (1) which exists as a result of 1 and 2.

Let $\sup_{0\leqq t<\infty} |\boldsymbol{x}(t)| = M$, $\varepsilon > 0$, given, and choose $T > 0$ as large as required for

$$\left(\int_T^t F^q(s, M) \, \mathrm{d}s\right)^{1/q} < \frac{\varepsilon}{2B}, \qquad t \geqq T \tag{34}$$

From (24) we choose $T_2 > T$ such that

$$\int_0^T |r(t, s)|^p \, \mathrm{d}s < \frac{\varepsilon}{2\left(\int_0^T F^q(s, M) \, \mathrm{d}s\right)^{1/q}}, \qquad t \geqq T_2 \tag{35}$$

Then, using (8), (25), (22), Hölder's inequality, (34) and (35), we have

$$|\boldsymbol{x}(t) - \boldsymbol{y}(t)| \leqq \int_0^T |\mathbf{r}(t, s)| \, F(s, |\boldsymbol{x}(s)|) \, \mathrm{d}s +$$

$$+ \int_T^t |\mathbf{r}(t, s)| \, F(s, |\boldsymbol{x}(s)|) \, \mathrm{d}s \leqq$$

$$\leqq \frac{\varepsilon}{2\left(\int_0^T F^q(s, M) \, \mathrm{d}s\right)^{1/q}} \left(\int_0^T F^q(s, M) \, \mathrm{d}s\right)^{1/q} + B \frac{\varepsilon}{2B} = \varepsilon$$

for $t \geqq T_2$.

This proves our statement.

4. Let $\boldsymbol{u}(t) \in BC$ be a solution of (1). We shall show that there is a solution $\boldsymbol{v}(t) \in BC$ of (2), such that

$$\lim_{t \to \infty} (\boldsymbol{u}(t) - \boldsymbol{v}(t)) = 0$$

Consider, as a definition, that

$$\boldsymbol{v}(t) = \boldsymbol{u}(t) - \int_0^t \mathbf{r}(t, s)\, \boldsymbol{g}(s, \boldsymbol{u}(s))\, \mathrm{d}s$$

It is immediate that $\boldsymbol{v}(t)$ is a solution of (2).

Using (22), (25) and Hölder's inequality, it follows that

$$|\boldsymbol{v}(t)| \leqq \|\boldsymbol{u}\| + B\left(\int_0^T F^q(s, |\boldsymbol{u}(s)|)\, \mathrm{d}s\right)^{1/q}$$

Assuming $\sup_{0 \leqq t < \infty} |\boldsymbol{u}(t)| = N$, we have, from (26), that

$$|\boldsymbol{v}(t)| \leqq \|\boldsymbol{u}\| + B\left(\int_0^\infty F^q(s, N)\, \mathrm{d}s\right)^{1/q} < \infty$$

From (23), $\boldsymbol{v}$ is continuous on $0 \leqq t < \infty$ and consequently belongs to BC. Since both solutions are now bounded, the final part of the proof is similar to what was done in 3.

Call $LL_1(D)$ the set of all measurable functions α on a set D such that the seminorms $\int_\Sigma |\alpha(t)|\, \mathrm{d}t$ are finite, for every compact subset Σ of D.

Consider now equations (4) and (5). In this situation we can define a resolvent $\mathbf{R}(t, s)$, associated to $\mathbf{A}(t)$ and to $\mathbf{B}(t, s)$, as the solution of the initial value problem.

$$\left|\begin{array}{l} \dfrac{\partial \mathbf{R}}{\partial t}(t, s) = \mathbf{A}(t)\, \mathbf{R}(t, s) + \displaystyle\int_s^t \mathbf{B}(t, u)\, \mathbf{R}(u, s)\, \mathrm{d}u \\ \mathbf{R}(s, s) = I, \qquad 0 \leqq s \leqq t \end{array}\right. \tag{36}$$

where I is the identy $n \times n$ matrix. For $s > t \geqq 0$, define $\mathbf{R}(t, s) = 0$. If $\mathbf{A} \in LL_1(R^+)$ and $\mathbf{B} \in LL_1(R^+ \times R^+)$, where $R^+ = [0, \infty)$, it is easy to see from the equation below

$$\mathbf{R}(t, s) = I + \int_s^t \left[\mathbf{A}(u) + \int_u^t \mathbf{B}(\sigma, u)\, \mathrm{d}\sigma\right] \mathbf{R}(u, s)\, \mathrm{d}u$$

which is the equivalent to (36), that $\mathbf{R}(t, s)$ exists and is continuous on (t, s), for $0 \leqq s \leqq t$ and (36) holds almost everywhere for $0 \leqq s \leqq t$. Moreover, under several hypothesis, including those from Theorem 3 below, (4) is equivalent to Volterra's equation,

$$\mathbf{x}(t)=\mathbf{y}(t)+\int_0^t \mathbf{R}(t, s)\,(\mathbf{G}\mathbf{x})\,(s)\,\mathrm{d}s \tag{37}$$

where $\mathbf{y}(t)$ is the solution of (5) given by

$$\mathbf{y}(t)=\mathbf{R}(t, 0)\mathbf{x}_0+\int_0^t \mathbf{R}(t, s)\,F(s)\,\mathrm{d}s \tag{38}$$

The perturbation $\mathbf{G}$ is a continuous mapping from BC to $LL_1(R^+)$ such that

$$|(\mathbf{G}\psi)\,(t)|\leqq F(t, \|\psi\|)\,, \qquad (0\leqq t<\infty) \tag{39}$$

where $F(t, u)$ is a continuous function of (t, u), for $0\leqq t<\infty$ and $u\geqq 0$, nondecreasing with respect to u, with $F(t,k)\geqq 0$, and $F(t,k)\in L_q(0,\infty)$, for every $k\geqq 0$, where $\frac{1}{p}+\frac{1}{q}=1$.

The use of (37) in place of (8) and (39) instead of (25) makes it possible to proof the Theorem below in the same manner as carried out of Theorem 3.

Theorem 3. Consider $A(t)\in LL_1(R^+)$ and $B(t, s)\in LL_1(R^+\times R^+)$; suppose, now, that the resolvent, defined by (36) satisfies the hypothesis (22) and (24). If the perturbation $(\mathbf{G}\mathbf{x})\,(t)$ satisfies (39) and (27) then system (4) and (5) are asymptotically equivalent.

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SÚHRN

ASYMPTOTICKÁ EKVIVALENCIA INTEGRÁLNYCH ROVNÍC VOLTERROVHO TYPU

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Článok sa zaoberá asymptotickou ekvivalenciou riešení dvoch integrálnych rovníc Volterrovho typu

$$\boldsymbol{x}(t) = \boldsymbol{f}(t) + \int_0^t \boldsymbol{a}(t, s)\,[\boldsymbol{x}(s) + \boldsymbol{g}(s, \boldsymbol{x}(s))]\,\mathrm{d}s$$

$$\boldsymbol{y}(t) = \boldsymbol{f}(t) + \int_0^t \boldsymbol{a}(t, s)\,\boldsymbol{y}(s)\,\mathrm{d}s$$

Nadväzuje na práce J. A. Nohela a zovšeobecňuje jeho výsledky.

РЕЗЮМЕ

АСИМПТОТИЧЕСКАЯ ЭКВИВАЛЕНТНОСТЬ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ ВОЛЬТЕРРА

А. С. Лима, Бразилия

В работе исследуется асимптотическая эквивалентность решений двух интегральных уравнений типа Вольтерра:

$$\boldsymbol{x}(t) = \boldsymbol{f}t) + \int_0^t \boldsymbol{a}(t, s)\,[\boldsymbol{x}(s) + \boldsymbol{g}(s, \boldsymbol{x}(s))]\,\mathrm{d}s$$

$$\boldsymbol{y}(t) = \boldsymbol{f}(t) + \int_0^t \boldsymbol{a}(t, s)\boldsymbol{y}(\mathrm{s})\,\mathrm{d}s$$

Здесь обобщаются результаты работ Й. А. Нохеля.

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A NOTE ON THE ASYMPTOTIC EQUIVALENCE OF TWO SYSTEMS OF DIFFERENTIAL EQUATIONS

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I. Introduction

Since the theory on linear systems of differential equations

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$$

is well known and, to a certain extent, complete, many research workers on this field have been concerned with comparing the properties of their solutions with the ones of the derived system

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{f}(t, \mathbf{y})$$

obtained by adding a perturbation.

In this paper, in particular, we will study the solutions of a system of the type

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{f}(t, \mathbf{y}) + \mathbf{p}(t) \tag{1}$$

which contains one linear term, one nonlinear and one forcing term, this last one depending only on the independent variable; these solutions will be compared with the ones of the system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{p}(t) \tag{2}$$

obtained from (1) by omitting the nonlinear term.

The nature of this comparison can be stated as: to determine which conditions, besides the one of continuity, can be imposed upon the $n \times n$ matrix, $\mathbf{A}(t)$, and to the $n \times 1$ column vector, $\mathbf{f}(t, \mathbf{y})$, $0 \leqq t < \infty$, $|\mathbf{y}| < \infty$, so that there will be a correspondence, $\emptyset$, among the solutions $\mathbf{y}(t)$ of (1) and $\mathbf{x}(t)$ of (2) in such a way that if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are corresponding solutions, then

$$\lim_{t \to \infty} |\mathbf{y}(t) - \mathbf{x}(t)| = 0$$

Such a problem is known as "asymptotic equivalence between two systems of differential equations".

In this paper we will present a generalization of Brauer's theorem which appeared in [1]. This will be done by substituting the condition

$$|\boldsymbol{f}(t), \boldsymbol{y})| \leqq \lambda(t)|\boldsymbol{y}| \tag{3}$$

which the author uses for a perturbation $\boldsymbol{f}(t, \boldsymbol{y})$, for a more general condition

$$|\boldsymbol{f}(t, \boldsymbol{y})| \leqq \lambda(t)\, \boldsymbol{F}(|\boldsymbol{y}|) \tag{4}$$

The generalization is achieved by using Bihari's inequality [3] instead of Gronwal—Bellman's.

We shall also prove that:

a) to suppose the solutions of (2) are bounded, together with the other assumed conditions, is enough to show that all solutions of (1) exist on $[0, \infty)$ and are bounded on this interval.

b) the reciprocal of the implication presented by Brauer is also valid and on the whole interval $[0, \infty)$, what is an extension of Brauer—Wong's theorem [2].

c) there are functions that satisfy (4) but not (3), what will be shown by an example.

In order to make this paper self-contained we shall state the following two lemmas:

Lemma 1. (Brauer [1] — Lemma 1). Let us suppose α is a positive constant and $\lambda(t)$ a continuous function, non-negative for $0 \leqq t < \infty$ such that

$$\int_0^\infty \lambda(s)\, \mathrm{d}s < \infty$$

or

$$\lim_{t \to \infty} \lambda(t) = 0$$

Then

$$\lim_{t \to \infty} \mathrm{e}^{-\alpha t} \int_0^t \mathrm{e}^{\alpha s}\, \lambda(s)\, \mathrm{d}s = 0$$

Lemma 2. (Bihari, [3]). Let $u(t) \geqq 0$ and $f(t) \geqq 0$ be continuous functions for $t_0 \leqq t < \infty$.

Let $c > 0$ and $F(u)$ be a continuous, and non-decreasing function on $0 \leqq u \leqq \leqq \bar{u} < \infty$ and positive for $0 < u < \bar{u} \leqq \infty$.

Let also

$$\psi(u) = \int_c^u \frac{\mathrm{d}v}{F(v)}, \qquad 0 \leqq u < \bar{u}, \; c > 0$$

Now, if

$$u(t) \leqq c + \int_{t_0}^{t} f(s)\, F(u(s))\, \mathrm{d}s \tag{5}$$

and

$$\int_{t_0}^{t} f(s)\, \mathrm{d}s \leqq \psi(\bar{u}-), \qquad t_0 \leqq t < \infty \tag{6}$$

then

$$u(t) \leqq \psi^{-1}\left\{\int_{t_0}^{t} f(s)\, \mathrm{d}s\right\}, \qquad t_0 \leqq t < \infty \tag{7}$$

Note. If $\bar{u} = \infty$ and $\psi(\bar{u}) = \infty$, the inequality (7) holds without restrictions.

II. A generalization of Brauer's result [1]

Theorem. If the matrix **A** is constant, all the solutions of (2) are bounded on $[0, \infty)$ and $\boldsymbol{f}(t, \boldsymbol{y})$ satisfies

$$|\boldsymbol{f}(t, \boldsymbol{y})| \leqq \lambda(t)\, F(|\boldsymbol{y}|), \qquad |\boldsymbol{y}| < \infty \tag{8}$$

where:

a) $$\lambda(t) \geqq 0, \qquad \lambda(t) \in C[0, \infty)$$

and

$$\int_0^{\infty} \lambda(s)\, \mathrm{d}s < \infty$$

b) $F(u)$ is a continuous and non-decreasing function for $0 \leqq u < \infty$, and positive for $0 < u < \infty$ and such that the function defined by

$$\int_H^u \frac{\mathrm{d}v}{F(v)} = \psi(u), \qquad H > 0, \ 0 \leqq u < \bar{u}$$

shows the property

$$\psi(\infty) = \int_H^{\infty} \frac{\mathrm{d}v}{F(v)} = \infty$$

Then:

I. every solution $\boldsymbol{y}(t)$ of (1) given by Cauchy's condition $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$, $t_0 \in [0, \infty)$, exists and is bounded on $[0, \infty)$;

II. the equations (1) and (2) are asymptotically equivalent, that is, to every solution $\boldsymbol{y}(t)$ of (1) corresponds one solution $\boldsymbol{x}(t)$ of (2) such that

$$\lim_{t\to\infty} |\boldsymbol{y}(t) - \boldsymbol{x}(t)| = 0 \tag{9}$$

and the converse is also true, that is, to every solution $\boldsymbol{x}(t)$ of (2) corresponds one solution $\boldsymbol{y}(t)$ of (1) such that (9) is satisfied.

Proof. The theorem assumes that the solution of (2) are bounded on $0 \leqq t < \infty$. Then using the fact that the general solution of (2) can be written as

$$\boldsymbol{x}(t) = \boldsymbol{u}(t) + \boldsymbol{x}_0(t)$$

where $\boldsymbol{u}(t)$ is the general solution of the system

$$\boldsymbol{u}' = \mathbf{A}\boldsymbol{u} \tag{10}$$

and $\boldsymbol{x}_0(t)$ is a particular solution of (2), we have that

$$\boldsymbol{u}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_0(t)$$

is bounded; this means that all solutions of (10) are bounded on $0 \leqq t < \infty$, which is equivalent to say that there is a constant $M > 0$ such that

$$|\mathbf{U}(t)| \leqq M, \quad |\mathbf{U}(t-s)| = |\mathbf{U}(t)\, U^{-1}(s)| \leqq M \tag{11}$$

for $0 \leqq s \leqq t$ and where $\mathbf{U}(t)$ is a fundamental matrix of (10).

I. Let $\boldsymbol{y}(t)$ and $\boldsymbol{x}(t)$ be solutions of (1) and (2) respectively satisfying the initial condition $\boldsymbol{y}(t_0) = \boldsymbol{x}(t_0) = \boldsymbol{y}_0$, $t_0 \in [0, \infty)$.

From the continuity conditions we know that $\boldsymbol{y}(t)$ exists on $(t_0 - \delta_1, t_0 + \delta_2)$ (on $[t_0, t_0 + \delta_2)$ if $t_0 = 0$). We will first show that this solution exists and is bounded on $[t_0, \infty)$. Using the formula of variation of constants we find

$$\boldsymbol{x}(t) = \mathbf{U}(t)\boldsymbol{y}_0 + \int_{t_0}^{t} \mathbf{U}(t)\, \mathbf{U}^{-1}(s)\, \boldsymbol{p}(s)\, \mathrm{d}s$$

$$\boldsymbol{y}(t) = \mathbf{U}(t)\boldsymbol{y}_0 + \int_{t_0}^{t} \mathbf{U}(t)\, \mathbf{U}^{-1}(s)\, \boldsymbol{p}(s)\, \mathrm{d}s + \tag{12}$$

$$+ \int_{t_0}^{t} \mathbf{U}(t)\, \mathbf{U}^{-1}(s)\, \boldsymbol{f}(s, \boldsymbol{y}(s))\, \mathrm{d}s\,, \qquad t \in [t_0, t_0 + \delta_2)$$

where $\mathbf{U}(t)$, with $\mathbf{U}(t_0) = \mathbf{E}$ is the fundamental matrix of (10).

As a result of the hypothesis assumed above, we have,

$$|\boldsymbol{y}(t)| \leqq H + M \int_{t_0}^{t} \lambda(s)\, F(|\boldsymbol{y}(s)|)\, \mathrm{d}s\,, \qquad t \in [t_0, t_0 + \delta_2) \tag{13}$$

where H is a constant such that

$$|\boldsymbol{x}(t)| = \left|\mathbf{U}(t)\, \boldsymbol{y}_0 + \int_{t_0}^{t} \mathbf{U}(t)\, \mathbf{U}^{-1}(s)\, \boldsymbol{p}(s)\, \mathrm{d}s\right| \leqq H\,, \qquad t \in [0, \infty)$$

because all solutions of (2) were supposed bounded.

Using (13) and taking into account that all conditions of Bihari's lemma are satisfied, we have

$$
\begin{aligned}
|\boldsymbol{y}(t)| &\leqq \psi^{-1}\left\{M\int_{t_0}^{t}\lambda(s)\,\mathrm{d}s\right\}\leqq \\
&\leqq \psi^{-1}\left\{M\int_{0}^{\infty}\lambda(s)\,\mathrm{d}s\right\}, \qquad t\in[t_0,\, t_0+\delta_2)
\end{aligned} \tag{14}
$$

since ψ^{-1}, as inverse function of ψ, is non-decreasing.

From the fact that

$$
M\int_0^{\infty}\lambda(s)\,\mathrm{d}s<\infty=\int_H^{\infty}\frac{\mathrm{d}s}{F(s)}=\psi(\infty)\,, \qquad H>0
$$

it follows that the solution $\boldsymbol{y}(t)$ is bounded on $[t_0, t_0+\delta_2)$ and can, therefore, be extended.

From (14) it is clear that $\boldsymbol{y}(t)$ can be extended on $[t_0, \infty)$ and that on this interval it is bounded.

In order to prove that $\boldsymbol{y}(t)$ can also be extended on $[0, t_0)$ let us make the transformation

$$
t=t_0-\tau
$$

Then (1) becomes

$$
\frac{\mathrm{d}\boldsymbol{y}(t_0-\tau)}{\mathrm{d}\tau}=-\mathbf{A}\boldsymbol{y}(t_0-\tau)-\boldsymbol{f}(t_0-\tau,\,\boldsymbol{y}(t_0-\tau))-\boldsymbol{p}(t_0-\tau)
$$

Now let

$$
\boldsymbol{y}(t_0-\tau)=\boldsymbol{u}(\tau)\,, \qquad \boldsymbol{f}(t_0-\tau,\,\boldsymbol{y}(t_0-\tau))=-\boldsymbol{g}(\tau,\,\boldsymbol{u}(\tau))
$$

and

$$
\boldsymbol{p}(t_0-\tau)=-\boldsymbol{P}(\tau)
$$

Then

$$
\frac{\mathrm{d}u}{\mathrm{d}\tau}=-\mathbf{A}u+\boldsymbol{g}(\tau,\,\boldsymbol{u}(\tau))+\mathbf{P}(\tau) \tag{1'}
$$

For equation (2) we consider $\boldsymbol{x}(t_0-\tau)=\boldsymbol{v}(\tau)$ and find

$$
\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\tau}=-\mathbf{A}\boldsymbol{v}+\mathbf{P}(\tau) \tag{2'}
$$

As a result, the solution $\boldsymbol{y}(t)$ of (1) which exists on $(t_0-\delta_1, t_0]$ is just the solution $\boldsymbol{u}(\tau)$ of (1') which exists on $0\leqq\tau<\delta_1$.

Since

$$\boldsymbol{v}(\tau)=\boldsymbol{x}(t_0-\tau)=\boldsymbol{x}(t)$$

exists on $0 \leqq t < \infty$, it follows that $\boldsymbol{v}(\tau)$ exists and is bounded on $-\infty < \tau \leqq t_0$.

The matrix $-\mathbf{A}$ has then all the roots with non-negative real parts.

Let $\mathbf{G}(z)$ be the fundamental matrix of the equation

$$\frac{d\boldsymbol{z}}{d\tau}=-\mathbf{A}\boldsymbol{z}$$

such that $\mathbf{G}(0)=\mathbf{E}$, where $\mathbf{E}$ is the identity matrix. Then the solution $\boldsymbol{u}(\tau)$ satisfies the integral equation

$$\boldsymbol{u}(\tau)=\mathbf{G}(\tau)\boldsymbol{y}_0+\int_0^\tau \mathbf{G}(\tau)\,\mathbf{G}^{-1}(s)\,\boldsymbol{g}(s,\boldsymbol{u}(s))\,\mathrm{d}s+$$

$$+\int_0^\tau \mathbf{G}(\tau)\,\mathbf{G}^{-1}(s)\,\boldsymbol{P}(s)\,\mathrm{d}s \tag{15}$$

We already know that $\boldsymbol{u}(\tau)$ exists on $[0, \delta_1)$. We shall now prove that it also exists on $[0, t_0)$.

The function

$$\mathbf{G}(\tau)\boldsymbol{y}_0+\int_0^\tau \mathbf{G}(\tau)\,\mathbf{G}^{-1}(s)\,\boldsymbol{P}(s)\,\mathrm{d}s$$

as a solution of (2′) is bounded and, therefore, there is $K_1>0$ such that

$$|\mathbf{G}(\tau)+\int_0^\tau \mathbf{G}(\tau)\,\mathbf{G}^{-1}(s)\,\mathrm{d}s| \leqq K_1, \qquad \tau\in(-\infty, t_0]$$

We have also that for $\mathbf{G}(\tau)$ and $\mathbf{G}(\tau)\,\mathbf{G}^{-1}(s)$,

$$|\mathbf{G}(\tau)|\leqq N, \quad |\mathbf{G}(\tau)\,\mathbf{G}^{-1}(s)|=|\mathbf{G}(t-s)|\leqq N,$$

$$0\leqq s<\tau\leqq t_0$$

Using (8) it follows that

$$|\boldsymbol{g}(\tau,\boldsymbol{u})|\leqq\beta(\tau)\,F(|\boldsymbol{u}|)$$

where

$$\beta(\tau)=\lambda(t_0-\tau)\geqq 0, \qquad 0\leqq\tau\leqq t_0$$

Then from (15),

$$|\boldsymbol{u}(\tau)|\leqq|\mathbf{G}(\tau)\,\boldsymbol{y}_0+\int_0^\tau \mathbf{G}(\tau)\,\mathbf{G}^{-1}(s)\,\boldsymbol{P}(s)\,\mathrm{d}s|+$$

$$+N\int_0^\tau \beta(s)\,F(|\boldsymbol{u}(s)|)\,\mathrm{d}s \leqq$$

$$\leqq K_1 + N\int_0^\tau \beta(s)\,F(|\boldsymbol{u}(s)|)\,\mathrm{d}s$$

But from Bihari's lemma we have

$$|\boldsymbol{u}(\tau)| \leqq \psi^{-1}\left[\int_0^\tau \beta(s)\,\mathrm{d}s\right], \qquad \tau\in[0,\delta_1)$$

and consequently, since ψ^{-1} is monotonic

$$|\boldsymbol{u}(\tau)| \leqq \psi^{-1}\left[\int_0^{t_0} \beta(s)\,\mathrm{d}s\right], \qquad \tau\in[0,\delta_1)$$

As a result of this $\boldsymbol{u}(\tau)$ can be extended on $[0, t_0]$.

II. Since the solutions of (10) are bounded, the characteristic roots of **A** have their real parts negative or zero. Let us suppose that **A** has the canonical form and can be written as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1, & 0 \\ 0, & \mathbf{A}_2 \end{pmatrix}$$

where the characteristic roots of $\mathbf{A}_1$ have negative real parts and those of $\mathbf{A}_2$ have zero real parts. Then the fundamental matrix $\mathbf{U}(t)$ of (10) is

$$\mathbf{U}(t) = \begin{pmatrix} \mathbf{U}_1(t), & 0 \\ 0, & \mathbf{U}_2(t) \end{pmatrix} \tag{16}$$

where $\mathbf{U}_1(t)$ and $\mathbf{U}_2(t)$ are square blocks, $\mathbf{U}_1(t)$ containing all terms corresponding to the characteristic roots of **A** with negative real parts and $\mathbf{U}_2(t)$ the ones corresponding to the characteristic roots of **A** with zero real parts.

Now let us decompose the identity matrix into blocks of the same kind: $\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2$ where

$$\mathbf{I}_1 = \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & \\ & & 1 & 0 \\ \hline & & 0 & 0 \end{array}\right] \qquad \text{and} \qquad \mathbf{I}_2 = \left[\begin{array}{c|cc} 0 & 0 & \\ \hline 0 & 1 & \\ & & \ddots \end{array}\right]$$

Then

$$\mathbf{U}(t)\,\mathbf{U}^{-1}(s) = \mathbf{U}(t)\,\mathbf{I}_1\mathbf{U}^{-1}(s) + \mathbf{U}(t)\,\mathbf{I}_2\mathbf{U}^{-1}(s)$$

Since all solutions of (10) are bounded, to every characteristic root of **A** with zero real part corresponds a simple block and, consequently, there are constants

$K>0$ and $\alpha>0$ such that

$$
\begin{aligned}
&|\mathbf{U}(t)\,\mathbf{I}_1\mathbf{U}^{-1}(s)|\leqq Ke^{-\alpha(t-s)}, \qquad 0\leqq s\leqq t \\
&|\mathbf{I}_2\mathbf{U}^{-1}(s)|\leqq K, \qquad 0\leqq s<\infty
\end{aligned} \tag{17}
$$

Applying once more the method of variation of constants, we can find a constant vector $\boldsymbol{c}$, such that, the solution $\boldsymbol{y}(t)$ of (1) satisfies the equation

$$
\begin{aligned}
\boldsymbol{y}(t) = \mathbf{U}(t)\boldsymbol{c} &+ \int_0^t \mathbf{U}(t)\,\mathbf{U}^{-1}(s)\,\boldsymbol{p}(s)\,\mathrm{d}s + \\
&+ \int_0^t \mathbf{U}(t)\,\mathbf{I}_1\mathbf{U}^{-1}(s)\,\boldsymbol{f}(s,\boldsymbol{y}(s))\,\mathrm{d}s - \\
&- \int_t^\infty \mathbf{U}(t)\,\mathbf{I}_2\mathbf{U}^{-1}(s)\,\boldsymbol{f}(s,\boldsymbol{y}(s))\,\mathrm{d}s
\end{aligned} \tag{18}
$$

Note that each solution of this equation is a solution of (1) as well. From the formula of variation of constants it is possible to see that

$$\boldsymbol{c} = \boldsymbol{y}_0 + \int_0^\infty \mathbf{I}_2\mathbf{U}^{-1}(s)\,\boldsymbol{f}(s,\boldsymbol{y}(s))\,\mathrm{d}s$$

Yet

$$\left|\int_0^\infty \mathbf{I}_2\mathbf{U}^{-1}(s)\,\boldsymbol{f}(s,\boldsymbol{y}(s))\,\mathrm{d}s\right| \leqq K\int_0^\infty \lambda(s)\,F(|\boldsymbol{y}(s)|)\,\mathrm{d}s$$

and $|\boldsymbol{y}(s)|$ is bounded, $F(|\boldsymbol{y}(s)|)$ is monotonic in $|\boldsymbol{y}|$, therefore $F(|\boldsymbol{y}(s)|)$ is bounded on $0\leqq t<\infty$.

And since

$$\int_0^\infty \lambda(s)\,\mathrm{d}s<\infty$$

we have that

$$\int_0^\infty \mathbf{I}_2\mathbf{U}^{-1}(s)\,\boldsymbol{f}(s,\boldsymbol{y}(s))\,\mathrm{d}s$$

converges and (18) is meaningful.

We shall at present establish a correspondence Φ between the solutions of (1) given by (18) and those of (2) given by,

$$\Phi(\boldsymbol{y}(t)) = \boldsymbol{x}(t) = \mathbf{U}(t)\boldsymbol{c} + \int_0^t \mathbf{U}(t)\mathbf{U}^{-1}(s)\,\boldsymbol{p}(s)\,\mathrm{d}s\ .$$

This correspondence establishes the asymptotic equivalence between the solutions of (1) and (2), because,

$$\boldsymbol{y}(t)-\bar{\boldsymbol{x}}(t)=\int_0^t \boldsymbol{U}(t)\, \mathbf{I}_1\mathbf{U}^{-1}(s)\, \boldsymbol{f}(s,\boldsymbol{y}(s))\, \mathrm{d}s - $$

$$-\int_t^\infty \mathbf{U}(t)\, \mathbf{I}_2\mathbf{U}^{-1}(s)\, \boldsymbol{f}(s,\boldsymbol{y}(s))\, \mathrm{d}s \tag{19}$$

and from (17) we have

$$|\boldsymbol{y}(t)-\bar{\boldsymbol{x}}(t)| \leqq K\int_0^t \mathrm{e}^{-\alpha(t-s)}\lambda(s)\, F(|\boldsymbol{y}(s)|)\, \mathrm{d}s +$$

$$+MK\int_t^\infty \lambda(s)\, F(|\boldsymbol{y}(s)|)\, \mathrm{d}s$$

$F(|\boldsymbol{y}(s)|)$ is bounded on $0\leqq t<\infty$. Then the first integral on the righ-hand side of (19) approaches zero as $t\to\infty$ (Lemma 1) and the second integral also approaches zero due to $\lambda(t)$ being integrable.

Then

$$\lim_{t\to\infty} |\boldsymbol{y}(t)-\bar{\boldsymbol{x}}(t)| = 0$$

To prove that the converse also holds, we will initially show that to every solution $\boldsymbol{x}(t)$ of (2) corresponds a solution $\boldsymbol{y}(t)$ of (1) defined on a certain interval $[t_0, \infty)$, such that (9) is satisfied. From I) above we already know that such a solution $\boldsymbol{y}(t)$ can be extended to $[0, \infty)$.

Let us consider the Banach space B of all continuous and bounded functions on $[t_0, \infty)$, assuming values in R^n, with norm

$$\|\boldsymbol{f}\| = \sup_{t\geqq t_0} |\boldsymbol{f}(t)|$$

where t_0 will be fixed below.

Let $\boldsymbol{x}(t)$ be a given solution of (2). Since it is bounded, there is a constant ϱ such that $\|\boldsymbol{x}\|\leqq\varrho$.

Consider now the closed ball

$$B_\varrho = \left\{\omega\in B \,\middle|\, \|\omega\|\leqq\varrho\right\}$$

For $\varphi(t)\in B_{2\varrho}$ let us define the integral operator

$$G\varphi(t) = \boldsymbol{x}(t) + \int_{t_0}^t \mathbf{V}(t)\, \mathbf{I}_1\mathbf{V}^{-1}(s)\, \boldsymbol{f}(s,\varphi(s))\, \mathrm{d}s -$$

$$-\int_t^\infty \mathbf{V}(t)\, \mathbf{I}_2\mathbf{V}^{-1}(s)\, \boldsymbol{f}(s,\varphi(s))\, \mathrm{d}s\,; \quad t\geqq t_0$$

where $\mathbf{V}(t)$, with $\mathbf{V}(t_0)=\mathbf{E}$, is the fundamental matrix of (2).

Since $\mathbf{V}(t)=\mathbf{U}(t-t_0)$ and from (8), (11) and (17), we have

$$|G\varphi(t)|\leqq\varrho+K\int_{t_0}^{t} e^{-\alpha(t-s)}\lambda(s)\,F(|\varphi(s)|)\,ds+$$

$$+MK\int_{t}^{\infty}\lambda(s)\,F(|\varphi(s)|)\,ds\,,\qquad t\geqq t_0$$

or

$$|G\varphi(t)|\leqq\varrho+KF(2\varrho)\int_{t_0}^{t} e^{-\alpha(t-s)}\lambda(s)\,ds+$$

$$+MK\,F(2\varrho)\int_{t}^{\infty}\lambda(s)\,ds\leqq$$

$$\leqq\varrho+K\,F(2\varrho)\left[\int_{t_0}^{t}\lambda(s)\,ds+M\int_{t}^{\infty}\lambda(s)\,ds\right]=$$

$$=\varrho+K\,F(2\varrho)\max\{1,M\}\int_{t_0}^{\infty}\lambda(s)\,ds$$

But

$$\int_{t_0}^{\infty}\lambda(s)\,ds<\infty\,,\quad \text{then we can choose } t_0 \text{ so that}$$

$$\int_{t_0}^{\infty}\lambda(s)\,ds\leqq\frac{\varrho}{KF(2\varrho)\max\{1,M\}}$$

Therefore,

$$|G(\varphi)|\leqq 2\varrho\,,\quad \text{that is,}\quad GB_{2\varrho}\subset B_{2\varrho}$$

where $B_{2\varrho}$ is a closed and convex subset of the Banach space B.

Next we shall prove that G is continuous: let $\varphi_n(t)\in B_{2\varrho}$, $\varphi(t)\in B_{2\varrho}$, $n=1,2,\dots$ so that $\|\varphi_n(t)-\varphi(t)\|\to 0$, uniformly on $[t_0,\infty)$.

Then

$$G\varphi_n-G\varphi=\int_{t_0}^{t}\mathbf{V}(t)\,\mathbf{I}_1\mathbf{V}^{-1}(s)\,[\boldsymbol{f}(s,\varphi_n(s))-\boldsymbol{f}(s,\varphi(s))]\,ds-$$

$$-\int_{t}^{\infty}\mathbf{V}(t)\,\mathbf{I}_2\mathbf{V}^{-1}(s)\,[\boldsymbol{f}(s,\varphi_n(s))-\boldsymbol{f}(s,\varphi(s))]\,ds$$

or, taking the norm

$$|G\varphi_n-G\varphi|\leqq K\int_{t_0}^{t} e^{-\alpha(t-s)}\,|\boldsymbol{f}(s,\varphi_n(s))-\boldsymbol{f}(s,\varphi(s))|\,ds+$$

$$+\int_t^\infty MK\,|\boldsymbol{f}(s,\varphi_n(s))-\boldsymbol{f}(s,\varphi(s))|\,\mathrm{d}s \leqq$$

$$\leqq K\int_{t_0}^t |\boldsymbol{f}(s,\varphi_n(s))-\boldsymbol{f}(s,\varphi(s))|\,\mathrm{d}s+$$

$$+MK\int_t^\infty |\boldsymbol{f}(s,\varphi_n(s))-\boldsymbol{f}(s,\varphi(s))|\,\mathrm{d}s \leqq$$

$$\leqq K\max\{1,M\}\int_{t_0}^\infty |\boldsymbol{f}(s,\varphi_n(s))-\boldsymbol{f}(s,\varphi(s))|\,\mathrm{d}s$$

or

$$\|G\varphi_n-G\varphi\|\leqq K\max\{1,M\}\int_{t_0}^\infty |\boldsymbol{f}(s,\varphi_n(s))-\boldsymbol{f}(s,\varphi(s))|\,\mathrm{d}s$$

However

$$|\boldsymbol{f}(s,\varphi_n(s))-\boldsymbol{f}(s,\varphi(s))|\leqq 2\lambda(s)\,F(2\varrho)$$

where

$$\lim_{n\to\infty}|\boldsymbol{f}(s,\varphi_n(s))-\boldsymbol{f}(s,\varphi(s))|=0$$

and

$$\int_0^\infty \lambda(s)\,\mathrm{d}s<\infty\,,$$

as assumed, we have, using Lebesgue's theorem, that

$$\lim_{n\to\infty}\|G\varphi_n-G\varphi\|=0$$

This proves that G is continuous on $B_{2\varrho}$.

In order to apply Schauder—Tychonoff's fixed point theorem it is sufficient to prove that the $GB_{2\varrho}$ functions are uniformly bounded and equicontinuous on each closed interval contained in $[t_0, \infty)$.

Since $GB_{2\varrho}\subset B_{2\varrho}$, it follows that the $GB_{2\varrho}$ functions are uniformly bounded.

Let us now consider $t, t'\in[t_0,\infty)$, $t<t'$.

Then

$$G\varphi(t)-G\varphi(t')=\boldsymbol{x}(t)-\boldsymbol{x}(t')+\int_{t_0}^t \mathbf{V}(t)\,\mathbf{I}_1\mathbf{V}^{-1}(s)\,\boldsymbol{f}(s,\varphi(s))\,\mathrm{d}s-$$

$$-\int_t^\infty \mathbf{V}(t)\,\mathbf{I}_2\mathbf{V}^{-1}(s)\,\boldsymbol{f}(s,\varphi(s))\,\mathrm{d}s-\int_{t_0}^{t'} \mathbf{V}(t')\,\mathbf{I}_1\mathbf{V}^{-1}(s)\,\boldsymbol{f}(s,\varphi(s))\,\mathrm{d}s+$$

$$+\int_{t'}^{\infty} \mathbf{V}(t')\, \mathbf{I}_2\mathbf{V}^{-1}(s)\, \boldsymbol{f}(s, \varphi(s))\, \mathrm{d}s =$$

$$= \boldsymbol{x}(t) - \boldsymbol{x}(t') + \int_{t_0}^{t} [\mathbf{V}(t) - \mathbf{V}(t')]\, \mathbf{I}_1 V^{-1}(s)\, \boldsymbol{f}(s, \varphi(s))\, \mathrm{d}s -$$

$$- \int_{t}^{t'} \mathbf{V}(t')\, \mathbf{I}_1\mathbf{V}^{-1}(s)\, \boldsymbol{f}(s, \varphi(s))\, \mathrm{d}s - \int_{t}^{t'} \mathbf{V}(t)\, \mathbf{I}_2\mathbf{V}^{-1}(s)\, \boldsymbol{f}(s, \varphi(s))\, \mathrm{d}s -$$

$$- \int_{t'}^{\infty} [\mathbf{V}(t) - \mathbf{V}(t')]\, \mathbf{I}_2\mathbf{V}^{-1}(s)\, \boldsymbol{f}(s, \varphi(s))\, \mathrm{d}s$$

Therefore

$$|G\varphi(t) - G\varphi(t')| \leqq |\boldsymbol{x}(t) - \boldsymbol{x}(t')| +$$

$$+ \int_{t_0}^{t} |\mathbf{V}(t)\, \mathbf{I}_1\mathbf{V}^{-1}(s) - \mathbf{V}(t')\, \mathbf{I}_1\mathbf{V}^{-1}(s)|\, \lambda(s)\, F(|\varphi|)\, \mathrm{d}s +$$

$$+ K\int_{t}^{t'} \lambda(s)\, F(|\varphi(s)|)\, \mathrm{d}s + MK\int_{t}^{t'} \lambda(s)\, F(\varphi(s))\, \mathrm{d}s +$$

$$+ K\int_{t_0}^{\infty} |\mathbf{V}(t) - \mathbf{V}(t')|\, \lambda(s)\, F(|\varphi(s)|)\, \mathrm{d}s \leqq$$

$$\leqq |\boldsymbol{x}(t) - \boldsymbol{x}(t')| + F(2\varrho)\int_{t_0}^{\infty} |\mathbf{V}(t)\, \mathbf{I}_1\mathbf{V}^{-1}(s) - \mathbf{V}(t')\, \mathbf{I}_1\mathbf{V}^{-1}(s)|\, \lambda(s)\, \mathrm{d}s +$$

$$+ (1+M)KF(2\varrho)\int_{t}^{t'} \lambda(s)\, \mathrm{d}s + KF(2\varrho)\int_{t_0}^{\infty} \lambda(s)\, |\mathbf{V}(t) - \mathbf{V}(t')|\, \mathrm{d}s$$

Since $\boldsymbol{x}(t)$, $\mathbf{V}(t)\, \mathbf{I}_1\mathbf{V}^{-1}(s)$, $\mathbf{V}(t)$ and $\int_{t_0}^{t} \lambda(s)\, \mathrm{d}s$ are continuous, it follows that $G\varphi(t)$ are equicontinuous on every closed interval contained in $[t_0, \infty)$.

This implies that G has at least one fixed point in $B_{2\varrho}$, that is, there exists a $y \in B_{2\varrho}$ such that $G\boldsymbol{y} = \boldsymbol{y}$.

From the definition of the operator G, we have:

$$|\boldsymbol{y}(t) - \boldsymbol{x}(t)| \leqq \left| \int_{t_0}^{t} \mathbf{V}(t)\, \mathbf{I}_1\mathbf{V}^{-1}(s)\, \boldsymbol{f}(s, \boldsymbol{y}(s))\, \mathrm{d}s \right| +$$

$$+ \left| \int_{t}^{\infty} \mathbf{V}(t)\, \mathbf{I}_2\mathbf{V}^{-1}(s) \boldsymbol{f}(s, \boldsymbol{y}(s)\, \mathrm{d}s \right| \leqq$$

$$\leqq KF(2\varrho)\int_{t_0}^{t} \mathrm{e}^{-\alpha(t-s)}\lambda(s)\, \mathrm{d}s + MK\, F(2\varrho)\int_{t}^{\infty} \lambda(s)\, \mathrm{d}s$$

From Lemma 1 and due to the fact that $\lambda(t)$ is integrale, we have that

$$\lim_{t\to\infty}|\boldsymbol{y}(t)-\boldsymbol{x}(t)|=0$$

Note 1. Condition (8) together with

$$\psi(\infty)=\int_H^\infty \frac{dv}{F(v)}=\infty\,, \qquad H>0$$

assures that every solution $\boldsymbol{y}(t)$ of (1) exists and is bounded on $[0, \infty)$; however, if this last condition is not satisfied it might happen that not every solution exists on $[0, \infty)$, and yet, if it exists, it might not be bounded.

To show this fact let us consider the following example: the solutions of

$$y'=\frac{1}{t^2}y^2\,, \qquad t\geqq 1$$

are $y=0$ and $y=\dfrac{t}{1+ct}$

It is evident that not every solution exists on $[1, \infty)$.

In this example,

$$F(\varphi)=\varphi^2 \quad \text{and} \quad \int_H^\infty \frac{d\varphi}{\varphi^2}<\infty$$

Note 2. The function

$$f(t, y)=\lambda(t)\, y\sqrt{\ln y}\,, \qquad 1<y<\infty$$

satisfies condition (8) of our theorem, but does not satisfy the condition

$$|f(t, y)|\leqq\lambda(t)\,|y|$$

from Brauer's theorem [1].

In fact

$$F(u)=u\sqrt{\ln u}\,, \qquad 1<u<\infty$$

is a continuous, positive and non-decreasing function. Also,

$$\int_H^u \frac{dv}{v\sqrt{\ln v}}=2\sqrt{\ln v}-2\sqrt{\ln H}$$

where $H>1$, $1<u<\bar{u}$ and

$$\int_H^\infty \frac{dv}{v\sqrt{\ln v}}=\infty$$

On the other hand

$$f(t, y) = \lambda(t)\, y\sqrt{\ln y}$$

does not satisfy the condition of Brauer's theorem [1], since the inequality

$$|f(t, y)| = \lambda(t)\, |y|\, \sqrt{\ln y} \leqq \lambda(t)\, |y|$$

where

$$\int_0^\infty \lambda(t)\, dt < \infty, \text{ and } 1 < y < \infty$$

does not hold true.

Note 3. Let us now consider in two planes X and Y, the set of all initial values of the asymptotically equivalent solutions of (1) and (2) respectively, for $t = t_0$. From that it is possible to define the asymptotic equivalence among these initial values in the following way: "the initial values, $\mathbf{x}_0$ of X and $\mathbf{y}_0$ of Y are asymptotically equivalent if there is a solution $\mathbf{x}(t)$ of (2) with initial value $\mathbf{x}_0 = \mathbf{x}(t_0)$, and a solution $\mathbf{y}(t)$ of (1) with initial value $\mathbf{y}_0 = \mathbf{y}(t_0)$, such that $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are asymptotically equivalent".

From this definition the result of our theorem may be stated as follows: "There is asymptotic equivalence between the planes X and Y, $t_0 = 0$, that is, to every initial value $\mathbf{x}_0$ of X corresponds an initial value $\mathbf{y}_0$ of Y such that $\mathbf{x}_0$ and $\mathbf{y}_0$ are asymptotically equivalent".

In general, when we have asymptotic equivalence in the restricted sense, that is, when relation (9) is satisfied only among finite subsets of solutions of (1) and (2), it should be interesting to study the properties of the subsets of the asymptotically equivalent initial values with respect to their power, convexity, if they are open or closed, bounded or not, and so on.

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SÚHRN

POZNÁMKA K ASYMPTOTICKEJ EKVIVALENCII DVOCH DIFERENCIÁLNYCH SYSTÉMOV

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Autor sa v práci zaoberá asymptotickou ekvivalenciou dvoch diferenciálnych systémov

1. $\mathbf{y}' = \mathbf{A}y + \mathbf{f}(t, \mathbf{y}) + \mathbf{p}(t)$
2. $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{p}(t)$

za predpokladu konštantnosti matice $\mathbf{A}$, ohraničenosti všetkých riešení rovnice (2) a toho, že $|\mathbf{f}(t, \mathbf{y})| \leqq \lambda(t)\, F(|\mathbf{y}|)$.

РЕЗЮМЕ

ЗАМЕТКА О АСИМПТОТИЧЕСКОЙ ЭКВИВАЛЕНТНОСТИ ДВУХ СИСТЕМ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

А. С. Лима, Бразилия

В работе исследуется асимптотическая эквивалентность двух систем дифференциальных уравнений

1. $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{f}(t, \mathbf{y}) + \mathbf{p}(t)$
2. $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{p}(t)$

предполагая постоенность матрицы $\mathbf{A}$, ограниченность всех решений системы (2) и то, что

$|\mathbf{f}(t, \mathbf{y})| \leqq \lambda(t)\, F(|\mathbf{y}|)$.

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ON THE ASYMPTOTIC BEHAVIOR OF A PERTURBED LINEAR SYSTEM

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1. Consider the non-linear system

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \boldsymbol{f}(t, \boldsymbol{x}) \tag{1}$$

where $\boldsymbol{x}$ is an n-dimensional vector, $\mathbf{A}$ an $n \times n$ matrix, $\boldsymbol{f}(t, \boldsymbol{x})$ an n-dimensional vector function continuous in $(t, \boldsymbol{x})$ for $t \geqq 0$, and $|\boldsymbol{x}| < \infty$, $|\ |$ being any convenient vector norm.

Many papers are devoted to the asymptotic equivalence between the solutions of (1) and those of the linear system

$$\dot{\boldsymbol{y}} = \mathbf{A}\boldsymbol{y} \tag{2}$$

as for instance Weyl [13], Levinson [6] [7] Wintner [14] [15], Jakubovic [5], Brauer [1] [2], Strauss and Yorke [11], Brauer and Wong [3] [4], Onuchic [8]—[10], Švec [12]. We have based our ideas on Švec [12], where he proved the asymptotic equivalence between (1) and (2) under certain conditions concerning the matrix $\mathbf{A}$ and the non linear term $\boldsymbol{f}(t, \boldsymbol{x})$. He started with an asymptotic equivalence result between the systems

$$\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x} + \boldsymbol{f}(t) \tag{1'}$$

$$\boldsymbol{y} = \mathbf{A}(t)\boldsymbol{y} \tag{2'}$$

$\mathbf{A}(t)$ being $n \times n$ matrix function continuous on $[0, \infty)$, $\boldsymbol{f}(t)$ a continous function on $[0, \infty)$.

In most of the papers, the conditions on $\boldsymbol{f}(t)$ used by the authors are of type

$$\int_0^\infty |\boldsymbol{f}(t)| \, dt < \infty \quad \text{or} \tag{3}$$

$$\lim_{t \to \infty} \boldsymbol{f}(t) = 0 \quad \text{or} \tag{4}$$

$$\lim_{t\to\infty}\int_t^{t+1} \boldsymbol{f}(t)=0 \tag{5}$$

We observe that condition (3) as well as condition (4) implies (5) but condition (5) does not imply (3) nor (4).

In this paper we will introduce a new condition still more general than (5) and will prove the asymptotic behavior of all bounded solutions of (1) when the eigenvalues of **A** have negative real parts, requiring that $\boldsymbol{f}(t, \boldsymbol{x})$ obeys a condition of the form

$$|\boldsymbol{f}(t, \boldsymbol{x})| \leqq F(t, |\boldsymbol{x}|) \tag{6}$$

for each $t\in[0, \infty)$ and $|x|<\infty$.

The case when (6) and $\int_0^\infty F(t, c)<\infty$ are fullfilled is included in Theorem 1 of Brauer and Wong [4]. Our theorem generalizes not only this case but also the case when $\lim_{t\to\infty} F(t, c)=0$ and $\lim_{t\to\infty}\int_t^{t+1} F(t, c)=0$.

2. Without loss of generality we can suppose that **A** has the Jordan form.

Let $\mu_1<\mu_2<\ldots<\mu_s=-\alpha$, $\alpha>0$, be the distinct real parts of eigenvalues $\lambda_i(\mathbf{A})$ of **A** and let m_i be the maximum order of those blocks in **A** which correspond to eigenvalues with real parts μ_i. So

$\operatorname{Re}\lambda_j(\mathbf{A})\leqq -\alpha = \max_j \operatorname{Re}\lambda_j(\mathbf{A})$; denote $m^* = m_s$.

If $\mathbf{Y}(t)$, $\mathbf{Y}(0)=\mathbf{E}$ is the fundamental matrix of (2) then

$$|\mathbf{Y}(t)|\leqq a\, e^{-\alpha t} X_m(t) \leqq K\, e^{-\beta t}, \qquad t\geqq 0 \tag{7}$$

where $X_k(t)=\begin{cases} t^{k-1} & t\geqq 1 \\ 1 & 0\leqq t\leqq 1\end{cases}$ and $0<\beta<\alpha$.

Let us prove (7). Let $-\alpha<-\beta<0$.

Since $\lim_{t\to\infty}\dfrac{e^{-\alpha t}X_{m^*}(t)}{e^{-\beta t}}=0$ and the function $\dfrac{e^{-\alpha t}X_{m^*}(t)}{e^{-\beta t}}$

is continuous on $[0, \infty)$ it is also bounded $[0, \infty)$. Therefore, there exists $K>0$ such that

$$\frac{e^{-\alpha t}X_{m^*}(t)}{e^{-\beta t}}\leqq K$$

fol all $t\in[0, \infty)$ and (7) follows.

The following Lemma 1. has been used in many of the papers we refered and will be helpful in the proof of Lemma 2.

Lemma 1. Let σ be a positive constant and let $g(x)\geqq 0$ be continuous on

$0 \leqq t < \infty$ and such that either

a) $\int_0^\infty g(t)\,dt < \infty$ or

b) $\lim_{t\to\infty} g(t) = 0$ or

c) $\lim_{t\to\infty} \int_t^{t+1} g(t)\,dt = 0$

Then

$$\lim_{t\to\infty} e^{-\sigma t} \int_0^t e^{\sigma s}\, g(s)\,ds = 0$$

Proof. If a) or b) is satisfied see Brauer [2]; if c) is satisfied see Strauss and Yorke [11].

Our Lemma 2 generalizes Lemma 1 and will be used in the proof of our Theorem 1.

Lemma 2. Let $\gamma(t) \geqq 0$ be a continuous function on $[0, \infty)$, $\{G_n(t)\}$ a sequence of continuous functions on $[0, \infty)$ such that

$$G_i(t) = \int_t^{t+1} G_{i-1}(s)\,ds\,, \qquad i = 1, 2, \dots \tag{8}$$

where $G_0(t) = \gamma(t)$. Let k be the first natural number such that

$$G_k(t) \to 0 \quad \text{as} \quad t \to \infty \tag{9}$$

and σ a positive constant.

Then

$$\lim_{t\to\infty} e^{-\sigma t} \int_0^t e^{\sigma s} \gamma(s)\,ds = 0 \tag{10}$$

Proof. If $k = 1$, by Lemma 1

$$\lim_{t\to\infty} e^{-\sigma t} \int_0^t e^{\sigma s} G_1(s)\,ds = 0$$

Let $k > 1$. Using Lemma 3.4 and Lemma 3.5 of Strauss and Yorke [11] we get

$$\int_{t_0}^t G_{i-1}(s)\,ds \leqq \int_{t_0-1}^t G_i(s)\,ds \tag{11}$$

for all $t \geqq t_0 \geqq 1$, $i = 1, 2, \dots$.

and

$$\int_{t_0}^{t} e^{\sigma s} G_{i-1}(s)\, ds \leqq \int_{t_0-1}^{t} e^{\sigma(s+1)} G_i(s)\, ds \tag{12}$$

for all $\sigma > 0$, $t \geqq t_0 \geqq 1$, $i = 1, 2, \ldots$.

Then, we will have using (12)

$$\begin{aligned} e^{-\sigma t}\int_0^t e^{\sigma s} G_{i-1}(s)\, ds &= e^{-\sigma t}\int_0^1 e^{\sigma s} G_{i-1}(s)\, ds + e^{-\sigma t}\int_1^t e^{\sigma s} G_{i-1}(s)\, ds \leqq \\ &\leqq e^{-\sigma t}\int_0^1 e^{\sigma s} G_{i-1}(s)\, ds + e^{\sigma}\, e^{-\sigma t}\int_1^t e^{\sigma s} G_i(s)\, ds \end{aligned} \tag{13}$$

for $i = 1, 2, \ldots, t \geqq 1$.

The first term in the last expression obviously tends to zero for $i = 1, 2, \ldots$ while the second tends to zero for $i = k$ (by Lemma 1). So, for $i = k$ in (13) we get

$$\lim_{t\to\infty} e^{-\sigma t}\int_0^t e^{\sigma s} G_{k-1}(s)\, ds = 0$$

Then from (13) successively we get

$$\lim_{t\to\infty} e^{-\sigma t}\int_0^t e^{\sigma t} G_{i-1}(s)\, ds = 0 \qquad \text{for } i = k-1,\ k-2, \ldots, 1$$

3. **Theorem 1.** Let

$$G_i(t, c) = \int_t^{t+1} G_{i-1}(s, c)\, ds\ , \qquad i = 1, 2, \ldots$$

be continuous functions on $[0, \infty)$ for all $c \geqq 0$ where $G_0(t, c) = F(t, c)$ and $F(t, u)$ is continuous in $t \geqq 0$, $u \geqq 0$ and nondecreasing in u, for $u \geqq 0$. Suppose that there exists a natural number k such that

$$G_k(t, c) \to 0 \text{ as } t \to \infty \text{ for all } c \geqq 0 \tag{11}$$

Suppose that in (1), $\boldsymbol{f}(t, \boldsymbol{x})$ satisfies

$$|\boldsymbol{f}(t, \boldsymbol{x})| \leqq F(t, |\boldsymbol{x}|) \tag{6}$$

for each $t \in [0, \infty)$ and $|\boldsymbol{x}| < \infty$ and that the eigenvalues of $\mathbf{A}$ have negative real parts. Then

a) There exists at least one bounded solution of (1) defined on

$$[t_0, \infty),\ t_0 \geqq 0 .$$

b) Each bounded solution of (1) has the property: $\lim_{t\to\infty} \boldsymbol{x}(t) = 0$.

Proof. a) Let $\mathbf{Y}(t)$, $\mathbf{Y}(0) = \mathbf{E}$ be the fundamental matrix of (2). Then using the variation of constants we get

$$x(t)=Y(t-t_0)x_0+\int_{t_0}^{t} Y(t-s)\, f(s, x(s))\, ds\ , \ t>t_0,\ \ x(t_0)=x_0$$

We will prove that this integral equation and consequently (1) has at least one bounded solution.

Let

$$S_0=\{\Phi\in C_0[t_0,\infty)\,|\lim_{t\to\infty}\Phi(t)=0; \qquad \|\Phi\|=\sup_{t\in[t_0,\infty)}|\Phi(t)|\}$$

Then it is obvious that S_0 is a Banach space.

Define on S_0 the operator

$$T\Phi=Y(t-t_0)x_0+\int_{t_0}^{t} Y(t-s)\, f(s, \Phi(s))\, ds$$

and consider the subset $S_{0\varrho}$ where

$$S_{0\varrho}=\{\Phi\in S_0|\,\|\Phi\|\leqq\varrho\ ,\quad 2\varrho>K|x_0|\}$$

We claim:

1. $TS_{0\varrho}\subset S_{0\varrho}$

For using (7), (6) and the monotonicity of F we have for $\Phi\in S_{0\varrho}$

$$|T\Phi|\leqq|Y(t-t_0)|\ |x_0|+\int_{t_0}^{t}|Y(t-s)|\ |f(s, \Phi(s))|\ ds\leqq$$

$$\leqq K\ |x_0|\ e^{-\beta(t-t_0)}+K\int_{t_0}^{t} e^{-\beta(t-s)}F(s, \varrho)\, ds$$

Now using the fact that $\beta>0$ and Lemma 2, we can choose t_0 such that for $t\geqq t_0$ this expression becomes $\leqq\varrho$ and therefore

$$\|T\Phi\|\leqq\varrho$$

2. T is continuous on S_0.

Let be Φ_n, $\Phi\in S_0$, $\|\Phi_n-\Phi\|\to0$ as $n\to\infty$. Then $\Phi_n(t)\to\Phi(t)$ uniformly on the interval $[t_0, \infty)$. For $T\Phi_n-T\Phi$ we have

$$|T\Phi_n-T\Phi|\leqq\int_{t_0}^{t}|Y(t-s)|\ |f(s, \Phi_n(s))-f(s, \Phi(s))|\ ds\leqq$$

$$\leqq K\int_{t_0}^{t} e^{-\beta(t-s)}\ |f(s, \Phi_n(s))-f(s, \Phi(s))|\ ds\ ,\qquad t\geqq t_0$$

Using the continuity of $f(t, x)$ and the uniform convergence of $\Phi_n(t)\to\Phi(t)$ on $[t_0, \infty)$ we have that given $\varepsilon>0$, there exists an integer $n_0(\varepsilon)$ such that for $n\geqq n_0(\varepsilon)$.

$$|\boldsymbol{f}(s, \Phi_n(s)) - \boldsymbol{f}(s, \Phi(s))| < \varepsilon \quad \forall s \in [t_0, \infty)$$

Then

$$K\int_{t_0}^{t} e^{-\beta(t-s)}|\boldsymbol{f}(s, \Phi_n(s)) - \boldsymbol{f}(s, \Phi(s))| \, ds \leqq K\varepsilon \int_{t_0}^{t} e^{-\beta(t-s)} \, ds \leqq \frac{K\varepsilon}{\beta},$$

$$n \geqq n_0(\varepsilon)$$

Therefore

$$\|T\Phi_n - T\Phi\| \leqq \frac{K\varepsilon}{\beta}, \qquad n \geqq n_0(\varepsilon)$$

and this proves the continuity of T on S_0.

3. $TS_{0\varrho}$ are equicontinuous on every interval $[t_0, t_1]$:

Let $t < t'$, $t, t' \in [t_0, t_1]$, then

$$T\Phi(t') - T\Phi(t) = (\mathbf{Y}(t' - t_0) - \mathbf{Y}(t - t_0))x_0 +$$

$$+ \int_{t_0}^{t} (\mathbf{Y}(t' - s) - \mathbf{Y}(t - s))f(s, \Phi(s)) \, ds +$$

$$+ \int_{t}^{t'} \mathbf{Y}(t' - s) \, \boldsymbol{f}(s, \Phi(s)) \, ds$$

Then

$$|T\Phi(t') - T\Phi(t)| \leqq |\mathbf{Y}(t' - t_0) - \mathbf{Y}(t - t_0)| \, |\boldsymbol{x}_0| +$$

$$+ \int_{t_0}^{t} |\mathbf{Y}(t' - s) - \mathbf{Y}(t - s)| \, |F(s, \varrho)| \, ds +$$

$$+ \int_{t}^{t'} |Y(t' - s)| \, |F(s, \varrho)| \, ds$$

This last expression does not depend on Φ and the continuity of $\mathbf{Y}(t)$ and the continuity of the integral assure the equicontinuity of the functions of $TS_{0\varrho}$.

Now, from the fact that $TS_{0\varrho} \subset S_{0\varrho}$ it follows that the functions of $TS_{0\varrho}$ are uniformly bounded.

The conditions for using Schauder's fixed point theorem are so satisfied. Consequently the operator T has a fixed $\bar{\Phi}$ on $S_{0\varrho}$, i. e. $T\Phi T\bar{\Phi} = \bar{\Phi}$ which means that $\bar{\Phi}(t)$ is a bounded and defined solution of (1) on $[t_0, \infty)$, and $\lim_{t\to\infty} \bar{\Phi}(t) = 0$.

b) Suppose now $\mathbf{x}(t)$, $\mathbf{x}(t_0) = \mathbf{x}_0$, is a bounded solution of (1), i.e., there exists M such that $|\mathbf{x}(t)| \leqq M$, $t \geqq t_0$. Then this solution satisfies the integral equation

$$\mathbf{x}(t)=\mathbf{Y}(t-t_0)\mathbf{x}_0+\int_{t_0}^{t}\mathbf{Y}(t-s)\,\mathbf{f}(s,\mathbf{x}(s))\,\mathrm{d}s\,,\qquad t\geqq t_0$$

and conversely, each solution $\mathbf{x}(t)$ of this integral equation is a solution of (1) satisfying the condition $\mathbf{x}(t_0)=\mathbf{x}_0$.

Then

$$|\mathbf{x}(t)|\leqq|\mathbf{Y}(t-t_0)|\;|\mathbf{x}_0|+\int_{t_0}^{t}|\mathbf{Y}(t-s)|\;|\mathbf{f}(s,\mathbf{x}(s))|\;\mathrm{d}s\leqq$$

$$\leqq K\,\mathrm{e}^{-\beta(t-t_0)}|\mathbf{x}_0|+K\int_{t_0}^{t}\mathrm{e}^{-\beta(t-s)}F(s,M)\,\mathrm{d}s$$

Since $\beta>0$, the first term in this last expression tends to zero and by Lemma 2, also the second. Therefore, $\mathbf{x}(t)\to 0$ as $t\to\infty$ and this completes the proof.

As a consequence of theorem 1 we can state the following

Corollary. Let $\{G_n(t)\}$ be as in Lemma 2 where $G_0(t)=f(t)$; suppose there exists a natural number k such that

$$G_k(t)\to 0 \text{ as } t\to\infty$$

and $\mathbf{A}$ is as in Theorem 1. Then all solutions of the equation

$$\dot{\mathbf{x}}=\mathbf{A}x+\mathbf{f}(t)$$

converge to zero as $t\to\infty$.

The theorem says that for each solution $\mathbf{x}(t)$ of (1) defined on $[t_0,\infty)$ either it is unbounded or if it is bounded, has limit zero as $t\to\infty$.

So, there exists an asymptotic equivalence between the set of all bounded solutions of (1) and the set of all solutions of (2) since $\operatorname{Re}\lambda_j(\mathbf{A})<0$.

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SÚHRN

O ASYMPTOTICKOM CHOVANÍ SA PERTURBOVANÉHO SYSTÉMU

M. G. Cerqueira, Brazília

V práci sú uvedené podmienky, ktoré zaručujú existenciu ohraničených riešení diferenciálneho systému $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(t, \mathbf{x})$, $t \geqq 0$ a ich konvergenciu k nule.

РЕЗЮМЕ

ОБ АСИМПТОТИЧЕСКОМ ПОВЕДЕНИИ ВОЗМУЩЕННОЙ ЛИНЕЙНОЙ СИСТЕМЫ

М.Г. Серкейра, Бразилия

В работе приведены условия, которые обеспечивают существование ограниченных решений системы дифференциальных уравнений

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(t, \mathbf{x}), \quad t = 0$$

и стремление их к нулю при $t \to +\infty$.

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EXTENSION OF MEASURES

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Let X be a set and R a ring of subsets of X. If μ is a positive measure on R, then μ can be extended to the outer measure μ^* on the hereditary σ-ring H generated by R. Also, the restriction of μ^* to the σ-ring $J\mu(R)$ of the μ^*-measurable sets of H is a complete measure [2, p. 47].

Now we define a class $P(\mathscr{A})$ which plays a central role in the development.

Definition. If X is a space and $\mathscr{A}$ a nonvoid class of subsets of X, then $P(\mathscr{A})$ is the class of the sets $E \subset X$ such that $E - A \in \mathscr{A}$ for every $A \in \mathscr{A}$ $(A \neq \emptyset)$.

The following properties of $P(\mathscr{A})$ are evident.

1. If $A - B \in \mathscr{A}$ for every A, $B \in \mathscr{A}$, then $\mathscr{A} \subset P(\mathscr{A})$.
2. If $E \in P(\mathscr{A})$ is disjoint from some set $A \in \mathscr{A}$, then $E \in \mathscr{A}$.

Let μ be an extended nonnegative real valued set function defined on the class $\mathscr{A}$ of subsets of X such that $\emptyset \in \mathscr{A}$ and $\mu(\emptyset) = 0$. Then we introduce the following

Definition. We define the d-variation $\bar{\mu}$ of μ on $P(\mathscr{A})$ by the equality

$$\bar{\mu}(E) = \sup \mu(E - A)$$

supremum being taken for $A \subset E$, $A \in \mathscr{A}$, and $A \neq \emptyset$ if $E \bar{\in} \mathscr{A}$.

Also if there is no $A \in \mathscr{A}$, $A \neq \Phi$, $A \subset E$, $E \bar{\in} \mathscr{A}$, then $\bar{\mu}(E) = 0$.

The following properties of $\bar{\mu}$ follow immediately from the above definition:

1. $\bar{\mu}(A) = \mu(A)$ for every $A \in \mathscr{A}$, if μ is monotone;
2. $0 \leqq \bar{\mu}(E) \leqq \infty$;
3. $\bar{\mu}(\emptyset) = 0$;
4. $\bar{\mu}(E) \leqq \bar{\mu}(F)$ if $E \subset F$ and μ is monotone.

In what follows when we consider a ring $\mathscr{A}$, we mean $\mathscr{A} \neq \{\Phi\}$.

Here we show that if $\mathscr{A}$ is a ring and μ a measure on $\mathscr{A}$, then μ^* can be extended from the σ-ring $J_\mu(\mathscr{A})$ of μ^*-measurable sets of the hereditary σ-ring generated by $\mathscr{A}$ to a complete measure $\bar{\mu}^*$ on the σ-ring $P(J_\mu(A))$. Thus a measure μ on a ring $\mathscr{A}$ is extended to a complete measure $\bar{\mu}^*$ on the σ-ring $P(J_\mu(A))$ which contains $J_\mu(A)$. Also we extend μ on $\mathscr{A}$ to a measure $\bar{\mu}$ on the σ-ring $P(\mathscr{A})$ containing A and then obtain the complete measure $\bar{\mu}^*$ on the σ-ring $J_{\bar{\mu}}(P(\mathscr{A}))$ of the $\bar{\mu}^*$-measurable sets of the hereditary σ-ring generated by $P(\mathscr{A})$.

Theorem 1. If $\mathscr{A}$ is a ring (σ-ring), then $P(\mathscr{A})$ is a ring (σ-ring) containing $\mathscr{A}$.

Proof. Let $\mathscr{A}$ be a ring and $E, F \in P(\mathscr{A})$. Then $E-A \in \mathscr{A}$, $F-A \in \mathscr{A}$ for every $E, F \in \mathscr{A}$ and $\mathscr{A} \neq \emptyset$. Now $(E \cup F)-A = (E-A) \cup (F-A) \in \mathscr{A}$. So $E \cup F \in P(\mathscr{A})$.

Also $(E-F)-A = E-(F \cup A) = E-[(F-A) \cup A] \in \mathscr{A}$.

So $E-F \in P(\mathscr{A})$. Hence $P(\mathscr{A})$ is a ring.

Next, let $\mathscr{A}$ be a σ-ring and $E_1, E_2, \ldots \in P(\mathscr{A})$. Then $E_i - A \in \mathscr{A}$ for every $A \in \mathscr{A}$ and $A \neq \emptyset$, $i = 1, 2, \ldots$. Now $(E_1 \cup E_2 \cup \ldots) - A = (E_1 - A) \cup (E_2 - A) \cup \cup \ldots \in A$.

So $\bigcup_{i=1}^{\infty} E_i \in P(\mathscr{A})$. Hence $P(\mathscr{A})$ is a σ-ring.

That $P(\mathscr{A})$ contains $\mathscr{A}$ follows from property 1 of the class $P(\mathscr{A})$.

This completes the proof.

Example. Let $\mathscr{A} = \{\emptyset, A\}$. The smallest ring containing $\mathscr{A}$ is $R = \{\emptyset, A\}$. But $P(\mathscr{A})$ contains Φ, A and all subsets of A. Thus $P(\mathscr{A})$ is not necessarily equal to the smallest ring containing $\mathscr{A}$.

Theorem 2. Let $\mathscr{A}$ be closed under finite unions. If μ is countably additive on $\mathscr{A}$, then $\bar{\mu}$ is countably additive on $P(\mathscr{A})$.

Proof. That $\bar{\mu}$ is superadditive may be shown as in [1, p. 34].

On the other hand, let $E_1, E_2, \ldots$ be disjoint sets of $P(\mathscr{A})$ and $E = \bigcup_{n=1}^{\infty} E_n$, where $E \in P(\mathscr{A})$.

Then for $A \in \mathscr{A}$, $A \subset E$ and $A \neq \Phi$ if $E \bar{\in} A$, we have $\mu(E-A)$

$$= \mu\left[\bigcup_{n=1}^{\infty} E_n - A\right]$$

$$= \mu\left[\bigcup_{n=1}^{\infty} (E_n - A)\right]$$

$$= \sum_{n=1}^{\infty} \mu(E_n - A)$$

$$\leqq \sum_{n=1}^{\infty} \bar{\mu}(E_n)$$

Hence

$$\bar{\mu}(E) \leqq \sum_{n=1}^{\infty} \bar{\mu}(E_n)$$

Consequently

$$\bar{\mu}(E) = \sum_{n=1}^{\infty} \bar{\mu}(E_n)$$

i.e.

$$\bar{\mu}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \bar{\mu}(E_n)$$

This completes the proof.

Now from the properties of $\bar{\mu}$ and from the above two theorems we obtain the following

Theorem 3. If μ is a measure on a ring (σ-ring) $\mathscr{A}$, then $\bar{\mu}$ is a measure on the ring (σ-ring) $P(\mathscr{A})$ which extends μ.

We denote by $\bar{\mu}^*$ the d-variation of the restriction of μ^* to the σ-ring $J_\mu(A)$. For the remaining part we suppose that μ is a measure on a ring $\mathscr{A}$ of subsets of X.

Theorem 4. The d-variation $\bar{\mu}^*$ of the restriction of μ^* to the σ-ring $J_\mu(A)$ is a complete measure on the σ-ring $P(J_\mu(\mathscr{A}))$ which extends μ.

Proof. That $\bar{\mu}^*$ is a measure on $P(J_\mu(\mathscr{A}))$ and that $\bar{\mu}^*$ extends μ follow from Theorem 3.

Let $E \in P(J_\mu(A))$, $\bar{\mu}^*(E)=0$ and $F \subset E$. If $A \in J_\mu(\mathscr{A})$, then $E-A \in J_\mu(\mathscr{A})$, $A \neq \Phi$. Also $\mu^*(E-A)=0$, since $0 \leqq \mu^*(E-A) \leqq \mu^*(E)=0$. Again $F-A \subset \subset E-A$ so that $\mu^*(F-A) \leqq \mu^*(E-A)$. Hence $\mu^*(F-A)=0$ and consequently $F-A \in J_\mu(\mathscr{A})$. Therefore $F \in P(J_\mu(\mathscr{A}))$. Thus $\bar{\mu}^*$ is a complete measure on $P(J_\mu(\mathscr{A}))$.

This completes the proof.

Now we extend μ from the ring $\mathscr{A}$ to the ring $P(\mathscr{A})$. We denote by $\bar{\mu}^*$ the outer measure on the hereditary σ-ring generated by $P(\mathscr{A})$ induced by $\bar{\mu}$ on $P(\mathscr{A})$. Here we denote the σ-ring of the $\bar{\mu}^*$-measurable sets by $J_{\bar{\mu}}(P(\mathscr{A}))$.

Theorem 5. The restriction of $\bar{\mu}^*$ to the σ-ring $J_{\bar{\mu}}(P(\mathscr{A}))$ is a complete measure which extends μ.

This theorem follows from Theorem C of [2, p. 47].

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SÚHRN

ROZŠÍRENIE MIERY

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Nech $P(\mathscr{A})$ je systém všetkých podmnožín E priestoru X pre ktoré platí implikácia

$$A \in \mathscr{A},\ A \neq \Phi \Rightarrow E - A \in \mathscr{A}\ .$$

Práca sa zaoberá vlastnosťami systému $P(\mathscr{A})$ a otázkami rozšírenia na $P(\mathscr{A})$ miery definovanej na $(\mathscr{A})$

РЕЗЮМЕ

ПРОДОЛЖЕНИЕ МЕРЫ

М. Пал, Индия

Пусть $P(\mathscr{A})$ система всех подмножеств E пространства X для которых справедливо:

$$A \in \mathscr{A},\quad A \neq \Phi \Rightarrow \mathrm{E} - \mathrm{A} \in \mathscr{A}\ .$$

В работе изучаются свойства системы $P(\mathscr{A})$ и вопрос продолжения на $P(\mathscr{A})$ меры, определенной на $\mathscr{A}$.

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ASYMPTOTICKÉ VLASTNOSTI RIEŠENÍ NELINEÁRNYCH DIFERENCIÁLNYCH ROVNÍC VYŠŠÍCH RÁDOV

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V tejto práci budeme hovoriť o asymptotických a oscilatorických vlastnostiach riešení diferenciálnej rovnice 3. rádu tvaru

$$y''' + p(x)y'' + q(x)y' + r(x)h(y) = f(x) \tag{1}$$

a rovnice 4. rádu tvaru

$$y^{(4)} + q(x)y' + r(x)h(y) = f(x) \tag{2}$$

kde $p(x)$, $q(x)$, $r(x)$, $f(x)$ a $h(y)$ sú spojité funkcie pre každé $x \in \langle x_0, \infty)$ a $y \in (-\infty, \infty)$, pričom $x_0 \in (-\infty, \infty)$.

Uvedieme výsledky, ktoré sú doplnením a do istej miery zovšeobecnením niektorých výsledkov uvedených v prácach [1] až [6].

Budeme hovoriť o riešeniach $y(x)$ rovnice (1), resp. (2), ktoré existujú na intervale $\langle x_0, \infty)$.

Uvažujme najprv o rovnici (1), kde $f(x) \equiv 0$, *t. j.* o rovnici tvaru

$$y''' + p(x)y'' + q(x)y' + r(x)h(y) = 0 \tag{3}$$

pričom nech pre každé $y \in (-\infty, \infty)$ sa $\operatorname{sgn} h(y) = \operatorname{sgn} y$. Platí:

Veta 1. Nech je $p(x) \in C^1\langle x_0, \infty)$, $h(y) \in C^2(-\infty, \infty)$ a nech pre každé $x \in \langle x_0, \infty)$ a $y \in (-\infty, \infty)$ platí:

$$p(x) \geqq 0, p'(x) \leqq 0, \quad 2p'(x) \leqq 2q(x) \leqq p'(x), \quad r(x) \geqq 0$$

$$h'(y) \geqq 0, \quad h(y)h''(y) - 2h'^2(y) \leqq 0$$

pričom $p^2(x) + r^2(x) \equiv 0$ nech neplatí na žiadnom čiastočnom intervale intervalu $\langle x_0, \infty)$.

Ak pre ľubovoľné kladné konštanty A a B je

$$\lim_{x\to\infty} (A + Bx - \int_{x_0}^{x} R(t)\, dt) = -\infty \tag{4}$$

kde $R(x)=\int_{x_0}^{x} r(t)\,\mathrm{d}t$, potom riešenie $y(x)$ rovnice (3) je oscilatorické alebo $\lim_{x\to\infty} y(x)=\lim_{x\to\infty} y'(x)=0$.

Dôkaz. Nech je $y(x)$ riešenie rovnice (3) a nech nie je oscilatorické. Potom existuje také číslo $x_1 \geqq x_0$, že pre každé $x \geqq x_1$ je $y(x) \neq 0$. Nech je napr. $y(x)>0$. Potom z rovnice (3) po delení funkciou $h(y)$ a integrovaní od x_1 do $x \geqq x_1$ dostávame:

$$\frac{y''(x)}{h(y(x))}-\frac{1}{2}\int_{x_1}^{x}(h(y)h''(y)-2h'^2(y))\frac{y'^3(t)}{h^3(y)}\,\mathrm{d}t+p(x)\frac{y'(x)}{h(y)}+$$

$$+\int_{x_1}^{x}(q(t)-p'(t))\frac{y'(t)}{h(y)}\,\mathrm{d}t \leqq K_0-\int_{x_1}^{x} r(t)\,\mathrm{d}t\,,$$

z čoho po integrovaní od x_1 do $x \geqq x_1$ vyplýva

$$\frac{y'(x)}{h(y(x))}-\frac{1}{2}\int_{x_1}^{x}(x-t)\,(h(y)h''(y)-2h'^2(y))\frac{y'^3(t)}{h^3(y)}\,\mathrm{d}t+$$

$$+\int_{x_1}^{x} p(t)\frac{y'(t)}{h(y)}\,\mathrm{d}t+\int_{x_1}^{x}(x-t)\,(q(t)-p'(t))\frac{y'(t)}{h(y)}\,\mathrm{d}t \leqq$$

$$\leqq K_1+K_0x-\int_{x_1}^{x} R(t)\,\mathrm{d}t$$

Z posledného vzťahu vzhľadom na predpoklady vety vyplýva, že neexistuje také číslo $x_2 \geqq x_1$, aby pre každé $x \geqq x_2$ bolo $y'(x) \geqq 0$. Preto platí:

1. $y'(x) \leqq 0$ pre každé $x \geqq x_2 \geqq x_1$, alebo
2. pre každé $x_2 \geqq x_1$ existuje také $x \geqq x_2$, že je $y'(x)>0$ a také $\bar{x} \geqq x_2$, že $y'(\bar{x})<0$.

Dokážeme, že 2. prípad nenastane. Predpokladajme, že 2. prípad nastane, existujú také čísla ξ_1 a ξ_2, že $y'(\xi_1)=y'(\xi_2)=0$ a $y'(x)<0$ pre každé $x \in (\xi_1, \xi_2)$, pričom $\xi_1 \geqq x_1$. Po vynásobení rovnice (3) funkciou $y'(x)$ a integrovaní od ξ_1 do ξ_2 dostávame

$$-\int_{\xi_1}^{\xi_2} y''^2(t)\,\mathrm{d}t+\int_{\xi_1}^{\xi_2}(q(t)-\frac{1}{2}p'(t))y'^2(t)\,\mathrm{d}t+$$

$$+\int_{\xi_1}^{\xi_2} r(t)h(y)y'(t)\,\mathrm{d}t=0$$

čo je spor s predpokladom. Bude teda $y'(x) \leqq 0$ pre každé $x \geqq x_2 \geqq x_1$.

Teraz dokážeme, že existuje také číslo $x_3 \geqq x_2$, že pre každé $x \geqq x_3$ bude $y''(x)>0$. Je zrejmé, že neexistuje $x_3 \geqq x_2$ tak, aby pre každé $x \geqq x_3$ bolo $y''(x) \leqq 0$. Totiž pre tieto x je $y'(x) \leqq 0$ čo je spor s predpokladom $y(x)>0$. Nech teda neexistuje také číslo $x_3 \geqq x_2$, že pre každé $x \geqq x_3$ je $y''(x) > 0$. Potom existujú čísla ξ_1 a ξ_2, že $y''(\xi_1) = y''(\xi_2) = 0$ a $y''(x) > 0$ pre každé $x \in (\xi_1, \xi_2)$, kde $\xi_1 \geqq x_2$. Z rovnice (3) vyplýva

$$\int_{\xi_1}^{\xi_2} p(t) y''^2(t)\, dt + \int_{\xi_1}^{\xi_2} q(t) y'(t) y''(t)\, dt + \\ + \int_{\xi_1}^{\xi_2} r(t) h(y) y''(t)\, dt = 0$$

čo je opäť spor. Bude preto existovať také $x_3 \geqq x_2$, že pre každé $x \geqq x_3$ bude $y''(x) > 0$.

Zistili sme, že ak je $y(x) > 0$ pre každé $x \geqq x_1 \geqq x_0$, potom existuje také číslo $x_3 \geqq x_1$, že je pre každé $x \geqq x_3$ $y'(x) \leqq 0$ a $y''(x) > 0$. Pre $y(x)$ potom platí:

$$\lim_{x \to \infty} y(x) = L \geqq 0$$

Dokážeme, že $L = 0$. Nech je $L > 0$.

Z rovnice (3) dostávame pre každé $x \geqq x_3$:

$$y'''(x) \leqq -r(x) h(y)$$

čiže

$$y''(x) \leqq y''(x_3) - h(L) \int_{x_3}^{x} r(t)\, dt$$

z čoho vzhľadom na predpoklad (4) dostaneme

$$\lim_{x \to \infty} y''(x) = -\infty \quad \text{a teda aj} \quad \lim_{x \to \infty} y'(x) = \lim_{x \to \infty} y(x) = -\infty$$

čo je spor. Preto $L = 0$ a tiež $\lim_{x \to \infty} y'(x) = 0$.

Ak predpokladáme, že je $y(x) < 0$ pre každé $x \geqq x_1 \geqq x_0$, podobne dokážeme, že bude $y'(x) \geqq 0$ a $y''(x) < 0$, z čoho opäť vyplýva tvrdenie vety.

Vo vete 1 na funkciu $h(y)$ sa kladú dosť silné predpoklady. Dá sa dokázať, že platí:

Veta 2. Nech sú splnené predpoklady vety 1, pričom namiesto predpokladov o funkcii $h(y)$ nech pre každé $y \in (-\infty, \infty)$ a $y \neq 0$ platí

$$\frac{h(y)}{y} \geqq \varepsilon > 0$$

kde ε je kladné číslo. Potom riešenie $y(x)$ rovnice (3) je oscilatorické alebo $\lim_{x \to \infty} y(x) = \lim_{x \to \infty} y'(x) = 0$.

Dôkaz. Nech je $y(x)>0$ pre každé $x \geqq x_1 \geqq x_0$. Po vynásobení rovnice (3) funkciou $\frac{1}{y(x)}$ a integrovaní od x_1 do $x \geqq x_1$ dostávame:

$$\frac{y''(x)}{y(x)} + \int_{x_1}^{x} \frac{y'^3(t)}{y^3(t)} \, dt + p(x) \frac{y'(x)}{y(x)} +$$

$$+ \int_{x_1}^{x} (q(t) - p'(t)) \frac{y'(t)}{y(t)} \, dt \leqq K_0^* - \varepsilon \int_{x_1}^{x} r(t) \, dt$$

z čoho po integrovaní od x_1 do $x \geqq x_1$ vyplýva

$$\frac{y'(x)}{y(x)} + \int_{x_1}^{x} (x-t) \frac{y'^3(t)}{y^3(t)} \, dt + \int_{x_1}^{x} (x-t) p(t) \frac{y'(t)}{y(t)} \, dt +$$

$$+ \int_{x_1}^{x} (x-t)\,(q(t) - p'(t)) \frac{y'(t)}{y(t)} \, dt \leqq K_1^* + K_0^* x - \varepsilon \int_{x_1}^{x} R(t) \, dt$$

Podobne ako v dôkaze vety 1 sa ukáže, že existuje také číslo $x_2 \geqq x_1$, že pre každé $x \geqq x_2$ platí: $y'(x) \leqq 0$, $y''(x) > 0$. Z rovnice (3) potom vyplýva

$$y'''(x) \leqq -\varepsilon y(x) r(x) \leqq -\varepsilon L r(x)$$

kde $L = \lim_{x \to \infty} y(x)$. Ak $L > 0$, dostaneme z poslednej nerovnosti spor. Bude teda $\lim_{x \to \infty} y(x) = 0$ a tiež $\lim_{x \to \infty} y'(x) = 0$.

Ak predpokladáme, že je $y(x) < 0$ pre každé $x \geqq x_1 \geqq x_0$, potom sa dokáže, že $\lim_{x \to \infty} y(x) = \lim_{x \to \infty} y'(x) = 0$ analogicky ako v dôkaze vety 1.

Predpokladajme v ďalšom, že v rovnici (1) je $p(x) \equiv 0$, t. j. nech je daná rovnica

$$y''' + q(x) y' + r(x) h(y) = f(x) \tag{5}$$

Platí:

Lemma. Nech je $q(x) \in C^1 \langle x_0, \infty)$ a nech pre každé $x \in \langle x_0, \infty)$ platí:

$$q(x) \geqq 0, \quad q'(x) + |f(x)| \leqq 0, \quad r(x) \geqq 0$$

Ak $y(x)$ je riešenie rovnice (5), pre ktoré platí

$$F_1(x_0) + \frac{1}{2} \int_{x_0}^{\infty} |f(t)| \, dt \leqq 0 \tag{6}$$

kde $F_1(x) = y(x) y''(x) - \frac{1}{2} y'^2(x) + \frac{1}{2} q(x) y^2(x)$, potom nulové body $y(x)$ a $y'(x)$ sa oddeľujú.

Dôkaz. Pozri [5].

Veta 3. Nech sú splnené predpoklady lemmy a predpoklad (4) vety 1 a nech naviac je pre každé $x \in \langle x_0, \infty)$ a $y \in (-\infty, \infty)$, $y \neq 0$

$$f(x) \leqq 0 \quad \text{a} \quad \frac{h(y)}{y} \geqq \varepsilon > 0$$

potom riešenie $y(x)$ rovnice (5), pre ktoré platí (6) je oscilatorické alebo:

1. $\lim_{x \to \infty} y(x) = 0$ ak $y(x) > 0$,

2. $\lim_{x \to \infty} \sup y(x) = 0$ ak $y(x) < 0$.

Dôkaz. Nech sú splnené predpoklady vety a nech $y(x)$ je riešenie rovnice (5), pre ktoré platí vzťah (6) a nech nie je oscilatorické. Potom je $y(x) > 0$ alebo $y(x) < 0$ pre každé $x \geqq x_1 \geqq x_0$.

1. Nech je $y(x) > 0$ pre každé $x \geqq x_1 \geqq x_0$. Z rovnice (5) dostávame

$$\frac{y'''(x)}{y(x)} + q(x) \frac{y'(x)}{y(x)} = -r(x) \frac{h(y)}{y(x)} + \frac{f(x)}{y(x)}$$

teda

$$\frac{y'''(x)}{y(x)} + q(x) \frac{y'(x)}{y(x)} \leqq -\varepsilon r(x)$$

Ak dvakrát integrujeme od x_1 do $x \geqq x_1$, dostávame

$$\frac{y'(x)}{y(x)} + \int_{x_1}^{x} (x-t) \frac{y'^3(t)}{y^3(t)} \, dt + \int_{x_1}^{x} q(t) \, (x-t)^2 \frac{y'(t)}{y(t)} \, dt \leqq$$

$$\leqq K_2 + K_3 x - \varepsilon \int_{x_1}^{x} R(t) \, dt$$

z čoho vzhľadom na predpoklad (4) vyplýva, že nemôže existovať také číslo $x_2 \geqq x_1$, že pre každé $x \geqq x_2$ bude $y'(x) \geqq 0$. Bude preto:

1. $y'(x) \leqq 0$ pre každé $x \geqq x_2 \geqq x_1$ alebo

2. pre každé $x_2 \geqq x_1$ existuje také $x \geqq x_2$, že je $y'(x) > 0$ a také $\bar{x} \geqq x_2$, že je $y'(\bar{x}) < 0$.

Je zrejmé, že vzhľadom na lemu 2. prípad nenastane. Teda pre každé $x \geqq x_2$ je $y'(x) \leqq 0$. Nech je $\lim_{x \to \infty} y(x) = L > 0$. Z rovnice (5) vyplýva

$$y''(x) + q(x)y(x) = y''(x_2) + q(x_2)y(x_2) +$$

$$+ \int_{x_2}^{x} q'(t)y(t) \, dt + \int_{x_2}^{x} f(t) \, dt - \int_{x_2}^{x} r(t)h(y) \, dt$$

teda

$$y''(x) \leqq K_4 - \varepsilon L \int_{x_2}^{x} r(t)\, dt$$

z čoho vyplýva, že $\lim_{x\to\infty} y''(x) = -\infty$, teda pre dostatočne veľké y bude $y(x) < 0$, čo je spor s predpokladom, teda $\lim_{x\to\infty} y(x) = 0$.

2. Nech je $y(x) < 0$ pre každé $x \geqq x_1 \geqq x_0$.

Z rovnice (5) vyplýva

$$\frac{y'''(x)}{y(x)} + q(x)\frac{y'(x)}{y(x)} \leqq -\varepsilon r(x) + \left|\frac{f(x)}{y(x)}\right| \tag{7}$$

Nech je $\limsup_{x\to\infty} y(x) = L < 0$. Potom existuje také číslo $x_2 \geqq x_1$, že pre každé $x \geqq x_2$ bude $y(x) \leqq \frac{L}{2} = L_1 < 0$, teda $|y(x)| \geqq |L_1|$. Zo vzťahu (7) pre každé $x \geqq x_2$ je teda

$$\frac{y'''(x)}{y(x)} + q(x)\frac{y'(x)}{y(x)} \leqq \frac{|f(x)|}{|L_1|} - \varepsilon r(x)$$

z čoho podobne ako v 1. prípade dokážeme, že bude pre každé $x \geqq x_3 \geqq x_2$ $y'(x) \geqq 0$. Z rovnice (5) potom dostávame

$$y''(x) \geqq y''(x_3) + q(x_3)y(x_3) + \int_{x_3}^{x} f(t)\, dt - \varepsilon L_1 \int_{x_3}^{x} r(t)\, dt$$

z čoho vyplýva, že $\lim_{x\to\infty} y''(x) = +\infty$, a teda aj $\lim_{x\to\infty} y(x) = +\infty$, čo je spor s predpokladom. Musí teda byť $\limsup_{x\to\infty} y(x) = L = 0$.

Tým je veta dokázaná.

Je zrejmé, že platí:

Veta 4. Nech sú splnené predpoklady vety 3, pričom namiesto predpokladu $f(x) \leqq 0$ nech je $f(x) \geqq 0$. Potom riešenie $y(x)$ rovnice (5), pre ktoré platí (6) je oscilatorické alebo:

1. $\liminf_{x\to\infty} y(x) = 0$ ak $y(x) > 0$,

2. $\lim_{x\to\infty} y(x) = 0$ ak $y(x) < 0$.

Teda platí nasledujúca veta:

Veta 5. Ak sú splnené predpoklady vety 4, pričom je $f(x) \equiv 0$, potom riešenie $y(x)$ rovnice (5), pre ktoré platí (6) je oscilatorické alebo $\lim_{x\to\infty} y(x)=0$.

Doteraz sme predpokladali, že pre rovnicu (1) je $r(x)\geqq 0$ a $\operatorname{sgn} h(y)=\operatorname{sgn} y$. V ďalšej časti práce predpokladáme, že pre každé $x\in\langle x_0,\infty)$ a $y\in(-\infty,\infty)$ je $\operatorname{sgn} r(x)=\operatorname{sgn} h(y)y$ pre rovnicu (1), resp. (2). Pritom $\operatorname{sgn} a=1$ ak $a\geqq 0$ a $\operatorname{sgn} a=-1$ ak $a<0$. Potom platí:

Veta 6. Nech pre každé $x\in\langle x_0,\infty)$ a $y\in(-\infty,\infty)$ je

$$q(x)\geqq 0\,,\qquad q'(x)+|f(x)|\leqq 0$$

Ak je

$$\int_{x_0}^{\infty} q(t)\,\mathrm{d}t=+\infty \tag{8}$$

potom riešenie $y(x)$ rovnice (5), pre ktoré platí (6) je oscilatorické alebo $\lim_{x\to\infty} y(x)=0$.

Dôkaz. Nech je $y(x)$ riešenie rovnice (5), ktoré spĺňa (6) a nech nie je oscilatorické. Potom existuje také $x_1\geqq x_0$, že pre každé $x\geqq x_1$ je $y(x)\neq 0$. Násobme rovnicu (5) funkciou $y(x)$ a integrujme od x_0 do $x\geqq x_0$. Vzhľadom na predpoklady vety dostaneme

$$y''(x)y(x)-\frac{1}{2}y'^2(x)+\frac{1}{2}q(x)y^2(x)\leqq F_1(x_0)+\frac{1}{2}\int_{x_0}^{x}|f(t)|\,\mathrm{d}t$$

teda

$$y''(x)y(x)-y'^2(x)\leqq y''(x)y(x)-\frac{1}{2}y'^2(x)\leqq -\frac{1}{2}q(x)y^2(x)$$

z čoho pre každé $x\geqq x_1$ vyplýva

$$\frac{y''(x)y(x)-y'^2(x)}{y^2(x)}=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y'(x)}{y(x)}\right]\leqq -\frac{1}{2}q(x)$$

Integrovaním od x_1 do $x\geqq x_1$ dostávame

$$\frac{y'(x)}{y(x)}\leqq\frac{y'(x_1)}{y(x_1)}-\frac{1}{2}\int_{x_1}^{x}q(t)\,\mathrm{d}t \tag{9}$$

Vzhľadom na predpoklad (8) z toho vyplýva, že pre každé $x\geqq x_2\geqq x_1$ je $y(x)y'(x)<0$. Nech je $y(x)>0$. Potom je $y'(x)<0$ pre každé $x\geqq x_2$, čiže existuje $\lim_{x\to\infty} y(x)=L\geqq 0$. Nech je $L>0$. Potom je pre každé $x\geqq x_2$

$$\frac{y'(x)}{L}<\frac{y'(x)}{y(x)}\leqq\frac{y'(x_1)}{y(x_1)}-\frac{1}{2}\int_{x_1}^{x}q(t)\,\mathrm{d}t$$

teda $\lim_{x\to\infty} y'(x) = -\infty$, čo je spor s predpokladom, že $y(x)>0$ pre každé $x \geqq x_1$.

Ak je $y(x)<0$ pre každé $x \geqq x_1$, potom zo vzťahu (9) vyplýva, že $y'(x)>0$ pre každé $x \geqq x_2 \geqq x_1$ a teda existuje $\lim_{x\to\infty} y(x) = L \leqq 0$. Analogicky ako v prvom prípade sa ukáže, že $L=0$.

Veta 7. Ak sú splnené predpoklady vety 6, potom každé riešenie $y(x)$ rovnice (2), pre ktoré platí

$$F_2(x_0) + \frac{1}{2}\int_{x_0}^{\infty} |f(t)|\, dt \leqq 0 \tag{10}$$

kde $F_2(x) = y'''(x)y(x) - y'(x)y''(x) + \frac{1}{2} q(x)y^2(x)$, je oscilatorické.

Dôkaz. Nech je $y(x)$ ľubovoľné riešenie rovnice (2), pre ktoré platí (10) a nech nie je oscilatorické. Potom existuje také číslo $x_1 \geqq x_0$, že $y(x) \neq 0$ pre každé $x \geqq x_1$. Platí:

$$y'''(x)y(x) - y''(x)y'(x) + \frac{1}{2} q(x)y^2(x) \leqq$$

$$\leqq F_2(x_0) + \frac{1}{2}\int_{x_0}^{x} |f(t)|\, dt$$

teda pre každé $x \geqq x_1$ je

$$\frac{d}{dx}\left[\frac{y''(x)}{y(x)}\right] \leqq -\frac{1}{2} q(x)$$

z čoho po integrovaní od x_1 do $x \geqq x_1$ dostávame

$$\frac{y''(x)}{y(x)} \leqq \frac{y''(x_1)}{y(x_1)} - \frac{1}{2}\int_{x_1}^{x} q(t)\, dt$$

Teda existuje také $x_2 \geqq x_1$, že pre každé $x \geqq x_2$ je $y''(x)y(x)<0$. Nech je $y(x)>0$. Potom je $y''(x)<0$ a môžu nastať dva prípady:

1. $y'(x)>0$ pre každé $x \geqq x_2$,
2. existuje také číslo ξ, že je $y'(\xi)<0$.

Je zrejmé, že ak nastane 2. prípad, potom existuje také číslo x_3, že $y(x_3)=0$, čo je spor s predpokladom. Nech je teda $y'(x)>0$ pre každé $x \geqq x_2$. Potom pre každé $x \geqq x_2$ platí:

$$\frac{y''(x)}{y(x_2)} \leqq \frac{y''(x)}{y(x)} \leqq \frac{y''(x_1)}{y(x_1)} - \frac{1}{2}\int_{x_1}^{x} q(t)\, dt$$

potom vzhľadom na predpoklad (8) platí, že $\lim_{x\to\infty} y''(x) = -\infty$, čo je spor s predpokladom, že $y(x)>0$ pre každé $x \geqq x_1$.

Podobne sa dokáže, že $y(x)<0$ neplatí pre $x \geqq x_1$.

Tým je veta dokázaná.

Poznámka. Ak je $r(x) \geqq 0$ a $h(y)=y$, potom vety 6 a 7 sú totožné s vetami 1 a 2 v práci [6].

Ako vidieť, vo vete 6 a 7 sa predpokladá, že

$$\int_{x_0}^{\infty} |f(t)| \, \mathrm{d}t < \infty .$$

Uvedieme postačujúce podmienky na to, aby riešenie $y(x)$ rovnice (5) bolo oscilatorické, alebo $\lim_{x \to \infty} y(x)=0$ a riešenie $y(x)$ rovnice (2) bolo oscilatorické, pričom nebudeme žiadať konvergenciu integrálu $\int_{x_0}^{\infty} |f(t)| \, \mathrm{d}t$.

Veta 8. Nech pre každé $x \in \langle x_0, \infty)$ a $y \in (-\infty, \infty)$ platí:

$$q(x)>0 , \qquad q'(x)<0$$

Ak platí (8), potom riešenie $y(x)$ rovnice (5), pre ktoré je

$$F_1(x_0) - \frac{1}{2}\int_{x_0}^{\infty} \frac{f^2(t)}{q'(t)} \, \mathrm{d}t \leqq 0 \tag{11}$$

je oscilatorické alebo $\lim_{x \to \infty} y(x)=0$.

Dôkaz. Nech $y(x)$ je riešenie rovnice (5), pre ktoré platí (11) a nech nie je oscilatorické. Nech je $y(x) \neq 0$ pre každé $x \geqq x_1 \geqq x_0$. Potom platí:

$$y''(x)y(x) - \frac{1}{2}y'^2(x) + \frac{1}{2}q(x)y^2(x) \leqq$$

$$\leqq F_1(x_0) + \frac{1}{2}\int_{x_0}^{x} q'(t)y^2(t) \, \mathrm{d}t + \int_{x_0}^{x} f(t)y(t) \, \mathrm{d}t$$

Keďže je pre $a<0$

$$ay^2 + by \leqq -\frac{b^2}{4a}$$

dostávame z poslednej nerovnosti vzťah

$$y''(x)y(x) - \frac{1}{2}y'^2(x) + \frac{1}{2}q(x)y^2(x) \leqq$$

$$\leqq F_1(x_0) - \frac{1}{2}\int_{x_0}^{x} \frac{f^2(t)}{q'(t)} \, \mathrm{d}t$$

z čoho podobne ako v dôkaze vety 6 vyplýva tvrdenie vety 8.

Je zrejmé, že platí:

Veta 9. Nech sú splnené predpoklady vety 8 a nech $y(x)$ je riešenie rovnice (2), pre ktoré platí

$$F_2(x_0) - \frac{1}{2}\int_{x_0}^{\infty} \frac{f^2(t)}{q'(t)}\,\mathrm{d}t \leqq 0$$

Potom je $y(x)$ oscilatorická funkcia.

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SUMMARY

ASYMPTOTIC NONLINEAR DIFFERENTIAL EQUATIONS

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Nonlinear differential equations of order 3 and 4 are investigated and conditions stated for the solution $y(x)$ to be oscilatory or for $\lim_{x\to\infty} y(x) = \lim_{x\to\infty} y(x) = 0$ to hold.

РЕЗЮМЕ

АСИМПТОТИЧЕСКИЕ СВОЙСТВА РЕШЕНИЙ НЕЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВЫСШИХ ПОРЯДКОВ

В. Шолтес и П. Шолтес, Кошице

В этой работе мы занимаемся нелинейным дифференциальным уравнением третьего и четвертого порядков. Здесь приведены условия, при которых решение $y(x)$ колеблется или

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} y(x) = 0.$$

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RÝCHLOSŤ RASTU MNOŽSTVA INFORMÁCIE

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V jednej zo svojich prác uviedol Rényi zaujímavú vlastnosť množstva informácie. Našim cieľom bude zovšeobecnenie tohto výsledku. Pre tento účel si určíme podmienky a zavedieme pojmy, ktoré potrebujeme na uskutočnenie uvedeného cieľa.

Nech $\xi_1, \ldots, \xi_n, \ldots$, je postupnosť náhodných premenných, ktorých rozdelenie závisí od parametra Θ. Možná hodnota tohto parametra je určená množinou $\mathcal{H} = \{\Theta_1, \ldots, \Theta_K\}$, kde $\Theta_i \neq \Theta_j$ pre $i \neq j$. Predpokladajme, že na $\mathcal{H}$ je dané apriórne rozdelenie $p_1 = P(\Theta = \Theta_1), \ldots, p_K = P(\Theta = \Theta_K)$ tak, že $p_i > 0$. Náhodné premenné ξ_n sú nezávislé a identicky rozdelené pri podmienkach $\Theta = \Theta_1, \ldots, \Theta = \Theta_K$. Nech $f_1(x), \ldots, f_K(x)$ sú hustoty rozdelenia premenných ξ_n pri podmienkach $\Theta = \Theta_1, \ldots, \ldots, \Theta = \Theta_K$. Predpokladajme, že $f_i(x) \neq f_j(x)$ pre $i \neq j$, na množine nenulovej miery. Ďalej uvažujme, že je daný výber $\eta_n = (\xi_1, \ldots, \xi_n)$.

Definícia 1. Entrópiou $H(\Theta)$ parametra Θ budeme rozumieť výraz

$$H(\Theta) = p_1 \log \frac{1}{p_1} + \ldots + p_K \log \frac{1}{p_K} \tag{1}$$

a podmienenou entrópiou $H(\Theta \mid \eta_n)$, parametra Θ, pri podmienke η_n, zase výraz

$$H(\Theta \mid \eta_n) = P(\Theta = \Theta_1 \mid \eta_n) \log \frac{1}{P(\Theta = \Theta_1 \mid \eta_n)} + \ldots +$$

$$+ P(\Theta = \Theta_K \mid \eta_n) \log \frac{1}{P(\Theta = \Theta_K \mid \eta_n)} \tag{2}$$

pričom log sa chápe pri základe 2.

Definícia 2. Množstvo informácie $I_n = (\eta_n, \Theta)$, ktoré obsahuje výber η_n, týkajúci sa parametra Θ budeme označovať

$$I_n = H(\Theta) - E[H(\Theta \mid \eta_n)] \tag{3}$$

kde $E[H(\Theta \mid \eta_n)]$ je strednou hodnotou $H(\Theta \mid \eta_n)$.

Rényi v [2] ukázal, že pre $K=2$, existujú také konštanty $A>0$ a $\lambda \in (0, 1)$, že platí

$$0 \leqq H(\Theta) - I_n = E[H(\Theta \mid \eta_n)] \leqq A\lambda^n \tag{4}$$

V ďalšej časti ukážeme platnosť tohto tvrdenia pre $K>2$. Pri dôkaze použijeme nasledujúcu lemu.

Lema. Nech $K \geqq 2$ je prirodzené číslo a nech $\alpha \in (0, 1)$. Potom existujú také konštanty $C_{i\alpha}$, ktoré závisia od α, že pre $(p_1, \ldots, p_K) \geqq 0$, $\sum_{i=1}^{K} p_i = 1$ platí

$$p_1 \log \frac{1}{p_1} + \ldots + p_K \log \frac{1}{p_K} \leqq \sum_{\substack{i=1 \\ i \neq j}}^{K} C_{i\alpha} p_i^\alpha \tag{5}$$

pre $j = 1, \ldots, K$.

Dôkaz. Nech $\alpha \in (0, 1)$ je pevné.

Postupujme úplnou indukciou vzhľadom na K. Pre $K=2$ lemu dokázal Rényi (pozri [2]). Nech tvrdenie platí pre $K>2$. Nech $p_1, \ldots, p_{K+1}$ sú kladné čísla také, že $\sum_{i=1}^{K+1} p_i = 1$.

Nech $m \in \{1, \ldots, K+1\}$. Ak $p_m = 1$, tak je tvrdenie zrejmé.

Ak $p_m < 1$, tak $1 - p_m = \sum_{\substack{i=1 \\ i \neq m}}^{K+1} p_i > 0$.

Podľa indukčných predpokladov existujú také konštanty $C_{i\alpha}$, kde $i \neq m$ a $i = 1, \ldots, K+1$, že pre $\frac{p_i}{1-p_m}$ platí

$$\sum_{\substack{i=1 \\ i \neq m}}^{K+1} \frac{p_i}{1-p_m} \log \frac{1-p_m}{p_i} \leqq \sum_{\substack{i=1 \\ i \neq j}}^{K+1} \frac{C_{i\alpha} p_i^\alpha}{(1-p_m)^\alpha} \tag{6}$$

$(j = 1, \ldots, K+1)$.

Úpravou (6) dostávame

$$\sum_{\substack{i=1 \\ i \neq m}}^{K+1} p_i \log \frac{1}{p_i} + \left(\sum_{\substack{i=1 \\ i \neq m}}^{k+1} p_i \right) \log (1-p_m) \leqq$$

$$\leqq \sum_{\substack{i=1 \\ i \neq m \\ i \neq j}}^{K+1} C_{i\alpha} p_i^\alpha (1-p_m)^{1-\alpha} \leqq \tag{7}$$

$$\leqq \sum_{\substack{i=1\\ i\neq m\\ i\neq j}}^{K+1} C_{i\alpha} p_i^{\alpha} \qquad \text{pre } j=1, \dots, K+1$$

Na základe [2] vieme, že existuje $C_{m\alpha}$, pre ktoré platí

$$p_m \log \frac{1}{p_m} + (1-p_m) \log \frac{1}{(1-p_m)} \leqq C_{m\alpha} p_m^{\alpha} \tag{8}$$

Sčítaním (7) a (8) dostaneme

$$\sum_{i=1}^{K+1} p_i \log \frac{1}{p_i} \leqq \sum_{\substack{i=1\\ i\neq j}}^{k+1} C_{i\alpha} p_i^{\alpha}, \qquad j=1, \dots, K+1$$

čo bolo treba dokázať.

Veta. Nech $f_i(x)>0$ pre $i=1, \dots, K$, kde $K\geqq 2$. Potom existujú také konštanty $A>0$ a $\lambda \in (0, 1)$ závisiace od $f_i(x)$ a p_i pre $i=1, \dots, K$, že

$$0 \leqq H(\Theta) - I_n \leqq A\lambda^n \tag{9}$$

kde $\lambda = \min\limits_{\alpha\in(0,1)} \max\limits_{i\neq j} \int_{-\infty}^{\infty} f_i^{\alpha}(x) f_j^{1-\alpha}(x)\, dx$

Dôkaz. Je zrejmé, že platí

$$\begin{aligned} H(\Theta) - I_n &= p_1 E[H(\Theta \mid \eta_n) \mid \Theta = \Theta_1] + \dots + \\ &+ p_K E[H(\Theta \mid \eta_n) \mid \Theta = \Theta_K] \end{aligned} \tag{10}$$

Nech $j \in \{1, \dots, K\}$. Potom na základe uvedenej lemy pre každé $\alpha \in (0, 1)$ platí

$$\begin{aligned} E[H(\Theta \mid \eta_n) \mid \Theta = \Theta_j] &\leqq E\left\{\left[\sum_{\substack{i=1\\ i\neq j}}^{K} C_{i\alpha} P^{\alpha}(\Theta = \Theta_i \mid \eta_n)\right] \middle| \Theta = \Theta_j\right\} = \\ &= \sum_{\substack{i=1\\ i\neq j}}^{K} C_{i\alpha} E[P^{\alpha}(\Theta = \Theta_i \mid \eta_n) \mid \Theta = \Theta_j] \end{aligned} \tag{11}$$

Podľa Bayesovej vety o podmienenej pravdepodobnosti a na základe predpokladov vety dostávame

$$P(\Theta = \Theta_i \mid \eta_n) = \frac{p_i \prod_{t=1}^{n} f_i(\xi_t)}{\sum_{q=1}^{K} p_q \prod_{t=1}^{n} f_q(\xi_t)} \leqq \frac{p_i \prod_{t=1}^{n} f_i(\xi_t)}{p_j \prod_{t=1}^{n} f_j(\xi_t)} \qquad \text{pre } i \neq j \tag{12}$$

Využitím nezávislosti ξ_t pri podmienke $\Theta = \Theta_j$ a vzťahu (12) dostávame

$$E[P^{\alpha}(\Theta = \Theta_i \mid \eta_n) \mid \Theta = \Theta_j] \leqq \left(\frac{p_i}{p_j}\right)^{\alpha} \left[\int_{-\infty}^{\infty} f_i^{\alpha}(x) f_j^{1-\alpha}(x)\, dx\right]^n \quad \text{pre } i \neq j . \tag{13}$$

Položme

$$\int_{-\infty}^{\infty} f_i^{\alpha}(x) f_j^{1-\alpha}(x)\, dx = \lambda_{ij}(\alpha)\,, \quad \text{pre } i \neq j \tag{14}$$

Potom dostávame

$$E[H(\Theta \mid \eta_n) \mid \Theta = \Theta_j] \leqq \sum_{\substack{i=1 \\ i \neq j}}^{K} \left(\frac{p_i}{p_j}\right)^{\alpha} C_{i\alpha} \lambda_{ij}^n(\alpha) \tag{15}$$

Použitím (15) na ohraničenie (10) dostávame

$$0 \leqq H(\Theta) - I_n \leqq \sum_{\substack{i=1 \\ i \neq j}}^{K} p_i\, p_j^{1-\alpha} C_{i\alpha} \lambda_{ij}^n(\alpha) \tag{16}$$

kde $j = 1, \ldots, K$.

Nech $\lambda_{i^*j^*}(\alpha) = \max_{i \neq j} \lambda_{ij}(\alpha)$ pre $i, j = 1, \ldots, K$. (17)

Je zrejmé, že potom platí

$$\sum_{\substack{i=1 \\ i \neq j}}^{K} p_i^{\alpha}\, p_j^{1-\alpha} C_{i\alpha} \lambda_{ij}^n(\alpha) \leqq \lambda_{i^*j^*}^n(\alpha) \sum_{\substack{i=1 \\ i \neq j}}^{K} p_i^{\alpha} p_j^{1-\alpha} C_{i\alpha} \quad \text{kde } j = 1, \ldots, K . \tag{18}$$

Funkcia $\lambda_{i^*j^*}(\alpha)$ je konvexnou funkciou premennej α, pre ktorú platí, $\lambda_{i^*j^*}(0) = \lambda_{i^*j^*}(1) = 1$. To znamená, že $\min_{\alpha} \lambda_{i^*j^*}(\alpha) = \lambda_{i^*j^*}(\alpha^*) = \lambda < 1$, pričom $\alpha^* \in (0, 1)$, kde $i^* \neq j^*$ a funkcie $f_{i^*}(x), f_{j^*}(x)$ podľa predpokladu nie sú skoro všade rovnaké.

Na základe predchádzajúcich úvah, potom dostávame

$$0 \leqq H(\Theta) - I_n \leqq A\lambda^n$$

kde $A = \sum_{\substack{i=1 \\ i \neq j}}^{K} p_i^{\alpha^*} p_j^{1-\alpha} C_{i\alpha^*}$ pre $j = 1, \ldots, K$, pričom je zrejmé, že $A > 0$ a $\lambda \in (0, 1)$.

Tým je naša veta dokázaná.

Uvedená veta tvrdí, že množstvo informácie týkajúce sa neznámej hodnoty parametra Θ, rozširovaním výberu rastie. To znamená, že (9) je možné chápať, ako akýsi obraz o rýchlosti rastu množstva informácie, vzťahujúcej sa na Θ.

Dôsledok. Nech platia predpoklady uvedené v predchádzajúcej vete. Potom platí

$$\lim_{n \to \infty} I_n = H(\Theta) \tag{19}$$

Pravdivosť vzťahu (19) vyplýva priamo z tvrdenia uvedenej vety. Hovorí nám, že rozširovaním výberu množstvo informácie, vzťahujúce sa na neznámu hodnotu parametra Θ, konverguje k entrópii tohto parametra.

Touto cestou si dovoľujem vysloviť svoju úprimnú vďaku doc. RNDr. J. Černému, CSc., za cenné pripomienky.

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SUMMARY

SPEED OF A GROWTH OF THE AMMOUNT OF INFORMATION

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In this paper the existence of a boundary is shown for the ammount of information, in the sample η_n containing the parameter Θ, where $\Theta \in \{\Theta_1, \ldots, \Theta_K\}$. This result is a nontrivial generalization one Rényi's result presented in [2].

РЕЗЮМЕ

СКОРОСТЬ РОСТА МНОЖЕСТА ИНФОРМАЦИИ

П. Карас, Братислава

В работе указано существование ограничения множества информации, находящегося во выборе η_n касающегося параметра Θ, где $\Theta \in \{\Theta_1, \ldots, \Theta_k\}$. Этот результат является нетривияльным обобщением одного утверждения Реньи указанного в [2].

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O RIEŠENÍ OKRAJOVÝCH ÚLOH PRE LINEÁRNE PARABOLICKÉ DIFERENCIÁLNE ROVNICE VYŠŠÍCH RÁDOV

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1. Úvod

V článku [1] sa vyšetruje riešenie lineárnej diferenciálnej rovnice $D_x^4 u(x, t) + D_t u(x, t) = \varphi(x, t)$ v obdĺžniku $(0, 1) \times (0, T)$, so začiatočnou podmienkou na $(0, T)$ a okrajovými podmienkami na úsečkách, ležiacich na priamkach $x = 0$ a $x = 1$ metódou Greenových funkcií.

Táto práca sa zaoberá dvomi zovšeobecnenými Dirichletovými úlohami pre nemohogénnu lineárnu parabolickú rovnicu párneho rádu $D_x^{2n} u(x, t) + (-1)^n D_t u(x, t) = \varphi(x, t)$ $(D_x = \partial/\partial x)$. Úloha sa rieši metódou Greenových funkcií a je skonštruované explicitné riešenie uvažovaných úloh. Táto metóda súčasne poskytuje možnosť vyšetrenia hladkosti v bodoch (0, 0) a (1, 0).

2. Formulácia úlohy

Nech R^n je n-rozmerný euklidovský priestor. Nech $\Omega_0 = (0, 1) \times (0, T)$, $\Omega = (0, 1) \times (0, T)$, $\Omega_1 = (-\infty, \infty) \times (0, \infty)$. Budeme skúmať nasledujúce úlohy

$$D(u, x, t) = D_x^{2n} u(x, t) + (-1)^n D_t u(x, t) = \varphi(x, t), \qquad (x, t) \in \Omega \tag{1}$$

$$u(x, 0) = g(x), \qquad 0 < x < 1 \tag{2}$$

$$D_x^{2s+j-1} u(0, t) = a_{2s,j}(t),$$

$$D_x^{2s+j-1} u(1, t) = b_{2s,j}(t), \qquad t \in (0, T\rangle, \; s = 0, 1, \ldots, n-1 \text{ a } j = 1, 2. \tag{3j}$$

Funkcie $\varphi(x, t)$, $g(x)$, $a_{2s,j}(t)$ a $b_{2s,j}(t)$ sú reálne funkcie z určitých tried, ktoré budú definované ďalej.

Reálna funkcia $u(x, t)$ je riešením úlohy (1), (2), (3j) pre $j = 1, 2$, ak

$u \in C_0[\bar{\Omega} - \{(0, 0), (1, 0)\}]$, $D_x^\nu u \in C_0(\Omega)$ pre $\nu = 1, 2, \ldots, 2n-1$, $D_t u \in C_0(\Omega)$ a spĺňa podmienky (1), (2), (3j).

Nech $A \subset R^1$ a $f(x)$ je reálna funkcia na A. Nech B je uzavretá podmnožina množiny A. Ak ku každému takto definovanému B existuje konštanta $K(B)$, ktorá závisí len od B taká, že

$$|f(x) - f(y)| \leqq K(B)\,|x-y|^{\frac{m}{2n}+\varepsilon}$$

pre všetky $(x, y) \in B$, $0 < \varepsilon + \dfrac{m}{2n} < 1$, kde $m = 0, 1, \ldots, 2n-2$, potom funkciu $f(x)$ nazývame lokálne hölderovsky spojitú na A s daným exponentom. Množinu všetkých takýchto funkcií budeme označovať $S_m(x, A)$. Ak funkcia f závisí od parametra λ, tak funkcia $f(x\,;\lambda)$ sa nazýva lokálne hölderovsky spojitá vzhľadom na x na A a rovnomerne vzhľadom na λ, ak

$$|f(x\,;\lambda) - f(y\,;\lambda)| \leqq K(B)\,|x-y|^{\frac{m}{2n}+\varepsilon}$$

na B. Tieto funkcie budeme označovať $S_m(x, A\,;\lambda)$. Množinu všetkých funkcií $v(x, t)$ spojitých na $\bar{A}$ so spojitými deriváciami $D_x^m v(x, t)$ a $D_t v(x, t)$ na A, kde m je prirodzené číslo, označíme $N_m(A)$.

Fundamentálnym riešením rovnice $L(u\,; x, t) = 0$ na $\bar{\Omega}$ nazývame spojitú funkciu $\Gamma(x, t\,; \xi, \tau)$ pre $(x, t\,; \xi, \tau) \in \bar{\Omega} \times \bar{\Omega}$ a $(x, t) \neq (\xi, \tau)$, ktorá má derivácie $D_t\Gamma$, $D_x\Gamma$, $D_x^2\Gamma$, $\ldots$, $D_x^{2n}\Gamma$ a integrál

$$u(x, t) = \int_0^T \mathrm{d}\tau \int_0^1 \Gamma(x, t\,; \xi, \tau)\varphi(\xi, \tau)\,\mathrm{d}\xi$$

je riešením rovnice $L(u\,; x, t) = \varphi(x, t)$ na Ω pre každú funkciu $\varphi \in C_0(\bar{\Omega}) \cap S_0[x, (0, 1)\,; t]$.

Spojitá funkcia $G_j(x, t\,; \xi, \tau)$ na $\bar{\Omega} \times \bar{\Omega}$, $t > \tau$, ktorá má prvú deriváciu vzhľadom na t a derivácie vzhľadom na x do $2n$-tého rádu sa nazýva Greenova funkcia úlohy (1), (2), (3j), $j = 1{,}2$ ak

$$G_j(x, t\,; \xi, \tau) = \Gamma(x, t\,; \xi, \tau) + v_j(x, t\,; \xi, \tau)$$

kde Γ je fundamentálne riešenie rovnice $L(u\,; x, t) = 0$ na $\bar{\Omega}$ a funkcia v_j spĺňa nasledujúce podmienky:

a) $L(v_j\,; x, t) = 0$ pre $t > \tau$;

b) $v_j|_{t=\tau} = 0$ pre $(x\,; \xi, \tau) \in \langle 0, 1 \rangle \times \bar{\Omega}$ ak aspoň jeden z bodov x, ξ leží v otvorenom intervale $(0, 1)$.

c) $D_x^{2s+j-1} G_j|_{x=0} = D_x^{2s+j-1} G_j|_{x=1} = 0$, $\quad s = 0, 1, \ldots, n-1$, $j = 1, 2$

3. Vlastnosti fundamentálneho riešenia

Definujme funkciu $\Gamma_\nu(x, t; \xi, \tau)$, $\nu = 0, 1, \ldots$ pre $(x, t; \xi, \tau) \in \Omega_1 \times \Omega_1$, $(x, t) \neq (\xi, \tau)$ nasledovne

$$\Gamma_\nu(x, t; \xi, \tau) = \begin{cases} k_\nu(x, t; \xi, \tau), & \text{ak } 0 \leqq \tau < t \\ 0, & \text{ak } \tau \geqq t \end{cases} \tag{4}$$

kde $k_\nu(x, t; \xi, \tau) = \dfrac{(-\mathrm{i})^\nu}{2\pi} \displaystyle\int_{-\infty}^{\infty} \varrho^\nu \exp\{-\mathrm{i}\varrho(x-\xi) - \varrho^{2n}(t-\tau)\, \mathrm{d}\varrho$ (5)

(i znamená imaginárnu jednotku).

V nasledujúcich lemách sú uvedené niektoré vlastnosti spojitosti, diferencovateľnosti a integrovateľnosti funkcie Γ_0, o ktorej je dokázané, že tvorí fundamentálne riešenie rovnice $L(u; x, t) = 0$. V ďalších úvahách podstatne využijeme odhad z [3], str. 310

$$|D_x^\nu \Gamma_0(x, t; \xi, \tau)| \leqq C_1(\nu)\, (t-\tau)^{-\frac{1+\nu}{2n}} \exp\left\{-C_2\left[\frac{(x-\xi)^{2n}}{t-\tau}\right]\right\}^{\frac{1}{2n-1}} \tag{6}$$

pre $(x, t; \xi, \tau) \in \Omega_1 \times \Omega_1$, $\tau < t$. C_1 závisí od ν a $C_2 > 0$ je absolútna konštanta. Tento odhad môžeme transformovať na tvar

$$|D_x^\nu \Gamma_0(x, t; \xi, \tau)| \leqq$$

$$\leqq C_1(\nu) \frac{|x-\xi|^{2n\mu-\nu-1}}{(t-\tau)^\mu} \left[\frac{(x-\xi)^{2n}}{t-\tau}\right]^{\frac{1+\nu-2n\mu}{2n}} \exp\left\{-C_2\left[\frac{(x-\xi)^{2n}}{t-\tau}\right]\right\}^{\frac{1+\nu-2n\mu}{2n}} \leqq$$

$$\leqq K(\nu) \frac{|x-\xi|^{2n\mu-\nu-1}}{(t-\tau)^\mu} \tag{7}$$

pre $(x, t; \xi, \tau) \in \Omega_1 \times \Omega_1$, $\tau < t$, $\xi \neq \tau$ a $\mu \leqq \dfrac{1+\nu}{2n}$, $\nu = 0, 1, 2, \ldots$, $K(\nu)$ je konštanta, ktorá závisí len od ν.

V článku [2], str. 861—862 je dokázané tvrdenie

$$\begin{aligned} \int_{-\infty}^{0} \Gamma_0(x, 1; 0, 0)\, \mathrm{d}x &= \int_0^\infty \Gamma_0(x, 1; 0, 0)\, \mathrm{d}x = \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \Gamma_0(x, 1; 0, 0)\, \mathrm{d}x = \frac{1}{2} \end{aligned} \tag{8}$$

Lema 1. Nech $\nu = 0, 1, 2, \ldots$ a $A = \{(x, t; \xi, \tau) \in \Omega_1 \times \Omega_1 : (x, t) \neq (\xi, \tau)\}$. Potom:

a) Funkcia $\Gamma_\nu(x, t; \xi, \tau)$ je spojitá na A a má nasledujúce vlastnosti: $\Gamma_\nu = D_x^\nu \Gamma_0$, $D_x^{2n\nu}\Gamma_0 = (-1)^{\nu(n-1)} D_t^\nu \Gamma_0 = (-1)^n D_\tau^\nu \Gamma_0 = D_\xi^{2n\nu}\Gamma_0$ a $D_x^\nu \Gamma_0 = (-1)^n D_\xi^\nu \Gamma_0$.

b) $\Gamma_\nu(x, t; \xi, \tau)$ je skoro rovnomerne ohraničená na A t. j., že ku každému $\delta > 0$ existuje $N(\delta) > 0$ také, že $|\Gamma_\nu(x, t; \xi, \tau)| < N(\delta)$ na A pre $(x-\xi)^2 + (t-\tau)^2 \geqq \delta^2$.

c) Ak $\nu = 0, 1, \ldots, 2n-1$ potom integrál $\int_\Omega \int |\Gamma_0(x, t; \xi, \tau)| \, d\xi \, d\tau$ je rovnomerne konvergentný vzhľadom na $P(x, t) \in \Omega_1$, t. j. ku každému $\varepsilon > 0$, existuje $\delta > 0$ tak, že

$$\int_{\Omega \cap S(P, \delta)} < D_x^\nu \Gamma_0(x, t; \xi, \tau)| \, d\xi \, d\tau < \varepsilon$$

pre všetky $P \in \Omega_1$ ($S(P, \delta)$ označíme kruh $(\xi - x)^2 + (\tau - t)^2 \leqq \delta^2$).

Lema 2. Nech $f \in C_0(\bar{\Omega})$. Potom integrál

$$I_\nu(x, t; \tau) \equiv \int_0^1 \Gamma_\nu(x, t; \xi, \tau) f(\xi, \tau) \, d\xi$$

pre $\nu = 0, 1, 2, \ldots$ má nasledujúce vlastnosti:

a) I_ν je spojitý na $\bar{\Omega} \times \langle 0, T \rangle$, $\tau \neq t$ a

$$D_x^\nu I_0(x, t; \tau) = I_\nu(x, t; \tau) \tag{9}$$

pre $(x, t; \tau) \in (0, 1) \times (0, T\rangle \times \langle 0, T\rangle$, $t \neq \tau$.

b) $D_t I_0(x, t; \tau)$ je spojitá na $\langle 0, 1\rangle \times (0, T) \times \langle 0, T\rangle$, $\tau \neq t$ a

$$D_t I_0(x, t; \tau) = (-1)^{n+1} I_{2n}(x, t; \tau) \tag{10}$$

pre $(x, t; \tau) \in \Omega_0 \times \langle 0, T\rangle$, $\tau \neq t$.

c) $\lim_{t \to \tau^+} I_0(x, t; \tau) = f(x, \tau)$ a $\lim_{\tau \to t^-} I_0(x, t; \tau) = f(x, t)$ v akomkoľvek obdĺžniku $\langle a, b\rangle \times (0, T)$, kde $0 < a < b < 1$.

Lema 3. Nech

$$T_\nu(x, t) = \int_0^t \int_0^1 \Gamma_\nu(x, t; \xi, \tau) f(\xi, \tau) \, d\xi \, d\tau$$

Potom

a) Pre $f \in C_0(\bar{\Omega})$ a $\nu = 0, 1, 2, \ldots, 2n-1$ T_ν je spojitá na $\bar{\Omega}$ a

$$D_x^\nu T_0(x, t) = T_\nu(x, t) \quad \text{pre} \quad (x, t) \in (0, 1) \times \langle 0, T\rangle \tag{11}$$

b) Pre $f \in C_0(\bar{\Omega}) \cap S_0[x, (0, 1); t]$, $T_{2n}(x, t)$ je spojitá na $\bar{\Omega}$ a

$$D_x^{2n} T_0(x, t) = T_{2n}(x, t) \quad \text{pre} \quad (x, t) \in (0, 1) \times \langle 0, T) \tag{12}$$

c) Pre $f \in C_0(\bar{\Omega}) \cap S_0[x, (0, 1); t]$ derivácia $D_t T_0$ je spojitá a

$$D_t T_0(x,t)=f(x,t)-\int_0^t\int_0^1 \Gamma_{2n}(x,t;\xi,\tau)f(\xi,\tau)\,d\xi\,d\tau=$$
$$=f(x,t)+(-1)^{n+1}T_{2n}(x,t)\quad \text{pre}\quad (x,t)\in\Omega\,. \tag{13}$$

$\Gamma_0(x,t;\xi,\tau)$ je fundamentálne riešenie rovnice $L(u;x,t)=0$ na $\bar{\Omega}$. (14)

Lema 4. Nech

$$J_\nu(x,t;\xi)=\int_0^t f(\tau)\Gamma_\nu(x,t;\xi,\tau)\,d\tau \tag{15}$$

Potom

a) Pre ľubovoľnú funkciu $f\in C_0(\langle 0,1\rangle)$ a $\nu=0,1,2,\ldots$ je integrál J_ν spojitý na $\bar{\Omega}\times\langle 0,1\rangle$, $\xi\neq x$ a platí rovnosť

$$D_x^\nu J_0(x,t;\xi)=J_\nu(x,t;\xi)$$
$$\text{pre } (x,t;\xi)\in(0,1)\times\langle 0,T\rangle\times\langle 0,T\rangle\,,\qquad \xi\neq x \tag{16}$$

b) Pre funkciu $f\in C_0(\langle 0,T\rangle)$ je D_tJ_ν spojitá a platí:

$$D_tJ_\nu(x,t;\xi)=\int_0^t f(\tau)D_t\Gamma_\nu(x,t;\xi,\tau)\,d\tau=$$
$$=(-1)^{n+1}J_{\nu+2n}(x,t;\xi) \tag{17}$$
$$\text{pre } (x,t;\xi)\in\Omega\times\langle 0,1\rangle\,,\qquad \xi=x$$

Dôkaz týchto liem možno urobiť pomocou odhadov (6) a (7) ako v [1].

Lema 5. a) Pre ľubovoľný pevný bod $y\in(0,1\rangle$ a $(x,t)\in(-\infty,\infty)\times(0,T\rangle$ platí

$$p(x,y)-\int_0^y \Gamma_0(x,t;\xi,0)\,d\xi=$$
$$=(-1)^n\int_0^t D_\xi^{2n-1}\Gamma_0(x,t;y,\tau)\,dt-(-1)^n\int_0^t D_\xi^{2n-1}\Gamma_0(x,t;0,\tau)\,d\tau\,,$$

kde

$$p(x,y)=\begin{cases}1 & \text{ak}\quad 0<x<y\\ 1/2 & \text{ak}\quad x=0\quad\text{alebo}\quad x=y\\ 0 & \text{ak}\quad x<0\quad\text{alebo}\quad y<x\end{cases} \tag{18}$$

Nech $J_\nu(x,t;z)$ znamená integrál z lemy 4. Potom

b) Pre každé $t\in(0,T\rangle, z\in\langle 0,1)$ platí:

1. $\lim\limits_{x\to z^+} J_{2n-1}(x,t;z)\;=\;(-1)^n\dfrac{1}{2}f(t)$ ak $f\in S_0[t,(0,T\rangle]$ a

2. $\lim\limits_{x\to z^+} J_{2n+2k-1}(x, t; z)=0$ ak $f\in S_{2k}[t, (0, T\rangle]$
a $k=1, 2, \ldots, n-1$.

c) Pre každé $t\in(0, T\rangle$, $z\in(0, 1\rangle$ platí:

1. $\lim\limits_{x\to z^-} J_{2n-1}(x, t; z)=(-1)^{n+1}\frac{1}{2}f(t)$ ak $f\in S_0[t, (0, T\rangle]$ a

2. $\lim\limits_{x\to z^-} J_{2n+2k-1}(x, t; z)=0$ ak $f\in S_{2k}[t, (0, T\rangle]$ a
$k=1, 2, \ldots, n-1$.

d) Nech funkcia $f\in C_0(\langle 0, T\rangle)$, potom pre každé $t\in(0, T\rangle$ a $k=1, \ldots, n-1$ platí:

1. $\lim\limits_{x\to z^+} J_{2k-1}(x, t; z)=0$ keď $z\in\langle 0, 1)$ a

2. $\lim\limits_{x\to z^-} J_{2k-1}(x, t; z)=0$ keď $z\in(0, 1\rangle$.

Dôkaz. a) Pre $x\neq y$, $x\neq 0$ integrály v (18) konvergujú na základe odhadu (7) a pre $x=0$ alebo $x=y$ konvergencia vyplýva z platnosti identity $D_\xi^{2n-1}\Gamma_0(0, t; 0, \tau)=D_\xi^{2n-1}\Gamma_0(x, t; y, \tau)=0$. Nech $y\in(0, 1\rangle$ je pevný bod. Podľa lemy 1. a platí $D_\tau\Gamma_0(x, t; \xi, \tau)=(-1)^n D_\xi^{2n}\Gamma_0(x, t; \xi, \tau)$. Potom $\int_0^y D_\tau\Gamma_0(x, t; \xi, \tau)\,d\xi=(-1)^n\int_0^y D_\xi^{2n}\Gamma_0(x, t; \xi, \tau)\,d\xi=(-1)^n[D_\xi^{2n-1}\Gamma_0(x, t; y, \tau)-$ $-D_\xi^{2n-1}\Gamma_0(x, t; 0, \tau)]$, $t\neq\tau$. Túto rovnicu integrujeme vzhľadom na τ od 0 po $t-\varepsilon$, $\varepsilon>0$

$$(-1)^n\int_0^{t-\varepsilon}[D_\xi^{2n-1}\Gamma_0(x, t; y, \tau)-D_\xi^{2n-1}\Gamma_0(x, t; 0, \tau]\,d\tau=$$

$$=\int_0^{t-\varepsilon}\left[\int_0^y D_\tau\Gamma_0(x, t; \xi, \tau)\,d\xi\right]d\tau=\int_0^y\Gamma_0(x, t; \xi, t-\varepsilon)\,d\xi-\int_0^y\Gamma_0(x, t; \xi, 0)\,d\xi\,,$$

pre $(x, t)\in\bar{\Omega}$

Použitím transformácie $\dfrac{x-\xi}{\sqrt[2n]{\varepsilon}}=-z$ dostaneme

$$\int_0^y\Gamma_0(x, t; \xi, t-\varepsilon)\,d\xi=\int_{-w_1}^{w_2}\Gamma_0(z, 1; 0, 0)\,dz\,,$$

kde $w_1=\dfrac{x}{\sqrt[2n]{\varepsilon}}$, $w_2=\dfrac{y-x}{\sqrt[2n]{\varepsilon}}$

Z uvedeného a z (8), ak $\varepsilon \to 0^+$ dostaneme

$$U = \lim_{\varepsilon \to 0^+} \int_{-w_1}^{w_2} \Gamma_0(z, 1; 0, 0)\, dz = 1\,,$$

ak $0 < x < y$. Ak $x = 0$, potom $w_1 = 0$, $w_2 \to \infty$ a $U = \frac{1}{2}$. Ak $x = y$, potom $w_1 \to \infty$, $w_2 = 0$ a $U = \frac{1}{2}$. Ak $x < 0$, potom $w_1 \to -\infty$, $w_2 \to \infty$ a $U = 0$. Ak $y < x$, potom $w_1 \to \infty$, $w_2 \to -\infty$ a $U = 0$. Tým je vzťah (18) dokázaný.

1\. b. Funkciu $f(t)$ môžeme spojite rozšíriť na interval $\langle 0, T \rangle$.

$$J_{2n-1}(x, t; z) = \int_0^t f(\tau) \Gamma_{2n-1}(x, t; z, \tau)\, d\tau =$$

$$= -\int_0^t [f(t) - f(\tau)] \Gamma_{2n-1}(x, t; z, \tau)\, d\tau + f(t) \int_0^t \Gamma_{2n-1}(x, t; z, \tau)\, d\tau.$$

Položme $[f(t) - f(\tau)] \Gamma_{2n-1}(x, t; z, \tau) = H$.

Potom, pre $0 \leqq z < x < 1$, ak použijeme na H odhad (6), z predpokladu pre f máme

$$|H| \leqq K(t - \tau)^{\varepsilon - 1}\,, \qquad K > 0$$

Teda limita integrálu $\int_0^t H\, d\tau$ pre $x \to z^+$ sa rovná nule. Použitím vzťahu (18), pre $y = z \neq 0$ a identity $D_\xi^{2n-1} \Gamma_0(z, t; z, \tau) = 0$, dostaneme

$$\lim_{x \to z^+} \int_0^t \Gamma_{2n-1}(x, t; z, \tau)\, d\tau = -\lim_{x \to z^+} \int_0^t D_\xi^{2n-1} \Gamma_0(x, t; z, \tau)\, d\tau =$$

$$= \lim_{x \to z^+} \Big[-(-1)^n p(x, z) + (-1)^n \int_0^z \Gamma_0(x, t; \xi, 0)\, d\xi - \int_0^t D_\xi^{2n-1} \Gamma_0(x, t; 0, \tau)\, d\tau \Big] =$$

$$= (-1)^n \int_0^z \Gamma_0(z, t; \xi, 0)\, d\xi - \int_0^t D_\xi^{2n-1} \Gamma_0(z, t; 0, \tau)\, d\tau\,.$$

Opätovným použitím vzťahu (18) pre výraz

$$\int_0^t D_\xi^{2n-1} \Gamma_0(z, t; 0, \tau)\, d\tau$$

dostaneme

$$\lim_{x \to z^+} \int_0^t \Gamma_{2n-1}(x, t; z, \tau)\, d\tau =$$

$$= (-1)^n \int_0^z \Gamma_0(z, t; \xi, 0)\, d\xi + (-1)^n p(z, z) - (-1)^n \int_0^z \Gamma_0(z, t; \xi, 0)\, d\xi -$$

$$- \int_0^t D_\xi^{2n-1} \Gamma_0(z, t; z, \tau)\, d\tau = (-1)^n \frac{1}{2}.$$

Ak $z = 0$, potom podľa (18) platí:

$$\lim_{x \to 0^+} \int_0^t \Gamma_{2n-1}(x, t; 0, \tau)\, d\tau = -\lim_{x \to 0^+} \int_0^t D_\xi^{2n-1} \Gamma_0(x, t; 0, \tau)\, d\tau =$$

$$= \lim_{x \to 0^+} \left\{ (-1)^n p(x, y) - (-1)^n \int_0^y \Gamma_0(x, t; \xi, 0)\, d\xi - \int_0^t D_\xi^{2n-1} \Gamma_0(x, t; y, \tau)\, d\tau \right\} =$$

$$= (-1)^n - (-1)^n \int_0^y \Gamma_0(0, t; \xi, 0)\, d\xi - \int_0^t D_\xi^{2n-1} \Gamma_0(0, t; y, \tau)\, d\tau =$$

$$= (-1)^n - (-1)^n \frac{1}{2}.$$

Z predchádzajúcich úvah vyplýva tvrdenie 1. b.

2. b.

$$J_{2n+2k-1}(x, t; z) =$$

$$= -\int_0^t [f(t) - f(\tau)] \Gamma_{2n+2k-1}(x, t; z, \tau)\, d\tau + f(t) \int_0^t \Gamma_{2n+2k-1}(x, t; z, \tau)\, d\tau.$$

Položme $[f(t) - f(\tau)] \Gamma_{2n+2k-1}(x, t, z, \tau) = Q$. Potom pre $0 \leqq z < x < 1$ na intervale $0 \leqq \tau < t \leqq T$ z použitia odhadu (6) platí

$$|Q| \leqq |f(t) - f(\tau)|\, K_1 (t - \tau)^{-\frac{n+k}{n}}$$

Podľa predpokladu na funkciu f je $|Q| \leqq K(t-\tau)^{\varepsilon - 1}$, odkiaľ $\lim_{x \to z^+} \int_0^t Q\, d\tau = 0$.
Ďalej vyšetrime limitu druhého člena predchádzajúcej rovnosti:

$$\lim_{x \to z^+} f(t) \int_0^t \Gamma_{2n+2k-1}(x, t; z, \tau)\, d\tau = \lim_{x \to z^+} f(t) \int_0^t D_x^{2n+2k-1} \Gamma_0(x, t; z, \tau)\, d\tau =$$

$$= \lim_{x \to z^+} f(t) \int_0^t (-1)^n D_\tau [D_x^{2k-1} \Gamma_0(x, t; z, \tau)]\, d\tau =$$

$$= (-1)^{n+1} \lim_{x \to z^+} f(t) D_x^{2k-1} \Gamma_0(x, t; z, 0) = 0.$$

Tvrdenia 1. c a 2. c sa dokážu analogicky ako tvrdenia 1. b a 2. b.

1. d. Použitím odhadu (6) dostaneme

$$|J_{2k-1}(x,t;z)| \leqq$$

$$\leqq \int_0^t |f(\tau)|\, C_1(2k-1)\,(t-\tau)^{-\frac{1+2k-1}{2n}} \,.\, \exp\left\{-C_2\left[\frac{(x-z)^{2n}}{t-\tau}\right]^{\frac{1}{2n-1}}\right\} \mathrm{d}\tau \leqq$$

$$\leqq K\,.\,C_1(2k-1)\int_0^t (t-\tau)^{-\frac{k}{n}}\,\mathrm{d}\tau < \infty,$$

čo znamená, že integrál konverguje pre každé $k = 1, 2, \ldots, n-1$ a pre každé $x \neq z$ z danej množiny. Teda prechodom s limitou za integrál dostaneme

$$\lim_{x\to z^+}\int_0^t f(\tau)\Gamma_{2k-1}(x,t;z,\tau) = \int_0^t \lim_{x\to z^+} f(\tau)\Gamma_{2k-1}(x,t;z,\tau)\,\mathrm{d}\tau = 0$$

Tvrdenie 2. d dokážeme analogicky ako 1. d.

4. Greenova funkcia

Veta 1. Funkcia

$$G_j(x,t;\xi,\tau) = \sum_{k=-\infty}^{\infty} [\Gamma_0(x,t;\xi+2k,\tau) + (-1)^j\Gamma_0(x,t;-\xi+2k,\tau)] \tag{19j}$$

pre $j = 1, 2$ a jej derivácie $D_\xi^\nu G_j$ pre $\nu = 1, 2, \ldots$ a $(x,t) \neq (\xi,\tau)$, $(x,t;\xi,\tau) \in \bar{\Omega}\times\bar{\Omega}$ sú spojité. Funkcia G_j je Greenova funkcia úlohy (1), (2), (3j).

Dôkaz. Vyšetrime konvergenciu radu

$$u_0^{(j)}(x,t;\xi,\tau) = (-1)^j\Gamma_0(x,t;-\xi,\tau) + \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} [\Gamma_0(x,t;\xi+2k,\tau) + (-1)^j\Gamma_0(x,t;-\xi+2k,\tau)] \tag{20j}$$

Môžeme teda zapísať $u_0^{(j)}(x,t;\xi,\tau) = G_j(x,t;\xi,\tau) - \Gamma_0(x,t;\xi,\tau)$. Pre $0 \leqq x \leqq 1$, $0 \leqq \xi \leqq 1$ a $k = \pm 1, \pm 2, \pm 3, \ldots$ je $|x \mp \xi - 2k| \geqq 2|k| - |\pm\xi - x| > 2|k| - 2$.

Z odhadu (6) dostaneme

$$|D_x^\nu \Gamma_0(x,t;\pm\xi+2k,\tau)| \leqq$$

$$\leqq C_1(\nu)\,(t-\tau)^{\frac{-1+\nu}{2n}} \exp\left\{-2^{\frac{2n}{2n-1}} C_2\left[\frac{(|k|-1)^{2n}}{t-\tau}\right]^{\frac{1}{2n-1}}\right\}$$

$$\text{pre} \quad \nu = 0, 1, \ldots \quad \text{a} \quad \tau < t .$$

Číselný rad

$$4C_1(\nu)\alpha^{-\frac{1+\nu}{2n}} \sum_{l=1}^{\infty} \exp\left\{-2^{\frac{2n}{2n-1}} C_2 T^{\frac{-1}{2n-1}} (l-1)^{\frac{2n}{2n-1}}\right\}$$

je konvergentná majoranta pre

$$\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} [D_x^\nu \Gamma_0(x, t; \xi + 2k, \tau) + (-1)^j D_x^\nu \Gamma_0(x, t; -\xi + 2k, \tau)]$$

$$j = 1, 2 \cdot \text{ ak } \quad 0 < \alpha < t - \tau \leqq T \tag{21}$$

Tým sme ukázali, že funkcia $D_x^\nu u_0^{(j)}$ je spojitá na $[(0, 1\rangle \times (0, T] \times \times [\langle 0, 1\rangle \times \langle 0, T\rangle]$ pre $\tau < t$. Pretože $D_\xi^\nu \Gamma_0(x, t; -\xi, \tau)$ je spojitá pre $(x, t) \neq \neq (\xi, \tau)$ na $\bar{\Omega} \times \bar{\Omega}$, nasledujúca majoranta radu (21) pre $k \neq \pm 1$

$$s(t, \tau) = 4C_1(\nu)\,(t-\tau)^{-\frac{1+\nu}{2n}} \sum_{l=2}^{\infty} \exp\left\{-2^{\frac{2n}{2n-1}} C_2(t-\tau)^{\frac{-1}{2n-1}} (l-1)\right\} =$$

$$= 4C_1(\nu)\,(t-\tau)^{-\frac{1+\nu}{2n}} \cdot \frac{q}{1-q},$$

kde $q = \exp\left\{-2^{\frac{2n}{2n-1}} C_2(t-\tau)^{\frac{-1}{2n-1}}\right\}$

má limitu rovnajúcu sa nule, ak $t \to \tau^+$. Teda všetky derivácie $D_\xi^\nu u_0^{(j)}$ sú spojité na $\Omega \times \bar{\Omega}$. Vzhľadom na lemu 1. a a rozklad $G_j = u_0^{(j)} + \Gamma_0$ je aj $D_\xi^\nu G_j$ na $\bar{\Omega} \times \bar{\Omega}$ spojitá, ak $(x, t) \neq (\xi, \tau)$ pre $\nu = 0, 1, 2, \ldots$ čím je prvá časť vety dokázaná.

Z predchádzajúcich úvah vyplýva, že rovnica $L(u_0^{(j)}; x, t) = 0$ nie je splnená iba v bodoch $(0, t; 0, t)$, $(1, t; 1, t)$ z $\bar{\Omega} \times \bar{\Omega}$ a vlastnosť a Greenovej funkcie je splnená. Ak x alebo ξ sú z otvoreného intervalu $(0, 1)$, potom z odhadu (6) dostaneme vlastnosť b, $\lim_{t \to \tau^+} u_0^{(j)}(x, t; \xi, \tau) = 0$. Vlastnosť c vyplýva z identity

$$D_x^\nu G_j(x, t; \xi, \tau) = \frac{(-i)^\nu}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \varrho^\nu \{[\cos \varrho(x - \xi - 2k) +$$

$$+(-1)^j \cos \varrho(x+\xi-2k)]-$$

$$-\mathrm{i}[\sin \varrho(x-\xi-2k)+(-1)^j \sin \varrho(x+\xi-2k)]\} \exp\{-\varrho^{2n}(t-\tau)\}\, \mathrm{d}\varrho \tag{22}$$

ktorú obdržíme derivovaním výrazu (19j) a použitím Eulerovej identity. Dôkaz vety je urobený.

Poznámka 1. Na množine $\bar{\Omega}\times\bar{\Omega}$ Greenova funkcia G_j obsahuje tri singulárne členy $\Gamma_0(x, t; \xi, \tau)$, $\Gamma_0(x, t; -\xi, \tau)$, $\Gamma_0(x, t; -\Gamma+2, \tau)$ a funkcia $u_0^{(j)}$ je spojitá na $\bar{\Omega}\times\Omega$ a $\Omega\times\bar{\Omega}$.

Veta 2. Nech $G_j(x, t; \xi, \tau)$ je Greenova funkcia z (19j). Potom pre $u\in N_{2n}(\bar{\Omega})$ platí nasledujúca identita:

$$u(x, t)=(-1)^n \sum_{k=0}^{2n-1}(-1)^k \int_0^t D_\xi^k u(1, \tau) D_\xi^{2n-k-1} G_j(x, t; 1, \tau)\, \mathrm{d}\tau-$$

$$-(-1)^n \sum_{k=0}^{2n-1}(-1)^k \int_0^t D_\xi^k u(0, \tau) D_\xi^{2n-k-1} G_j(x, t; 0, \tau)\, \mathrm{d}\tau+$$

$$+\int_0^1 u(\xi, 0) G_j(x, t; \xi, 0)\, \mathrm{d}\xi+ \tag{23}$$

$$+(-1)^n \int_0^t \int_0^1 G_j(x, t; \xi, \tau) L(u; \xi, \tau)\, \mathrm{d}\xi\, \mathrm{d}\tau \quad \text{na } \Omega$$

Dôkaz. Vzhľadom na rozklad $G_j=u_0^{(j)}+\Gamma_0$ a lemy 2. a, 3. a, 4. a sú všetky integrály v (23) spojité funkcie na Ω. Nech $u(x, t)$ a $v(x, t)$ sú ľubovoľné funkcie z $N_{2n}(\bar{\Omega})$. Uvažujme identitu

$$uM(v; \xi, \tau)-vL(u; \xi, \tau)=$$

$$=D_\xi\left[\sum_{k=0}^{2n-1}(-1)^k D_\xi^k u D_\xi^{2n-k-1} v\right]-(-1)^n D_\tau(u \,.\, v)$$

kde $M(v; \xi, \tau)=D_\xi^{2n} v-(-1)^n D_\tau v$.

Po integrácii na Ω a použití Greenovej formuly dostaneme

$$\iint_\Omega [uM(v; \xi, \tau)-vL(u; \xi, \tau)]\, \mathrm{d}\xi\, \mathrm{d}\tau=$$

$$=\int_{\partial\Omega}\left[\sum_{k=0}^{2n-1}(-1)^k D_\xi^k u D_\xi^{2n-k-1} v\, \mathrm{d}\tau+(-1)^n u \,.\, v\, \mathrm{d}\xi\right]$$

kde $\partial\Omega$ je hranica oblasti Ω. Uvažujme kladne orientovaný obdĺžnik $\Omega' = (0\leqq \xi\leqq 1) \times (0 \leqq \tau \leqq t-\varepsilon)$, $\varepsilon>0$ s vrcholmi v bodoch $A_1(0, 0)$, $A_2(1, 0)$, $M_1(0, t-\varepsilon)$ a $M_2(1, t-\varepsilon)$. V tomto obdĺžniku položme $v(\xi, \tau) = G_j(x, t; \xi, \tau)$. Z Greenovej formule po jednoduchej úprave dostaneme

$$-\iint_{\Omega'} G_j(x,t;\xi,\tau)L(u;\xi,\tau)\,d\xi\,d\tau =$$

$$\sum_{k=0}^{2n-1}(-1)^k\int_0^{t-\varepsilon} D_\xi^k u(1,\tau)D_\xi^{2n-k-1}G_j(x,t;1,\tau)\,d\tau -$$

$$-\sum_{k=0}^{2n-1}(-1)^k\int_0^{t-\varepsilon} D_\xi^k u(0,\tau)D_\xi^{2n-k-1}G_j(x,t;0,\tau)\,d\tau +$$

$$+(-1)^n\int_0^1 u(\xi,0)G_j(x,t;\xi,0)\,d\xi -$$

$$-(-1)^n\int_0^1 u(\xi,t-\varepsilon)\,[\Gamma_0(x,t;\xi,t-\varepsilon)+u_0^{(j)}(x,t;t-\varepsilon)]\,d\xi\,. \tag{24}$$

Označme $I(\varepsilon)=\int_0^1 u(\xi,t-\varepsilon)\Gamma_0(x,t;\xi,t-\varepsilon)\,d\xi$. Transformáciou $-z=$ $=(x-\xi)\varepsilon^{\frac{-1}{2n}}$ dostaneme

$$I(\varepsilon)=\int_{-w_1}^{w_2} u(x+z\sqrt[2n]{\varepsilon},t-\varepsilon)\Gamma_0(z,1;0,0)\,dz$$

kde $(x,t)\in\Gamma_0$, $\omega_1=\frac{x}{\sqrt[2n]{\varepsilon}}$ a $\omega_2=\frac{1-x}{\sqrt[2n]{\varepsilon}}$. Vzhľadom na vetu o strednej hodnote možno písať

$$I(\varepsilon)-u(x,t)=\int_{-w_1}^{w_2} u(x+z\sqrt[2n]{\varepsilon},t-\varepsilon)\Gamma_0(z,1;0,0)\,dz -$$

$$-u(x,t)\int_{-\infty}^{\infty}\Gamma_0(z,1;0,0)\,dz =$$

$$=\int_{-w_1}^{w_2}[u(x+z\sqrt[2n]{\varepsilon},t-\varepsilon)-u(x,t)]\Gamma_0(z,1;0,0)\,dz -$$

$$-u(x,t)\left(\int_{-\infty}^{\infty}-\int_{-w_1}^{w_2}\right)\Gamma_0(z,1;0,0)\,dz =$$

$$=\sqrt[2n]{\varepsilon}\int_{-w_1}^{w_2} zD_\xi u(x+\Theta z\sqrt[2n]{\varepsilon},t-\Theta\varepsilon)\Gamma_0(z,1;0,0)\,dz -$$

$$-\varepsilon\int_{-w_1}^{w_2} D_\tau u(x+\Theta z\sqrt[2n]{\varepsilon},t-\Theta\varepsilon)\Gamma_0(z,1;0,0)\,dz -$$

$$-u(x, t)\left(\int_{-\infty}^{\infty} - \int_{-w_1}^{w_2}\right) \Gamma_0(z, 1; 0, 0)\, dz\,,$$

kde $0<\Theta<1$.

Ak označíme $N=\max(|u|, |D_\xi u|, |D_\tau u|)$ ($\bar{\Omega}$ je kompaktná množina, na ktorej existuje maximum), potom

$$|I(\varepsilon)-u(x, t)|<N \sqrt[2n]{\varepsilon}\int_{-\infty}^{\infty} |z\Gamma_0(z, 1; 0, 0)|\, dz+\varepsilon N\int_{-\infty}^{\infty} |\Gamma_0(z, 1; 0, 0)|\, dz+$$

$$+N\left|\left(\int_{-\infty}^{-w_1}+\int_{w_2}^{\infty}\right)\Gamma_0(z, 1; 0, 0)\, dz\right|.$$

(Posledné integrály konvergujú na základe odhadu; (6)). Potom limita pravej strany pre $\varepsilon\to 0^+$ je nulová a zo vzťahu (24) pre $\varepsilon\to 0^+$ dostaneme tvrdenie vety.

Poznámka 2. Ak namiesto funkcie G_j vo vzorci (23) dáme funkciu $H(x, t; \xi, \tau) = \Gamma_0+h$, kde h má nasledujúce vlastnosti:

a) $L(h; x, t)=0$ pre $t>\tau$,

b) $\lim_{x\to\tau^+} h(x, t; \xi, \tau)=0$ pre $(x; \xi, \tau)\in\langle 0, 1\rangle\times\bar{\Omega}$, ak aspoň jeden z bodov x alebo ξ leží v otvorenom intervale $(0, 1)$, potom tento vzorec zostane platný.

5. Riešenie okrajovej úlohy

Nasledujúca veta dáva formulu pre explicitné vyjadrenie riešenia úlohy (1), (2), (3j).

Veta 3. Nech pravá strana v (1), $\varphi(x, t)\in C_0(\bar{\Omega})\cap S_0[x, (0, 1); t]$, $g(x)\in C_0(\langle 0, 1\rangle)$ a okrajové funkcie $a_{2s, j}(t)\in S_{2n-2s-2}[t, (0, T\rangle]$ a $b_{2s, j}(t)\in S_{2n-2s-2}[t, (0, T\rangle]$, pre $s=0, 1, \ldots, n-1$, $j=1, 2$. Potom funkcia

$$u_j(x, t)=(-1)^{n+j}\int_0^t\left[\sum_{l=1}^{n} a_{2n-2l, j}(\tau)D_\xi^{2l-j}G_j(x, t; 0, \tau)\right]d\tau+$$

$$+(-1)^{n+j+1}\int_0^t\left[\sum_{l=1}^{n} b_{2n-2l, j}(\tau)D_\xi^{2l-j}G_j(x, t; 1, \tau\right]d\tau+ \qquad (25j)$$

$$+\int_0^1 g(\xi)G_j(x, t; \xi, 0)\, d\xi+(-1)^n\int_0^t\int_0^1 G_j(x, t; \xi, \tau)\varphi(\xi, \tau)\, d\xi\, d\tau$$

$j=1, 2$ je riešením úlohy (1), (2), (3j).

Dôkaz. Na základe predpokladu existuje spojité rozšírenie funkcií $a_{2n-2l, j}(t)$ a $b_{2n-2l, j}(t)$ pre $l=1, 2, \ldots, n$, $j=1, 2$ na uzavretý interval $\langle 0, T\rangle$. Funkcie

$D_\xi^\nu \Gamma_0(x, t; -\xi, \tau)$ a $D_\xi^\nu \Gamma_0(x, t; \pm\xi + 2k, \tau)$, pre $k = \pm 1, \pm 2, \pm 3, \ldots$ a $\nu = 0, 1, 2, \ldots$ majú spojité derivácie podľa x a t na $\Omega \times \bar{\Omega}$ ľubovoľného rádu a spĺňajú rovnicu $L(u; x, t) = 0$ na Ω_0 pre každý pevný bod $(\xi, \tau) \in \bar{\Omega}$. Podľa formúl (10), (13), (17) a derivácii fundamentálnych riešení Γ_o, derivácie $D_\xi^\nu \Gamma_0(x, t; \xi, \tau)$ spĺňajú tiež rovnicu (1) na Ω. Z uvedeného tvrdenia a z rozkladu $D_\xi^\nu G_j = D_\xi^\nu \Gamma_0 + D_\xi^\nu u_0^j$ vyplýva, že $u_j(x, t)$ spĺňa rovnicu (1) pre $j = 1, 2$. Nech $t \to 0^+$ v (25j), potom podmienku (2) dostaneme z lemy 2. c, 3. a, 4. a prechodom limity pod integrál

$$\lim_{t \to 0^+} u_1(x, t) = \lim_{t \to 0^+} \int_0^1 g(\xi) G_1(x, t; \xi, 0)\, d\xi =$$

$$= \lim_{t \to 0^+} \int_0^1 g(\xi) \Gamma_0(x, t; \xi, o d\xi - \lim_{t \to 0^+} \int_0^1 g(\xi) \Gamma_0(x, t; -\xi, 0)\, d\xi +$$

$$+ \lim_{t \to 0^+} \int_0^1 g(\xi) \Big[\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} (\Gamma_0(x, t; \xi + 2k, 0) - \Gamma_0(x, t; -\xi + 2k, 0) \Big] d\xi = g(x)\,.$$

Analogicky sa ukáže, že $u_2(x, t)$ spĺňa podmienku (2).

Skôr, ako dokážeme, že (25_j) spĺňa (3_j), $j = 1, 2$ upravme (19_j). Derivovaním podľa x a ξ (prechod derivácie za sumačný znak je možný, lebo rad konverguje rovnomerne spolu s deriváciami — pozri dôkaz vety 1) obdržíme

$$D_x^\nu D_\xi^{\nu_1} G_j(x, t; \xi, \tau) =$$

$$= \sum_{k=-\infty}^{\infty} \Big[\frac{(-i)^\nu (i)^{\nu_1}}{2\pi} \int_{-\infty}^{\infty} \varrho^{\nu+\nu_1} \exp\{-i\varrho(x - \xi - 2k) - \varrho^{2n}(t - \tau)\}\, d\varrho +$$

$$+ (-1)^j \frac{(-i)^\nu (i)^{\nu_1}}{2\pi} \int_{-\infty}^{\infty} \varrho^{\nu_1+\nu} \exp\{-i\varrho(x + \xi - 2k) - \varrho^{2n}(t - \tau)\}\, d\varrho \Big]$$

a použitím Eulerovho vzorca dostaneme

$$D_x^\nu D_\xi^{\nu_1} G_j(x, t; \xi, \tau) =$$

$$= \frac{(-i)^{\nu+\nu_1}}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \varrho^{\nu+\nu_1} \{[(-1)^{\nu_1} \cos \varrho(x - \xi - 2k) + (-1)^j \cos \varrho(x + \xi - 2k) - \tag{26}$$

$$- i[(-1)^\nu \sin \varrho(x - \xi - 2k) + (-1)^j \sin \varrho(x + \xi - 2k)]\} \exp\{-\varrho^{2n}(t - \tau)\}\, d\varrho$$

Nech $j = 1$

1. Dokážme, že $\lim_{x \to 0^+} D_x^{2s} u_1(x, t) = (-1)^{n+1} \lim_{x \to 0^+} \int_0^t \sum_{l=1}^{n} a_{2n-2l,1}(\tau)$.

$$D_x^{2s} D_\xi^{2l-1} G_1(x, t; 0, \tau)\, d\tau + (-1)^n \lim_{x \to 0^+} \int_0^t \sum_{l=1}^{n} b_{2n-2l,1}(\tau) D_x^{2s} D_\xi^{2l-1} G_1(x, t; 1, \tau)\, d\tau +$$

$$+ \lim_{x \to 0^+} \int_0^1 g(\xi) D_x^{2s} G_1(x, t; \xi, 0)\, d\xi +$$

$$+(-1)^n \lim_{x\to 0^+}\int_0^t\int_0^1 D_x^{2s}G_1(x,t;\xi,\tau)\varphi(\xi,\tau)\,d\xi\,d\tau = a_{2s,1}(t)$$

pre $s=0, 1, \dots, n-1$.

Dôkaz urobíme tak, že budeme vyšetrovať hodnoty jednotlivých členov v poslednej rovnici.

1. a. Vzhľadom na (19j)

$$D_x^{2s}D_\xi^{2l-1}G_1(x,t;0,\tau) = -2\sum_{k=-\infty}^{\infty}\Gamma_{2s+2l-1}(x,t;2k,\tau)\,.$$

Ak $2s+2l-1=2n-1$, potom podľa lemy 5. b. 1 je

$$(-1)^n\, 2\lim_{x\to 0^+}\int_0^t a_{2n-2l,1}(\tau)\Gamma_{2s+2l-1}(x,t;0,\tau)\,d\tau = a_{2s,1}(t)$$

Ak $1\leqq 2s+2l-1<2n-1$, resp. $2n-1 < 2s+2l-1 \leqq 4n-3$, potom podľa lemy 5. d. 1 resp. lemy 5. c. 1 platí

$$\lim_{x\to 0^+}\int_0^t a_{2n-2l,1}(\tau)\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\Gamma_{2s+2l-1}(x,t;2k,\tau)\,d\tau = 0$$

1. b. Vzhľadom na lemu 4. a

$$\lim_{x\to 0^+}\int_0^t b_{2n-2l,1}(\tau)D_x^{2s}D_\xi^{2l-1}G_1(x,t;1,\tau)\,d\tau =$$

$$=\int_0^t b_{2n-2l,1}(\tau)D_x^{2s}D_\xi^{2l-1}G_1(0,t;1,\tau)\,d\tau$$

Použitím vzťahu (26) dostaneme

$$D_x^{2s}D_\xi^{2l-1}G_1(0,t;1,\tau)=$$

$$=\frac{(-i)^{2s+2l-1}}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\varrho^{2s+2l-1}\{[-\cos\varrho(-1-2k)-\cos\varrho(1-2k)]-$$

$$-i[-\sin\varrho(-1-2k)-\sin\varrho(1-2k)]\}\exp\{-\varrho^{2n}(t-\tau)\}\,d\varrho$$

Prvá časť poslednej rovnice má nulovú hodnotu, lebo je to integrál nepárnej funkcie na symetrickom intervale. Druhá časť poslednej rovnice je funkčný rad, ktorého súčet sa rovná nule. Teda daná limita sa rovná 0 pre $t\in(0, T\rangle$.

1. c. Ďalej máme dokázať, že $\lim_{x\to 0^+}\int_0^1 g(\xi)D_x^{2s}G_1(x,t;\ \xi,0)\,d\xi = 0$. Podľa lemy 2. a $\lim_{x\to 0^+}\int_0^1 g(\xi)D_x^{2s}G_1(x,t;\xi,0)\,d\xi = \int_0^1 g(\xi)D_x^{2s}G_1(0,t;\xi,0)\,d\xi$ a z (26)

derivácia

$$D_x^{2s}G_1(0,t;\xi,0)=\frac{(-i)^{2s}}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\varrho^{2s}\{[\cos\varrho(-\xi-2k)]-$$

$$-\cos\varrho(\xi-2k)-i(\sin\varrho(-\xi-2k)-\sin\varrho(\xi-2k)]\}\exp\{-\varrho^{2n}(t-\tau)\}\,d\varrho$$

sa rovná nule.

1. d. Podľa lemy 3. a je

$$\lim_{x\to0^+}\int_0^t\int_0^1 D_x^{2s}G_1(x,t;\xi,\tau)\varphi(\xi,\tau)\,d\xi\,d\tau=$$

$$=\int_0^t\int_0^1 D_x^{2s}G_1(0,t;\xi,\tau)\varphi(\xi,\tau)\,d\xi\,d\tau=0$$

lebo výraz $D_x^{2s}G_1(0,t;\xi,\tau)=0$ z tých istých dôvodov ako v 1. c.

2. Ďalej dokážeme, že

$$\lim_{x\to1^-}D_x^{2s}u_1(x,t)=$$

$$=(-1)^{n+1}\lim_{x\to1^-}\int_0^t\sum_{l=1}^n a_{2n-2l,1}(\tau)$$

$$D_x^{2s}D_\xi^{2l-1}G_1(x,t;0,\tau)\,d\tau+$$

$$+(-1)^n\lim_{x\to1^-}\int_0^t\sum_{l=1}^n b_{2n-2l,1}(\tau)D_x^{2s}D_\xi^{2l-1}G_1(x,t;1,\tau)\,d\tau+$$

$$+\lim_{x\to1^-}\int_0^1 g(\xi)D_x^{2s}G_1(x,t;\xi,0)\,d\xi+$$

$$+(-1)^n\lim_{x\to1^-}\int_0^t\int_0^1 D_x^{2s}G_1(x,t;\xi,\tau)\varphi(\xi,\tau)\,d\xi\,d\tau=b_{2s,1}(t)\,,$$

pre $s=0,1,\dots,n-1$.

Stačí ak vyšetríme hodnoty jednotlivých limít v poslednej rovnici.

2. a. Dokážme, že

$$\lim_{x\to1^-}\int_0^t\sum_{l=1}^n a_{2n-2l,1}(\tau)D_x^{2s}D_\xi^{2l-1}G_1(x,t;0,\tau)\,d\tau=0$$

Skutočne podľa lemy 4.a

$$\lim_{x\to1^-}\int_0^t a_{2n-2l,1}(\tau)D_x^{2s}D_\xi^{2l-1}G_1(x,t;0,\tau)\,d\tau=$$

$$=\int_0^t a_{2n-2l,1}(\tau)D_x^{2s}D_\xi^{2l-1}G_1(1,t;0,\tau)\,d\tau$$

a zo vzťahu (26)

$$D_x^{2s} D_\xi^{2l-1} G_1(1, t; 0, \tau) =$$

$$= \frac{(-i)^{2s+2l-1}}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \varrho^{2s+2l-1} \{[-\cos \varrho(1-2k) - \cos \varrho(1-2k)] -$$

$$- i[-\sin \varrho(1-2k) - \sin \varrho(1-2k)]\} \exp\{-\varrho^{2n}(t-\tau)\} \, d\varrho = 0$$

z tých istých dôvodov ako v časti 1. b.

2. b. Vzhľadom na (19_j)

$$D_x^{2s} D_\xi^{2l-1} G_1(x, t; 1, \tau) =$$

$$= \sum_{k=-\infty}^{\infty} [-\Gamma_{2s+2l-1}(x, t; 1+2k, \tau) - \Gamma_{2s+2l-1}(x, t; -1+2k, \tau)] .$$

Odtiaľ dostaneme

$$(-1)^n \lim_{x \to 1^-} \int_0^t b_{2n-2l,1}(\tau) D_x^{2s} D_\xi^{2l-1} G_1(x, t; 1, \tau) \, d\tau =$$

$$= (-1)^{n+1} 2 \lim_{x \to 1^-} \int_0^t b_{2n-2l,1}(\tau) \Gamma_{2s+2l-1}(x, t; 1, \tau) \, d\tau +$$

$$+ (-1)^{n+1} \lim_{x \to 1^-} \int_0^t b_{2n-2l,1}(\tau) \Bigg[\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \Gamma_{2s+2l-1}(x, t; 1+2k, \tau) +$$

$$+ \sum_{\substack{k=-\infty \\ k \neq 1}}^{\infty} \Gamma_{2s+2l-1}(x, t; -1+2k, \tau) \Bigg] d\tau .$$

Ďalej vidíme, že

$$A = 2 \lim_{x \to 1^-} \int_0^t b_{2n-2l,1}(\tau) \Gamma_{2s+2l-1}(x, t; 1, \tau) \, d\tau = b_{2s,1}(t)$$

ak $2s + 2l - 1 = 2n - 1$ podľa lemy 5. c. 1. $A = 0$, ak $1 \leqq 2s + 2l - 1 < 2n - 1$, resp. $2n - 1 < 2s + 2l - 1 \leqq 4n - 3$ podľa lemy 5. d. 2, resp. 5. c. 2.

$$\lim_{x \to 1^-} \int_0^t b_{2n-2l,1}(\tau) \Bigg[\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \Gamma_{2s+2l-1}(x, t; 1+2k, \tau) +$$

$$+ \sum_{\substack{k=-\infty \\ k \neq 1}}^{\infty} \Gamma_{2s+2l-1}(x, t; -1+2k, \tau) \Bigg] d\tau = 0 ,$$

lebo výrazy $\Gamma_{2s+2l-1}$ tvoria integrály nepárnej funkcie na symetrickom intervale.

2. c. Podľa lemy 2. a

$$\lim_{x\to 1^-}\int_0^1 g(\xi)D_x^{2s}G_1(x, t; \xi, 0)\,d\xi = \int_0^1 g(\xi)D_x^{2s}G_1(1, t; \xi, 0)\,d\xi$$

Vzhľadom na (26) výraz $D_x^{2s}G_1(1, t; \xi, 0)$ nadobúda nulovú hodnotu ako v časti 1. c. Teda

$$\lim_{x\to 1^-}\int_0^1 g(\xi)D_x^{2s}G_1(1, t; \xi, 0)\,d\xi = 0 \quad \text{pre } s = 0, 1, \dots, n-1 \text{ a } t \neq 0 .$$

2. d: Záverom vidíme, že

$$\lim_{x\to 1^-}\int_0^t\int_0^1 D_x^{2s}G_1(x, t; \xi, \tau)\varphi(\xi, \tau)\,d\xi\,d\tau = 0 ,$$

lebo výraz $D_x^{2s}G_1(1, t; \xi, \tau)$ možno upraviť tak, ako v časti 2. c. Tým je dôkaz vety pre $j = 1$ skončený. Dôkaz vety pre $j = 2$ sa vykoná analogicky.

Tým sme dokázali, že (25_j) spĺňa (1), (2), (3_j) pre $j = 1, 2$ a teda je riešením danej úlohy.

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SUMMARY

ON SOLVING THE BOUNDARY VALUE PROBLEM IN LINEAR PARABOLIC DIFFERENTIAL EQUATIONS OF HIGHER ORDERS

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The article deals with solving the boundary value problem (1), (2), (3j) for linear parabolic equations of an even order in the rectangle $\langle 0, 1\rangle \times \langle 0, T\rangle$. The main result is in the explicit representation of the solution of this problem by the metod of Green's functions.

РЕЗЮМЕ

О РЕШЕНИИ КРАЕВОЙ ЗАДАЧИ ДЛЯ ЛИНЕЙНЫХ ПАРАБОЛИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВЫСШИХ ПОРЯДКОВ

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Работа посвящена решению краевой задачи (1), (2), (3j) для линейных дифференциальных параболических уравнений четного порядка в прямоугольнике $\langle 0, 1 \rangle \times \langle 0, T \rangle$. Главным результатом является явное выражение этой задачи при помощи метода функции Грина.

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VÝPOČET STREDNEJ HODNOTY POKUSOV $E_\Theta(n)$ GAMA-ROZDELENIA VYUŽITÍM SEKVENČNÉHO KRITÉRIA POMERU PRAVDEPODOBNOSTI V PRÍPADE, ŽE $E_\Theta(Z)=0$

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1. Úvod

Je známe, že použitím sekvenčného kritéria pomeru pravdepodobnosti potrebujeme na dosiahnutie výsledku podstatne menej pozorovaní, ako pri použití klasického Neymanovho—Pearsonovho testu. Wald [1] uviedol vzťahy pre strednú hodnotu pokusov $E_\Theta(n)$ v prípade, že náhodný výber sa vykoná z normálneho rozdelenia pravdepodobnosti $N(\Theta; \sigma^2)$, kde disperzia σ^2 je známa. Testoval hypotézu H_0: $\Theta = \Theta_0$ proti alternatíve H_1: $\Theta = \Theta_1$. V tejto poznámke je uvedený výsledok, ktorý zodpovedá gama-rozdeleniu pravdepodobnosti Ga(x, r, σ), definovanému hustotou

$$f(x, r, \sigma) = \frac{1}{\sigma^r \Gamma(r)} e^{-(x/\sigma)} x^{r-1} \qquad (0 < x < \infty; r > 0; \sigma > 0) \tag{1}$$

Testujeme hypotézu

$$H_0: \sigma = \sigma_0$$

proti

$$H_1: \sigma = \sigma_1 \qquad (\sigma_0 < \sigma_1) \tag{2}$$

za predpokladu, že parameter r je známy.

2. Sekvenčné kritérium pomeru pravdepodobnosti

Hypotéza (2) sa testuje tak, že vyšetrujeme náhodnú premennú

$$Z_k = \sum_{i=1}^{k} z_i \tag{3}$$

kde

$$z_i = \ln \frac{f(x_i, \sigma_1)}{f(x_i, \sigma_0)} \qquad (i = 1, 2, \ldots)$$

Ak $\ln B < Z_k < \ln A$ $(k = 1, 2, \ldots, n)$, treba urobiť $(n+1)$. pozorovanie. Ak pre $k = n$ je $Z_n \leqq \ln B$, prijímame hypotézu H_0 a ak $Z_n \geqq \ln A$, prijímame hypotézu H_1. Konštanty A a B volíme približne $A \approx \frac{1-\beta}{\alpha}$ a $B \approx \frac{\beta}{1-\alpha}$, kde α a β sú chyby 1. a 2. druhu, ktoré určíme tak, aby pre operačnú charakteristiku platilo:

$$L(\sigma_0) = 1 - \alpha \quad \text{a} \quad L(\sigma_1) = \beta \tag{4}$$

Hovoríme, že takto konštruované sekvenčné kritérium pomeru pravdepodobnosti má silu $(\alpha\,;\sigma_0;\beta\,;\sigma_1)$. Wald [1] ukázal, že strednú hodnotu pokusov $E_\sigma(n)$ možno odhadnúť vzťahom

$$E_\sigma(n) \approx \begin{cases} \dfrac{L(\sigma)\ln B + [1 - L(\sigma)]\ln A}{E_\sigma(z)} & \text{ak} \quad E_\sigma(z) \neq 0 \\[2ex] -\dfrac{\ln A \ln B}{E_\sigma(z^2)} & \text{ak} \quad E_\sigma(z) = 0 \end{cases} \tag{5}$$

kde

$$z = \ln \frac{f(x, \sigma_1)}{f(x, \sigma_0)} \tag{6}$$

a

$$L(\sigma) \approx \frac{A^h - 1}{A^h - B^h} \tag{7}$$

pričom $h = h(\sigma) \neq 0$ je nenulové riešenie rovnice

$$\int_0^\infty \left(\frac{f(x, \sigma_1)}{f(x, \sigma_0)} \right)^h f(x, \sigma)\, \mathrm{d}x = 1 \tag{8}$$

Veta 1. Nech $(x_1, x_2, \ldots)$ je postupnosť nezávislých náhodných premenných, každá s gama-rozdelením pravdepodobnosti $\mathrm{Ga}(x, r, \sigma)$, kde parameter r je známy. Testujme hypotézu $H_0 : \sigma = \sigma_0$ proti hypotéze $H_1 : \sigma = \sigma_1$ $(\sigma_0 < \sigma_1)$, pričom využijeme sekvenčné kritériá pomeru pravdepodobnosti sily $(\alpha\,;\sigma_0;\beta\,;\sigma_1)$. Potom ak po n-tom pozorovaní platí

$$h_0 + ns < \sum_{i=1}^n x_i < h_1 + ns \tag{9}$$

treba vykonať $(n+1)$. pozorovanie. V opačnom prípade prijímame hypotézu H_1, ak $\sum_{i=1}^{n} x_i \geqq h_1 + ns$, alebo H_0, ak $\sum_{i=1}^{n} x_i \leqq h_0 + ns$. Konštanty h_0, h_1 a s sme označili

$$h_0 = \sigma_0\sigma_1 \frac{\ln B}{\sigma_1 - \sigma_0}; \quad h_1 = \sigma_0\sigma_1 \frac{\ln A}{\sigma_1 - \sigma_0}; \quad s = r \frac{\ln \frac{\sigma_1}{\sigma_0}}{\sigma_1 - \sigma_0} \sigma_0\sigma_1 \tag{10}$$

Dôkaz. Podľa (3) $z_i = \ln \frac{\sigma_0^r}{\sigma_1^r} + x_i\left(\frac{1}{\sigma_0} - \frac{1}{\sigma_1}\right)$. Ak

$$\ln B < n \ln \frac{\sigma_0^r}{\sigma_1^r} + \sum_{i=1}^{n} x_i\left(\frac{1}{\sigma_0} - \frac{1}{\sigma_1}\right) < \ln A \tag{11}$$

treba urobiť $(n+1)$. pozorovanie. Po úprave dostaneme

$$\frac{\ln B + n \ln \frac{\sigma_1^r}{\sigma_0^r}}{\frac{1}{\sigma_0} - \frac{1}{\sigma_1}} < \sum_{i=1}^{n} x_i < \frac{\ln A + n \ln \frac{\sigma_1^r}{\sigma_0^r}}{\frac{1}{\sigma_0} - \frac{1}{\sigma_1}} \tag{12}$$

čo je vzhľadom na označenie (10) a vzťah (9) to, čo sme mali dokázať.

V práci [2] je ukázané, že riešením vzťahu (8) dostaneme

$$\sigma = \frac{\left(\frac{\sigma_0}{\sigma_1}\right)^h - 1}{h(\sigma_0 - \sigma_1)} \sigma_0\sigma_1 \tag{13}$$

Ako vidno $h(\sigma)$ a teda ani $L(\sigma)$ dané vzťahom (7) nezávisí od parametra r. Ľahko sa dá ukázať, že pre $h = +\infty$; 1; 0; -1; $-\infty$ vo vzťahu (13) zodpovedajú hodnoty $\sigma = 0$; σ_0; $\frac{r}{s}$; σ_1; $+\infty$ a tejto príslušnej dvojici $(h; \sigma)$ zodpovedajú vo vzťahu (7) tieto hodnoty operačnej charakteristiky:

$$L(0) = 1; \quad L(\sigma_0) = 1 - \alpha; \quad L\left(\frac{s}{r}\right) = \frac{h_1}{h_1 - h_0};$$

$$L(\sigma_1) = \beta; \quad L(\infty) = 0.$$

Veta 2. Nech platia rovnaké predpoklady ako vo vete 1. Potom strednú hodnotu pokusov $E_\sigma(n)$ v prípade $\mathrm{Ga}(x, r, \sigma)$ možno odhadnúť vzťahom

$$E_\sigma(n) \approx \begin{cases} \dfrac{L(\sigma)(h_0 - h_1) + h_1}{\sigma r - s} & \text{pre } \sigma \neq \dfrac{s}{r} \\[2ex] -\dfrac{h_0 h_1}{s^2} r & \text{pre } \sigma = \dfrac{s}{r}. \end{cases} \tag{14}$$

Dôkaz. Podľa (6)

$$E_\sigma(z)=E\left[\ln\frac{\sigma_0^r}{\sigma_1^r}+x\left(\frac{1}{\sigma_0}-\frac{1}{\sigma_1}\right)\right]=$$

$$=\ln\frac{\sigma_0^r}{\sigma_1^r}+\left(\frac{1}{\sigma_0}-\frac{1}{\sigma_1}\right)E(x)=r\left(\ln\frac{\sigma_0}{\sigma_1}+\sigma\frac{\sigma_1-\sigma_0}{\sigma_0\sigma_1}\right),$$

pretože stredná hodnota $\mathrm{Ga}(x, r, \sigma)$ je $E(x)=\sigma r$. Dosadením tohto výsledku do prvej časti vzťahu (5) a využitím označenia (10) dostaneme prvú časť vzťahu (14). Potom podľa druhej časti vzťahu (5) treba najprv nájsť

$$E_{\delta=(s/r)}(z^2)=\int_0^\infty\left[r\ln\frac{\sigma_0}{\sigma_1}+x\left(\frac{1}{\sigma_0}-\frac{1}{\sigma_1}\right)\right]^2\frac{1}{\Gamma(r)\left(\frac{s}{r}\right)^r}x^{r-1}\,\mathrm{e}^{-(rx/s)}\,\mathrm{d}x=$$

$$=\frac{r^rd^2}{s^{r+2}\Gamma(r)}\int_0^\infty(x-s)^2x^{r-1}\,\mathrm{e}^{-(rx/s)}\,\mathrm{d}x=\frac{\mathrm{d}^2}{r}=r\left(\ln\frac{\sigma_1}{\sigma_0}\right)^2.$$

Pri výpočte $E_{\sigma=(s/r)}(z^2)$ sme označili $\ln\frac{\sigma_1^r}{\sigma_0^r}=d$ a vzhľadom na vzťah (10) $\frac{1}{\sigma_0}-\frac{1}{\sigma_1}=\frac{d}{s}$.

Po substitúcii $u=\frac{rx}{s}$ a využitím identity $\Gamma(r+1)\;=\;r\Gamma(r)$ sme dostali

a) $\int_0^\infty x^{r+1}\,\mathrm{e}^{-(rx/s)}\,\mathrm{d}x=\frac{s^{r+2}}{r^{r+2}}\Gamma(r+2)=\frac{s^{r+2}}{r^r}\left(1+\frac{1}{r}\right)\Gamma(r)$;

b) $-2s\int_0^\infty x^r\,\mathrm{e}^{-(rx/s)}\,\mathrm{d}x=-2\frac{s^{r+2}}{r^{r+1}}\Gamma(r+1)=-2\frac{s^{r+2}}{r^r}\Gamma(r)$;

c) $s^2\int_0^\infty x^{r-1}\,\mathrm{e}^{-(rx/s)}\,\mathrm{d}x=\frac{s^{r+2}}{r^r}\Gamma(r)$.

Dosadením $E_{\sigma=(s/r)}(z^2)$ do druhej časti vzťahu (5) a využitím označenia (10) dostaneme po jednoduchej úprave druhú časť vzťahu (14), čo sme mali dokázať.

Numerická poznámka. Pretože podľa predpokladu $\sigma_0<\sigma_1$ môžeme označiť

$$\sigma_1=k\sigma_0\qquad(k>1)\tag{15}$$

Potom vzťahy (10) budú v tvare

$$h_0=\frac{k}{k-1}\sigma_0\ln B;\quad h_1=\frac{k}{k-1}\sigma_0\ln A;\quad s=\frac{kr}{k-1}\sigma_0\ln k\qquad(k>1)\tag{16}$$

Po dosadení (16) do (14) a využitím (4) dostaneme pre hodnoty parametra $\sigma=\sigma_0$, $\sigma=\sigma_1$ a $\sigma=\frac{s}{r}$ tieto približné odhady strednej hodnoty pokusov $E_\sigma(n)$, ktoré sa hodnotia na numerický výpočet

$$E_{\sigma=\sigma_0}(n) \approx \frac{(1-\alpha)(\ln B - \ln A) + \ln A}{(k-1-k\ln k)r} k \tag{17}$$

$$E_{\sigma=\sigma_1}(n) \approx \frac{\beta(\ln B - \ln A) + \ln A}{(k-1-\ln k)r} \tag{18}$$

$$E_{\sigma=(s/r)}(n) \approx -\frac{\ln A \ln B}{r(\ln k)^2} \tag{19}$$

Porovnaním výsledkov s prácou [3], kde bola stredná hodnota pokusov $E_\Theta(n)$ numericky vypočítaná pre Rayleighovo rozdelenie, vidíme, že pre $r=1$, t. j. pre exponenciálne rozdelenie, ktorého hustota je v tvare $f(x)=\frac{1}{\sigma}e^{-x/\sigma}$ $(x>0,\ \sigma>0)$ — je stredná hodnota pokusov rovnaká ako pre Rayleighovo rozdelenie.

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РЕЗЮМЕ

ВЫЧИСЛЕНИЕ СРЕДНЕГО ОБЪЕМА ВЫБОРКИ $E_\Theta(n)$ ГАМАРАСПРЕДЕЛЕНИЯ ПРИ ПОСЛЕДОВАТЕЛЬНОМ КРИТЕРИИ ОТНОШЕНИЯ ВЕРОЯТНОСТЕЙ, ГДЕ $E_\Theta(z)=0$

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В работе рассматривается совокупность гамма-распределения с плотностью распределения

$$f(x, r, \sigma) = \frac{1}{\sigma^r \Gamma(r)} e^{-(x/r)} x^{r-1} \qquad (0<x<\infty;\ r>0;\ \sigma>0)$$

На основании последовательного критерия отношения вероятностей исследуются гипотезы $H_0: \sigma = \sigma_0$ против $H_1: \sigma = \sigma_1 (\sigma_0 < \sigma_1)$ при данных ошибках α и β 1-го и 2-го рода, предполагая, что параметр r является известным.

В работе предлагается приблизительное выражение для среднего объема выборки $E_\sigma(n)$, где $E_\sigma(z) = 0$. Для значения параметров $\sigma = \sigma_0$, $\sigma = \sigma_1$ и $\sigma = \frac{S}{r}$ даны приблизительные формулы, которые могут быть использованы для вычисления $E_\sigma(n)$.

SUMMARY

THE CALCULATION OF THE AVERAGE SAMPLE NUMBER $E_\Theta(n)$ TO SEQUENTIAL SAMPLING FROM GAMA POPULATION, IF $E_\Theta(z) = 0$

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The present note describes sequential probability ratio test for the gama population defined by density

$$f(x, r, \sigma) = \frac{1}{\sigma^r \Gamma(r)} e^{-(x/\sigma)} x^{r-1} \qquad (0 < x < \infty;\ r > 0;\ \sigma > 0)$$

The hypothesis to be tested is: $H_0: \sigma = \sigma_0$ against $H_1: \sigma = \sigma_1$ $(\sigma_0 < \sigma_1)$, if α is probability of the first kind of error and β of the second one, assuming that r is known. In the paper an approximate formula for the average sample number is given $E_\sigma(n)$, if $E_\sigma(z) = 0$. For parameters $\sigma = \sigma_0$, $\sigma = \sigma_1$ and $\sigma = \frac{s}{r}$ are given approximate formulas, which are suitable for numerical calculations of $E_\sigma(n)$.

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APLIKÁCIA SEKVENČNÉHO KRITÉRIA POMERU PRAVDEPODOBNOSTI NA RAYLEIGHOVO ROZDELENIE

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Definícia 1. Hovoríme, že náhodná premenná x má Rayleighovo rozdelenie pravdepodobnosti Ra (x, σ^2), ak hustota rozdelenia pravdepodobnosti $f(x)$ takejto náhodnej premennej je v tvare

$$f(x)=\begin{cases} 0 & \text{pre} \quad -\infty < x < 0 \\ \dfrac{x}{\sigma^2} e^{-(x^2/2\sigma^2)} & \text{pre} \quad 0 \leqq x < \infty; \ \sigma > 0 \end{cases} \tag{1}$$

Jediným neznámym parametrom $\mathrm{Ra}(x, \sigma^2)$ je σ^2. Po veľmi jednoduchých výpočtoch sa dá ukázať, že

$$E(x)=\sigma\sqrt{\frac{\pi}{2}}; \quad E(x^2)=2\sigma^2; \quad D(x)=\frac{4-\pi}{2}\sigma^2 \tag{2}$$

V praxi sa s týmto rozdelením stretávame napríklad pri skúmaní výstrednosti strojárskych súčiastok, pri hodnotení vzdialenosti zásahu cieľa pri strelbe, v elektrotechnike pri sledovaní spoľahlivosti elektróniek, v kontrole akosti, a pod. Úlohou je zistiť, ktorá z hypotéz

$$H_0: \sigma^2 = \sigma_0^2$$

$$(\sigma_0^2 < \sigma_1^2) \tag{3}$$

proti

$$H_1: \sigma^2 = \sigma_1^2$$

je správna. Klasicky bola táto úloha riešená nasledujúcim spôsobom [3]:

Majme usporiadaný náhodný výber

$$x_{(p)} \leqq x_{(p+1)} \leqq \ldots \leqq x_{(q)} \qquad (x_{(0)}=0,\ 0 \leqq p \leqq q \leqq n)$$

ktorý sa urobil zo základného súboru s $\mathrm{Ra}(x, \sigma^2)$ (ak $p=0$, potom $q=n$). Potom náhodná premenná

$$t=\sum_{i=p+1}^{q}\frac{x_{(i)}^2-x_{(p)}^2}{2}+\frac{n-q}{2}[x_{(q)}^2-x_{(p)}^2] \tag{4}$$

má gama-rozdelenie $\Gamma(q-p,\sigma^2)$ a H_0 zamietame, ak $t\geqq c$, kde $c=\frac{\sigma_0^2}{2}\chi_{1-\alpha}^2(2q-2p)$ je kritická oblasť tohto testu a $\chi_{1-\alpha}^2(2q-2p)$ je kvantil χ^2-rozdelenia s $\nu=2q-2p$ stupňami voľnosti. Silofunkcia $1-\beta(\sigma^2)$ kritickej oblasti je

$$1-\beta(\sigma^2)=P\left(\frac{2t}{\sigma^2}\geqq\frac{\sigma_0^2}{\sigma^2}\chi_{1-\alpha}^2(2q-2p)\right) \tag{5}$$

kde náhodná premenná $\frac{2t}{\sigma^2}$ má χ^2-rozdelenie s $\nu=2q-2p$ stupňami voľnosti.

Určme rozsah výberu $q-p$ pre dané prirodzené číslo $2(q-p)$, $\alpha<1-\beta$ a $\sigma^2=k\sigma_0^2$ $(k>1)$ potrebný na to, aby $1-\beta(\sigma^2)\geqq 1-\beta$. Z (5) vyplýva, že $\beta(k\sigma_0^2)=P\left(\frac{2t}{\sigma^2}\leqq\frac{1}{k}\chi_{1-\alpha}^2(2q-2p)\right)\leqq\beta$, takže n nájdeme tak, aby platilo

$$\frac{\chi_{1-\alpha}^2(2q-2p)}{\chi_\beta^2(2q-2p)}\leqq k \tag{6}$$

V tejto práci je úloha (3) riešená využitím sekvenčného kritéria pomeru pravdepodobnosti, ktoré skonštruoval A. Wald (1945). V práci je tiež ukázané, že keď riešime úlohu (3) sekvenčným kritériom pomeru pravdepodobnosti, potrebujeme na rozhodovanie podstatne menší počet pozorovaní, ako pri klasickom postupe využitím vzťahu (6).

Procedúra sekvenčného kritéria pomeru pravdepodobnosti je nasledovná: Nech $f(x,\Theta)$ je hustota rozdelenia pravdepodobnosti náhodnej premennej x s neznámym parametrom $\Theta\in\Omega$. Potom $H_0:\Theta=\Theta_0$ proti $H_1:\Theta=\Theta_1$ $(\Theta_0<\Theta_1)$ testujeme tak, že vyšetrujeme náhodnú premennú

$$z_i=\ln\frac{f(x_i,\Theta_1)}{f(x_i,\Theta_0)} \quad (i=1,2,\dots) \tag{7}$$

a

$$Z_k=\sum_{i=1}^{k}z_i \tag{8}$$

Potom ak $\ln B<Z_k<\ln A$ $(k=1,2,\dots,n)$ treba urobiť $(n+1)$. pozorovanie. Ak pre $k=n$ je $Z_n\leqq\ln B$ prijímame hypotézu H_0 a ak $Z_n\geqq\ln A$ prijímame hypotézu H_1. Konštanty A a B možno približne voliť

$$A \approx \frac{1-\beta}{\alpha} \quad \text{a} \quad B \approx \frac{\beta}{1-\alpha} \tag{9}$$

kde α a β sú chyby 1. a 2. druhu. Volíme ich tak, aby pre operačnú charakteristiku $L(\Theta)$ platilo

$$L(\Theta_0) = 1 - \alpha \quad \text{a} \quad L(\Theta_1) = \beta \tag{10}$$

Definícia 2. Sekvenčné kritérium pomeru pravdepodobnosti vyhovujúce podmienkam (10) budeme nazývať sekvenčné kritérium sily (α, Θ_0, β, Θ_1).

Lemma 1 (Wald [1]). Nech z_i ($i = 1, 2, \ldots$) je postupnosť nezávislých náhodných premenných, každá s rovnakým rozdelením pravdepodobností ako náhodná premenná

$$z = \ln \frac{f(x, \Theta_1)}{f(x, \Theta_0)} \tag{11}$$

Nech rozptyl $D(z) > 0$. Potom pravdepodobnosť, že sekvenčné kritérium pomeru pravdepodobnosti sily (α, Θ_0, β, Θ_1) sa skončí po konečnom počte krokov sa rovná jednej.

Lemma 2. (Wald [1]). Nech x je náhodná premenná s hustotou rozdelenia pravdepodobnosti $f(x, \Theta)$, $\Theta \in \Omega$. Nech náhodná premenná z definovaná vzťahom (11) vyhovuje nasledujúcim podmienkam:

1. Stredná hodnota $E(z) \neq 0$ a existuje.
2. Pre ľubovoľné reálne h existuje stredná hodnota funkcie $g(h) = E(e^{zh})$ a existujú aspoň jej prvé dve derivácie v bode h.
3. Existuje také číslo $0 < \varepsilon < 1$, že platí $P(e^z \geqq 1 + \varepsilon) > 0$ a $P(e^z \leqq 1 - \varepsilon) > 0$.

Potom pre každé Θ existuje hodnota $h = h(\Theta) \neq 0$, že platí

$$\int_{-\infty}^{\infty} \left[\frac{f(x, \Theta_1)}{f(x, \Theta_0)}\right]^h f(x, \Theta)\, dx = 1 \tag{12}$$

t. j.

$$E(e^{zh(\Theta)}) = 1 \tag{13}$$

Lemma 3. (Wald [1]). Nech platia predpoklady lemy 2. Potom približná hodnota operačnej charakteristiky $L(\Theta)$ sekvenčného kritéria pomeru pravdepodobnosti sily (α, Θ_0, β, Θ_1) je daná vzťahom

$$L(\Theta) \approx \frac{A^{h(\Theta)} - 1}{A^{h(\Theta)} - B^{h(\Theta)}} \tag{14}$$

kde $h(\Theta)$ vyhovuje podmienke (12) a A a B je dané vzťahom (9). Približný odhad strednej hodnoty pokusov $E_\Theta(n)$ je daný vzťahom

$$E_\Theta(n) \approx \begin{cases} \dfrac{L(\Theta)\ln B + [1 - L(\Theta)]\ln A}{E_\Theta(z)} & \text{ak } E_\Theta(z) \neq 0 \\ -\dfrac{\ln A \ln B}{E_\Theta(z^2)} & \text{ak } E_\Theta(z) = 0 \end{cases} \tag{15}$$

kde náhodná premenná z je daná vzťahom (11).

Veta 1. Nech $(x_1, x_2, \ldots)$ je postupnosť nezávislých náhodných premenných, každá s $\mathrm{Ra}(x, \sigma^2)$. Nech $H_0: \sigma^2 = \sigma_0^2$ a $H_1: \sigma^2 = \sigma_1^2$ $(\sigma_0^2 < \sigma_1^2)$. Určme pomocou sekvenčného kritéria pomer pravdepodobnosti sily $(\alpha, \sigma_0^2, \beta, \sigma_1^2)$. Ak po n-tom pozorovaní platí

$$\sum_{i=1}^{n} x_i^2 \leqq r_n = h_1 + ns \tag{16}$$

prijímame H_0. Ak po n-tom pozorovaní platí

$$\sum_{i=1}^{n} x_i^2 \geqq r_n = h_1 + ns \tag{17}$$

prijímame H_1. V prípade, že platí

$$a_n < \sum_{i=1}^{n} x_i^2 < r_n \tag{18}$$

treba vykonať $(n+1)$. pozorovanie, kde konštanty h_0, h_1 a s sme označili

$$h_0 = \sigma_0^2\sigma_1^2 \frac{2\ln B}{\sigma_1^2 - \sigma_0^2}; \qquad h_1 = \sigma_0^2\sigma_1^2 \frac{2\ln A}{\sigma_1^2 - \sigma_0^2};$$

$$s = \sigma_0^2\sigma_1^2 \frac{\ln \dfrac{\sigma_1^4}{\sigma_0^4}}{\sigma_1^2 - \sigma_0^2} \tag{19}$$

Dôkaz. Podľa (7) platí $z_i = 2\ln\dfrac{\sigma_0}{\sigma_1} + \dfrac{1}{2}x_i^2\left(\dfrac{1}{\sigma_0^2} - \dfrac{1}{\sigma_1^2}\right)$. Potom $(n+1)$. pozorovanie treba vykonať vtedy, ak platí nerovnosť

$$\ln B < 2n\ln\frac{\sigma_0}{\sigma_1} + \left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right)\sum_{i=1}^{n} x_i^2 < \ln A\,.$$

Po úprave dostaneme

$$\frac{2\ln B + n\ln\dfrac{\sigma_1^4}{\sigma_0^4}}{\dfrac{1}{\sigma_0^2} - \dfrac{1}{\sigma_1^2}} < \sum_{i=1}^{n} x_i^2 < \frac{2\ln A + n\ln\dfrac{\sigma_1^4}{\sigma_0^4}}{\dfrac{1}{\sigma_0^2} - \dfrac{1}{\sigma_1^2}}$$

čo je vzhľadom na označenie (19) nerovnosť (18). Úplne rovnako sa dokážu aj nerovnosti (16) a (17).

Veta 2. Nech platia rovnaké predpoklady ako vo vete 1. Potom približná hodnota operačnej charakteristiky $L(\sigma^2)$ má tvar

$$L(\sigma^2)\frac{\left(\frac{1-\beta}{\alpha}\right)^h-1}{\left(\frac{1-\beta}{\alpha}\right)^h-\left(\frac{\beta}{1-\alpha}\right)^h} \tag{20}$$

kde σ^2 a h sú viazané vzťahom

$$\sigma^2=\frac{\sigma_0^2\sigma_1^2\left[\left(\frac{\sigma_0}{\sigma_1}\right)^{2h}-1\right]}{h(\sigma_0^2-\sigma_1^2)} \tag{21}$$

Pre

$$h=+\infty;\ 1;\ 0;\ -1;\ -\infty \tag{22}$$

sú hodnoty σ^2 resp. $L(\sigma^2)$ dané vzťahmi

$$\sigma^2=0;\ \sigma^2=\sigma_0^2;\ \sigma^2=\frac{s}{2};\ \sigma^2=\sigma_1^2;\ \sigma^2=+\infty \tag{23}$$

resp.

$$L(0)=1;\ L(\sigma_0^2)=1-\alpha;\ L\left(\frac{s}{2}\right)=\frac{h_1}{h_1-h_0};\ L(\sigma_1^2)=\beta;\quad L(\infty)=0 \tag{24}$$

Hodnoty h_0, h_1 a s sú dané vzťahmi (19).

Dôkaz. Podľa lemy 2 platí

$$1=\frac{1}{\sigma^2}\left(\frac{\sigma_0}{\sigma_1}\right)^{2h}\int_0^\infty\left[\frac{e^{-(x^2/2\sigma_1^2)}}{e^{-(x^2/2\sigma_0^2)}}\right]^h x\,e^{-(x^2/2\sigma^2)}\,dx=$$

$$=\frac{1}{\sigma^2}\left(\frac{\sigma_0}{\sigma_1}\right)^{2h}\int_0^\infty x\,e^{-x^2/2)(h/\sigma_1^2)-h/\sigma_0^2+1/\sigma^2)}dx=\frac{\left(\frac{\sigma_0}{\sigma_1}\right)^{2h}}{\sigma^2\left(\frac{h}{\sigma_1^2}-\frac{h}{\sigma_0^2}+\frac{1}{\sigma^2}\right)}$$

Riešením tejto rovnice pre neznámu σ^2 nájdeme vzťah (11), čo sme mali dokázať. Dosadením hodnôt (22) do (21) dostaneme hodnoty σ^2 dané vzťahom (23) a dosadením týchto hodnôt do (20) dostaneme hodnoty operačnej charakteristiky $L(\sigma^2)$ dané vzťahom (24).

Poznámka. Pomocou bodov $(\sigma^2; L(\sigma^2))$ možno zostrojiť ľahko graf operačnej charakteristiky. V mnohých praktických prípadoch sa dá vystačiť s piatimi bodmi, ktorých súradnice sú dané vzťahmi (23) a (24). V prípade, že by v praxi bolo nutné brať do úvahy aj ďalšie body operačnej charakteristiky (t. j. $\sigma^2 \in (0; \infty)$ a $L(\sigma^2) \in \langle 0; 1 \rangle$, je výhodné označiť

$$\frac{h}{2\sigma_1^2} - \frac{h}{2\sigma_0^2} = t\,, \quad \text{resp.} \quad h = \frac{-2t}{\dfrac{1}{\sigma_0^2} - \dfrac{1}{\sigma_1^2}} \tag{25}$$

Potom sa dá ľahko ukázať, že vzťahy (20) resp. (21) budú v tvare

$$L(\sigma^2) \approx \frac{e^{-th_1} - 1}{e^{-th_1} - e^{-th_0}} \tag{26}$$

resp.

$$\sigma^2 = \frac{e^{ts} - 1}{2t} \tag{27}$$

kde h_0, h_1 a s sú dané vzťahmi (19).

Veta 3. Nech platia rovnaké predpoklady ako vo vete 1. Potom strednú hodnotu pokusov $E_{\sigma^2}(n)$ $\mathrm{Ra}(x, \sigma^2)$ možno odhadnúť vzťahom

$$E_{\sigma^2}(n) \approx \begin{cases} \dfrac{L(\sigma^2)\,(h_0 - h_1) + h_1}{2\sigma^2 - s}, & \text{ak} \quad \sigma^2 \neq \dfrac{s}{2} \\[2ex] -\dfrac{h_0 h_1}{s^2}, & \text{ak} \quad \sigma^2 = \dfrac{s}{2} \end{cases} \tag{28}$$

kde $L(\sigma^2)$ je dané vzťahom (20) a h_0, h_1 a s vzťahom (19).

Dôkaz. Podľa (11)

$$z = \ln \frac{\dfrac{x}{\sigma_1^2}\, e^{-(x^2/2\sigma_1^2)}}{\dfrac{x}{\sigma_0^2}\, e^{-(x^2/2\sigma_0^2)}} = \ln \frac{\sigma_0^2}{\sigma_1^2} + \frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) x^2\,.$$

Podľa (2) ale $E(x^2) = 2\sigma^2$. Preto $E_{\sigma^2}(n) = \ln \dfrac{\sigma_0^2}{\sigma_1^2} + \left(\dfrac{1}{\sigma_0^2} - \dfrac{1}{\sigma_1^2}\right)\sigma^2$. Využitím lemy 3 dostaneme po jednoduchých úpravách prvú časť vzťahu (28). Ak $\sigma^2 = \dfrac{s}{2}$, potom

$$E_{\sigma^2 = (s/2)}(z^2) = \frac{2}{s}\int_0^{\infty} \left[\ln \frac{\sigma_0^2}{\sigma_1^2} + \frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) x^2\right]^2 x\, e^{-(x^2/s)}\, dx$$

Označme $\ln \frac{\sigma_1^2}{\sigma_0^2} = d$. Pretože $\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) = \frac{d}{s}$ bude mať predchádzajúci integrál tvar

$$E_{\sigma^2=(s/2)}(z^2) = \frac{2d^2}{s^3}\int_0^\infty (x^2 - s)^2 x \, e^{-(x^2/s)} \, dx = d^2 = \left(\ln \frac{\sigma_1^2}{\sigma_0^2}\right)^2$$

Dosadením tohto výsledku do druhej časti vzťahu (15) a po menšej úprave využitím označenia (19) dostaneme druhú časť vzťahu (28), čo sme mali dokázať.

Numerická poznámka. Pretože podľa predpokladu $\sigma_0^2 < \sigma_1^2$ a teda

$$\sigma_1^2 = k\sigma_0^2 \qquad (k > 1) \tag{29}$$

možno využitím vzťahu (28) numericky ľahko vyčísliť strednú hodnotu pokusov $E_{\sigma^2}(n)$. Po dosadení vzťahu (29) do (19) dostaneme

$$h_0 = \frac{2k - \sigma_0^2 \ln B}{k-1}; \quad h_1 = \frac{2k\sigma_0^2 \ln A}{k-1}; \quad s = \frac{2k\sigma_0^2 \ln k}{k-1} \qquad (k > 1) \tag{30}$$

Po dosadení (30) do (28) a využitím (24) dostaneme po jednoduchej úprave pre hodnoty parametra $\sigma^2 = \sigma_0^2$, $\sigma^2 = \sigma_1^2$ a $\sigma^2 = \frac{s}{2}$ tieto približné odhady $E_{\sigma^2}(n)$, ktoré sú vhodné pre numerický výpočet

$$E_{\sigma^2=\sigma_0^2}(n) \approx \frac{(1-\alpha)(\ln B - \ln A) + \ln A}{k - 1 - k \ln k} k \tag{31}$$

$$E_{\sigma^2=\sigma_1^2}(n) \approx \frac{\beta(\ln B - \ln A) + \ln A}{k - 1 - \ln k} \tag{32}$$

$$E_{\sigma^2=s/2}(n) \approx -\frac{\ln A \ln B}{(\ln k)^2} \tag{33}$$

kde A a B je dané vzťahom (9).

V *tab. 1* je stredná hodnota pokusov $E_{\sigma^2}(n)$ vypočítaná podľa vzťahov (31), (32) a (33) pre $k = 1, 5; 2; 2{,}5$ a 3 a všetky možnosti $\alpha = 0{,}01$; $0{,}05$ a $\beta = 0{,}01$; $0{,}05$. Všetky výsledky sú zaokrúhlené k nasledujúcemu celému číslu.

Bolo ukázané [1], že v prípade náhodného výberu zo základného súboru s normálnym rozdelením $N(\Theta, \sigma^2)$, kde stredná hodnota Θ je známa, treba k rozhodnutiu (3) pomocou sekvenčného kritéria pomeru pravdepodobnosti urobiť približne dvojnásobný počet pokusov.

Pre porovnanie sekvenčného kritéria pomeru pravdepodobnosti s klasickým testom (4) je v *tab. 1* pre hodnotu parametra $\sigma^2 = \sigma_1^2$ vypočítaný rozsah náhodného výberu n podľa vzťahu (6), kde sme položili $p = 0$ a $q = n$ (t. j. do úvahy sa berú všetky pozorovania). Na výpočet boli použité tabuľky [4]. Z *tab. 1* vidno, že k rozhodnutiu (3) treba klasickým testom urobiť približne trojnásobný počet pozorovaní, ako sekvenčným testom. Pri sekvenčnom teste je pre parameter $\sigma^2 = \frac{s}{2}$ (všeobecne pre $E_\Theta(z) = 0$) hodnota $E_{\sigma^2=(s/2)}(n)$ maximálna zo všetkých σ^2. Ako vidno z *tab. 1*, je $E_{\sigma^2=(s/2)}(n)$ menšie ako každé príslušné n odhadnuté klasickým testom.

Tabuľka 1

Numerický výpočet $E_{\sigma^2}(n)$ sekvenčného kritéria pomeru pravdepodobnosti Ra (x, σ^2)

$k = \frac{\sigma_1^2}{\sigma_0^2}$		1,5		2		2,5		3	
$E_{\sigma^2}(n)$	α / β	0,01	0,05	0,01	0,05	0,01	0,05	0,01	0,05
$E_{\sigma_0^2}(n)$	0,01	63	58	24	22	15	14	11	10
	0,05	41	37	16	14	10	9	7	7
$E_{\sigma_1^2}(n)$	0,01	48 (133)	31 (100)	15 (46)	10 (35)	8 (27)	5 (21)	5 (19)	4 (15)
	0,05	45 (95)	29 (67)	14 (32)	9 (23)	8 (19)	5 (14)	5 (13)	3 (10)
$E_{s/2}(n)$	0,01	129	83	44	29	26	17	18	12
	0,05	83	53	29	19	17	11	12	8

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РЕЗЮМЕ

ПРИМЕНЕНИЕ ПОСЛЕДОВАТЕЛЬНОГО КРИТЕРИЯ ОТНОШЕНИЯ ВЕРОЯТНОСТЕЙ К РАСПРЕДЕЛЕНИЮ РЭЛЕЯ

И. Гарай, Братислава

А. Вальд (1948 г.) разработал применение последовательного критерия отношения вероятностей к биноминальному распределению. В данной работе приведены подобные результаты для совокупности с плотностью распределения Рэлея

$$f(x)=\frac{x}{\sigma^2}\,e^{-(x^2/2\sigma^2)} \qquad (0\leqq x<\infty;\ \sigma>0)$$

В работе рассматриваются гипотезы $H_0::\sigma^2=\sigma_0^2$ в отношении шении к $H_1:\sigma^2=\sigma_1^2$ $(\sigma_0<\sigma_1^2)$ при данных ошибках α и β 1-го и 2-го рода. В работе предлагается приблизительное выражение для оперативной характеристики $L(\sigma^2)$ и среднего объема выборки $E_{\sigma^2}(n)$, для которого приводится таблица 1 ряда значений $k=\sigma_1^2/\sigma_0^2$ ($k=1{,}5$; 2; 2,5; 3) и различных $\alpha,\beta=0{,}01$, 0,05.

SUMMARY

THE USE OF THE SEQUENTIAL PROBABILILITY RATIO TEST SAMPLING FROM RAYLEIGH POPULATION

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A. Wald (1948) has used the sequential probability ratio test for the binomial and normal distribution. The purpose of the present paper is to obtain similar results for the Rayleigh population defined by density

$$f(x)=\frac{x}{\sigma^2}\,e^{-(x^2/2\sigma^2)} \qquad (0\leqq x<\infty;\ \sigma>0)$$

In the paper is tested: $H_0:\sigma^2=\sigma_0^2$ against $H_1:\sigma^2=\sigma_1^2(\sigma_0^2<\sigma_1^2)$, if α is the probability of the first kind of error and β of the second one. We have obtained on approximate formula for operating characteristic curve $L(\sigma^2)$ and the average sample number $E_{\sigma^2}(n)$. The *table 1* gives approximate values of $E_{\sigma^2}(n)$ for various values of $k=\frac{\sigma_1^2}{\sigma_0^2}$ ($k=1{,}5$; 2; 2,5; 3) and α,β = 0,01; 0,05).

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A GENERALIZATION OF THE FINITE ELEMENT METHOD FOR QUASI-PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

In this paper we apply the finite element method to obtain the solution of the boundary value problem for some type of quasi-parabolic partial differential equations. These equations are known for example in the theory of viscoelastic shallow shells and plates. According to the idea of J. Brilla [6], [9] we perform the Laplace transformation and then solve the relevant associated boundary value problem by the finite element method. The inverse transformation of exact solution of such problems are mostly very complicated and the inverse transformation can be found only numerically.

In our case the proper method for finding the inverse transformation is the method of the expansion of the exact solution in the Dirichlet series.

2. Formulation of a problem

Let Ω be the bounded domain $\Omega \subset E_2$, where E_2 is 2-dimensional Euclidean space. Further let S be a boundary of the domain Ω. Let us solve the equation

$$Aw + \frac{\partial}{\partial t} Bw = f \qquad (2.1)$$

with the boundary condition

$$Lw = 0 \quad \text{on} \quad S \qquad (2.2)$$

and the initial condition

$$w = 0 \quad \text{for} \quad t = 0 \qquad (2.3)$$

After using Laplace transformation in the form

$$\tilde{w}(x, y, p) = \int_0^\infty w(x, y, t)\, e^{-pt}\, dt$$

the equation (2.1) will be assumed in the form

$$C\tilde{w} = \tilde{f} \tag{2.4}$$

where a tildą means Laplace transformation and $C = \tilde{A} + p\tilde{B}$. Let C be a positive definite elliptic operator of the 4^{th} order for every real positive parameter p in $L_2(\Omega)$, i. e.

$$(\tilde{w}, C\tilde{w}) \geqq \gamma^2 \|\tilde{w}\|^2$$

where $(,)$, $\|.\|$ denote scalar product and the norm in $L_2(\Omega)$ respectively, and γ is a real positive number.

According to [12] the solution of the equations (2.4) minimizes the functional

$$\Phi(\tilde{w}) = (C\tilde{w}, \tilde{w}) - 2(\tilde{w}, \tilde{f}) \tag{2.5}$$

and vice versa. There exists just one element minimizing (2.5) in the energy space and this unique element will be called the weak solution of the equation (2.4).

If the coefficients and the right side of the equation (2.4) are continuous in the sence of Hölder with the positive exponent $\alpha < 1$, then after [2] p. 144 for every real positive parameter p the weak solution of the equation (2.4) is Hölder continuous with its derivatives up the 4^{th} order inclusive and therefore is continuous on Ω.

3. The solution of the associated problem by the finite element method

The finite element method is a successful application of the Ritz-Galerkin technique. The basic idea is now well established: given the problem of minimizing a functional $\Phi(\tilde{w})$ over an infinite-dimensional space V, the Ritz method minimizes $\Phi(\tilde{w})$ over a finite dimensional subspace $V_N \subset V$, and the finite element method is based on the convenient choice of V_N. In our case V will be the subspace of Sobolev space $W_2^{(2)}(\Omega)$ generated by all functions from $W_2^{(2)}(\Omega)$, which satisfy the main (also stable) boundary conditions of the associated problem. Further, the space V_N will be a set of all m-times continuously-differentiable functions satisfying the main boundary conditions of the associated problem, which are polynomials of the s-th order on the individual elements of the given division of the domain Ω.

If $\{g_i(x, y)\}_{i=1,2,\dots,N}$ is a base of the space V_N then every function $\tilde{w}(x, y) \in V_N$ can be expressed in the form

$$\tilde{w}(x, y, p)=\sum_{i=1}^{N}\delta_i(p)g_i(x, y) \qquad (3.1)$$

where the real numbers $\delta_i(p)$ ($i = 1, 2, \ldots, N$) depending on the parameter p of the Laplace transformation are the coordinates of the function $\tilde{w}$ in the base $\{g_i(x, y)\}_{i=1,2,\ldots,N}$.

The aim of the finite element method is the deriving of the matrix $\mathbf{K}$ and the vector $\mathbf{F}$, where

$$\mathbf{K}=(K_{ij}(p)), \qquad i, j = 1, 2, \ldots, N \qquad (3.2)$$

$$\mathbf{F}=(F_i(p))^T, \qquad i = 1, 2, \ldots, N \qquad (3.3)$$

by means of which the functional (2.3) can be written in the form

$$\Phi=\frac{1}{2}\boldsymbol{\Delta}^T\mathbf{K}\boldsymbol{\Delta}-\boldsymbol{\Delta}^T\mathbf{F} \qquad (3.4)$$

with $\boldsymbol{\Delta}$ being the vector of the coordinates of the function $\tilde{w}(x, y, p)$ in the base $\{g_i(x, y)\}_{i=1,2,\ldots,N}$.

$$\boldsymbol{\Delta}=(\delta_i(p))^T; \qquad i = 1, 2, \ldots, N \qquad (3.5)$$

The superscript T indicates transposition.

With respect to the positive definitness of our problem the functional (3.4) is minimized exactly for one vector of coordinates satisfying the necessary minimum conditions

$$\frac{\partial\Phi}{\partial\delta_i}=0 \qquad \text{for } i = 1, 2, \ldots, N \qquad (3.6)$$

With respect to (3.4) and (3.6) our minimized vector of coordinates is the solution of the system of linear algebraic equations

$$\mathbf{K}\boldsymbol{\Delta}=\mathbf{F} \qquad (3.7)$$

With regard to (3.2), (3.3) the solution of the system (3.7) is dependent on the parameter p. We have showed that by the finite element method it is possible to find the solution $\tilde{w}$ of an associated problem for each real positive parameter p in every node.

4. The inverse transformation of the associated problem

The inverse Laplace transformation is very complicated and practically it is only possible to determine this transformation numerically.

In this chapter we shall find the exact solution in the form of Dirichlet series.

With respect to the form of the Dirichlet series it necessary to make the next assumptions: for every point $(x', y') \in \Omega$ the solution w of the equation (2.1) has properties

$$\lim_{t \to \infty} w(x', y', t) = k \tag{4.1}$$

where k is a real number and $w(x', y', t)$ is increasing function. For example, the assumption (4.1) is usual satisfied in the theory of viscoelastic shallow shells and plates.

Let $\tilde{w}(x, y, p)$ be the Laplace transform of a function $w(x, y, t)$. We can write

$$w(x, y, t) = w_0(x, y) + \Delta w(x, y, t) \tag{4.2}$$

where

$$w_0(x, y) = w(x, y, \infty) \tag{4.3}$$

The values of the function w will be determined approximately in the nodes of the given division of Ω. The following notation will be used for the individual nodal point (x_0, y_0)

$$w(x_0, y_0, t) = w(t)$$

$$w_0(x_0, y_0) = w_0$$

In our case the Dirichlet series

$$\Delta w_D(t) = \sum_{i=1}^{n} S_i \, e^{-t/\gamma_i} \tag{4.4}$$

can be used as a resonable approximation to the solution $\Delta w(t)$.

The present method makes use of this series in which the γ_i are prescribed positive constants, and the S_i are unspecified coefficients to be calculated by minimizing the total square error between Δw and Δw_D. This total square error is

$$E^2 = \int_0^\infty [\Delta w(t) - \Delta w_D(t)]^2 \, dt$$

From the minimizing condition we receive n relations

$$-\frac{1}{2}\frac{\partial E^2}{\partial S_i} = 0 = \int_0^\infty [\Delta w(t) - \Delta w_D(t)] \, e^{-(t/\gamma_i)} \, dt; \qquad i = 1, 2, \ldots, n \tag{4.5}$$

With respect to the Laplace transformation from (4.5) we have

$$[p \, . \, \Delta \tilde{w}_D(p)]_{p=(1/\gamma_i)} = [p \Delta \tilde{w}(p)]_{p=(1/\gamma_i)}; \qquad i = 1, 2, \ldots, n$$

and we can write explicitly

$$\sum_{j=1}^{n} \frac{S_j}{1 + \frac{\gamma_i}{\gamma_j}} = [p \Delta \tilde{w}(p)]_{p=(1/\gamma_i)}; \qquad i = 1, 2, \ldots, n \tag{4.6}$$

From the relations (4.2) it follows

$$p\Delta\tilde{w}(p)=p\tilde{w}(p)-w_0 \tag{4.7}$$

where from the properties of the Laplace transformation, we have

$$w_0=\lim_{p\to 0}\{p\tilde{w}(p)\}$$

We can see that the total square error is minimized by collocating the p-multiplied transform of the Dirichlet series (4.4) and an associated solution, $p\Delta\tilde{w}(p)$, at n points $p=\frac{1}{\gamma_i}$. With this solution given numerically or graphically for $0\leqq p<\infty$, suitable values of γ_i can be prescribed by inspection. Of course, it is possible to determine the parameters γ_i by minimizing the square error with respect to each one, but this problem leads to a nonlinear set of equations. It is very difficult to solve this system of equations.

5. Numerical example

In order to ilustrate our method we shall investigate simply supported rectangular concrete plate reinforced in two perpendicular directions with sides 600×420 cm with constant thickness $h=16$ cm loaded by uniform load $q=$ $=1$ kpcm^{-2}. According to [5], [6] the equation of this plate can be written in terms of Laplace transformation as follows

$$d_{11}(p)\frac{\partial^4\tilde{w}}{\partial x^4}+2(d_{12}(p)+d_{33}(p))\frac{\partial^4\tilde{w}}{\partial x^2\,\partial y^2}+d_{22}(p)\frac{\partial^4\tilde{w}}{\partial y^4}=\frac{12q}{h^3} \tag{5.1}$$

where $d_{ij}(p)$ $(i, j=1, 2, 3)$ are elements of a matrix **D**

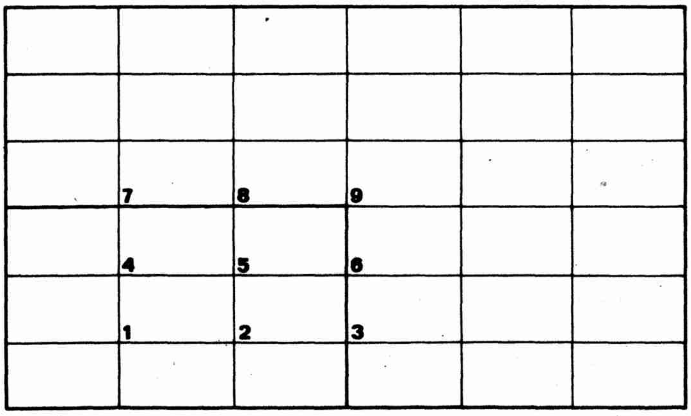

Fig. 1

$$\mathbf{D}=\frac{10^5}{p+\frac{1}{6}}\begin{pmatrix} 4{,}54p+0{,}382\,, & 0{,}783p+0{,}0662\,, & 0 \\ 0{,}783p+0{,}0662\,, & 4{,}72p+0{,}412\,, & 0 \\ 0 & 0 & 1{,}953p+0{,}165 \end{pmatrix}$$

The matrix $\mathbf{D}$ is derived in [7].

The boundary conditions are

$$\tilde{w}\Big|_S=0\,,\quad \frac{\partial^2\tilde{w}}{\partial x^2}\Big|_S=\frac{\partial^2\tilde{w}}{\partial y^2}\Big|_S=0 \tag{5.2}$$

The problem (5.1), (5.2) being positive definite has a unique solution which we can obtain by minimizing the functional Φ (2.3). This functional after using Green's formula has a form

$$\Phi=\int_\Omega\int\left[\frac{h^3}{24}\boldsymbol{a}^{\mathrm{T}}(\tilde{w})\,\mathbf{D}\boldsymbol{a}(\tilde{w})-\tilde{q}\tilde{w}\right]\mathrm{d}\Omega \tag{5.3}$$

where $\boldsymbol{a}(\tilde{w})$ is a vector defined by the relation

$$\boldsymbol{a}(\tilde{w})=\left(\frac{\partial^2\tilde{w}}{\partial x^2},\ \frac{\partial^2\tilde{w}}{\partial y^2},\ 2\frac{\partial^2\tilde{w}}{\partial x\,\partial y}\right)^T$$

In (4.7) we need the values of the expression $p\tilde{w}$ and so it is better to write the functional (5.3) in the form

$$\Phi=\int_\Omega\int\left[\frac{h^3}{24p^2}\boldsymbol{a}^{\mathrm{T}}(p\tilde{w})\,\mathbf{D}\boldsymbol{a}(p\tilde{w})-\frac{p\tilde{w}}{p^2}\right]\mathrm{d}\Omega \tag{5.4}$$

In our version of finite element method a rectangular element and a bicubic polynomial is used. Then a function $p\tilde{w}$ has on every rectangle of our division of Ω the form

$$p\tilde{w}(x,y,p)=\sum_{i=0}^{3}\sum_{j=0}^{3}C_{ij}(p)x^iy^j \tag{5.5}$$

In each node of our division we prescribed four values

$$p\tilde{w},\ \frac{\partial(p\tilde{w})}{\partial x},\ \frac{\partial(p\tilde{w})}{\partial y},\ \frac{\partial^2(p\tilde{w})}{\partial x\,\partial y}\,.$$

Then the polynomial (5.5) is determined uniquely on every rectangle and the function, which is on every rectangle a polynomial of this type (5.5), is 1-times continuous differentiable on $\bar{\Omega}$ [11].

With respect to the given division of our domain Ω and from properties of the functional Φ the expression (5.4) can be written in the form

$$\Phi=\sum_{k=1}^{m}\frac{1}{p^2}\int_{\Omega_k}\int\left[\frac{h^3}{24}\boldsymbol{a}^{\mathrm{T}}(p\tilde{w})\,\mathbf{D}\boldsymbol{a}(p\tilde{w})-p\tilde{w}\right]\mathrm{d}x\,\mathrm{d}y \tag{5.6}$$

where m is a number of individual rectangles Ω_K of our division of the rectangle Ω.

Setting

$$\Phi_k^{(1)} = \frac{1}{p^2}\int_{\Omega_k}\int\left[\frac{h^3}{24}\,\mathbf{a}^T(p\tilde{w})\,\mathbf{D}\mathbf{a}(p\tilde{w})\right]\mathrm{d}x\,\mathrm{d}y$$

$$\Phi_k^{(2)} = \frac{1}{p^2}\int\int_{\Omega_k} p\tilde{w}\,\mathrm{d}x\,\mathrm{d}y \tag{5.7}$$

we can express the functional (5.6) in the form

$$\Phi = \sum_{k=1}^{m}(\Phi_k^{(1)} - \Phi_k^{(2)})$$

By the transformation

$$x = x_0 + a\xi\,; \qquad y = y_0 + b\eta \tag{5.8}$$

the rectangle Ω_k is transformed to the unit rectangle Ω_0 *(fig. 2)*, where a, b are sides of the rectangle Ω_k. Denoting

$$\mathbf{\Theta} = \begin{pmatrix} 1, & 0, & 0 \\ 0, & 0, & 1 \\ 0, & 2, & 0 \end{pmatrix} \qquad \mathbf{\Psi} = \begin{pmatrix} a^{-2}, & 0, & 0 \\ 0, & a^{-2}b^{-1}, & 0 \\ 0, & 0, & b^{-2} \end{pmatrix}$$

we can write, according to the transformation (5.8), the expression $\Phi_k^{(1)}$ in the form

$$\Phi_k^{(1)} = \frac{h^3 ab}{24p^2}\int_{\Omega_0}\int \mathbf{b}^T(p\tilde{w})\mathbf{\Psi}^T\mathbf{\Theta}^T\mathbf{D}\mathbf{\Theta}\mathbf{\Psi}\mathbf{b}(p\tilde{w})\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{5.9}$$

where

$$\mathbf{b}(\tilde{w}) = \left(\frac{\partial^2\tilde{w}}{\partial\xi^2},\ \frac{\partial^2\tilde{w}}{\partial\xi\,\partial\eta},\ \frac{\partial^2\tilde{w}}{\partial\eta^2}\right)^T \tag{5.10}$$

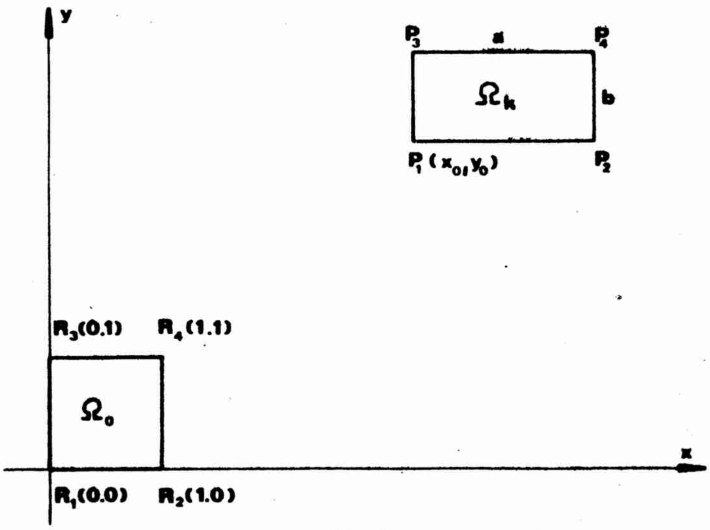

Fig. 2

Setting
$$\tilde{\mathbf{D}}=\frac{h^3}{12}ab\,\mathbf{\Psi}^T\mathbf{\Theta}^T\mathbf{D}\mathbf{\Theta}\mathbf{\Psi}=[\tilde{d}_{ij}(p)]_{i,j=1,2,3}$$

according to (5.10), the expression (5.9) will be in the form

$$\Phi_k^{(1)}=\frac{1}{2p^2}\sum_{i=1}^{3}\sum_{j=1}^{3}\tilde{d}_{ij}(p)\int_{\Omega_0}\int\frac{\partial^2(pw)}{\partial\xi^{3-i}\partial\eta^{i-1}}\frac{\partial^2(p\tilde{w})}{\partial\xi^{3-j}\,\partial\eta^{j-1}}\,d\xi\,d\eta \tag{5.11}$$

Now we introduce the known algorithm to calculate the integrals in (5.11) without numerical integrations in a digital computer.

Let P_1, P_2, P_3, P_4 and R_1, R_2, R_3, R_4 be the verteces of the rectangles Ω_k and Ω_0 resp. The functions $p\tilde{w}(\xi, \eta, p)$, $p\tilde{w}(x, y, p)$ are uniquely determined on rectangles Ω_0, Ω_k by the vectors $\tilde{\mathbf{\Delta}}$, $\mathbf{\Delta_k}$, resp.:

$$\tilde{\mathbf{\Delta}}=\Big(p\tilde{w}(R_1),\ \frac{\partial(p\tilde{w}(R_1))}{\partial\xi},\ \frac{\partial(p\tilde{w}(R_1))}{\partial\eta},\ \frac{\partial^2(p\tilde{w}(R_1))}{\partial\xi\,\partial\eta},\ p\tilde{w}(R_2),\ \ldots,\ \frac{\partial^2(p\tilde{w}(R_4))}{\partial\xi\,\partial\eta}\Big)^T \tag{5.12}$$

$$\mathbf{\Delta_k}=\Big(p\tilde{w}(P_1),\ \frac{\partial(p\tilde{w}(P_1))}{\partial x},\ \frac{\partial(p\tilde{w}(P_1))}{\partial y},\ \frac{\partial^2(p\tilde{w}(P_1))}{\partial x\,\partial y},\ p\tilde{w}(P_2),\ \ldots,\ \frac{\partial^2(p\tilde{w}(P_4))}{\partial x\,\partial y}\Big)^T \tag{5.13}$$

According to (5.12), (5.13) and to (5.8) we have

$$\Delta=\mathbf{L}\Delta_{\mathbf{k}} \tag{5.14}$$

where

$$\mathbf{L}=\begin{pmatrix}\mathbf{H}, & & & \\ & \mathbf{H}, & & \\ & & \mathbf{H}, & \\ & & & \mathbf{H}\end{pmatrix}\quad \mathbf{H}=\begin{pmatrix}1, & & & \\ & a, & & \\ & & b, & \\ & & & ab\end{pmatrix}. \tag{5.15}$$

We can write

$$p\tilde{w}(\xi, \eta, p)=\mathbf{c}^T\mathbf{g} \tag{5.16}$$

with

$$\mathbf{c}=(c_1, c_2, \ldots, c_{16})^T$$

$$\mathbf{g}(\xi, \eta)=(1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \ldots, \xi^3\eta^3)^T \tag{5.17}$$

where the components of the vector $\mathbf{g}$ are the terms of the given bicubic polynomial and the components of the vector $\mathbf{c}$ are their coefficients.

Denoting

$$\mathbf{S}=\Big(\mathbf{g}(R_1),\ \frac{\partial\mathbf{g}(R_1)}{\partial\xi},\ \frac{\partial\mathbf{g}(R_1)}{\partial\eta},\ \frac{\partial^2\mathbf{g}(R_1)}{\partial\xi\,\partial\eta},\ \mathbf{g}(R_2),\ \ldots,\ \frac{\partial^2\mathbf{g}(R_4)}{\partial\xi\,\partial\eta}\Big)^T$$

the vector **c** can be expressed in the form

$$\mathbf{c} = \mathbf{S}^{-1}\tilde{\boldsymbol{\Delta}} \tag{5.18}$$

The bicubic polynomial is uniquely defined on the rectangle Ω_0 and that is why the matrix **S** is the regular one and the matrix $\mathbf{S}^{-1}$ exists.

According to (5.14), (5.16), (5.18) the expression $p\tilde{w}(\xi, \eta, p)$ can be written in the form

$$p\tilde{w}(\xi, \eta, p) = \boldsymbol{\Delta}_k^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}(\mathbf{S}^{-1})^{\mathrm{T}}\mathbf{g} \tag{5.19}$$

Then relation (5.11) will have the form

$$\Phi_k^{(1)} = \frac{1}{2p^2}\boldsymbol{\Delta}_k^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\tilde{d}_{ij}(p)\mathbf{A}_{ij}\right)\mathbf{L}\boldsymbol{\Delta}_k \tag{5.20}$$

where

$$\mathbf{A}_{ij} = (\mathbf{S}^{-1})^{\mathrm{T}}\int_{\Omega_0}\int \mathbf{G}_i\mathbf{G}_j^{\mathrm{T}}\, d\xi\, d\eta\ \mathbf{S}^{-1} \tag{5.21}$$

$$\mathbf{G}_i(\xi, \eta) = \frac{\partial^2\mathbf{g}(\xi, \eta)}{\partial\xi^{3-i}\,\partial\eta^{i-1}} \quad \text{for } i = 1, 2, 3$$

For the integrals occuring in (5.21) the next formula is used

$$\int_{\Omega_0}\int \xi^k\eta^e\, d\xi\, d\eta = \frac{1}{(k+1)\,(l+1)}$$

Setting

$$\mathbf{K}_k = (K_k^{ij}(p))_{i,\,j=1,\,2,\,\dots,\,16} = \mathbf{L}^{\mathrm{T}}\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\mathrm{d}_{ij}(p)\mathbf{A}_{ij}\right)\mathbf{L}$$

table 1

Node No.	W_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}
1	0,764	$5{,}32.10^{-6}$	$-6{,}51.10^{-5}$	$6{,}71.10^{-4}$	$-7{,}52.10^{-3}$	$-3{,}97.10^{-1}$	$6{,}23.10^{-2}$	$-5{,}74.10^{-2}$	$5{,}70.10^{-2}$	$-5{,}68.10^{-2}$	$5{,}60.10^{-2}$	$-4{,}56.10^{-2}$
2	1,245	$8{,}62.10^{-6}$	$-1{,}05.10^{-4}$	$1{,}09.10^{-3}$	$-1{,}22.10^{-2}$	$-0{,}65.10^{-1}$	$1{,}01.10^{-1}$	$-9{,}30.10^{-2}$	$9{,}23.10^{-2}$	$-9{,}20.10^{-2}$	$9{,}02.10^{-2}$	$-7{,}38.10^{-2}$
3	1,404	$9{,}69.10^{-6}$	$-1{,}18.10^{-4}$	$1{,}22.10^{-3}$	$-1{,}37.10^{-2}$	$-7{,}29.10^{-1}$	$1{,}13.10^{-1}$	$-1{,}05.10^{-1}$	$1{,}04.10^{-1}$	$-1{,}04.10^{-1}$	$1{,}01.10^{-1}$	$-8{,}30.10^{-2}$
4	1,293	$9{,}04.10^{-6}$	$-1{,}10.10^{-4}$	$1{,}14.10^{-3}$	$-1{,}28.10^{-2}$	$-6{,}74.10^{-1}$	$1{,}06.10^{-1}$	$-9{,}75.10^{-2}$	$9{,}67.10^{-2}$	$-9{,}64.10^{-2}$	$9{,}45.10^{-2}$	$-7{,}73.10^{-2}$
5	2,121	$1{,}47.10^{-5}$	$-1{,}80.10^{-4}$	$1{,}85.10^{-3}$	$-2{,}08.10^{-2}$	$-1{,}10$	$1{,}72.10^{-1}$	$-1{,}59.10^{-1}$	$1{,}58.10^{-1}$	$-1{,}57.10^{-1}$	$1{,}54.10^{-1}$	$-1{,}26.10^{-1}$
6	2,395	$1{,}66.10^{-5}$	$-2{,}02.10^{-4}$	$2{,}09.10^{-3}$	$-2{,}34.10^{-2}$	$-1{,}25$	$1{,}94.10^{-1}$	$-1{,}79.10^{-1}$	$1{,}77.10^{-1}$	$-1{,}77.10^{-1}$	$1{,}73.10^{-1}$	$-1{,}42.10^{-1}$
7	1,480	$1{,}04.10^{-5}$	$-1{,}27.10^{-4}$	$1{,}31.10^{-3}$	$-1{,}46.10^{-2}$	$-7{,}71.10^{-1}$	$1{,}21.10^{-1}$	$-1{,}12.10^{-1}$	$1{,}11.10^{-1}$	$-1{,}10.10^{-1}$	$1{,}08.10^{-1}$	$-8{,}86.10^{-2}$
8	2,432	$1{,}69.10^{-5}$	$-2{,}06.10^{-4}$	$2{,}13.10^{-3}$	$-2{,}39.10^{-2}$	$-1{,}27$	$1{,}98.10^{-1}$	$-1{,}82.10^{-1}$	$1{,}81.10^{-1}$	$-1{,}80.10^{-1}$	$1{,}77.10^{-1}$	$-1{,}45.10^{-1}$
9	2,749	$1{,}90.10^{-5}$	$-2{,}33.10^{-4}$	$2{,}40.10^{-3}$	$-2{,}69.10^{-2}$	$-1{,}43$	$2{,}23.10^{-1}$	$-2{,}05.10^{-1}$	$2{,}04.10^{-1}$	$-2{,}03.10^{-1}$	$1{,}99.10^{-1}$	$-1{,}63.10^{-1}$

table 2

t	0	0.1	0.2	0.3	0.5	1
w(t)	1.349	1.435	1.478	1.492	1.498	1.513
t	2	3	5	10	19	30
w(t)	1.585	1.677	1.860	2.201	2.514	2.660
t	40	50	60	70	90	∞
w(t)	2.707	2.725	2.732	2.736	2.740	2.749

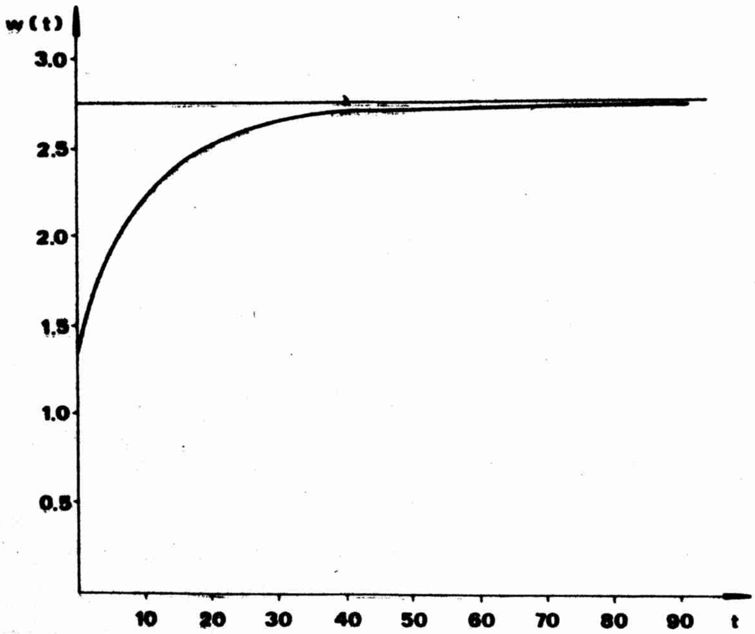

Fig. 3

the relation (5.20) can be expressed in the form

$$\Phi_k^{(1)} = \frac{1}{2p^2} \mathbf{\Delta}_k^T \mathbf{K}_k \mathbf{\Delta}_k \tag{5.22}$$

According to the transformation (5.8) and to (5.19) the expression (5.7) we can write in the form

$$\Phi_k^{(2)} = \frac{1}{p^2} \mathbf{\Delta}_k^T \mathbf{F}_k \tag{5.23}$$

where

$$\mathbf{F}_k = ab\,\mathbf{L}^T(\mathbf{S}^{-1})^T \int_{\Omega_0}\int \mathbf{g}(\xi, \eta)\, d\xi\, d\eta$$

Making use of (5.22) and (5.23) we have the functional Φ in the form

$$\Phi = \frac{1}{p^2}\sum_{k=1}^{m}\left(\frac{1}{2}\boldsymbol{\Delta}_k^T\mathbf{K}_k\boldsymbol{\Delta}_k - \boldsymbol{\Delta}_k^T\mathbf{F}_k\right)$$

If we have equidistanced division of our domain Ω then matrixes $\mathbf{K}_k$ and $\mathbf{F}_k$ are equal for every element Ω_k. Let $\boldsymbol{\Delta}$ (3.5) be the vector of all prescribed values in the nodes. All components of the vector $\boldsymbol{\Delta}$ are among the components of the vectors $\boldsymbol{\Delta}_k$ and vice versa. From the necessary minimum condition (3.6) we receive the finally system of linear algebraic equations (3.7).

In our case the best rate of convergence of $O(l^2)$, where $l = \max(a, b)$ [3], [10]. With regard to a symetry the values $p\tilde{w}$ were calculated only for a quarter of our plate for 12 values of a parameter p: 0, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10, 10^2, 10^3, 10^4, 10^5. In (4.6) were put $p = \frac{1}{\gamma_i}$. From numerical experiments we can see that for the other combination of the parameters p, the numerical results are approximately the same. Then the solution $w(t)$ for every node has a form

$$w(t) = w_0 + \sum_{i=1}^{11} S_i \, e^{-t \, . \, 10^{i-6}}$$

Coefficients S_i ($i = 1, 2, \ldots, 11$) and w_0 for the nodes of the given division of the domain Ω are in the *table 1*. A graph and values of a function $w(t)$ in the middle of the plate (node 9) are in the *fig. 3* and *table 2*, resp.

The numerical results were done by the digital computer CDC 3300 in VVS OSN in Bratislava.

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РЕЗЮМЕ

ОБОБЩЕНИЕ МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ ДЛЯ РЕШЕНИЯ КВАЗИПАРАБОЛИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ЧАСТНЫМИ ПРОИЗВОДНЫМИ

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В этой статье автор применяет метод конечных элементов для решения квазипараболического дифференциального уравнения с частными производными (2.1) при краевых и начальных условиях (2.2), (2.3). После преобразования Лапласа при некоторых условиях можем решить соответствующую ассоциированную задачу методом конечных элементов. Удобным методом для обратной трансформации Лапласа является метод развития точного решения в ряд Дирихле. Применение этого метода показано на примере вязкоупругой свободно опертой прямоугольной плиты при равномерно распределенной нагрузке.

SÚHRN

ZOVŠEOBECNENIE METÓDY KONEČNÝCH PRVKOV PRE RIEŠENIE KVÁZI-PARABOLICKÝCH PARCIÁLNYCH DIFERENCIÁLNYCH ROVNÍC

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Článok sa zaoberá aplikáciou metódy konečných prvkov na riešenie kvázi-parabolickej parciálnej diferenciálnej rovnice (2.1) pri okrajových resp. počiatočných podmienkach (2.2), (2.3). Po vykonaní Laplaceovej transformácie za určitých predpokladov môžeme riešiť príslušný pridružený problém metódou konečných prvkov. Ako najvhodnejšia metóda na inverznú Laplaceovu transformáciu je použitá metóda rozvoja presného riešenia do Dirichletovho radu. Použitie uvedenej metódy je ilustrované na príkladeväzkopružnej kĺbovite uloženej rovnomerne zaťaženej obdĺžnikovej dosky.

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A NOTE ON ONE-SIDED BASES OF SEMIGROUPS

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The structure of semigroups containing one-sided bases is investigated in [1]. At the end of the paper two theorems are stated and proved. The first of them gives a necessary and sufficient condition for $S-\mathscr{A}$ to be a maximal left ideal of the semigroup S, where $\mathscr{A}$ denotes the union of all right bases of S. The last theorem gives a necessary and sufficient condition for $S-\mathscr{A}$ to be the so called *left ideal* L^*. However, I have found out now that both of them are mutually equivalent. The purpose of the paper is to prove it.

Definition ([5]). We say that a subset A of a semigroup S is a right base of S if $A\cup SA=S$ and there exists no proper subset $B\subseteq A$ for which $B\cup SB=S$.

A quasi-ordering is introduced in S as follows: $a\leqq b$ means $a\cup Sa\subseteq b\cup Sb$.

Lemma 1 ([5]). A non-empty subset A of a semigroup S is a right base of S if and only if A satisfies the following conditions:

1. for any $x\in S$ there exists $a\in A$ such that $x\leqq a$,
2. for any two distinct elements $a, b\in A$ neither $a\leqq b$, nor $b\leqq a$.

Lemma 2 ([1]). Let $\mathscr{A}\neq\emptyset$ be the union of all right bases of a semigroup S. If $L=S-\mathscr{A}$ is non-empty, then L is a proper left ideal of S.

The notion of a maximal proper ideal is used in the usual sense ([3]). We say that a semigroup S contains a left ideal L^*, if L^* is such a maximal proper left ideal in which every proper left ideal of S is contained ([4]).

We say that an element $a\in S$ is left invertible if $Sa=S$.

Lemma 3 ([2]). Let a semigroup S contain at least one left invertible element. Then S contains the ideal L^* and the complement of this ideal is the set of all left invertible elements.

Theorem. Let $\emptyset\neq\mathscr{A}\neq S$, where $\mathscr{A}$ is the union of all right bases of S. Then the following statements are equivalent.

1. $(S-\mathscr{A})$ is a maximal proper left ideal of S.
2. For every element $a\in\mathscr{A}$, $\mathscr{A}\subseteq(a)_L=a\cup Sa$.
3. $S-\mathscr{A}=L^*$.

4. Every right base of S is one-element set and one of the following conditions holds:

a) Any right base of S is formed by a left invertible element.

b) The semigroup S contains only one right base $A=\{a\}$ and we have: $a \cup Sa = S$, but $a \notin Sa$.

Proof. 1 $\Leftrightarrow$ 2. Let $L = S - \mathcal{A}$ be a maximal proper left ideal of a semigroup S. Then $\mathcal{A} \neq S$. Let $a \in \mathcal{A}$. If $\mathcal{A} \subseteq a \cup Sa$ does not hold, then $(S-\mathcal{A}) \cup (a)_L$ as union of two left ideals is a left ideal of S, but a proper one. Then $S-\mathcal{A}$ is not a maximal left proper ideal, which is a contradiction to the assumption.

Let $\mathcal{A} \subseteq a \cup Sa$ for any $a \in \mathcal{A}$, and $\mathcal{A} \neq S$. We have to prove that $S-\mathcal{A}$ is a maximal proper left ideal of S. Let $S-\mathcal{A} \subseteq L'$, where L' is a left ideal of S and $S-\mathcal{A} \neq L'$. Then $L' \cap \mathcal{A} \neq \emptyset$. Let $a \in L' \cap \mathcal{A}$, so $a \in L'$. It follows that $Sa \subseteq SL' \subseteq$ $\subseteq L'$, $a \cup Sa \subseteq L'$. From there, and according to the assumption we obtain $\mathcal{A} \subseteq$ $\subseteq a \cup Sa \subseteq L'$. Consequently, $\mathcal{A} \subseteq L'$, $S-\mathcal{A} \subseteq L'$, therefore $S = L'$.

1 $\Leftrightarrow$ 3. Let $S-\mathcal{A}$ be a maximal left ideal of S, so for any $a \in \mathcal{A}$, $\mathcal{A} \subseteq a \cup Sa$. Let M be a left ideal of S which is not contained in $S-\mathcal{A}$. Then there exists some $a \in M$ with $a \notin S-\mathcal{A}$, hence $a \in \mathcal{A}$. By the statement 2. it holds $\mathcal{A} \subseteq a \cup Sa \subseteq M$ and because $\mathcal{A}$ is the union of right bases, for any right base $A \subseteq \mathcal{A}$, $a \in A$, it holds $A \subset M$, $SA \subset M$, together $S = A \cup SA \subseteq M$, hence $M = S$. We get $S-\mathcal{A} = L^*$.

If $S-\mathcal{A} = L^*$, then evidently $S-\mathcal{A}$ is a maximal left ideal of S.

3 $\Leftrightarrow$ 4. Let $S-\mathcal{A} = L^*$. Then $S-\mathcal{A}$ is a maximal proper left ideal. 2. implies $\mathcal{A} \subseteq a \cup Sa$ for every $a \in \mathcal{A}$. But moreover, every left ideal of S is included in $S-\mathcal{A}$, hence there are only two possibilities: either $a \cup Sa \subseteq S-\mathcal{A}$, or $a \cup Sa = S$. The first possibility cannot hold, because at least $a \notin S-\mathcal{A}$. Therefore, the other possibility must hold, so $a \cup Sa = S$ for any $a \in \mathcal{A}$. Thus $\{a\}$ is a right base of S. And as $\{a\}$ is an arbitrary element of $\mathcal{A}$, all right bases are one-element sets. Therefore only the following three cases are possible:

a) $a \cup Sa = S$, $a \in Sa$ for any element $a \in \mathcal{A}$. (It means that every element $a \in \mathcal{A}$ is left invertible.)

b) $a \cup Sa = S$, $a \notin Sa$ for any element $a \in \mathcal{A}$.

c) $a \cup Sa = S$, $a \in Sa$ for some element $a \in \mathcal{A}$, but $b \cup Sb = S$, $b \notin Sb$ for another element $b \in \mathcal{A}$, $b \neq a$. We shall show that if $S-\mathcal{A}$ is a maximal proper left ideal of S then the case c) cannot occur and in the case b) the semigroup S contains only one such a base.

Let us assume that in the case b) a semigroup S contains at least two right bases: $A_1 = \{a_1\}$, $A_2 = \{a_2\}$ such that $a_1 \cup Sa_1 = S$, $a_1 \notin Sa_1$, $a_2 \cup Sa_2 = S$, $a_2 \notin Sa_2$. Then $S-\mathcal{A} \subseteq Sa_1 \subseteq S$, where $S-\mathcal{A} \neq Sa_1$, because $a_2 \in Sa_1$. But it means that $S-\mathcal{A}$ is not a maximal proper left ideal, and this is a contradiction. If the case c) occurs, then again $S-\mathcal{A}$ is not a maximal proper left ideal, which is again a contradiction to the assumption.

Let us assume that all right bases of S are one-element bases and that one of the conditions a) and b) holds. If a) holds, then the statement 3. follows from Lemma 3. If b) holds, then $S-\{a\}=S-\mathscr{A}=L$ is a left ideal. It is evident that it is a maximal proper left ideal. We show that every proper left ideal of S is included in L. Let L_1 be a left ideal of S which is not included in L. Then evidently $a \in L_1$, therefore $Sa \subseteq SL_1 \subseteq L_1$. But, since $a \in L_1$, $S = a \cup Sa \subseteq L_1$, therefore $L_1 = S$. It means that $L = L^*$.

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SÚHRN

POZNÁMKA O JEDNOSTRANNÝCH BÁZACH V POLOGRUPÁCH

I. Fabrici, Bratislava

Cieľom tejto poznámky je dôkaz vety, ktorá udáva ekvivalentnosť tvrdení, že $S-\mathscr{A}$ je taký maximálny ľavý ideál pologrupy S, ktorý obsahuje každý vlastný ľavý ideál ($\mathscr{A}$ je množinový súčet všetkých pravých báz).

РЕЗЮМЕ

ЗАМЕТКА ОБ ОДНОСТОРОННЫХ БАЗИСАХ В ПОЛУГРУППАХ

И. Фабрици, Братислава

В этой заметке доказана теорема об эквивалентности утверждений, когда $S-\mathscr{A}$ такой максимальный левый идеал полугруппы S, в котором содержится всякий собственный левый идеал ($\mathscr{A}$ — объединение всех правых базисов).

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ON SEMIHOMEOMORPHISMS AND RELATED MAPPINGS

TIBOR NEUBRUNN, Bratislava

Introduction

Several topological properties preserve under more general mappings than are those of homeomorphisms. The somewhat continuous and semi-continuous functions as studied in [1], [2] may serve as examples. This paper contains some results which coplete or generalize some of the known results concerning these functions.

N. Levine [2] defined a semi-open set in a topological space X as a set A such that there exists an open set O so that $O \subset A \subset \bar{O}$ where $\bar{E}$ is the closure of E. He defined a function $f: X \to Y$ (X, Y topological spaces) to be semi-continuous if $f^{-1}(G)$ is semi-open for any open $G \subset Y$. The function $f: X \to Y$ is said to be semi-open if for any open $U \subset X$ the set $f(U)$ is semi-open in Y.

S. Gene Crossley and S. K. Hildebrandt in [3] defined irresolute and pre-semi-open functions. A function $f: X \to Y$ is said to be pre-semi-open (irresolute) if $f(A)$ is semi-open in Y ($f^{-1}(B)$ is semi-open in X) for any semi-open $A \subset X$ (any semi-open $B \subset Y$).

A one-to-one mapping $f: X \to Y$ is said to be semi-homeomorphism between X and Y if it is onto, irresolute and pre-semi-open.

Kempisty in [4] introduced a notion of quasicontinuous function. A function $f: X \to Y$ is said to be quasicontinuous at $x_0 \in X$ if for any open set V such that $f(x_0) \in V$ and any open U such that $x_0 \in U$ there exists a non-empty open set $G \subset U$ such that $f(G) \subset V$. If f is quasicontinuous at each point $x \in X$, it is said to be quasicontinuous in X.

A function $F: X \to Y$ is said to be somewhat continuous if for $V \subset Y$, V open in Y and such that $f^{-1}(V) \neq \emptyset$ there exists an open set $W \subset X$ such that $\emptyset \neq W \subset f^{-1}(V)$. It is said somewhat continuous on $X_0 \subset X$ if the restriction $f \mid X_0$ is somewhat continuous.

Results. The first is already a known result proved by A. Nebrunnová [6].

Lemma. A function $f: X \to Y$ is semi-continuous if and only if it is quasicontinuous.

Theorem 1. A function $f: X \to Y$ is semi-continuous if and only if it is somewhat continuous on every open subset of X.

Proof. Let f be semi-continuous and $\emptyset \neq G \subset X$ an open set. Let $V \subset Y$ be such open set that $f^{-1}(V) \cap G$ is nonempty. The set $G \cap f^{-1}(V)$ is semi-open as an intersection of a semi-open and open sets (see [2]). Since it is also nonempty it contains a nonempty open subset. Hence f is somewhat continuous on G.

Now let f be somewhat continuous on every open subset of X. Let $X_0 \in X$ and $V \subset Y$ open such that $f(x_0) \in V$. For any open U such that $x_0 \in U$ we have $f^{-1}(V) \cap U \neq \emptyset$. From somewhat continuity on U the existence of an open subset $W \neq 0$, $W \subset f^{-1}(V) \cap U$ follows. Hence $f(W) \subset V$. The quasicontinuity of f at x_0 is proved. Since x_0 is arbitrary point, f is quasicontinuous. Hence (see Lemma) f is semicontinuous.

Definition. A function $f: X \to Y$ is said to be somewhat open with respect to an open set $V \subset Y$ if for any open set $U \subset X$ such that $f(U) \cap V \neq \emptyset$ there exists a nonempty open set $G \subset f(U) \cap V$. It is said to be somewhat open if it is somewhat open with respect to $V = Y$. A one-to-one somewhat open and somewhat continuous function $f: X \to Y$ is said to be a somewhat homeomorphism.

Theorem 2. A function $f: X \to Y$ is semi-open if and only if it is somewhat open with respect to any open subset of Y.

Proof. Let f be semi-open. Let $U \subset X$, $V \subset Y$ be open sets such that $f(U) \cap V \neq 0$. Since $f(U)$ is semi-open and V open, the intersection $f(U) \cap V$ is semi-open. Hence the existence of an non-void open set $G \subset f(U) \cap V$ follows. Hence f is somewhat open with respect to V.

Now let f be somewhat open with respect to any open $V \subset Y$. Let $U \subset X$, $U \neq 0$ be open. Let $y \in f(U)$ and V any neighbourhood of J. We have $f(U) \cap V \neq 0$. Hence there exists a nonempty open set $G \subset f(U) \cap V$. So $y \in \overline{\operatorname{int} f(U)}$. The inclusion $f(U) \subset \overline{\operatorname{int} f(U)}$ is proved. Hence $f(U)$ is semi-open.

Theorem 3. If $f: X \to Y$ is one-to-one open and somewhat continuous then f is irresolute.

Proof. Let $A \subset Y$ be semi-open. Let $x \in f^{-1}(A)$. Then $y = f(x) \in A$. For any open U containing x the set $f(U)$ is open and contains y. Hence $f(U) \cap A \neq \emptyset$. Since $f(U)$ is open and A semi-open the existence of an open set $G \neq \emptyset$ follows such that $G \subset f(U) \cap A$. We have

$$f^{-1}(G) \subset f^{-1}[f(U) \cap A] \subset f^{-1}[f(U)] = U$$

The somewhat continuity implies that there exists an open set $V \subset f^{-1}(G)$, $V \neq \emptyset$. Hence $V \subset U$, $V \subset f^{-1}(A)$. Since U is an arbitrary neighbourhood of x we have $x \in \overline{\operatorname{int} f^{-1}(A)}$. So $f^{-1}(A)$ is semi-open and Theorem is proved.

The first of the following examples shows that the assumption of f being one-to-one is essential. The second shows the essentiality of the fact that f is open.

The third — one shows that somewhat continuity does not follow from the fact that f is open and one-to-one.

Example 1. Let $X=(0, 1)\cup(2, 3)$ with the topology of the real line. Let $Y=\{0, 1\}$ with the discrete topology.

$$\text{Define } f(x)=\begin{cases} 0 & \text{if } x\in(0, 1), \quad \text{or} \quad x=\frac{5}{2} \\ 1 & \text{if } \; x\in(2, 3), \quad x\neq\frac{5}{2} \end{cases}.$$

Example 2. Let $X=Y=(-\infty, +\infty)$ with the usual topology. Let $a\neq b$, $a, b\in(-\infty, +\infty)$. Put

$$g(x)=x \text{ if } x\neq a,\ x\neq b;\ g(a)=b,\ g(b)=a.$$

Example 3. Let X be the space of all the rational numbers with the topology of real line and Y the space of positive integers with the discrete topology. Any one-to-one function $f\colon X\to Y$ has the desired properties.

Corollary 1. If $f\colon X\to Y$ is one-to-one open and somewhat continuous then f is semi-continuous.

Corollary 2. In the set of all one-to-one open functions $f{:}X\to Y$ the somewhat continuous and the semi-continuous functions coincide.

Theorem 4. Let $f\colon X\to Y$ be continuous and somewhat open then f is pre-semi-open.

Proof. Let $A\subset X$ be semi-open. Let $y\in f(A)$. We have $y=f(x)$, $x\in A$. Let V be any open set containing y. The continuity in x implies the existence of an open set U such that $x\in U$, $f(U)\subset V$. Since A is semi-open, U open, $U\cap A\neq\emptyset$, there exists $W\neq\emptyset$, W open and $W\subset U\cap A$. Since f is somewhat open there exists an open set G, $G\neq 0$, $G\subset f(W)\subset f(U\cap A)\subset f(U)\cap f(A)\subset f(A)\cap V$. Hence $y\in\overline{\operatorname{int}(f(A))}$. So $f(A)\subset\overline{\operatorname{int} f(A)}$. The set $f(A)$ is semi-open and the theorem is proved.

Corollary 1 (See also [2]). If f is continuous and semi-open then f is pre-semi-open. (See Theorem 2)

Corollary 2. In the set of all continuous functions $f{:}X\to Y$ the somewhat open and the semi-open functions coincide.

Note 1. The condition of continuity in Theorem 4 can not be removed. As an example serves example 2. The condition of f being somewhat open can not be removed as well. The example 3 shows it when instead of f the inverse f^{-1} is considered.

Note 2. Theorem 3 is a consequence of Theorem 4 and in case of one-to-one mappings they are equivalent.

It follows from the evident facts that in case of a one-to-one function f we have

a) f is continuous if and only if f^{-1} is open;

b) f is somewhat continuous if and only if f^{-1} is somewhat open;

c) f is irresolute if and only if f^{-1} is pre-semi-open.

Note 3. Any semi-homeomorphism is a somewhat homeomorphism as can be easily deduced from the corresponding definitions. The example 2 shows that a somewhat homeomorphism need not be a semihomeomorphism.

Note 4. In [5] there were studied one-to-one mappings between X an Y which are semi-open and continuous. They were also called semi-homeomorphism. (We shall not call them semi-homeomorphism since the notion is used in an another sense).

It may be shown that a semi-homeomorphism f need not satisfy the condition of being continuous hence need not belong to the mentioned mappings studied in [5].

Example 4. Let $X = Y = \{a, b, c\}$. Define the topology in X as $\{\emptyset, \{a\}, \{a, b\}, X\}$ and the topology in Y as $\{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. The identity function $f: X \to Y$ is a semi-homeomorphism but it is not continuous.

If we consider the mappings defined by N. Biswas then for X compact and Y Hausdorff they are homeomorphisms hence also semi-homeomorphisms. It is not difficult to give an example showing that they are not homeomorphisms in general case. The inverse of the function considered in Example 4 is of this kind. What I do not know is whether there exists a one-to-one continuous and semi-open function without being a semi-homeomorphism.

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SÚHRN

O SEMIHOMEOMORFIZMOCH A PRÍBUZNÝCH ZOBRAZENIACH

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Rôzne zovšeobecnenia pojmu spojitosti dávajú možnosť študovať vlastnosti homeomorfizmov, ktorí k nim patria.

Táto práca obsahuje výsledky týkajúce sa rozličných takýchto homeomorfizmov a vzťahov medzi nimi.

РЕЗЮМЕ

О СЕМИГОМЕОМОРФИЗМАХ И С НИМИ СВЯЗАННЫХ ОТОБРАЖЕНИЯХ

Т. Неубрунн, Братислава

Многие обобщения понятия непрерывности дают возможность исследовать соответствующие им гомеоморфизмы. Настоящая работа представляет результаты, касающиеся различных таких гомеоморфизмов.

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ON PERIODIC SOLUTIONS OF THE FOURTH ORDER LINEAR SYSTEMS

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In the paper a fourth order linear system is considered all coefficients of which are not periodic functions. Some necessary and sufficient conditions are established for the existence of a periodic solution of such a system.

1. Consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{Q}(t)\mathbf{x}(t) \tag{1}$$

where $\mathbf{x}(t)$ is a real n-dimensional vector function, $\mathbf{Q}(t) = (q_{ik}(t))$ is a real continuous $n \times n$ matrix function for $t \in (-\infty, \infty)$.

The following two lemmas have been in a modified form and without proof given in the paper [1].

Lemma 1. A necessary condition for the existence of a non-trivial periodic solution of the system (1) with period T is that for every real number t the identity

$$\mathbf{D}(t, T) = \begin{vmatrix} q_{11}(t+T) - q_{11}(t), & \dots, & q_{1n}(t+T) - q_{1n}(t) \\ \vdots & & \\ q_{n1}(t+T) - q_{n1}(t), & \dots, & q_{nn}(t+T) - q_{nn}(t) \end{vmatrix} \equiv 0 \tag{2}$$

hold.

Proof. Let $(x_1(t), \dots, x_n(t))$ be a non-trivial solution of (1) such that for every t $x_i(t+T) = x_i(t)$, $\dot{x}_i(t+T) = \dot{x}_i(t)$, $i = 1, 2, \dots, n$ is true, i.e. $\mathbf{x}(t+T) = \mathbf{x}(t)$, $\dot{\mathbf{x}}(t+T) = \dot{\mathbf{x}}(t)$. Then $\dot{\mathbf{x}}(t+T) = \mathbf{Q}(t+T)\,\mathbf{x}(t+T) = \mathbf{Q}(t+T)\,\mathbf{x}(t) = \dot{\mathbf{x}}(t) = \mathbf{Q}(t)\,\mathbf{x}(t)$ from where we obtain $[\mathbf{Q}(t+T) - \mathbf{Q}(t)]\,\mathbf{x}(t) \equiv 0$. This result and the above assumption that $\mathbf{x}(t)$ is a nontrivial solution lead to $\det[\mathbf{Q}(t+t) - \mathbf{Q}(t)] = \mathbf{D}(t, T) \equiv 0$, which establishes the lemma.

Remark 1. The abovementioned condition is not sufficient as the linear differential equation

$$\dot{\mathbf{x}}_1(t) = \mathbf{x}_1(t)$$

shows. Its general solution is $\mathbf{x}_1(t) = C \exp t$.

Lemma 2. Let the coefficients $q_{ik}(t)$ of the matrix function $\mathbf{Q}(t)$ satisfy the following conditions

$$q_{ik}(t+T)=q_{ik}(t) \text{ for } i \leqq k,\ i, k=1, 2, \ldots, n \tag{3}$$

$$q_{j+1,j}(t+T)-q_{j+1,j}(t) \neq 0 \text{ for } j=1, 2, \ldots, n-1 \tag{4}$$

for $t \in (-\infty, \infty)$. Then the system (1) has a non-trivial T-periodic solution if and only if

$$\int_0^T q_{nn}(s)\, ds = 0 \tag{5}$$

$$q_{jn}(t) \equiv 0 \quad \text{for} \quad j=1, 2, \ldots, n-1 \tag{6}$$

This solution is of the form $x_1(t)=x_2(t)=\ldots = x_{n-1}(t)\equiv 0$, $x_n(t)= = C \exp\int_0^t q_{nn}(s)\, ds$, $C \neq 0$.

Proof. I. Let $(x_1(t), \ldots, x_n(t))$ be a non-trivial T-periodic solution of the system (1). Then obviously

$$\sum_{k=1}^{n} [q_{ik}(t+T)-q_{ik}(t)]x_k(t) \equiv 0 \quad \text{for} \quad i=1, 2, \ldots, n\,.$$

According to the assumptions we have

$$[q_{21}(t+T)-q_{21}(t)]x_1(t) \equiv 0\,,$$
$$\Rightarrow x_1(t) \equiv 0$$

$$[q_{31}(t+T)-q_{31}(t)q_{31}(t)]x_1(t)+[q_{32}(t+T)-q_{32}(t)]x_2(t)=0\,,$$
$$\Rightarrow x_2(t) \equiv 0$$

$\vdots$

$$[q_{n-1,1}(t+T)-q_{n-1,1}(t)]x_1(t)+\ldots+[q_{n-1,n-2}(t+T)-q_{n-1,n-2}(t)]x_{n-2}(t) \equiv 0\,,$$
$$\Rightarrow x_{n-2}(t) \equiv 0$$

$$[q_{n1}(t+T)-q_{n1}(t)]x_1(t)+\ldots+[q_{n,n-1}(t+T)-q_{n,n-1}(t)]x_{n-1}(t) \equiv 0\,,$$
$$\Rightarrow x_{n-1}(t) \equiv 0$$

Therefore in the considered case the system (1) reduces to

$$\dot{x}_n(t)=q_{nn}(t)x_n(t)$$

which gives after the integration the expression $x_n(t) = C \exp\int_0^t q_{nn}(s)\, ds$. Because we assume $x_n(t+T)=x_n(t)$, the equality $\int_0^T q_{nn}(s)\, ds=0$ must be fulfilled, i.e. (5) is true. Substituing the obtained expression into (1) we arrive at

$$0 = \dot{x}_1(t) = q_n(t)\, x_n(t)\,, \qquad \Rightarrow q_{ln}(t) \equiv 0$$

$$\vdots$$

$$0 = \dot{x}_{n-1}(t) = q_{n-1,n}(t)\, x_n(t)\,, \quad \Rightarrow q_{n-1,n}(t) \equiv 0 \quad .$$

i.e. (6) is true, too.

II. The sufficient condition is obvious.

Remark 2. Analogically the following lemma can be proved.

Lemma 2′. Let the coefficients $q_{ik}(t)$ of the matrix $\mathbf{Q}(t)$ satisfy the following conditions

$$q_{ik}(t+T) = q_{ik}(t) \quad \text{for} \quad i \geqq k,\ i, k = 1, 2, \ldots, n \tag{3'}$$

$$q_{j,j+1}(t+T) - q_{j,j+1}(t) \neq 0 \quad \text{for} \quad j = 1, 2, \ldots, n-1 \tag{4'}$$

for $t \in (-\infty, \infty)$. Then the system (1) has a non-trivial T-periodic solution if and only if

$$\int_0^T q_{11}(s)\, \mathrm{d}s = 0 \tag{5'}$$

$$q_{j,1}(t) = 0 \quad \text{for} \quad j = 2, 3, \ldots, n \tag{6'}$$

is true. This solution is of the form $x_1(t) = C \exp \int_0^t q_{11}(s)\, \mathrm{d}s$, $x_2(t) = \ldots = x_n(t) \equiv 0$.

2. Consider now the system

$$\dot{\mathbf{x}}(t) = \mathbf{P}(t)\mathbf{x}(t) \tag{1$_4$}$$

where $\mathbf{x}(t)$ is a real 4^{th}-dimensional vector function, $\mathbf{P}(t) = (p_{ik}(t))$ is a real continuous 4×4 matrix function for $t \in (-\infty, \infty)$, the coefficients of which are not all periodic functions with period T.

If $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ is a non-trivial T-periodic solution of that system, then by Lemma 1 the rank of the matrix $[\mathbf{P}(t+T) - \mathbf{P}(t)]$ is smaller then four. Clearly we also have

$$\sum_{k=1}^{4} [p_{ik}(t+T) - p_{ik}(t)] x_k(t) \equiv 0 \quad \text{for} \quad i = 1, 2, 3, 4 \tag{7$_4$}$$

a) Let us suppose that for every real number t the inequality

$$p_{44}(t+T) - p_{44}(t) \neq 0 \tag{8}$$

is true. Using this in (7$_4$) we get the relation

$$x_4(t) = \varphi_{41}(t)x_1(t) + \varphi_{42}(t)x_2(t) + \varphi_{43}(t)x_3(t) \tag{9}$$

where

$$\varphi_{4i}(t) = -\frac{p_{4i}(t+T) - p_{4i}(t)}{p_{44}(t+T) - p_{44}(t)}, \qquad i = 1, 2, 3$$

We see that under the above formulated assumption the considered solution $\boldsymbol{x}(t)$ statisfies the third order system

$$\dot{\boldsymbol{x}}(t) = \mathbf{Q}(t)\, \boldsymbol{x}(t) \tag{1_3}$$

where $\mathbf{Q}(t) = (q_{ik}(t))$ is a real continuous 3×3 matrix-function, the coefficients of which are defined by the relation

$$q_{ik}(t) = p_{ik}(t) + p_{i4}(t)\varphi_{4k}(t) \quad \text{for} \quad i, k = 1, 2, 3 \quad \text{and} \quad t \in (-\infty, \infty) \tag{10}$$

If we apply Lemma 2 to the system (1_3), we obtain a necessary and sufficient condition for the existence of a non-trivial T-periodic solution of this system provided that (3) and (4) for $n = 3$ are valid. The condition means that both (5) and (6) are fulfilled, where $q_{ik}(t)$ are determined by (10). The periodic solution has the components

$$x_1(t) = x_2(t) \equiv 0\,, \qquad x_3(t) = C \exp \int_0^t q_{33}(s)\, \mathrm{d}s \tag{11}$$

Having assumed (3), (4) (5) and (6), the fourth component $x_4(t)$ will be T-periodic if and only if the function $\varphi_{43}(t)$ is T-periodic.

Now, let us state the conditions which guarantee that the functions $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ obtained in this way form a T-periodic solution of the system (1_4).

Making use of (9) and (11) from the system (1_4) we obtain

$$\begin{aligned}
0 \equiv \dot{x}_1(t) &= p_{13}(t)x_3(t) + p_{14}(t)x_4(t) = \\
&= [p_{13}(t) + p_{14}(t)\varphi_{43}(t)]x_3(t) = q_{13}(t)x_3(t)\,, \\
0 \equiv \dot{x}_2(t) &= p_{23}(t)x_3(t) + p_{24}(t)x_4(t) = \\
&= [p_{23}(t) + p_{24}(t)\varphi_{43}(t)]x_3(t) = q_{23}(t)x_3(t)\,, \\
\dot{x}_3(t) &= p_{33}(t)x_3(t) + p_{34}(t)x_4(t) = \\
&= p_{33}(t) + p_{34}(t)\varphi_{43}(t)]x_3(t) = q_{33}(t)x_3(t)\,, \\
\dot{x}_4(t) &= p_{43}(t)x_3(t) + p_{44}(t)x_4(t) = \\
&= [p_{43}(t) + p_{44}(t)\varphi_{43}(t)]x_3(t)
\end{aligned}$$

from where we succesively arrive at the identities $p_{13}(t) + p_{14}(t)\varphi_{43}(t) = q_{13}(t) \equiv \equiv 0$, $p_{23}(t) + p_{24}(t)\varphi_{43}(t) = q_{23}(t) \equiv 0$, i.e. (6) is true, $x_3(t) = C \exp \int_0^t [p_{33}(s) + p_{34}(s)\varphi_{43}(s)]\, \mathrm{d}s = C \exp \int_0^t q_{33}(s)\, \mathrm{d}s$, which with regard to the equality $x_3(t+T) =$

$=x_3(t)$ gives that $\int_0^T q_{33}(s)\,ds = 0$, i.e. (5) holds, $\dot{x}_4(t) = C[p_{43}(t) + p_{44}(t)\varphi_{43}(t)] \exp\int_0^t q_{33}(s)\,ds = \dot{\varphi}_{43}(t)x_3(t) + \varphi_{43}(t)\dot{x}_3(t) = \dot{\varphi}_{43}(t)C\exp\int_0^t q_{33}(s)\,ds + \varphi_{43}(t)q_{33}(t)C\exp\int_0^t q_{33}(s)\,ds$. After some calculations we come to the identity

$$p_{43}(t) + p_{44}(t)\varphi_{43}(t) - \dot{\varphi}_{43}(t) - \varphi_{43}(t)q_{33}(t) \equiv 0 \tag{12}$$

The above considerations imply the following

Theorem 1. Let (3) and (4) for $n=3$, where $q_{ik}(t)$, $i, k = 1, 2, 3$ are defined by the relations (10) and (8) be true. Then a necessary and sofficient condition for the existence of a non-trivial T-periodic solution of the system (1_4) is that

1. the assumptions (5) and (6) be true,
2. the function $\varphi_{43}(t)$ be T-periodic, and
3. the identity (12) be satisfied.

This solution has the components

$$x_1(t) = x_2(t) \equiv 0\,,$$
$$x_3(t) = C\exp\int_0^t [p_{33}(s) + p_{34}(s)\varphi_{43}(s)]\,ds\,,$$
$$x_4(t) = C\varphi_{43}(t)\exp\int_0^t [p_{33}(s) + p_{34}(s)\varphi_{43}(s)]\,ds\,.$$

b) Let us suppose that there exists at least one second-order subdeterminant of the matrix of the system

$$\sum_{k=1}^{3} [q_{ik}(t+T) - q_{ik}(t)]x_k(t) \equiv 0 \quad \text{for} \quad i = 1, 2, 3 \tag{7_3}$$

which is different from zero for every real number t. Here $x_1(t)$, $x_2(t)$, $x_3(t)$ are again the first three components of the above considered non-trivial T-periodic solution $\boldsymbol{x}(t)$ of the system (1_4). Let the mentioned subdeterminant be in the right lower corner. We denote it as $\mathbf{Q}_{11}(t)$, i.e. the following inequality

$$\mathbf{Q}_{11}(t) = \begin{vmatrix} q_{22}(t+T) - q_{22}(t), & q_{23}(t+T) - q_{23}(t) \\ q_{32}(t+T) - q_{32}(t), & q_{33}(t+T) - q_{33}(t) \end{vmatrix} \neq 0$$
$$\text{for } t \in (-\infty, \infty) \tag{13}$$

is supposed. Then

$$x_2(t)=\frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)}x_1(t)\,, \qquad x_3(t)=\frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)}x_1(t) \tag{14}$$

may be written, where $\mathbf{A}_i(t)$, $i=1, 2$ can be calculated by the known method. Substituing the expressions (14) into (1_3) and integrating we obtain

$$x_1(t)=C\exp\int_0^t B(s)\,ds\,, \qquad C\neq 0 \tag{15}$$

where

$$B(t)=\frac{q_{11}(t)\mathbf{Q}_{11}(t)+q_{12}(t)\mathbf{A}_1(t)+q_{13}(t)\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)} \tag{16}$$

In the paper [1] (p. 1011) the following Lemma has been proved.

Lemma 3. Let (2) for $n=3$ and (13) be assumed. Then the system (1_3) has a non-trivial T-periodic solution if and only if the following condition are true:

1. The functions

$$\frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)}, \quad \frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)}, \quad B(t) \tag{17}$$

are T-periodic;

2. the equality

$$\int_0^T B(s)\,ds=0 \tag{18}$$

holds;

3. the coefficients $q_{ik}(t)$ defined by the relation (10) satisfy the identities

$$\begin{gathered}\dot{\mathbf{A}}_1(t)\mathbf{Q}_{11}(t)-\mathbf{A}_1(t)\dot{\mathbf{Q}}_{11}(t)+\mathbf{A}_1(t)B(t)\mathbf{Q}_{11}(t)-q_{21}(t)[\mathbf{Q}_{11}(t)]^2-\\ -q_{22}(t)\mathbf{A}_1(t)\mathbf{Q}_{11}(t)-q_{23}(t)\mathbf{A}_2(t)\mathbf{Q}_{11}(t)\equiv 0;\\ \dot{\mathbf{A}}_2(t)\mathbf{Q}_{11}(t)-\mathbf{A}_2(t)\dot{\mathbf{Q}}_{11}(t)+\mathbf{A}_2(t)B(t)\mathbf{Q}_{11}(t)-q_{31}(t)[\mathbf{Q}_{11}(t)]^2-\\ -q_{32}(t)\mathbf{A}_1(t)\mathbf{Q}_{11}(t)-q_{33}(t)\mathbf{A}_2(t)\mathbf{Q}_{11}(t)\equiv 0\quad .\end{gathered} \tag{19}$$

This solution is defined by the relation (15) and (14).

If moreover the assumption (8) is satisfied, then it (9) is true, too and hence, by virtue of Lemma 3 to the T-periodicity of $x_4(t)$ is sufficient, that the functions $\varphi_{4i}(t)$, $i=1, 2, 3$ be T-periodic.

Proceeding as in Section a) we determine the conditions which imply that the functions determined by the relations (15), (14) and (9) form a T-periodic solution of the system (1_4). Substituing the expressions (15), (14) and (9) into (1_4) we obtain that

$$\dot{x}_1(t) = p_{11}(t) C \exp \int_0^t B(s)\, ds + p_{12}(t) \frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)} C \exp \int_0^t B(s)\, ds +$$

$$+ p_{13}(t) \frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)} C \exp \int_0^t B(s)\, ds + p_{14}(t) \left[\varphi_{41}(t) + \right.$$

$$\left. + \varphi_{42}(t) \frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)} + \varphi_{43}(t) \frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)} \right] C \exp \int_0^t B(s)\, ds =$$

$$= B(t) C \exp \int_0^t B(s)\, ds$$

$$\dot{x}_2(t) = p_{21}(t) C \exp \int_0^t B(s)\, ds + p_{22}(t) \frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)} C \exp \int_0^t B(s)\, ds +$$

$$+ p_{23}(t) \frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)} C \exp \int_0^t B(s)\, ds + p_{24}(t) \left[\varphi_{41}(t) + \right.$$

$$\left. + \varphi_{42}(t) \frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)} + \varphi_{43}(t) \frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)} \right] C \exp \int_0^t B(s)\, ds =$$

$$= \frac{\dot{\mathbf{A}}_1(t)\mathbf{Q}_{11}(t) - \mathbf{A}_1(t)\dot{\mathbf{Q}}_{11}(t)}{[\mathbf{Q}_{11}(t)]^2} C \exp \int_0^t B(s)\, ds + \frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)} B(t) C \exp \int_0^t B(s)\, ds$$

$$\dot{x}_3(t) = \left[p_{31}(t) + p_{32}(t) \frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)} + p_{33}(t) \frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)} \right] C \exp \int_0^t B(s)\, ds +$$

$$+ p_{34}(t) \,.\, \left[\varphi_{41}(t) + \varphi_{42}(t) \frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)} + \varphi_{43}(t) \frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)} \right] C \exp \int_0^t B(s)\, ds =$$

$$= \left[\frac{\mathbf{A}_2(t)\mathbf{Q}_{11}(t) - \mathbf{A}_2(t)Q_{11}(t)}{[\mathbf{Q}_{11}(t)]^2} + \frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)} \right] B(t) C \int_0^t \exp B(s)\, ds$$

$$\dot{x}_4(t) = \left[p_{41}(t) + p_{42}(t) \frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)} + p_{43}(t) \frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)} \right] C \exp \int_0^t B(s)\, ds +$$

$$+ p_{44}(t) \left[\varphi_{41}(t) + \varphi_{42}(t) \frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)} + \varphi_{43}(t) \frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)} \right] C \exp \int_0^t B(s)\, ds =$$

$$= [\dot{\varphi}_{41}(t) + \varphi_{41}(t) B(t)] \,.\, C \exp \int_0^t B(s)\, ds +$$

$$+ \left[\dot{\varphi}_{42}(t) \frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)} + \varphi_{42}(t) \frac{\dot{\mathbf{A}}_1(t)\mathbf{Q}_{11}(t) - \mathbf{A}_1(t)\dot{\mathbf{Q}}_{11}(t)}{[\mathbf{Q}_{11}(t)]^2} + \right.$$

$$+\varphi_{42}(t)\frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)}B(t)\Big]C\exp\int_0^t B(s)\,ds+$$

$$+\Big[\dot{\varphi}_{43}(t)\frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)}+\varphi_{43}(t)\frac{\dot{\mathbf{A}}_2(t)\mathbf{Q}_{11}(t)-\mathbf{A}_2(t)\dot{\mathbf{Q}}_{11}(t)}{[\mathbf{Q}_{11}(t)]^2}+$$

$$+\varphi_{43}(t)\frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)}.B(t)\Big]C\exp\int_0^t B(s)\,ds$$

must hold.

The obtained equalities may be rewritten in a more convenient form.

Thus we get

Theorem 2. Let (2), (8) and (13) be satisfied. Then a sufficient condition for the existence of a non-trivial T-periodic solution of the system (1_4) is that

1. the assumptions (17), (18) and (19) be fulfilled;
2. the functions $\varphi_{4i}(t)$ be T-periodic for $i=1, 2, 3$;
3. the identity

$$\begin{aligned}&[\dot{\mathbf{A}}_1(t)\mathbf{Q}_{11}(t)-\mathbf{A}_1(t)\dot{\mathbf{Q}}_{11}(t)]\varphi_{42}(t)+[\dot{\mathbf{A}}_2(t)\mathbf{Q}_{11}(t)-\mathbf{A}_2(t)\dot{\mathbf{Q}}_{11}(t)]\varphi_{43}(t)+\\&+[\dot{\varphi}_{41}(t)+\varphi_{41}(t)B(t)-p_{41}(t)-p_{44}(t)\varphi_{41}(t)]\,[\mathbf{Q}_{11}(t)]^2+\\&+[\dot{\varphi}_{42}(t)+\varphi_{42}(t)B(t)-p_{42}(t)-p_{44}(t)\varphi_{42}(t)]\mathbf{A}_1(t)\mathbf{Q}_{11}(t)+\\&+[\dot{\varphi}_{43}(t)+\varphi_{43}(t)B(t)-p_{43}(t)-p_{44}(t)\varphi_{43}(t)]\mathbf{A}_2(t)\mathbf{Q}_{11}(t)\equiv 0\end{aligned}\tag{20}$$

be true. If this condition is satisfied, the functions

$$\begin{aligned}&x_1(t)=C\exp\int_0^t B(s)\,ds\,,\\&x_2(t)=\frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)}C\exp\int_0^t B(s)\,ds\,,\\&x_3(t)=\frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)}C\exp\int_0^t B(s)\,ds\,,\\&x_4(t)=C\Big[\varphi_{41}(t)+\varphi_{42}(t)\frac{\mathbf{A}_1(t)}{\mathbf{Q}_{11}(t)}+\varphi_{43}(t)\frac{\mathbf{A}_2(t)}{\mathbf{Q}_{11}(t)}\Big]\exp\int_0^t B(s)\,ds\end{aligned}\tag{21}$$

form a T-periodic solution of the system (1_4). Here $C\neq 0$ is a constant.

3. Let us suppose that the inequality

$$\mathbf{P}_1(t)=\begin{vmatrix}p_{33}(t+T)-p_{33}(t), & p_{34}(t+T)-p_{34}(t)\\ p_{43}(t+T)-p_{43}(t), & p_{44}(t+T)-p_{44}(t)\end{vmatrix}=0$$

$$\text{for}\quad t\in(-\infty,\infty)\tag{22}$$

is true. Then, using (7_4), for each T-periodic solution $(x_1(t), x_2(t), x_3(t), x_4(t))$ of the system (1_4) we obtain

$$x_3(t)=\frac{\mathbf{A}_{13}(t)}{\mathbf{P}_1(t)}x_1(t)+\frac{\mathbf{A}_{23}(t)}{\mathbf{P}_1(t)}x_2(t)$$
$$x_4(t)=\frac{\mathbf{A}_{14}(t)}{\mathbf{P}_1(t)}x_1(t)+\frac{\mathbf{A}_{24}(t)}{\mathbf{P}_1(t)}x_2(t) \qquad (23)$$

where $\mathbf{A}_{i3}(t)$, $\mathbf{A}_{i4}(t)$, $i=1, 2$ can be calculated by the known method. Substituing (23) into the system (1_4) we get

$$\dot{\mathbf{x}}(t)=\mathbf{R}(t)\mathbf{x}(t) \qquad (1_2)$$

where $\mathbf{R}(t)\equiv(r_{ik}(t))$ is a 2×2 matrix-function, the coefficients of which are determined by the relation

$$r_{ik}(t)=p_{ik}(t)+p_{i3}(t)\frac{\mathbf{A}_{k3}(t)}{\mathbf{P}_1(t)}+p_{i4}(t)\frac{\mathbf{A}_{k4}(t)}{\mathbf{P}_1(t)}, \qquad i, k=1, 2 \qquad (24)$$

Hence, if the conditions of Theorems 1 and 2 of the paper [1] are satisfied, we come to $x_1(t)$, $x_2(t)$ which form a non-trivial T-periodic solution of the system (1_2). If moreover the functions (23) are also T-periodic, then substituing them into the system (1_4) we can obtain the conditions for the existence of a non-trivial T-periodic solution of the considered system (1_4).

4. Let us suppose that the inequality

$$\mathbf{P}_2(t)=\begin{vmatrix} p_{22}(t+T)-p_{22}(t), & p_{23}(t+T)-p_{23}(t), & p_{24}(t+T)-p_{24}(t) \\ p_{32}(t+T)-p_{32}(t), & p_{33}(t+T)-p_{33}(t), & p_{34}(t+T)-p_{34}(t) \\ p_{42}(t+T)-p_{42}(t), & p_{43}(t+T)-p_{43}(t), & p_{44}(t+T)-p_{44}(t) \end{vmatrix}\neq0 \qquad (25)$$

is true on the entire real line. Then from the system (7_4) for a T-periodic solution $(x_1(t), x_2(t), x_3(t), x_4(t))$ we obtain

$$x_2(t)=\frac{\mathbf{B}_1(t)}{\mathbf{P}_2(t)}x_1(t), \quad x_3(t)=\frac{\mathbf{B}_2(t)}{\mathbf{P}_2(t)}x_1(t), \quad x_4(t)\frac{\mathbf{B}_3(t)}{\mathbf{P}_2(t)}x_1(t) \qquad (26)$$

where $\mathbf{B}_i(t)$, $i=1, 2, 3$ can be found by the Cramer rule. Substituing (26) into the system (1_4) we arrive at

$$\dot{x}_1(t)=\left[p_{11}(t)+p_{12}(t)\frac{\mathbf{B}_1(t)}{\mathbf{P}_2(t)}+p_{13}(t)\frac{\mathbf{B}_2(t)}{\mathbf{P}_2(t)}+\right.$$

$$\left.+p_{14}(t)\frac{\mathbf{B}_3(t)}{\mathbf{P}_2(t)}\right]x_1(t)=\mathbf{A}(t)x_1(t)\,,$$

which gives after the integration

$$x_1(t)=C\exp\int_0^t \mathbf{A}(s)\,\mathrm{d}s \tag{27}$$

This implies that the functions $x_i(t)$, $i=1, 2, 3, 4$ are T-periodic if the functions

$$\frac{\mathbf{B}_1(t)}{\mathbf{P}_2(t)},\quad \frac{\mathbf{B}_2(t)}{\mathbf{P}_2(t)},\quad \frac{\mathbf{B}_3(t)}{\mathbf{P}_2(t)},\quad \mathbf{A}(t) \tag{28}$$

are T-periodic and the equality

$$\int_0^T \mathbf{A}(s)\,\mathrm{d}s=0 \tag{29}$$

is valid. Substituing the functions determined by the relations (27) and (26) into the system (1_4) after some calculations we obtain the identities

$$\begin{aligned}&\dot{\mathbf{B}}_i(t)\mathbf{P}_2(t)-\mathbf{B}_i(t)\dot{\mathbf{P}}_2(t)+\mathbf{B}_i(t)\mathbf{A}(t)\mathbf{P}_2(t)-\\&-p_{i+1,1}(t)\,[\mathbf{P}_2(t)]^2-p_{i+1,2}(t)\mathbf{B}_1(t)\mathbf{P}_2(t)-\\&-p_{i+1,3}(t)\mathbf{B}_2(t)\mathbf{P}_2(t)-p_{i+1,4}(t)\mathbf{B}_3(t)\mathbf{P}_2(t)\equiv 0\end{aligned} \tag{30}$$

for $i=1, 2, 3$.

Thus, we have proved the following

Theorem 3. Let for the coefficients of the system (1_4) the relations (2) and (25) hold. Then a necessary and sufficient condition for the existence of a non-trivial T-periodic solution of the system (1_4) is that

1. the equality (29) be true;
2. the functions (28) be T-periodic;
3. the identities (30) for the coefficients of the system (1_4) hold.

If the condition is satisfied, the relations (27) and (26) define this periodic solution.

Remark 3. This result can be without difficulty extended to the general n-th order system (1). Thus the following theorem is true.

Theorem 4. Let for the coefficients of the system (1) the identity (2) be true. Denote $\mathbf{Q}_{11}(t)$ the $(n-1)$-st order subdeterminant of the matrix of the system (7_n)

which is situated in the right lower corner. Let $\mathbf{Q}_{11}(t) \neq 0$ for every real number t. Then a necessary and sufficient condition for the existence of a non-trivial T-periodic solution of the system (1) is

1. the T-periodicity of the functions $\dfrac{\mathbf{B}_j(t)}{\mathbf{Q}_{11}(t)}$, $j = 1, 2, \ldots, n-1$;

$$\mathbf{A}(t) = \frac{p_{11}(t)\mathbf{Q}_{11}(t) + \sum_{k=2}^{n} p_{1k}(t)\mathbf{B}_{k-1}(t)}{\mathbf{Q}_{11}(t)}$$

where $\mathbf{B}_j(t)$, $j = 1, 2, \ldots, n$ can be found in the known way;

2. the equality $\int_0^T \mathbf{A}(s)\,\mathrm{d}s = 0$; and

3. the identities

$$\dot{\mathbf{B}}_j(t)\mathbf{Q}_{11}(t) - \mathbf{B}_j(t)\dot{\mathbf{Q}}_{11}(t) + \mathbf{B}_j(t)\mathbf{A}(t)\mathbf{Q}_{11}(t) -$$

$$- p_{j+1,1}(t)\,[\mathbf{Q}_{11}(t)]^2 - \sum_{k=2}^{n} p_{j+1,k}(t)\mathbf{B}_{k-1}(t)\mathbf{Q}_{11}(t) \equiv 0\,,$$

$$j = 1, 2, \ldots, n-1;$$

If the condition is satisfied, the functions

$$x_1(t) = C \exp \int_0^t \mathbf{A}(s)\,\mathrm{d}s, \quad x_k(t) = \frac{\mathbf{B}_{k-1}(t)}{\mathbf{Q}_{11}(t)}\, x_1(t) \quad \text{for } k = 2, 3, \ldots, n$$

form a T-periodic solution.

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SÚHRN

O PERIODICKÝCH RIEŠENIACH LINEÁRNYCH SYSTÉMOV ŠTVRTÉHO RÁDU

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V práci sa skúma lineárny systém štvrtého rádu, ktorého koeficienty nie sú súčasne všetky T-periodické funkcie a určujú sa niektoré nevyhnutné a postačujúce podmienky existencie T-periodického riešenia tohto systému.

РЕЗЮМЕ

О ПЕРИОДИЧЕСКИХ РЕШЕНИЯХ ЛИНЕЙНЫХ СИСТЕМ ЧЕТВЕРТОГО ПОРЯДКА

Владислав Роса, Братислава

В статьи рассматривбаетсця линейная система четвертого порядка, все коэффициенты которой неявляются одновременно Т-периодическими функциями и установливаются некоторые необходимые и достаточные условия существования Т-периодических решений етой системы.

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GENERALIZED SYMMETRIC MEANS (G.S.M.'S) AND BIPOLYKAYS OF DEGREE THE 5th

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The g.s.m.'s of degrees 3rd, 4th have been developed by R. Hooke in [1]. This paper contains all distinct g.s.m.'s and bipolykays of degree the 5th.

I. Preliminaries

The symbol $\sum^{=}$ shall mean the sum over all subscripts that follow with the restriction that differently primed subscripts remain unequel Symmetric means and polykays have been priviously denoted by brackets and parentheses respectively [3], e.g. the symmetric means $\langle ab \rangle = \frac{1}{N(N-1)} \sum^{\neq} x_i^a x_j^b$ where $i = 1, 2, \ldots, N$, and corresponding polykay is denoted (ab). In contract to this "primary" notation Hooke [1] used a "secondary" notation, where the entries $a, b, \ldots, d$ of a symmetric mean $ab \ldots d$ of degree m form a partition of the integer m. It will be convenient to represent such a partition in terms of m distinct symbols, so that the secondary notation for $ab \ldots d$ will be $p_1p_2 \ldots p_a, \ldots, s_1s_2 \ldots s_d$ where commas are used to separate the parts of the partition, and the lengths of the part are the positive integers $a, b, \ldots, d$, whise sum is m. Thus in the secondary notation the comma separates the parts of the partition with $a, b, \ldots, d$ denoting the lengths of the partitions. Two partitions are said to be equivalent, or not distinct if they are identical, except possibly for the order of parts and the order of symbols within a part. For $m = 2$ the only are 11 and 2 itself, or in the secondary notation p_1, q_1 and p_1p_2, respectively, which in this case will be denoted by p, q and pq. Henceforth let α denote the partition pq and β the partition p, q. Let $\gamma, \delta, \ldots$ denote the arbitrary partitions. A subpartition of a partition γ may be formed by inserting one or more commas between the letters of γ. Thus for $m = 2$, β is a subpartition of α. The following implicit definition of the polykays is given by Hooke:

Definition 1. The polykays of degree m are defined by the equations

$$\langle \gamma \rangle = (\gamma) + \sum (\delta_\gamma)$$

where the summation is over all the distinct subpartitions δ_γ of γ.

Two symmetric means, or polykays, are equivalent, or not distinct, if the partitions representing them can be made equivalent by renaming the symbols.

Since any symmetric polynomial function can be expressed as a linear combination of symmetric means, it follows from the definition just given that it can also be expressed as a linear combination of polykays.

Thus, for example, the polykays of degree three are defined by the equations

$$\begin{aligned}
\langle p, q, r \rangle &= (p, q, r) \\
\langle p, qr \rangle &= (p, qr) + (p, q, r) \\
\langle pqr \rangle &= (pqr) + (p, qr) + (q, pr) + (r, pq) + (p, q, r)
\end{aligned}$$

These may be solved to give

$$\begin{aligned}
(p, q, r) &= \langle p, q, r \rangle \\
(p, qr) &= \langle p, qr \rangle - \langle p, q, r \rangle \\
(pqr) &= \langle pqr \rangle - \langle p, qr \rangle - \langle q, pr \rangle - \langle r, pq \rangle + 2\langle p, q, r \rangle
\end{aligned}$$

or in Tukey's primary notation

$$\begin{aligned}
k_{111} &= \langle 111 \rangle \\
k_{12} &= \langle 12 \rangle - \langle 111 \rangle \\
k_3{}' &= \langle 3 \rangle - 3\langle 12 \rangle + 2\langle 111 \rangle
\end{aligned}$$

In introducing bipolykays Hooke [1] defined generalized symmetric means to be the averages of monomial functions over a matrix, i.e., a polynomial of the form

$$\frac{1}{N} \sum_{r, s, \ldots, v} x_{rs}^{a_{rs}} \ldots x_{tv}^{a_{tv}}$$

where the symbol $\sum^*$ indicates the sum over all the subscripts with the restriction that subscripts represented by different letters must remain unequal throughout the summation, and N is the number of terms in the summation. The general term

$$x_{rs}^{a_{rs}} \ldots x_{tv}^{a_{tv}}$$

contains m factors, of which a_{rs} are equal to x_{rs}, etc., the degree of the symmetric mean being $m = a_{rs} + \ldots + a_{tv}$. To each factor a different symbol is assigned and the resulting set of symbols partitioned in two ways-once by rows and once by columns. Hence the secondary notation for the g.s.m. is an ordered pair $\langle \alpha/\beta \rangle$ of partitions α and β, each on the same set of symbols. For example

$$\begin{bmatrix} 2 & 1 \\ - & 1 \end{bmatrix} = \frac{1}{r(r-1)c(c-1)} \sum{}^* x_{ij}^2 x_{ik} x_{vk} = \langle pqs, t/pq, st \rangle$$

Following Hooke's formalization of the bipolykays we now define a noncommutative "dot-multiplication" for symmetric means as follows:

$$\langle \alpha \rangle \cdot \langle \beta \rangle = \langle \alpha/\beta \rangle$$

if α and β consist of the same symbols, $\langle \alpha \rangle \cdot \langle \beta \rangle = 0$ otherwise.

This noncommutative multiplication can be extendes by distributivity to provide dot-product of linear combinations of symmetric means.

Definition 2. The bipolykay (α/β), where α and β are partitions of the same set of symbols, is

$$(\alpha/\beta) = (\alpha) \cdot (\beta)$$

is being understood that (α) and (β) are expressed as sums of symmetric means before the dot-product is taken.

Example. Consider the bipolykay $\begin{pmatrix} 1 & 1 \\ - & 1 \end{pmatrix}$.

(The primary notation for a bipolykay (α/β) is the same as that for the g.s.m. $\langle \alpha/\beta \rangle$, with $\langle \ \rangle$'s replaced by parentheses). This becomes, in the secondary notation

$$\begin{pmatrix} 1 & 1 \\ - & 1 \end{pmatrix} = (pq, s/p, qs) = (pq, s) \cdot (p, qs) =$$

$$= [\langle pq, s \rangle - \langle s, p, q \rangle] \cdot [\langle qs, p \rangle - \langle p, q, s \rangle] =$$

$$= \langle s, pq/p, qs \rangle - \langle s, p, q/p, qs \rangle - \langle s, pq/p, q, s \rangle + \langle s, p, q/p, q, s \rangle =$$

$$\begin{bmatrix} 1 & 1 \\ - & 1 \end{bmatrix} - \begin{bmatrix} 1 & - \\ 1 & - \\ - & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & - \\ - & - & 1 \end{bmatrix} + \begin{bmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{bmatrix}$$

II. Formulas for g.s.m.'s and bipolykays

In this section, tables will be presented which make possible the use of bipolykays up through the degree 5. The distinct g.s.m.'s of degrees 1, 2, 3 and 4 were listed in [1], [4] and were denoted by t's for degree, 3, f's for degree 4:

$$t_4 = \begin{bmatrix} 1 & 1 & 1 \\ - & - & - \end{bmatrix}, \quad t_7 = \begin{bmatrix} 2 & - \\ - & 1 \end{bmatrix}, \quad \text{etc.}$$

$$f_6 = \begin{bmatrix} 1 & 1 & 1 & - \\ - & - & - & 1 \end{bmatrix}, \quad f_{18} = \begin{bmatrix} 2 & - & - \\ - & 1 & 1 \end{bmatrix}, \quad \text{etc.}$$

For degrees 3 and 4 he uses [1] a denotation analogous to that used above for g.s.m.'s letting T's stand for bipolykays of degree 3 and F's for bipolykays of degree 4. Thus

$$T_4 = \begin{pmatrix} 1 & 1 & 1 \\ - & - & - \end{pmatrix}, \quad T_7 = \begin{pmatrix} 2 & - \\ - & 1 \end{pmatrix}, \quad \text{etc.}$$

$$F_6 = \begin{pmatrix} 1 & 1 & 1 & - \\ - & - & - & 1 \end{pmatrix}, \quad F_{18} = \begin{pmatrix} 2 & - & - \\ - & 1 & 1 \end{pmatrix}, \quad \text{etc.}$$

Those of degree 5 require also more space and so they be denoted by p's with subscripts as follows:

$$p_1 = \begin{vmatrix} 1 & - & - & - & - \\ - & 1 & - & - & - \\ - & - & 1 & - & - \\ - & - & - & 1 & - \\ - & - & - & - & 1 \end{vmatrix}, \quad p_2 = \begin{vmatrix} 1 & 1 & - & - & - \\ - & - & 1 & - & - \\ - & - & - & 1 & - \\ - & - & - & - & 1 \end{vmatrix},$$

$$p_3 = \begin{vmatrix} 1 & - & - & - \\ 1 & - & - & - \\ - & 1 & - & - \\ - & - & 1 & - \\ - & - & - & 1 \end{vmatrix}, \quad p_4 = \begin{vmatrix} 1 & 1 & - & - & - \\ - & - & 1 & 1 & - \\ - & - & - & - & 1 \end{vmatrix}$$

$$p_5 = \begin{vmatrix} 1 & - & - \\ 1 & - & - \\ - & 1 & - \\ - & 1 & - \\ - & - & 1 \end{vmatrix}, \quad p_6 = \begin{vmatrix} 1 & 1 & 1 & - & - \\ - & - & - & 1 & - \\ - & - & - & - & 1 \end{vmatrix},$$

$$p_7 = \begin{vmatrix} 1 & - & - \\ 1 & - & - \\ 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{vmatrix}, \quad p_8 = \begin{vmatrix} 1 & 1 & 1 & 1 & - \\ - & - & - & - & 1 \end{vmatrix}$$

$$p_9 = \begin{vmatrix} 1 & - \\ 1 & - \\ 1 & - \\ 1 & - \\ - & 1 \end{vmatrix}, \quad p_{10} = \begin{vmatrix} 1 & 1 & - & - \\ 1 & - & - & - \\ - & - & 1 & - \\ - & - & - & 1 \end{vmatrix},$$

$$p_{11}=\begin{vmatrix}1&1&-&-\\-&-&1&-\\-&-&-&1\\-&-&-&1\end{vmatrix},\qquad p_{12}=\begin{vmatrix}2&-&-&-\\-&1&-&-\\-&-&1&-\\-&-&-&1\end{vmatrix}$$

$$p_{13}=\begin{vmatrix}1&1&-&-\\1&-&1&-\\-&-&-&1\end{vmatrix},\qquad p_{14}=\begin{vmatrix}1&1&-\\1&-&-\\-&1&-\\-&-&1\end{vmatrix},$$

$$p_{15}=\begin{vmatrix}1&1&1&-\\1&-&-&-\\-&-&-&1\end{vmatrix},\qquad p_{16}=\begin{vmatrix}1&1&-\\1&-&-\\1&-&-\\-&-&1\end{vmatrix}$$

$$p_{17}=\begin{vmatrix}1&1&-\\1&1&-\\-&-&1\end{vmatrix},\qquad p_{18}=\begin{vmatrix}2&-&-&-\\-&1&1&-\\-&-&-&1\end{vmatrix},\qquad p_{19}=\begin{vmatrix}2&-&-\\-&1&-\\-&1&-\\-&-&1\end{vmatrix},\qquad p_{20}=\begin{vmatrix}2&1&-&-\\-&-&1&-\\-&-&-&1\end{vmatrix}$$

$$p_{21}=\begin{vmatrix}2&-&-\\1&-&-\\-&1&-\\-&-&1\end{vmatrix},\qquad p_{22}=\begin{vmatrix}2&1&1&-\\-&-&-&1\end{vmatrix},\qquad p_{23}=\begin{vmatrix}2&-\\1&-\\1&-\\-&1\end{vmatrix},\qquad p_{24}=\begin{vmatrix}2&1&-\\-&1&-\\-&-&1\end{vmatrix}$$

$$p_{25}=\begin{vmatrix}2&-&-\\1&1&-\\-&-&1\end{vmatrix},\qquad p_{26}=\begin{vmatrix}2&1&-\\1&-&-\\-&-&1\end{vmatrix},\qquad p_{27}=\begin{vmatrix}2&-&-\\-&2&-\\-&-&1\end{vmatrix},\qquad p_{28}=\begin{vmatrix}3&-&-\\-&1&-\\-&-&1\end{vmatrix}$$

$$p_{29}=\begin{vmatrix}2&2&-\\-&-&1\end{vmatrix},\qquad p_{30}=\begin{vmatrix}2&-\\2&-\\-&1\end{vmatrix},\qquad p_{31}=\begin{vmatrix}3&1&-\\-&-&1\end{vmatrix},\qquad p_{32}=\begin{vmatrix}3&-\\1&-\\-&1\end{vmatrix}$$

$$p_{33}=\begin{vmatrix}4&-\\-&1\end{vmatrix},\qquad p_{34}=\begin{vmatrix}1&1&1&-&-\\-&-&-&1&1\end{vmatrix},\qquad p_{35}=\begin{vmatrix}1&-\\1&-\\1&-\\-&1\\-&1\end{vmatrix},\qquad p_{36}=\begin{vmatrix}1&1&-&-\\1&-&-&-\\-&-&1&1\end{vmatrix}$$

$$p_{37}=\begin{vmatrix}1&1&-\\1&-&-\\-&-&1\\-&-&1\end{vmatrix},\quad p_{38}=\begin{vmatrix}1&1&1\\1&-&-\\-&1&-\end{vmatrix},\quad p_{39}=\begin{vmatrix}1&1&-\\1&-&1\\1&-&-\end{vmatrix},\quad p_{40}=\begin{vmatrix}1&1&-\\1&-&1\\-&1&-\end{vmatrix}$$

$${}_{41}=\begin{vmatrix}2&-&-\\-&1&1\\-&1&-\end{vmatrix},\quad p_{42}=\begin{vmatrix}1&1&-\\-&-&1\\-&-&1\\-&-&1\end{vmatrix},\quad p_{43}=\begin{vmatrix}1&-&-&-\\1&-&-&-\\-&1&1&1\end{vmatrix},\quad p_{44}=\begin{vmatrix}2&-&-\\1&-&-\\-&1&1\end{vmatrix}$$

$$p_{45}=\begin{vmatrix}2&1&-\\-&-&1\\-&-&1\end{vmatrix},\quad p_{46}=\begin{vmatrix}1&1&-\\1&-&1\\-&-&1\end{vmatrix},\quad p_{47}=\begin{vmatrix}1&1&-\\1&-&-\\-&1&1\end{vmatrix},\quad p_{48}=\begin{vmatrix}1&1&1\\1&-&-\\1&-&-\end{vmatrix}$$

$$p_{49}=\begin{vmatrix}1&1&1&-\\1&-&-&1\end{vmatrix},\quad p_{50}=\begin{vmatrix}1&1\\1&-\\1&-\\-&1\end{vmatrix},\quad p_{51}=\begin{vmatrix}2&-&-&-\\-&1&1&1\end{vmatrix},\quad p_{52}=\begin{vmatrix}2&-\\-&1\\-&1\\-&1\end{vmatrix}$$

$$p_{53}=\begin{vmatrix}2&1&-&-\\-&-&1&1\end{vmatrix},\quad p_{54}=\begin{vmatrix}2&-\\1&-\\-&1\\-&1\end{vmatrix},\quad p_{55}=\begin{vmatrix}1&1&1&1\\1&-&-&-\end{vmatrix},\quad p_{56}=\begin{vmatrix}1&1\\1&-\\1&-\\1&-\end{vmatrix}$$

$$p_{57}=\begin{vmatrix}1&1&1&1&1\\-&-&-&-&-\end{vmatrix},\quad p_{58}=\begin{vmatrix}1&-\\1&-\\1&-\\1&-\\1&-\end{vmatrix},\quad p_{59}=\begin{vmatrix}2&1&1&1\\-&-&-&-\end{vmatrix},\quad p_{60}=\begin{vmatrix}2&-\\1&-\\1&-\\1&-\end{vmatrix}$$

$$p_{61}=\begin{vmatrix}3&-&-\\-&1&1\end{vmatrix},\quad p_{62}=\begin{vmatrix}3&-\\-&1\\-&1\end{vmatrix},\quad p_{63}=\begin{vmatrix}2&-&-\\-&2&1\end{vmatrix},\quad p_{64}=\begin{vmatrix}2&-\\-&2\\-&1\end{vmatrix}$$

$$p_{65}=\begin{vmatrix}1&1&1\\1&1&-\end{vmatrix},\quad p_{66}=\begin{vmatrix}1&1\\1&1\\1&-\end{vmatrix},\quad p_{67}=\begin{vmatrix}2&-&-\\1&1&1\end{vmatrix},\quad p_{68}=\begin{vmatrix}2&1\\-&1\\-&1\end{vmatrix}$$

$$p_{69}=\begin{vmatrix}2&1&-\\1&-&1\end{vmatrix},\quad p_{70}=\begin{vmatrix}2&1\\1&-\\-&1\end{vmatrix},\quad p_{71}=\begin{vmatrix}2&1&-\\-&1&1\end{vmatrix},\quad p_{72}=\begin{vmatrix}2&-\\1&1\\-&1\end{vmatrix}$$

$p_{73}=\begin{vmatrix}2&1&1\\1&-&-\end{vmatrix}$, $p_{74}=\begin{vmatrix}2&1\\1&-\\1&-\end{vmatrix}$, $p_{75}=\begin{vmatrix}2&1&1\\-&1&-\end{vmatrix}$, $p_{76}=\begin{vmatrix}2&-\\1&1\\1&-\end{vmatrix}$

$p_{77}=\begin{vmatrix}3&-\\-&2\end{vmatrix}$, $p_{78}=\begin{vmatrix}3&1&1\\-&-&-\end{vmatrix}$, $p_{79}=\begin{vmatrix}3&-\\1&-\\1&-\end{vmatrix}$, $p_{80}=\begin{vmatrix}3&-\\1&1\end{vmatrix}$

$p_{81}=\begin{vmatrix}3&1\\-&1\end{vmatrix}$, $p_{82}=\begin{vmatrix}2&-\\2&-\\1&-\end{vmatrix}$, $p_{83}=\begin{vmatrix}2&2&1\\-&-&-\end{vmatrix}$, $p_{84}=\begin{vmatrix}2&1\\1&1\end{vmatrix}$

$p_{85}=\begin{vmatrix}2&1\\2&-\end{vmatrix}$, $p_{86}=\begin{vmatrix}2&2\\1&-\end{vmatrix}$, $p_{87}=\begin{vmatrix}2&1\\-&2\end{vmatrix}$, $p_{88}=\begin{vmatrix}3&-\\2&-\end{vmatrix}$

$p_{89}=\begin{vmatrix}3&2\\-&-\end{vmatrix}$, $p_{90}=\begin{vmatrix}3&1\\1&-\end{vmatrix}$, $p_{91}=\begin{vmatrix}4&1\\-&-\end{vmatrix}$, $p_{92}=\begin{vmatrix}4&-\\1&-\end{vmatrix}$

$p_{93}=\begin{vmatrix}5&-\\-&-\end{vmatrix}$

For degree 5 we use a notation analogous to that used above for g.s.m.'s letting P's stand bipolykays of degree 5.

$$\text{Thus}\quad P_1=\begin{pmatrix}1&-&-&-&-\\-&1&-&-&-\\-&-&1&-&-\\-&-&-&1&-\\-&-&-&-&1\end{pmatrix},\quad P_2=\begin{pmatrix}1&1&-&-&-\\-&-&1&-&-\\-&-&-&1&-\\-&-&-&-&1\end{pmatrix},\quad \ldots\ \text{etc.}$$

The formulas for bipolykays of degree 5 are following:

$P_1 = p_1$
$P_2 = p_2 - p_1$
$P_3 = p_3 - p_1$
$P_4 = p_4 - 2p_2 + p_1$
$P_5 = p_5 - 2p_3 + p_1$
$P_6 = p_6 - 3p_2 + 2p_1$
$P_7 = p_7 - 3p_3 + 2p_1$
$P_8 = p_8 - 4p_6 - 3p_4 + 12p_2 - 6p_1$
$P_9 = p_9 - 4p_7 - 3p_5 + 12p_3 - 6p_1$

$P_{10}=p_{10}-p_3-p_2+p_1$
$P_{11}=p_{11}-p_3-p_2+p_1$
$P_{12}=p_{12}-p_3-p_2+p_1$
$P_{13}=p_{13}-2p_{10}-p_4+p_3+2p_2-p_1$
$P_{14}=p_{14}-2p_{10}-p_5+2p_3+p_2-p_1$
$P_{15}=p_{15}-p_{11}-2p_{10}-p_6+2p_3-2p_1+3p_2$
$P_{16}=p_{16}-p_{11}-2p_{10}-p_7+3p_3+2p_2-2p_1$
$P_{17}=p_{17}-2p_{14}-2p_{13}+4p_{10}+p_5+p_4-2p_3-2p_2+p_1$
$P_{18}=p_{18}-p_{12}-p_{11}-p_4+p_3+2p_2-p_1$
$P_{19}=p_{19}-p_{12}-p_{11}-p_5+2p_3+p_2-p_1$
$P_{20}=p_{20}-p_{12}-2p_{10}-p_6+2p_3+3p_2-2p_1$
$P_{21}=p_{21}-p_{12}-2p_{10}-p_7+3p_3+2p_2-2p_1$
$P_{22}=p_{22}-2p_{20}-p_{18}-2p_{15}-2p_{13}+2p_{12}+2p_{11}+8p_{10}-p_8+4p_6+3p_4-$
$-6p_3-12p_2+6p_1$
$P_{23}=p_{23}-2p_{21}-p_{19}-2p_{16}-2p_{14}+2p_{12}+2p_{11}+8p_{10}-p_9+4p_7+3p_5-$
$-12p_3-6p_2+6p_1$
$P_{24}=p_{24}-p_{20}-p_{19}-p_{15}-2p_{14}+p_{12}+p_{11}+4p_{10}+p_6+2p_5-4p_3-3p_2+2p_1$
$P_{25}=p_{25}-p_{21}-p_{18}-p_{16}-2p_{13}+p_{12}+p_{11}+4p_{10}+p_7+2p_4-3p_3-4p_2+2p_1$
$P_{26}=p_{26}-p_{21}-p_{20}-2p_{16}-2p_{15}+p_{12}+2p_{11}+6p_{10}+2p_7+2p_6-6p_3-$
$-6p_2+4p_1$
$P_{27}=p_{27}-2p_{19}-2p_{18}+2p_{12}+2p_{11}+p_5+p_4-2p_3-2p_2+p_1$
$P_{28}=p_{28}-3p_{21}-3p_{20}+3p_{12}+6p_{10}+2p_7+2p_6-6p_3-6p_2+4p_1$
$P_{29}=p_{29}-p_{27}-4p_{24}-2p_{22}+4p_{20}+4p_{19}+2p_{18}-2p_{17}+4p_{15}+8p_{14}+$
$+4p_{13}-4p_{12}-4p_{11}-16p_{10}+p_8-4p_6-6p_5-3p_4+12p_3+12p_2-6p_1$
$P_{30}=p_{30}-p_{27}-4p_{25}-2p_{23}+4p_{21}+2p_{19}+4p_{18}-2p_{17}+4p_{16}+4p_{14}+$
$+8p_{13}-4p_{11}-4p_{12}-16p_{10}+p_9-4p_7-3p_5-6p_4+12p_3+12p_2-6p_1$
$P_{31}=p_{31}-p_{28}-3p_{26}-3p_{25}-3p_{22}+6p_{21}+6p_{20}+3p_{18}+6p_{16}+6p_{15}+$
$+6p_{13}-6p_{12}-6p_{11}-24p_{10}+2p_8-6p_7-8p_6-6p_4+18p_3+24p_2-12p_1$
$P_{32}=p_{32}-p_{28}-3p_{26}-3p_{24}-3p_{23}+6p_{21}+6p_{20}+3p_{19}+6p_{16}+6p_{15}+$
$+6p_{14}-6p_{12}-6p_{11}-24p_{10}+2p_9-8p_7-6p_6-6p_5+24p_3+18p_2-12p_1$
$P_{33}=p_{33}-4p_{32}-4p_{31}-3p_{30}-3p_{29}+4p_{28}+3p_{27}+12p_{26}+12p_{25}+12p_{24}+$
$+12p_{23}+12p_{22}-24p_{21}-24p_{20}-12p_{19}-12p_{18}+6p_{17}-24p_{16}-$
$-24p_{15}-24p_{14}-24p_{13}+24p_{12}+24p_{11}+96p_{10}-6p_9-6p_8+24p_7+$
$+24p_6+18p_5+18p_4-72p_3-72p_2+36p_1$
$P_{34}=p_{34}-p_6-3p_4+5p_2-2p_1$
$P_{35}=p_{35}-p_7-3p_5+5p_3-2p_1$
$P_{36}=p_{36}-p_{11}-p_{10}-p_4+p_3+2p_2-p_1$
$P_{37}=p_{37}-p_{11}-p_{10}-p_5+2p_3+p_2-p_1$
$P_{38}=p_{38}-2p_{37}-2p_{15}-p_{14}+2p_{11}+4p_{10}+p_6+2p_5-4p_3-3p_2+2p_1$
$P_{39}=p_{39}-2p_{36}-2p_{16}-p_{13}+2p_{11}+4p_{10}+p_7+2p_4-3p_3-4p_2+2p_1$
$P_{40}=p_{40}-p_{37}-p_{36}-p_{14}-p_{13}+p_{11}+3p_{10}+p_5+p_4-2p_3-2p_2+p_1$

$$P_{41}=p_{41}-p_{37}-p_{36}-p_{19}-p_{18}+p_{12}+2p_{11}+p_{10}+p_5+p_4-2p_3-2p_2+p_1$$
$$P_{42}=p_{42}-3p_{11}-p_7+3p_3+2p_2-2p_1$$
$$P_{43}=p_{43}-3p_{11}-p_6+2p_3+3p_2-2p_1$$
$$P_{44}=p_{44}-p_{42}-2p_{36}-p_{21}-p_{18}+p_{12}+3p_{11}+2p_{10}+p_7+2p_4-3p_3-4p_2+2p_1$$
$$P_{45}=p_{45}-p_{43}-2p_{37}-p_{20}-p_{19}+p_{12}+3p_{11}+2p_{10}+p_6+2p_5-4p_3-3p_2+2p_1$$
$$P_{46}=p_{46}-p_{37}-p_{36}-p_{14}-p_{13}+p_{11}+3p_{10}+p_5+p_4-2p_3-2p_2+p_1$$
$$P_{47}=p_{47}-p_{37}-p_{36}-p_{14}-p_{13}+p_{11}+3p_{10}+p_5+p_4-2p_3-2p_2+p_1$$
$$P_{48}=p_{48}-p_{43}-p_{42}-2p_{16}-2p_{15}+5p_{11}+4p_{10}+2p_7+2p_6-6p_3-6p_2+4p_1$$
$$P_{49}=p_{49}-p_{36}-p_{34}-p_{15}-2p_{13}+p_{11}+4p_{10}+p_6+3p_4-2p_3-5p_2+2p_1$$
$$P_{50}=p_{50}-p_{37}-p_{35}-p_{16}-2p_{14}+p_{11}+4p_{10}+p_7+3p_5-5p_3-2p_2+2p_1$$
$$P_{51}=p_{51}-p_{43}-p_{34}-3p_{18}+2p_{12}+3p_{11}+p_6+3p_4-2p_3-5p_2+2p_1$$
$$P_{52}=p_{52}-p_{42}-p_{35}-3p_{19}+2p_{12}+3p_{11}+p_7+3p_5-5p_3-2p_2+2p_1$$
$$P_{53}=p_{53}-2p_{36}-p_{34}-p_{20}-p_{18}+p_{12}+2p_{11}+2p_{10}+p_6+3p_4-2p_3-5p_2+2p_1$$
$$P_{54}=p_{54}-2p_{37}-p_{35}-p_{21}-p_{19}+p_{12}+2p_{11}+2p_{10}+p_7+3p_5-5p_3-2p_2+2p_1$$
$$P_{55}=p_{55}-p_{43}-3p_{36}-3p_{15}+6p_{11}+6p_{10}-p_8+4p_6+3p_4-6p_3-12p_2+6p_1$$
$$P_{56}=p_{56}-p_{42}-3p_{37}-3p_{16}+6p_{11}+6p_{10}-p_9+4p_7+3p_5-12p_3-6p_2+6p_1$$
$$P_{57}=p_{57}-10p_{34}-5p_8+20p_6+15p_4-30p_2+9p_1$$
$$P_{58}=p_{58}-10p_{35}-5p_9+20p_7+15p_5-30p_3+9p_1$$
$$\begin{aligned}P_{59}=p_{59}&-p_{57}-2p_{55}-3p_{53}-p_{51}-6p_{49}+2p_{43}+6p_{36}+10p_{34}-3p_{22}+\\&+6p_{20}+3p_{18}+12p_{15}+6p_{13}-3p_{12}-9p_{11}-18p_{10}+5p_8-20p_6-\\&-15p_4+9p_3+30p_2-9p_1\end{aligned}$$
$$\begin{aligned}P_{60}=p_{60}&-p_{58}-2p_{56}-3p_{54}-p_{52}-6p_{50}+2p_{42}+6p_{37}+10p_{35}-3p_{23}+\\&+6p_{21}+3p_{19}+12p_{16}+6p_{14}-3p_{12}-9p_{11}-18p_{10}+5p_9-20p_7-\\&-15p_5+30p_3+9p_2-9p_1\end{aligned}$$
$$\begin{aligned}P_{61}=p_{61}&-3p_{53}-3p_{44}+2p_{42}+6p_{36}+2p_{34}-p_{28}+3p_{21}+3p_{20}+3p_{18}-\\&-3p_{12}-6p_{11}-6p_{10}-2p_7-2p_6-6p_4+6p_3+10p_2-4p_1\end{aligned}$$
$$\begin{aligned}P_{62}=p_{62}&-3p_{54}-3p_{45}+2p_{43}+6p_{37}+2p_{35}-p_{28}+3p_{21}+3p_{20}+3p_{19}-\\&-3p_{12}-6p_{11}-6p_{10}-2p_7-2p_6-6p_5+10p_3+6p_2-4p_1\end{aligned}$$
$$\begin{aligned}P_{63}=p_{63}&-p_{53}-p_{51}-p_{45}+p_{43}-2p_{41}+2p_{37}+2p_{36}+p_{34}-p_{27}+p_{20}+\\&+3p_{19}+4p_{18}-3p_{12}-5p_{11}-2p_{10}-p_6-2p_5-3p_4+4p_3+5p_2-2p_1\end{aligned}$$
$$\begin{aligned}P_{64}=p_{64}&-p_{54}-p_{52}-p_{44}+p_{42}-2p_{41}+2p_{37}+2p_{36}+p_{35}-p_{27}+4p_{19}+\\&+3p_{18}+p_{21}-3p_{12}-5p_{11}-2p_{10}-p_7-3p_5-2p_4+5p_3+4p_2-2p_1\end{aligned}$$
$$\begin{aligned}P_{65}=p_{65}&-2p_{49}-p_{47}-p_{40}-p_{38}+2p_{37}+2p_{36}+p_{34}-p_{17}+2p_{15}+3p_{14}+\\&+4p_{13}-2p_{11}-8p_{10}-p_6-2p_5-3p_4+4p_3+5p_2-2p_1\end{aligned}$$
$$\begin{aligned}P_{66}=p_{66}&-2p_{50}-p_{46}-p_{40}-p_{39}+2p_{37}+2p_{36}+p_{35}-p_{17}+2p_{16}+4p_{14}+\\&+3p_{13}-2p_{11}-8p_{10}-p_7-3p_5-2p_4+5p_3+4p_2-2p_1\end{aligned}$$
$$\begin{aligned}P_{67}=p_{67}&-p_{51}-2p_{49}-p_{48}-p_{44}+p_{43}+p_{42}+2p_{36}+2p_{34}-2p_{25}+2p_{21}+3p_{18}+\\&+2p_{16}+2p_{15}+4p_{13}-2p_{12}-5p_{11}-8p_{10}-2p_7-2p_6-6p_4+6p_3+\\&+10p_2-4p_1\end{aligned}$$
$$\begin{aligned}P_{68}=p_{68}&-p_{52}-2p_{50}-p_{48}-p_{45}+p_{43}+p_{42}+2p_{37}+2p_{35}-2p_{24}+2p_{20}+\\&+p_{19}+2p_{16}+2p_{15}+4p_{14}-2p_{12}-5p_{11}-8p_{10}-2p_7-2p_6-6p_5+\end{aligned}$$

$+10p_3+6p_2-4p_1$

$P_{69}=p_{69}-p_{53}-2p_{49}-2p_{39}+4p_{36}+2p_{34}-p_{26}-p_{25}+p_{21}+p_{20}+p_{18}+$
$+4p_{16}+2p_{15}+4p_{13}-p_{12}-4p_{11}-10p_{10}-2p_7-2p_6-6p_4+6p_3+$
$+10p_2-4p_1$

$P_{70}=p_{70}-p_{54}-2p_{50}-2p_{38}+4p_{37}+2p_{35}-p_{26}-p_{24}+p_{21}+p_{20}+p_{19}+4p_{15}+$
$+4p_{14}+2p_{16}-p_{12}-4p_{11}-10p_{10}-2p_6-6p_5-2p_7+10p_3+6p_2-4p_1$

$P_{71}=p_{71}-p_{53}-p_{49}-2p_{47}-p_{41}+2p_{37}+3p_{36}+p_{34}-p_{24}+p_{20}+p_{19}+p_{18}+$
$+p_{15}+2p_{14}+2p_{13}-p_{12}-3p_{11}-6p_{10}-p_6-2p_5-3p_4+4p_3+5p_2-2p_1$

$P_{72}=p_{72}-p_{54}-p_{50}-2p_{46}-p_{41}+3p_{37}+2p_{36}+p_{35}-p_{25}+p_{21}+p_{19}+p_{18}+$
$+p_{16}+2p_{14}+2p_{13}-p_{12}-3p_{11}-6p_{10}-p_7-3p_5-2p_4+5p_3+4p_2-2p_1$

$P_{73}=p_{73}-2p_{55}-2p_{48}-p_{44}+2p_{43}+2p_{42}-2p_{39}+6p_{36}-2p_{26}-p_{22}+2p_{21}+$
$+2p_{20}+p_{18}+8p_{16}+8p_{15}+2p_{13}-2p_{12}-14p_{11}-20p_{10}+2p_8-6p_7-$
$-8p_6-6p_4+18p_3+24p_2-12p_1$

$P_{74}=p_{74}-2p_{56}-2p_{48}-p_{45}+2p_{43}+2p_{42}-2p_{38}+6p_{37}-2p_{26}-p_{23}+2p_{21}+$
$+2p_{20}+p_{19}+8p_{16}+8p_{15}+2p_{14}-2p_{12}-14p_{11}-20p_{10}+2p_9-8p_7-$
$-6p_6-6p_5+24p_3+18p_2-12p_1$

$P_{75}=p_{75}-p_{55}-p_{46}-p_{45}+p_{43}-p_{41}-p_{40}-2p_{38}+6p_{37}+3p_{36}-p_{24}-p_{22}+$
$+2p_{20}+2p_{19}+p_{18}+5p_{15}+4p_{14}+2p_{13}-2p_{12}-8p_{11}-14p_{10}+p_8-$
$-4p_6-6p_5-3p_4+12p_3+12p_2-6p_1$

$P_{76}=p_{76}-p_{56}-p_{47}-p_{44}+p_{42}-p_{41}-p_{40}-2p_{39}+3p_{37}+6p_{36}-p_{25}-p_{23}+$
$+2p_{21}+p_{19}+2p_{18}+5p_{16}+2p_{14}+4p_{13}-2p_{12}-8p_{11}-14p_{10}+p_9-$
$-4p_7-3p_5-6p_4+12p_3+12p_2-6p_1$

$P_{77}=p_{77}-3p_{64}-3p_{63}-p_{62}-p_{61}+3p_{54}+3p_{53}+2p_{52}+2p_{51}+3p_{45}+3p_{44}-$
$-2p_{43}-2p_{42}+6p_{41}-6p_{37}-6p_{36}-2p_{35}-2p_{34}+p_{28}+3p_{27}-3p_{21}-$
$-3p_{20}-9p_{19}-9p_{18}+7p_{12}+12p_{11}+6p_{10}+2p_7+2p_6+6p_5+6p_4-$
$-10p_3-10p_2+4p_1$

$P_{78}=p_{78}-3p_{73}-6p_{69}-3p_{67}-p_{61}-3p_{59}+2p_{57}+6p_{55}+9p_{53}+3p_{51}+$
$+18p_{49}+6p_{48}+3p_{44}-6p_{43}-3p_{42}+6p_{39}-18p_{36}-20p_{34}-2p_{31}+$
$+2p_{28}+12p_{26}+6p_{25}+9p_{22}-18p_{20}-9p_{21}-9p_{18}-36p_{15}-18p_{16}-$
$-18p_{13}+9p_{12}+27p_{11}+54p_{10}-10p_8+9p_7+40p_6+30p_4-27p_3-$
$-60p_2+18p_1$

$P_{79}=p_{79}-3p_{74}-6p_{70}-3p_{68}-p_{62}-3p_{60}+2p_{58}+6p_{56}+9p_{54}+3p_{52}+$
$+18p_{50}+6p_{48}+3p_{45}-3p_{43}-6p_{42}+6p_{38}-18p_{37}-20p_{35}-2p_{32}+$
$+2p_{28}+12p_{26}+6p_{24}+9p_{23}-18p_{21}-9p_{20}-9p_{19}-36p_{16}-18p_{15}-$
$-18p_{14}+9p_{12}+27p_{11}+54p_{10}-10p_9+40p_7+9p_6+30p_5-60p_3-$
$-27p_2+18p_1$

$P_{80}=p_{80}-3p_{76}-3p_{71}-3p_{69}-p_{61}+2p_{56}+6p_{53}+6p_{49}+6p_{46}+3p_{44}-$
$-2p_{42}+3p_{41}+6p_{39}-6p_{37}-18p_{36}-6p_{34}-p_{32}+p_{28}+3p_{26}+3p_{25}+$
$+3p_{24}+3p_{23}-6p_{21}-6p_{20}-3p_{19}-6p_{18}-12p_{16}-6p_{15}-6p_{14}-$
$-12p_{13}+6p_{12}+18p_{11}+36p_{10}-2p_9+8p_7+6p_6+6p_5+18p_4-24p_3-$
$-30p_2+12p_1$

$$P_{81}=p_{81}-3p_{75}-3p_{72}-3p_{70}-p_{62}+2p_{55}+6p_{54}+6p_{50}+6p_{47}+3p_{45}-$$
$$-2p_{43}+3p_{41}+6p_{38}-18p_{37}-6p_{36}-6p_{35}-p_{31}+p_{28}+3p_{26}+3p_{25}+$$
$$+3p_{24}+3p_{22}-6p_{21}-6p_{20}-6p_{19}-3p_{18}-6p_{16}-12p_{15}-12p_{14}-6p_{13}+$$
$$+6p_{12}+18p_{11}+36p_{10}-2p_{8}+6p_{7}+8p_{6}+18p_{5}+6p_{4}-30p_{3}-24p_{2}+12p_{1}$$
$$P_{82}=p_{82}-4p_{76}-4p_{72}-4p_{66}-2p_{64}-2p_{60}+p_{58}+4p_{56}+6p_{54}+2p_{52}+$$
$$+12p_{50}+2p_{47}+4p_{46}+4p_{44}-4p_{42}+4p_{41}+2p_{40}+8p_{39}-12p_{37}-12p_{36}-$$
$$-10p_{35}-p_{30}+p_{27}+8p_{25}+6p_{23}-12p_{21}-6p_{19}-6p_{18}+2p_{17}-24p_{16}-$$
$$-12p_{14}-12p_{13}+6p_{12}+18p_{11}+36p_{10}-5p_{9}+20p_{7}+15p_{5}+9p_{4}-30p_{3}-$$
$$-18p_{2}+9p_{1}$$
$$P_{83}=p_{83}-4p_{75}-4p_{71}-4p_{65}-2p_{63}-2p_{59}+p_{57}+4p_{55}+6p_{53}+2p_{51}+$$
$$+12p_{49}+4p_{47}+2p_{46}+4p_{45}-4p_{43}+4p_{41}+2p_{40}+8p_{38}-12p_{37}-$$
$$-12p_{36}-10p_{34}-p_{29}+p_{27}+8p_{24}+6p_{22}-12p_{20}-6p_{19}-6p_{18}+2p_{17}-$$
$$-24p_{15}-12p_{14}-12p_{13}+6p_{12}+18p_{11}+36p_{10}-5p_{8}+20p_{6}+9p_{5}+15p_{4}-$$
$$-18p_{3}-30p_{2}+9p_{1}$$
$$P_{84}=p_{84}-p_{72}-p_{71}-p_{70}-p_{69}-2p_{66}-2p_{65}+p_{54}+p_{53}+4p_{50}+4p_{49}+$$
$$+2p_{47}+2p_{46}+p_{41}+2p_{40}+2p_{39}+2p_{38}-6p_{37}-6p_{36}-2p_{35}-2p_{34}+$$
$$+p_{26}+p_{25}+p_{24}-p_{21}-p_{20}-p_{19}-p_{18}+2p_{17}-4p_{16}-4p_{15}-8p_{14}-8p_{13}+$$
$$+p_{12}+6p_{11}+18p_{10}+2p_{7}+2p_{6}+6p_{5}+6p_{4}-10p_{3}-10p_{2}+4p_{1}$$
$$P_{85}=p_{85}-2p_{76}-p_{74}-2p_{69}-2p_{67}-2p_{65}-p_{63}+2p_{56}+2p_{53}+2p_{51}+$$
$$+8p_{49}+2p_{48}+2p_{47}+p_{45}+2p_{44}-2p_{43}-2p_{42}+2p_{41}+2p_{40}+4p_{39}+$$
$$+2p_{38}-6p_{37}-12p_{36}-6p_{34}-p_{30}+p_{27}+2p_{26}+6p_{25}+3p_{23}-6p_{21}-$$
$$-2p_{20}-3p_{19}-8p_{18}+2p_{17}-12p_{16}-8p_{15}-6p_{14}-16p_{13}+6p_{12}+$$
$$+18p_{11}+36p_{10}-2p_{9}+8p_{7}+6p_{6}+6p_{5}+18p_{4}-24p_{3}-30p_{2}+12p_{1}$$
$$P_{86}=p_{86}-2p_{75}-p_{73}-2p_{70}-2p_{68}-2p_{66}-p_{64}+2p_{55}+2p_{54}+2p_{52}+8p_{50}+$$
$$+2p_{48}+2p_{46}+2p_{45}+p_{44}-2p_{43}-2p_{42}+2p_{41}+2p_{40}+2p_{39}+4p_{38}-$$
$$-12p_{37}-6p_{36}-6p_{35}-p_{29}+p_{27}+2p_{26}+6p_{24}+3p_{22}-2p_{21}-6p_{20}-$$
$$-8p_{19}-3p_{18}+2p_{17}-8p_{16}-12p_{15}-16p_{14}-6p_{13}+6p_{12}+18p_{11}+$$
$$+36p_{10}-2p_{8}+6p_{7}+8p_{6}+18p_{5}+6p_{4}-30p_{3}-24p_{2}+12p_{1}$$
$$P_{87}=p_{87}-2p_{72}-2p_{71}-p_{68}-p_{67}-p_{64}-p_{63}+2p_{54}+2p_{53}+p_{52}+p_{51}+$$
$$+2p_{50}+2p_{49}+p_{48}+2p_{47}+2p_{46}+p_{45}+p_{44}-p_{43}-p_{42}+4p_{41}-6p_{37}-$$
$$-5p_{36}-2p_{35}-2p_{34}+p_{27}+2p_{25}+2p_{24}-2p_{21}-2p_{20}-5p_{19}-5p_{18}-2p_{16}-$$
$$-2p_{15}-4p_{14}-5p_{13}+4p_{12}+9p_{11}+12p_{10}+2p_{7}+2p_{6}+6p_{5}+6p_{4}-$$
$$-10p_{3}-10p_{2}+4p_{1}$$
$$P_{88}=p_{88}-3p_{87}-3p_{85}-6p_{84}-3p_{82}-2p_{80}-p_{79}-p_{77}+12p_{76}+3p_{74}+$$
$$+12p_{72}+6p_{71}+6p_{70}+12p_{69}+3p_{68}+6p_{67}+12p_{66}+6p_{65}+6p_{64}+$$
$$+3p_{63}+p_{62}+2p_{61}+5p_{60}-2p_{58}-10p_{56}-15p_{54}-9p_{53}-5p_{52}-$$
$$-3p_{51}-30p_{50}-18p_{49}-6p_{48}-4p_{47}-12p_{46}-3p_{45}-12p_{44}+3p_{43}+10p_{42}-$$
$$-12p_{41}-8p_{40}-24p_{39}-6p_{38}+30p_{37}+36p_{36}+20p_{35}+9p_{34}+2p_{32}+$$
$$+3p_{30}-2p_{28}-3p_{27}-12p_{26}-24p_{25}-6p_{24}-15p_{23}+30p_{21}+9p_{20}+$$
$$+15p_{19}+18p_{18}-6p_{17}+60p_{16}+18p_{15}+30p_{14}+36p_{13}-15p_{12}-$$
$$-45p_{11}-90p_{10}+10p_{9}-40p_{7}-9p_{6}-30p_{5}-27p_{4}+60p_{3}+45p_{2}-18p_{1}$$

$$
\begin{aligned}
P_{89} = {} & p_{89} - 3p_{87} - 3p_{86} - 6p_{84} - 3p_{83} - 2p_{81} - p_{78} - p_{77} + 12p_{75} + 3p_{73} + \\
& + 6p_{72} + 12p_{71} + 12p_{70} + 6p_{69} + 6p_{68} + 3p_{67} + 6p_{66} + 12p_{65} + 3p_{64} + 6p_{63} + \\
& + 2p_{62} + p_{61} + 5p_{59} - 2p_{57} - 10p_{55} - 9p_{54} - 15p_{53} - 3p_{52} - 5p_{51} - \\
& - 18p_{50} - 30p_{49} - 6p_{48} - 12p_{47} - 4p_{46} - 12p_{45} - 3p_{44} + 10p_{43} + 3p_{42} - \\
& - 12p_{41} - 8p_{40} - 6p_{39} - 24p_{38} + 36p_{37} + 30p_{36} + 9p_{35} + 20p_{34} + 2p_{31} + \\
& + 3p_{29} - 2p_{28} - 3p_{27} - 12p_{26} - 6p_{25} - 24p_{24} - 15p_{22} + 9p_{21} + 30p_{20} + \\
& + 18p_{19} + 15p_{18} - 6p_{17} + 18p_{16} + 60p_{15} + 36p_{14} + 30p_{13} - 15p_{12} - \\
& - 45p_{11} - 90p_{10} + 10p_{8} - 9p_{7} - 40p_{6} - 27p_{5} - 30p_{4} + 45p_{3} + 60p_{2} - 18p_{1} \\
P_{90} = {} & p_{90} - 3p_{76} - 3p_{75} - 3p_{74} - 3p_{73} + 6p_{56} + 6p_{55} + 6p_{48} + 3p_{47} + p_{46} + \\
& + 3p_{45} + 3p_{44} - 6p_{43} - 6p_{42} + 3p_{41} + 2p_{40} + 6p_{39} + 6p_{38} - 18p_{37} - 18p_{36} - \\
& - p_{32} - p_{31} + p_{28} + 9p_{26} + 3p_{25} + 3p_{24} + 6p_{23} + 6p_{22} - 12p_{21} - 12p_{20} - \\
& - 6p_{19} - 6p_{18} - 30p_{16} - 30p_{15} - 12p_{14} - 12p_{13} + 12p_{12} + 48p_{11} + \\
& + 84p_{10} - 6p_{9} - 6p_{8} + 24p_{7} + 24p_{6} + 18p_{5} + 18p_{4} - 72p_{3} - 72p_{2} + 36p_{1} \\
P_{91} = {} & p_{91} - 4p_{90} - 6p_{85} - 3p_{83} - 4p_{80} - 4p_{78} + 12p_{76} + 12p_{75} + 12p_{74} + \\
& + 12p_{73} + 12p_{71} + 24p_{69} + 12p_{67} + 12p_{65} + 6p_{63} + 4p_{61} + 12p_{59} - 6p_{57} - \\
& - 12p_{56} - 24p_{55} - 36p_{53} - 12p_{51} - 72p_{49} - 24p_{48} - 12p_{47} - 2p_{46} - \\
& - 12p_{45} - 12p_{44} + 24p_{43} + 12p_{42} - 12p_{41} - 10p_{40} - 24p_{39} - 24p_{38} + \\
& + 36p_{37} + 72p_{36} + 60p_{34} - p_{33} + 8p_{32} + 8p_{31} + 3p_{30} + 3p_{29} - 8p_{28} - 3p_{27} - 48p_{26} - \\
& - 24p_{25} - 24p_{24} - 18p_{23} - 36p_{22} + 36p_{21} + 72p_{20} + 18p_{19} + 36p_{18} - 6p_{17} + \\
& + 72p_{16} + 144p_{15} + 36p_{14} + 72p_{13} - 36p_{12} - 108p_{11} - 216p_{10} + \\
& + 9p_{9} + 30p_{8} - 36p_{7} - 120p_{6} - 27p_{5} - 90p_{4} + 108p_{3} + 180p_{2} - 54p_{1} \\
P_{92} = {} & p_{92} - 4p_{90} - 6p_{86} - 3p_{82} - 4p_{81} - 4p_{79} + 12p_{76} + 12p_{75} + 12p_{74} + 12p_{73} + \\
& + 12p_{72} + 24p_{70} + 12p_{68} + 12p_{66} + 6p_{64} + 4p_{62} + 12p_{60} - 6p_{58} - 24p_{56} - \\
& - 12p_{55} - 36p_{54} - 12p_{52} - 72p_{50} - 24p_{48} - 2p_{47} - 12p_{46} - 12p_{45} - 12p_{44} + \\
& + 12p_{43} + 24p_{42} - 12p_{41} - 10p_{40} - 24p_{39} - 24p_{38} + 72p_{37} + 36p_{36} + \\
& + 60p_{35} - p_{33} + 8p_{32} + 8p_{31} + 3p_{30} + 3p_{29} - 8p_{28} - 3p_{27} - 48p_{26} - 24p_{25} - \\
& - 24p_{24} - 36p_{23} - 18p_{22} + 72p_{21} + 36p_{20} + 36p_{19} + 18p_{18} - 6p_{17} + 144p_{16} + \\
& + 72p_{15} + 72p_{14} + 36p_{13} - 36p_{12} - 108p_{11} - 216p_{10} + 30p_{9} + 9p_{8} - \\
& - 120p_{7} - 36p_{6} - 90p_{5} - 27p_{4} + 180p_{3} + 108p_{2} - 54p_{1} \\
P_{93} = {} & p_{93} - 5p_{92} - 5p_{91} + 20p_{90} - 10p_{89} - 10p_{88} + 30p_{87} + 30p_{86} + 30p_{85} + \\
& + 60p_{84} + 30p_{83} + 30p_{82} + 20p_{81} + 20p_{80} + 20p_{79} + 20p_{78} + 10p_{77} - 120p_{76} - \\
& - 120p_{75} - 60p_{74} - 60p_{73} - 120p_{72} - 120p_{71} - 120p_{70} - 120p_{69} - \\
& - 60p_{68} - 60p_{67} - 120p_{66} - 120p_{65} - 60p_{64} - 60p_{63} - 20p_{62} - 20p_{61} - \\
& - 60p_{60} - 60p_{59} + 24p_{58} + 24p_{57} + 120p_{56} + 120p_{55} + 180p_{54} + \\
& + 180p_{53} + 60p_{52} + 60p_{51} + 360p_{50} + 360p_{49} + 120p_{48} + 120p_{47} + 120p_{46} + \\
& + 120p_{45} + 120p_{44} - 120p_{43} - 120p_{42} + 240p_{41} + 240p_{40} + 240p_{39} + \\
& + 240p_{38} - 720p_{37} - 720p_{36} - 240p_{35} - 240p_{34} + 5p_{33} - 40p_{32} - 40p_{31} - \\
& - 30p_{30} - 30p_{29} + 40p_{28} + 60p_{27} + 240p_{26} + 240p_{25} + 240p_{24} + 180p_{23} + \\
& + 180p_{22} - 360p_{21} - 360p_{20} - 360p_{19} - 360p_{18} + 120p_{17} - 720p_{16} - \\
& - 720p_{15} - 720p_{14} - 720p_{13} + 360p_{12} + 1080p_{11} + 2160p_{10} - 120p_{9} - \\
& - 120p_{8} + 480p_{7} + 480p_{6} + 720p_{5} + 720p_{4} - 1440p_{3} - 1440p_{2} + 576p_{1}
\end{aligned}
$$

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SÚHRN

VŠEOBECNÉ SYMETRICKÉ PRIEMERY A BIPOLYKAYSY 5. RÁDU

F. Lamoš, Bratislava

V tejto práci sú na jednej strane vypočítané všeobecné symetrické priemery 5. rádu, a na druhej strane sú nájdené vzťahy, ktoré spájajú tieto priemery so zodpovedajúcimi bipolykaysami.

РЕЗЮМЕ

ОБЩИЕ СИММЕТРИЧЕСКИЕ СРЕДНИЕ И БИПОЛИКЕЙСИ (BIPOLYKAIS) 5. ПОРЯДКА

Ф. Ламош, Братислава

В этой работе, с одной стороны вычислены общие симметрические средние 5. порядка, и с другой стороны найдены соотношения, связывающие эти средние с соответствующими биполикейсами.

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LOWER ESTIMATE OF THE MINIMUM OF ENERGY FUNCTIONAL BY MEANS OF THE DUALITY THEOREM

IGOR BOCK, Bratislava

We show in this paper that by means of the duality theorem the lower estimate of the minimum of the energy functional corresponding to the positive definite partial differential operator can be obtained. In the case of the partial differential equation

$$D_1 \frac{\partial^4 u}{\partial x^4} + 2D_3 \frac{\partial^4 u}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 u}{\partial y^4} = a(x, y)$$

with the homogeneous Dirichlet's boundary datas we construct the maximizing sequence. We can estimate then the minimum with sufficiently small error.

1. Duality theorem

By means of the duality theorem we can convert the minimum problem of one functional to the maximum problem of the adjoint functional on the adjoint domain.

Definition 1. Let f, $-g$ be the real convex functionals defined on the convex subsets C resp. D of the real, or the complex Hilbert space H. The equations

$$\begin{aligned} f^*(x^*) &= \sup_{x \in C} [\operatorname{Re}(x^*, x) - f(x)] \\ g^*(x^*) &= \inf_{x \in D} [\operatorname{Re}(x^*, x) - g(x)] \end{aligned} \tag{1.1}$$

define the adjoint functionals f^*, g^* with the domains

$$\begin{aligned} C^* &= \{x^* \in H: f^*(x^*) < \infty\} \\ D^* &= \{x^* \in H: g^*(x^*) < -\infty\} \end{aligned} \tag{1.2}$$

Duality theorem 1 [2]. Let the sets $C \cap D \neq 0$, $C^* \cap D^* \neq 0$. If there exists a point $x_1 \in (\text{int } C) \cap D$, in which f is continuous, or a point $x_2 \in C \cap (\text{int } D)$, in which g is continuous, then

$$\inf_{C \cap D} [f(x) - g(x)] = \max_{C^* \cap D^*} [g^*(x^*) - f^*(x^*)] \tag{1.3}$$

More general theorem is proved in the paper [7]. Instead of Hilbert space the locally convex linear topological space over real numbers and the subsets C^*, D^* of the dual space are in [7]. We can apply this general theorem in the case of complex Hilbert space dealing H as real Hilbert space with scalar product $[x, y] = \operatorname{Re}(x, y)$.

2. The application of the duality theorem at the operator $A = T^*BT$

Let A be the linear symmetric positive definite operator defined on the dense linear subset M of the real, or complex Hilbert space H with the values in H. It is well known, that the point $x_0 \in M$, which is the solution of the equation

$$Ax = a, \quad x \in M, \quad a \in H \tag{2.1}$$

minimizes the quadratic functional

$$\varphi(x) = (Ax, x) - 2\operatorname{Re}(a, x) \tag{2.2}$$

and on the contrary the point which minimizes the functional (2.2) on the set M, is the solution of the equation (2.1).

Let γ^2 be the lower bound of the operator A, defined by the inequality

$$(Ax, x) \geq \gamma^2(x, x), \qquad \forall x \in M \tag{2.3}$$

We define the energetical scalar product and the energetical norm by the equations

$$[x, y] = (Ax, y), \quad |||x|||^2 = (Ax, y), \quad x, y \in M \tag{2.4}$$

If $x_0 \in M$ is the solution of the equation (2.1) and $\mu = -(Ax_0, x_0)$ is the minimal value of the functional (2.2) on M, then

$$|||x - x_0|||^2 = \varphi(x) - \mu, \qquad x \in M \tag{2.5}$$

and by the inequality (2.3)

$$|||x - x_0|||^2 = \gamma^{-2}(\varphi(x) - \mu) \tag{2.6}$$

If the equation (2.1) has no solution in M, then there exists its generalized solution, which minimizes the functional

$$\varphi(x) = |||x|||^2 - 2\,\mathrm{Re}(a, x) \tag{2.7}$$

on the Hilbert space H_A the completion of the set M in the energetical norm $|||\,.\,|||$. The generalized solution x_0 is defined by the equation

$$(x, a) = [x, x_0], \qquad x \in H_A \tag{2.8}$$

which arises from the Riesz theorem on representation of a linear bounded functional ([6]; 110). From the equation (2.8) we obtain the equation

$$\mu = \min_{x \in H_A} \varphi(x) = -|||x_0|||^2 \tag{2.9}$$

The equation (2.5) and the inequality (2.6) can be continued on the energetical space H_A. If $m \leqq \mu$, then $\varphi(x) - m$ is the estimate of the error (2.5), or $\gamma^{-2}(\varphi(x) - m)$ is the estimate of the quadratic error (2.6).

It is possible in the case of the operator $A = T^*BT$ to construct an adjoint functional whose maximal value coincides with the minimal value of the original functional. The maximing sequence for the adjoint functional enables us to obtain a lower estimate of the minimum μ with sufficiently small error. We show in the third part one possible way of the construction of the maximizing sequence.

Let H_1 be the Hilbert space with the scalar product $(u, v)_1$ and the norm $\|u\|_1$. Let T be the linear operator mapping the linear subset $D_T \subset H$ into H_1. Let B be the linear symmetric positive definite operator mapping the space H_1 into itself, Then there exists the inverse operator B^{-1} with the same characteristic. We assume the relations

$$M \subset D_T, \quad B(T(M)) \subset D_{T^*}, \quad a = T^*c, \quad c \in D_T \tag{2.10}$$

where D_{T^*} is the domain of the adjoint operator T^* defined by the equation

$$(Tu, v)_1 = (u, T^*v), \qquad u \in D_T, \ v \in D_{T^*}. \tag{2.11}$$

As a result of positive definiteness and boundedness of the operator B, the relations (2.10), (2.11) we find that the operator $A = T^*BT$ is positive definite if and only if there exists a constant $\varkappa > 0$ that

$$\|Tu\|_1 \geqq \varkappa \|u\|, \qquad u \in M \tag{2.12}$$

The problem of solving the equation $Ax = a$ is in this case equivalent with the problem of minimizing the functional

$$\begin{aligned}\varphi(x) &= (Ax, x) - 2\,\mathrm{Re}(a, x) = (T^*BTx, x) - 2\,\mathrm{Re}(T^*c, x) = \\ &= (BTx, Tx)_1 - 2\,\mathrm{Re}(c, Tx), \qquad x \in M\end{aligned} \tag{2.13}$$

We can extend the functional $\varphi(x)$ to the functional

$$\Phi(w)=(Bw, w)_1-2\,\mathrm{Re}(c, w)_1\,, \qquad w\in H_1 \tag{2.14}$$

Let L be the closure of the set $T(M)$ in the space H_1, $L^{\perp}$ the orthogonal complement of the set L in H_1. From boundedness and positive definiteness of the operator B follow the relations for the minimum of the functionals $\varphi(x)$, $\Phi(w)$

$$\mu=\inf_{x\in M}\varphi(x)=\inf_{w\in T(M)}\Phi(w)=\min_{w\in L}\Phi(w) \tag{2.15}$$

The next theorem on a lower estimate of the constant μ is proved in [2] by means of the duality theorem 1:

Theorem 2. a) $\mu=\min_{w\in L}\Phi(w)=\max_{w\in c+L^{\perp}}[-(B^{-1}v, v)_1]$

b) The point w_0 minimizes the functional Φ on the set L iff $v_0=Bw_0$ maximizes the functional $-(B^{-1}v, v)_1$ on $c+L^{\perp}$.

3. The generalization of Trefftz Method on the equation of orthotropic plate

We can employ the Theorem 2. on the estimate of the minimum of the energy functional corresponding to the elliptic partial differential equation of 2m-th order

$$Au=a_0u+\sum_{k=1}^{m}(-1)^k\sum_{|i|=|j|=k}\partial i_1\ldots$$

$$\ldots\partial_{i_k}(a_{j_1\ldots j_k}^{i_1\ldots i_k}\,\partial_{j_1}\ldots\partial_{j_k}u)=a(\bar{x})$$

where $u(\bar{x})=u(x_1,\ldots,x_n)$ is the function defined on the region $G\subset E_n$ with Lipschitz boundary. It is possible in the case of Dirichlet's homogeneous boundary conditions

$$u(\bar{x})=\partial_{i_1}\ldots\partial_{i_k}u(\bar{x})=0\,,\quad \forall\,(i_1,\ldots,i_k)$$

$$k=1,\ldots,m-1;\quad \bar{x}\in\partial G$$

to estimate the constant μ defined in (2.9) using the maximizing sequence for the functional defined on the set of solutions of the equation $Au=a$ which need not to satisfy any boundary conditions. It is the generalization of the Trefftz method employed on the equation with Laplace, or biharmonic operator.

We shall deal with the equation for the deflection of thin orthotropic plates

$$Du=D_1\frac{\partial^4u}{\partial x^4}+2D_3\frac{\partial^4u}{\partial x^2\,\partial y^2}+D_2\frac{\partial^4u}{\partial y^4}=$$

$$= D_1 u'''' + 2D_3 u''^{\cdot\cdot} + D_2 u^{\cdot\cdot\cdot\cdot} = a(x, y), \qquad a \in L_2(G) \tag{3.1}$$

with Dirichlet's boundary conditions, which express the clamped boundary of the plate

$$u/\partial G = u'/\partial G = \dot{u}/\partial G = 0 \tag{3.2}$$

We denote $u' = \frac{\partial u}{\partial x}$, $\dot{u} = \frac{\partial u}{\partial y}$ classical and generalized derivatives. The coefficients D_i are assumed constant and positive. The domain of the operator D is the set $M = C_0^\infty(G)$ of infinite differentiable functions with compact support in G. Using the positiveness of the coefficients and Fridrichs inequality ([6], 296) we can deduce easily that the operator D is positive definite on $C_0^\infty(G)$. The energetical norm has the form

$$|||u|||_0^2 = \int_G (D_1 u''^2 + 2D_3 u'^2 + D_2 u^{\cdot\cdot 2})\, dG, \qquad u \in C_0^\infty(G)$$

which is equivalent with the norm in the space $W_2^{(2)}(G)$, Sobolev space of square integrable functions, which posses the first and second derivatives in the generalized sense in $L_2(G)$. Hence the energetical space H_D coincides with the space $\overset{o}{W}{}_2^{(2)}(G)$ the closure of the set $C_0^\infty(G)$ in the space $W_2^{(2)}(G)$.

Let us consider the mapping

$$Tu = [u'', u', \ddot{u}], \qquad u \in C_0^\infty$$

with the values in $H_1 = L_2(G) \times L_2(G) \times L_2(G)$.

Let $\bar{w} = [w_1, w_2, w_3]$ be the vector-function with the functions

$$w_i \in W_{2,\mathrm{loc}}^{(2)}(G) \cap L_2(G)\,.\ (u \in W_{2,\mathrm{loc}}^{(r)}(G)$$

if $u \in W_2^{(r)}(G)$ for arbitrary compact subset $K \subset G$, ([1], 145)). We denote

$$M_1 = \{\bar{w} = [w_1, w_2, w_3] \in H_1,\ w_i \in W_{2,\mathrm{loc}}^{(2)}(G) \cap L_2(G),$$
$$w_1'' + w_2' + \ddot{w}_3 \in L_2(G)\}$$

Integrating by parts we arrive at

$$(Tu, \bar{w})_{H_1} = \int_G (u'' w_1 + u'' w_2 + \ddot{u} w_3)\, dG =$$
$$= \int_G u(w_1'' + w_2' + \ddot{w}_3)\, dG, \qquad u \in C_0^\infty(G)$$

Then

$$M_1 \subset D_{T^*}, \quad T^*\tilde{w} = w_1'' + w_2^{\cdot\cdot} + \ddot{w}_3 , \qquad \tilde{w} \in M_1$$

As basic set M we can use also the set

$$M_0 = \{u \in C^{(4)}(\bar{G}), \ u/\partial G = u'/\partial G = \mathring{u}/\partial G = 0\}$$

Using the theorem on traces in the space $W_2^{(2)}(G)$ ([8], 90) we can deduce that the completion of the set M_0 coincides with the space $\mathring{W}_2^{(2)}(G)$.

The operator B defined by $B\tilde{w} = [D_1 w_1, \ 2D_3 w_2, \ D_2 w_3]$, $\tilde{w} \in H_1$ is positive definite and depicting H_1 into H_1. It is obvious that

$$B(T(M)) \subset D(T^*), \ Du = T^*BTu , \qquad u \in M$$

The operators D, T^*, B, T fulfill all asumptions of the 2. chapter and so there exists just one function minimizing the functional

$$\varphi(u) = |||u|||_D^2 - 2(a, u) = \int_G (D_1 u''^2 + 2D_3 u'^{\cdot 2} + D_2 u^{\cdot\cdot 2} - 2au) \, dG \quad (3.3)$$

in the space $\mathring{W}_2^{(2)}(G)$. Moreover after the theory on differentiability of weak solutions of elliptic equations ([1], 146) $u_0 \in W_{2,\mathrm{loc}}^{(4)}(G)$ and

$$D_1 u_0'''' + 2D_3 u_0''^{\cdot\cdot} + D_2 u_0^{\cdot\cdot\cdot\cdot} = a(x, y) , \qquad (x, y) \in G \quad (3.4)$$

(The equation (3.4) is valid with omiting the set of zero measure). Using the equations (2.7), (2,9), (3.3) and the theorem 2. we arrive at

Theorem 3.

$$\mu = \min_{u \in \mathring{W}_2^{(2)}} \left[\int_G (D_1 u''^2 + 2D_3 u'^{\cdot 2} + D_2 u^{\cdot\cdot 2} - 2au) \, dG \right] =$$

$$= \max_{\substack{w \in D_{T^*} \\ T^*w = a}} \left[- \int_G \left(D_1^{-1} w_1^2 + \frac{1}{2} D_3^{-1} w_2^2 + D_2^{-1} w_3^2 \right) dG \right] = \quad (3.5)$$

$$= \max_{\substack{v \in W_{2,\mathrm{loc}}^{(4)}(G) \cap W_2^{(2)}(G) \\ Dv = a}} \left[- \int_G (D_1 v''^2 + 2D_3 v'^{\cdot 2} + D_2 v^{\cdot\cdot 2}) \, dG \right]$$

It is possible to construct the maximizing sequence for the functional $-d(v) = -\int_G (D_1 v''^2 + 2D_3 v'^{\cdot 2} + D_2 v^{\cdot\cdot 2}) dG$ in the same way as by the Ritz method.

If we know some solution $v_0 \in W_{2,\mathrm{loc}}^{(4)}(G) \cap W_2^{(2)}(G)$ of the equation (3.4), we can setting $v = v_0 + w$ obtain from the equation (3.5) the equation

$$m = \min_{\substack{w \in C^{(4)}(G) \cap W_2^{(2)}(G) \\ Dw = o}} \left[\int_G (D_1 w''^2 + 2D_3 w'^{\cdot 2} + D_2 w^{\cdot\cdot 2}) dG + \right.$$

$$+2\int_G (D_1 v_0'' w'' + 2D_3 v_0^{\prime\cdot} w^{\prime\cdot} + D_2 v_0^{\cdot\cdot} w^{\cdot\cdot})dG\Big] \tag{3.6}$$

with

$$m = -\mu - \int_G (D_1 v_0''^2 + 2D_3 v_0^{\prime\cdot 2} + D_2 v_0^{\cdot\cdot 2})dG$$

We can consider the set $C^{(4)}(G)$ instead of $W^{(4)}_{2,\,loc}(G)$ because after the theory on differentiation of the weak solutions of elliptic equations ([1], 144) every solution of the equation $Dw=0$ posses all derivatives in G. The functional in (3.6) has the form

$$F(w) = d(w, w) + 2l(w), \qquad w \in P$$
$$P = \{w \in C^{(4)}(G) \cap W_2^{(2)}(G), \qquad Dw = 0\} \tag{3.7}$$

Instead of the set P we consider the factor set $\mathcal{P} = P/N$, assuming N the set of all constants and linear polynomials. We introduce the scalar product and the norm on the set $\mathcal{P}$ by

$$[w_1, w_2] = \int_G (D_1 w_1'' w_2'' + 2D_3 w_1^{\prime\cdot} w_2^{\prime\cdot} + D_2 w_1^{\cdot\cdot} w_2^{\cdot\cdot})dG = d(w_1, w_2) \tag{3.8}$$

$$|||w|||_d^2 = \int_G (D_1 w''^2 + 2D_3 w^{\prime\cdot 2} + D_2 w^{\cdot\cdot 2})dG = d(w, w) \tag{3.9}$$

It is obvious that the bilinear form $d(w_1, w_2)$ has on $\mathcal{P}$ all properties of a scalar product. We denote H_d the Hilbert space, which is the completion of the space $\mathcal{P}$ in the norm $|||\cdot|||_d = \sqrt{d(\cdot)}$.

The functional $l(w) = \int_G (D_1 v_0'' w'' + 2D_3 v_0^{\prime\cdot} w^{\prime\cdot} + D_2 v_0^{\cdot\cdot} w^{\cdot\cdot})dG$ is linear and bounded in the energetical norm $|||\cdot|||_d$. If we find the complete system of functions in the space H_d we can use the Ritz method for a construction of a minimizing sequence for the functional $F(w)$. It is sufficient to find the complete system in $\mathcal{P}$, because H_d is the completion of the set $\mathcal{P}$. We use the expression of a solution of the equation $Dw=0$ by means of the analytical functions. The equation $Dw=0$ determines the characteristic equation

$$D_1 + 2D_3 \alpha^2 + D_2 \alpha^4 = 0 \tag{3.10}$$

All roots of the equation (3.10) are not real because the coefficients D_i are positive. There are three cases which depend on discriminant of the equation (3.10).

I. If $D_3^2 - D_1D_2 = 0$, the equation has two double conjugate complex roots $\mu_1, \bar{\mu}_1$.

II. If $D_3^2 - D_1D_2 > 0$, the equation has four different roots with zero real part $\mu_1, \bar{\mu}_1, \mu_2, \bar{\mu}_2$.

III. If $D_3^2 - D_1D_2 < 0$, the equation has four roots with the real part different from zero $\mu_1, \bar{\mu}_1, \mu_2, \bar{\mu}_2$. Moreover $\mu_1 = -\mu_2$.

After the theory of orthotropic plates ([3], 54) we cean every solution of the equation.

$$D_1 w'''' + 2D_3 w''^{\cdot\cdot} + D_2 w^{\cdot\cdot\cdot\cdot} = 0, \; w \in C^{(4)}(G) \tag{3.11}$$

express in the case I. in the form

$$w = \operatorname{Re}[w_1(z_1) + \bar{z}_1 w_2(z_1)] \tag{3.12}$$

and in the cases II., III. in the form

$$w = \operatorname{Re}[w_1(z_1) + w_2(z_2)] \tag{3.13}$$

where $z_1 = x + \mu_1 y$, $z_2 = x + \mu_1 y$, w_1 is a holomorphic function of the variable z_1 in the region S_1 and w_2 is a holomorphic function of the variable z_2 in the region S_2. The regions S_1 and S_2 arise from the region G by the afine transformations of coordinates defined by the equations

$$\begin{aligned} x_1 &= x + \operatorname{Re}\mu_1 y\,, & x_2 &= x + \operatorname{Re}\mu_2 y \\ y_1 &= \operatorname{Im}\mu_1 y\,, & y_2 &= \operatorname{Im}\mu_2 y \end{aligned} \tag{3.14}$$

In the case I. it is possible by means of the transformation of coordinates

$$x_1 = \sqrt[4]{D_1}x\,, \qquad y_1 = \sqrt[4]{D_2}y$$

to transform the equation (3.11) to the biharmonic equation. We consider the cases II. and III. By means of the representation (3.13) and the theorem on approximation of holomorphic functions by polynomials we construct a complete system of solution of the equation (3.11) in the energetical norm (3.9).

In order to obtain further properties of the functions $w_1(z_1)$, $w_2(z_2)$ we recall the proof of (3.13) in [3] in a slightly changed way. We rewrite the equation (3.11) in the form

$$Dw = D^{(1)}D^{(2)}D^{(3)}D^{(4)}w = 0\,, \qquad w \in C^{(4)}(G) \tag{3.15}$$

with

$$D^{(1)}=\frac{\partial}{\partial y}-\mu_1\frac{\partial}{\partial x}, \qquad D^{(2)}=\frac{\partial}{\partial y}-\bar{\mu}_1\frac{\partial}{\partial x}$$

$$D^{(3)}=\frac{\partial}{\partial y}-\mu_2\frac{\partial}{\partial x}, \qquad D^{(4)}=\frac{\partial}{\partial y}-\bar{\mu}_2\frac{\partial}{\partial x}$$

The function $\varphi=D^{(2)}D^{(3)}D^{(4)}w$ satisfies the equation $D^{(1)}\varphi=0$. According to Cauchy—Rieman equations in variables (x_1, y_1) is $\varphi(z_1)$ holomorphic in the region S_1 and we can set $\varphi(z_1) = f_1^{(''''')}(z_1)$, where $f_1^{(''')}(z_1)$ is the third derivative of the function f according z_1. The function $\frac{1}{\mu_1-\bar{\mu}_1}\cdot f_1^{(''')}(z_1)$ is a particular solution of the equation $D^{(2)}f=f_1^{('''')}(z_1)$. Every solution of that equation can be expressed as a sum of particular solution and the solution of homogeneous equation. Then we have

$$D^{(3)}D^{(4)}w=\frac{1}{\mu_1-\bar{\mu}_1}f_1^{(''')}(z_1)+f_2^{(''')}(\bar{z}_1) \tag{3.16}$$

where $f_2^{(''')}(\bar{z}_1)$ is a holomorphic function of the variable $\bar{z}_1$. Repeating that procedure we arrive at

$$\begin{aligned} w=&\frac{1}{(\mu_1-\bar{\mu}_1)(\mu_1-\mu_2)(\mu_1-\bar{\mu}_2)}f_1(z_1)+\\ &+\frac{1}{(\bar{\mu}_1-\mu_2)(\bar{\mu}_1-\bar{\mu}_2)}f_2(\bar{z}_1)+\frac{1}{\mu_2-\bar{\mu}_2}f_3(z_2)+f_4(\bar{z}_2)\end{aligned} \tag{3.17}$$

One can deduce easily from Cauchy—Rieman equation, that $\overline{f_2(\bar{z}_1)}$ is a holomorphic function of variable z_1 in the region S_1 and the function $\overline{f_4(\bar{z}_2)}$ is holomorphic of variable z_2 in S_2.

The function w is real and so we can express it in the form (3.13), where

$$w_1(z_1)=\frac{1}{(\mu_1-\bar{\mu}_2)(\mu_1-\mu_2)}\left(\frac{1}{\mu_1-\bar{\mu}_1}f_1(z_1)+\overline{f_2(z_1)}\right)$$

$$w_2(z_2)=\frac{1}{\mu_2-\bar{\mu}_2}f_3(z_2)+\overline{f_4(z_2)}$$

One can comparing with (3.16) see easily, that

$$w_1^{(4)}(z_1)=\frac{1}{(\mu_1-\bar{\mu}_2)(\mu_1-\mu_2)}D^{(3)}D^{(4)}w$$

After differentiation of the equation (3.17) we obtain

$$w_2^{(4)}(z_2)=\frac{1}{(\mu_1-\mu_2)(\bar{\mu}_1-\mu_2)}D^{(1)}D^{(2)}w$$

Using the assumption $w\in W_2^{(2)}(G)$ one can deduce easily that

$$w_1^{('')}(z_1)\in L_2(G)\,,\qquad w_2^{('')}(z_2)\in L_2(G)$$

and after the transformations of coordinates

$$w_1^{('')}(z_1)\in L_2(S_1)\,,\qquad w_2^{('')}(z_2)\in L_2(S_2)$$

According to the theorem on approximation of holomorphic functions by the polynomials ([5], 126) we can every function $w\in L_2(S)$ and holomorphic in the region S approximate by complex polynomials in $L_2(S)$. The functions $w_1^{('')}(z_1)$ and $w_2^{('')}(z_2)$ fullfil the assumptions of the approximation theorem. Then there exist for an arbitrary $\varepsilon_1>0$ the polynomials $P_{1n}(z_1)$ and $P_{2m}(z_2)$ that

$$\begin{aligned}&\int_G \left|w_1^{(4)}(z_1)-P^{('')}{}_{1n}(z_1)\right|^2 dG<\varepsilon_1\\&\int_G |w_2^{('')}(z_2)-P_{2m}^{('')}(z_2)|^2\,dG<\varepsilon_1\end{aligned}\qquad(3.18)$$

From the Cauchy—Riemann equations we have the next relations for the 2. derivatives of the functions $w_1(z_1)$ and $w_2(z_2)$.

$$\begin{aligned}\frac{\partial^2\,\mathrm{Re}\,w_j(z_j)}{\partial x_j^2}&=\mathrm{Re}\,w_j^{('')}(z_j)\\\frac{\partial^2\,\mathrm{Re}\,w_j(z_j)}{\partial x_j\,\partial y_j}&=-\mathrm{Im}\,w_j^{('')}(z_j)\\\frac{\partial^2\,\mathrm{Re}\,w_j(z_j)}{\partial y_j^2}&=-\mathrm{Re}\,w_j^{('')}(z_j)\,,\qquad j=1,2\end{aligned}\qquad(3.19)$$

Using the equations (3.14), (3.19) we arrive at relations

$$\begin{aligned}&[\mathrm{Re}\,w_j(z_j)]''=\mathrm{Re}\,w_j^{('')}(z_j)\\&[\mathrm{Re}\,w_j(z_j)]^{\prime\cdot}=\mathrm{Re}\,\mu_j\,\mathrm{Re}w\,{}^{('')}_j(z_j)-\mathrm{Im}\,\mu_j\,\mathrm{Im}\,w_j^{('')}(z_j)\\&[\mathrm{Re}\,w_j(z_j)]^{\cdot\cdot}=(\mathrm{Re}\,\mu_j)^2\,\mathrm{Re}\,w_j^{('')}(z_j)-(\mathrm{Im}\,\mu_j)^2\,\mathrm{Re}\,w_j^{('')}(z_j)-\\&\qquad -2\,\mathrm{Re}\,\mu_j\,\mathrm{Im}\,\mu_j\,\mathrm{Im}\,w_j^{('')}(z_j)\end{aligned}\qquad(3.20)$$

Employing the equations (3.9), (3.13), (3.18), (3.20) we find for an arbitrary $\varepsilon>0$ the polynomials $P_{1n}(z_1)$, $P_{2m}(z_2)$ that

$$|||w-\mathrm{Re}\,(P_{1n}(z_1)+P_{2m}(z_2)|||_d<\varepsilon\,,\qquad w\in P\qquad(3.21)$$

The polynomials Re $(P_{1n}(z_1)+P_{2m}(z_2))$ are not considering their linear parts whose second derivatives vanish, the linear combinations of the polynomials

$$\begin{aligned} \varphi_{1n} &= \text{Re}\,(x+\mu_1 y)^n\,, \qquad \psi_{1n} = \text{Im}\,(x+\mu_1 y)^n\,, \\ \varphi_{2n} &= \text{Re}\,(x+\mu_2 y)^n\,, \qquad \psi_{2n} = \text{Im}\,(x+\mu_2 y)^n\,, \end{aligned} \qquad n>1 \tag{3.22}$$

which are not zero (in the factor set $\mathscr{P}$) solutions of the equation $Dw=0$. We show that it is possible to choose from the system (3.22) the maximal linear independent system which is complete in the energetical norm (3.9).

The asumption II. The coordinate functions are in the form

$$\begin{aligned} \varphi_{1n} &= \text{Re}\,(x+i\varrho_1 y)^n\,, \qquad \psi_{1n} = \text{Im}\,(x+i\varrho_1 y)^n\,, \\ \varphi_{2n} &= \text{Re}\,(x+i\varrho_2 y)^n\,, \qquad \psi_{2n} = \text{Im}\,(x+i\varrho_2 y)^n \\ n &= 2, 3, \ldots, |\varrho_1| \neq |\varrho_2|, \; \varrho_1 \neq 0, \; \varrho_2 \neq 0 \end{aligned} \tag{3.22'}$$

The polynomials (3.22′) are for every n homogeneous of the n-th degree. It is sufficient for the linear independence of the system (3.22′) to show the linear independence only for the quadruples of the polynomials (3.22) for every n.

If $n=2$, then

$$\begin{aligned} \varphi_{12} &= x^2-\varrho_1^2 y^2\,, \qquad \psi_{12} = \varrho_1 xy \\ \varphi_{22} &= x^2-\varrho_2^2 y^2\,, \qquad \psi_{22} = \varrho_2 xy \end{aligned}$$

It is easy to find out that the polynomials φ_{12}, ψ_{12}, φ_{22} are linear independent. The polynomials $\{x^{n-k}y^k\}_{k=0}^n$ are linear independent. The polynomials φ_{1n}, φ_{2n}, ψ_{1n}, ψ_{2n} are the linear combinations of the polynomials $\{x^{n-k}y^k\}_{k=0}^n$ and so for their linear independence is necessary and sufficient that the rank of their matrix of coefficients is 4. Sufficient for it is that at least one subdeterminant of the 4-th order is different from zero.

$$(x+\varrho_j iy)^n = x^n + n\varrho_j ix^{n-1}y - \binom{n}{2}\varrho_j^2 x^{n-2}y^2 - \\ -\binom{n}{3}\varrho_j^3 ix^{n-3}y^3 + \ldots$$

The minor determinant formed from the first four collumns of the matrix of coefficients of polynomials φ_{1n}, φ_{2n}, ψ_{1n}, ψ_{2n} is

$$\begin{vmatrix} 1 & 0 & -\binom{n}{2}\varrho_1^2 & 0 \\ 1 & 0 & -\binom{n}{2}\varrho_2^2 & 0 \\ 0 & n\varrho_1 & 0 & -\binom{n}{3}\varrho_1^3 \\ 0 & n\varrho_2 & 0 & -\binom{n}{3}\varrho_2^3 \end{vmatrix} = n\binom{n}{2}\binom{n}{3}\varrho_1\varrho_2(\varrho_1^2-\varrho_2^2)\neq 0$$

and so for $n>2$ is the system $\{\varphi_{1n}, \varphi_{2n}, \psi_{1n}, \psi_{2n}\}$ linear independent. Then the polynomials (3.22′) with ommiting the function ψ_{12}, or ψ_{22} form complete system of functions in the factor set $\mathscr{P}$ with the energetical scalar product (3.8) and the norm (3.9).

The assumption III. The coordinate functions are in the form

$$\varphi_{1n} = \text{Re}[x+(a+bi)y]^n\ , \qquad \psi_{1n} = \text{Im}[x+(a+bi)y]^n\ ,$$
$$\varphi_{2n} = \text{Re}[x-(a+bi)y]^n\ , \qquad \psi_{2n} = \text{Im}[x-(a+bi)y]^n\ ,$$
$$n=2, 3, \ldots, a\neq 0\ , b\neq 0 \tag{3.22″}$$

We can find out the linear independence of the system (3.22″) in the same way as for the system (3.22′) in the case II.

If $n=2$, then

$$\varphi_{12}=x^2+2axy+(a^2-b^2)y^2,\ \psi_{12}=2bxy$$
$$\varphi_{22}=x^2-2axy+(a^2-b^2)y^2,\ \psi_{22}=-2bxy$$

We can see that in this system there are just two linear independent functions. We can take $\{\varphi_{12}, \varphi_{22}\}$, or $\{\varphi_{12}, \psi_{12}\}$.

If $n>2$, then we must find out the rank of the matrix of the coefficients.

$$x\pm(a+bi)y]^n = x^n \pm n(a+bi)x^{n-1}y + \binom{n}{2}(a+bi)^2x^{n-2}y^2 \pm$$
$$\pm\binom{n}{3}(a+bi)^3x^{n-3}y^3+\ldots$$

The minor determinant from the first four columns is

$$\begin{vmatrix} 1 & na & \binom{n}{2}(a^2-b^2) & \binom{n}{3}(a^3-3ab^2) \\ 1 & -na & \binom{n}{2}(a^2-b^2) & -\binom{n}{3}(a^3-3ab^2) \\ 0 & nb & 2\binom{n}{2}ab & \binom{n}{3}(3a^2b-b^3) \\ 0 & -nb & 2\binom{n}{2}ab & -\binom{n}{3}(3a^2b-b^3) \end{vmatrix} = 16n\binom{n}{2}\binom{n}{3}a^2b^2(a^2+b^2) \neq 0$$

Then the quadruple $\{\varphi_{1n}, \varphi_{2n}, \psi_{1n}, \psi_{2n}\}$ is linear independent for every $n>2$ and the system of polynomials (3.22″) without the polynomials ψ_{12}, ψ_{22}, or φ_{22}, ψ_{22} is complete in the factor set $\mathcal{P}$.

We can use the Ritz's method for the construction of the minimizing sequence $\{w_n\}$ for the functional (3.7). The coordinate functions are the polynomials defined in (3.22). The sequence $v_n = v_0 + w_n$ is the maximizing sequence for Treffz's functional

$$-d(v) = -\int_G (D_1 v''^2 + 2D_3 v'^{\cdot 2} + D_2 v^{\cdot\cdot 2})\,dG$$

defined on the set of all solutions from the space $W^{(4)}_{2,\mathrm{loc}}(G) \cap W^{(2)}_2(G)$. The sequence of corresponding values of Trefftz's functional indicates the lower estimates for the minimum of the energy functional (3.3).

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SÚHRN

DOLNÝ ODHAD MINIMA FUNKCIONÁLA ENERGIE POMOCOU VETY O DUALITE

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Táto práca poukazuje na to, ako pomocou vety o dualite možno získať dolný odhad minima funkcionálu energie zodpovedajúceho parciálnej diferenciálnej rovnici

$$D_1 \frac{\partial^4 u}{\partial x^4} + 2D_3 \frac{\partial^4 u}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 u}{\partial y^4} = a(x, y)$$

s homogénnymi Dirichletovými okrajovými podmienkami. Problém minima v oblasti definície je upravený na problém maxima adjugovaného funkcionálu na množine všetkých riešení diferenciálnej rovnice. Pomocou vety o aproximácii v strede analytickej funkcie polynómami bola zostrojená maximalizačná postupnosť.

РЕЗЮМЕ

НИЖНЯЯ ОЦЕНКА МИНИМУМА ФУНКЦИОНАЛА ЭНЕРГИИ С ПОМОЩЬЮ ТЕОРЕМЫ О ДВОЙСТВЕННОСТИ

И. Бок, Братислава

В этой работе мы получили, с помощью теоремы о двойственности, нижнюю оценку минимума функционала энергии дифференциального уравнения

$$D_1 \frac{\partial^4 u}{\partial x^4} + 2D \frac{\partial^4 u}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 u}{\partial y^4} = a(x, y)$$

с однородными краевыми условиями Дирихле. Мы перевели проблему минимума на области определения в проблему максимума сопряженного функционала на множестве всех решений дифференциального уравнения. С помощью теоремы о приближении в среднем аналитических функций многочленами была построена максимизирующая последовательность.

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ON SETS OF VALUES OF SOME ARITHMETICAL FUNCTIONS

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In this paper we shall study the sets of values of functions σ, $\sigma+\tau$, $\sigma+\varphi$, where $\sigma(n)$ denotes the sum of all natural divisors of n, $\tau(n)$ the number of all natural divisors of n and φ is the Euler's function.

If A is a set of positive integers, then the number of all elements $a\in A$, $a\leqq n$ will be denoted by $A(n)$. If there exists the number

$$\delta(A)=\lim_{n\to\infty}\frac{A(n)}{n}$$

it will be called the *natural density* of the set A. The number

$$\delta_2(A)=\limsup_{n\to\infty}\frac{A(n)}{n}$$

will be called the *upper density* of the set A.

Let us consider the set of all values of the function σ, that is, the set $M=\{\sigma(1), \sigma(2), \dots\}$. In [2] H. J. Kanold has proved that $\delta(M)=0$. We shall give another proof, which is partly different from Kanold's one.

Theorem 1. The set M of all values of the function σ has the natural density zero.

Proof. If $n=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_s^{\alpha_s}$ is the standard form of the integer $n>1$, then

$$\sigma(n)=\prod_{k=1}^{s}(1+p_k+\dots+p_k^{\alpha_k}).$$

Let $\varepsilon>0$ be given and $\varepsilon'=\varepsilon/2$. There exists a positive integer k such that $2^{-k}<\varepsilon'$. Let N_1 denote the set of all these positive integers in the standard forms of which there are at least $k+1$ primes with odd exponents α_i and let $N_2=N-N_1$. Put $\sigma(N_1)=A$, $\sigma(N_2)=B$, hence $M=A\cup B$.

For an odd α_i and $p_i\neq 2$ the number $1+p_i+p_i^2+\dots+p_i^{\alpha_i}$ is divisible by 2. Evidently, if $n\in N_1$ then $2^k\,|\,\sigma(n)$ and hence

$$A(n) \leqq \frac{n}{2^k} < n\varepsilon' \tag{1}$$

Every element of the set N_2 has the form $p_1 p_2 \dots p_r p_{r+1}^{\alpha_{r+1}} p_{r+2}^{\alpha_{r+2}} \dots p_{r+s}^{\alpha_{r+s}}$ where $p_1, p_2, \dots, p_r$ are distinct primes, $s \geqq 0$ is an integer, α_i are even natural numbers, $0 \leqq r \leqq k$. According to [1], p. 256, we have $\delta(N_2) = 0$. For each positive integer we have $\sigma(n) \geqq n$. Hence if $x \in N_2$ and $\sigma(x) \leqq n$, then $x \leqq n$. This yields $B(n) \leqq N_2(n)$. Since $\delta(N_2) = 0$, there exists n_0 such that for $n \geqq n_0$ the inequality $N_2(n) < n\varepsilon'$ holds. Hence

$$B(n) \leqq N_2(n) < n\varepsilon' \tag{2}$$

for $n \geqq n_0$.

From (1) and (2) we obtain $\delta_2(A) \leqq \varepsilon'$ and $\delta_2(B) \leqq \varepsilon'$. So $\delta_2(M) \leqq \delta_2(A) + \delta_2(B) \leqq \varepsilon' + \varepsilon' = \varepsilon$. Since $\varepsilon > 0$ was arbitrary chosen, we have $\delta_2(M) = \delta(M) = 0$ and the theorem is proved.

Now we shall prove two theorems about natural densities of the sets of all values of the functions $\sigma + \tau$, $\sigma + \varphi$.

Theorem 2. The set F of all values of the function $\sigma + \tau$ has the natural density zero.

Proof. Let N_1, N_2 have the same sense as in the preceeding proof. It is well-known that

$$\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_s + 1), \quad \text{if} \quad n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$$

is the standard form of n, and therefore $2^k \mid \tau(n)$ if $n \in N_1$.

Let $\varepsilon > 0$ be given and $\varepsilon' = \varepsilon/2$. We can chose a positive integer k so that $2^{-k} < \varepsilon'$. Put $f(n) = \sigma(n) + \tau(n)$ for $n = 1, 2, \dots, f(N_1) = F_1$ and $f(N_2) = F_2$. Hence $F = F_1 \cup F_2$.

For every $n \in N_1$ we have $2^k \mid (\sigma(n) + \tau(n))$ and hence

$$F_1(n) \leqq \frac{n}{2^k} < n\varepsilon' \tag{3}$$

If $\sigma(x) + \tau(x) \leqq n$ for some $x \in N_2$ then $x \leqq n$. Since the natural density of N_2 is zero, there exists such an n_0 that for each $n \geqq n_0$ we have $N_2(n) < n\varepsilon'$. Hence

$$F_2(n) \leqq N_2(n) < n\varepsilon' \tag{4}$$

holds for every $n \geqq n_0$.

From (3) and (4) we obtain $\delta_2(F_1) \leqq \varepsilon'$, $\delta_2(F_2) \leqq \varepsilon'$ and so $\delta_2(F) \leqq \delta_2(F_1) + \delta_2(F_2) \leqq \varepsilon' + \varepsilon' = \varepsilon$. Since ε was arbitrary chosen, we have $\delta_2(F) = \delta(F) = 0$.

Theorem 3. The set G of all values of the function $\sigma + \varphi$ has the natural density zero.

Proof. Let N_1, N_2 have the same meaning as in the proof of Theorem 1. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$ is the standard form of n, then

$$\varphi(n) = \frac{n}{p_1 p_2 \dots p_s} (p_1 - 1)(p_2 - 1) \dots (p_s - 1)$$

(see [1], p. 256)

Let $\varepsilon > 0$ be given and $\varepsilon' = \varepsilon/2$. Then we can choose k so that $2^{-k} < \varepsilon'$. Put $g(n) = \sigma(n) + \varphi(n)$ for $n = 1, 2, \dots$, $g(N_1) = G_1$ and $g(N_2) = G_2$. If $x \in N_1$, then $2^k \mid g(x)$ and hence

$$G_1(n) \leqq \frac{n}{2^k} < n\varepsilon' \tag{5}$$

Further $g(x) > x$ for every positive integer x and therefore $g(x) \leqq n$ implies $x \leqq n$. So $G_2(n) \leqq N_2(n)$. Since $\delta(N_2) = 0$ there exists such an n_0 that for each $n \geqq n_0$ we have $N_2(n) < n\varepsilon'$. This together with $G_2(n) \leqq N_2(n)$ gives

$$G_2(n) \leqq N_2(n) < n\varepsilon' \tag{6}$$

for each $n \geqq n_0$.

From (5) and (6) it follows $\delta_2(G) \leqq \delta_2(G_1) + \delta_2(G_2) \leqq \varepsilon' + \varepsilon' = \varepsilon$ and so we have $\delta_2(G) = \delta(G) = 0$, as ε was choosen arbitrary.

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SÚHRN

O MNOŽINÁCH HODNÔT NIEKTORÝCH ARITMETICKÝCH FUNKCIÍ

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V práci je dokázané, že prirodzená hustota množiny hodnôt každej z funkcií σ, $\sigma + \tau$, $\sigma + \varphi$ je nulová ($\sigma(n)$ znamená súčet a $\tau(n)$ počet prirodzených deliteľov čísla $n \geqq 1$, φ je Eulerova funkcia).

РЕЗЮМЕ

О МНОЖЕСТВАХ ЗНАЧЕНИЙ НЕКОТОРЫХ АРИФМЕТИЧЕСКИХ ФУНКЦИЙ

П. Беро, Братислава

В работе доказано, что натуральная плотность множества значений каждой из функций σ, $\sigma+\tau$, $\sigma+\varphi$ равняется нулю ($\sigma(n)$ означает сумму и $\tau(n)$ число натуральных делителей числа $n \geqq 1$, φ есть функция Эйлера).

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A CHARACTERISTIC PROPERTY OF BANACH SPACES

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In the theory of series in Banach spaces it is well known the following Orlicz's theorem (see [2], p. 86): The series $\sum_{n=1}^{\infty} u_n$ of elements of Banach space is unconditionally convergent if and only if for each increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers the series $\sum_{k=1}^{\infty} u_{n_k}$ is convergent. T. Šalát has raised the problem to give a characterization of Banach spaces by using the above mentioned theorem. The answer on this problem is given in the next Theorem 1 and Theorem 2.

Further we shall use the following well known characteristic property of Banach spaces (see [1], p. 47).

Lemma. A normed linear space X is complete if and only if every absolutely convergent series of elements of X is convergent.

Theorem 1. Let X be a normed linear space. Then X is complete if and only if the unconditional convergence of an arbitrary series $\sum_{n=1}^{\infty} u_n$, $u_n \in X$, implies the convergence of each series $\sum_{k=1}^{\infty} u_{n_k}$, where $\{n_k\}_{k=1}^{\infty}$ is any increasing sequence of natural numbers.

Proof. The fact, that the completnees of the space X implies the part "only if" of our theorem is a consequence of the Orlicz's theorem.

If X is not complete, then there exists, according to Lemma, an absolutely convergent series $\sum_{n=1}^{\infty} x_n$, $x_n \in X$, which is not convergent in X. Let αX be the completnees of the space X (see [1], p. 47). Since αX is a Banach space and for the series $\sum_{n=1}^{\infty} x_n$ the Cauchy condition is fulfilled it is, according to Cauchy theorem (see [2], p. 86), convergent in αX. Hence $\sum_{n=1}^{\infty} x_n = x \in \alpha X - X$. Let us construct the series $\sum_{n=1}^{\infty} u_n$ by the following way: $u_{2k-1} = x_k$, $u_{2k} = -x_k$ $(k = 1, 2, \ldots)$. The series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent and hence it is unconditionally convergent in αX. Since $\sum_{n=1}^{2N} u_n = 0$ for each $N = 1, 2, \ldots$ we have $\sum_{n=1}^{\infty} u_n = 0 \in X$. The sum of any unconditionally convergent series does not depend on the order of its terms (see [2], p. 87). Consequently $\sum_{n=1}^{\infty} u_{\tau(n)} = 0$, where τ is any permutation of the set of natural numbers. Hence $\sum_{n=1}^{\infty} u_n$ is unconditionally convergent in X and the series $\sum_{k=1}^{\infty} u_{2k-1}$ $\left(\text{or } \sum_{k=1}^{\infty} u_{2k}\right)$ is not convergent in X.

Theorem 2. Let X be a normed linear space and let $\sum_{n=1}^{\infty} u_n$ be a series of elements of X. Then the convergence of each of series $\sum_{k=1}^{\infty} u_{n_k}$, where $\{n_k\}_{k=1}^{\infty}$ is an arbitrary increasing sequence of natural numbers, implies the unconditional convergence of $\sum_{n=1}^{\infty} u_n$.

Proof. Let us assume that for each increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers we have $\sum_{k=1}^{\infty} u_{n_k} \in X$. If we put $n_k = k$, $k = 1, 2, \ldots$, then obviously $\sum_{n=1}^{\infty} u_n \in X$. From the Orlicz's theorem it follows the validity of our theorem in Banach spaces, hence it holds in αX. Therefore $\sum_{n=1}^{\infty} u_{\tau(n)} \in \alpha X$, where τ is any permutation of the set of natural numbers. The sum of any unconditionally convergent series does not depend on the order of its terms, and so $\sum_{n=1}^{\infty} u_{\tau(n)} = \sum_{n=1}^{\infty} u_n \in X$. Consequently, the series $\sum_{n=1}^{\infty} u_n$ is unconditionally convergent in X.

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SÚHRN

JEDNA CHARAKTERISTICKÁ VLASTNOSŤ BANACHOVÝCH PRIESTOROV

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V práci sa dokazuje toto tvrdenie: Nech X je normovaný lineárny priestor. Potom X je úplný priestor vtedy a len vtedy, ak z bezpodmienečnej konvergencie ľubovoľného radu $\sum_{n=1}^{\infty} u_n$, $u_n \in X$, vyplýva konvergencia radu $\sum_{k=1}^{\infty} u_{n_k}$, kde $\{n_k\}_{k=1}^{\infty}$ je ľubovoľná rastúca postupnosť prirodzených čísel.

РЕЗЮМЕ

ОДНО ХАРАКТЕРИСТИЧЕСКОЕ СВОЙСТВО ПРОСТРАНСТВ БАНАХА

П. Костырко, Братислава

В работе доказывается утверждение: Если X нормированное линейное пространство, то X является полным пространством тогда и только тогда, если из безусловной сходимости всякого ряда

$$\sum_{n=1}^{\infty} u_n, \quad u_n \in X,$$

вытекает сходимость ряда

$$\sum_{k=1}^{\infty} u_{n_k}, \quad \text{где} \quad \{n_k\}_{k=1}^{\infty}$$

любая возрастающая последовательность натуральных чисел.

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ON COMPACT MEASURES ON SEMIRINGS

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It is known that every compact measure defined in a field is σ-additive. See [1], p. 118. Marczewski in his proof uses a continuity of the measure from above as a characterization of σ-additivity which in general case does not hold for semirings. See [2], p. 44. In paper we prove the assertion for semirings.

Definition. Let μ be a measure (non negative additive setfunction) defined on a semiring m. Let F be a class of subsets of X. We say that F approximates m with respect to the measure μ, if for every set $E \in m$ and for every $\eta > 0$ there exists a set $P \in F$ and a set $D \in m$ such that

$$D \subset P \subset E \quad \text{and} \quad \mu(E) - \mu(D) < \eta$$

Theorem. Let μ be a measure defined on a semiring m. Let F^* be a class closed under finite intersections and such that every decreasing sequence of sets belonging to F has non-void intersection. Let F^* approximates m with respect to μ. Then μ is σ-additive.

Proof. The non-negative function μ can be extended from the semiring m into the least field n over m denoted $n(m)$.

Let $\bar{\mu}$ be this extension. We will show that the system $F_1 = \left\{ \bigcup_{i=1}^{n} A_i,\ a_i \in F^*,\ n \text{ positive integer},\ A_i \text{ disj.} \right\}$ approximates $n(m)$ with rexpect to μ.

Since F^* approximates m with rexpect to μ, then for every a set $E_i \in m$ and for every $\eta_i > 0$ there exist a set $P_i \in F^*$ and a set $D_i \in m$ such that

$$D_i \subset P_i \subset E_i \quad \text{and} \quad \mu(E_i) - \mu(D_i) < \eta_i$$

Every a set $R \in n(m)$ can be written as $R = \bigcup_{i=1}^{n} E_i$, $E_i \in m$, E_i disjoint. See [2], p. 30. So for every a set $R \in n(m)$ and for every $\eta > 0$ there exists a set $\bigcup_{i=1}^{n} P_i \in F_1$ such that

$$\bigcup_{i=1}^{n} D_i \subset \bigcup_{i=1}^{n} P_i \subset \bigcup_{i=1}^{n} E_i$$

and

$$\bar{\mu}\left(\bigcup_{i=1}^{n} E_i\right) - \bar{\mu}\left(\bigcup_{i=1}^{n} D_i\right) = \sum_{i=1}^{n} \bar{\mu}(E_i) - \bar{\mu}(D_i) =$$
$$= \sum_{i=1}^{n} \mu(E_i) - \mu(D_i) < \sum_{i=1}^{n} \eta_i < \eta$$

It suffices to show that F_1 is a compact class, i.e. for every sequence $E_k \in F_1$, $E_k \neq \emptyset$, $E_m \supset E_{m+1}$ $(m = 1, 2, \ldots)$ we have $\bigcap_{k=1}^{\infty} E_k \neq \emptyset$.

Let

$$E_m = \bigcup_{i=1}^{n_m} D_i^m \in F^* \qquad (i = 1, 2, \ldots, n_m)$$

choose first $D_{i_1}^1$ in such a way $D_{i_1}^1 \cap E_m \neq \emptyset$ for infinite many m. If such $D_{i_1}^1$ does not exist then $D_{i_1}^1 \cap E_m = \emptyset$ for sufficiently large m and every i, thus $E_m = E_1 \cap E_m = = \bigcup_{i=1}^{n_1} (D_{i_1}^1 \cap E_m) = \emptyset$ which is contradiction with the supposition $E_m \neq \emptyset$.

Similarly we choose $D_{i_2}^2$ such $D_{i_1}^1 \cap D_{i_2}^2 \cap E_m = \emptyset$ for infinitely many indices m. Using this method we construct a sequence

$$D_{i_1}^1, \; D_{i_2}^2, \; D_{i_3}^3, \ldots$$

such that

$$C_k = D_{i_1}^1 \cap D_{i_2}^2 \cap \ldots \cap D_{i_k}^k = \emptyset \qquad (k = 1, 2, \ldots)$$

Since

$$C_k \in F^* E_k \supset D_{i_k}^k \supset C_k \supset C_{k+1} \qquad (k = 1, 2, \ldots)$$

it holds

$$\emptyset \neq \sum_{k=1}^{\infty} C_k \subset \sum_{k=1}^{\infty} E_k$$

The class F^* is a compact class according to μ. See [1], p. 115.

Example. Finally we present an example of a compact measure on a semiring.

Let X be the real line then the class of all bounded intervals closed from the left and open from the right is a semiring. See [2], p. 27.

Put

$$\mu\langle a, b) = \sum_{n_i \in \langle a, b)} \frac{1}{2^{|n_i|}}$$

where $n_i \in N'$ is the set of all integers. μ is a non negative additive function. The compact class F approximating given semiring with respect to μ is the class of all bounded closed intervals on X. See [3], p. 153.

For every set $\langle a, b)$ and for every $\eta > 0$ there exists a set $\langle a, b - \eta)$ such that

$$\langle a, b - \eta) \subset \langle a, b - \eta \rangle \subset \langle a, b)$$

and

$$\mu \langle a, b) - \mu \langle a, b - \eta) < \eta.$$

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SÚHRN

KOMPAKTNÉ MIERY NA POLOOKRUHOCH

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V práci je zavedený pojem kompaktnej (additívnej) miery na polookruhu. Ukazuje sa, že Marczewskeho vetu o tom, že každá kompaktná (additívna) miera, definovaná na okruhu je σ-additívna, možno dokázať aj na polookruhu. V práci je podaný príklad kompaktnej miery, definovanej na polookruhu.

РЕЗЮМЕ

КОМПАКТНЫЕ МЕРЫ НА ПОЛУКОЛЬЦАХ

С. Горна, Братислава

В работе введено определение компактной (аддитивной) меры на полукольце. Показано, что теорему Марчевского о том, что каждая компактная (аддитивная) мера, определенная на кольце, является σ-аддитивной, можно доказать тоже на полукольце. В работе введен пример компактной меры, определенной на полукольце.

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Úvod

Axioma výberu na ľubovoľnom systéme S neprázdnych množín zaručuje existenciu tzv. výberovej funkcie (v ďalšom skrátene v. f.), t. j. takej funkcie $g: S \to \cup\{X; X \in S\}$, pre ktorú platí, že $\forall X \in S(g(X) \in X)$. Vzniká otázka, kedy na systéme S existuje prostá výberová funkcia (v ďalšom skrátene p. v. f.), t. j. taká v. f. g, pre ktorú navyše platí, že $\forall X_1, X_2 \in S\,(X_1 \neq X_2 \to g(X_1) \neq g(X_2))$. Nevyhnutnou podmienkou existencie p. v. f. na S je, aby platilo $\forall S' \subset S\,(\overline{\overline{\cup\{X; X \in S'\}}} \geqq \bar{\bar{S}}')$ (kde $\bar{\bar{S}}'$ je kardinálne číslo k S'). Táto podmienka však nie je postačujúca, pretože napr. systém $\{\{x_1, x_2, \ldots, x_n, \ldots\}, \{x_1\}, \{x_2\}, \ldots, \{x_n\}, \ldots\}$ (kde $x_i \neq x_j$ pre $i \neq j$) ju splňuje a pritom na ňom neexistuje p. v. f.

V tejto práci ukážeme, že uvedená podmienka je pre konečný systém a pre systém zložený z konečných množín postačujúca k existencii p. v. f. Ďalej preskúmame štruktúru systémov, na ktorých existuje práve jedna p. v. f.

Veta 1. Nech pre systém S platí:

1. $$\bar{\bar{S}} < \aleph_0,$$

2. $$\forall S' \subset S\,(\overline{\overline{\cup\{X; X \in S'\}}} \geqq \bar{\bar{S}}').$$

Potom na S existuje p. v. f.

Dôkaz. Nech systém S splňuje predpoklady vety 1. Potom S neobsahuje prázdnu množinu. Usporiadajme množiny z S do prostej postupnosti $X_1, X_2, \ldots, X_n$ a vyberajme z nich postupne prvky $x_1, x_2, \ldots, x_i, \ldots$ tak, aby $x_1 \in X_1$, $x_2 \in X_2 - \{x_1\}$, $\ldots, x_i \in X_i - \{x_1, x_2, \ldots, x_{i-1}\} \ldots$ Ak po k krokoch dôjdeme k takej množine X_{k+1}, že $X_{k+1} - \{x_1, x_2, \ldots, x_k\} = \emptyset$, postupujeme ďalej takto:

Utvorme

$$S_1 = \{X_i; 1 \leqq i \leqq k,\ x_i \in X_{k+1}\} \quad \text{a} \quad A_1 = \cup\{X_i; X_i \in S_1\}$$

Ak $A_1 \subset \{x_1, x_2, \ldots, x_k\}$, utvorme

$$S_2 = \{X_i\,;\, 1 \leqq i \leqq k,\ x_i \in A_1\} \quad \text{a} \quad A_2 = \bigcup \{X_i\,;\, X_i \in S_2\}$$

Ak $A_2 \subset \{x_1, x_2, \ldots, x_k\}$, utvorme

$$S_3 = \{X_i\,;\, 1 \leqq i \leqq k,\ x_i \in A_2\} \quad \text{a} \quad A_3 = \cup \{X_i\,;\, X_i \in S_3\}$$

atď.

Ak $A_{j-1} \subset \{x_1, x_2, \ldots, x_k\}$, utvorme

$$S_j = \{X_i\,;\, 1 \leqq i \leqq k,\ x_i \in A_{j-1}\} \quad \text{a} \quad A_j = \cup \{X_i\,;\, X_i \in S_j\}$$

atď.

Zrejme $X_{k+1} \subset A_1 \subset A_2 \subset \ldots \subset A_{j-1} \subset A_j \subset \ldots$

Ďalej $X_{k+1} \neq A_1 \neq A_2 \neq \ldots \neq A_{j-1} \neq A_j \neq \ldots$, pretože $\bar{\bar{S}}_1 = \bar{\bar{X}}_{k+1}$ a ak označíme $S' = S_1 \cup \{X_{k+1}\}$, bude $\bar{\bar{S}}' = \bar{\bar{S}}_1 + 1$ a $A_1 = \cup \{X_i\,;\, X_i \in S'\}$. Pritom podľa predpokladu vety 1 musí byť $\bar{\bar{A}}_1 \geqq \bar{\bar{S}}'$, t. j. $\bar{\bar{A}}_1 \geqq \bar{\bar{X}}_{k+1} + 1$, teda $A_1 \neq X_{k+1}$.

Podobne z konštrukcie S_j vyplýva, že $\bar{\bar{S}}_j = \bar{\bar{A}}_{j-1}$ a teda, ak označíme $S' = S_j \cup \{X_{k+1}\}$, bude $\bar{\bar{S}}' = \bar{\bar{A}}_{j-1} + 1$ a $A_j = \cup \{X_i\,;\, X_i \in S'\}$. Pritom podľa 2. predpokladu vety 1 musí byť $\bar{\bar{A}}_j \geqq \bar{\bar{S}}'$, t. j. $\bar{\bar{A}}_j \geqq \bar{\bar{A}}_{j-1} + 1$, teda $A_j \neq A_{j-1}$.

Z $X_{k+1} \subsetneqq A_1 \subsetneqq A_2 \subsetneqq \ldots \subsetneqq A_{j-1} \subsetneqq A_j \subsetneqq \ldots$ vyplýva, že po konečnom počte krokov musíme dôjsť k takému A_m, pre ktoré platí, že $A_m \not\subset \{x_1, x_2, \ldots, x_k\}$.

Vyberme teraz $x_{k+1} \in A_m - \{x_1, x_2, \ldots, x_k\}$.

Ďalej vyberme

$X_{i_m} \in S_m$	tak, aby	$x_{k+1} \in X_{i_m}$,
$X_{i_{m-1}} \in S_{m-1}$	tak, aby	$x_{i_m} \in X_{i_{m-1}}$,
$X_{i_{m-2}} \in S_{m-2}$	tak, aby	$x_{i_{m-1}} \in X_{i_{m-2}}$, atď.

až $X_{i_1} \in S_1$ tak, aby $x_{i_2} \in X_{i_1}$.

Môžeme to urobiť, pretože ak $X_i \in S_j$, potom podľa definície S_j pre predtým vybraté $x_i \in X_i$ musí platiť, že $x_i \in A_{j-1} = \cup \{X_i\,;\, X_i \in S_{j-1}\}$.

Potom prestavajme postupnosť $X_1, X_2, \ldots, X_n$ tak, že X_{i_j} posunieme na miesto $X_{i_{j+1}}$, pre $j = 1, 2, \ldots, m-1$; a X_{k+1} na miesto X_{i_1} a X_{i_m} na miesto X_{k+1}. Pre takto prestavanú postupnosť bude platiť, že $x_1 \in X_1$, $x_2 \in X_2$, ..., $x_k \in X_k$, $x_{k+1} \in X_{k+1}$.

Opakovaním postupu môžeme teda zostrojiť postupnosť $x_1, x_2, \ldots, x_n$ tak, aby $x_i \neq x_j$ pre $i \neq j$ a $x_1 \in X_1$, $x_2 \in X_2$, ..., $x_n \in X_n$. Tým je dôkaz vety 1 skončený.

Veta 2. Nech pre systém S platí:

1. $\forall X \in S (\bar{\bar{X}} < \aleph_0)$

2. $\forall S' \subset S(\overline{\overline{\cup\{X; X \in S'\}}} \geqq \overline{\overline{S'}})$.

potom na S existuje p. v. f.

Dôkaz. Nech systém S splňuje predpoklady vety 2. Nech M je množina všetkých takých dvojíc (S', g'), pre ktoré platí:

a) $S' \subset S$,

b) g' je p. v. f. na S',

c) g' sa dá rozšíriť (tak, že bude opäť p. v. f.) na každé S'', pre ktoré platí, že $S' \subset S'' \subset S$ a $\overline{\overline{S'' - S'}} < \aleph_0$.

Potom platí:

A. Množina M je neprázdna.

Dôkaz. Nech $S' \subset S$ a $\overline{\overline{S'}} < \aleph_0$. Nech $H = \{S''; S' \subset S'' \subset S, \overline{\overline{S''}} < \aleph_0\}$. Pre každé $S'' \in H$ nech $G(S'')$ je množina všetkých tých funkcií g', ktoré sú p. v. f. na S' a ktoré sa dajú rozšíriť na S''. Potom $\cap\{G(S''); S'' \in H\} \neq \emptyset$, pretože:

a') $\forall S'' \in H(0 < \overline{\overline{G(S'')}} < \aleph_0)$ (ľavá nerovnosť vyplýva z toho, že podľa vety 1, na $S' \cup S''$ existuje p. v. f., a pravá z toho, že S' je vytvorený konečným počtom konečných množín).

b') $\forall S''$, $S''' \in H(G(S'') \cap G(S''') \neq \emptyset)$, pretože $S'' \cup S''' \in H$ a $G(S'' \cup S''') \subset G(S'') \cap G(S''')$.

Ak teraz $g' \in \cap\{G(S''); S'' \in H\}$, potom $(S', g') \in M$. Tým je dôkaz tvrdenia **A** skončený.

Usporiadajme teraz množinu M takýmto spôsobom:

Ak (S', g'), $(S'', g'') \in M$, potom $(S', g') < (S'', g'')$ vtedy, ak:

a'') $S' \subset S''$,

b'') $g''/S' = g'$ (kde g''/S' je zúženie g'' na S').

Ďalej platí:

B. Množina M pri tomto usporiadaní má maximálne prvky.

Dôkaz. Nech M_1 je nejaký neprázdny reťazec z M. Nech $S^\circ = \cup\{S'; (S', g') \in M_1\}$. Nech g° je funkcia definovaná na S° rovnosťou $\forall (S', g') \in M_1$ $(g^\circ/S' = g')$. Potom $(S^\circ, g^\circ) \in M$, pretože

a) $S^\circ \subset S$,

b) g° je p. v. f. na S° (čo vyplýva z definície g°),

c) g° sa dá rozšíriť na každý S'', pre ktorý platí $S^0 \subset S'' \subset S$, $\overline{\overline{S'' - S^0}} < \aleph_0$, pretože ak pre každé $(S', g') \in M_1$ bude $G(S')$ označovať množinu všetkých tých p. v. f. na $S'' - S^0$, ktoré vzniknú rozšírením g' na $S' \cup (S'' - S^0)$, potom $\cap\{G(S'); (S', g') \in M_1\} \neq \emptyset$, pretože:

a') $\forall (S', g') \in M_1$ $(0 < \overline{\overline{G(S')}} < \aleph_0)$ (kde ľavá nerovnosť vyplýva z definície M (bod c) a z toho, že $\overline{\overline{S'' - S^0}} < \aleph_0$).

b') $\forall (S', g')$, $(S''', g''') \in M_1((S', g') < (S''', g''') \to G(S''') \subset G(S'))$,

t. j. $\{G(S');\ (S', g') \in M_1\}$ tvorí vzhľadom k inklúzii reťazec.

Ak teraz $g \in \cap\{G(S');\ (S', g') \in M_1\}$, potom g^0 rozšírená na S'' tak, že $\forall X \in S'' - S^0(g^0(X) = g(X))$, bude p. v. f. na S''. T. j. $(S^0, g^0) \in M$. Pritom pre každé $(S', g') \in M_1$ bude $(S', g') < (S^0, g^0)$, t. j. (S^0, g^0) je horným ohraničením reťazca M_1. Teda každý reťazec z M je zhora ohraničený a preto (veta Kuratowského—Zornova) v M existujú maximálne prvky. Tým je dôkaz tvrdenia **B** ukončený.

Nech teraz (S^0, g^0) je maximálnym prvkom v M. Potom platí:

C. $S^0 = S$.

Dôkaz. Ak by $S - S^0 \neq \emptyset$, potom by existoval S' tak, že $S' \subset S - S^0$, $0 < \bar{\bar{S}}' < \aleph_0$. Ak označíme $H = \{S'';\ S' \subset S'' \subset S - S^0, \bar{\bar{S}}'' < \aleph_0\}$ a ak $G(S'')$ je množina všetkých tých funkcií g', ktoré sú p. v. f. na S' a ktoré sa dajú rozšíriť na $S^0 \cup S''$ tak, že na S^0 sú rovné g^0, potom $\cap\{G(S'');\ S'' \in H\} \neq \emptyset$ a ak $g \in \cap\{G(S'');\ S'' \in H\}$, potom g^0 môžeme rozšíriť na $S^0 \cup S'$ tak, že položíme $\forall X \in S'(g^0(X) = g(X))$ a pritom bude $(S^0 \cup S', g^0) \in M$ práve vďaka tomu, že g^0 sme rozšírili tak, aby $g^0/S' \in \cap\{G(S'');\ S'' \in H\}$. Keďže $S' \neq \emptyset$, bude $(S^0, g^0) \subsetneqq (S^0 \cup S', g^0)$, čo je spor s maximálnosťou (S^0, g^0). Tým je dôkaz tvrdenia **C** ukončený.

Z rovnosti $S^0 = S$ pre maximálny prvok (S^0, g^0) vyplýva, že na S existuje p. v. f., napr. g^0.

Tým je veta 2 dokázaná.

Poznámka 1. V tejto práci budeme pokladať $\emptyset$ za p. v. f. na $S = \emptyset$ (aby sme vo vetách nemuseli robiť predpoklad, že $S \neq \emptyset$). Zrejme potom $(\emptyset, \emptyset) \in M$, t. j. dôkaz tvrdenia **A** v dôkaze vety 2 by sme mohli vynechať.

Poznámka 2. Pre systémy S zložené iba z konečných množín je podmienka uvedená na začiatku ekvivalentná s touto podmienkou

$$\forall S' \subset S,\ \bar{\bar{S}}' < \aleph_0\ (\overline{\overline{\cup\{X;\ X \in S'\}}} \geqq \bar{\bar{S}}')$$

Odtiaľ vyplýva, že v systéme, zloženom iba z konečných množín, existujú maximálne podsystémy, na ktorých existujú p. v. f.

Poznámka 3. Predchádzajúce vety 1, 2 môžeme takto zovšeobecniť:

Nech pre systém množín $\{X_t;\ t \in T\}$ platí, že

$$\forall T' \subset T\ (\overline{\overline{\cup\{X_t;\ t \in T'\}}} \geqq \bar{\bar{T}}')$$

Potom ak $\bar{\bar{T}} < \aleph_0$, alebo ak pre každé $t \in T$ je $\bar{\bar{X}}_t < \aleph_0$, existuje funkcia g: $\{X_t;\ t \in T\} \to \cup\{X_t;\ t \in T\}$ tak, že platí:

$$\forall t \in T(g(X_t) \in X_t)$$

$$\forall t_1, t_2 \in T(t_1 \neq t_2 \to g(X_{t_1}) \neq g(X_{t_2}))$$

Túto funkciu g opäť nazveme p. v. f. na $\{X_t;\ t \in T\}$.

Podobne všetky ďalšie vety, uvedené v tejto práci, platia tiež pre systémy s indexami. Aby sme sa vyhli zložitosti pri označovaní, budeme používať iba systémy bez indexov.

Poznámka 4. Z predchádzajúceho vidieť, že ak sme od systémov požadovali veľmi špeciálne vlastnosti, potom sa nám podarilo (veta 1, 2) nájsť nutnú a postačujúcu podmienku k existencii p. v. f. Ak však prejdeme len k trošku všeobecnejším systémom, potom o existencii p. v. f. nevieme nič povedať (pozri príklad 1). (Okrem vety 3, ktorá je takmer triviálna.) Jediná veta, ktorá hovorí o vzťahu medzi všeobecnými systémami a p. v. f. (ktorú sa nám podarilo dokázať) je veta 4.

Podobne, ak sme na druhej strane od p. v. f. požadovali veľmi špeciálne vlastnosti, potom sa nám podarilo (veta 5) úplne opísať štruktúru systémov, na ktorých takéto funkcie existujú.

Príklad 1. V tomto príklade je opísaný systém množín s konštantnou mohutnosťou, ktorý splňuje podmienku uvedenú na začiatku a na ktorom neexistuje p. v. f.

Nech ω_0 je prvé nekonečné a ω_1 prvé transfinitné ordinálne číslo a $\aleph_1$ prvé transfinitné kardinálne číslo. Nech $\{x_\alpha\}_{1\leqq\alpha<\omega_1}$ je prostá transfinitná postupnosť. Definujme:

$$X_\alpha = \{x_{\alpha'}\,;\, 1\leqq\alpha'\leqq\alpha\}\,,\qquad X'_\alpha = \{x_{\alpha'}\,;\, 2\leqq\alpha'\leqq\alpha\}$$

$$S_1 = \{X_\alpha\,;\,\omega_0\leqq\alpha<\omega_1\}\,,\qquad S_2 = \{X'_\alpha\,;\,\omega_0\leqq\alpha<\omega_1\}$$

$$S = S_1\cup S_2$$

Potom platí

a) Na systéme S_1 môžeme definovať p. v. f. rovnosťou $g(X_\alpha)=x_\alpha$. Pritom pre každú p. v. f. g na S_1 platí, že

$$\overline{\overline{\cup\{X_\alpha\,;\,X_\alpha\in S_1\}-g(S_1)}}\leqq\aleph_0$$

Dôkaz. Nech $S_1(g)=\{X_\alpha\,;\,g(X_\alpha)\neq x_\alpha,\ X_\alpha\in S_1\}$. Ku každému $X_\alpha\in S_1(g)$ môžeme zostrojiť postupnosť x_{α_1}, x_{α_2}, x_{α_3}, ... tak, že $g(X_\alpha)=x_{\alpha_1}$, $g(X_{\alpha_1})=x_{\alpha_2}$, $g(X_{\alpha_2})=x_{\alpha_3}$, ...

Pritom $\alpha_1>\alpha_2>\alpha_3>\ldots$, pretože z vlastnosti g vyplýva, že $x_{\alpha_1}\neq x_{\alpha_2}$, $x_{\alpha_2}\in X_{\alpha_1}$ a z definície $X_{\alpha_1} = \{x_{\alpha'}\,;\,1\leqq\alpha'\leqq\alpha_1\}$ vyplýva, že $\alpha_2<\alpha_1$, atď.

Z nerovnosti $\alpha_1>\alpha_2>\alpha_3>\ldots$ vyplýva, že postupnosť x_{α_1}, x_{α_2}, x_{α_3}, ... musí byť konečná, t. j. musí existovať také x_{α_n}, že $g(X_{\alpha_{n-1}})=x_{\alpha_n}$ a ku x_{α_n} nie je definovaná množina X_{α_n}, t. j. $\alpha_n<\omega_0$, t. j. $x_{\alpha_n}\in\{x_\alpha\,;\,1\leqq\alpha<\omega_0\}$.

Ďalej z predchádzajúceho vyplýva, že ak ku každému $x_\alpha\in\{x_\alpha\,;\,1\leqq\alpha<\omega_0\}$ zostrojíme postupnosť množín X_{α_1}, X_{α_2}, X_{α_3}, ... (ktorá môže byť prázdna, konečná, nekonečne spočítateľná tak, že $X_{\alpha_1} = g^{-1}(x_\alpha)$, $X_{\alpha_2} = g^{-1}(x_{\alpha_1})$, $X_{\alpha_3} = g^{-1}(x_{\alpha_2})$, ..., potom tieto postupnosti vyčerpajú všetky množiny z $S_1(g)$ a teda $\overline{\overline{S_1(g)}} \leqq \aleph_0$, t. j. skoro pre všetky, až na spočítateľný počet, $X_\alpha \in S_1$ bude $g(X_\alpha) = x_\alpha$ a teda $\overline{\overline{\cup\{X_\alpha ; X_\alpha \in S_1\} - g(S_1)}} \leqq \aleph_0$. Tým je dôkaz tvrdenia a) ukončený.

b) Zrejme $\forall X \in (\bar{\bar{X}} = \aleph_0)$. Z tvrdenia a) vyplýva, že na systéme S neexistuje p. v. f., pretože $\bar{\bar{S}}_2 = \aleph_1$ a každá p. v. f. na S_1 nechá iba spočítateľne veľa prvkov z $\cup\{X ; X \in S\}$ neobsadených. Pritom systém S splňuje podmienku $\forall S' \subset$ $\subset S(\overline{\overline{\cup\{X ; X \in S'\}}} \geqq \bar{\bar{S}}')$, pretože:

Buď $\overline{\overline{\cup\{X ; X \in S'\}}} = \aleph_0$ a potom existuje α_0 tak, že $\cup\{X ; X \in S\} \subset X_{\alpha_0}$. Odtiaľ vyplýva, že $S' \subset \{X_\alpha ; \omega_0 \leqq \alpha \leqq \alpha_0\} \cup \{X'_\alpha ; \omega_0 \leqq \alpha \leqq \alpha_0\}$, t. j. $\bar{\bar{S}}' \leqq \aleph_0$.

Alebo $\cup\{X ; X \in S'\} = \aleph_1$, potom keďže $\bar{\bar{S}} = \aleph_1$, opäť bude $\bar{\bar{S}}' \leqq \aleph_1$ (presnejšie $\bar{\bar{S}}' = \aleph_1$). Teda podmienka platí.

T. j. systém S sa skladá z množín s konštantnou mohutnosťou, spĺňa na začiatku uvedenú podmienku a pritom na ňom neexistuje p. v. f.

Veta 3. Nech pre S platí, že $\forall X \in S$ $(\bar{\bar{X}} = \text{const} = \bar{\bar{S}})$. Potom na S existuje p. v. f.

Dôkaz. Nech S splňuje predpoklady vety 3. Nech ω je prvé také ordinálne číslo, ku ktorému existuje množina A tak, že $\bar{\bar{A}} = \omega$ a $\bar{\bar{A}} = \bar{\bar{S}}$. Potom množiny z S môžeme usporiadať do prostej zovšeobecnenej postupnosti $\{X_\alpha\}_{1 \leqq \alpha < \omega}$. (Zovšeobecnená postupnosť je taká postupnosť, ktorej indexy sú ordinálne čísla.) Nech $\bar{g}$ je v. f. na systéme všetkých neprázdnych podmnožín množiny $\cup\{X ; X \in S\}$. Definujme funkciu g na $\{X_\alpha\}_{1 \leqq \alpha < \omega}$ indukciou takto:

1. $g(X_1) = \bar{g}(X_1)$,
2. Ak je g definovaná na $\{X_{\alpha'}\}_{1 \leqq \alpha' < \alpha}$, potom položme $g(X_\alpha) = \bar{g}(X_\alpha - \{g(X_{\alpha'}) ; 1 \leqq \alpha' < \alpha\})$.

Keďže $\alpha < \omega$, bude $\overline{\overline{\{g(X_{\alpha'}) ; 1 \leqq \alpha' < \alpha\}}} < \bar{\bar{S}}$ a keďže $\bar{\bar{X}}_\alpha = \bar{\bar{S}}$, bude $X_\alpha - \{g(X_{\alpha'}) ; 1 \leqq \alpha' < \alpha\} \neq \emptyset$, teda takáto definícia g je možná.

Pritom zrejme g je p. v. f. na S. Tým je dôkaz vety 3 skončený.

Príklad 2. Z predchádzajúcej vety vyplýva, že napr. na systéme všetkých neprázdnych otvorených intervalov na priamke existuje p. v. f., t. j. že existuje prostá reálna funkcia $f(x, y)$ tak, že pre každé reálne x, y z nerovnosti $x < y$ vyplýva nerovnosť $x < f(x, y) < y$. Nájsť príklad takejto funkcie nebude asi ľahké.*

Veta 4. Nech pre systém S platí, že

1. $\forall S' \subset S$ $(\overline{\overline{\cup\{X ; X \in S'\}}} \geqq \bar{\bar{S}}')$.

Potom v S existuje podsystém S^0 tak, že:

2. $\bar{\bar{S}}^0 = \bar{\bar{S}}$,

2'. na S^0 existuje p.v.f.

* Takáto funkcia je zostrojená v: Problem 6004 [1974, 1121], Amer. Math. Monthly 83 (1976), 575.

Dôkaz. Nech S splňuje podmienku 1. Nech M je množina všetkých takých dvojíc (S', g'), že:

a) $S' \subset S$,

b) g' je p. v. f. na S'.

potom $M \neq \emptyset$.

Definujme na M usporiadanie $<$ tak, že ak (S', g'), $(S'', g'') \in M$, potom $(S', g') < (S'', g'')$ vtedy, ak:

a') $S' \subset S''$,

b') $g' = g''/S'$.

Nech M_1 je nejaký neprázdny reťazec z M. $S^0 = \cup\{S'; (S', g') \in M_1\}$. Definujme g^0 na S^0 rovnosťou $\forall (S', g') \in M_1\ (g^0/S' = g')$. Potom

$$(S^0, g^0) \in M$$

Teda M má maximálne prvky. Nech (S^0, g^0) je nejaký maximálny prvok v M. Potom $g^0(S^0) \supset \cup\{X; X \in S - S^0\}$, pretože ak by $X \in S - S^0$ a $X \not\subset g^0(S^0)$, potom by sme mohli g^0 predĺžiť na $S^0 \cup \{X\}$ a potom by $(S^0, g^0) \leqq (S^0 \cup \{X\}, g^0)$, čo je spor s maximálnosťou (S^0, g^0).

Odtiaľ a z predpokladu vety a to aj z toho, že g^0 je prostá, dostaneme, že $\bar{\bar{S}}^0 = \overline{\overline{g^0(S^0)}} \geqq \overline{\overline{\cup\{X; X \in S - S^0\}}} \geqq \overline{\overline{S - S^0}}$. Ďalej, buď $\bar{\bar{S}} < \aleph_0$ a potom veta 4 platí, alebo $\bar{\bar{S}} \geqq \aleph_0$ a vtedy $\bar{\bar{S}} = \max\{\bar{\bar{S}}^0, \overline{\overline{S - S^0}}\}$ a z predchádzajúceho dostaneme, že $\bar{\bar{S}} = \bar{\bar{S}}^0$. Keďže g^0 je p.v.f. na S^0, veta 4 opäť platí. Tým je dôkaz vety 4 skončený.

Veta 5. Na systéme S existuje práve jedna p. v. f. vtedy a len vtedy, ak množiny z S sa dajú usporiadať do zovšeobecnenej postupnosti $\{X_\alpha\}_\alpha$ tak, aby

$$\forall \alpha (\overline{\overline{X_\alpha - \cup\{X_{\alpha'}; \alpha' < \alpha\}}} = 1)$$

Dôkaz. Nech množiny z S sú usporiadané do postupnosti $\{X_\alpha\}_\alpha$ tak, že platí podmienka vety. Zrejme potom je to prostá postupnosť. Definujme na S funkciu g^0 takto

$$\forall \alpha (\{g^0(X_\alpha)\} = X_\alpha - \cup\{X_{\alpha'}; \alpha' < \alpha\})$$

Je zrejmé, že g^0 je p. v. f. na S. Pritom platí, že

$$\forall \alpha (\{g^0(X_{\alpha'}); \alpha' < \alpha\} = \cup\{X_{\alpha'}; \alpha' < \alpha\})$$

Nech teraz g je nejaká p. v. f. na S. Potom

1. $g(X_1) = g^0(X_1)$ (keďže $\bar{\bar{X}}_1 = 1$),

2. ak $\forall \alpha' < \alpha (g(X_{\alpha'}) = g^0(X_{\alpha'}))$, potom tiež $g(X_\alpha) = g^0(X_\alpha)$, pretože $\{g(X_{\alpha'}); \alpha' < \alpha\} = \{g^0(X_{\alpha'}); \alpha' < \alpha\} = \cup\{X_{\alpha'}; \alpha' < \alpha\}$ (a teda keďže g je p. v. f.) musí byť $g(X_\alpha) \in X_\alpha - \cup\{X_{\alpha'}; \alpha' < \alpha\}$. Podľa predpokladu vety taký prvok je len jeden a ten sa rovná $g^0(X_\alpha)$.

Z 1. a 2. predpokladu a z indukcie vyplýva, že $g = g^0$, t. j. na S existuje práve jedna p. v. f.

Nech na druhej strane na S existuje práve jedna p. v. f. Označme ju g_0. Zrejme $g_0(S) = \cup\{X; X \in S\}$. Ďalej platí:

A. Systém S (ak $S \neq \emptyset$) obsahuje aspoň jednu jednoprvkovú množinu.

Dôkaz. Ak by S neobsahoval jednoprvkové množiny, potom by sme mohli z S vybrať nekonečnú postupnosť množín $X_1, X_2, \dots, X_n, \dots$ takýmto spôsobom:

Vyberme $X_1 \in S$	a označme $g_0(X_1) = x_1$.
Vyberme $x_2 \in X_1 - \{x_1\}$	a označme $X_2 = g_0^{-1}(x_2)$.
Vyberme $x_3 \in X_2 - \{x_2\}$	a označme $X_3 = g_0^{-1}(x_3)$, atď.
Vyberme $x_{n+1} \in X_n - \{x_n\}$	a označme $X_{n+1} = g_0^{-1}(x_{n+1})$, atď.

(Keďže g_0 je zobrazenie na, existuje pre každé x_i, $g_0^{-1}(x_i)$.)

Pritom musí platiť, že $x_i \neq x_j$ $(i \neq j)$ pretože ak by $x_i = x_j$ a $x_{i+1} \neq x_{i+2} \neq \dots \neq x_{j-1}$, potom na množinách $X_i, X_{i+1}, \dots, X_{j-1}$ by sme mohli zmeniť definíciu g_0 tak, že by sme priradili $X_i \to x_{i+1}$, $X_{i+1} \to x_{i+2}, \dots, X_{j-1} \to x_i$. Takto vzniknutá funkcia by bola opäť p. v. f. na S, čo je spor s predpokladom o existencii jedinej takejto funkcie.

Keďže $x_i \neq x_j$ $(i \neq j)$, môžeme zmeniť definíciu g_0 na množinách $X_1, X_2, \dots, X_n, \dots$ tak, že priradíme $X_1 \to x_2$, $X_2 \to x_3, \dots, X_n \to x_{n+1}, \dots$ Takto vzniknutá funkcia je opäť p. v. f. na S, rôzna od g_0, čo je spor s predpokladom.

Podobne, ak na systéme označenom indexami $\{X_t; t \in T\}$ existuje práve jedna p. v. f., potom (ak $T \neq \emptyset$) $\{X_t; t \in T\}$ obsahuje jednoprvkové množiny.

B. Nech w je v. f. na množine všetkých neprázdnych podsystémov z S. Definujme postupnosť $\{X_\alpha\}_\alpha$ indukciou takto

1. $X_1 = w(\{X; \overline{\overline{X}} = 1, X \in S\})$.

2. Ak máme definované $\{X_{\alpha'}\}_{\alpha' < \alpha}$, potom položme $X_\alpha = w(\{X; \overline{\overline{X - \cup\{X_{\alpha'}; \alpha' < \alpha\}}} = 1, X \in S\})$, ak $\{X; \overline{\overline{X - \cup\{X_{\alpha'}; \alpha' < \alpha\}}} = 1, X \in S\} \neq \emptyset$. Ak je predchádzajúca množina prázdna, položme $\{X_\alpha\}_\alpha \equiv \{X_{\alpha'}\}_{\alpha' < \alpha}$.

Z definície $\{X_\alpha\}_\alpha$ vyplýva, že systém $\{X - \cup\{X_\alpha;_\alpha X \in S\}$ neobsahuje jednoprvkové množiny.

Ďalej definovaná postupnosť $\{X_\alpha\}_\alpha$ má vlastnosť z vety 5. Teda podľa prvej časti dôkazu na $\{X_\alpha\}_\alpha$ existuje práve jedna p. v. f. Označme ju g'. Potom $g'(\{X_\alpha\}_\alpha) = \cup\{X_\alpha; \alpha\}$ a keďže na S (podľa predpokladu) existuje práve jedna p. v. f., musí tiež na systéme označenom indexami $\{(X - \cup\{X_\alpha; \alpha\})_x; X \in S - \{X_\alpha\}_\alpha\}$ existovať práve jedna p. v. f.

Podľa A ak $S - \{X_\alpha\}_\alpha \neq \emptyset$, potom systém $\{(X - \cup\{X_\alpha; \alpha\})_x; X \in S - \{X_\alpha\}_\alpha\}$ musí obsahovať jednoprvkové množiny. Z predchádzajúceho teda vyplýva, že $S = \{X_\alpha\}_\alpha$. Tým je dôkaz vety 5 skončený.

Poznámka 5. Nech na S existuje p. v. f. Potom existuje podsystém $S_1^0 \subset S$, ktorý je najväčší spomedzi všetkých tých podsystémov, na ktorých existuje práve jedna p. v. f., a existuje podsystém $S_2^0 \subset S$, ktorý je najväčší spomedzi všetkých tých podsystémov $S' \subset S$, na ktorých existuje p. v. f. a pre každú p. v. f. g' na S' platí $g'(S') = \cup\{X; X \in S'\}$. Zrejme $S_1^0 \subset S_2^0$ a pre každú p. v. f. g na S platí, že $g/S_1^0 = \text{const}$ a $g(S_2^0) = \text{const}$.

Záver

K ľubovoľnému systému S ($\emptyset \notin S$) priraďme systém $m(S)$ takýmto spôsobom

$$S(x) = \{X; X \in S, x \in X\}$$

$$m(S) = \{S(x); x \in \cup\{X; X \in S\}\}$$

Ďalej nech $P(S)$, $Q(S)$, $R(S)$ označujú tieto výroky

$P(S)$ ... na S existuje p. v. f.
$Q(S)$ ... v S existuje také $S' \subset S$, že na S' existuje p. v. f. g' tak, že $g'(S') = \cup\{X; X \in S\}$.
$R(S)$ ... na S existuje p. v. f. g tak, že $g(S) = \cup\{X; X \in S\}$.

Potom platí

a) $P(S) \equiv Q(m(S))$,
b) $Q(S) = P(m(S))$,
c) $R(S) = R(m(S))$.

Z a) vyplýva, že namiesto hľadania podmienok potrebných k existencii p. v. f., stačí nám hľadať podmienky potrebné k platnosti $Q(S)$, čo však nie je o nič ľahšie.

Z b) a z predtým odvodených podmienok pre existenciu p. v. f. môžeme pre platnosť $Q(S)$ odvodiť tieto podmienky:

A. Nevyhnutnou podmienkou k platnosti $Q(S)$ je, aby

1'. $\forall Y \subset \cup\{X; X \in S\}$ $(\overline{\overline{\{X; X \in S, X \cap Y \neq \emptyset\}}} \geqq \bar{\bar{Y}})$. (Pretože $m(S)$ musí splňovať podmienku uvedenú na začiatku.)

B. Ak pre S platí:

1. $\overline{\overline{\cup\{X; X \in S\}}} < \aleph_0$,

1'. $\forall Y \subset \cup\{X; X \in S\}$ $(\overline{\overline{\{X; X \in S, X \cap Y \neq \emptyset\}}} \geqq \bar{\bar{Y}})$, potom pre S platí $Q(S)$. (Keďže potom $m(S)$ splňuje podmienky vety 1.)

C. Ak pre S platí:

1. $\forall S' \subset S, \bar{\bar{S'}} \geqq \aleph_0$ $(\cap\{X; X \in S'\} = \emptyset)$,

1'. $\forall Y \subset \cup\{X; X \in S\}$ $(\overline{\overline{\{X; X \in S, X \cap Y \neq \emptyset\}}} \geqq \overline{\overline{Y}})$, potom pre S platí $Q(S)$. (Keďže potom $m(S)$ splňuje podmienky vety 2.)

Z a), b) vyplýva ekvivalencia

$$P(S) \equiv P(m(m(S)))$$

Zdalo by sa, že pomocou tejto ekvivalencie by sme mohli nájsť nové podmienky k existencii p. v. f. a to takýmto spôsobom: Nech $L(S) \rightarrow P(S)$. Potom $L(m(m(S))) \rightarrow P(m(m(S)))$. Z predchádzajúcej ekvivalencie dostaneme $L(m(m(S))) \rightarrow P(S)$. Keďže však systémy S a $m(m(S))$ (ak S rozlišuje body) sú homeomorfné a všetky (tu ukázané) podmienky $L(S)$ sa homeomorfizmom zachovávajú, bude $L(S) = L(m(m(S)))$, t. j. nedostaneme nič nové.

Pritom dva systémy S, V nazývame homeomorfnými, ak existuje prostá funkcia $f: \cup\{X; X \in S\} \xrightarrow{\text{na}} \cup\{Y; Y \in V\}$ tak, že $F(S) = V$. Podobne sa definuje homeomorfizmus pre systémy označené indexami. Zrejme že ak S a V sú homeomorfné, potom $P(S) \equiv P(V)$.

Uvedený spôsob odvodzovania nových podmienok sa dá úspešne použiť v prípade c), pretože ak $L(S) \rightarrow R(S)$, potom $L(m(S)) \rightarrow R(m(S))$ a vzhľadom k c) $L(m(S)) \rightarrow R(S)$. T. j. každá postačujúca podmienka $L(S)$ k platnosti $R(S)$ sa dá zdvojiť (avšak za predpokladu, že $L(S) \not\equiv L(m(S))$, čo pre všeobecné $L(S)$ platí). To isté platí i pre nutnú podmienku, t. j. ak $R(S) \rightarrow L(S)$, potom $R(S) \rightarrow$ $\rightarrow L(m(S))$.

Nech napr. $L(S)$ je výrok utvorený predpokladmi vety 5 (t. j. výrok, ktorý tvrdí, že množiny z S môžeme usporiadať do postupnosti $\{X_\alpha\}_\alpha$ tak, že $\forall_\alpha(\overline{\overline{X_\alpha - \cup\{X_{\alpha'}; \alpha'<\alpha\}}} = 1)$). Podľa vety 5 bude $L(S) \rightarrow R(S)$. Teda platí tiež $L(m(S)) \rightarrow R(S)$. Pritom $L(m(S))$ predstavuje tento výrok: Množiny z S sa dajú usporiadať do postupnosti $\{X_\alpha\}_\alpha$ tak, že $\forall_\alpha(\overline{\overline{X_\alpha - \cup\{X_{\alpha'}; \alpha'>\alpha\}}} = 1)$.

Teda ak S má predchádzajúcu vlastnosť, potom na S existuje p. v. f., ktorá zobrazuje S na $\cup\{X; X \in S\}$ (čo však ihneď vidieť i bez predchádzajúceho).

Podobne, nech napr. $L(S)$ je výrok ekvivalentný s podmienkou v úvode. Potom $R(S) \rightarrow L(S)$ a teda tiež $R(S) \rightarrow L(m(S))$. T. j. k platnosti $R(S)$ sú nevyhnutné obidve podmienky.

Tento postup vytvárania nových podmienok nemôžeme ďalej predĺžiť, pretože z homeomorfnosti S a $m(m(S))$ vyplýva ekvivalencia $L(S) \equiv L(m(m(S)))$ (pre všetky dostatočne všeobecné $L(S)$).

Nakoniec treba podotknúť, vzhľadom k tomu, že dôkazy i tak jednoduchých viet ako sú vety 1, 2, sú značne dlhé (a ktoré sa nám nepodarilo skrátiť), že nájsť ďalšie systémy na ktorých existujú p. v. f. nebude asi ľahké.

Veta 1 je tzv. P. Hallova „manželská" veta. Iné dôkazy viet 1, 2 možno nájsť v učebnici M. Hall: Combinatorial theory, ktorá vyšla v roku 1967 vo Walthame. Kniha bola preložená do ruštiny.

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SUMMARY

INJECTIVE SELECTION — FUNCTION

O. Strauch, Bratislava

In this paper we deal with functions about which the axiom of the choice talks.

РЕЗЮМЕ

ВЗАИМНО ОДНОЗНАЧНЫЕ ВИБОРОЧНЫЕ ФУНКЦИИ

О. Штраух, Братислава

В этой статье мы занимаемся функциами о которых говорит аксиома выбора.

UNIVERSITAS COMENIANA
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
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ÜBER EINE RANDWERTAUFGABE VIERTER ORDNUNG IN VIER PUNKTEN

MICHAL ČVERČKO, Košice

1. Es seien $a_1<a_2<a_3<a_4$ reelle Zahlen. Da die homogene lineare Randaufgabe

$$\left.\begin{array}{ll} y^{(4)}=0 & \text{auf } \langle a_1, a_4\rangle \\ y(a_i)=0, & i=1,2,3,4 \end{array}\right\} \qquad \text{(UH)}$$

keine nicht triviale Lösung hat, existieren laut [1] die auf den Intervallen $\langle a_1, a_4\rangle \times (a_k, a_{k+1})$ definierten Greenschen Funktionen $G_k(x, t)$, $k=1, 2, 3$ derart, dass die einzige Lösung $y(x)$ der nichthomogenen linearen Randwertaufgabe

$$\left.\begin{array}{ll} y^{(4)}=h(x), & x\in\langle a_1, a_4\rangle \\ y(a_i)=0, & i=1,2,3,4 \end{array}\right\} \qquad \text{(UN)}$$

wo $h(x)\in C_0\,(\langle a_1, a_4\rangle)$ ist, in der Form

$$y(x)=\sum_{k=1}^{3}\int_{a_k}^{a_{k+1}} G_k(x, t)h(t)\,dt$$

ausgedrückt werden kann.

Mit Hilfe dieser Beziehung und in Anlehnung an die Fixpunktsätze, finden wir hinreichende Bedingungen für die Existenz und Unizität der Lösungen der nichtlinearen Randwertaufgabe

$$\left.\begin{array}{ll} y^{(4)}=f(x, y, y', y'', y'''), & x\in\langle a_1\, a_4\rangle \\ y(a_i)=0, & i=1,2,3,4 \end{array}\right\}$$

wo die Funktion f später bestimmt wird.

Die Funktionen G_k erweitern wir auf die einzige Greensche Funktion G dieser Aufgabe, welche auf dem Viereck $\langle a_1, a_4\rangle \times \langle a_1, a_4\rangle$ definiert sein wird und mit deren Hilfe man die Lösung $y(x)$ der Aufgabe (UN) folgend ausdrückt:

$$y(x)=\int_{a_1}^{a_4} G(x, t)h(t)\,dt$$

Weiter werden wir folgende Bezeichnungen benutzen. R wir die Menge aller reellen Zahlen bedeuten, $J_k=\langle a_k, a_{k+1}\rangle$, $k=1, 2, 3$, weiter $J=\langle a_1, a_4\rangle$, $J_t^*=\langle a_1, t)$, $J_t^{**}=(t, a_4\rangle$. Das Innere der Menge S bezeichnen wir mit S^0 und die abgeschlossene Hülle der Menge S bezeichnen wir mit $\bar{S}$. Die Intervalle $J\times J_k^0$ bezeichnen wir P_k, $k=1, 2, 3$ und $Q=J\times J$. Weiter

$$P_{k1}=\{(x, y)\in P_k \mid x\leqq y\}, \qquad k=1, 2, 3$$

$$P_{k2}=\{(x, y)\in P_k \mid x>y\}, \qquad k=1, 2, 3$$

G_x, eventuell $G_x(., t)$ wird die Ableitung der Funktion $G(., t)$ bedeuten, d. h. die partielle Ableitung nach dem ersten Argument der Funktion $G(., .)$.

1.1. Die Funktionen G_k, $k=1, 2, 3$ sind mit folgenden Anforderungen ([1], S. 50) eindeutig bestimmt: Für $t\in J_k^0$

1. $G_k(., t)$ ist im Intervall J definiert, also $G_k: P_k\to R$,
2. $G_k(., t)$, $G_{kx}(., t)$, $G_{kxx}(., t)\in C_0(J)$,
3. $G_{kxxx}(., t)$ ist stetig auf den Intervallen J_t^*, J_t^{**} und im Punkt t ist ein Sprung gleich eins:

$$G_{kxxx}(t+0, t)-G_{kxxx}(t-0, t)=1$$

4. $G_k(., t)$ entspricht der Gleichung $y^{(4)}=0$ auf den Intervallen J_t^* und J_t^{**},
5. $G_k(a_i, t)=0$ für $i=1, 2, 3, 4$.

1.2. Nehmen wir $t\in J_1^0$. Da die Funktionen $1, x, x^2, x^3$ ein Fundamentalsystem von Lösungen der Gleichung $y^{(4)}=0$ bilden, muss $G_1(., t)$ gemäss der Bedingung (4) auf den Intervallen J_t^* und J_t^{**} ein Polynom dritten Grades sein und muss laut (5) die Form

$$G_1(x, t)=\begin{cases}(x-a_1)\,[d_1(t)x^2+d_2(t)x+d_3(t)]\,, & \text{für } x\in J_t^*\\ d_4(t)\,(x-a_2)\,(x-a_3)\,(x-a_4)\,, & \text{für } x\in J_t^{**}\end{cases}$$

haben. Die Bedingungen (2) und (3) ermöglichen ein System von vier Gleichungen mit den Unbekannten $d_i(t)$, $i=1, 2, 3, 4$ zusammenzustellen. Durch die Lösung desselben erhalten wir

$$d_1(t)=\frac{1}{6}\left[\frac{1}{a}(t-a_1)^3-1\right]$$

$$d_2(t)=\frac{1}{6}\left[\frac{1}{a}(a_1-a_2-a_3-a_4)\,(t-a_1)^3+3t-a_1\right]$$

$$d_3(t)=\frac{1}{6}\left[\frac{1}{a}\,(a_1(a_1-a_2-a_3-a_4)+a_2a_3+a_2a_4+a_3a_4)\,.\right.$$
$$\left.\,.\,(t-a_1)^3-a_1^2+3a_1t-3t^2\right]$$

$$d_4(t)=\frac{1}{6a}\,(t-a_1)^3$$

$a \quad =(a_2-a_1)\,(a_3-a_1)\,(a_4-a_1)$ ist.

1.3. Es sei $t\in J_2^0$. Die Bedingungen (4) und (5) werden erfüllt, wenn

$$G_2(x,t)=\begin{cases}(x-a_1)\,(x-a_2)\,[h_1(t)x+h_2(t)]\,, & \text{für } x\in J_t^*\\ (x-a_3)\,(x-a_4)\,[h_3(t)x+h_4(t)]\,, & \text{für } x\in J_t^{**}\end{cases}$$

und aus (2) und (3) folgt, dass

$$h_1(t)=\frac{1}{6d}\,\{(a_1+a_2-a_3-a_4)t^3-3(a_1a_2-a_3a_4)t^2-$$
$$-[a_3a_4(a_1+a_2)-a_1a_2(a_3+a_4)]t-a_3a_4(a_3a_4-a_1a_2)+$$
$$+(a_3+a_4)\,[(a_1+a_2)a_3a_4-(a_3+a_4)a_1a_2]\}$$

$$h_2(t)=-\frac{1}{6d}\,\{[(a_3+a_4)\,(a_1+a_2-a_3-a_4)+a_3a_4-a_1a_2]t^3-$$
$$-3a_3a_4(a_1+a_2-a_3-a_4)t^2-3a_3a_4(a_3a_4-a_1a_2)t+$$
$$+a_3a_4[(a_1+a_2)a_3a_4-(a_3+a_4)a_1a_2]\}$$

$$h_3(t)=\frac{1}{6}+h_1(t)$$

$$h_4(t)=-\frac{1}{6d}\,\{[(a_1+a_2)\,(a_1+a_2-a_3-a_4)-a_1a_2+a_3a_4]t^3-$$
$$-3a_1a_2(a_1+a_2-a_3-a_4)t^2-3a_1a_2(a_3a_4-a_1a_2)t+$$
$$+a_1a_2[(a_1+a_2)a_3a_4-(a_3+a_4)a_1a_2]\}$$

wo $d=(a_4-a_1)\,(a_4-a_2)\,(a_3-a_1)\,(a_3-a_2)$.

1.4. Wenn $t\in J_3^0$, muss gemäss (4) und (5)

$$G_3(x,t)=\begin{cases}b_4(t)\,(x-a_1)\,(x-a_2)\,(x-a_3)\,, & \text{für } x\in J_t^*\\ (x-a_4)\,[b_1(t)x^2+b_2(t)x+b_3(t)]\,, & \text{für } x\in J_t^{**}\end{cases}$$

sein.

Mit Rücksicht auf (2) und (3) ist

$$b_1(t)=\frac{1}{6b}\,(t-a_4)^3+\frac{1}{6}$$

$$b_2(t)=\frac{1}{6b}(a_4-a_1-a_2-a_3)(t-a_4)^3-\frac{1}{6}(3t-a_4)$$

$$b_3(t)=\frac{1}{6b}[a_1a_2+a_1a_3+a_2a_3-a_4(a_1+a_2+a_3-a_4)\,.$$

$$.\,(t-a_4)^3]+\frac{1}{6}(3t^2-3a_4t+a_4^2)$$

$$b_4(t)=\frac{1}{6b}(t-a_4)^3$$

wo $b=(a_4-a_1)(a_4-a_2)(a_4-a_3)$.

2. Bei der Bestimmung von $G_k(x, t)$, $k=1, 2, 3$ beschränkten wir uns auf ein solches x, t, wo $x\in J_t^*$ J_t^{**} und $t\in J_k^0$ ist. Die für $G_k(x, t)$ gewonnenen Ausdrücke haben Sinn auch für $t\in\{a_1, a_2, a_3, a_4\}$ und auch für die Punkte (t, t), $t\in J$. Wir können diese auch bei Erweiterung des Definitionsgebietes der Funktionen G_k auf $\bar{P}_k$ verwenden. Die Funktionen G_k werden dann zusammen mit G_{kx}, G_{kxx} auf $\bar{P}_k$ stetig sein. Weiter können wir uns durch direkte Einsetzung davon überzeugen, dass

$$G_1(x, a_2)=G_2(x, a_2)\,, \qquad \text{für } x\in J$$

$$G_2(x, a_3)=G_3(x, a_3)\,, \qquad \text{für } x\in J$$

und deshalb ist die Funktion $G\colon Q\to R$, welche durch die Beziehung

$$G(x, t)=G_k(x, t) \quad \text{für} \quad (x, t)\in\bar{P}_k\,, \quad k=1, 2, 3$$

definiert ist, stetig auf Q. Dasselbe gilt auch für G_x, G_{xx}. Es ist ersichtlich, dass die Funktion $G_{xxx}(.\,, t)$ auf J_t^* und J_t^{**} stetig ist und im Punkte t einen Sprung gleich eins hat.

2.1. Hilfssatz 1. Für $(x, t)\in(J_1^0\cup J_3^0)\times J$ ist $G(x, t)<0$ und für $(x, t)\in$ $\in J_2^0\times J$ ist $G(x, t)>0$.

Beweis. Diese Behauptung folgt direkt aus dem Hilfssatz (4.2) und der darauffolgenden Bemerkung in [3], S. 80—81.

2.2. Hilfssatz 2. Es sei y_1 die Lösung der Randwertaufgabe

$$\left.\begin{array}{ll} y^{(4)}=1 & \text{auf } J \\ y(a_i)=0\,, & i=1, 2, 3, 4 \end{array}\right\}$$

Für $x\in J$ gilt dann

$$|y_1(x)|\leq\frac{9}{2048}(a_4-a_1)^4$$

Beweis. Es ist leicht festzustellen, dass für $x \in J$

$$y_1(x) = \frac{1}{24}(x-a_1)(x-a_2)(x-a_3)(x-a_4).$$

Bei festem x, a_1, a_4 ist dieser Ausdruck eine Funktion von (a_2, a_3). Bezeichnen wir diese als F.

Wenn $a_1 < x < a_2$, ist $F_{a_2} < 0$, $F_{a_3} < 0$ und daher

$$0 \geqq F(a_2, a_3) = \frac{1}{24}(x-a_1)(x-a_4)^3.$$

Wenn $a_2 < x < a_3$, ist $F_{a_2} < 0$, $F_{a_3} > 0$ und daher

$$0 \leqq F(a_2, a_3) \leqq \frac{1}{24}(x-a_1)^2(x-a_4)^2.$$

Wenn $a_3 < x < a_4$, ist $F_{a_2} > 0$, $F_{a_3} > 0$ und daher

$$0 \geqq F(a_2, a_3) \geqq \frac{1}{24}(x-a_1)^3(x-a_4).$$

Für $f_1(x) = \frac{1}{24}(x-a_1)(x-a_4)^3$, $x \in J_1$ gilt

$$|f_1(x)| \leqq \frac{27}{24\,.\,64\,.\,4}(a_4-a_1)^4$$

und eine gleiche Abschätzung gilt auch für $f_3(x) = \frac{1}{24}(x-a_1)^3\,.\,(x-a_4)$ und $x \in J_3$.

Für $f_2(x) = \frac{1}{24}(x-a_1)^2(x-a_4)^2$ und $x \in J_2$ gilt

$$|f_2(x)| \leqq \frac{1}{24\,.\,16}(a_4-a_1)^4$$

Für $|y_1(x)|$ gilt also die angeführte Abschätzung.

2.3. Hilfssatz 3. Für $x \in J$ ist

$$\int_J |G(x,t)|\,\mathrm{d}t \leqq \frac{9}{2048}(a_4-a_1)^4$$

Beweis. Die Funktion y_1 ist die Lösung der angeführten Randwertaufgabe mit der rechten Seite identisch gleich eins und deshalb ist für $x \in J$

$$y_1(x) = \int_J G(x,t)\,.\,1\,\mathrm{d}t$$

Auf Grund des Hilfssatzes 1 erhalten wir daraus, dass für $x \in J$

$$|y_1(x)| = \int_J |G(x, t)| \, dt$$

ist und deshalb können wir bei Abschätzung dieses Integrals den Hilfssatz 2 anwenden.

3. Aus dem Ausdruck $G_k(x, t)$ ist ersichtlich, dass die Funktion G und deren Ableitungen G_x, G_{xx}, G_{xxx} auf dem Viereck $J \times J$ begrenzt sind. Bezeichnen wir

$$\begin{aligned}
K_0 &= \sup \{|G(x, t)| : (x, t) \in Q\} \\
K_1 &= \sup \{|G_x(x, t)| : (x, t) \in Q\} \\
K_2 &= \sup \{|G_{xx}(x, t)| : (x, t) \in Q\} \\
K_3 &= \sup \{|G_{xxx}(x, t)| : (x, t) \in [Q\{(t, t) : t \in J\}]\} \\
K &= \max \{K_i \mid i = 0, 1, 2, 3\} \\
\tilde{K}_i &= (a_4 - a_1)K_i, \quad i = 0, 1, 2, 3 \\
\tilde{K} &= (a_4 - a_1)K
\end{aligned}$$

3.1. Satz 1. Die Funktion $f: J \times R^4 \to R$ sei stetig und es sei

$$|f(x, y, y', y'', y''')| \leqq m \quad \text{für} \quad (x, y, y', y'', y''') \in J \times R^4$$

Dann hat die Randwertaufgabe

$$\left.\begin{aligned} y^{(4)} &= f(x, y, y', y'', y''') \quad \text{auf } J \\ y(a_i) &= 0, \quad i = 1, 2, 3, 4 \end{aligned}\right\} \tag{U}$$

eine solche Lösung y, dass $|y^{(s)}(x)| \leqq m\tilde{K}_s$, $s = 0, 1, 2, 3$.

Beweis. D sei der Banachsche Raum aller Funktionen die mit stetiger dritter Ableitung im Intervall y versehen sind. Wobei die Norm der Funktion h durch die Beziehung

$$\|h\| = \max_{s=0, 1, 2, 3} (\max_{t \in J} |h^{(s)}(t)|)$$

definiert ist. Nehmen wir eine beliebige Funktion h aus der Kugel

$$S = \{h \in D \mid \|h\| \leqq m\tilde{K}\}$$

Für diese Funktion setzen wir $Th = y$, wo y die einzige Lösung der Aufgabe

$$\left.\begin{aligned} y^{(4)} &= f(x, h(x), h'(x), h''(x), h'''(x)), \quad x \in J \\ y(a_i) &= 0, \quad i = 1, 2, 3, 4 \end{aligned}\right\}$$

ist. Damit ist der Operator T auf S definiert, welcher S auf S abbildet, da

$$y(x) = \int_J G(x, t) f(t, h(t), h'(t), h''(t), h'''(t)) \, dt$$

und also

$$\|y\| \leqq \tilde{K}m$$

Für $h_1, h_2 \in S$ sei $Th_1 = y_1$, $Th_2 = y_2$

Dann ist

$$\|y_1 - y_2\| \leqq K\int_J |f(t, h_1(t), h_1'(t), h_1''(t), h_1'''(t)) -$$

$$- f(t, h_2(t), h_2'(t), h_2''(t), h_2'''(t))| \, dt$$

und da die Funktion f stetig ist, bei $\|h_1 - h_2\| \to 0$ auch $\|y_1 - y_2\| \to 0$.

Der Operator T ist also stetig.

Für die Funktion y aus der Menge der Werte des Operators T, d. h. für $y = Th$ mit einen gewissen $h \in S$ haben wir $|y^{(4)}(x)| \leqq m$ für $x \in J$.

Für die Funktion $y \in TS$ haben wir also

$$\begin{aligned}
y(x_2) - y(x_1) &= y'(\xi_1)(x_2 - x_1) && \text{für irgendein } \xi_1 \in (x_1, x_2)\\
y'(x_2) - y'(x_1) &= y''(\xi_2)(x_2 - x_1) && \text{für irgendein } \xi_2 \in (x_1, x_2)\\
y''(x_2) - y''(x_1) &= y'''(\xi_3)(x_2 - x_1) && \text{für irgendein } \xi_3 \in (x_1, x_2)\\
y''''(x_2) - y'''(x_1) &= y^{(4)}(\xi_4)(x_2 - x_1) && \text{für irgendein } \xi_4 \in (x_1, x_2)
\end{aligned}$$

und also

$$|y^{(s)}(x_2) - y^{(2)}(x_1)| \leqq \tilde{K}_s \,.\, m \,.\, |x_2 - x_1| \,, \qquad s = 0, 1, 2$$

$$|y^{(3)}(x_2) - y^{(3)}(x_1)| \leqq m|x_2 - x_1|$$

Die Funktionen $y \in TS$ sind also derart, dass y, y', y'', y''' gleichmässig beschränkt und gleichgradig stetig sind. Nach dem Satz von Arzela hat die Menge der Werte des Operators T eine kompakte abgeschlossene Hülle. Nach dem Satz von Schauder ([2], S. 476) hat der Operator T einen Fixpunkt und unsere Aufgabe hat also eine Lösung.

3.2. Satz 1 A. Die Funktion $F: J \times \langle -r, r\rangle^4 \to R$ sei stetig und es sei

$$|f(x, y, y', y'', y''')| \leqq m$$

für $(x, y, y', y'', y''') \in J \times \langle -r, r\rangle^4$ und dabei ist $r \geqq m\tilde{K}$. Dann hat die Randwertaufgabe (U) eine solche Lösung y, dass

$$|y^{(s)}(x)| \leqq m\tilde{K}_s \,, \qquad s = 0, 1, 2, 3$$

Beweis. Beim Beweis des Satzes 1 benutzten wir nur einen Teil des Definitionsgebietes der Funktion f, wenn mit dem Operator T nur die Funktionen h mit der Norm $\|h\| \leqq m\tilde{K}$ abbildeten. Es ist also möglich aus dem vorhergehenden Satz eine Schlussfolgerung nur aus den Voraussetzungen dieses Satzes ausgehend abzuleiten.

3.3. Satz 1 B. Die Funktion $f: J \times \langle -r, r\rangle \to R$ sei stetig und es sei $|f(x, y)| \leqq m$ auf $J \times \langle -r, r\rangle$, wo $r \geqq mk$ und

$$k = \frac{9}{2048}(a_4 - a_1)^4$$

Dann hat die Randwertaufgabe

$$\left.\begin{array}{ll} y^{(4)} = f(x, y) & \text{auf } J \\ y(a_i) = 0, & i = 1, 2, 3, 4 \end{array}\right\}$$

eine solche Lösung y, dass $|y(x)| \leqq mk$ für $x \in J$ ist.

Beweis. Es sei $\tilde{D}$ der Banachsche Raum von stetigen auf J definierten Funktionen mit der Norm $\|h\| = \max \{|h(x)| : x \in J\}$ Auf der Kugel $S = \{h \in \tilde{D}: \|h\| \leqq mk\}$ definieren wir den Operator T indem wir $Th = y$ setzen, wo y die einzige Lösung der Aufgabe

$$\left.\begin{array}{ll} y^{(4)} = f(x, h(x)), & x \in J \\ y(a_i) = 0, & i = 1, 2, 3, 4 \end{array}\right\}$$

ist, und also $y(x) = \int_J G(x, t) f(t, h(t))\, dt, \ x \in J$.

Daraus erhalten wir, dass $\|y\| \leqq mk$ und also $TS \subset S$. Es sei $h_1, h_2 \in S$ und $Th_1 = y_1$, $Th_2 = y_2$. Dann ist

$$\|y_1 - y_2\| \leqq K_0 \int_J |f(t, h_1(t)) - f(t, h_2(t))|\, dt$$

Da f eine stetige Funktion ist, ist bei $\|h_1 - h_2\| \to 0$, auch $\|y_1 - y_2\| \to 0$, der Operator T ist also stetig. Es sei $y \in TS$, $y = Th$ und $x_1, x_2 \in J$. Dann

$$|y(x_1) - y(x_2)| \leqq K_1 m |x_1 - x_2|$$

Die Klasse der Funktionen TS ist also gleichgradig stetig und ausserdem gleichmässig beschränkt und hat also gemäss dem Arzela-schen Satz eine kompakte abgeschlossene Hülle. Es sind also die Voraussetzungen des Schauder-schen Satzes erfüllt und nach diesem hat T einen Fixpunkt und gerade das ist die Lösung unserer Aufgabe.

3.4. Satz 2. Die Funktion $f: J \times R^4 \to R$ sei stetig und soll die Lipschitz-sche Bedingung

$$|f(x, y_1, y_1', y_1'', y_1''') - f(x, y_2, y_2', y_2'', y_2''')| \leqq$$

$$\leqq \sum_{k=0}^{3} \vartheta_k |y_1^{(k)} - y_2^{(k)}|$$

mit den Konstanten ϑ_i, $i = 0, 1, 2, 3$ und zwar so kleinen, dass $\tilde{K} \sum_0^3 \vartheta_k < 1$ erfüllen.

Dann hat die Randwertaufgabe

$$\left.\begin{array}{l} y^{(4)}=f(x,y,y',y'',y''') \quad \text{auf} \quad J \\ y(a_i)=0, \qquad i=1,2,3,4 \end{array}\right\} \tag{U}$$

eine einzige Lösung. Diese Lösung kann als Limes der Folge $(y_n)_1^\infty$ der Lösungen der linearen Aufgaben

$$\left.\begin{array}{l} y_n^{(4)}=f(x,y_{n-1}(x),\ y'_{n-1}(x),\ y''_{n-1}(x),\ y'''_{n-1}(x)) \\ y_n(a_i)=0, \qquad i=1,2,3,4 \end{array}\right\} \tag{U$_n$}$$

gewonnen werden, wobei $y_0\equiv 0$.

Beweis. Es soll D dieselbe Bedeutung wie im Beweis des Satzes 1 haben. $r>0$ sei hinreichend gross. Den Operator T definieren wir auf der Kugel

$$S=\{h\in D|\ \|h\|\leqq r\}$$

so, wie im Beweis des Satzes 1.

Wenn $y_0=T0$ und $|f(x,0,0,0,0)|=m_0$ für $x\in J$, dann

$$y_0(x)=\int_J G(x,t)f\,(t,0,0,0,0)\,\mathrm{d}t$$

und

$$\|y_0\|\leqq \tilde{K}m_0$$

Weiter, wenn $y_1=Th_1$, $y_2=Th_2$ für h_1, $h_2\in S$, dann

$$y_1(x)-y_2(x)=\int_J G(x,t)\,[f(t,h_1(t),h_1'(t),h_1''(t),h_1'''(t))-$$
$$-f(t,h_2(t),\ldots,h_2'''(t))]\,\mathrm{d}t$$

und

$$\|y_1-y_2\|\leqq K\sum_{i=0}^{3}\vartheta_i\|h_1-h_2\|$$

Wenn $\tilde{K}m_0\leqq r\left(1-\tilde{K}\sum_0^3\vartheta_i\right)$ dann erfüllt der Operator T die Voraussetzungen des Banachschen Fixpunktsatzes ([3], S. 475) und es existiert also die Funktion y für welche

$$y(x)=(Ty)\,(x)=\int_J G(x,t)f(t,.,.,.)\,\mathrm{d}t$$
$$y(a_i)=0, \qquad i=1,2,3,4$$

gilt.

Es existiert also eine Lösung unserer Randwertaufgabe. Laut demselben Satz ist diese Lösung ein Limes der angeführten Folge von Funktionen.

3.5. Satz 2 A. Die Funktion $f: J \times \langle -r, r\rangle^4 \to R$ sei stetig und soll die folgende Lipschitz-sche Bedingung erfüllen:

$$|f(x, y_1, y_1', y_1'', y_1''') - f(x, y_2, y_2', y_2'', y_2''')| \leqq$$

$$\leqq \sum_{k=0}^{3} \vartheta_k |y_1^{(k)} - y_2^{(k)}|$$

mit derartigen Konstanten ϑ_i, $i = 0, 1, 2, 3$, dass $\tilde{K}\sum_0^3 \vartheta_k < 1$, und r eine beliebige Zahl ist, welche der Ungleichheit

$$\tilde{K} m_0 \leqq r\left(1 - \tilde{K}\sum_{i=0}^{3} \vartheta_i\right)$$

entspricht, wo $m_0 = \max\{|f(x, 0, 0, 0, 0)| : x \in J\}$. Die Randwertaufgabe (U) hat dann eine einzige Lösung, welche der Limes der Folge $(y_n)_1^\infty$ von Lösungen der linearen Aufgaben ist, wobei $y_0 \equiv 0$.

Beweis. Beim Beweis des Satzes 2 nehmen wir den Wert der Funktion F nur in den Punkten (x, y, y', y'', y''') wo $(y, y', y'', y''') \in \langle -r, r\rangle^4$ ist und stützten uns nur tatsächlich auf die Voraussetzungen dieses Satzes.

3.6. Satz 2 B. Die Funktion $f: J \times \langle -r, r\rangle^4 \to R$ sei stetig und erfülle die Lipschitz-sche Bedingung mit Rücksicht auf die zweite bis fünfte Veränderliche mit den Konstanten ϑ_i, $i = 0, 1, 2, 3$ welche die Bedingung erfüllen:

$$\tilde{K}\sum_0^3 \vartheta_k < 1$$

wobei r eine beliebige Zahl ist, die der Ungleichheit $r = \tilde{K}m$ entspricht, und

$$m = \sup\{|f(x, y, y', y'', y''')| : (x, y, y', y'', y''') \in$$

$$\in J \times \langle -r, r\rangle^4\}$$

Dann hat die Randwertaufgabe (U) eine einzige Lösung, welche der Limes der Folge $(y_n)_1^\infty$ von Lösungen der linearen Aufgaben (U_n) ist, wobei $y_0 \equiv 0$.

Beweis. Es sollen D, S, T dieselbe Bedeutung wie im Beweis des Satzes 2 haben. Wen $h \in S$ und $Th = y$, dann ist

$$\|y\| \leqq \tilde{K}m \leqq r \quad \text{und also} \quad TS \subset S$$

Weiter wenn h_1, $h_2 \in S$ und $Th_1 = y_1$, $Th_2 = y_2$, dann $\|y_1 - y_2\| \leqq \tilde{K} \cdot \sum_{i=0}^{3} \vartheta_i \cdot$ $\cdot \|h_1 - h_2\|$, wobei $\tilde{K}\sum_0^3 \vartheta_i < 1$. Der Operator T erfüllt also die Voraussetzungen des

Banach-schen Satzes ([2], S. 475) und laut dieses Satzes hat T einen einzigen Fixpunkt und dieser ist die Lösung unserer Aufgabe (U). Nach demselben Satz kann diese Lösung auf die angeführte Weise gewonnen werden.

3.7. Satz 2 C. Die Funktion $f: J \times \langle -r, r\rangle \to R$ sei stetig und erfülle die Lipschitz-sche Bedingung

$$|f(x, y_1) - f(x, y_2)| \leqq L|y_1 - y_2|$$

mit einer solchen Konstante L, dass $0 < kL < 1$, wo $k = \frac{9}{2048}(a_4 - a_1)^4$, r der Bedingung $km_0 \leqq r(1 - kL)$ entspricht, und $m_0 = \max\{|f(x, 0)| : x \in J\}$. Dann hat die Randwertaufgabe

$$\left.\begin{array}{ll} y^{(4)} = f(x, y), & x \in J \\ y(a_i) = 0, & i = 1, 2, 3, 4 \end{array}\right\} \tag{U}$$

eine einzige Lösung, welche der Limes der Folge $(y_n)_1^\infty$ von Lösungen der linearen Aufgaben

$$\left.\begin{array}{ll} y_n^{(4)} = f(x, y_{n-1}(x)), & x \in J \\ y_n(a_i) = 0, & i = 1, 2, 3, 4 \end{array}\right\} \tag{U$_n$}$$

ist, wobei $y_0 \equiv 0$.

Beweis. Es sollen $\tilde{D}$, S, T dieselbe Bedeutung wie im Beweis des Satzes 1 B haben. Wenn $y_0 = T0$, dann

$$y_0(x) = \int_J G(x, t) f(t, 0)\, dt, \qquad x \in J$$

und laut Hilfssatz 3

$$\|y_0\| \leqq km_0$$

Wenn $h_i \in S$, $Th_i = y_i$, $i = 1, 2$, dann ist

$$\|y_1 - y_2\| \leqq kL\|h_1 - h_2\|$$

wobei gemäss der Voraussetzung $0 < kL < 1$ ist. Wenn noch die Bedingung $km_0 \leqq r(1 - kL)$ erfüllt ist, dann erfüllt der Operator T alle Voraussetzungen des Banachschen Satzes und laut diesen hat T einen einzigen Fixpunkt. Dieser Punkt ist die Lösung der gegebenen Randwertaufgabe und kann nach demselben Satz auf die angeführte Weise gewonnen werden.

3.8. Satz 2 D. Die Funktion $f: J \times \langle -r, r\rangle \to R$ sei stetig und erfülle die Lipschitz-sche Bedingung mit Rücksicht auf die zweite Veränderliche mit der Konstante L, die Bedingung $0 < kL < 1$ erfüllt und wo $k = \frac{9}{2048}(a_4 - a_1)^4$. Dabei soll r der Ungleichheit $r \geqq km$ entsprechen, wo

$$m = \sup \{|f(x, y)| : (x, y) \in J \times \langle -r, r\rangle\} .$$

Dann hat die Randwertaufgabe eine einzige Lösung, welche der Limes der Folge $(y_n)_1^\infty$ von Lösungen der linearen Randwertaufgabe (U_n) ist, wobei $y_0 \equiv 0$.

Beweis. $\tilde{D}$ habe dieselbe Bedeutung wie im Beweis des Satzes 1 B. Wir definieren den Operator T auf der Kugel $S = \{h \in \tilde{D} \mid \|h\| \leqq r\}$ so wie im Beweis des Satzes 1 B. Gemäss Hilfssatz 3 erhalten wir für $y = Th \|y\| \leqq km \leqq r$, so dass $TS \subset S$. Wenn $y_i = Th_i$, $i = 1, 2$, dann $\|y_2 - y_1\| \leqq kL \|h_2 - h_1\|$, wobei laut der Voraussetzung $0 < kL < 1$ ist. Der Operator T erfüllt also die Voraussetzungen des Banach-schen Fixpunktsatzs und nach diesem Satz hat T einen einzigen Fixpunkt. Dieser Punkt ist die Lösung der Aufgabe (U) und nach demselben Satz ist dieser Punkt der Limes der angeführten Folge.

4. Die Sätze 1, 1 A, 2, 2 A, 2 B können für den Fall der Randwertaufgabe

$$\left.\begin{array}{ll} y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), & x \in \langle a_1, a_2\rangle \\ y(a_i) = 0, & i = 1, 2, \ldots, n \end{array}\right\}$$

wo n eine beliebige natürliche Zahl ist, leicht verallgemeinert werden.

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SÚHRN

O OKRAJOVEJ ÚLOHE ŠTVRTÉHO RÁDU V ŠTYROCH BODOCH

M. Čverčko, Košice

V práci sa rieši nelineárna okrajová úloha v štyroch bodoch pre nelineárnu diferenciálnu rovnicu 4. rádu. Pomocou Greenovej funkcie sa prejde k zodpovedajúcej nelineárnej integrálno-diferenciálnej rovnici a táto sa rieši použitím Banachovej a Schauderovej vety o pevnom bode.

РЕЗЮМЕ

О КРАЕВОЙ ЗАДАЧЕ ЧЕТВЕРТОГО ПОРЯДКА В ЧЕТЫРЕХ ТОЧКАХ

М. Чверчко, Кошице

В работе решается нелинейная краевая задача в четырей точках для нелинейного дифференциального уравнения 4-ого порядка. С помощью функции Грина эта проблема сводится к одному интегро-дифференциальному уравнению, существование решения которого доказывается на основании теоремы Банаха и Шаудера о неподвижной точке.

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