ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE

MATHEMATICA

XXXII

1975

SLOVENSKÉ PEDAGOGICKÉ NAKLADATEĽSTVO
BRATISLAVA

ACTA FACULTATIS NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA

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Vydáva Slovenské pedagogické nakladateľstvo v Bratislave, Sasinkova 5, č. tel.; 645-51. Povolilo Povereníctvo kultúry číslom 2265/56-IV/1.

(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXXII, 1975)

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE

MATHEMATICA XXXII – 1975

Об одной постоянной литтльвуда в теории дзета-функции римана

ЯН МОЗЕР, Братислава

Пусть $\frac{1}{2}+i\gamma'$, $\frac{1}{2}+i\gamma''$ соседние нули функции $\zeta(s)$. Литтльвуд в работе [2] нашел (в предположении справедливости гипотезы Римана) оценку

$$\gamma''-\gamma' < \frac{A}{\ln\ln\gamma'}, \quad \gamma' > e^2, \tag{1}$$

используя соответствующую оценку (см. [3]) для функции $S(t)$.

Методом А. ЗЕЛЬБЕРГА нетрудно получить (см. Добавление I.) оценку

$$|S(t)| < 5{,}6\,\frac{\ln t}{\ln\ln t}, \quad t \geqq e^{16}. \tag{2}$$

Если использовать, с одной стороны, оценку (2), и, с другой, подходящие соседние нули функции $\zeta\left(\frac{1}{2}+it\right)$ в окрестности значения $t_0 = 10\,000\,019$, (t_0 – простое число, см. [4]), то получается

Т е о р е м а. Если $\gamma > e^{16}$, то

$$2 < A < 80\,. \tag{3}$$

Прежде чем приступить к доказательству, введем некоторые обозначения.

Положим:

$$f_1(t) = \frac{1}{4}\,\mathrm{arctg}\,\frac{1}{2t} + \frac{t}{4}\,\ln\left(1 + \frac{1}{4\,t^2}\right), \quad t > 0,$$

$$\varphi(u) = \int_0^u \left([v] - v + \frac{1}{2}\right) dv,$$

$$f_2(t) = \mathfrak{Im}\left\{\int_0^\infty \frac{\varphi(u)\;du}{\left(u + \frac{1}{4} + i\,\frac{t}{2}\right)^2}\right\}, \quad t \geq 0,$$

(4) $$f(t) = f_1(t) + f_2(t),$$

(5) $$L(t) = \frac{1}{2\pi}\,t\ln t - \frac{1 + \ln 2\pi}{2\pi}\,t + \frac{7}{8}$$

Нетрудно получить оценку

(6) $$|f(t + x) - f(t)| < 2{,}75\,\frac{x}{t^2}, \quad t \geq 1, \quad x > 0.$$

Доказательство теоремы

(a) Пусть $\frac{1}{2} + i\gamma'$, $\frac{1}{2} + i\gamma''$ соседние нули функции $\zeta(s)$.

Пусть дальше

$$t' = \gamma' + \frac{1}{2}\,\frac{\gamma'' - \gamma'}{10^3}, \quad t'' = \gamma'' - \frac{1}{2}\,\frac{\gamma'' - \gamma'}{10^3},$$

т. е.

$$\gamma' < t' < t'' < \gamma'', \quad t'' - t' = \frac{999}{1000}\,(\gamma'' - \gamma')$$

Используя известное соотношение ([1], стр. 209)

$$N(t) = L(t) + S(t) + f(t),$$

получается

$$L(t'') - L(t') = -\left[S(t'') - S(t') + f(t'') - f(t')\right].$$

Из этого, используя (2) (полагая $5{,}6 = \alpha$) и (6), получается

$$\gamma'' - \gamma' < \frac{1000}{999}\,4\pi\alpha\,\frac{\ln\gamma''}{\ln\frac{\gamma'}{2\pi}}\,\frac{1}{\ln\ln\gamma'} + 5{,}5\pi\,\frac{\gamma'' - \gamma'}{\gamma'^2\,\ln\frac{\gamma'}{2\pi}}$$

Так как

$$\frac{\ln \gamma''}{\ln \frac{\gamma'}{2\pi}} < 1 + \frac{\ln 2\pi}{\ln \frac{\gamma'}{2\pi}} + \frac{\gamma'' - \gamma'}{\gamma' \ln \frac{\gamma'}{2\pi}} ,$$

то

$$\gamma'' - \gamma' < \frac{A(\gamma')}{\ln\ln \gamma'} ,$$

где

$$A(\gamma') = \frac{\frac{1000}{999} \, 4\pi\alpha \left(1 + \frac{\ln 2\pi}{\ln \frac{\gamma'}{2\pi}}\right)}{1 - \frac{1000}{999} \frac{4\pi\alpha}{\gamma' \ln \frac{\gamma'}{2\pi} \ln\ln \gamma'} - \frac{5{,}5\pi}{\gamma'^2 \ln \frac{\gamma'}{2\pi}}}$$

Наконец, при $\gamma' > e^{16}$

$$A(\gamma') < 79{,}8 < 80 .$$

(δ) Положим:

$$\chi(s) = \pi^{s - \frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$$

$$\vartheta(t) = -\frac{1}{2} \arg \chi\left(\frac{1}{2} + it\right) ,$$

$$Z(t) = e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right) = Z(2\pi\tau), \; t = 2\pi\tau.$$

Изучая по схеме Титчмарша - Лемера, [6], (используя быстродействующую вычислительную машину) поведение функции $Z(2\pi\tau)$ в окрестности числа

$$\tau_0 = \frac{t_0}{2\pi} \doteq 1\,591\,552{,}454$$

получилась приложенная таблица (см. также график) . Так что получены нули τ', τ'' функции $Z(2\pi\tau)$ для которых

$$\tau'' > 1\,591\,552{,}409$$

$$\tau' < 1\,591\,552{,}288$$

$$\tau'' - \tau' > 0{,}121$$

т. е.

(7) $\gamma'' - \gamma' > 2\pi \cdot 0{,}121 > 6{,}283 \cdot 0{,}121 > 0{,}76.$

Так как

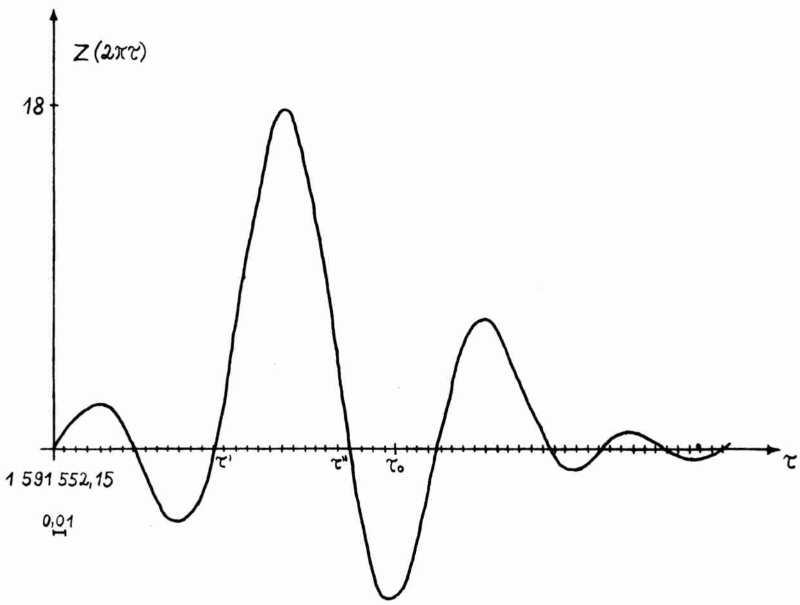
$$e^{16} < 9.10^6 < 10\ 000\ 019\ ,$$

$$\ln\ln\gamma' > \ln\ln 10^7 \quad 2{,}778$$

то, используя(1)и(7), получается

$$A > (\gamma'' - \gamma')\ln\ln\gamma' > 2{,}11 > 2.$$

τ	$Z(2\pi\tau)$
1 591 552,287	- 0,150 109
,288	0,127 388
,289	0,413 266
,290	0,707 268
,30	4,027 373
,31	7,777 516
,32	11,513 566
,33	14,752 955
,34	17,047 026
,35	18,051 322
,36	17,582 063
,37	15,649 087
,38	12,459 341
,39	8,389 867
,400	3,934 334
,406	1,300 327
,407	0,874 814
,408	0,454 522
,409	0,039 946
1 591 552,410	- 0,368 427



Obr. 1

Д о б а в л е н и е I.

Если составить внимательно численный комментарий изложеного в[1], стр.360-362, то получается оценка

$$|S(t)| < 5,6\ \frac{\ln t}{\ln\ln t}\ ,\quad t \geqq e^{16}.$$

Положим:

$$s = o+it,\ 4 \leqq x \leqq t^2,\ x = \sqrt{\ln t},\ t \geqq e^{16},\ o_1 = \frac{1}{2} + \frac{1}{\ln x}\ ,\ o \geqq o_1$$

$$\Lambda_x(n) = \begin{cases} \dfrac{\Lambda(n)\ \ln \frac{x^2}{n}}{\ln x}\ , & x \leqq n \leqq x^2, \\ \Lambda(n)\ , & 1 \leqq n < x, \end{cases}$$

($\Lambda(n)$ - функция фон Мангольдта)

Приведем только оценки двух сумм содержащих $\Lambda_x(n)$, а именно, имеет место

Л е м м а.

$$\frac{1}{\ln x}\sum_{n\leqq x^2}\frac{\Lambda_x(n)}{n^{\sigma_1}} < 1{,}736\ x,$$

$$\sum_{n\leqq x^2}\frac{\Lambda_x(n)}{n^{\sigma_1}\ln n} < 1{,}368\ x.$$

Прежде всего

(1) $$\Lambda_x(n) = \Lambda(n)\left[2 - \frac{\ln n}{\ln x}\right] \leqq \Lambda(n), \quad x \leqq n \leqq x^2,$$

(2) $$\frac{1}{n^{\frac{1}{\ln x}}} = \frac{1}{e^{\frac{\ln n}{\ln x}}} < \begin{cases} 1, & 1 \leqq n \leqq x, \\ \frac{1}{e}, & x \leqq n \leqq x^2, \end{cases}$$

(3) $$\frac{\Lambda(n)}{\ln x} \leqq \begin{cases} 1, & 1 \leqq n \leqq x, \\ 2, & x \leqq n \leqq x^2. \end{cases}$$

Пусть теперь x – целое. Тогда

$$\frac{1}{\ln x}\sum_{n\leqq x^2}\frac{\Lambda_x(n)}{n^{\sigma_1}} < \sum_{2\leqq n\leqq x}\frac{1}{\sqrt{n}} + \frac{2}{e}\sum_{x+1\leqq n\leqq x^2}\frac{1}{\sqrt{n}} <$$

$$< \int_1^x w^{-\frac{1}{2}}\,dw + \frac{2}{e\sqrt{x+1}} + \frac{2}{e}\int_{x+1}^{x^2} w^{-\frac{1}{2}}\,dw = 2(\sqrt{x} - 1) +$$

$$+ \frac{2}{e\sqrt{x+1}} + \frac{4}{e}\left(x - \sqrt{x+1}\right) < \frac{4}{e}x + \left(2 - \frac{4}{e}\right)\sqrt{x} =$$

$$= \left[\frac{4}{e} + \left(2 - \frac{4}{e}\right)\frac{1}{\sqrt{x}}\right]\sqrt{x} \leqq \left(1 + \frac{2}{e}\right)x < 1{,}736\ x.$$

Тот же самый результат получается и в случае нецелого x. Подобным образом поступается и в случае второй суммы.

Добавление II.

О нулях Лемера дзета-функции Римана

"Но, разумеется, даже доказательство гипотезы Римана неисчерпало бы до конца всю теорию функции $\zeta(s)$. Многие тонкие свойства $\zeta(s)$ и тогда останутся невыясненными."

Е.К. ТИТЧМАРШ, [1], стр. 331.

ЛЕМЕР, (5), изучая 10 000 первых нулей функции $\zeta(s)$, нашел пару нулей

$$(1)\quad \frac{1}{2} + i\,7005{,}0629\ ;\ \frac{1}{2} + i\,7005{,}1006$$

разность ординат которых составляет 0,0377. Пара (1) представляет соседние нули $\zeta(s)$ с наименьшей разностью ординат среди первых 10000 нулей $\zeta(s)$. К тому же, в промежутке

$$\langle\ 7005{,}0629\ ;\ 7005{,}1006\ \rangle$$

функция $|Z(t)|$ достигает наименьшего локального максимума относительно промежутка

$$\mathcal{I}_1 = \langle 0\ ;\ 7005{,}1006\ \rangle,$$

а именно,

$$\min\left\{\max_{\mathcal{I}_1} |Z(t)|\right\} \doteq 0{,}003\,967\,5.$$

В работе [6], ЛЕМЕР, продолжая свои исследования, нашел пару нулей

$$(2)\quad \frac{1}{2} + i\,17\,143{,}7319\ ;\ \frac{1}{2} + i\,17\,143{,}7673$$

разность ординат которых составляет 0,0354. Пара (2) представляет соседние нули $\zeta(s)$ с наименьшей разностью ординат среди первых 25 000 нулей. К тому же, в промежутке

$<17\,143{,}7319 ; 17\,143{,}7673>$

функция $|Z(t)|$ достигает наименьшего локального максимума относительно промежутка

$$\mathfrak{J}_2 = <0 ; 17\,143{,}7673>,$$

а именно,

$$\min \left\{ \max_{\mathfrak{J}_2} |Z(t)| \right\} \doteq 0{,}002\,153\,36.$$

Заметим, что разности ординат нескольких соседних с парой(2) пар нулей $\zeta(s)$, попадают приблизительно в промежуток $<0{,}6; 1{,}2>$, в то время как разность ординат нулей пары (2) составляет лишь 0,0354.

Напомним что ЛЕМЕР в своих работах [5], [6], кроме численной проверки гипотезы Римана (в соответствующих промежутках), изучал закон Грама (исключения из этого закона). Это привело его к детальному изучению первых 25000 нулей $\zeta(s)$, и, последнее обстоятельство привело его к открытию пар (1) и (2) нулей функции $\zeta(s)$.

МЕЛЛЕР, в работе [7], сообщая о своих вычислениях связанных с проверкой гипотезы Римана, отмечает положение дел в окрестности ординат нулей пары (2) потому, что здесь, для обнаружения перемены знака функции $Z(2\pi\tau)$, должен был использовать более точную формулу для последней. Заметим что МЕЛЛЕР незнал о работе [6] Лемера.

Просмотривая эти работы, наиболее поразительным показывается уже самый факт существования пар (1) и (2)

Конечно, если предположить, что ординаты нулей $\frac{1}{2}+i\gamma, \gamma>0$, функции $\zeta(s)$ занумерованы так, что кратный нуль считается один раз, то в существовании числа

$$\min_{n\leq N}(\gamma_{n+1} - \gamma_n) = m(N)$$

или, в частности, числа

$$\min_{n\leq 25000}(\gamma_{n+1} - \gamma_n) = 0{,}0354$$

(N - целое положительное), нет ничего удивительного.

Однако, если к тому же известно, что разности ординат нескольких соседних с парой (2) пар нулей $\zeta(s)$, приблизительно от 16 до 32 раз превосходят число 0,0354 , то такое обстоятельство, вероятно заслуживает внимания.

Позволим себе привести некоторые заметки в этом направлении.

1. Назовем пары (1), (2) нулей функции $\zeta(s)$, парами Лемера.

2. Естественно предположить, что существует бесконечно много пар Лемера.

3. Однако, предшествующее утверждение преждевременно, так как отсутствует определение пар ЛЕМЕРА.

4. Пробуя характеризовать пару ЛЕМЕРА на основе приведенных численных данных, возможно было бы сказать так: разность ординат нулей пары ЛЕМЕРА "существенно" меньша арифметического среднего разностей ординат "нескольких" соседних пар нулей функции $\zeta(s)$. Здесь в кавычки вставены понятия нуждающееся в уточнении (отсутствует метрическая характеристика этих понятии), так что сказанное выше не есть определение.

5. Более подходящим показывается определение следующего типа: соседние нули $\frac{1}{2}+i\gamma'$, $\frac{1}{2}+i\gamma''$ функции $\zeta(s)$ назовем нулями Лемера (пара Лемера) если

$$\gamma''-\gamma' < \exp\left\{-1,75\,\frac{\ln\gamma'}{(\ln\ln\gamma')^2}\right\}$$

[Число 1,75 получилось из соотношения

$$\exp\left\{-A\,\frac{\ln\gamma'}{(\ln\ln\gamma')^2}\right\} = 0,0377$$

при $\gamma' = 7005,0629$. функция

$$\exp\left\{-1,75\,\frac{\ln\gamma'}{(\ln\ln\gamma')^2}\right\}$$

разумеется, имеет только наводящее значение, хотя она и согласованна с первыми двумя парами ЛЕМЕРА.]

6. В этом случае, гипотеза о бесконечности множества пар Лемера означает утверждение, что приведенное выше неравенство имеет бесконечно много решении на множестве соседних нулей.

7. Если $\frac{1}{2}+i\gamma'$, $\frac{1}{2}+i\gamma''$, $\gamma'<\gamma''$ соседние нули функции $\zeta(s)$, то введем обозначение

$$\Delta(\gamma',\gamma'') = \gamma''-\gamma'.$$

Для изучавшего нули функции $\zeta(s)$ известно, что с возрастанием γ' величина $\Delta(\gamma',\gamma'')$ сложным образом осцилирует (стремясь, конечно, к нулю при $\gamma'\to+\infty$). Возникает вопрос. Неубывает ли $\Delta(\gamma',\gamma'')$ монотонно с возрастанием γ' именно на множестве пар Лемера?

8. В этой обстановке было бы весьма желательным открытие нес-

колькух следующих пар Лемера, что, как возможно думать, совершенно под силу современной вычислительной технике. Открытие таковых (помимо углубления численных знаний в этой области) послужило бы толчком для предварительных аналитических конструкций относящихся к этому удивительному явлению на критической прямой.

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On a constant of Littlewood in the theory of the Riemann zeta function

JÁN MOSER

S u m m a r y

In this remark we have an estimation from above and from below of one constant relative to the vertical distribution of zeros of the Riemann zeta function.

Besides int he appendix the attention focus on the very interesting pairs of zeros of the Riemann zeta function, which have been discovered by D.H. Lehmer.

O jednej Littlewoodovej konštante v teórii Riemannovej dzeta funkcie

JÁN MOSER

S ú h r n

V tejto poznámke je odhad zhora aj zdola pre jednu konštantu, ktorá sa týka vertikálneho rozloženia nulových bodov Riemannovej dzeta funkcie.

Popritom v dodatku autor obracia pozornosť na veľmi zaujímavé dvojice nulových bodov Riemannovej dzeta funkcie, ktoré objavil D.H. LEHMER.

Došlo: 4. decembra 1972,
do nakladateľstva 8.júla 1974

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(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXXII, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXXII - 1975

Antihomographies. The representation of antiinvolutions in the three-dimensional euclidean space

ZITA SKLENÁRIKOVÁ, Bratislava

1. Let S_1 be a one-dimensional projective space over the complex field K /briefly: a complex line/. Let $x \in S_1$; then $x = (x_1, x_2)$ denotes that complex numbers x_1, x_2 are homogenous co-ordinates of the point x in a certain co-ordinate system σ of the complex line S_1. The notion of a homography is supposed to be known. [1] . Likewise the notions of double elements and an involutory pair of a transformation.

2. The first part of our paper deals with antihomography transformations on a complex line, the special case of which are antiinvolutions. The results of this part are known,([1] , [2;1,2]). The essential significance of the paper is to work up the subject by the united method of the using of homogenous co-ordinates. For the reason of entirety in this part there are introduced some known theorems without their proofs. In the second part there is studied the representation of the antiinvolutions in the three-dimensional Euclidean space through RIEMANN transformation. In this part the outline of the transformation mentioned in [1] is made more precisely and there are some other consequent results introduced.

I. Antihomographies and antiinvolutions

1. Definitions, equations

Definition I,1. A point transformation $\bar{H} : S_1 \longrightarrow S_1$ is called the antihomography, if there exists a regular matrix over the complex field $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and in S_1 a co-ordinate system $\sigma(x_1, x_2)$ such that the point $x' = (x_1', x_2')$ corresponds to the point $x = (x_1, x_2)$ when

$$x_1'(a\bar{x}_1 + b\bar{x}_2) + x_2'(c\bar{x}_1 + d\bar{x}_2) = 0. \qquad (1)$$

Remarks. 1. The equation (1) can be written in the matrix form

$$(x_1' \; x_2')\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = 0 \qquad (1')$$

2. The equation (1') is equivalent to:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} \qquad (2)$$

3. There will be used the notation $x' = \bar{H}(x)$.

Theorem I,1. An antihomography can never be a homography.

Proof. Let $\bar{H}$ be an antihomography which is the homography H on S_1 as well. Let $\bar{H}, H$ have the next equations in the certain co-ordinate system $\sigma(x_1, x_2)$, respectively:

$$\bar{H} : (x') = A(\bar{x}), \quad H : (x') = B(x),$$

where A,B are regular matrices over the complex field. Then for all $x \in S_1$ it holds : $\rho A(\bar{x}) = B(x)$, $\rho \in K$, $\rho \neq 0$, consequently for all x with x_1, x_2 real. From it follows: $\rho A = B$. H has now an equation $(x') = \rho A(x)$. Then for all $x \in S_1$ the following equalities hold:

$$-c(\nu x_1 - \bar{x}_1) - d(\nu x_2 - \bar{x}_2) = 0$$
$$a(\nu x_1 - \bar{x}_1) + b(\nu x_2 - \bar{x}_2) = 0,$$

$\nu \in K, \ \nu \neq 0, A = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$.

But $|A| \neq 0$, i e. it holds for all $x \in S_1$: $\bar{x}_1 = \nu x_1$, $\bar{x}_2 = \nu x_2$; which is false.

The multiplication in the set of the antihomographies and the product of a homography and an antihomography are defined obviously.

Theorem I,2. The set of all homographies and antihomographies on S_1 is a group.

The proof follows immediately from the expression of the transformation in the form (2) .

Theorem I,3. Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be double elements of the antihomography $\bar{H}$ given by (1).

Then $\bar{H}$ can be introduced by an equation

$$b^o \begin{vmatrix} x_1' & x_2' \\ \beta_1 & \beta_2 \end{vmatrix} \begin{vmatrix} \bar{x}_1 & \bar{x}_2 \\ \bar{\alpha}_1 & \bar{\alpha}_2 \end{vmatrix} + c^o \begin{vmatrix} x_1' & x_2' \\ \alpha_1 & \alpha_2 \end{vmatrix} \begin{vmatrix} \bar{x}_1 & \bar{x}_2 \\ \bar{\beta}_1 & \bar{\beta}_2 \end{vmatrix} = 0, \qquad (3)$$

where $b^o = \alpha_1 (a \bar{\beta}_1 + b \bar{\beta}_2) + \alpha_2 (c \bar{\beta}_1 + d \bar{\beta}_2)$

$$c^o = \beta_1 (a \bar{\alpha}_1 + b \bar{\alpha}_2) + \beta_2 (c \bar{\alpha}_1 + d \bar{\alpha}_2).$$

The proof is evident.

Remarks. 1. In (3) there is $c^o \neq 0$, $b^o \neq 0$, as the matrix $BA\bar{B}^{-1}$ is regular.

2. If in (3) there is b^o, $c^o \in K$, $b^o \neq 0$, and $\alpha \neq \beta$ then (3) is the equation of an antihomography with double elements α, β.

Theorem I,4. Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be an involutory pair of the antihomography $\bar{H}$ given by (1). Then $\bar{H}$ can be introduced by an equation

$$^o a \begin{vmatrix} x_1' & x_2' \\ \beta_1 & \beta_2 \end{vmatrix} \begin{vmatrix} x_1 & x_2 \\ \bar{\beta}_1 & \bar{\beta}_2 \end{vmatrix} + {}^o d \begin{vmatrix} x_1' & x_2' \\ \alpha_1 & \alpha_2 \end{vmatrix} \begin{vmatrix} x_1 & x_2 \\ \bar{\alpha}_1 & \bar{\alpha}_2 \end{vmatrix} = 0, \qquad (4)$$

where $^{o}a = \alpha_1 (a\bar{\alpha}_1 + b\bar{\alpha}_2) + \alpha_2 (c\bar{\alpha}_1 + d\bar{\alpha}_2)$

$^{o}c = \bar{\beta}_1 (a\bar{\beta}_1 + b\bar{\beta}_2) + \beta_2 (c\bar{\beta}_1 + d\bar{\beta}_2)$.

The proof is evident.

R e m a r k s . 1. In (4) there is $^{o}a \neq 0$, $^{o}d \neq 0$.

2. Conversely, if $^{o}a \neq 0$, $^{o}d \neq 0$ are the arbitrary complex numbers and $\alpha \neq \beta$ then (4) is the equation of an antihomography with the involutory pair (α, β).

D e f i n i t i o n I,2. An antihomography $\bar{H}$ for which $\bar{H}^2$ is the identity transformation is said to be an involution.

T h e o r e m I,5. Let $\bar{H}$ be an antihomography given by (1). Then $\bar{H}$ is an involution if and only if:

$a' = d' = 0$, $b' = -c' \neq 0$, where $a' = -a\bar{c} + b\bar{a}$, $b' = -a\bar{d} + b\bar{b}$,

$c' = -c\bar{c} + d\bar{a}$, $d' = -c\bar{d} + b\bar{d}$.

The proof is by direct computation of the product $\bar{H}^2$.

T h e o r e m I,6. Let the antihomography $\bar{H}$ given by (1) be an antiinvolution. Then there exists such a complex number $k \neq 0$, that for the coefficients $a^{+} = ka$, $b^{+} = kb$, $d^{+} = kd$, $c^{+} = kc$ it holds: a^{+}, d^{+} are real and $b^{+} = \bar{c}^{+}$. (5)

Conversely, if numbers a,d from (1) are real and $b = \bar{c}$, then $\bar{H}$ given by (1) is an antiinvolution.

P r o o f . Let $\bar{H}$ be the antiinvolution given by (1) . By Theorem I,5 there exists a real number $\rho \neq 0$ such that

$$|\rho| = 1, \ \bar{a} = \rho a, \ \bar{b} = \rho c, \ \bar{c} = \rho b, \ \bar{d} = \rho d. \qquad (6)$$

Let $k \in K$, $k \neq 0$. Then $\overline{ak} = (\rho \bar{k}/k)\, ak$. When we choose k such that $\rho \cdot \bar{k}/k = 1$, then

$$\overline{ak} = a\,k, \ \overline{dk} = d\,k, \ \overline{bk} = c\,k, \ \overline{ck} = b\,k,$$

i.e. a^{+} and d^{+} are real and $b^{+} = \bar{c}^{+}$. An equation of the form (1) with coefficients a^{+}, b^{+}, c^{+}, d^{+} is obviously the equation of the antiinvolution $\bar{H}$.

The converse assertion can be verified by direct computation.

R e m a r k . If in the next the antiinvolution $\bar{H}$ is given by (1), a, d would be assumed real and $c = \bar{b}$.

2. D o u b l e p o i n t s o f a n t i i n v o l u t i o n s

Let $\bar{H}$ be the antiinvolution given by (1) .

a/ Let $a = 0$, $d \neq 0$. By choosing the convenient co-ordinate system, that is by using the linear transformation

$$B : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{b} \\ \frac{1}{d} & -\frac{1}{d} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} ,$$

the equation of $\bar{H}$ will have the form $y_1' \bar{y}_1 - y_2' \bar{y}_2 = 0$. For double elements it will be obtained the equation

$$y_1 \bar{y}_1 - y_2 \bar{y}_2 = 0. \qquad (7)$$

From (7) it is obvious that the set of double elements $\bar{H}$ is infinite.

b/ Let $a = 0$, $d = 0$. By using the linear transformation of the co-ordinate system $\sigma(x_1,x_2)$ into $\sigma'(y_1,y_2)$: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{b} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

the equation of $\bar{H}$ will take the form $y_1' \bar{y}_2 + y_2' \bar{y}_1 = 0$. For double elements the equation

$$y_1 \bar{y}_2 + y_2 \bar{y}_1 = 0 \qquad (8)$$

will be obtained.

From (8) it is obvious that the set of double elements of $\bar{H}$ is infinite again.

c/ Let $a \neq 0$. The equation of $\bar{H}$ can be expressed in the form

$$a\left(x_1' + \frac{\bar{b}}{a}x_2'\right) \cdot \left(\bar{x}_1 + \frac{b}{a}\bar{x}_2\right) + \frac{ad - b\bar{b}}{a}\, x_2' \bar{x}_2 = 0 \quad (9)$$

and analougously the equation for double elements. By use of the linear transformation $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -\bar{b} \\ 0 & a \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ the eqation of $\bar{H}$ will have the form $x_1' \bar{x}_1 + \Delta x_2' \bar{x}_2 = 0$. For double elements it will be obtained:

$$x_1 \bar{x}_1 + \Delta x_2 \bar{x}_2 = 0, \qquad \Delta = ad - b\bar{b} \neq 0, \ \Delta \in R. \qquad (10)$$

From (10) it is obvious that in the case $\Delta < 0$ the set of double elements of $\bar{H}$ is infinite and in the case $\Delta > 0$ $\bar{H}$ has no double element.

D e f i n i t i o n I,3. An antiinvolution $\bar{H}$ which has at least one double point is called the antiinvolution of the first species, an antiinvolution which has no double point is called the antiinvolution of the second species.

T h e o r e m I,7. The antiinovlution $\bar{H}$ given by (1) is of the first species /second species/ if and only if

$$\Delta = \begin{vmatrix} a & b \\ \bar{b} & d \end{vmatrix} < 0 \quad \left(\text{if and only if } \Delta = \begin{vmatrix} a & b \\ \bar{b} & d \end{vmatrix} > 0\right).$$

The proof is evident from the previous considerations, see a/ - c/.

3. R e p r e s e n t a t i o n o f a n t i i n v o l u - t i o n s i n C a u c h y ' s p l a n e

T h e o r e m I,8. An antiinvolution of the first species is represented in the Cauchy's plane by an inversion of the ratio positive or by an axial symmetry. An antiinvolution of the second species is there represented by an inversion of the ratio negative.

P r o o f . Let us denote $z = \dfrac{x_1}{x_2}$ for $x_2 \neq 0$, $z' = \dfrac{x_1'}{x_2'}$ for $x_2' \neq 0$ and let $z = X + iY$, $z' = X' + iY'$ $(X, Y, X', Y' \in R)$. The representative of the point $x = (x_1, x_2) \in S_1$ is for $x_2 \neq 0$ the point of the Cauchy's plane with the rectangular co-ordinates (X,Y) and for $x_2 = 0$ the ideal point.

a/ Let $a = 0$, $b = b_1 + i\, b_2$. The representation of the antiinvolution is then the transformation

$$X' = X \frac{-b_1^2 + b_2^2}{b_1^2 + b_2^2} + 2\,Y \frac{b_1\, b_2}{b_1^2 + b_2^2} - \frac{d\, b_1}{b_1^2 + b_2^2}$$

$$Y' = X \frac{2\, b_1 b_2}{b_1^2 + b_2^2} + Y \frac{b_1^2 - b_2^2}{b_1^2 + b_2^2} + \frac{d\, b_2}{b_1^2 + b_2^2} \tag{11}$$

with the addition that the point at infinity of the Cauchy's plane is self-corresponding. To the set of double elements there corresponds the locus

$$2\, b_1\, X - 2\, b_2\, Y + d = 0 \tag{11'}$$

complemented by the ideal point.

By (11) there is defined the axial symmetry with the axis (11') in the Cauchy's plane. Conversely, let

$$X' = \frac{-\alpha^2 + \beta^2}{\alpha^2 + \beta^2} X - \frac{2\alpha\beta}{\alpha^2 + \beta^2} Y - \frac{2\alpha\beta}{\alpha^2 + \beta^2}$$

$$Y' = \frac{-2\alpha\beta}{\alpha^2 + \beta^2} X + \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} Y - \frac{2\beta\gamma}{\alpha^2 + \beta^2} ;$$

be the axial symmetry with the axis $\alpha X + \beta Y + \gamma = 0$.

This axial symmetry corresponds to the antiinvolution $\bar{H}$ with the equation

$$x_1'(b\bar{x}_2) + x_2'(\bar{b}\bar{x}_1 + d\bar{x}_2) = 0, \text{ where } b = \frac{\alpha}{2} - i\frac{\beta}{2}$$

and $d = \gamma$.

b/ Let $a \neq 0$. The representation of the antiinvolution (9) is then the transformation

$$\left(z' + \frac{\bar{b}}{a}\right) \cdot \left(\bar{z} + \frac{b}{a}\right) = -\frac{ad - b\bar{b}}{a^2} \qquad (12)$$

with the addition that the ideal point of the Cauchy's plane corresponds to the point $z = -\frac{\bar{b}}{a}$, and conversely.

The set of double elements is represented by the locus

$$\left(z + \frac{\bar{b}}{a}\right) \cdot \left(\bar{z} + \frac{b}{a}\right) = -\frac{ad - b\bar{b}}{a^2} \qquad (12')$$

By (12) there is defined in the Cauchy's plane the circle inversion with the centre $z_0 = -\frac{\bar{b}}{a}$ and the ratio $\frac{b\bar{b} - ad}{a^2} =$

$= -\frac{\Delta}{a^2}$.

By (12') the representation of the set of double elements of the antiinvolution $\bar{H}$ given by (1) with $\Delta < 0$ is the circle with the centre $z_0 = -\frac{\bar{b}}{a}$ and the radius $r = \sqrt{-\frac{ad - bb}{a^2}}$.

Conversely, let $(z' + z_0)(\bar{z} + \bar{z}_0) = k$, $k \in R$; be the circle inversion in the Cauchy's plane. This inversion represents just one antiinvolution $\bar{H}$; its equation can be written in the form

$$x_1'(\bar{x}_1 + \bar{z}_0\bar{x}_2) + x_2'(z_0\bar{x}_1 + [-k + z_0\bar{z}_0]\,\bar{x}_2) = 0$$

$$\Delta = \begin{vmatrix} 1 & \bar{z}_o \\ z_o & -k + z_o\bar{z}_o \end{vmatrix} = -k;$$

hence in the case $k > 0$ $\bar{H}$ is the antiinvolution of the first species, in the case $k < 0$ $\bar{H}$ is of the second species.

II. Representation of antiinvolutions in the three-dimensional Euclidean space

1. Let $\bar{E}_3$ be the real three-dimensional Euclidean space completed by the ideal plane. Let Σ (X,Y,Z,T) be the co-ordinate system for the expression of the homogenous co-ordinates such that its restriction in E_3 is the rectangular co-ordinate system Σ (x,y,z) for the expression of the non-homogenous co-ordinates. Let $M \in \bar{E}_3$; then M = (x,y,z) , or M = (X,Y,Z,T) denote that x,y,z, are the rectangular co-ordinates of the point M in Σ (x,y,z) and X,Y,Z,T are its homogenous co-ordinates in Σ (X,Y,Z,T) . Let G be the unit sphere in $\bar{E}_3$ whose centre O is the origin. Symbols int G and ext G denote the sets of points interior and exterior with respect to the sphere G respectively.

T h e o r e m II,1. Let M be the set of all antiinvolutions of the first species on S_1. Let $A \in M$ and A be given by (1) from the part I., with a and d real, and $b = \bar{c}$. Then the mapping given by

$$X = -2b_1, \quad Y = 2b_2, \quad Z = a - d, \quad T = a + d \qquad (1)$$

is a bijective transformation of M into ext G.

P r o o f . a/ It follows from Theorem I,7 that $ad - b_1^2 - b_2^2 = \Delta < 0$.

Let A be the pooint in $\bar{E}_3$ which corresponds by (1) to the antiinvolution A. Let A = (X,Y,Z,T) . Then

$$X^2 + Y^2 + Z^2 - T^2 = 4(b_1^2 + b_2^2) + (a - c)^2 + (a + c)^2 = -4\Delta > 0,$$

whence $A \in$ ext G.

b/ Let $A = (X,Y,Z,T)$ be a point of $\bar{E}_3$ in ext G. Then A corresponds to the antiinvolution A given by (1) from the part I., with

$$a = \frac{T+Z}{2}, \quad d = \frac{T-Z}{2}, \quad b_1 = -\frac{X}{2}, \quad b_2 = \frac{Y}{2} \qquad (2).$$

Consequently, $\Delta = ad - (b_1^2 + b_2^2) = \frac{1}{4}(T^2 - X^2 - Y^2 - Z^2) < 0$, hence $A \in M$.

Yet it is to be proved that for each representation of the point $A \in \bar{E}_3$ there exists just one antiinvolution A.

Let $A' = (X',Y',Z',T')$ be another representation of A. Then there exists a real number $\rho \neq 0$ such that $X' = \rho X$, Y' $Y' = \rho Y$, $Z' = \rho Z$, $T' = \rho T$, and it follows $a' = \rho a$, $d' = \rho d$, $b_1' = \rho b_1$, $b_2' = \rho b_2$, $\Delta' = \rho^2 \Delta$, where a', b', c', d' are the coefficients from the equation of the antiinvolution A', to which A' corresponds. Hence $A' = A$.

<u>T h e o r e m   II,2.</u> Let M' be the set of all antiinvolutions of the second species on S_1. Let $B \in M'$ and B be given by (1) from the part I., with a and d real, and $b = \bar{c}$. Then the mapping given by (1) from Theorem II,1 is a bijective transformation of M' into int G.

The proof is analogical to the proof of Theorem II,1.

<u>R e m a r k . A point in</u> E_3 which corresponds by Theorems II,1; II,2 to an antiinvolution of the first or of the second species will be called the representative of this antiinvolution in $\bar{E}_3$.

2. Let P be a point in $\bar{E}_3$, $P \notin G$. P induces a transformation of G on itself in the following sense.

Let $\pi : G \longrightarrow G$ be the involutory transformation of G on itself defined as follows:

For all $X \in G$, $X \longmapsto X' \in PX \cap G$ so that a/ $X' \neq X$ if card $(PX \cap G) = 2$ and b/ $X' = X$ if card $(PX \cap G) = 1$.

R e m a r k s . 1. There will be used the notation $X' = \mathcal{T}(X)$ or $X' = P(X)$.

2. $\mathcal{T}$ is an involutory transformation which for $P \in$ ext G has infinitely many double points (the common circle of G and its tangent cone of vertex P) and for $P \in$ int G has no double point.

Since there exists the bijective transformation of G into S_1 /the RIEMANN mapping by means of the stereographic projection/, $\mathcal{T}$ may be considered the transform of a transformation φ on S_1.

T h e o r e m II,3. The transformation φ on S_1 is an antiinvolution the representative of which in E_3 is the point P.

P r o o f . First it will be showed what is the stereographic projection of the transformation $\mathcal{T}$ from the point I = (0,0,-1,1) upon the plane Z = 0. The stereographic projection from the point I will be denoted τ. There are two possibilities for $\tau(P)$:

a/ $\tau(P)$ is a non-ideal point of the plane Z = 0. It is easily checked that for an arbitraty couple of points $X \in G$ and $X' = \mathcal{T}(X)$ it holds:

$$\overline{\tau(P)\tau(X)} \cdot \overline{\tau(P)\tau(X')} = |k| , \ k \in R,$$

k is constant for a given point P. Also the sterographic projection of the transformation $\mathcal{T}$ in this case there is the circle inversion with the centre $\tau(P)$ and the ratio +k and -k, respectively with the addition that the ideal point of the Cauchy's plane corresponds to the point $\tau(P)$.

b/ $\tau(P)$ is the ideal point of the plane Z = 0. Then for an arbitrary couple of points $X \in G$ and $X' = \mathcal{T}(X)$ it holds that the points $\tau(X), \tau(X')$ are the coresponding points in the axial symmetry with the axis in the line of intersection of the polar plane od P and the plane Z = 0. In this case the stereographic projection of $\mathcal{T}$ is the axial symmetry with the addition that the ideal point of the Cauchy's plane is self--corresponding point of this transformation. Both assertions were proved by use of the knowledge of the stereographic projection of a circle and the property of four harmonic points on a line.

Both transformations by Theorem I,8 represent antiinvolutions on S_1. Further it will be showed that the representative of the antiinvolution in E_3 obtained in this way is just P. Let $P = (X^o, Y^o, Z^o, T^o)$. Then by a simple computation $\tau(P) = (X^o, Y^o, 0^o, Z^o + T^o)$. It follows that $\tau(P)$ is a non-ideal point just when $Z^o + T^o \neq 0$.

a/ First there will be considered all points $P \in \bar{E}_3$ such that $Z^o + T^o \neq 0$. Two cases will be differentiated:

a_1/ $P \notin z$ and a_2/ $P \in z$. /z - the coordinate axis/.

a_1/ Let $P \notin z$ and $\{L, L'\} = OP \cap G$. Then $L = \left(\frac{X^o}{\mathcal{H}}, \frac{Y^o}{\mathcal{H}}, \frac{Z^o}{\mathcal{H}}\right)$, $L' = \left(-\frac{X^o}{\mathcal{H}}, \frac{-Y^o}{\mathcal{H}}, -\frac{Z^o}{\mathcal{H}}\right)$ and $\tau(L) = \left(\frac{X^o}{Z^o + \mathcal{H}}, \frac{Y^o}{Z^o + \mathcal{H}}, 0\right)$,

$\tau(L') = \left(\frac{X^o}{Z^o - \mathcal{H}}, \frac{Y^o}{Z^o - \mathcal{H}}, 0\right)$, where $\mathcal{H} = \sqrt{X^{o\,2} + Y^{o\,2} + Z^{o\,2}}$.

Let k be the ratio of the corresponding circle inversion. To express $|k|$ there will be used

$$\overline{\tau(P) \cdot \tau(L)}^2 = \frac{(T^o - \mathcal{H})^2 \left[(X^o)^2 + (Y^o)^2\right]}{(Z^o + T^o)^2 \; (Z^o + \mathcal{H}^2)}$$

$$\overline{\tau(P) \; \tau(L')}^2 = \frac{(T^o + \mathcal{H})^2 \left[(X^o)^2 + (Y^o)^2\right]}{(Z^o + T^o)^2 \; (Z^o - \mathcal{H})^2} \; ; \text{ then}$$

$$|k| = \overline{\tau(P) \, \tau(L)} \, . \, \overline{\tau(P) \tau(L')} = \frac{\left|(X^o)^2 + (Y^o)^2 + (Z^o)^2 - (T^o)^2\right|}{(Z^o + T^o)^2} .$$

If $P \in$ ext G, the circle inversion with the centre $\tau(P)$ has the ratio positive; if $P \in$ int G, the ratio is negative; generally

$$k = \frac{(X^o)^2 + (Y^o)^2 + (Z^o)^2 - (T^o)^2}{(Z^o + T^o)^2}$$

and the equation of the corresponding circle inversion is

$$\left(z' - \frac{X^o + iY^o}{Z^o + T^o}\right)\left(\bar{z} - \frac{X^o - iY^o}{Z^o + T^o}\right) = \frac{(X^o)^2 + (Y^o)^2 + (Z^o)^2 - (T^o)^2}{(Z^o + T^o)^2} .$$

By Theorem I,8 it follows that this inversion corresponds to an antiinvolution on S_1 given by

$$x_1' \, (a\bar{x}_1 + b\bar{x}_2) + x_2' \, (\bar{b}\bar{x}_1 + d\bar{x}_2) = 0, \text{ where}$$

$a = T^o + Z^o$, $b = - X^o + iY^o$, $d = T^o - Z^o$, and this is by (2) from Theorem II,1 the antiinvolution, the representative of which in $\bar{E}_3$ is the point P.

a_2/ Let $P \in z$, so $P = (0,0,Z^o,T^o)$. The point $(1,0,0,1)$ can be chosen for L. Further steps and the conclusion are the same as in a_1/ .

b/ Now there will be considered all $P \in \bar{E}_3$ such that $T^o + Z^o = 0$. The line of intersection of the polar plane of P and the plane $Z = 0$ has an equation $XX^o + YY^o + Z^o = 0$.

By Theorem I,8 the corresponding axial symmetry has the equations:

$$X' = \frac{-(X^o)^2 + (Y^o)^2}{(X^o)^2 + (Y^o)^2} X - \frac{2\,X^oY^o}{(X^o)^2 + (Y^o)^2} Y - \frac{2\,X^oY^o}{(X^o)^2 + (Y^o)^2}$$

$$Y' = \frac{-2X^oY^o}{(X^o)^2 + (Y^o)^2} X + \frac{(X^o)^2 - (Y^o)^2}{(X^o)^2 + (Y^o)^2} Y - \frac{2Y^oZ^o}{(X^o)^2 + (Y^o)^2},$$

and this symmetry corresponds to an antiinvolution $\bar{H}$ given by

$$x_1' (b\bar{x}_2) + x_2' (b\bar{x}_1 + b\bar{x}_2) = 0,$$

where $b = \frac{X^o}{2} - i\frac{Y^o}{2}$, and $d = Z^o$.

Since $-2b_1 = -X^o$, $2b_2 = -Y^o$, $a - d = -Z^o$, $a + d = Z^o = -T^o$,

by (2) from Theorem II,1 it follows that the representative of this antiinvolution in $\bar{E}_3$ is the point P.

The Theorem is proved for all $P \in \bar{E}_3$.

3. Some other properties following from geometrical representation of antiinvolutions

<u>Theorem II,4.</u> Let $\bar{H}_o, \bar{H}_1$ be distinct antiinvolutions of the same species. Let P_o, P_1 be their representatives in $\bar{E}_3$. Then the product $\bar{H}_1\bar{H}_o$ is a homography which:

a/ has just two distinct double points if and only if the line P_oP_1 intersects G in two discincts points, or does not intersect it at all;

b/ is parabolic when the line P_oP_1 is a tangent to G. And conversely.

<u>Proof.</u> Let $P_oP_1 \cap G = \{M, N \neq M\}$. Thus M,N correspond to the points $m,n \in S_1$ such that m,n is a common involutory pair of $\bar{H}_o$ and $\bar{H}_1$. If $P_oP_1 \cap G = \emptyset$, then conjugate polar to the line P_oP_1 /which respect to G/ intersects G in two distinct points M,N. M,N correspond to the points $m,n \in S_1$; m,n

are double points both of $\bar{H}_o$ and $\bar{H}_1$. If $P_oP_1 \cap G = \{M\}$, then M corresponds to the point $m \in S_1$, m is a common double point of $\bar{H}_o$ and $\bar{H}_1$. The converse assertion would be proved indirectly.

<u>Theorem II,5.</u> Let $\bar{H}_o \neq \bar{H}_1$ be antiinvolutions of the different species. Then there exists just one common involutory pair of $\bar{H}_o$ and $\bar{H}_1$.

<u>Proof.</u> Let P_o, P_1 be the representatives of $\bar{H}_o$, $\bar{H}_1$. The line P_oP_1 intersects G in two distinct points.

<u>Theorem II,6.</u> Let M,N be distinct points of G. Let M, N correspond to the points $m,n \in S_1$. Then:

1/ The set of all points $P \in \bar{E}_3$ which are representatives of antiinvolutions $\bar{H}$ with common double points m,n is a conjugate polar to a line P_oP_1 with respect to G.

The proof follows immediately from the representation in $\bar{E}_3$.

<u>Theorem II,7.</u> Let T be a point of G and let it correspond to a point $t \in S_1$. Then the set of all points $P \in \bar{E}_3$ which are representatives of antiinvolutions $\bar{H}$ with common double point t is a tangent plane of G in T /without the point T/. The proof follows immediately from the representation in $\bar{E}_3$.

<u>Theorem II,8.</u> Let $\bar{H}_o \neq \bar{H}_1$ be antiinvolutions. Let $P_o \neq P_1$ be their representatives in $\bar{E}_3$. Then $\bar{H}_1\bar{H}_o \neq \bar{H}_o\bar{H}_1$ if and only if the points P_o, P_1 are conjugate poles with respect to G.

<u>Remark.</u> The property mentioned in Theorem II,8 is equivalent to the property that $\bar{H}_1\bar{H}_o$ is an involution.

<u>Proof.</u> The homography $\bar{H}_1\bar{H}_o$ is an involution just when it has at least one involutory pair ([1]), and it is the case iff there exist four points M,M',N,N' on G such that:

1. they are distinct one to another,
2. they are incident with a plane,
3. $M' = P_o(M)$, $N = P_1(M')$, $N' = P_o(N)$, $M_1 = P_1(N')$.

If the points P_o,P_1 are conjugate poles with respect to G, then the existence of four such points with the properties

1. - 3. follows from the polar properties of quadrics. Conversely, if there exist four points with the properties

1. - 3. then the points P_o,P_1 as intersection points of opposite sides of a plane quadrangle inscribed into a circle /in which the plane entersects G/ are the conjugate poles with respect to G.

Consequences. 1. The centre O of G represents an antiinvolution commutative with all antiinvolutions such that their representative is an ideal point in $\bar{E}_3$.

2. Let one of antiinvolutions $\bar{H}_o,\bar{H}_1$ be of the second species. The necessary condition for the product $\bar{H}_1\bar{H}_o$ to be an involution is that the other antiinvolution is of the first species.

3. Let $\bar{H}_o$ be an antiinvolution and let P_o be its representative in $\bar{E}_3$. The set of all points $P \in \bar{E}_3$ which are the representatives of antiinvolutions $\bar{H}$ such that $\bar{H}\,\bar{H}_o$ is an involution, is the polar plane π_o of the point P_o with respect to G /without eventual intersection points π_o and G/.

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Antihomografie. Reprezentácia antiinvolúcií v trojrozmernom Euklidovom priestore

ZITA SKLENÁRIKOVÁ

R e s u m é

V práci autorka skúma zobrazenie antiinvolúcií komplexnej projektívnej priamky do rozšíreného trojzmerného Euklidovho priestoru. Vlastnosti antiinvolúcií a vzťahy medzi nimi sú reprezentované názornými geometrickými vzťahmi v Euklidovom priestore.

Антигомографии. Представление антиинволюций в трехмерном Евклидовом пространстве

ЗИТА СКЛЕНАРИКОВА

Р е з ю м е

В работе рассмотрено отображение антиинволюций комплексной проективной прямой в пополненое трехмерное Евклидово пространство. Свойства инволюций и отношения между ними представлены наглядными геометрическими отношениями в Евклидовом пространстве.

Received: April 13, 1973,
in press July 8, 1974.

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(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXXII, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXXII – 1975

O ohraničenosti riešení diferenciálnej rovnice druhého rádu s oneskoreným argumentom

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V tomto článku sa budeme zaoberať vyšetrovaním niektorých postačujúcich podmienok ohraničenosti riešení diferenciálnych rovníc s oneskoreným argumentom. Budeme uvažovať o riešeniach diferenciálnej rovnice

/1/ $$x''(t) + \sum_{i=0}^{m} a_i(t)\, x(h_i(t)) = 0,$$

ktoré spĺňajú začiatočné podmienky:

/2/ $$x(t) = \varphi(t) \text{ pre } t \in E_{to}, \quad x(t_o) = x_o, \quad x'(t_o^+) = x_o'.$$

Ďalej budeme predpokladať, že $a_i(t), h_i(t)$, $i = 0, 1, 2, \ldots m$, sú spojité funkcie na intervale $J = \langle t_o, \infty)$ a pre každé $t \in J$ platí $h_o(t) = t$, $h_i(t) \leqq t$, $i = 1, 2, \ldots m$, $\varphi(t)$ je spojitá a ohraničená funkcia na množine E_{to}, $x_o = \varphi(t_o)$ a x_o' je ľubovoľné reálne číslo. Množinu E_{to} nazývame začiatočnou množinou a je definovaná takto : $E_{to} = \bigcup_{i=1}^{m} E^i_{to}$, kde $E^i_{to} = \{s;\ s = h_i(t) < t_o,\ t \in J\} \cup \{t_o\}$.

Funkcia $u(t)$ je riešením začiatočnej úlohy /1/, /2/, na intervale J, ak platí:

$$u''(t) + \sum_{i=1}^{m} a_i(t)\, u(h_i(t)) = 0,$$

$u(t_0) = x_0$, $u'(t_0^+) = x_0'$ a $u(h_i(t)) = \varphi(h_i(t))$ pre $h_i(t) < t_0$, $i = 1, 2, \dots m$/.

Pri vyšetrovaní riešení diferenciálnej rovnice /1/ budeme vychádzať z vlastností riešení diferenciálnej rovnice

/3/ $\quad x''(t) + a_0(t)\, x(t) = 0.$

D e f i n í c i a . Nech $x(t)$ je definovaná a po čiastkách spojitá funkcia na intervale J. Pre každé $t \in J$ definujeme funkciu $X(t)$ predpisom

$$X(t) = \begin{cases} x(t_0) & \text{pre } t = t_0 \\ \sup\limits_{s \in \langle t_0, t\rangle} x(s) & \text{pre } t > t_0 . \end{cases}$$

Funkcia $X(t)$ je neklesajúca. Toto tvrdenie vyplýva priamo z definície $X(t)$.

L e m a 1. Ak $x(t)$ je neklesajúca funkcia na intervale J, potom $X(t) = x(t)$ pre každé $t \in J$.

D ô k a z . Pre každé také $s, t \in J$, že $t_0 \leq s \leq t$, je $x(t_0) \leq x(s) \leq x(t)$. Potom $\sup\limits_{s \in \langle t_0, t\rangle} x(s) = x(t)$, čím je tvrdenie lemy 1 dokázané.

L e m a 2 . Ak $x(t)$ je nerastúca funkcia na intervale J, potom $X(t) = x(t_0)$ pre každé $t \in J$.

D ô k a z . Pre každé $t \in J$ platí $x(t) \leq x(t_0)$. Potom $\sup\limits_{s \in \langle t_0, t\rangle} x(s) = x(t_0)$ a tvrdenie lemy 2 je dokázané.

L e m a 3 . Ak $x(t) \in C(J)$, potom aj $X(t) \in C(J)$.

D ô k a z . Predpokladajme, že funkcia $X(t)$ je nespojitá v čísle $t_1 \in (t_0, \infty)$. Označme $A = X(t_1) = \sup\limits_{s \in \langle t_0, t_1\rangle} x(s)$.

Ak $x(t_1) < A$, potom existuje také okolie $O(t_1)$ čísla t_1, že pre všetky $t \in O(t_1)$ je $x(t) < A$, a preto $X(t) = A$ pre $t \in O(t_1)$. Znamená to však, že $X(t)$ je spojitá v čísle t_1, čo je v spore s predpokladom.

Nech $x(t_1) = A$. Vypočítame $\lim\limits_{t \to t_1^-} X(t)$ a $\lim\limits_{t \to t_1^+} X(t)$.

Pre každé $t \leq t_1$ $(t > t_0)$ platí $x(t) \leq X(t) \leq X(t_1) = A$, a preto $\lim\limits_{t \to t_1^-} X(t) = A$.

Nech $\lim\limits_{t \to t_1^+} X(t) = Y > A$. Ku každému $\varepsilon > 0$ existuje také $\delta > 0$, že pre všetky $t \in \langle t_1, t_1 + \delta \rangle |x(t) - A| \leq \varepsilon$. Ak $B = \max\limits_{t \in \langle t_1, t_1 + \delta' \rangle} x(t)$ potom $A < Y \leq B$. Zo spojitosti $x(t)$ a poslednej nerovnosti vyplýva, že existuje také $\bar{t} \in (t_1, t_1 + \delta\rangle$, pre ktoré $x(\bar{t}) = Y$. Ak zvolíme $\varepsilon = \frac{Y - A}{2}$, dostaneme

$$Y - A = |x(\bar{t}) - A| \leq \varepsilon = \frac{Y - A}{2},$$

čo nie je možné. Preto $\lim\limits_{t \to t_1^+} X(t) = A$.

Dokázaná spojitosť funkcie $X(t)$ v čísle t_1 je v spore s predpokladom. Tým je lema 3 dokázaná.

L e m a 4. Nech po čiastkach spojitá funkcia $x(t)$ definovaná na intervale J je v každom bode nespojitosti spojitá sprava alebo zľava /prípadne s výnimkou čísla t_0/. Nech pre každé $\bar{t} \in J$ je $\lim\limits_{t \to \bar{t}^-} X(t) \geq \lim\limits_{t \to \bar{t}^+} x(t)$. Potom funkcia $X(t)$ je na intervale J spojitá.

D ô k a z. Stačí dokázať spojitosť funkcie $X(t)$ v bodoch nespojitosti funkcie $x(t)$.

Nech t_1 je bodom nespojitosti funkcie $x(t)$ a $\lim_{t \to t_1^-} x(t) = x(t_1)$

Ak $X(t_1) > x(t_1)$ a $\lim_{t \to t_1^+} x(t) < X(t_1)$, potom existujú také čísla $\delta_1 > 0$, $\delta_2 > 0$, že pre každé $(t \in t_1-\delta_1, t_1+\delta_2$ $X(t) = X(t_1)$, z čoho plynie spojitosť funkcie $X(t)$ v čísle t_1.

Ak $X(t_1) > x(t_1)$ a $\lim_{t \to t_1^+} x(t) = X(t_1)$, potom existuje $\delta > 0$ také, že $X(t) = X(t_1)$ pre $t \in (t_1 - \delta, t_1>$. Spojitosť sprava vyplýva zo spojitosti funkcie $x(t)$ na istom intervale $(t_1, t_1 + \delta)$ $\delta > 0$ a podmienky, že $\lim_{t \to t_1^+} x(t) = X(t_1)$.

Ak $X(t_1) = x(t_1)$ a $\lim_{t \to t_1^+} x(t) < X(t_1)$, potom $X(t) = X(t_1)$ pre $t \in <t_1, t_1 + \delta_2)$. Pre každé $t \in (t_1 - \delta_1, t_1>$ je $x(t) \leq X(t) \leq X(t_1) = x(t_1)$, takže $\lim_{t \to t_1^-} X(t) = X(t_1)$. Tým je dokázané, že funkcia $X(t)$ je v čísle t_1 spojitá.

V prípade, že $\lim_{t \to t_1^+} x(t) = x(t_1)$, dôkaz o spojitosti $X(t)$ urobíme analogickým spôsobom ako v predchádzajúcej časti. V čísle t_o je funkcia $X(t)$ spojitá sprava. Toto tvrdenie vyplýva z predchádzajúcej časti dôkazu, pretože $X(t_o) \geq \lim_{t \to t_o^+} x(t)$.

L e m a 5. Nech $x(t) \in C(J)$, $h(t) \in C(J)$ a nech pre každé $t \in J$ je $x(t) \geq 0$, $h(t) \leq t$ a

$$x_h(t) = \begin{cases} x(h(t)), & \text{ak } h(t) \geq t_o \\ 0, & \text{ak } h(t) < t_o \end{cases}$$

Potom funkcia $X_h(t) = \sup_{s \in \langle t_o, t \rangle} x_h(s)$ má najviac jeden bod nespojitosti.

D ô k a z . Najprv lemu dokážeme za predpokladu, že $h(t_o) = t_o$.

Ak $h(t) \geq t_o$ $(h(t) \leq t_o)$ na intervale $\langle t_o, T \rangle$ $(T \leq \infty)$, potom $x_h(t) = x(h(t))$ $(X_h(t) = x(t_o))$, z čoho vyplýva spojitosť funkcie $X_h(t)$ na intervale $\langle t_o, T \rangle$.

Nech $\bar{t} > t_o$ je ľubovoľný izolovaný bod nespojitosti funkcie $x_h(t)$ a nech nenastane predchádzajúci prípad. Potom existujú také kladné čísla $\varepsilon_1, \varepsilon_2$, že pre každé $t \in (\bar{t} - \varepsilon_1, \bar{t})$ je $h(t) < t_o$ $[h(t) > t_o]$ a pre všetky $t \in (\bar{t}, \bar{t} + \varepsilon_2)$ je $h(t) > t_o$ $[h(t) < t_o]$. Funkcia $x_h(t)$ je v čísle $\bar{t}$ spojitá sprava [zľava] a $\lim_{t \to \bar{t}+} x_h(t) =$

$$= x(h(\bar{t})) = x(t_o) = X_h(t_o) \leq \lim_{t \to \bar{t}-} X_h(t)$$

$$\left[\lim_{t \to \bar{t}+} x_h(t) = 0 \leq x(t_o) = X_h(t_o) \leq \lim_{t \to \bar{t}-} X_h(t)\right]$$

Potom podľa lemy 4 je funkcia $X_h(t)$ spojitá v čísle $\bar{t}$.

Nech $\bar{t} > t_o$ je hromadný bod bodov nespojitosti funkcie $x_h(t)$. Predpokladajme, že $X_h(\bar{t}) > x(t_o)$. Označme $\max_{t \in \langle t_o, \bar{t} \rangle} h(t) = H$ a $\max_{u \in \langle t_o, H \rangle} x(u) = L$. Funkcie $x(u)$ a $h(t)$ sú spojité, a preto existujú také čísla $u_1 \in \langle t_o, H \rangle$, $t_1 \in \langle t_o, \bar{t})$, pre ktoré $x(u_1) = x(h(t_1)) = L$. Potom na intervale $\langle t_1, \bar{t} \rangle$ je $X_h(t) = L$. K číslu $\varepsilon = u_1 - t_o$ existuje také číslo $\delta > 0$, že pre všetky $t \in \langle \bar{t}, \bar{t} + \delta)$ $|h(t) - h(\bar{t})| = |h(t) - t_o| < \varepsilon = u_1 - t_o$, odkiaľ vyplýva, že $h(t) < u_1$. Preto $X_h(t) = L$ pre $t \in \langle \bar{t}, \bar{t} + \delta)$. Tým sme dokázali, že funkcia $X_h(\bar{t})$ je spojitá v čísle $\bar{t}$.

Ak $X_h(\bar{t}) = x(t_o)$, potom je $X_h(t) = x(t_o)$ pre $t \in \langle t_o, \bar{t} \rangle$ a stačí dokázať, že $\lim_{t \to \bar{t}+} X_h(t) = x(t_o)$. Zo spojitosti funkcií $x(u)$ a $h(t)$ vyplýva: Ku každému číslu $\mathcal{E} > 0$, existuje také číslo $\delta > 0$, že pre všetky $u \in \langle t_o, t_o + \delta)$ je $|x(u) - x(t_o)| < \mathcal{E}$ a pre každé $\delta > 0$ existuje také číslo $\eta > 0$, že pre všetky $t \in \langle \bar{t}, \bar{t} + \eta)$ je $|h(t) - h(\bar{t})| < \delta$. Označme $\sup_{u \in \langle t_o, t_o + \delta)} x(u) = L_1$. Potom pre každé $t \in \langle \bar{t}, \bar{t} + \eta)$

$$|X_h(t) - X_h(\bar{t})| = |X_h(t) - x(t_o)| \leq |L_1 - x(t_o)| \leq \mathcal{E}$$

Z poslednej nerovnosti vyplýva, že $\lim_{t \to \bar{t}+} X_h(t) = x(t_o)$, čo bolo treba dokázať.

Predpokladajme teraz, že $h(t_o) < t_o$. Nech $h(t_1) = t_o$ a $(h)t < t_o$ pre $t \in \langle t_o, t_1)$. Z definície funkcie $X_h(t)$ vyplýva, že $X_h(t) = 0$ pre $t \in \langle t_o, t_1)$ a $X_h(t_1) = x(t_o)$. Spojitosť funkcie $X_h(t)$ na intervale $\langle t_1, \infty)$ dokážeme podobne ako v predchádzajúcom odseku. Potom $\lim_{t \to t_1^+} X_h(t) = x(t_o)$ a $\lim_{t \to t_1^-} X_h(t) = 0$. Ak $x(t_o) \neq 0$ je funkcia $X_h(t)$ nespojitá v čísle t_1. Ak $x(t_o) = 0$, je $X_h(t)$ spojitá v čísle t_1.

Tým sme dokázali tvrdenie lemy.

<u>L e m a 6.</u> Nech $x(t) \in C(J)$, $b_i(t) \in C(J)$, $h_i(t) \in C(J)$, $i = 1, 2, \dots m$. Nech pre každé $t \in J$ $x(t) \geq 0$, $b_i(t) \geq 0$, $h_i(t) \leq t$, $i = 1, 2, \dots m$ a

$$/4/ \qquad x(t) \leq C + \int_{t_o}^{t} \sum_{i=1}^{m} b_i(s)\, x_{hi}(s)\, ds ,$$

kde C je nezáporné číslo. Potom platí

/5/ $$x(t) \leq C \exp \int_{t_o}^{t} \sum_{i=1}^{m} b_i(s)\, ds$$

pre každé $t \in J$.

D ô k a z . Z vlastností funkcií $x_{hi}(t)$ a $X(t)$ a predpokladov lemy vyplýva, že $x_{hi}(t) \leq X_{hi}(t) \leq X(t)$ pre $t \in J$. Funkcia na pravej strane vzťahu /4/ je neklesajúca, a preto

$$\sup_{s \in \langle t_o, t \rangle} \left[C + \int_{t_o}^{s} \sum_{i=1}^{m} b_i(u)\, x_{hi}(u)\, du \right] =$$

$$= C + \int_{t_o}^{t} \sum_{i=1}^{m} b_i(s)\, x_{hi}(s)\, ds.$$

Potom platí

$$X(t) = \sup_{s \in \langle t_o, t \rangle} x(s) \leq C + \int_{t_o}^{t} \sum_{i=1}^{m} b_i(s)\, x_{hi}(s) ds \leq$$

$$\leq C + \int_{t_o}^{t} \sum_{i=1}^{m} b_i(s)\, X_{hi}(s)\, ds \leq$$

$$\leq C + \int_{t_o}^{t} \sum_{i=1}^{m} b_i(s)\, X(s)\, ds.$$

Ak použijeme Bellmanovu lemu, dostaneme

$$x(t) \leqq X(t) \leqq C \exp \int_{t_o}^{t} \sum_{i=1}^{m} b_i(s)\,ds .$$

č. b. t. d.

V e t a 1. Ak všetky riešenia diferenciálnej rovnice /3/, ich prvé derivácie sú ohraničené na intervale J a $\int_{t_o}^{\infty} \sum_{i=1}^{m} |a_i(t)|\,dt < \infty$, potom všetky riešenia diferenciálnej rovnice /1/ spĺňajúce počiatočné podmienky /2/ sú aj so svojimi prvými deriváciami ohraničené na intervale J.

D ô k a z . Nech $u_1(t)$, $u_2(t)$ je taký fundamentálny systém riešení diferenciálnej rovnice /3/, pre ktorý wronskián $W(u_1, u_2) = 1$. Každé riešenie rovnice /1/ môžeme napísať v tvare

$$/6/ \quad x(t) = c_1 u_1(t) + c_2 u_2(t) + \int_{t_o}^{t} \left[u_1(t) u_2(s) - u_2(t) u_1(s) \right] \cdot \sum_{i=1}^{m} a_i(s)\, x\big(h_i(s)\big)\, ds,$$

kde konštanty c_1, c_2 sú určené začiatočnými podmienkami /2/ a funkciami $u_1(t)$, $u_2(t)$; $x\big(h_i(t)\big) = \varphi\big(h_i(t)\big)$, $i = 1, 2, \dots m$, ak $h_i(t) < t_o$.

Označme

$$x_{hi}(t) = \begin{cases} x\big(h_i(t)\big), & \text{ak } t_o \leqq h_i(t), \quad t \in J \\ 0, & \text{ak } h_i(t) < t_o, \quad t \in J \end{cases}$$

a

$$\varphi_{hi}(t) = \begin{cases} 0 & , \text{ak } t_0 \leqq h_i(t),\ t \in J \\ \varphi(h_i(t)) & , \text{ak } h_i(t) < t_0,\ t \in J. \end{cases}$$

Potom pre rovnicu /6/ dostaneme vyjadrenie

$$/7/ \quad x(t) = c_1 u_1(t) + c_2 u_2(t) + \int_{t_0}^{t} \left[u_1(t)\, u_2(s) - \right.$$

$$\left. - u_2(t)\, u_1(s) \right] . \sum_{i=1}^{m} a_i(s) \left[x_{hi}(s) + \varphi_{hi}(s) \right] ds.$$

Ak uvážime, že funkcie $u_1(t)$, $u_2(t)$, $\varphi(t)$ sú ohraničené, tak zo /7/ plynie

$$/8/ \quad |x\ t\ | \leqq K_1 + K_2 \int_{t_0}^{t} \sum_{i=1}^{m} |\ a_i(s)|\, |x_{hi}(s)|\, ds,$$

kde $|c_1 u_1(t) + c_2 u_2(t)| + \int_{t_0}^{t} |u_1(t)\ u_2(s) -$

$$- u_2(t)\ u_1(s)| . \sum_{i=1}^{m} a_i(s)|\,|\varphi_{hi}(s)|\, ds \leqq K_1$$

a

$$|u_1(t)\ u_2(s) - u_2(t)\ u_1(s)| \leqq K_2$$

pre každé t, $s \in J$. Podľa lemy 6 z /8/ vyplýva

$$/9/ \quad |x(t)| \leqq K_1 e^{K_2 \int_{t_o}^{t} \sum_{i=1}^{m} |a_i(s)|\, ds} < K,$$

čím je dokázané, že riešenie $x(t)$ je ohraničené na intervale J.

Treba ešte dokázať, že funkcia $x'(t)$ je ohraničená na intervale J. Zo /7/ vyplýva

$$x'(t) = c_1 u_1'(t) + c_2 u_2'(t) + \int_{t_o}^{t} \left[u_1'(t)\, u_2(s) - \right.$$

$$\left. - u_2'(t)\, u_1(s) \right] \cdot \sum_{i=1}^{m} a_i(s) \left[x_{hi}(s) + \varphi_{hi}(s) \right] ds,$$

odkiaľ

$$|x'(t)| \leqq A_1 + A_2 \int_{t_o}^{t} \sum_{i=1}^{m} |a_i(s)|\, ds \leqq C,$$

kde $|c_1 u_1'(t) + c_2 u_2'(t)| = A_1$

$$|u_1'(t)\, u_2(s) - u_2'(t)\, u_1(s)| \cdot M \leqq A_2$$

pre každé $t, s \in J$ a $M = \max \left\{ K; \sup_{t \in E_{t_o}} \varphi(t) \right\}$.

Tým je veta 1 dokázaná.

Podobne dokážeme nasledujúce tvrdenie.

T v r d e n i e : 1. Ak všetky riešenia diferenciálnej rovnice /3/ sú ohraničené a $a_i(t) = a_i$, $i = 1, 2, \dots m$, potom pre riešenie $x(t)$ diferenciálnej rovnice /1/ spĺňajúcej začiatočné podmienky /2/ platí

$$|x(t)| \leqq K_1 e^{K_2 \sum_{i=1}^{m} |a_i| (t - t_o)}$$

pre každé $t \in J$.

D ô k a z . Pre riešenie $x(t)$ rovnice (1), spĺňajúce začiatočné podmienky (2) sme v dôkaze vety 1 dostali vzťah /7/

$$x(t) = c_1 u_1(t) + c_2 u_2(t) + \int_{t_o}^{t} \left[u_1(t) u_2(s) - u_2(t) u_1(s) \right] \cdot \sum_{i=1}^{m} a_i(s) \left[x_{hi}(s) + \varphi_{hi}(s) \right] ds.$$

Pretože $u_1(t)$, $u_2(t)$ sú ohraničené funkcie v intervale J, zo /7/ plynie

$$/10/ \quad |x(t)| \leqq K_1^* + K_2 \int_{t_o}^{t} \sum_{i=1}^{m} |a_i(s)| \left[|x_{hi}(s)| + |\varphi_{hi}(s)| \right] ds,$$

kde $|c_1 u_1(t) + c_2 u_2(t)| \leqq K_1^*$ a $|u_1(t) u_2(s) - u_2(t) u_1(s)| \leqq K_2$

pre každé $t, s \in J$. Ďalej platí

$$/11/ \quad |x(t)| \leqq \sup_{s \in \langle t_o, t \rangle} |x(s)| = X(t) = K_1^* + K_2 \int_{t_o}^{t} \sum_{i=1}^{m} |a_i(s)|$$

$$[|x_{hi}(s)|+|\varphi_{hi}(s)|]ds \leq K_1^* + K_2 \int_{t_o}^{t} \sum_{i=1}^{m} |a_i(s)|[X(s) + \varphi_o]ds,$$

kde $\varphi_o = \sup_{t \in E_{t_o}} |\varphi(t)|$. Jednoduchou úpravou z /11/ dostaneme

$$X(t) + \varphi_o \leq K_1 + K_2 \int_{t_o}^{t} \sum_{i=1}^{m} |a_i(s)| [X(s) + \varphi_o] ds,$$

odkiaľ podľa Bellmanovej lemy

$$|x(t)| \leq X(t) + \varphi_o \leq K_1 e^{K_2 \int_{t_o}^{t} \sum_{i=1}^{m} |a_i(s)| ds} \leq$$

$$= K_1 e^{K_2 \sum_{i=1}^{m} |a_i| (t - t_o)},$$

kde $K_1 = K_1^* + \varphi_o$.

Analogicky sa dokážu tvrdenie 2 a tvrdenie 3.

<u>T v r d e n i e 2 .</u> Ak všetky riešenia diferenciálnej rovnice /3/ sú ohraničené, $a_i(t) = \frac{a_i}{t}$, $i = 1, 2, \ldots m$, a $t_o > 0$, potom pre riešenie $x(t)$ diferenciálnej rovnice /1/ spĺňajúcej začiatočné podmienky /2/ platí

$$|x(t)| \leq K_1 \left(\frac{t}{t_o}\right)^{K_2 \sum_{i=1}^{m} |a_i|}$$

pre každé $t \in J$.

T v r d e n i e 3. Ak všetky riešenia diferenciálnej rovnice /3/ sú ohraničené,

$$a_i(t) = \frac{a_i}{t^{\alpha}}, \quad i = 1, 2, \dots m, \alpha > 1 \text{ a } t_0 > 0, \quad \text{potom}$$

pre riešenie $x(t)$ diferenciálnej rovnice /1/, spĺňajúcej počiatočné podmienky /2/ platí

$$|x(t)| \leq K_1 e^{\frac{K_2}{1-\alpha} \sum_{i=1}^{m} |a_i| \left(\frac{1}{t^{\alpha-1}} - \frac{1}{t_0^{\alpha-1}} \right)}$$

pre každé $t \in J$.

P o z n á m k a. Predpoklady vety 1 sú postačujúce podmienky ohraničenosti riešenia lineárnej nehomogénnej diferenciálnej rovnice druhého rádu, ak $h_i(t) < t_0$, $i = 1, 2, \dots m$, pre každé $t \in J$ a $\varphi(t) \not\equiv 0$ pre $t \in E_{t_0}$.

Ak $h_i(t) \equiv t$, $i = 1, 2, \dots m$, pre každé $t \in J$, potom predpoklady vety 1 sú postačujúce podmienky ohraničenosti riešení lineárnej homogénnej diferenciálnej rovnice druhého rádu.

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On the Boundedness of the Solutions of the Linear Second - Order Delay Differential Equation

FRANTIŠEK ŠIŠOLÁK

S u m m a r y

In this paper the sufficient conditions of boundedness of solutions of the differential equation

$$/1/ \quad x''(t) + \sum_{i=1}^{m} a_i(t)\, x\left(h_i(t)\right) = 0$$

are investigated, provided that all solutions of the differential equation

$$/3/ \quad x''(t) + a_0(t)\, x(t) = 0$$

are bounded in the interval $I = \langle t_0, \infty)$.

In the introductory part of the paper some propeties of the function

$$X(t) = \begin{cases} x(t_0) & \text{for } t = t_0 \\ \sup\limits_{s \in \langle t_0, t\rangle} x(s) & \text{for all } t > t_0 \end{cases}$$

are investigated.

Došlo: 5. októbra 1972,
do nakladateľstva 8. júla 1974.

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О ограничености решений линейного дифференйиального уровнения второго порядка с запаздывающим аргументом

ФРАНТИШЕК ШИШОЛАК

Р е з ю м е

В статьи изучаются достаточные условия ограничености решений дифференциального уравнения

(1) $x''(t) + \sum_{i=0}^{m} a_i(t)\, x(h_i(t)) = 0$

Предполагается, что все решения дифференциального уравнения

(3) $x''(t) + a_0(t)\, x(t) = 0$

ограниченны в интервале $J = \langle t_0, \infty)$.

В начале статьи расматриваются некоторые свойства функции

$$X(t) = \begin{cases} x(t_0) & \text{для} \quad t = t_0 \\ \sup\limits_{s \in \langle t_0, t \rangle} x(s) & \text{для} \quad t > t_0 \end{cases}$$

(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXXII, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXXII - 1975

On some problems in the elementary theory of numbers

LÁSZLÓ VOJTECH, Nitra

In this paper we are going to deal with some problems in the elementary theory of numbers. Consequently the paper can be divided into four sections.

In the first section we are going to give the generalization of a problem contained in the Amer. Math. Monthly. In the second section a certain property of Legendre's symbol will be dealt with. In the third section we will show that 52 is not a member of the progression $\{\sigma(n)-n\}_{n=1}^{\infty}$. In the last section we shall formulate several problems in connection with primitive λ roots (mod m).

1

In [1] the following problem has been given for solving: Prove that for every natural number $n > 2$ the inequality

$$\varphi(n^2) + \varphi(n^2 + 2n + 1) < 2n^2 \quad \text{holds,}$$

where φ denotes Euler's function.

We are going to examine a more general problem and prove the following two theorems.

Theorem 1,1. Let n, k be natural numbers. Then for every even $n \geqq 4(k-1)$ and for every odd $n \geqq 2(k-1)$ the inequality $\varphi(n^k) + \varphi[(n+1)^k] < 2n^k$ holds.

•

Theorem 1,2. If n,k are natural numbers and $n > k\ (k+1)$, then

$$\sum_{i=0}^{k} \varphi[(n+i)^2] < (k+1)\ n^2 .$$

It is evident that the statement formulated in the version of the original problem is a species case of theorem 1,2. The proof of the theorem 1,1 we are going to build on the following lemmas.

Lemma 1,1. For all natural numbers $k > 1$, $m > 1$ the inequality $\varphi(m^k) \leqq m^{k-1}\ (m-1)$ holds.

Lemma 1,2. Let k be a natural and m an even natural number. Then

$$\varphi(m^k) \leqq \frac{1}{2}\ m^k .$$

The lemmas 1,1 and 1,2 follow easily from the basic properties of Euler's function.

Lemma 1,3. Let n,h be natural numbers and $n \geqq 2h$, then

$$(n+1)^h < 2n^h .$$

Proof. By the virtue of binomial theorem

$$(n+1)^h = n^h + \sum_{i=1}^{h} \binom{h}{i}\ n^{h-i} .$$

Since for every $i \geqq 1$ we have $\binom{h}{i} \leqq \frac{h^i}{2^{i-1}}$, hence we obtain for $n \geqq 2h$ the inequality

$$(n+1)^h \leqq n^h + \sum_{i=1}^{h} \frac{\left(\frac{n}{2}\right)^i\ n^{h-i}}{2^{i-1}} = n^h + n^h \sum_{i=1}^{h} \frac{1}{2^{2i-1}} < 2n^h .$$

Lemma 1,4. If n,h are natural numbers and $n \geqq 4h$, then then

$$(n+1)^h < \frac{3}{2}\ n^h .$$

P r o o f . From the inequality

$$(n + 1)^h \leqq n^h + \sum_{i=1}^{h} \frac{h^i \, n^{h-i}}{2^{i-1}},$$

on the basis of the assumption of the lemma we obtain

$$(n + 1) \leqq n^h + \frac{1}{2} n^h \sum_{i=1}^{h} \frac{1}{2^{3i-2}} < \frac{3}{2} n^h .$$

P r o o f o f T h e o r e m 1,1. Let us first suppose that n is an even natural number and $n \geqq 4(k - 1)$, where $k \geqq 1$ is a natural number. The by virtue of lemman 1,1; 1,2 and 1,4 we obtain

$$\varphi(n^k) + \varphi[(n + 1)^k] \leqq \frac{1}{2} n^k + (n + 1)^{k-1} . n < 2n^k .$$

Let now n be an odd number, $n \geqq 2(k - 1)$ and let k be an arbitrary natural number. Then by means of lemmas 1,1; 1,2 and 1,3 we have

$$\varphi(n^k) + \varphi[(n + 1)^k] \leqq n^k - n^{k-1} + \frac{1}{2}(n + 1)^{k-1} . (n + 1) < 2n^k.$$

P r o o f o f t h e o r e m 1,2. Let us put

$$S(\alpha, \beta) = \sum_{i=0}^{k} \varphi[(n + i)^2] ,$$

where α (β) equals 0 when n (k) is even and equals 1 when n (k) is odd. First we examine the case $S(0,0)$. From Lemmas 1,1 and 1,2 it follows that

$$S(0,0) \leqq \frac{1}{2} n^2 + (n + 1) n + \frac{1}{2} (n + 2)^2 + \ldots + \frac{1}{2} (n + k)^2 .$$

If we carry out the operations on the right side of this inequality and arrange the summands according to the powers of number n, we obtain

$$S(0,0) \leqq \left[\frac{1}{2} \left(\frac{k}{2} + 1 \right) + \frac{k}{2} \right] n^2 + [(1 + 5 + 9 + \dots$$

$$+ (2k - 3) + (2 + 4 + 6 + \dots + k)] \; n + \left(\frac{2^2}{2} + \frac{4^2}{2} + \dots \right.$$

$$\left. + \frac{k^2}{2} \right) + 3 \cdot 2 + 5 \cdot 4 + \dots + (k - 1)(k - 2) .$$

It is easy to see that the coefficient of n^2 is $\frac{3k + 2}{4}$ and that of n is $\frac{3k^2}{4}$. Let us denote the absolute member with the symbol A. Clearly

$$A = \frac{2^2}{2} \left[1^2 + 2^2 + \dots + \left(\frac{k}{2} \right)^2 \right] + [(1 + 2)\, 2 + (1 + 4)\, 4 + \dots$$

$$\dots (1 + k - 2)(k - 2)] .$$

Whence

$$A = 2 \left[1^2 + 2^2 + \dots + \left(\frac{k}{2} \right)^2 \right] + 2^2 \left[1^2 + 2^2 + \dots \left(\frac{k - 2}{2} \right)^2 \right] +$$

$$+ [2 + 4 + \dots + (k - 2)] .$$

Now we use the formula $\sum_{i=1}^{m} i^2 = \frac{1}{6} m (m + 1)(2m + 1)$, and so we obtain

$$A = \frac{1}{12} k (k + 1)(k + 2) + \frac{1}{6} k (k - 2)(k - 1) + \frac{1}{4} k(k - 2) = \frac{k^3}{4}$$

and thus we arrive at the inequality

$$S(0,0) \leqq \frac{3k + 2}{4} n^2 + \frac{3k^2}{4} n + \frac{k^3}{4} .$$

Similarly we obtain the following inequalities:

$$S(0,1) \leqq \frac{3k + 3}{4} n^2 + \frac{3k^2 + 2k - 1}{4} n + \frac{k^3 + k^2 - k - 1}{4}$$

$$S(1,0) = \frac{3k+4}{4} n^2 + \frac{3k^2+2k-4}{4} n + \frac{k^3+k^2-k}{4}$$

$$S(1,1) = \frac{3k+3}{4} n^2 + \frac{3k^2-3}{4} n + \frac{k^3+1}{4} .$$

None of the sums

$$S(0,0) + \left(\frac{2}{4} n^2 + \frac{2k-4}{4} n + \frac{k^2-k}{4} \right)$$

$$S(0,1) + \left(\frac{1}{4} n^2 - \frac{3}{4} n + \frac{1}{4} \right) \qquad (1)$$

$$S(1,0) + 0$$

$$S(1,1) + \left(\frac{1}{4} n^2 + \frac{2k-1}{4} n + \frac{k^2-k-1}{4} \right)$$

is greater than

$$\frac{3k+4}{4} n^2 + \frac{3k^2+2k-1}{4} n + \frac{k^3+k^2-k}{4} .$$

Since $n > k(k+1)$, $k \geqq 1$, the second summands in (1) are nonnegative numbers and thus for each pair (α, β) of numbers 0,1 we obtain

$$S(\alpha,\beta) \leqq \frac{3k+4}{4} n^2 + \frac{3k^2+2k-4}{4} n + \frac{k^3+k^2-k}{4} \qquad (2)$$

Now it is sufficient to prove that under the assumption of $n > k\ (k+1)$ the right-hand side of the last inequality is smaller that $(k+1)\ n^2$. Evidently

$$n \geqq k^2 + k + 1, \qquad (3)$$

and it is easily seen that

$$kn - (3k^2 + 2k - 4) \geqq k^3 - 2k^2 - k + 4 > 0 \qquad (4)$$

From the relationships (3) and (4) it follows

$$n\left[kn - (3k^2 + 2k - 4)\right] \geqq (k^2 + k + 1)(k^3 - 2k - k + 4) \qquad (5)$$

Evidently the inequality

$$(k^2 + k + 1)(k^3 - 2k^2 - k + 4) > k^3 + k^2 - k \qquad (6)$$

is true, since (6) is equivalent to the inequality

$$k^3[k(k-1) - 3] + 4(k+1) > 0,$$

whose validity can be easily seen for every natural $k \geq 1$. From relations (5) and (6) it follows

$$\frac{k}{4} n^2 > \frac{3k^2 + 2k - 4}{4} n + \frac{k^3 + k^2 - k}{4},$$

adding to both sides of the last inequality the number $\frac{3k+4}{4} n^2$, we find that the right - hand side of the inequality (2) is smaller than $(k + 1) n^2$.

2

It is known that for each odd prime p and every natural number a, (a, p) = 1 the equality

$$\sum_{x=1}^{p} \left(\frac{x}{p}\right) \left(\frac{x+a}{p}\right) = -1 \qquad (1)$$

holds, where $\left(\frac{x}{p}\right)$ denotes Legendre's symbol (see [4] p. 78). In the relation (1) the symbol $\left(\frac{x}{p}\right)$ is defined also for $p \mid x$ and in this case we put $\left(\frac{x}{p}\right) = 0$. The next theorem given several well-known fundamental properties of Legendre's symbol which we shall need later on.

Theorem 2.1. Let p be an odd prime and let a, b denote itegers relatively prime to p. Then

(a) $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod p$,

(b) $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$,

(c) $a \equiv b \pmod p$ implies that $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$,

(d) $\left(\frac{a^2}{p}\right) = 1, \left(\frac{1}{p}\right) = 1, \left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$.

P r o o f . See [3] p. 70.

In the following we are going to examine the sums of the form

$$S(a,p) = \sum_{x=1}^{p} \left(\frac{x}{p}\right)\left(\frac{x+a}{p}\right)\left(\frac{x+2a}{p}\right) ,$$

where p is an odd prime, a, p = 1.

L e m m a 2,1. Let p be an odd prime and let a be a natural number, (a,p) = 1. Then

$$S(a,p) = \left(\frac{a}{p}\right) S(1,p) .$$

P r o o f . It holds evidently

$$S(a,p) = \sum_{x=1+a}^{p+a} \left(\frac{x-a}{p}\right)\left(\frac{x}{p}\right)\left(\frac{x+a}{p}\right) = \sum_{x=1+a}^{p+a} \left(\frac{x}{p}\right)\left(\frac{x^2-a^2}{p}\right). \quad (2)$$

If $x \equiv y \pmod p$, then it holds on the basis of (c) that

$$\left(\frac{x}{p}\right)\left(\frac{x^2-a^2}{p}\right) = \left(\frac{y}{p}\right)\left(\frac{y^2-a^2}{p}\right). \quad (3)$$

Since $\{1+a, 2+a, \ldots, p+a\}$ and $\{1,2, \ldots, p\}$ are complete residue systems (mod p), it follows from (2) and (3) that

$$S(a,p) = \sum_{x=1}^{p} \left(\frac{x}{p}\right)\left(\frac{x^2-a^2}{p}\right). \quad (4)$$

From the fact that $\{a, 2a, \ldots, pa\}$ is a complete residue system (mod p) and in view of properties (c) nad (b) of Legendre´s symbol, from the relation (4) we obtain

$$S(a,p) = \sum_{x=1}^{p} \left(\frac{xa}{p}\right)\left(\frac{x^2a^2-a^2}{p}\right) = \sum_{x=1}^{p} \left(\frac{x}{p}\right)\left(\frac{a}{p}\right)\left(\frac{x^2-1}{p}\right)\left(\frac{a^2}{p}\right)$$

Whence on the basis of (d) we obtain

$$S(a,p) = \left(\frac{a}{p}\right) \sum_{x=1}^{p} \left(\frac{x}{p}\right)\left(\frac{x^2-1}{p}\right) = \left(\frac{a}{p}\right) S(1,p),$$

since by (4)

$$S(1,p) = \sum_{x=1}^{p} \left(\frac{x}{p}\right)\left(\frac{x^2-1}{p}\right).$$

In the following we are going to examine the sum $S(1,p)$, to which the following theorem relates.

Theorem 2,2. Let p be a prime number. Then

$$S(1,p) \equiv (-1)^{\frac{p+3}{4}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}} \pmod p, \text{ if } p \equiv 1 \pmod 4$$

and

$$S(1,p) = 0, \text{ if } p \equiv 3 \pmod 4.$$

In proving the first part of theorem 2,2 we shall proceed in a way similar to [4] p.95 exerciese 6β, and use also the following known lemma (see [4] p.95).

Lemma 2,2. Let n be a natural number and $S_n = 1^n + 2^n + \dots + (p-1)^n$, where p is a prime number. Then

$$S_n \equiv -1 \pmod p, \text{ if } (p-1) \mid n$$

and

$$S_n \equiv 0 \pmod p, \text{ if } (p-1) \nmid n.$$

Proof of theorem 2,2.

Let us first assume that $p \equiv 1 \pmod 4$. Then by (a) we obtain

$$S(1,p) \equiv \sum_{x=1}^{p-1} x^{\frac{p-1}{2}} (x^2-1)^{\frac{p-1}{2}} \pmod p.$$

By the binomial theorem it follows from the above congruence that

$$S(1,p) \equiv \sum_{x=1}^{p-1} \Big[x^{\frac{3}{2}(p-1)} - \binom{\frac{p-1}{2}}{1} x^{\frac{3}{2}(p-1)-2} + \binom{\frac{p-1}{2}}{2} x^{\frac{3}{2}\,p-1\,-4} -$$

$$- \dots + x^{\frac{p-1}{2}} \Big] (\bmod\ p),$$

whence by a simple modification we obtain

$$S(1,p) \equiv \sum_{x=1}^{p-1} x^{\frac{3}{2}(p-1)} - \binom{\frac{p-1}{2}}{1} \sum_{x=1}^{p-1} x^{\frac{3}{2}(p-1)-2} + \binom{\frac{p-1}{2}}{2} \sum_{x=1}^{p-1} x^{\frac{3}{2}(p-1)}$$

$$\dots + \sum_{x=1}^{p-1} x^{\frac{p-1}{2}} \ (\bmod\ p).$$

Since among of the exponents $\frac{3}{2}(p-1)$, $\frac{3}{2}(p-1)-2, \dots, \frac{1}{2}(p-1)$ only the number $\frac{3}{2}(p-1)-2 \cdot \frac{p-1}{4} = p-1$ is divisible by the number $p-1$, it follows from the lemma 2,2 that

$$S(1,p) \equiv -1^{\frac{p-1}{4}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}} \sum_{x=1}^{p-1} x^{p-1} \equiv (-1)^{\frac{p+3}{4}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}} (\bmod\ p)$$

Now let $p \equiv 3 \ (\bmod\ 4)$. On the basis of the definition of Legendre's symbol we have

$$S(1,p) = \sum_{x=2}^{p-2} \left(\frac{x}{p}\right)\left(\frac{x^2-1}{p}\right) .$$

For every natural number x, $2 \leqq x \leqq p-2$ there exists such a natural number y_x from the smallest positive reduced residue system $(\bmod\ p)$ that

$xy_x \equiv 1 \pmod p$. (5)

Then on the basis of properties (b), (c) and (d) of Theorem 2,1 we obtain

$$S(1,p) = \sum_{x=1}^{p-2} \left(\frac{x-1}{p}\right)\left(\frac{x^2+x}{p}\right)\left(\frac{xy_x}{p}\right) = \sum_{x=2}^{p-2} \left(\frac{x-1}{p}\right)\left(\frac{xy_x+y_x}{p}\right)\left(\frac{x^2}{p}\right)$$

6

It follows from the congruence (5) that $xy_x + y_x \equiv 1 + y_x \pmod p$. Then on the basis of (c) and (d) from the relation (6) we obtain

$$S(1,p) = \sum_{x=2}^{p-2} \left(\frac{x-1}{p}\right)\left(\frac{1+y_x}{p}\right).$$

Hence on the basis of (b) and (c) we have

$$S(1,p) = \sum_{x=2}^{p-2} \left(\frac{x-1+xy_x-y_x}{p}\right) = \sum_{x=2}^{p-2} \left(\frac{x-y_x}{p}\right), \quad (7)$$

since $x - 1 + xy_x - y_x \equiv x - y_x \pmod p$. It is known from the theory of quadratic congruences that the solutions of the congruence $x^2 \equiv 1 \pmod p$ are 1 and $p-1$. Therefore $x \neq y_x$. Since for every natural x, $2 \leqq x \leqq p-2$, the congruence (5) has only one solution, we obtain

$$y_{y_x} = x, \quad \{2,3,\ldots,p-2\} = \{y_2, y_3, \ldots, y_{p-2}\} \quad (8)$$

It follows from (8) that in the set of all differences of the form $x - y_x$ $(2 \leqq x \leqq p-2)$ there exist $\frac{p-3}{2}$ pairs of the form $(c - y_c, y_c - c)$. On the basis of the assumption $p \equiv 3 \pmod 4$ and properties (b), (d) we get

$$\left(\frac{c-y_c}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{y_c-c}{p}\right) = -\left(\frac{y_c-c}{p}\right).$$

From this and from the relation (7) there follows straightaway the second assertion of Theorem 2,2.

3

It is well known (see [2] p.252), that every even number k, $2 < k \leqq 50$ is a term of the progression

$$\{\sigma(n) - n\}_{n=1}^{\infty} \qquad (1)$$

It follows from the well-known Goldbach's hypothesis that every even number greater that 6 is the sum of two different prime numbers. This hypothesis has not been proved or disproved. Let us suppose the validity of this hypothesis. Then it is possible to prove, that every odd natural number grater than 7 is a term of the progression (1) (see [2] p.252). We are going to show it.

Let $m > 7$ be an odd natural number. Then $m - 1 = p + q$, where p,q are different odd prime numbers. Since

$$\sigma(pq) - pq = p + q + 1 = m, \qquad (2)$$

we can see that m is a term of the progression (1). On the basis of (2) by a direct calculation one can check that every odd natural number k, $7 < k < 50$, is a term of the progression (1). The numbers 2 and 5 are not terms of the progression (1) (see [2] p.242). Since $\sigma(4) - 4 = 3$, $\sigma(8) - 8 = 7$, we see that, except 2 and 5 every natural number $k \leqq 50$ is a term of the progression (1).

So far it is not known if there exist infinitely many natural numbers which are not terms of the progression (1) (see [2] p.252). In the next we shall prove that 52 is not a term of the progression (1). For this we shall need several auxiliary theorems.

<u>Lemma 3,1.</u> $\sigma(n)$ is odd if and only if $n = t^2$, or $n = 2t^2$ where t is a natural number.

Proof. See [2] p. 227.

Lemma 3,2. $\sigma(n) - n$ is even if and only if $n = t^2$, where t is an odd natural number, or if n is even, but $n \neq h^2$ and $n \neq 2h^2$, where h is a natural number (see [2] p. 252 exercise 1).

Proof. $\sigma(n) - n$ is even if and only if the numbers $\sigma(n)$ and n are even or odd simultaneously.

a/ In virtue of lemma 3,1 $\sigma(n)$ and n are odd if and only if $n = t^2$, where t is an odd natural number.

b/ If $\sigma(n)$ and n are even natural numbers, then from the lemma 3,1 the second assertion follows easily, since $\sigma(n)$ is even if and only if $n \neq h^2$ and $n \neq 2h^2$, where h is a natural number.

Lemma 3,3. If n is an even natural number, then $\sigma(n) - n \geqq \frac{n}{2}$

Proof. See [2] p. 252, exercise 2.

In virtue of Lemma 3,3, the even numbers $n > 104$ do not satisfy the equation $\sigma(n) - n = 52$.

Lemma 3,4. If in the standard form of a natural number n there appear at least two different prime numbers with greater exponents than 1, then $\sigma(n) - n \geqq 55$, if further n is odd, then $\sigma(n) - n \geqq 178$.

Proof. Let

$$n = \prod_{i=1}^{r} p_i^{\alpha_i}$$

where $r \geqq 2$, $\alpha_1 \geqq 2$, $\alpha_2 \geqq 2$ and $p_1 < p_2 < \dots < p_r$. Then

$$\sigma(n) - n = \left(\prod_{i=1}^{r} \sum_{j=0}^{\alpha_i} p_i^j\right) - n \geqq p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r} +$$

$$+ p_1^{\alpha_1}(p_2^{\alpha_2-1} + p_2^{\alpha_2-2} + \dots + 1) + p_1^{\alpha_1-1}(p_2^{\alpha_2} + p_2^{\alpha_2-1} + \dots + 1)$$

$$+ \dots + (p_2^{\alpha_2} + p_2^{\alpha_2-1} + \dots + 1) - n \geqq 2^2 (3 + 1) +$$

$$+ 2(3^2 + 3 + 1) + (3^2 + 3 + 1) = 55.$$

If now $p_1 \geqq 3$, $p_2 \geqq 5$, then $\sigma(n) - n \geqq 3^2(5 + 1) +$
$+ 3(5^2 + 5 + 1) + (5^2 + 5 + 1) = 178.$

It follows from Lemma 3,2 that it is sufficient to examine such odd n which are the squares of natural number.

L e m m a 3,5. Let $n = t^2$, where t is an odd natural number. Then $\sigma(n) - n \neq 52$.

P r o o f . On the basis fof Lemma 3,4 clearly $\sigma(n) - n \neq 52$ if t has at least two different prime divisors. Consequently, let $t = p^{\alpha}$, where p is an odd number and $\alpha \geqq 1$. Now we shall prove that if $n = p^{2\alpha}$, then $\sigma(n) - n \neq 52$. If namely

$$\sigma(p^{2\alpha}) - p^{2\alpha} = \frac{p^{2\alpha+1} - 1}{p - 1} - p^{2\alpha} = 52,$$

then after simple modifications we obtain

$$p(52 - p^{2\alpha-1}) = 3.17,$$

where either $p = 3$, or $p = 17$. Then in the first case $3^{2\alpha-1} = 35$ and in the second case $17^{2\alpha-1} = 49$. None of these equalities can be valid. Thus the proof of the lemma is completed. It follows from lemmas 3,2 and 3,5 that for an odd n $\sigma(n) - n \neq 52$. For even n that in virtue of Lemma 3,2 can satisfy the equation $\sigma(n) - n = 52$, the following lemma is valid.

L e m m a 3,6. If n is an even natural number, then

$$\sigma(n) - n \neq 52.$$

Proof. In virtue of lemma 3,3 it is sufficient to examine only even numbers $n \leqq 104$. From the table in [2] p. 252 it can be seen, that if $n < 100$, then $\sigma(n) - n \neq 52$. Among of the remaining even numbers none satisfies the equation $\sigma(n) - n = 52$, which we can easily find. Then it follows from lemmas 3,2 and 3,6 that even n's do not satisfy the equation $\sigma(n) - n = 52$. Consequently, $k = 52$ is not a term of the progression (1).

Remark. By analogical procedures it can be shown that 88 is not a term of the progression (1). If n is an odd natural number, then in virtue of lemmas 3,2 and 3,4 it is sufficient to examine the numbers of the form $n = p^{2\alpha}$, where p is an odd prime number. Similarly like in Lemma 3,5, it is possible to prove that $n = p^{2\alpha}$ does not satisfy the equation $\sigma(n) - n = 88$. In virtue of Lemma 3,3, the even numbers $n > 176$ do not satisfy the equation $\sigma(n) - n = 88$. From the table in [2] p. 252, it is possible to see that if $n < 100$, then $\sigma(n) - n \neq 88$. By direct calculation one can find out that the even numbers n, $100 \leqq n \leqq 176$, do not satisfy equation $\sigma(n) - n = 88$. Let us remark that every even k, $52 < k < 88$, is a term of progression (1).

4

Let λ denote an arithmetical function defined in the following way:

1. If $m = 2^{\alpha}$ and $\alpha = 0, 1$, or 2, then $\lambda(m) = \varphi(m)$.
2. If $m = 2^{\alpha}$ and $\alpha \geqq 3$, then $\lambda(m) = \frac{\varphi(m)}{2}$
3. If $m = p^{\alpha}$, p is an odd number, then $\lambda(p^{\alpha}) = \varphi(p^{\alpha})$.
4. If $m = 2^{\alpha} p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, p_i $(i = 1, 2, \ldots, r)$ are distinct odd prime numbers, then

$$\lambda(m) = [\lambda(2^{\alpha}), \lambda(p_1^{\alpha_1}), \ldots, \lambda(p_r^{\alpha_r})] \text{ (See [5] p. 99)}.$$

$\lambda(m)$ is such smallest natural number that for every a, $(a, m) = 1$ it holds that

$$a^{\lambda(m)} \equiv 1 \pmod m .$$

See [2] p. 189-192). The number belonging to the exponent $\lambda(m) \pmod m$ is called the primitive λ root $\pmod m$. It is known that for every natural $m > 1$ there exists the primitive λ root $\pmod m$ (see [5] p. 112 - 113).

It is easy to see that every natural a, $a \not\equiv 1 \pmod m$, $(a, m) = 1$, belongs to the exponent $k > 1 \pmod m$, that divides $\lambda(m)$. The assertion that every natural $k > 1$, $k \setminus \lambda(m)$ is an exponent to which at least one number of the reduced residue system $\pmod m$ belongs follows from the fact, that number $b^{\frac{\lambda(m)}{k}}$ belongs to the exponent $k \pmod m$, where b is a primitive λ root $\pmod m$.

Let p be an odd prime number and let $\{t_1, t_2, \ldots, t_n\}$, $n = \varphi(p-1)$, be a reduced residue system $(\bmod (p-1))$. If g is a primitive root $\pmod p$ then no two elements of the set $\{g^{t_1}, g^{t_2}, \ldots, g^{t_n}\}$ are congruent $\pmod p$ and each of these elements is a primitive root $\pmod p$. In [6] the problem has been raised, if there exist such h, $(h,p) = 1$, that h is not a primitive root $\pmod p$, but at the same time the elements of the set $\{h^{t_1}, h^{t_2}, \ldots, h^{t_n}\}$ are in pairs incongruent $\pmod p$. It has been shown in solving of this problem (see [6]) that the answer to this question is positive if and only if there exists such a proper divisor d of the number $p - 1$ that is not a divisor of any difference $t_i - t_j$, $i \neq j$ $i, j = 1, 2, \ldots, n$. It has been shown in [6] that the last condition is satisfied exactly by all prime numbers $p \equiv 3 \pmod 4$.

In the next we are going to formulate similar questions in connection with primitive λ roots $\pmod m$.

Let $m > 2$ be a natural number. Then the elements of the set $\{b^1, b^2, \ldots, b^{\lambda(m)}\}$ are in pairs incongruent (mod m) if and only if b is a primitive λ root (mod m). Further let $R = \{r_1, r_2, \ldots, r_{\varphi(\lambda(m))}\}$ be the smallest positive reduced residue system (mod $\lambda(m)$). If b is primitive λ root (mod m), then no two elements of the set $\{b^{r_1}, b^{r_2}, \ldots, b^{r_{\varphi(\lambda(m))}}\}$ are congruent (mod m) and all of them are primitive λ roots (mod m).

In the next let a denote an arbitrary natural number relatively prime to m, which is not a primitive λ root (mod m). Let us put $A_a = \{a^{r_1}, a^{r_2}, \ldots, a^{r_{\varphi(\lambda(m))}}\}$,

where $R = \{r_1, r_2, \ldots, r_{\varphi(\lambda(m))}\}$ has its preceding meaning.

Theorem 4,1. Let $m > 2$ be a natural number. Then the elements of the set A_a are for each a, $(a, m) = 1$, a is not a primitive λ root (mod m), in pairs incongruent (mod m) if and only if there exists such a proper divisor of the number $\lambda(m)$ that does not divide the difference of any two different elements of the set R.

Proof. 1. Let d be an arbitrary proper divisor of the number $\lambda(m)$. Let us suppose the existence of such two different elements r_i, r_j $r_i > r_j$ of the set R, that

$$d \setminus r_i - r_j \qquad (1)$$

The wo prove that for a suitable a, $(a,m) = 1$, at least two elements of the set A_a are congruent (mod m). Let us choose for a a natural number, $(a, m) = 1$, which belongs to the exponent d (mod m) consequently, a is not a primitive λ root (mod m). From the relation (1) we obtain

$$a^{r_i - r_j} \equiv 1 \pmod{m}.$$

If we multiply the last mentioned congruence by the number a^{r_j}, then

$$a^{r_i} \equiv a^{r_j} \pmod{m}.$$

2. Now let a^{r_u}, a^{r_v} be two different elements of the set A_a, where $(a,m) = 1$, a is not a primitive λ root $(\bmod\ m)$. Suppose that

$$a^{r_u} \equiv a^{r_v} \pmod{m}, \qquad (2)$$

where $r_u > r_v$. Let us divide both sides of the congruence (2) by the number a^{r_v}. We obtain

$$a^{r_u - r_v} \equiv 1 \pmod{m}. \qquad (3)$$

Since a is not a primitive λ root $(\bmod\ m)$, the number a belongs to the exponent k which is the proper divisor of the number $\lambda(m)$. Hence, in virtue of (3) we obtain

$$k \setminus r_u - r_v .$$

Thus the proof is completed.

Let n be a natural number and the elements of the set $S = \{s_1, s_2, \ldots, s_{\varphi(n)}\}$ constitute the least positive reduced residue system $(\bmod\ n)$. Let us denote by the symbol N the set of all the natural numbers n, for which the following statement is valid: n has a proper divisor which does not divide any two different elements of the set S. The structure of the set N will be described in the following theorem.

<u>T h e o r e m 4,2.</u> A natural number n is an element of the set N if and only if $n \equiv 2 \pmod{4}$.

<u>P r o o f .</u> 1. Let $n \equiv 2 \pmod{4}$. Then n has the form $n = 4k + 2$, where $k \geqq 0$. We are going to show that the number $d = 2k + 1$, which is a proper divisor of the number n, does not divide the difference of any two different elements of the

set S. If for some $i \neq j$ an even number $s_i - s_j$ were divisible by the number d, then $|s_i - s_j| \geqq 4k + 2$, and hence it would follow that some of the numbers s_i, s_j is greater than $4k + 2$. It is a contradiction to the definition of numbers belonging to S.

2. If $n \not\equiv 2 \pmod 4$, then we distinguish two cases.

a. Let $n = 4k$ $(k \geq 1)$. We shall prove that to each proper divisor d of the number n there exist at least two different elements of the set S, the difference of which is divisible by the number d.

If d is an odd proper divisor of the number n, $d = 2r + 1$ $(r \geq 1)$, then clearly $d \backslash k$. Since $(2k+1, 4k) = 1$, hence it follows that $2k + 1$ and 1 are such elements of the set S, the difference of which is divisible by the number d.

Now let us consider the even d, $d = 2r$ $(r \geqq 1)$, $d \backslash n$. In the case of an odd r, $2r \backslash 2k$, and similarly as before, also now $d = 2r$ divides the difference $2k + 1$ and 1. If r is even, $r = 2t$ $(t \geqq 1)$, then $d = 4t$, and clearly t is a proper divisor of the number k. Let us put $k = tv$, where $v \geqq 2$. If v is even, $v = 2u$, then the numbers $a = 2k + 1 = 4tu + 1$ and 1 are elements of the set S and $d \backslash a - 1$. Let v be odd. Let us put

$$v = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_\ell^{\alpha_\ell},$$

where $\alpha_i \geqq 1$, q_i $(i = 1, 2, 3, \ldots, \ell)$ are odd prime numbers $\ell \geqq 1$.

Since $q_\ell \geqq 3$, we obtain

$$4k = 4tv \geqq 4t \frac{v}{q_\ell^{\alpha_\ell}} \cdot 3 > 8tq_1^{\alpha_1} q_2^{\alpha_2} \cdots q_{\ell-1}^{\alpha_{\ell-1}} + 1 = b_1 \quad (4)$$

It follows from inequality (4) that

$$b_2 = 4tq_1^{\alpha_1} q_2^{\alpha_2} \cdots q_{\ell-1}^{\alpha_{\ell-1}} + 1 < 4k.$$

We shall prove that at least one of the numbers b_1, b_2 is relatively prime to the number 4k. Suppose that

$$(b_1, 4k) = d_1 > 1, \quad (b_2, 4k) = d_2 > 1 .$$

Then $d_1 = q_\ell^{\beta_\ell}$, $1 \leqq \beta_\ell \leqq \alpha_\ell$ and $d_2 = q_\ell^{\gamma_\ell}$, $1 \leqq \gamma_\ell \leqq \alpha_\ell$.
Hence $q_\ell \setminus b_1$, $q_\ell \setminus b_2$, and consequently $q_\ell \setminus 2b_2 - b_1 = 1$.

It is contrary to the fact that q_ℓ is a prime number. Consequently either 1, b_1, or 1, b_2, belong to S and their difference is clearly divisible by the number d.

b. Let n be an odd natural number whose standard form is

$$n = p_1^{\delta_1} p_2^{\delta_2} \dots p_w^{\delta_w} ,$$

where $\delta_i \geqq 1$ $(i = 1, 2, \dots, w)$, $w \geqq 1$. Then the proper divisor d of the number n has the form

$$d = p_1^{\rho_1} p_2^{\rho_2} \dots p_w^{\rho_w} , \tag{5}$$

where $0 \leqq \rho_i \leqq \delta_i$ $(i = 1, 2, \dots, w)$ and at least for one of the exponents ρ_i $(i = 1, 2, \dots, w)$ the inequality $\rho_i < \delta_i$ holds.

Let us define the natural number D as follows:

1. $D = d$, if $\rho_i \geqq 1$ for each i $(i = 1, 2, \dots, w)$.

2. $D = d\, p_{j_1}^{\delta_{j_1}} \dots p_{j_{h-1}}^{\delta_{j_{h-1}}}$, if $\rho_{j_1} = \dots = \rho_{j_h} = 0$,

where $1 \leqq j_\ell \leqq w$ $(\ell = 1, 2, \dots, h)$, $w > h \geqq 1$, and all the other exponents in (5) are $\geqq 1$.

It is easy to see that $d \setminus D$ and D is the proper divisor of the number n. Then analogically to the conclusion a.

it will be proved that either 1, D + 1 or 1, 2D + 1 are elements of the set S, and their diference is divisible by the number d.

The following theorem is an easy consequence of the theorems 4,1 and 4,2.

T h e o r e m 4,3. Let $m > 2$ be a natural number. Then no two elements of the set A_a for each a, $(a,m) = 1$, a is not a primitive λ root (mod m), are congruent (mod m) if and only if $\lambda(m) \equiv 2 \pmod 4$.

R E F E R E N C E S

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O niektorých problémoch z elementárnej teórie čísiel

LÁSZLÓ VOJTECH

S ú h r n

Článok je rozdelený do štyroch častí.

V prvej časti sú dve zovšeobecnenia istého problému z Amer. Math. Monthly.

V druhej časti autor vyšetruje súčty tvaru

$$\sum_{x=1}^{p} \left(\frac{x}{p}\right)\left(\frac{x+a}{p}\right)\left(\frac{x+2a}{p}\right),$$

kde p je nepárne prvočíslo a $\left(\frac{x}{p}\right)$ značí Legendreov symbol.

V tretej časti autor dokazuje, že 52 nie je členom postupnosti

$$\{\sigma(n) - n\}_{n=1}^{\infty} .$$

V štvrtej časti rozriešil otázku, kedy sú prvky množiny $\{a^{r_1}, a^{r_2}, \dots, a^{r_{\varphi(\lambda(m))}}\}$ po dvoch inkogruentné (mod m), kde a, m sú prirodzené čísla, (a, m) = 1, m > 2, a nie je primitívnym λ koreňom (mod m) a $\{r_1, r_2, \dots, r_{\varphi(\lambda(m))}\}$ je najmenší kladný redukovaný zvyškový systém (mod $\lambda(m)$).

О некоторых проблемах элементарной теории чисел

ВОЙТЕХ ЛАСЛО

Р е з ю м е

В данной статье автор занимается некоторыми проблемами элементарной теории чисел. Статья разделена на четыре части.

В первой части приводятся два обобщения одной проблемы из АММ.

Во второй части исследованы суммы вида

$$\sum_{x=1}^{p}\left(\frac{x}{p}\right)\left(\frac{x+a}{p}\right)\left(\frac{x+2a}{p}\right)$$

где p нечетное простое число, а $\left(\frac{x}{p}\right)$ обозначает символ Лежандра.

В третьей части приводится доказательство того, что 52 не является членом последовательности $\{\sigma(n) - n\}_{n=1}^{\infty}$.

В четвертой части решается вопрос, когда элементы множества $\{a^{r_1}, a^{r_2}, \dots, a^{r_{\varphi(\lambda(m))}}\}$ несравнимы между собой по модулю m, где a, m являются натуральными числами, (a, m) = 1, m>2, а не первообразным корнем по модулю m и множество $\{r_1, r_2, \dots, r_{\varphi(\lambda(m))}\}$ является наименьшой положительной приведенной системой вычетов по модулю $\lambda(m)$

Received: February 12, 1973, in press July 8, 1974.

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(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXXII, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXXII – 1975

On asymptotic properties and distribution of zeros of solution of y" = q(t)y

MIROSLAV BARTUŠEK

1.1. Consider a differential equation

$$(q) \qquad y'' = q(t)\,y\,, \quad q \in C^0[a,b)\,, \quad b \leqq \infty,$$

where $C^n[a,b)$ (n being a non-negative integer) is the set of all continuous funcitons having continuous derivatives up to and including the order n on $[a,b)$. Let y_1 be a non-trivial solution of (q) vanishing at $t \in [a,b)$ and y_2 a non-trivial one the derivative of which vanishes at t. If $\varphi(t)$ is the first zero of y_1 lying on the right of t, then φ is called the basic central dispersion of the 1-st kind (briefly, dispersion of the 1-st kind). Let $q(t) < 0$, $t \in [a,b)$. If $\Psi(t)$, resp. $X(t)$, resp. $\omega(t)$ is the first zero of y_2', resp. y_1', resp. y_2, then Ψ, resp. X, resp. ω is called the basic central dispersion of the 2-nd, resp. 3-rd, resp. 4-th kind (briefly, dispersion of the 2-nd, resp. 3-rd, resp. 4-th kind).

In all the work we will deal only with oscillatoric($t \to b_-$) differential equations (i.e. every non-trivial solution has infinitely many zeros on every interval of the form $[t_0, b)$, $t_0 \in [a, b)$).

The properties of dispersions can be found in [4]. Let δ be the dispersion of the k-th kind, k = 1, 2, 3, 4. Then δ has these properties ($q(t) < 0$ on $[a,b)$ if k = 2,3,4):

$$
(1)\quad
\begin{array}{lll}
1/\ \delta \in C^3\,[a,b) & \text{if} & k = 1\\
\quad\ \delta \in C^1\,[a,b) & \text{if} & k = 2, 3, 4\\
2/\ \delta(t) > t & \text{on} & [a,b)\\
3/\ \delta'(t) > 0 & \text{on} & [a,b)\\
4/\ \lim\limits_{t\to b-} \delta(t) = b. & &
\end{array}
$$

Let n be a positive integer. If δ_n is the n-th iterate of the dispersion δ of the k-th kind, then δ_n has the same properties (1), see [4] §13.

We shall need another properties of dispersions. Let $q(t)<0$, $t\in [a,b)$. Let φ_n, resp. ψ_n, resp. χ_n, resp. ω_n be the n-th iterate of the dispersion φ, resp. ψ, resp χ, resp. ω of the 1-st, resp. 2-nd, resp, 3-rd, resp. 4-th kind of (q) and let y be a non-trivial solution of (q). Then we have for $t \in [a,b)$ (see [4] §13) :

$$
(2)\quad \varphi_n'(t) = y^2(\varphi_n(t)) / y^2(t) \qquad \text{for} \quad y(t)\neq 0
$$

$$
= y'^2(t) / y'^2(\varphi_n(t)) \qquad \text{for} \quad y(t)= 0
$$

/this relation is valid even if the assumption $q(t)<0$, $t\in [a,b)$ is omitted);

$$
(3)\quad \varphi'(t) = q(t_1) / q(t_3), \quad t < t_1 < \chi(t) < t_3 < \varphi(t)
$$

$$
\psi_n'(t) = \frac{q(t)}{q(\psi_n(t))}\,\frac{y'^2(\psi_n(t))}{y'^2(t)} \qquad \text{for} \quad y'(t) \neq 0
$$

$$
(4)\qquad = \frac{q(t)}{q(\psi_n(t))}\cdot\frac{y^2(t)}{y^2(\psi_n(t))} \qquad \text{for} \quad y'(t) = 0
$$

$$
(5)\quad \psi'(t) = q(t)\cdot q(t_4) \,/\, \big(q(\psi(t))\cdot q(t_2)\big),
$$

$$
t < t_2 < \omega(t) < t_4 < \psi(t)
$$

$$
(6)\quad \chi_n'(t) = - y'^2(\chi_n(t)) \,/\, \big(q(\chi_n(t))\cdot y^2(t)\big)
$$

for $y(t) \neq 0$

$$= -y'^2(t) / \left(q\left(\chi_n(t)\right) \cdot y^2\left(\chi_n(t)\right)\right)$$

for $y(t) \neq 0$

(7) $\chi'(t) = q(t_1) / q\left(\chi(t)\right)$

(8) $\omega'(t) = q(t) / q(t_2)$

(9) $-\frac{1}{2}\frac{\varphi_n'''}{\varphi_n'} + \frac{3}{4}\frac{\varphi''^2}{\varphi'^2} + q(\varphi_n)\varphi_n'^2 = q(t).$

1.2. Some results giving us a certain review about the relations among the dispersion φ of the 1-st kind of (q) and the behaviour of solutions of (q) on $[a,b)$ are summed up in the following Theorem (see [1], [7], [8]):

T h e o r e m 1. Let (q), $q \in C^0[a, b]$, $b \leqq \infty$ be an oscillatoric $(t \to b_-)$ differential equation and φ_n the n-th iterate of its dispersion φ of the 1-st kind. Let $t_0 \in [a,b)$.

a/ Every solution of (q) is bounded on $[t_0, b)$ if and only if a constant N exists such that

(10) $\varphi_n'(x) \leqq N, \quad x \in [t_0, \varphi(t_0)), \quad n = 1, 2, 3, \ldots$

b/ Every solution of (q) tends to zero for $t \to b_-$ if, and only if

(11) $$\lim_{n\to\infty} \varphi_n'(x) = 0$$

uniformly for $x \in [t_0, \varphi(t_0))$.

1.3. We shall need the following lemma in our considerations. See [6] §14.3.1.

L e m m a 1. Let $q(t) \neq 0$ be continuous for $a \leqq t < b$. Introduce the new dependent variable v and independent variable s defined by

$v = y'$, $\quad ds = |q(t)|\,dt$, $\quad s(a) = a.$

Then the equation (q) and

$$v'' = Q(s)\,v \qquad \text{where} \qquad Q(s) = \frac{1}{q(t)}$$

are equivalent.

2. This paragraph is devoted to relations between q and the derivative of the dispersion φ of the 1-st kind of (q).

Theorem 2. Let n be a non-negative integer and $q \in C^n\,[a, \infty)$. Let φ be the dispersion of the 1-st kind of (q). If

$$\lim_{t\to\infty} q(t) = C, \quad -\infty < C < 0,$$

$$\lim_{t\to\infty} q^{(k)}(t) = 0, \quad k = 1, 2, \dots, n$$

(for $n = 0$ the last condition must be omitted), then

$$\lim_{t\to\infty} \varphi'(t) = 1, \tag{12}$$

$$\lim_{t\to\infty} \varphi^{(k)}(t) = 0, \quad k = 2, 3, \dots, n+3. \tag{13}$$

Proof. Let $\varepsilon > 0, \varepsilon < |C|$ be an arbitrary number. Then there exists a number t_0 such that for $t \geqq t_0$ we have $q(t) < 0$, $|q(t) - C| < \varepsilon$. Further, from Sturm Comparison Theorem and from $\lim\limits_{t\to\infty} q(t) = C$ it follows that (q) is oscillatoric on $[a, \infty)$. From this and from (3) we have

$$1 - \frac{2\varepsilon}{|C| + \varepsilon} = \frac{|C| - \varepsilon}{|C| + \varepsilon} \leqq \varphi'(t) \leqq \frac{|C| + \varepsilon}{|C| - \varepsilon} = 1 + \frac{2\varepsilon}{|C| - \varepsilon},$$

$$|1 - \varphi'(t)| \leqq \frac{2\varepsilon}{|C| - \varepsilon}, \quad t \geqq t_0$$

and thus $\lim\limits_{t\to\infty} \varphi'(t) = 1$.

It follows from this and from the differential equation (9) that

(14) $$\lim_{t\to\infty} (3\varphi''^2 - 2\varphi'''\varphi') = 0.$$

Assume that $\lim_{t\to\infty} \varphi''(t) = d \neq 0$. Then the function φ' is monotone and $\lim_{t\to\infty} \varphi'(t) = \pm\infty$. But it is in contradiction with (12).

Let the limit $\lim_{t\to\infty} \varphi''$ do not exist. Then there exists an infinite set of numbers $M = \{t : \varphi'''(t) = 0,\ t \in [a,\infty)\}$ such that the infinity is its accumulation point (the function φ''^2 has local maxima at some of these numbers). It follows from (14) that for $t \to \infty$, $t \in M$ we have $\varphi''^2(t) \to 0$. Thus $\lim_{t\to\infty} \varphi''^2(t) = 0$ and it conflicts with our assumptions. Also we can see that $\lim_{t\to\infty} \varphi'' = 0$ must be valid and remainder of the statement (13) for $k = 2,3$ follows from (14).

We shall prove (13) by means of the mathematical induction. For $k = 2,3$ the statement was proved. Let us assume that (13) is valid for $k = 2,3, \dots, m+2$ $(1 \leq m \leq n)$. Differentiation of (9) gives us the following relations:

$$\left[-\frac{1}{2}\frac{\varphi'''}{\varphi'} + \frac{3}{4}\frac{\varphi''^2}{\varphi'^2} \right]^{(m)} = \left[q(t) - q(\varphi)\varphi'^2 \right]^{(m)}$$

$$-\frac{1}{2}\frac{\varphi^{(m+3)}}{\varphi'} - \frac{1}{2}\sum_{\ell=0}^{m-1}\binom{m}{\ell}\varphi^{(\ell+3)}\left(\frac{1}{\varphi'}\right)^{(m-\ell)} + \frac{3}{4}\sum_{\ell=0}^{m}\binom{m}{\ell}\varphi^{(\ell+2)}\left(\frac{\varphi''}{\varphi'^2}\right)^{(m-\ell)} =$$

$$= q^{(m)}(t) - \sum_{\ell=1}^{m}\binom{m}{\ell}(\varphi'^2)^{(m-\ell)}\sum\frac{\ell!}{i!j!\dots h!}\,q^{(s)}(\varphi)\cdot\left(\frac{\varphi'}{1!}\right)^{i}\cdot$$

$$\cdot\left(\frac{\varphi''}{2!}\right)^{j}\dots\left(\frac{\varphi^{(r)}}{r!}\right)^{h} - (\varphi'^2)^{(m)}\,q(\varphi),$$

where the $\sum$ is taken over all non-negative integer solution of $i+2j+\ldots+rh = \ell$ and $i+j+\ldots+h = s$ (we use the Leibnitz Formula and the relation for the n-th derivative of a compound function, see [9] o.43). We put: $\overset{(k)}{\varphi} = \frac{d^k \varphi}{dt^k}$.

From this for $t \to \infty$ we get:

$$-\frac{1}{2} \lim_{t\to\infty} \overset{(m+3)}{\varphi} = \lim_{t\to\infty} \left(\overset{(m)}{q}(t) - \overset{(m)}{q}(\varphi)\varphi'^{\,m+2} \right) = 0.$$

Thus $\lim_{t\to\infty} \overset{(m+3)}{\varphi}(t) = 0$

and the statement (13) is proved.

T h e o r e m 3. Let n be a non-negative integer and (q), $q \in C^n[a,\infty), q < 0$ be an oscillatoris $(t\to\infty)$ differential equation. Let φ be its dispersion of the first kind. If

$$\lim_{t\to\infty} \overset{(k)}{q}(t) = 0, \qquad k = 0,1,2,\ldots,n$$

and φ' is bounded on $[a,\infty)$, then

$$\lim_{t\to\infty} \overset{(k)}{\varphi}(t) = 0 \qquad \text{for } k = 2,3,\ldots,n+3 .$$

P r o o f . As the function φ' is bounded on $[a,\infty)$, it follows from (9) that

$$\lim_{t\to\infty} (3\varphi''^2 - 2\varphi'''\varphi') = 0$$

and we can prove the statement of the theorem in a similar way as in Theorem 2.

R e m a r k 1. The assumption in Theorems 2,3 about the interval $[a,\infty)$ is essential because, an oscillatoric $(t\to b_-)$ differential equation (q), $q\in C^m[a,b)$, $b<\infty$ can not have the property: $\lim_{t\to b} q(t) = C > -\infty$. This follows from the Sturm Comparison Theorem.

R e m a r k 2 . The assumption in Theorem 3 about the boundedness of φ' is essential. This follows from the following example.

Let $\varphi(t) = e^t$, $t \in [0,\infty)$. Then $\lim\limits_{t\to\infty} \varphi^{(k)}(t) = \infty$, $k = 1,2,3,\dots$ and there exists an oscillatoric $(t \to \infty)$ differential equation (q) with the dispersion φ of the first kind (see [4] §15.10). From (9) we can see that

$$(15)\quad q(e^t) = \left(q(t) - \frac{1}{4}\right)e^{-2t}, \quad t \in [0,\infty)$$

must be valid. Let us denote $J_n = [a_n, a_{n+1})$, $a_0 = 0$, $a_n = e^{a_{n-1}}$, $n \geqq 1$. We shall prove the following inequality by means of mathematical induction:

$$(16)\quad \max_{t\in J_n} |q(t)| \leqq M \cdot e^{-2(n-1)} + \frac{n}{4} e^{2-2n}, \quad n = 1,2,3,\dots$$

where $M = \max\limits_{t\in[0,1]} |q(t)|$. For $n = 1$ the statement follows from (15). Let it be valid for $k = 1,2,3,\dots,n$. Then according to (15) we have:

$$\max_{t\in J_{n+1}} |q(t)| \leqq \left(M e^{-2(n-1)} + \frac{n}{4} e^{2-2n} + \frac{1}{4}\right)\frac{1}{a_{n+1}^2} \leqq \left(M e^{-2(n-1)} + \frac{n+1}{4}\right)\frac{1}{a_{n+1}^2} \leqq M \cdot e^{-2n} + \frac{n+1}{4} e^{-2n}$$

because $a_n \geqq e^1 = a_2$, $n \geqq 2$.

Also (16) is valid and from this we have

$$\lim_{t\to\infty} q(t) = 0.$$

T h e o r e m 4. Let (q), $q \in C^0[a,b)$, $q < 0$ be an oscillaroric $(t \to b_-)$ differential equation and φ its dispersion of the first kind. Let $t_o \in [a,b)$ and

$$f_1(t) \leqq q(t) \leqq f_2(t) < 0 \quad , \quad t \in [t_o, b),$$

where f_1, f_2 are functions.

a/ If the functions f_1, f_2 are non-increasing, then

$$\frac{f_2(t)}{f_1(\varphi(t))} \leqq \varphi'(t) \leqq \frac{f_1(\varphi(t))}{f_2(t)} \quad , \quad t \in [t_o, b).$$

b/ If the functions f_1, f_2 are non-decreasing, then

$$\frac{f_2(\varphi(t))}{f_1 \quad t} \leqq \varphi'(t) \leqq \frac{f_1(t)}{f_2(\varphi(t))} \quad , \quad t \in [t_o, b).$$

c/ If f_1 is non-decreasing and f_2 non-increasing, then

$$\frac{f_2(t)}{f_1(t)} \leqq \varphi'(t) \leqq \frac{f_1(t)}{f_2(t)} \quad , \quad t \in [t_o, b).$$

P r o o f . a/ It follows from (3) that

$$\varphi'(t) = \frac{q(t_1)}{q(t_3)} \geqq \frac{f_2(t_1)}{f_1(t_3)} \geqq \frac{f_2(t)}{f_1(\varphi(t))} ,$$

$$\varphi'(t) = \frac{q(t_1)}{q(t_3)} \leqq \frac{f_1(t_1)}{f_2(t_3)} \leqq \frac{f_1(\varphi(t))}{f_2(t)} ,$$

$$t < t_1 < t_3 < \varphi(t)$$

and the statement is valid in this case.

b/c/ In these cases the proof is similar.

R e m a r k 3. The following inequalities for dispersions were obtained by the author of [4] §13.13 for monotone funcitons q .

Let δ be the dispersion of the k-th kind, k = 1,2,3,4 of (q) . If (q) is non-decreasing (non-increasing), (q) oscillatoric on $[a,b)$, then

$$\delta'(t) \geqq 1 \quad (\delta'(t) \leqq 1), \qquad t \in [a,b).$$

T h e o r e m 5. Let (q), $q \in C^0[a,\infty)$, $q < 0$ be an oscillatoric $(t \to \infty)$ differential equation and δ its dispersion of the k-th kind, k = 1,2,3,4.

a/ If q is non-increasing, then

$$\frac{q(t)}{q(t+c)} \leqq \frac{q(t)}{q(\delta(t))} \leqq \delta'(t),$$

where $c = \delta(a) - a$, $t \in [a,\infty)$.

b/ If q is non-decreasing, then

$$\frac{q(t)}{q(\delta(t))} \geqq \delta'(t), \qquad t \in [a,\infty).$$

P r o o f . Let δ be the dispersion of the first kind and q be non-increasing. Then it follows from (3) that

$$\delta'(t) = \frac{q(t_1)}{q(t_3)} \geqq \frac{q(t)}{q(\delta(t))}, \quad t \in [a,\infty), \quad t < t_1 < \chi(t) < t_3 < \delta(t).$$

From Remark 3 we can see that the function $\delta(t) - t$ is non-incresing. Also, $\delta(t) \leqq t + \delta(a) - a$ for $t \geqq a$ and

$$\delta'(t) \geqq \frac{q(t)}{q(\delta(t))} \geqq \frac{q(t)}{q(t+c)}.$$

We can see that the theorem is valid in this case. The statement for q non-decreasing and the other dispersions can be proved by the same way. We must only use (5), (7) (8) instead of (3).

3.1. Theorem 1 given us the necessary and sufficient condition for every solution being bounded on $[a, b)$ (tending to zero for $t \to b_-$). The conditions (10) and (11) are not very suitable for applications. For this reason it is convenient to seek for the sufficient and / or necessary conditions on q or at least on the dispersion φ of the first kind for every solution being bounded or tending to zero for $t \to b_-$ (but conditions on φ ought to be much more simple than (11) is). This problem was given in [8]. Some results were derived in [1]. The following theorem concerns the above mentioned problem.

Theorem 6. Let (q), $q \in C^0[a, \infty)$, $q < 0$ be an oscillatoric $(t \to \infty)$ differential equation and φ its dispersion of the first kind.

a/ If every solution is bounded on $[a, \infty)$, then there exists a constant $M > 0$ such that

$$\varphi(t) - t \leqq M .$$

b/ If every solution tends to zero for $t \to \infty$, then

$$\lim_{t \to \infty} \left(\varphi(t) - t\right) = 0.$$

Proof. a/ Let every solution be bounded on $[a, \infty)$. Then it follows from Theorem 1 that

$$\varphi'_n(t) \leqq M_2 = \text{const} < \infty, \quad t \in [a, \varphi(a)) ,$$

$n = 1,2,3, \dots$

where φ_n is the n-th iterate of the function φ.

Let $t \in [\varphi(a), \infty)$ be an arbitrary number. There exist numbers n, $x \in [a, \varphi(a))$ such that $\varphi_n(x) = t$. By integration over the interval $(x, \varphi(x))$ we get

$$\varphi_{n+1}(x) - \varphi_n(x) \leqq M_2 \; \varphi(x) - x) \leqq M_2 \cdot \max_{x \in [a, \varphi(a)]} (\varphi(x) - x) =$$

$$= M_1 \cdot M_2, \quad \varphi(t) - t \leqq M, \quad M = \max(M_1, M_1 \cdot M_2)$$

and we can see that the statement is valid.

b/ The proof is similar as in a/. We must only use (11) instead of (10) .

R e m a r k 4 . The assumptions of Theorem 6a/ give us only the sufficient condition for $\varphi(t) - t \leq M$, $t \in [a, \infty)$. This assertion follows from two following results:

If $0 > -c^2 > q(t)$, $t \in [a, \infty)$, then the differences between consecutive zeros of an arbitrary solution of (q) are bounded (see [10] §7.4.2b)).

There exists a differential equation (q), $q \in C^0[a, \infty)$, $\lim_{t \to \infty} q(t) = -1$ such that it has an unbounded solution on $[a, \infty)$ (see [3] §6.5.).

R e m a r k 5 . Let (q), $q \in C^0[a, \infty)$, $q < 0$ be oscillatoric on $[a, \infty)$ and let every solution of (q) tend to zero for $t \to \infty$. Let φ be the dispersion of the first kind of (q) . Then wo can see that for the equation ($\bar{q}$) with the dispersion of the first kind $\bar{\varphi} = \varphi + \varepsilon$ ($\varepsilon > 0$ an arbitrary number there exists a solution not tending to zero for $t \to \infty$ and the equation ($\bar{\bar{q}}$) with the dispersion $\bar{\bar{\varphi}} = \varphi - \varepsilon$ of the first kind could be defined only on a finite interval (this follows from (1)).

3.2. L e m m a 2 . Let (q), $q \in C^0[a,b)$, $q < 0$ be oscilatoric on $[a,b)$ and φ_n be the n-th iterate of its dispersion φ of the first kind.

a/ Let there exist a solution of (q) tending to zero for $t \to b$ and let t_0 be its zero, $t_0 \in [a, \varphi(a))$. Then

$$\lim_{n \to \infty} \varphi_n'(t) = 0, \quad t \in [a, \varphi(a)), \quad t \neq t_0 .$$

b/ Let there exist a solution y bounded on $[a,b)$ and $t \in [a,b)$, $y(t) \neq 0$. Then there exists a constant $M > 0$ such that

$$\varphi_n'(t) \leq M, \quad n = 1,2,3,\ldots$$

P r o o f . The statement of Lemma follows directly from (2) and (1).

T h e o r e m 7 . Let (q), $q \in C^0 [a,b)$, $q < 0$ be an oscillatoric $(t \to b)$ differential equation. Let y be a non non-trivial solution of (q), $\{t_{k,y}\}$ the sequence of the extremants of y and $\{t'_{k,y}\}$ one of y', $t_{k,y} \in [a,b)$, $t'_{k,y} \in [a,b)$.

a/ Every solution of (q) tends to zero for $t \to b_-$ if, and only if

$$\lim_{k \to \infty} \left| \frac{y'(t'_{k,y})}{y'(t'_{0,y})} \right| = \infty$$

uniformly for every non-trivial solution y.

b/ Every solution of (q) is bounded on $[a,b)$ if, and only if there exists a constant $M > 0$ such that for an arbitrary solution /non-trivial/ y we have

$$| y'(t'_{k,y}) | \geqq M . | y'(t'_{0,y}) |$$

c/ The derivative of every solution of (q) tends to zero for $t \to b_-$ if, and only if

$$\lim_{k \to \infty} \left| \frac{y(t_{k,y})}{y(t_{0,y})} \right| = \infty$$

uniformly for every non-trivial solution of (q).

d/ The derivative of every solution of (q) is bounded on $[a,b)$ if, and only if there exists a constant M such that for an arbitrary non-trivial solution y of (q) we have

$$| y(t_{k,y}) | \geqq M . | y(t_{0,y}) | .$$

e/ Let $y, \bar{y}$ be linearly independent solutions of (q) and $\bar{y}$ be bounded on $[a,b)$. Then there exists a constant $M > 0$ such that

$$|y'(t'_{k,y})| \geqq M \cdot |y'(t'_{o,y})| .$$

f/ Let $y, \bar{y}$ be linearly independent solutions of (q) and $\bar{y}$ tend to zero for $t \to b_-$. Then

$$\lim_{k\to\infty} |y'(t'_{k,y})| = \infty .$$

P r o o f . a/ b/ The statement follows from Theorem 1 and (2) .

c/ d/ Lemma 1 applied to the cases a/ b/ given us the statement of the theorem.

e/ f/ The statement follows from Lemma 2 and (2) .

T h e o r e m 8 . Let (q) , $q \in C^0[a,b)$, $q < 0$ be oscillatoric on $[a,b)$. Let y be an arbitrary non-trivial solution of (q) and $\{t_k\}$ the sequence of the extremants of its derivative y' . Let X be the dispersion of the 3-rd kind of (q) (thus $\{X(t_k)\}$ is the sequence of the extremants of y) .

a/ If $0 > q(t) \geqq C = \text{const} > -\infty$, then

$$\left|\frac{y'(t_k)}{y(X(t_k))}\right| \leqq \sqrt{-C} .$$

b/ If $0 > C = \text{const} \geqq q(t) > -\infty$, then

$$\left|\frac{y'(t_k)}{y(X(t_k))}\right| \geqq \sqrt{-C} .$$

Particulary, if $\lim_{t\to b} q(t) = -C^2$, $0 \leqq C \leqq \infty$, then

$$\lim_{k\to\infty} \left| \frac{y'(t_k)}{y(\chi(t_k))} \right| = C.$$

P r o o f . It follows from (6), (7) that we have

$$\left| \frac{y'(t_k)}{y(\chi(t_k))} \right| = \sqrt{\chi'(t_k)\cdot|q(\chi(t_k))|} = \sqrt{|q(t_k')|},$$

where $t_k < t_k' < \chi(t_k)$. From this and from the assumptions ew get the statement of the theorem.

R e m a r k 6 . The particulary case of Theorem 8 was proved by Kneser and Ascoli (see [10] §7.4.2.) , but under stronger assumptions. They assumed: $q \in C^0[a,\infty)$, $q<0$, $q(t)$ has bounded variation and $\lim_{t\to\infty} q(t) = -\alpha^2 \neq 0$.

C o r o l l a r y 1 . Let $0 > c \geqq q(t) \geqq d > -\infty$, where c, d are constants . Let y be an arbitrary solution of (q). Then the functions y and y' are both bounded or both unbounded on $[a,b)$. The function y tends to zero for $t\to b$ if, and only if y' has the same property.

T h e o r e m 9 . Let (q), $q \in C^0[a,b)$, $q<0$ be oscillatoric on $[a,b)$.

a/ Let $0 > q(t) \geqq C = \text{const} > -\infty$. If there exists a solution y tending to zero for $t\to b$, then every solution of (q) linearly independent with y is unbounded.

b/ Let $\lim_{t\to b} q(t) = 0$. Then there exists a solution unbounded on $[a,b)$. Moreover, if y_1 and y_2 are linearly independent solutions and y_1 is bounded on $[a,b)$, then y_2 is unbounded and

$$\lim_{t\to b} y_1'(t) = 0, \quad y_2'(t_k) \geqq \text{const} > 0,$$

where $\{t_k\}$ is the sequence of the extremants of y_2'.

c/ Let $0 > C = \text{const} \geq q(t) > -\infty$. If there exists a solution y which derative y' tends to zero for $t \to b$, then the derivative of every solution of (q), linearly independent with y is unbounded.

d/ Let $\lim_{t \to b} q(t) = -\infty$. Then there exists a solution the derivative of which is unbounded on $[a,b)$. Moreover, if y_1 and y_2 are linearly independent solutions and y_1 is bounded on $[a,b)$, then y_2' is unbounded and

$$\lim_{t \to b} y_1(t) = 0, \qquad y_2(t_k) \geq \text{const} > 0,$$

where $\{t_k\}$ is the sequence of the extremants of y_2.

<u>P r o o f .</u> a/ The statement follows from Theorem 7f/ and Theorem 8a/.

b/ The first part of the statement is a consequence of Theorem 7b/ and Theorem 8. Let y_1 be a solution bounded on $[a,b)$ and y_2 linearly independent solution with y_1. Then it follows from Theorem 7e/ that

$$|y_2'(t_k)| \geq M \, |y_2'(t_o)| , \quad M = \text{const} > 0.$$

Finally, according to Theorem 8a/ we have $\lim_{t \to b} y_1'(t) = 0$.

c/d/ These cases can be proved by the application of Lemma 1 to a/ and b/.

<u>R e m a r k 7 .</u> We must notice a result of Prodi. See [5] §2.5.5. Let the function $q \in C^0[a, \infty)$, q be non-increasing and $\lim_{t \to \infty} q(t) = -\infty$. Then there exists a non-trivial solution of (q) tending to zero for $t \to \infty$.

<u>T h e o r e m 10.</u> Let (q), $q \in C^0[a,b)$, $q(t) < 0$, $t \in [a,b)$ be an oscillatoric $(t \to b\_)$ differential equation and φ, ψ its dispersions of the 1-st and 2-nd kind, resp. Consider the following assertions on $[a,b)$:

A/ The sequence of absolute values of local extremes of (the derivative of) an arbitrary solution of (q) is non-increasing.

B/ The sequence of absolute values of local extremes of the derivative of an arbitrary solution /of an arbitrary solution/ is non-dedreasing.

C/ $\frac{q(\psi(t))}{q(t)}\psi'(t) \geqq 1$ $\left(\varphi(t) - t\right.$ is non-decreasing$\left.\right)$.

D/ $\varphi(t) - t$ is non-increasing $\left(\frac{q(\psi(t))}{q(t)}\psi'(t) \leqq 1\right)$.

Then $A \Longleftrightarrow C \Longleftrightarrow B \Longleftrightarrow D$.

<u>P r o o f .</u> The function $\varphi(t) - t$ is non-increasing /non-decreasing/ iff $\varphi' \leqq 1$ $(\varphi' \geqq 1)$.
Let A/ be valid. Let $t_0 \in [a,b)$ be an arbitrary number and y a solution of (q) such that $y(t_0) = 0$. Then $t_k = \varphi(t_{k-1})$ where $\{t_k\}$ is the sequence of the extremants of y'. According to (2) we have

$$\varphi'(t_o) = \frac{y'^2(t_o)}{y'^2(t_1)} \geqq 1 .$$

As t_o is an arbitrary number, we can see that $\varphi' \geqq 1$. Let $\varphi' \geqq 1$. It follows from (2) that for an arbitrary solution y

$$\frac{y'^2(t_k)}{y'^2(t_{k+1})} = \varphi'(t_k) \geqq 1$$

holds where $y(t_k) = 0$. From this we can see that $\{|y'(t_k)|\}$ is non-increasing. So we proved that $A \Longleftrightarrow C$. The relation $B \Longleftrightarrow D$ we can prove in the same way, But we must use (4), too. The rest of the statement follows from the proved part of the theorem and from the relation

$$\frac{q(\psi(t))}{q(t)}\psi' t \leqq \frac{1}{\varphi'(t)} , \quad t \in [a,b)$$

being proved in [2] (Theorem 2) .

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O asymptotických vlastnostiach a rozložení koreňov riešení differenciálnej rovnice $y'' = q(t)y$

M. BARTUŠEK

Súhrn

V práci sa autor zaoberá štúdiom oscilatorickej diferenciálnej rovnice

(q) $y'' = q(t)y, \quad q \in C^0[a,b), \quad b \leqq \infty.$

Prvá časť práce obsahuje výsledky toho druhu, že z istých predpokladov o nosiči (q) sú odvodené niektoré vlastnosti základnej

centrálnej disperzie φ prvého druhu, charakterizujúce rozloženie koreňov netriviálnych riešení rovnice (q) . Vety 1 a 2 dávajú prvý člen asymptotického rozvoja derivácie funkcie φ za predpokladu, že (q) sa chová asymptoticky ako konstanta.

Výsledky druhej časti práce sa vzťahujú na problém: Nájsť nevyhnutné a (alebo) postačujúce podmienky na to, aby všetky netriviálne riešenia rovnice (q) (alebo ich derivácie) boli ohraničené, resp. neohraničené, resp. konvergovali k nule pre $t \to b\_$.

Об асимптотических свойствах и распределении нулей решений дифференциального уравнения $y'' = q(t)y$

М. БАРТУШЕК

Р е з ю м е

В этой работе мы занимаемся осциллирующим дифференциальным уравнением

$$(q) \qquad y'' = q(t)y, \quad q \in C^0 [a,b) , \ b \leq \infty.$$

Первая часть работы содержит результаты такого рода, что из каких-то предположений с функции q выводенны некоторые свойства фундаментальной центральной дисперсии φ первого рода, которая характеризует распределение нулей нетривиальных решений уравнения (q).
Теоремы 1 и 2 дают нам первый член асимтотического разложения производных функции φ когда $\lim_{t \to b} q(t) = c, \quad 0 \leq c < +\infty.$

Результаты второй части касаются проблемы: Найти необходимые и (или) достаточные условия для того, чтобы все нетривиальные решения уравнения (q) (или их производные) были ограниченные или неограниченные или стремились к нулю для $t \to b$.

Received: November 27. 1972,
in press July 8, 1974.

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(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXXII, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXXII – 1975

On a problem of Znám

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In the summer session on theory of numbers in Lubochňa /Slovakia, ČSSR/ in September 1972 Štefan Znám put the following problem.

Let n be a positive integer >1. Are there integers $x_i > 1$ $(1 \leq i \leq n)$ such that for each $1 \leq i \leq n$ the integer x_i is a proper divisor of $x_1 \dots x_{i} \cdot x_{i+1} \dots x_n + 1$?

From the Theorem proved in this paper it follows that for $2 \leq n \leq 4$ there are not such integers x_i . For $n = 5$ Mr. Jaroslav Janák found by means of computer the integers 2,3,11,23,31 satisfying the given condition.

Theorem. The only positive integral solutions $a \leq b \leq c \leq d$, x,y,z,w of the system of equations

(1) $abc + 1 = dx,$

(2) $abd + 1 = cy,$

(3) $acd + 1 = bz,$

(4) $bcd + 1 = aw,$

are given in the following table:

a	b	c	d	x	y	z	w
1	1	1	1	2	2	2	2
1	1	1	2	1	3	3	3
1	1	2	3	1	2	7	7
1	2	3	7	1	5	11	43
2	3	7	43	1	37	201	452

P r o o f . We find out directly that the values given in the table (5) are the solutions of equations (1) - (4).

Let $a \leq b \leq c \leq d$, x, y, z, w be positive integers satisfying the equations (1) - (4).
Then

(6) $(a,x) = (a,y) = (b,x) = (b,y) = 1$.

Thus $a^2b^2 \neq xy$. From (1) and (2) we get

$$(7) \quad c = \frac{ab + x}{xy - a^2b^2}, \qquad d = \frac{ab + y}{xy - a^2b^2} .$$

If $b = c$, then according to $(b,c) = 1$ $a = b = c = 1$ and we get two solutions $d = 1$, $x = y = z = w = 2$ and $d = 2$, $x = 1$, $y = z = w = 3$ given in the table (5).

Let $b < c$. By multiplication (1) and (2) we get $a^2b^2cd < cdxy = a^2b^2cd + abc + abd + 1 < a^2b^2cd + 2acd$ (since $b \leq d - 1$, $b \leq c - 1$ and $a \geq 1$), thus $a^2b^2 < xy < a^2b^2 + 2a$.
Hence there is an integer t such that

(8) $xy = a^2b^2 + t$, $1 \leq t \leq 2a - 1$.

From (6) it follows

(9) $(a,t) = (b,t) = 1$.

According to (7) and (8) it holds

$$(10) \quad c = \frac{ab + x}{t}, \qquad d = \frac{ab + y}{t} .$$

From (3), (4), (8) and (10) it follows: $t^2bz = t^2(acd + 1) = 2a^3b^2 + a^2bx + a^2by + at + t^2$ and $t^2aw = 2a^2b^3 + ab^2x + ab^2y + bt + t^2$, hence $b / t(a + t)$ and $a / t(b + t)$.

From (9) we get

(11) $b / (a + t)$; $a / (b + t)$.

Therefore there exists a positive integer k such that

(12) $kb = a + t$.

Since $k = \frac{a + t}{b} \leq \frac{3a - 1}{a} < 3$, it holds $k = 1$ or $k = 2$. There exists a positive integer l such that $la = b+t$. Then $akl = a + t(k + 1)$ and according to (9) we get four possibilities: $k = 1$, $a = 1$; $k = 1$, $a = 2$; $k = 2$, $a = 1$; $k = 2$, $a = 3$.

A/ Let $k = 1$, $a = 1$. From (8) and (12) we get $t = 1$, $b = 2$ and $xy = 5$. Since $x \leq y$, it holds $x = 1$ and $y = 5$.
Hence (10) gives $c = 3$ and $d = 7$.

B/ Let $k = 1$, $a = 2$. Then $1 \leq t \leq 3$. We will use then (12), (8) and (10) again. For $t = 1$ we get $b = 3$, $xy = 37$,
hence $x = 1$ and $y = 37$. Thus $c = 7$, $d = 43$.

For $t = 3$ we get $b = 5$, $xy = 103$, $x = 1$, $y = 103$ and thus $c = \frac{11}{3}$, which is a contradiction.

According to (9) we cannot have $t = 2$.

C/ Let $k = 2$, $a = 1$. Then by (8) and (12) $t = 1$, $b = 1$. $xy = 2$, $x = 1$, $y = 2$ and from (10) we have $c = 2$, $d = 3$.

D/ Let $k = 2$, $a = 3$. Then $1 \leq t \leq 5$ and from (9) it follows $t = 1,2,3,4,5$. Thus according to (12) $2b = 4,5,7,8$ so that $b = 2$ or $b = 4$. If $b = 2$, then a b. Therefore it holds $t = 5$ and $b = 4$, hence by (8) $xy = 149$, $x = 1$ and $y = 149$. According to (10) we have $c = \frac{13}{5}$, which is a contradiction.

By this the Theorem is proved.

O jednom Známovom probléme

LADISLAV SKULA

S ú h r n

ŠTEFAN ZNÁM položil roku 1972 nasledujúci problém: "Nech n je prirodzené číslo > 1. Existujú celé čísla $x_i > 1$ $(1 \leq i \leq n)$ tak, že pre každé $(1 \leq i \leq n)$ je číslo x_i vlastný[1] deliteľ čísla $x_1 \dots x_{i-1} \cdot x_{i+1} \dots x_n + 1$?" [1] V práci autor ukazuje, že pre $2 \leq n \leq 4$ také celé čísla x_i neexistujú a uvádza výsledok J. JANÁKA získaný počítačom, že pre $n = 5$ čísla 2,3,11, 23,31 vyhovujú daným podmienkam.

Об одной проблеме Знама

ЛАДИСЛАВ СКУЛА

Р е з ю м е

ШТЕФАН ЗНАМ сформулировал в 1972 г. следующую проблему: "Пусть n натуральное число больше 1. Существуют ли такие натуральные числа $x_i > 1$ $(1 \leq i \leq n)$ что для каждого $1 \leq i \leq n$ число x_i является собственным делителем числа $x_1 \dots x_{i-1} \cdot x_{i+1} \dots x_n + 1$?"

В работе показывается что для $2 \leq n \leq 4$ таких чисел x_i нет и для $n = 5$ приведен результат Я. ЯНАКА полученный с помощью вычислительной машины: числа 2, 3, 11, 23, 31 удовлетворяют данным условиям.

Received June 1, 1973,
in press July 8, 1974.

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(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXXII, 1975)
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MATHEMATICA XXXII – 1975

Über die nichtlineare Differentialgleichung in der Form $y''' + p(x)y'' + q(x)f(y') + r(x)h(y, y'') = 0$

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In den Arbeiten [1], [2], [3], [4] sowie in anderen werden die Fragen der Begrenztheit, Unbegrenztheit und Oszillationsfähigkeit bei den Lösungen der Differentialgleichung zweiter Ordnung untersucht. In der vorliegenden Arbeit beweisen wir mit Hilfe von ähnlichen Methoden einige hinreichende Bedingungen für die Unbegrenztheit und Oszillationsfähigkeit der nichtlinearen Differentialgleichung dritter Ordnung der Form

$$y''' + p(x)y'' + q(x)f(y') + r(x)h(y,y'') = 0, \tag{1}$$

wo $p(x)$, $q(x)$, $r(x) \in C^0 \langle a; \infty)$, $f(u) \in C^1(R_1)$, $h(y,v) \in C^0(R_2)$, wobei $a \in (-\infty; \infty)$, $R_1 = (-\infty; \infty)$, $R_2 = R_1 \times R_1$.

In der Arbeit wird vorteilhaft das Bellman-Gronwal-Lemma verwendet daher ist es nachstehend angeführt.

Lemma 1. Es sei c eine nichtnegative Konstante. Für $x \geq a$ sei $u(x) \geq 0$, $v(x) \geq 0$ und

$$u(x) \leq c + \int_a^x u(t)v(t)\,dt,$$

bzw.

$$u(x) \geq c + \int_a^x u(t)v(t)\,dt,$$

dann gilt für $x \geqq a$

$$u(x) \leqq c \exp \left\{ \int_a^x v(t)\, dt \right\},$$

bzw.

$$u(x) \geqq c \exp \left\{ \int_a^x v\; t\; dt \right\}.$$

S a t z 1. Es sei $q(x) \in C^1 \langle a; \infty)$ und es gelte für jedes $x \in \langle a; \infty)$, $u \in R_1$, $(y, v) \in R_2$:

a/ $q(x) \geqq k_1 > 0, \quad 0 \leqq h(y, v)\; v \leqq k_2 v^2$;

b/ $\int_a^\infty \{ -r(t) \}_+ dt = k_3,$

$\int_a^\infty \{ q'(t) \}_+ dt = k_4,$

$\int_a^\infty \{ -p(t) \}_+ dt = k_5,$

c/ $0 \leqq \int_0^u f(s)\, ds = F(u),$

wobei k_1, k_2, k_3, k_4, k_5 Konstanten sind und $\{z(x)\}_+$ den positiven Teil der Funktion $z(x)$ darstellt.

Dann ist die zweite Ableitung jeder Lösung $y(x)$ der Differentialgleichung (1) mit der Eigenschaft

$$k_6 = \frac{1}{2} y''^2(a) + q(a) F\left[y'(a) \right] \geqq 0$$

definiert und im Intervall $\langle a; \infty)$ begrenzt.

Wenn darüber hinaus $F(u) \geqq k_7$ für $|u| \geqq K$, K ist eine positive Konstante und

$$(2) \qquad \frac{k_6}{k_8} \exp\left\{\frac{1}{k_8}(k_2 k_3 + k_4 + k_5)\right\} < k_7 ,$$

wo

$$k_8 = \min\left\{\frac{1}{2}, k_1\right\} ,$$

dann ist auch die erste Ableitung der Lösung $y(x)$ im Intervall $\langle a; \infty)$ begrenzt.

B e w e i s . Die Lösung $y(x)$ der Differentialgleichung (1) sei in einem Intervall $\langle a; \bar{x})$, $\bar{x} \leqq \infty$ definiert.

Wenn man die Differentialgleichung (1) mit $y''(x)$ multipliziert und man nacheinander von a bis x integriert, dann erhält man nach der Umbildung mit Rücksicht auf die Voraussetzungen

$$\frac{1}{2} y''^2(x) + k_1 F[y'(x)] \leqq k_6 + \int_a^x \{q'(t)\}_+ F[y'(t)]\, dt +$$
$$+ \int_a^x k_2 \{-r(t)\}_+ y''^2(t)\, dt + \int_a^x \{-p(t)\}_+ y''^2(t) dt$$

woraus

$$y''^2(x) + F[y'(x)] \leqq \frac{k_6}{k_8} + \frac{1}{k_8} \int_a^x \left[\{q'(t)\}_+ + k_2\{-r(t)\}_+ + \{-p(t)\}_+\right] \left[F(y'(t)) + y''^2(t)\right] dt$$

folgt.

Aus der letzten Ungleichheit mit Rücksicht auf das Lemma 1 ergibt sich für $x \in \langle a; \bar{x})$

$$(3) \quad y''^2(x) + F[y'(x)] \leqq \frac{k_6}{k_8} \exp\{\frac{1}{k_8}(k_2k_3 + k_4 + k_5)\} .$$

Aus der Beziehung (3) ist ersichtlich, dass $y''(x)$ für ein beliebiges $y'(x)$ begrenzt ist. Erwägen wir $\bar{x} < \infty$, dann ist im Intervall $\langle a; \bar{x})$ auch $y'(x)$ begrenzt.

Im weiteren zeigen wir, dass $y'(x)$ im Intervall $\langle a; \infty)$ begrenzt ist. Man setze nun das Gegenteil voraus.

Es sei

$$\lim_{x \to \infty} \sup |y'(x)| = \infty .$$

Dann existiert aber eine solche Folge

$$\{x_n\}_{n=1}^{\infty}, \quad x_n \to \infty \qquad \text{für } n \to \infty,$$

dass

$$\lim_{n \to \infty} |y'(x_n)| = \infty.$$

Da gemäss (3)

$$F[y'(x)] \leqq \frac{k_6}{k_8} \exp\{\frac{1}{k_8}(k_2k_3 + k_4 + k_5)\}$$

ist, dann ist für $x = x_n$ und $n \to \infty$ ein Widerspruch mit der Voraussetzung (2).

Damit ist der Satz bewiesen.

<u>Satz 2</u>. Es sei $q(x) \in C^1 \langle a; \infty)$. Es sei für $x \in \langle a; \infty)$ und für jedes $u \in R_1$, $(y, v) \in R_2$ gelte:

$$1/ \quad 0 \leqq \int_0^u f(s)\,ds = F(u); \; h(y; v)\,v \geqq k_1 v^2 , \quad k_1 > 0;$$

$$2/ \quad q(x) \leqq 0, \quad p(x) + k_1 \{r(x)\}_+ \leqq 0,$$

$$q'(x) \geqq q(x)\left[-2p(x) - 2k_1 \{r(x)\}_+\right].$$

Jede Lösung $y(x)$ der Differentialgleichnug (1), welche im Intervall $\langle a;\infty)$ existiert und für welche im Punkte a

$$\frac{1}{2} y''^2(a) + q(a) F[y'(a)] = k_2 > 0$$

gilt, ist dann für $x \to \infty$ zusammen mit ihrer ersten Ableitung unbegrenzt.

B e w e i s . Multipliziert man die Differentialgleichung (1) mit $y''(x)$ und intergriert man nacheinander von a bis x, dann erhält man

$$\frac{1}{2} y''^2(x) + q(x) F[y'(x)] + \int_a^x p(t) y''^2(t) dt +$$

$$+ \int_a^x r(t) h[y(t), y''(t)] y''(t) dt =$$

$$= \frac{1}{2} y''^2(a) + q(a) F[y'(a)] + \int_a^x q'(t) F[y'(t) dt,$$

woraus mit Rücksicht auf die Voraussetzungen

$$\frac{1}{2} y''^2(x) + q(x) F[y'(x)] \geqq k_2 + \int_a^x q'(t) F[y'(t)] dt -$$

$$- \int_a^x [p(t) + k_1 \{r(t)\}_+ y''^2(t) dt \geqq k_2 -$$

$$- \int_a^x [2p(t) + 2k_1 \{r(t)\}_+]\left[\frac{y''^2(t)}{2} + q(t) F[y'(t)]\right] dt$$

folgt und nach Lemma 1

$$\frac{y''^2(x)}{2} + q(x)F[y'(x)] \geqq k_2 \exp\left\{\int_a^x [-2p(t) - 2k_1\{r(t)\}_+ dt \geqq\right.$$

$$\geqq k_2 ,$$

$$y''^2(x) \geqq 2k_2 ,$$

$$|y''(x)| \geqq \sqrt{2k_2}$$

gilt, woraus ersichtlich ist, dass $y'(x)$ und $y(x)$ für $x \to \infty$ unbegrenzt sind.

F o l g e r u n g 1 . Wenn die Voraussetzungen des Satzes 2 erfüllt sind und wenn ausserdem

$$-\int_a^\infty [p(t) + k_1\{r(t)\}_+]dt = \infty$$

gilt, dann ist auch die zweite Ableitung der Lösung $y(x)$ der Differentialgleichung (1), die im Intervall $\langle a;\infty)$ existiert, für $x \to \infty$ unbegrenzt.

S a t z 3 . Es gelte für alle $x \in \langle a;\infty)$, $u \in R_1$, $(y,v) \in R_2$

a/ $p(x) \leqq 0$, $q(x) \geqq 0$, $r(x) \leqq 0$, $r(x) \to 0$ für $x \to \infty$, $u f(u) > 0$ für $u \neq 0$, $kv^2 \geqq h(y,v)v \geqq 0$, $k > 0$.

Es sei für alle $u \in R_1$ und $x \in \langle x_0;\infty)$, wo $x_0 \geqq a$ genügend gross ist und es gelte

b/ $f'(u) \geqq k_1 > 0$.

Es sei

c/ $$-\int_a^\infty r(x) \exp\left\{\int_a^\infty p(s)\,ds\right\} dx = k_2 < \infty;$$

d/ $\int\limits_a^{\infty} q(x) \exp\left\{\int\limits_a^x p(s)\,ds\right\} dx = \infty$

k, k_1, k_2 sind dabei Konstanten.

Dann ist die erste Ableitung jeder Lösung der Differentialgleichung (1) die im Intervall $\langle a;\infty)$ existiert, im Intervall $\langle a;\infty)$ oszillatorisch.

B e w e i s . Es sei eine Lösung $y(x)$ der Differentialgleichung (1) im Intervall $\langle a;\infty)$ definiert. Man setze voraus, dass $y'(x)$ im Intervall $\langle a;\infty)$ unoszillatorisch ist. Es existiert also ein solches $x_1 \geqq a$, dass für $x \geqq x_1$ entweder $y'(x) > 0$ oder $y'(x) > 0$ ist. In unserem Falle sei $y'(x) > 0$ für $x \geqq x_1$. So ist mit Rücksicht auf die Voraussetzungen $f\left[y'(x)\right] > 0$. Für $x \in \langle x_1;\infty)$ folgt nun aus der Differentialgleichung (1)

$$(4)\qquad \frac{y'''\exp\left\{\int\limits_{x_1}^x p(t)\,dt\right\}}{f(y')} + \frac{p(x)\,y''\exp\left\{\int\limits_{x_1}^x p(t)\,dt\right\}}{f(y')} + \frac{r(x)\,h(y,y'')\exp\left\{\int\limits_{x_1}^x p(t)\,dt\right\}}{f(y')} = -q(x)\exp\left\{\int\limits_{x_1}^x p(t)\,dt\right\}.$$

Durch Integration der Beziehung (4) von x_1 bis x erhält man

$$(5)\qquad \frac{y''(x)\exp\left\{\int\limits_{x_1}^x p(t)\,dt\right\}}{f\left[y'(x)\right]} + \int\limits_{x_1}^x \frac{y''^2(t)\,f'\left[y'(t)\right]}{f^2\left[y'(t)\right]}\cdot \exp\left\{\int\limits_{x_1}^t p(s)\,ds\right\} dt + \int\limits_{x_1}^x \frac{r(t)\,h\left[y(t),\, y''(t)\right]}{f\left[y'(t)\right]}\,.$$

$$\exp\left\{\int_{x_1}^{t} p(s)\,ds\right\} dt = \frac{y''(x_1)}{f[y'(x_1)]} - \int_{x_1}^{x} q(t) \exp\left\{\int_{x_1}^{t} p(s)ds\right\} dt .$$

Da

$$\left[\frac{h(y,v)}{f(u)} - 1\right]^2 = \frac{h^2(y,v)}{f^2(u)} - \frac{2h(y,v)}{f(u)} + 1 \geqq 0$$

folgt mit Hinsicht auf die Voraussetzungen des Satzes aus der Beziehung (5)

$$(6) \quad \frac{y''(x)\exp\left\{\int_{x_1}^{x} p(t)\,dt\right\}}{f\; y'(x)} + \int_{x_1}^{x} \frac{y''^2(t)}{f^2\; y'(t)} \exp\left\{\int_{x_1}^{t} p(s)ds\right\}$$

$$\left[f'(y'(t)) + \frac{1}{2} k^2\, r(t)\, dt\right] \leqq \frac{y''(x_1)}{f[y'(x_1)]} -$$

$$- \frac{1}{2}\int_{x_1}^{x} r(t) \exp\left\{\int_{x_1}^{t} p(s)ds\right\} dt - \int_{x_1}^{x} q(t) .$$

$$\exp\left\{\int_{x_1}^{t} p(s)ds\right\} dt .$$

Da für $x \to \infty$ $r(x) \to 0$ ist und für $x \geqq x_0$ $f'[y'(x)] \geqq k_1 > 0$ ist, existiert ein solches $x_2 = \max\{x_0; x_1\}$, dass für $x \geqq x_2$ und jedes $y'(x)$

$$f'[y'(x)] + \frac{1}{2} k^2 \quad r(x) \geqq 0$$

ist.

Derart erhält man für $x \geqq x_2$

$$\frac{y''(x)}{f[y'(x)]} \exp\left\{\int_{x_2}^{x} p(t)\,dt\right\} = \frac{y''(x_2)}{f[y'(x_2)]} -$$

$$-\frac{1}{2}\int_{x_2}^{x} r(t)\exp\left\{\int_{x_2}^{t} p(s)\,ds\right\} dt - \int_{x_2}^{x} q(t)\,.$$

$$\exp\left\{\int_{x_2}^{t} p(s)\,ds\right\} dt$$

woraus ersichtlich ist, dass ein solches $x_3 \geqq x_2$ existiert, dass für $x \geqq x_3$

$$\frac{y''(x)}{f[y'(x)]} < 0$$

ist. Mit Rücksicht auf die Voraussetzungen des Satzes ist also $y''(x) < 0$ und auch

$$h[y(x),\ y''(x)] \leqq 0$$

für $x \geqq x_3$. Aus der Differentialgleichung (1) geht jetzt für $x \geqq x_3$ hervor

$$y'''(x) \leqq 0.$$

Durch zweifache Integration der letzten Ungleichheit von x_3 bis x erhält man

$$y'(x) \leqq y''(x_3)(x - x_3) + y'(x_3)\,.$$

Das bedeutet aber, dass ein solches $x_4 \geqq x_3$ existiert, dass für $x \geqq x_4$ $y'(x) < 0$ ist. Was ein Widerspruch mit der Voraussetzung ist.

Ähnlich ist der Beweis im Falle $y'(x) < 0$ für $x \geqq x_1$.

S a t z 4. Es gelte für alle $x \in \langle a; \infty)$, $u \in R_1$, $(y,v) \in R_2$

a/ $p(x) \leqq 0$, $q(x) \geqq 0$, $r(x) \geqq 0$, $r(x) \to 0$ für $x \to \infty$, $u\, f(u) > 0$ für $u \neq 0$, $0 \geqq h(y,v)v \geqq kv^2$, wo $k < 0$;

b/ es gelte für alle $u \in R_1$ und $x \in \langle x_0; \infty)$, wo $x_0 \geqq a$ genügend gross ist

$$f'(u) \geqq k_1 > 0 ;$$

c/ $\int_a^{\infty} r(x) \exp \left\{ \int_a^x p(t)\,dt \right\} dx = k_2 < \infty$;

d/ $\int_a^{\infty} q(x) \exp \left\{ \int_a^x p(t)\,dt \right\} dx = \infty$

wobei k, k_1 k_2 Konstanten sind.

Dann ist die erste Ableitung jeder Lösung der Differentialgleichung (1), die im Intervall $\langle a; \infty)$ existiert, im Intervall $\langle a; \infty)$ oszillatorisch.

B e w e i s . Da die Beziehung

$$\left[\frac{h(y,v)}{f(u)} + 1 \right]^2 = \frac{h^2(y,v)}{f^2(u)} + \frac{2h(y,v)}{f(u)} + 1 \geqq 0$$

gilt, ist mit Rücksicht auf die Voraussetzungen des Satzes die Ungleichung

$$-\frac{1}{2}\, r(x) - \frac{k_2}{2}\, r(x)\, v^2 \leqq r(x)\, h(y, v)$$

richtig.

Wenn wir weiter auf ähnliche Weise vorgehen wie beim Beweis des Satzes 3, dann erhalten wir aus der Differentialgleichung (1)

$$(7)\quad \frac{y''(x)}{f[y'(x)]} \exp\left\{\int_{x_2}^{x} p(t)\,dt\right\} + \int_{x_2}^{x} \frac{y''^2(t)}{f^2[y'(t)]}\left[f'(y'(t)) - \frac{1}{2}\, k_2\, r(t)\right] \exp\left\{\int_{x_2}^{t} p(s)\,ds\right\} dt = \frac{y''(x_2)}{f[y'(x_2)]} +$$

$$+ \frac{1}{2} \int_{x_2}^{x} r(t) \exp\left\{\int_{x_2}^{t} p(s)\,ds\right\} dt -$$

$$- \int_{x_2}^{x} q(t) \exp\left\{\int_{x_2}^{t} p(s)\,ds\right\} dt\,,$$

wo $x_2 = \max\{x_0; x_1\}$ wie im Beweis des Satzes 3. Folglich ist für $x \geqq x_2$ und für alle $y'(x)$

$$f'[y'(x)] - \frac{1}{2}\, k^2\, r(x) \geqq 0\,.$$

So geht mit Hinsicht auf die letzte Ungleichung aus der Beziehung (7) die Existenz eines solchen $x_3 \geqq x_2$ hervor, dass für $x \geqq x_3$

$$\frac{y''(x)}{f[y'(x)]} < 0$$

ist und daraus ist $y''(x) < 0$ für $x \geqq x_3$.

Der weitere Vorgang des Beweises ist ähnlich wie beim Satz 3.

Analog wie der Satz 3 und 4 werden auch folgende Sätze bewiesen:

S a t z 5 . Es sein die Voraussetzungen des Satzes 3 derart erfüllt, dass anstatt von $r(x) \to 0$ für $x \to \infty$ und anstatt von b/, c/, d/

$$\frac{1}{2} k^2 \, r(x) + f'(u) \geqq 0,$$

$$\int_a^{\infty} \left[q(x) + \frac{1}{2} r(x) \right] \exp \left\{ \int_a^x p(s) \, ds \right\} dx = \infty$$

ist.

Dann gilt die Behauptung des Satzes 3.

S a t z 6. Es seien die Voraussetzungen des Satzes 4 so erfüllt, dass anstatt von $r(x) \to 0$ für $x \to \infty$ und anstatt von b/, c/, d/

$$f'(u) - \frac{1}{2} k^2 \, r(x) \geqq 0,$$

$$\int_a^{\infty} \left[q(x) - \frac{1}{2} r(x) \right] \exp \left\{ \int_a^x p(t) \, dt \right\} dx = \infty.$$

Dann ist die erste Ableitung jeder Lösung der Gleichung (1), die im Intervall $\langle a; \infty)$ existiert, im Intervall $\langle a; \infty)$ oszillatorisch.

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O nelineárnej diferenciálnej rovnici v tvare $y''' + p(x)y'' + q(x)\, f(y') + r(x)\, h(y, y'') = 0$

LADISLAV MORAVSKÝ

S ú h r n

V práci autor dokázal postačujúce podmienky na to, aby prvá a druhá derivácia riešenia diferenciálnej rovnice

(1) $y''' + p(x)\, y'' + q(x)\, f(y') + r(x)\, h(y', y'') = 0$

bola neohraničená a podmienky, aby riešenie so svojou prvou a druhou deriváciou bolo ohraničené v $\langle a, \infty)$. Ďalej uviedol postačujúce podmienky, za ktorých prvá derivácia riešenia diferenciálnej rovnice (1) má nekonečne mnoho nulových bodov v intervale $\langle a, \infty)$.

Решение дифференциального уравнения $y''' + p(x)\, y'' + q(x) f(y') + r(x) h(y, y'') = 0$

ЛАДИСЛАВ МОРАВСКЫ

Р е з ю м е

В работе доказаны достаточные условия для того, чтобы решение дифференциального уравнения

$$y''' + p(x)\,y'' + q(x)f(y') + r(x)h(y, y') = 0$$

вместе с первой и второй производиными были ограничены в $\langle a; \infty)$. В дальнейшем показаны достаточные условия, для которых первая производная решения уравнения /1/ имеет бесконечное число нулевых точек.

Eingegangen am 25. VII. 1972,
in Verlag am 8. Juli 1974.

Adresse des Autors: Katedra matematiky VŠT, Košice, Švermova 3

(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXXII, 1975)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXXII - 1975

Planárne mapy s predpísanými stupňami vrcholov a oblastí

MARIÁN TRENKLER, Košice

Úvod

Nech $p_k(M)$ resp. $v_k(M)$ označuje počet k-uholníkových oblastí (k-uholníkov), resp. k-valentných vrcholov plenárnej mapy M so súvislým grafom.

Položme si nasledujúcu otázku: Ak máme danú dvojicu postupností $p = (p_1, p_2, p_3, \dots)$ a $v = (v_1, v_2, v_3, \dots)$ z celých nezáporných čísel, či existuje taká planárna mapa M, pre ktorú platí $p_k(M) = p_k$, $v_k(M) = v_k$ pre všetky $k \neq 4$. Keď takáto mapa M existuje, hovoríme, že dvojica postupností p, v je realizovateľná.

Z Eulerovej vety vyplýva nasledujúca nevyhnutná podmienka na realizovanie dvojice p, v

$$\sum_{k\geq 1} (4-k)(p_k + v_k) = 8. \qquad (1)$$

Ďalšia nutná podmienka je $\sum_{k\geq 1} kp_k \equiv 0 \pmod 2$. (2)

V tejto práci je daná nutná a postačujúca podmienka na realizovanie danej dvojice postupností p, v. V špeciálnom prípade, keď platí $p_k = v_k = 0$, pre k=1,2 odpoveď na už položenú otázku dávajú práce [1] a [2]. Na rozdiel od týchto prác pri použití konštrukcie opísanej v tejto práci mapa M obsahuje podstatne menší počet štvoruholníkov a štvorvalentných vrcholov.

Náš výsledok je sformulovaný v nasledujúcej vete:

V e t a . Dvojica postupností $p = (p_1, p_2, p_3, \dots)$, $v = (v_1, v_2, v_3, \dots)$ z celých nezáporných čísel je realizovateľná práve vtedy, keď spĺňa podmienky (1), (2) a odlišuje sa od dvojíc spĺňajúcich podmienky

$$p_k = v_k = 0 \text{ pre všetky } k \neq 0 \pmod 2 \text{ a } \sum_{k \equiv 2 (\mathrm{mod}\, 4)} v_k = 1 \pmod 2. \tag{3}$$

D ô k a z . Najskôr dokážeme, že dvojica postupností p, v spĺňajúca (1), (2), (3) nie je realizovateľná.

MALKEVITCH dokázal [3, str. 16], že dvojica p, v nie je realizovateľná, keď spĺňa (1), (2) a platí $p_k = v_k = 0$ pre všetky $k \neq 0 \pmod 2$ a $\sum_{k \equiv 2 (\mathrm{mod}\, 4)} v_k = 1$.

Predpokladajme, že daná dvojica postupností p, v spĺňajúca (1), (2), (3) je realizovateľná; z toho vyplýva, že existuje príslušná mapa M. V tejto mape spárujme navzájom všetky 2(mod4) - valentné vrcholy, okrem jedného. Pre každú dvojicu vrcholov, ktorá vznikla spárovaním, zvoľme jednu jednoduchú cestu, ktorá ich spája. Každú hranu tejto cesty doplníme dvoma novými hranami, ktorých vrcholy budú totožné s vrcholmi dopĺňanej hrany. Takýmto doplnením hrán zväčší sa násobnosť všetkých vrcholov zvolenej cesty o štyri, okrem dvojice 2(mod4) -valentných vrcholov, ktoré sa stanú 0(mod4) - valentné, pričom vzniknú len dvojuholníky. Takto vznikne mapa, ktorá podľa už uvedenej vety neexistuje.

Že podmienky postačujú, dokážeme tak, že pre každú dvojicu postupností p,v spĺňajúcu nevyhnutné podmienky, opíšeme konštrukciu mapy M, ktorá obsahuje predpísaný počet k-uholníkov a k-valentných vrcholov pre všetky $k \neq 4$, štvoruholníkov a štvorvalentných vrcholov.

I. N e c h p l a t í $\sum_{k\geq 5} v_k = 0$

I.1. $v_2 \equiv 0 \pmod 2$

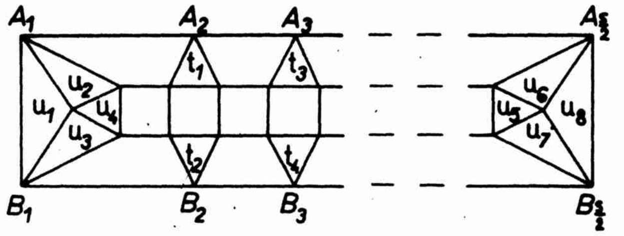
Pri konštrukcii vyjdeme podľa potreby z jednej zo štvorvalentných máp M_1, M_2 alebo M_3. Mapu M_1 použijeme vo všetkých prípadoch, keď $s \equiv 0 \pmod 2$ a súčasne neplatia obe nasledujúce podmienky $p_1 + p_3 = 0$, $v_1 \neq 0$, mapu M_2, keď $s \equiv 0 \pmod 2$ a nemôžeme použiť M_1 a mapu M_3 v prípade, keď $s \equiv 1 \pmod 2$, kde $s = \sum_{k\geq 5} (k-4) p_k$.

Mapa M_1 (obr.2) obsahuje jeden s-uholník $A_1 A_2 \ldots A_{\frac{s}{2}} B_{\frac{s}{2}} \ldots B_1$, 2s-5 štvoruholníky a s+8 trojuholníky, ktoré označíme podľa obrázku $u_1, u_2, \ldots, u_8, t_1, t_2, \ldots, t_s$. Mapa M_2 (obr.3) obsahuje jeden s-uholník $A_1 \ldots A_{\frac{s+2}{2}} B_{\frac{s}{2}} \ldots B_2$, 2s-7 štvoruholníky a s+8 trojuholníky $u_1, u_2, \ldots, u_6, t_1, \ldots t_{s+2}$. Mapa M_3 (obr.4) obsahuje s-uholník $A_1 A_2 \ldots A_{\frac{s+1}{2}} B_{\frac{s-1}{2}} \ldots B_1$, 2s-6 štvoruholníky a s+8 trojuholníky $u_1, \ldots u_7, t_1, \ldots, t_{s+1}$.

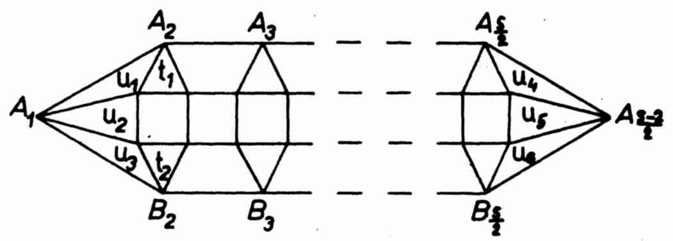
Ďalej vytvoríme z s-uholníka p_k k-uholníkov, $k\geq 5$. Najskôr opíšeme ich vytváranie v mapách M_1 a M_3.

Nech $k=2m$ a $p_k \neq 0$. Vyberieme bod R_1 na hrane $A_{m-1}A_m$ a bod R_2 na hrane $B_{m-1}B_m$. Spojením R_1 s R_2 novou hranou vznikne k-uholník $A_1 \ldots A_{m-1} R_1 R_2 B_{m-1} \ldots B_1$. Vrcholy R_1, R_2 sú trojvalentné; aby boli štvorvalentné spojíme ich cestou pretínajúcou len tri štvoruholníky tak, aby vznikli dva štvorvalentné vrcholy a tri štvoruholníky.

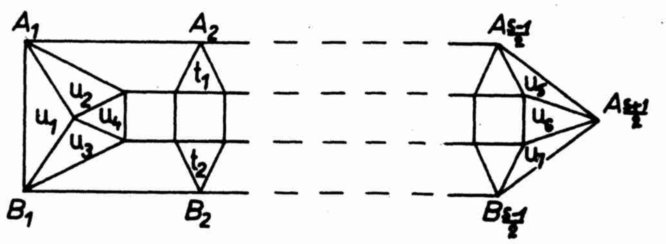
Steny s nepárnym počtom hrán budeme vytvárať po dvojiciach okrem jednej, keď s je nepárne. Nech $k=2m+1$ a $t=2n+1$, pričom $p_k \neq 0$ a $p_t \neq 0$. Pre jednoduchosť indexovania predpokladajme, že z s-uholníka začíname vytvárať k-uholník s t-uholníkom. Najskôr vytvorme $2(m+n-1)$-uholník



obr. 2

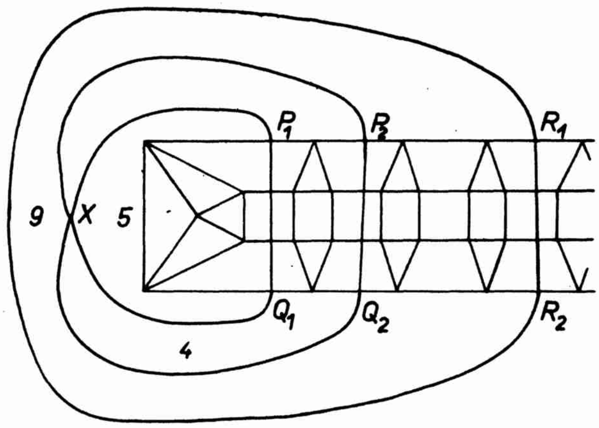


obr. 3



obr. 4

$A_1 \dots A_{m+n-2}R_1R_2B_{m+n-2} \dots B_1$. Vyberme bod P_1 na hrane $A_{m-1}A_m$, bod P_2 na $B_{m-1}B_m$, bod Q_1 na A_mA_{m+1}, a bod Q_2 na B_mB_{m+1}, resp. Q_1 na hrane A_mR_1 a Q_2 na B_mR_2, keď t=5. Spojme P_1 hranu s Q_2 a bod P_2 s Q_1 cestou dĺžky dva pretínajúcou hranou P_1Q_2. Takto vznikne dvojica nepárnouholníkov a dva štvoruholníky $XP_1A_mQ_1$ a $XP_2B_mQ_2$, kde X je spoločný vrchol cesty P_1Q_2 s P_2Q_1 . Rovnako ako vyššie vytvoríme z trojvalentných vrcholov P_1,P_2,Q_1,Q_2 štvorvalentné. Na obr. 4 je 5-uholník s 9-uholníkom.



obr.5

Takto vytvoríme všetky predpísané k-uholníky, $k \geq 5$.

Vytvráranie k-uholníkov, $k \geq 5$, v mape M_2 je podobné. Keď existuje také $k=2m+1$, že $p_k \neq 0$, potom vyberieme body R_1 na hrane $A_m A_{m+1}$ a R_2 na $B_m B_{m+1}$ a ich spojením hranou vznikne k-uholník a ďalší postup je rovnaký ako vyššie. Keď takéto k neexistuje, potom každá stena vznikne rovnako ako prvý nepárnouholník v mape M_1 .

Ďalej vytvoríme z trojuholníkov postupne v_3 trojvalentných vrcholov, p_2 dvojuholníkov, p_1 jednouholníkov a v_1 jednovalentných vrcholov. Pritom platí zásada, že okrem niekoľkých výnimiek, pozmieňame trojuholníky v nasledujúcom poradí $u_1, u_2, \ldots, u_i, t_1, t_2, \ldots, t_j, u_{i+1}, \ldots, u_k$, kde $i=4$, $j=s$, $k=8$, keď sme vychádzali z mapy M_1; $i=3$, $j=s+2$, $k=6$ pri M_2 a $i=4$, $j=s+1$, $k=7$ pri použití M_3.

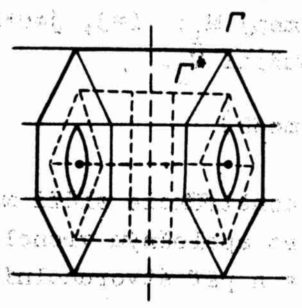
a_1/ $v_1=0$, $p_1 + p_3 \equiv 0 \pmod 2$

V tomto prípade za základ konštrukcie slúži mapa M_1.

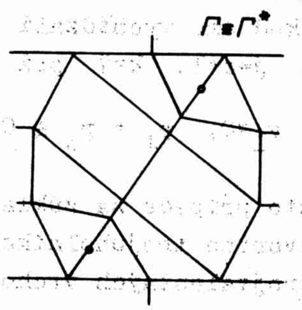
Zo štvorice trojuholníkov so spoločným vrcholom vznikne štvorica trojvalentných vrcholov a päť štvoruholníkov, keď spoločný vrchol nahradíme štvoruholníkom. Keď potrebujeme len dva takéto vrcholy, potom do tohto štvoruholníka doplníme uhlopriečku. Dvojicu trojvalentných vrcholov a tri štvoruholníky vytvoríme z dvojice trojuholníkov t_{2i-1}, t_{2i}, $i=1,2,\dots\frac{s}{2}$, keď na ich hranách, ktoré sú súčasne hranami jedného štvoruholníka, vyberieme dva body a tieto spojíme novou hranou.

Dvojuholník vytvoríme z dvojice trojvalentných vrcholov spojených hranou, keď ich spojíme ešte jednou hranou.

Dvojicu dvojvalentných vrcholov vytvoríme z dvoch dvojuholníkov, ktoré vznikli zo štvorice trojuholníkov so spoločným vrcholom, alebo štvorice trojuholníkov $t_i,\dots,t_{i+3}$, $i=1,3,\dots$, alebo s-1. Tieto dvojuholníky s 11+3x štvoruholníkmi (x je počet bodov vybratých na hrane $A_{\frac{i+3}{2}}A_{\frac{i+5}{2}}$ pri vytváraní k-uholníkov, $k\geq 5$,) tvoria súvislú časť mapy, ktorá je ohraničená grafovou kružnicou Γ. Na obr.6 je Γ nakreslená hrubou čiarou. Rozrežme mapu pozdĺž Γ a k časti obsahujúcej dvojuholníky vytvorme duálnu časť (čiarkovane nakreslená na obr.6). Označme Γ^* kružnicu z hrán spájajúcich vrcholy priradené dualizáciou tým štvoruholníkom, ktorých hrany boli hranami Γ. Pretože počet vrcholov Γ aj Γ^* je rovnaký, môžeme obe časti spojiť tak, aby vznikla mapa, v ktorej namiesto dvoch dvojuholníkov budú dva dvojvalentné vrcholy.(obr.7)

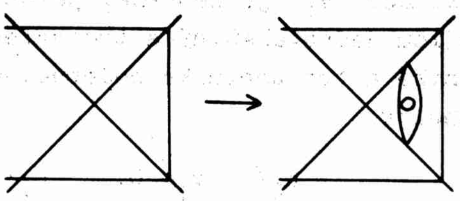


obr. 6

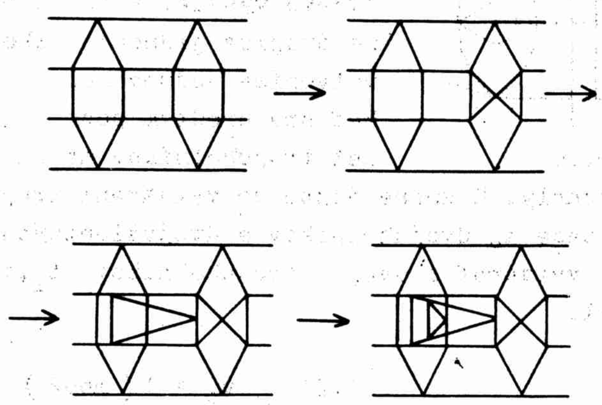


obr. 7

Jednouholník vytvoríme z trojice trojuholníkov so spoločným vrcholom tak, ako je to na obr. 8. Keď takáto trojica trojuholníkov nie je k dispozícii, vytvoríme ju postupne tak, ako je to na obrázku 9.



obr.8



obr.9

$a_2/$ $v_1 = 0$, $p_1 + p_3 \equiv 1 \pmod 2$

V tomto prípade za základ konštrukcie slúži mapa M_3.

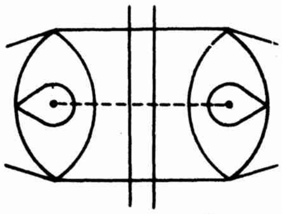
Postup konštrukcie je rovnaký ako v $a_1/$, iba naviac musíme opísať vytvorenie dvojice trojvalentných vrcholov z trojuholníkov u_6 a u_7. Tieto vzniknú spolu s dvoma štvoruholníkmi, keď spoločný vrchol týchto trojuholníkov nahradíme hranou.

$b_1/$ $v_1 \neq 0$, $p_1 + p_3 \neq 0$

V tomto prípade postupujeme rovnako ako v $a_1/$ alebo $a_2/$.

$\left[\frac{v_1}{2}\right]$ dvojíc jednovalentných vrcholov naviac vytvoríme z rovnakého počtu jednouholníkov tak, ako je to čiarkovane nakreslené na obr. 10. Pred vytvorením týchto dvojíc jednouholníkov nevytvoríme všetky jednouholníky, ale iba jeden, alebo jeden trojuholník. Keď v_1 je nepárne, potom aj v_3 je nepárne číslo. Dvojicu jednovalentný a trojvalentný vrchol vytvoríme z jednouholníka a trojuholníka podobne, ako dvojicu jednovalentných vrcholov.

b_2/ $v_1 \neq 0$, $p_1 + p_3 = 0$



obr.10

Pri konštrukcii vychádzame z mapy M_2. Postup konštrukcie je podobný ako vyššie. V tomto prípade vznikne dvojica jednouholníkov, z ktorých vytvoríme jednovalentné vrcholy, keď sme predtým pozmenili párny počet trojuholníkov na iné predpísané oblasti a vrcholy. K zmene dôjde vo vytváraní trojvalentných vrcholov, a teda aj dvojuholníkov a dvojvalentných vrcholov, ktoré budeme vytvárať z dvojíc trojuholníkov t_i, t_{i+2}, $i=1,2,\dots,s-1$.

I.2 $v_2 = 1 \pmod 2$

a/ $$\sum_{5 \leq k \equiv 1 \pmod 2} p_k \geq 2.$$

Keď $p_i \geq 2$ a $i = 1 \pmod 2$ definujme novú dvojicu postupností p', v' nasledujúcim spôsobom

$$p'_i = p_i - 2$$

$$p'_{i-1} = p_{i-1} + 2$$

$$p'_k = p_k \qquad \text{pre všetky } k \neq i,\ i-1,$$

$v'_2 = v_2 - 1$

$v'_t = v_t$ pre všetky $t \neq 2$,

resp. keď dva rovnaké nepárnouholníky nie sú predpísané, potom

$p'_i = p_i - 1$ kde $p_i \neq 0$, $p_j \neq 0$,

$p'_{i-1} = p_{i-1} + 1$ $i,j = 1 \pmod 2$

$p'_j = p_j - 1$

$p'_{j-1} = p_{j-1} + 1$

$p'_k = p_k$ pre všetky $k \neq i, i-1, j, j-1$,

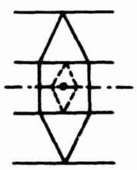
$v'_2 = v_2 - 1$

$v'_t = v_t$ pre všetky $t \neq 2$.

Nech platí súčasne $v'_1 \neq 0$ a $p'_1 + p'_3 = 0$. Najskôr zostrojíme mapu s p'_k k-uholníkmi a v'_k k-valentnými vrcholmi, pre všetky k, v ktorej dvojica (i-1) -uholníkov, resp. jeden (i-1)-uholník a (j-1) -uholník majú spoločnú hranu. Na tej vyberme bod, ktorý bude dvojvalentný vrchol.

Nech súčasne platí $v'_1 \neq 0$ a $p'_1 + p'_3 = 0$. Najskôr zostrojíme mapu, v ktorej budú všetky predpísané vrcholy okrem jedného d dvojvalentného a všetky predpísané steny s dvoma trojuholníkmi naviac. Tieto trojuholníky musia byť umiestnené tak, aby cesta pridaná pri vytváraní dvojice jednovalentných - cholov z dvoch jednouholníkov (bodkočiarkovaná na obr. 11) pretínala štvoruholník, ktorého dve hrany sú hranami trojuholníkov. Ďalší postup je nakreslený čiarkovane na obrázku 11.

obr.11

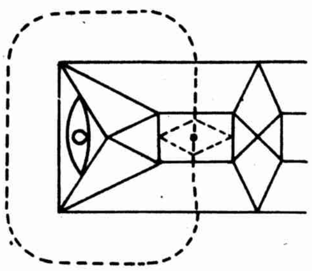


b/ $\sum_{5 \leq k \equiv 1 \pmod 2} p_k = 1.$

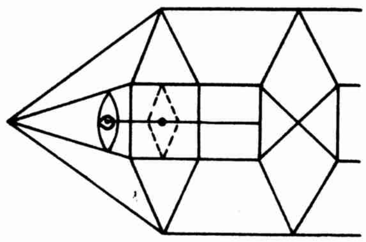
V tomto prípade postupujeme podobne ako v a/. Dvojvalentný vrchol vyberieme na hrane $A_s B_s$, ktorá pred vybratím tohto vrcholu bola spoločnou hranou (k-1) -uholníka, $p_k \neq 0$ a $k \equiv 1 \pmod 2$ a trojuholníka.

c/ $p_k = 0$ pre všetky $5 \leq k \equiv 1 \pmod 2$.

Keď $p_3 \geq 2$, resp. $v_3 \geq 2$, zostrojíme mapu, ktorá obsahuje všetky predpísané vrcholy a oblasti okrem jedného dvojvalentného vrchola a dvoma trojuholníkmi naviac. Tieto s ďalšími dvoma trojuholníkmi, resp. dvoma trojvalentnými vrcholmi vzniknú z trojuholníkov u_5, u_6, u_7, u_8. V oboch prípadoch môžeme dvojicu trojuholníkov zostrojiť tak, aby mali spoločnú hranu, na ktorej vyberieme potrebný vrchol. Keď platí $\left[\frac{p_3}{2}\right] + \left[\frac{v_3}{2}\right] = 0$, potom $p_1 \geq 1$ alebo $v_1 \geq 1$. Postupujeme rovnako ako vyššie, rozdielne bude len vytvorenie dvojvalentného vrchola. V prvom prípade je toto nakreslené čiarkovane na obrázku 12 a v druhom na obrázku 13.



obr.12



obr.13

II. $\sum_{k \geq 5} p_k = 0$

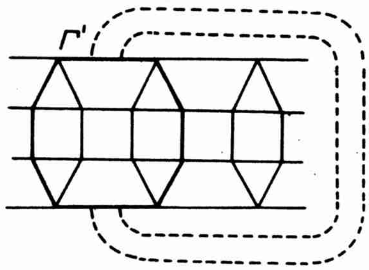
Najskôr zostrojíme mapu M^* , ktorá obsahuje v_k k-uholníky a p_k k-valentné vrcholy pre všetky k. Požadovanú mapu dostaneme, keď urobíme duálnu mapu k M^*.

III. $\sum_{k\geq 5} p_k \neq 0, \quad \sum_{k\geq 5} v_k \neq 0$

a/ Nech súčasne neplatí neplatí $p_2=v_2=0$, $p_3\leq 1$, $v_3\leq 1$ a $\sum_{k\geq 5}(k-4)\,p_k \not\equiv 2 \pmod 3$.

Najskôr rozložíme dané postupnosti na dve dvojice postupností $r=\{r_k\}$, $w=\{w_k\}$ a $r'=\{r'_k\}$, $w'=\{w'_k\}$ tak, aby platilo $p_k = r_k + r'_k$ a $v_k = w_k + w'_k$ pre všetky $k\neq 3$, $p_3 = r_3 + r'_3 - 8$, $r_3\geq 4$, $r'_3\geq 4$ a $v_3 = w_3 + w'_3$, pričom dvojica r,w má spĺňať podmienky prípadu I. a dvojica r', w' podmienky II.

Novo definované postupnosti možno realizovať podľa I. a II. Zostrojme príslušné mapy tak, aby v oboch mapách z $u_1,\dots,u_4$ resp. z $t_i,\dots,t_{i+3}$, $i=1,3,\dots$, alebo s-3, ostala štvorica trojuholníkov, ktorú použijeme pri spájaní oboch máp. Trojuholníky $u_1,\dots,u_4$ resp. $t_i,\dots,t_{i+3}$ spolu s 5+3x alebo 8+4x (x je počet bodov vybratých na hrane $aA_{\frac{i+3}{2}}A_{\frac{i+5}{2}}$ pri vytváraní k-uholníkov, $k\geq 5$,) určujú kružnicu Γ'. Je nakreslená hrubou čiarou na obr. 14. Keď vynecháme tú časť mapy, ktorá obsahuje trojuholníky, štvoruholníky a je ohraničená kružnicou Γ' vznikne stena, ktorej každý vrchol je trojvalentný. Keď je v oboch mapách počet hrán Γ' rovnaký, postupne stotožníme vrcholy kružníc Γ' oboch máp, a takto vykonáme spojenie. V opačnom prípade najskôr musíme zväčšiť počet vrcholov Γ' v jednej z dvojice máp o párny počet tak, aby bol v oboch rovnaký. Toto vykonáme pred vytváraním predpísaných trojvalentných vrcholov tak, že na Γ' vyberieme potrebný počet bodov a tieto po dvojiciach spojíme novými cestami, ktoré vytvoria len ďalšie štvoruholníky. Dve takéto cesty sú čiarkovane nakreslené na obr. 14.



obr.14

b/ Nech platí $p_2=v_2=0$, $p_3\leq 1$, $v_3\leq 1$ a $\sum_{k\geq 5}(k-4)p_k\not\equiv$ $\not\equiv 2(\mathrm{mod}\,3)$.

b_1/ $p_1 \neq 0$

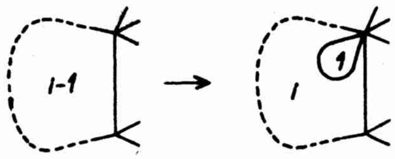
Keď $v_i\neq 0$ pre $i\geq 6$, definujme dvojicu postupností p', v' nasledovne

$$p'_1 = p_1 - 1$$
$$p'_k = p_k - 1 \qquad \text{kde } p_k\neq 0,\ k\geq 5$$
$$p'_{k-1} = p_{k-1} + 1$$
$$p'_j = p_j \qquad \text{pre všetky } j\neq 1,\ k-1, k$$
$$v'_i = v_i - 1$$
$$v'_{i-2} = v_{i-2} + 1$$
$$v'_t = v_t \qquad \text{pre všetky } t\neq i-2, i,$$

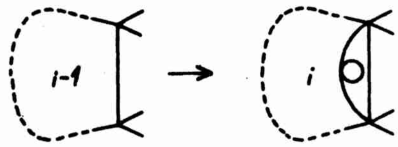
resp., keď $\sum_{i\geq 6} v_i = 0$

$$v'_5 = v_5 - 1$$
$$v'_t = v_t \qquad \text{pre všetky } t\neq 5.$$

Dvojica postupností p', v' spĺňa predpoklady predchádzajúcich prípadov, preto opísanou konštrukciou môžeme vytvoriť mapu, v ktorej jeden (i-2)-valentný vrchol je vrcholom (i-1)-uholníka, resp. tento má dva štvorvalentné vrcholy spojené hranou. V tejto mape doplníme chýbajúci jednouholník, zväčšíme násobnosť jednej strany a jedného resp. dvoch vrcholov tak, ako je to na obrázkoch 15 a 16.



obr.15



obr.16

b_2/ $p_1 = 0$

Požadovanú mapu dostaneme z príslušnej duálnej mapy, ktorú zostrojíme, keď použijeme b_1/.

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Planar maps with prescribed types of vertices and faces and 1-connected graph

MARIÁN TRENKLER

S u m m a r y

Let $p_k(M)$ denote the number of k-gonal faces and $v_k(M)$ denote the nember of k-valent vertices of the planar map M with 1-connected graph. The couple of sequences $p=(p_1,p_2,\dots)$ $v=(v_1,v_2,\dots)$ of non-negative integer numbers we call realisable if there exists a planar map M such that $p_k(M)=p_k$ and $v_k(M)=v_k$ for all $k\neq 4$. In our paper necessary and sufficient condition are given for realiasation of the couple p,v.

Планарные карты с определенной степенью вершин и областей и с 1 - связным графом

МАРИЯН ТРЕНКЛЕР

Р е з ю м е

Пысть $п_к(М)$ обозначает число к - угольников и $в_к(М)$ обозначает число вершин к - той степени планарной карты с 1 - связным графом. Пару последовательностей $п=(п_1,п_2,\dots)$ $в=(в_1,в_2,)$ неотрицательных целых чисел мы навываем реализуемой, если существует планарная карта с 1 - связным графом, как $п_к(М)=пк$ и $в_к(М)=вк$ для всех $к\neq 4$. В этой работе приводятся необходимые и достаточные условия для реализации пары п,в.

Došlo 7. decembra 1972,
do nakladateľstva 8. júla 1974.

Autorova adresa: Katedra matematiky, UJPŠ 04000, Košice, Komenského 14

(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXXII, 1975)

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXXII – 1975

O istom zobrazení normálnej racionálnej krivky K_n do projektívnej roviny P_2

ŠTEFAN NOVOTNÝ, Nitra

Cieľom tohto článku je predovšetkým zobraziť normálnu racionálnu krivku K_n projektívneho priestoru P_n, $n \geq 3$, zo stredu premietania O_{n-3} do projektívnej roviny P_2. Na základe tohto premietania môžeme riešiť základné incidenčné úlohy o normálnej racionálnej krivke K_n. Ide tu o také úlohy, ako sú: nájsť $(n+4)$-tý bod na krivke K_n, ktorá je určená $(n+3)$-mi základnými bodmi; určiť dotyčnicu v jednom zo základných bodov ku krivke K_n a pod. Takéto úlohy sú analogické s úlohami o jednoduchej kužeľosečke, ktoré riešime pomocou Pascalovej vety o 6-uholníku vpísanom do tejto kužeľosečky. Práve pomocou Pascalovej vety a istého zobrazenia Z, ktoré autor ďalej rozvádza, dajú sa takéto úlohy jednoducho vyriešiť.

Najprv však uvedieme niektoré definície, predtým už dokázané vety a lemmy, ktoré nám poslúžia v ďalšej časti.

Nech v projektívnom priestore P_n, $n \geq 3$, nad telesom komplexných čísel C, je daná normálna racionálna krivka K_n. $(n+4)$-uholník $(A_1A_2A_3\dots A_n\dots A_{n+4})$ s vrcholmi A_i, $i=1,\dots,n+4$, sa nazýva <u>vpísaným</u> do normálnej racionálnej krivky K_n, ak body A_i, A_j pre každé $i \neq j$, $i,j=1,\dots,n+4$, ležia na krivke K_n; priamky $A_{\bar{i}}A_{\overline{i+1}}$ /$\bar{i}=i \bmod (n+4)$; $i=1,\dots,n+4$/ nazývame <u>strany</u> uvedeného $(n+4)$-uholníka. Nadrovinu $(A_{\overline{i+1}}\dots A_{\overline{i+n}})$, určenú bodmi $A_{\bar{j}}$ $(\bar{j}=\overline{i+1},\dots,\overline{i+n})$ nazývame pritiľahlou k strane $(A_{\overline{i+n+2}}A_{\overline{i+n+3}})$. Platí Chaslesova.

<u>Veta 1.</u> V $(n+4)$-uholníku $(A_1A_2A_3\dots A_n\dots A_{n+4})$, ktorý je vpísaný do normálnej racionálnej krivky K_n projektívneho

priestoru P_n, vytínajú nadroviny $(A_{\overline{i+1}}A_{\overline{i+2}}A_{\overline{i+3}}\dots A_{\overline{i+n}})$, $(A_{\overline{i+2}}A_{\overline{i+3}}A_{\overline{i+4}}\dots A_{\overline{i+n+1}})$, $(A_{\overline{i+3}}A_{\overline{i+4}}A_{\overline{i+5}}\dots A_{\overline{i+n+2}})$, na protiľahlých stranách $(A_{\overline{i+n+2}}A_{\overline{i+n+3}})$, $(A_{\overline{i+n+3}}A_{\overline{i+n+4}})$, $(A_{\overline{i+n+4}}A_{\overline{i+1}})$ trojicu bodov ${}^{i_1}S_o$, ${}^{i_2}S_o$, ${}^{i_3}S_o$, ktoré spolu s vrcholmi $A_{\overline{i+3}},\dots A_{\overline{i+n}}$ ležia v jednej nadrovine ${}^{i}S_{n-1}$.

V skrátenom zápise veta znie:

${}^{i}S_{n-1} \equiv ({}^{i_1}S_o\ {}^{i_2}S_o\ {}^{i_3}S_o\ A_{\overline{i+3}}\dots A_{\overline{i+n}})$, kde

$$
\begin{aligned}
{}^{i_1}S_o &\equiv (A_{\overline{i+1}}A_{\overline{i+2}}A_{\overline{i+3}}\dots A_{\overline{i+n}}) \cap (A_{\overline{i+n+2}}A_{\overline{i+n+3}}),\\
{}^{i_2}S_o &\equiv (A_{\overline{i+2}}A_{\overline{i+3}}A_{\overline{i+4}}\dots A_{\overline{i+n+1}}) \cap (A_{\overline{i+n+3}}A_{\overline{i+n+4}}), \qquad (1)\\
{}^{i_3}S_o &\equiv (A_{\overline{i+3}}A_{\overline{i+4}}A_{\overline{i+5}}\dots A_{\overline{i+n+2}}) \cap (A_{\overline{i+n+4}}A_{\overline{i+1}}).
\end{aligned}
$$

Analytický spôsob dôkazu tejto vety pomocou úplnej matematickej indukcie je uverejnený v zborníku Acta technologica agriculturae V.; Nitra 1971 pod názvom "Dôkaz rozšírenia Pascalovej vety na racionálne krivky priestoru P_n", Š. NOVOTNÝ.

D e f i n í c i a 1 . Chaslesovým bodom S_o budeme rozumieť priesečník nadroviny $(A_{\overline{i+1}}\dots A_{\overline{i+n}})$ s protiľahlou stranou $(A_{\overline{i+n+2}}A_{\overline{i+n+3}})$ kde body $A_{\overline{j}}$, $\overline{j}=\overline{i+1},\dots,\overline{i+n},\dots,\overline{i+n+4}$ sú vrcholy $(n+4)$-uholníka $(A_1\dots A_n\dots A_{n+4})$ vpísaného do normálnej racionálnej krivky K_n.

D e f i n í c i a 2 . Chaslesovou priamkou S_1 budeme rozumieť takú priamku projektívneho priestoru P_n, ktorá prechádza dvomi rôznymi Chaslesovými bodmi S_o, S_o', pričom platí:

$$
\begin{aligned}
S_o &\equiv (A_{\overline{i+1}}A_{\overline{i+2}}A_{\overline{i+3}}\dots A_{\overline{i+n}}) \cap (A_{\overline{i+n+2}}A_{\overline{i+n+3}}), \qquad (2)\\
S_o' &\equiv (A_{\overline{i+n+2}}A_{\overline{i+n+3}}A_{\overline{i+3}}\dots A_{\overline{i+n}}) \cap (A_{\overline{i+1}}A_{\overline{i+2}}).
\end{aligned}
$$

D e f i n í c i a 3 . Chaslesovou nadrovinou ${}^{i}S_{n-1}$ rozumieme nadrovinu zo vzťahov (1) .

L e m m a 1. Chaslesova nadrovina $^1(^3,\ldots,^nS_{n-1})$ inciduje s vrcholmi $A_3,\ldots,A_n$ a so šiestimi Chaslesovými bodmi $^1S_o^1$, $^1S_o^2,\ldots,\ ^1S_o^6$, kde

$$
\begin{aligned}
^1S_o^1 &\equiv (A_1A_2A_3 \ldots A_n)\cap(A_{n+2}A_{n+3}),\\
^1S_o^2 &\equiv (A_2A_3A_4 \ldots A_{n+1})\cap(A_{n+3}A_{n+4}),\\
^1S_o^3 &\equiv (A_3A_4A_5 \ldots A_{n+2})\cap(A_{n+4}A_1),\\
^1S_o^4 &\equiv (A_{n+2}A_{n+3}A_3\ldots A_n)\cap(A_1A_2),\\
^1S_o^5 &\equiv (A_3\ldots A_nA_{n+3}A_{n+4})\cap(A_2A_{n+1}),\\
^1S_o^6 &\equiv (A_3\ldots A_nA_{n+4}A_1)\cap(A_{n+1}A_{n+2}).
\end{aligned}
\tag{3}
$$

L e m m a 2. Každá Chaslesova nadrovina inciduje práve s tromi Chaslesovými priamkami.

Dôkazy liemm 1,2 a definície 1,2,3 sú uverejnené v zborníku Acta technologica agriculturae IX.,Nitra pod názvom "Konfigurácie odvodené z Chaslesových podpriestorov", Š. NOVOTNÝ.

Pre n=2 veta 1 sa zmení na známu Pascalovu

V e t u 2. Tri dvojice protiľahlých strán 6-uholníka vpísaného do jednoduchej kužeľosečky pretínajú sa v troch bodoch, ktoré ležia na jednej priamke.

V skrátenom zápise veta znie:

$$^i\pi_1 \equiv (^{i_1}{}_1\pi_o\ {}^{i_2}{}_2\pi_o\ {}^{i_3}{}_3\pi_o),\ \text{kde}$$

$$
\begin{aligned}
^{i_1}{}_1\pi_o &\equiv (A_{\overline{i+1}}A_{\overline{i+2}})\cap(A_{\overline{i+4}}A_{\overline{i+5}}),\\
^{i_2}{}_2\pi_o &\equiv (A_{\overline{i+2}}A_{\overline{i+3}})\cap(A_{\overline{i+5}}A_{\overline{i+6}}),\\
^{i_3}{}_3\pi_o &\equiv (A_{\overline{i+3}}A_{\overline{i+4}})\cap(A_{\overline{i+6}}A_{\overline{i+1}}),\ \text{pričom}\ \bar{i} = i\ \text{mod}6,\\
&\qquad i=1,\ldots,6.
\end{aligned}
\tag{4}
$$

D e f i n í c i a 4 . Priesečníky ${}^{i}{}_{j}\pi_{0}$, j=1,2,3, protiľahlých strán 6-uholníka $(A_1 \dots A_6)$, ktorý je vpísaný do jednoduchej kužeľosečky K_2, budeme nazývať Pascalove body.

Označme (n-3) -rozmerný podpriestor určený ľubovoľnými (n-2) vrcholmi (n+4) -uholníka ($A_1 \dots A_n \dots A_{n+4}$) vpísaného do normálnej racionálnej krivky K_n projektívneho priestoru P_n ako $O_{n-3} = (A_{\bar{i}+3} \dots A_{\bar{i}+n})$, $\bar{i}$=i mod(n+4) , i=0,1,....,n, ...n+4, a nazývajme ho stred premietania. Body $A_{\bar{i}+3}, \dots, A_{\bar{i}+n}$ sú ľubovoľné (n-2) vrcholy daného (n+4)-uholníka. Dvojrozmernú rovinu P_2, ktorá neinciduje ani s jedným z bodov $A_{\overline{i+3}}, \dots, A_{\overline{i+n}}$, budeme nazývať priemetňa. Premietanie podpriestorov P_p, p=0,1,...,n-1, zo stredu premietania O_{n-3} do priemetne P_2 označujme Z.

V e t a 3 . V premietaní Z zobrazuje sa normálna racionálna krivka K_n do jednoduchej kužeľosečky K_2 a Chaslesova nadrovina S_{n-1}, incidujúca s podpriestorom O_{n-3} sa v tomto zobrazení premieta do Pascalovej priamky π_1.

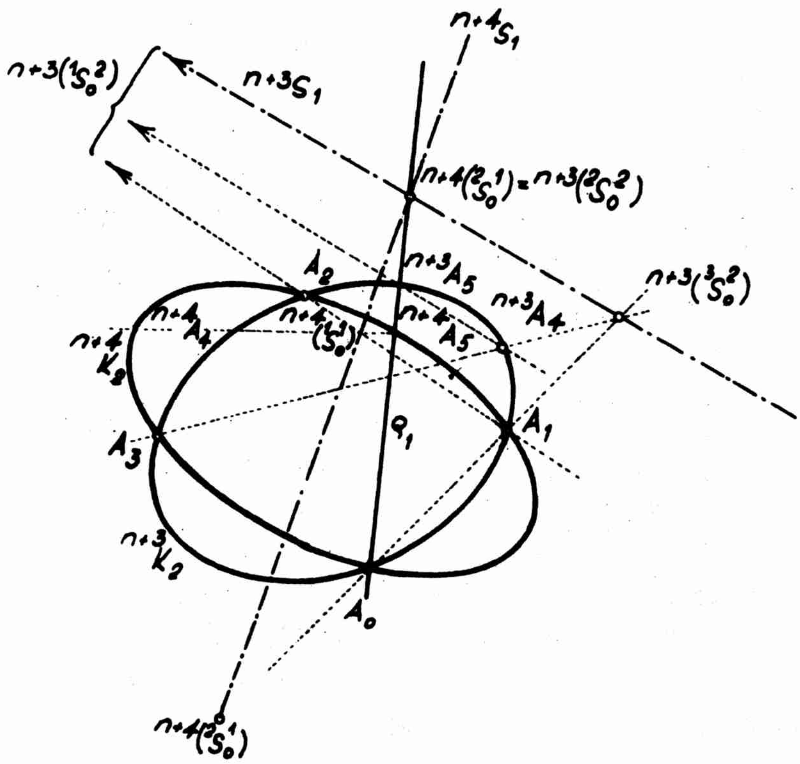
D ô k a z . 1. Označme $(A_1 \dots A_n \dots A_{n+4})$ ako (n+4) -uholník vpísaný do normálnej racionálnej krivky K_n projektívneho priestoru P_n a stred premietania $O_{n-3} \equiv (A_3 \dots A_n)$. Obrazy zvyšných 6 vrcholov v premietaní Z označme $\acute{A}_1$, $\acute{A}_2$, $\acute{A}_{n+1}$, $\acute{A}_{n+3}$, $\acute{A}_{n+4}$, $\acute{A}_{n+2}$.

2. Hľadajme ďalej obraz Chaslesovej nadroviny $S_{n-1} \equiv$ $\equiv ({}^{1}S_{0}^{1}\,{}^{1}S_{0}^{2}\,{}^{1}S_{0}^{3}\,{}^{1}S_{0}^{4}\;{}^{1}S_{0}^{5}\;{}^{1}S_{0}^{6} A_3 \dots A_n)$ uvedenej v lemme 1. Platí: $S_{n-1} \cap P_2 \equiv \pi_1$, kde π_1 je priamka a to preto, že podpriestory S_{n-1} a P_2 sa pretínajú práve v priamke.

3. Body ${}^{1}\pi_{0}^{4}$, ${}^{2}\pi_{0}^{5}$, ${}^{3}\pi_{0}^{6}$, kde ${}^{1}\pi_{0}^{4} \equiv (\acute{A}_1\acute{A}_2) \cap$ $\cap (\acute{A}_{n+2}\acute{A}_{n+3})$, ${}^{2}\pi_{0}^{5} \equiv (\acute{A}_2\acute{A}_{n+1}) \cap (\acute{A}_{n+3}\acute{A}_{n+4})$, ${}^{3}\pi_{0}^{6} \equiv$ $\equiv (\acute{A}_{n+1}\acute{A}_{n+2}) \cap (\acute{A}_{n+4}\acute{A}_1)$, ležia na priamke π_1 .

4. Z obrátenej vety k Pascalovej vete 2 vyplýva, že body $\acute{A}_1, \acute{A}_2, \acute{A}_{n+1}, \acute{A}_{n+2}, \acute{A}_{n+3}, \acute{A}_{n+4}$ ležia na nejakej jednoduchej kužeľosečke K_2 a $\mathcal{T}_1$ je vzhľadom na 6-uholník ($\acute{A}_1\acute{A}_2\acute{A}_{n+1}\acute{A}_{n+2}\acute{A}_{n+3}\acute{A}_{n+4}$), vpísaný do kužeľosečky K_2, Pascalovou priamkou.

__Úloha 1.__ V projektívnom priestore P_n, $n \geq 3$, sú dané body $A_1, \ldots, A_4, A_6, \ldots, A_n, \ldots, A_{n+4}$ v počte $(n+3)$ tak, že žiadne z nich v počte $(n+1)$ neležia v jednej nadrovine. Týmito $(n+3)$-mi bodmi je v priestore P_n určená normálna racionálna krivka K_n. V priestore P_n zvoľme bod X tak, aby s bodmi $A_6, \ldots, A_n, \ldots, A_{n+4}$ v počte $(n-1)$-určoval jednoznačne nadrovinu ϱ_{n-1}. Treba nájsť $(n+4)$-tý bod A_5 krivky K_n tak, aby ležal v nadrovine ϱ_{n-1}.



obr.17

<u>Riešenie</u> (Pozri obr. 17 , kde je znázornené riešenie pre $n=3$):

1. Považujme rovinu $\pi \equiv (A_1A_2A_3)$ za priemetňu a $(n-3)$ -rozmerné podpriestory ${}^iO_{n-3} \equiv (A_{i_7} \dots A_{i_n} A_{i_{n+1}} \dots A_{i_{n+4}})$, ktoré sú určené niektorou skupinou $(n-2)$ bodov z bodov $A_6, \dots, A_n, A_{n+1}, \dots, A_{n+4}$ (v počte $n-1$) za stredy premietania.

Vieme, že počet takýchto skupín je $j = \binom{n-1}{n-2} = n-1$. Ak vynecháme i-tý bod, $i=6,\dots,n,n+1,\dots,n+4$, z bodov $A_6,\dots,A_n, A_{n+1},\dots,A_{n+4}$, označíme stred premietania ${}^iO_{n-3}$. Takéto premietania budeme označovať ${}^iZ$.

2. Označme spoločný bod priemetne π a podpriestoru $P_{n-2} \equiv (A_6 \dots A_n A_{n+1} \dots A_{n+4})$ ako A_o.

3. Premietajme bod A_4 do priemetne π zo stredov premietania ${}^iO_{n-3}$ a jeho priemety označme ${}^iA_4$. Priemetom vynechaného bodu A_i, $i=6,\dots,n,n+1,\dots,n+4$, v týchto projekciách bude stále ten istý bod A_o.

4. Body A_1, A_2, A_3 v premietaniach ${}^iZ$ budú samodružné a spolu s bodmi A_o, ${}^iA_4$ určujú pre každé i jednoduchú bodovú kužeľosečku ${}^iK_2$. Totiž, žiadne tri body $A_k, A_r, A_s, k \neq r \neq s$, $k,r,s = 0,1,\dots 4$, z bodov A_o, A_1, A_2, A_3, A_4, neležia na jednej priamke. Keby nejaká takáto trojica bodov incidovala s jednou priamkou, potom v jednej nadrovine, určenej podpriestorom P_{n-2} a niektorým z bodov A_1, A_2, A_3, by ležal aspoň jeden ďalší z bodov $A_1, \dots, A_4, A_6, \dots, A_n, A_{n+1}, \dots, A_{n+4}$. Znamená to, že v jednej nadrovine by ležalo $(n+1)$ daných bodov, čo je v spore s predpokladom úlohy.

5. Nadrovina $\rho_{n-1} \equiv (A_6 \dots A_n A_{n+1} \dots A_{n+4} X)$ pretína priemetňu π v priamke π_1, ktorá prechádza bodom A_o.

6. Ak použijeme Chaslesovu vetu pre $(n+4)$ -uholník $(A_1 \dots A_n A_{n+1} \dots A_n)$ vpísaný do krivky K_n, vtedy pre $i=n+4$ platí:

$^{n+4}O_{n-3} \equiv (A_6 \dots A_n A_{n+1} \dots A_{n+4}),$

$(A_1A_2) \cap (A_4A_5A_6 \dots A_n A_{n+1} \dots A_{n+3}) \equiv {}^1S_o^1 ,$

$(A_2A_3) \cap (A_6A_7 \dots A_n A_{n+1} \dots A_{n+4} X) \equiv {}^2S_o^1 ,$

$(A_3A_4) \cap (A_6A_7 \dots A_n A_{n+1} \dots A_{n+4} A_1) \equiv {}^3S_o^1$, kde

${}^1S_{n-1} \equiv ({}^1S_o^1 \; {}^2S_o^1 \; {}^3S_o^1 \; A_6 \dots A_n A_{n+1} \dots A_{n+3})$ je Chaslesova nadrovina .

Zatiaľ čo Chaslesove body ${}^2S_o^1$, ${}^3S_o^1$ sú predchádzajúcimi vzťahmi určené, bod ${}^1S_o^1$ nie, pretože bod A_5 nie je daný. Avšak ${}^1S_{n-1}$ je n bodmi ${}^2S_o^1, {}^3S_o^1, A_6, \dots, A_n, A_{n+1}, \dots A_{n+3}$ dostatočne určená, pretože tieto body sú lineárne nezávislé. Nadrovina ${}^1S_{n-1}$ pretína priemetňu π v Pascalovej priamke ${}^{n+4}S_1 \equiv / {}^{n+4}({}^2S_o^1)$ ${}^{n+4}({}^3S_o^1)$ / pre 6-uholník $(A_oA_1A_2A_3 {}^{n+4}A_4 {}^{n+4}A_5)$ vpísaný do kužeľosečky ${}^{n+4}K_2$. Pascalove body ${}^{n+4}({}^2S_o^1), {}^{n+4}({}^3S_o^1)$ nájdeme pomocou Pascalovej vety 2 takto:

$(A_oA_1) \cap (A_3 {}^{n+4}A_4) \equiv {}^{n+4}({}^3S_o^1),$

$(A_2A_3) \cap \rho_1 \equiv {}^{n+4}({}^2S_o^1),$

$(A_1A_2) \cap {}^{n+4}S_1 \equiv {}^{n+4}({}^1S_o^1).$

Hľadaný bod ${}^{n+4}A_5 \equiv \rho_1 \cap / {}^{n+4}A_4 {}^{n+4}({}^1S_o^1) /$. Nájdený bod ${}^{n+4}A_5$ je priemetom bodu A_5 v premietaní ${}^{n+4}Z$. Z predchádzajúceho vyplýva, že bod A_5 leží v (n-2) -rozmernom podpriestore ${}^{n+4}X_{n-2} \equiv (A_6 \dots A_n A_{n+1} A_{n+2} A_{n+3} {}^{n+4}A_5)$, ktorý je obsiahnutý v nadrovine ρ_{n-1}.

7. Nech ďalej platí, že i=n+3. Vtedy ${}^{n+3}O_{n-3} \equiv (A_6 \dots A_n A_{n+1} A_{n+2} A_{n+4})$ je stred premietania. Toto premietanie bu-

deme označovať ${}^{n+3}Z$ a priemet bodu A_4 v ňom ako ${}^{n+3}A_4$. Opäť použijeme Chaslesovu vetu a budeme skúmať Chaslesovu nadrovinu ${}^2S_{n-1}$, incidujúcu so stredom premietania ${}^{n+3}O_{n-3}$. Platí:

$$(A_1A_2)\cap(A_4A_5A_6\ldots A_nA_{n+1}A_{n+2}A_{n+4}) \equiv {}^1S_o^2 \not\equiv {}^1S_o^1$$

$$(A_2A_3)\cap(XA_6\ldots A_nA_{n+1}A_{n+2}A_{n+3}A_{n+4}) \equiv {}^2S_o^2 \equiv {}^2S_o^1 ,$$

$$(A_3A_4)\cap(A_6\ldots A_nA_{n+1}A_{n+2}A_{n+4}A_{n+3}A_1) \equiv {}^3S_o^2 \equiv {}^3S_o^1 , \text{ pričom}$$

Chaslesova nadrovina ${}^2S_{n-1} \equiv ({}^1S_o^2\ {}^2S_o^2\ {}^3S_o^2A_6\ldots A_nA_{n+1}A_{n+2}A_{n+4}$ $A_{n+4}) \equiv {}^1S_{n-1}$.

Chaslesova nadrovina ${}^2S_{n-1}$ je n bodmi ${}^2S_o^2$, ${}^3S_o^2$, $A_6, \ldots, A_n$, $A_{n+1}, A_{n+2}, A_{n+4}$ dostatočne určená a pretína priemetňu π v Pascalovej priamke ${}^{n+3}S_1 \equiv /{}^{n+3}({}^2S_o^2)\ {}^{n+3}({}^3S_o^2)$ /, pričom ${}^{n+3}({}^3S_o^2) \equiv (A_3{}^{n+3}A_4)\cap(A_oA_1)$ a ${}^{n+3}({}^2S_o^2) \equiv {}^{n+4}({}^2S_o^1)$;

${}^{n+3}({}^1S_o^2) \equiv (A_1A_2)\cap{}^{n+3}S_1$ a ${}^{n+3}A_5 \equiv /\ \rho_1\cap{}^{n+3}A_4{}^{n+3}({}^1S_o^2)$ / je priemetom hľadaného bodu A_5 v premietaní ${}^{n+3}Z$.

Opäť platí, že bod A_5 leží v (n-2)-rozmernom podpriestore ${}^{n+3}X_{n-2} \equiv (A_6\ldots A_nA_{n+1}A_{n+2}A_{n+4}{}^{n+3}A_5) \not\equiv {}^{n+4}X_{n-2}$.

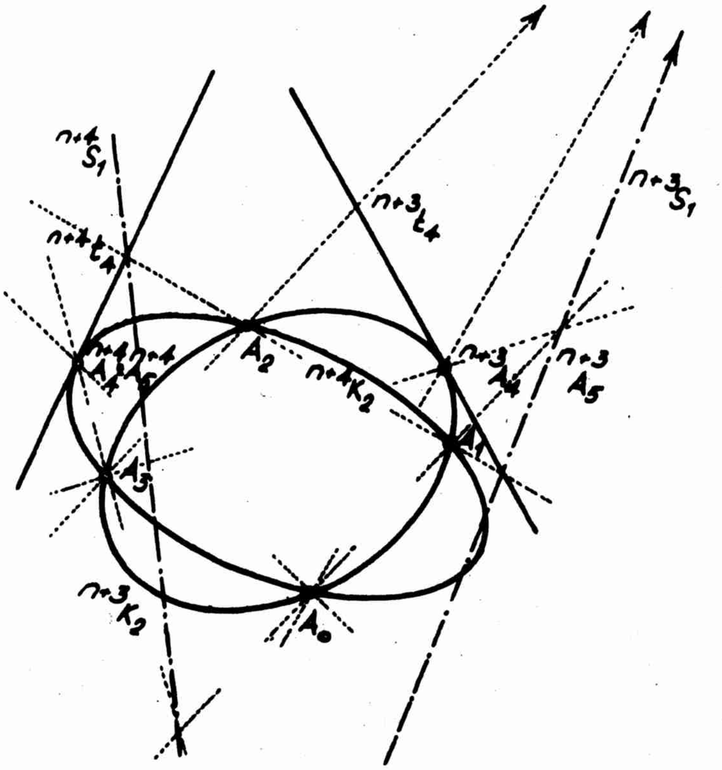
8. Takýmto spôsobom by sme mohli pokračovať dotiaľ, kým by sme nevylúčili každý bod raz z podpriestoru P_{n-2}. Ku každému vylúčeniu jedného bodu A_i, $i=6,\ldots,n,n+1,\ldots,n+4$ existuje premietanie ${}^iZ$ a priemet ${}^iK_2$ normálnej racionálnej krivky K_n. Každý takýto priemet je jednoduchá kužeľosečka, patriaca do zväzku kužeľosečiek

$$\mu_1{}^{n+4}K_2 + \mu_2{}^{n+3}K_2 = 0 , \qquad (\mu_1,\mu_2) \neq (0,0). \quad (5)$$

Ku každej kužeľosečke ${}^iK_2$ prislúcha Pascalova priamka ${}^iS_1$, incidujúca s pevným Pascalovým bodom ${}^{n+4}({}^2S_o^1) \equiv /(A_2A_3)$ $\cap\ \rho_1/$. Najpodstatnejšie je však to, že bod A_5 leží v ďalšom, novom (n-2)-rozmernom podpriestore ${}^iX_{n-2} \equiv (A_6\ldots A_{i-1}A_{i+1}\ldots$ $\ldots A_nA_{n+1}\ldots A_{n+4}{}^iA_5)$.

9. Vieme, že počet premietaní ^{i}Z je $i = n-1$, a teda aj podpriestorov $^{i}X_{n-2}$ je n-1, pričom každý takýto podpriestor leží v danej nadrovine ρ_{n-1} a obsahuje bod A_5. Nech $n' = n-1$. Potom $\rho_{n'}$ môžeme považovať za projektívny priestor a podpriestor $^{i}X_{n'-1}$ za nadrovinu tohto podpriestoru. Pretože nadrovín $^{i}X_{n'-1}$ je n, potom ony určujú hľadaný bod A_5.

<u>Úloha 2.</u> V projektívnom priestore P_n, $n \geq 3$, nad telesom komplexných čísel C sú dané body $A_1, \ldots, A_4, A_6, \ldots, A_n, A_{n+1}, \ldots, A_{n+4}$ v počte $(n+3)$ tak, že žiadne z nich v počte $(n+1)$ neležia v jednej nadrovine. V bode A_4 treba určiť dotyčnicu t k normálnej racionálnej krivke K_n, ktorá je danými bodmi určená.



obr.18

R i e š e n i e /pozri obr.18/.

1. Nech predovšetkým platí: $A_4 \equiv A_5$. Podobne, ako v predchádzajúcej úlohe, použijeme tie isté premietania ${}^{i}Z$, $i=6,\ldots,n,n+1,\ldots,n+4$, pomocou stredov premietania ${}^{i}O_{n-3} \equiv (A_{i_7} \ldots A_{i_n} A_{i_{n+1}} \ldots A_{i_{n+4}})$ do priemetne $\pi \equiv (A_1A_2A_3)$.
Podpriestor $P_{n-2} \equiv (A_6 \ldots A_n A_{n+1} \ldots A_{n+4})$ pretína priemetňu π v bode A_0.

2. Ak premietne bod A_4, potom v jednotlivých premietaniach ${}^{i}Z$ dostávame obrazy ${}^{i}A_4$ bodu A_4. V každom prípade bod ${}^{i}A_4$ spolu s bodmi A_0, A_1, A_2, A_3, X_4 určuje jednoduchú kužeľosečku ${}^{i}K_2$. Táto kužeľosečka je priemetom krivky K_n v premietaní ${}^{i}Z$.

3. Pomocou Pascalovej priamky známym spôsobom zostrojíme dotyčnicu ${}^{i}t_4$ v bode ${}^{i}A_4$ ku kužeľosečke ${}^{i}K_2$. Táto dotyčnica spolu so stredom premietania ${}^{i}O_{n-3}$ tvorí nadrovinu ${}^{i}\tau_{n-1}$, ktorá má dvojbodový styk s krivkou K_n v jej bode A_4. Nadrovina ${}^{i}\tau_{n-1}$ je súčasne tangenciálnou nadrovinou kvadratickej kužeľovej nadplochy ${}^{i}Q$ typu $(n-2)$, ktorá obsahuje krivku K_n ako podvarietu rozmeru 1.

4. Pretože počet premietaní ${}^{i}Z$ zo stredov ${}^{i}O_{n-3}$ je práve n-1, aj počet bodov ${}^{i}A_4$ a jednoduchých kužeľosečiek ${}^{i}K_2$ je n-1. Ak zostrojíme dotyčnicu ${}^{i}t_4$ v každom bode ${}^{i}A_4$ ku kužeľosečke ${}^{i}K_2$ zväzku (5), vtedy dostávame n-1 tangenciálnych nadrovín ${}^{i}\tau_{n-1}$ kvadratických kužeľových nadplôch ${}^{i}Q$. Spoločný priesek nadrovín ${}^{i}\tau_{n-1}$ je hľadaná dotyčnica t_4.

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One projection of the curve K_n in the projective space P_n, $n \geq 3$

ŠTEFAN NOVOTNÝ

Summary

Let Z be a projection in P_n, $n \geq 3$ with the centre O_{n-3} and image domain space P_2. A normal rational curve K_n of P_n is mapped by Z into a regular conic. Using the Pascal theorem about a normal rational curve K_n two problems, similar to those about the regulare conic, are solved.

Об одном отображении нормальной рациональной кривой K_n в проективном пространстве P_n, $n \geq 3$

СТЕФАН НОВОТНЫ

Резюме

В предлагаемой работе отображается нормальная рациональная кривая K_n проективного пространства P_n, $n \geq 3$, из центра O_{n-3} в проективную плоскость P_2. Такое отображение обозначается З. Образом кривой в этой отображении являетсяне коническое сечение. На основе теоремы Паскаля о норнальной рациональной кривой K_n и свойств отображения Z, решает автор две задачи о кривой K_n, которые аналогичны задачам о невырожденном коническом сечении K_2.

Došlo 11. mája 1972, do nakladateľstva 8. júla 1974.

Autorova adresa: Katedra matematiky VŠP, 949 01 Nitra, námestie 1. mája.

Acta

F. R. N. U. C

MATHEMATICA XXXII

Vydalo Slovenské pedagogické nakladateľstvo Bratislava - Schválené výmerom SÚKK č. 1197/I-OR-1973 - Prvé vydanie- Náklad 874 - Rukopis zadaný v novembri 1974 - Vytlačené v novembri 1975 - Papier 2 122 3, 80 g - Tlačili Nitrianske tlačiarne, n.p., Nitra - Tlačené ofsetom - Písmo strojopis - - Technická redaktorka Beata Boriskovová - 03/1 - Strán 132 - AH 5,16 - VH 5,68 /text 5,04, obr. 0,64/ - 67 - 388 - 75

Celý náklad prevzala Ústredná knižnica PFUK, Bratislava, ul. 29. augusta č. 5.

67 - 388 - 75
03/1