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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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A NEW PROOF OF EBERHARD'S THEOREM

STANISLAV JENDROĽ, Košice

1. Introduction

We shall call the sequence $(p_k : k \geq 3) = (p_3, p_4, \ldots)$ of non-negative integers 3-realizable if there exists a trivalent convex polyhedron P such that $p_k(P) = p_k$ for all $k \geq 3$ ($p_k(P)$ denotes the number of k-gonal faces of P). EULER'S relation for convex polyhedra leads to the following necessary condition for 3-realizability of the sequence $(p_k : k \geq 3)$ (cf. [1], [2], [3]):

$$(*) \qquad 3p_3 + 2p_4 + p_5 = 12 + \sum_{k \geq 7} (k-6)p_k$$

An interesting feature of this condition is the absence of requirements concerning p_6. The following question suggests itself:

Given a sequence $(p_k : 3 \leq k \neq 6)$ satisfying the condition (*), do there exist values of p_6 for which the sequence $(p_k : k \geq 3)$ is 3-realizable?

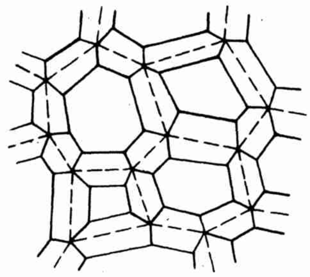
An answer is given by EBERHARD'S teorem (cf. [1], [2], [3]):

Theorem: For every sequence $(p_k : 3 \leq k \neq 6)$ of non-negative integers which satisfies (*) there exist values of p_6 for which the sequence $(p_k : k \geq 3)$ is 3-realizable.

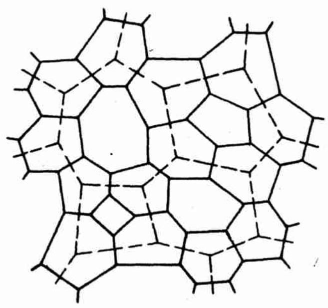
The original EBERHARD'S proof of this theorem (cf. [1]) was rather lengthy and difficult to follow; the same is true of the proof given in [3]. The best known proof is due to GRÜNBAUM [2]. It is easier to follow, but still rather long. The proof given in the present paper is shorter than any of these and uses only elementary methods.

2. Definitions and notations

Definition 1: We shall use the term operation $\mathcal{E}$ to denote a transformation of a map M into a map M^* which replaces the edges of the map by hexagons (cf. [2], p. 263). After operation $\mathcal{E}$ is performed on M, an adjacent r-gon R and s-gon S are replaced in M^* by an r-gon R' and s-gon S' separated by a hexagon (see fig. 1; here and in the sequel broken lines denote the edges of the original map).



Definition 2: We shall use the term operation μ to denote a transformation of a map M into a map M^{**} which replaces vertices of M by hexagons (see [2], p. 265). An adjacent s-gon S and t-gon T in M are transformed into an s-gon S'' and t-gon T'' in M^{**} separated by two adjacent hexagons in such a way that they are joined by a single edge (see fig. 2).



Both operations are fundamental to the following construction, since in both the faces of M are preserved and only hexagons are added.

A pair of adjacent faces of which one is an m-gon and the other a pentagon (here and in the sequel m denotes a natural number ≥ 6) we shall call configuration A, an adjacent m-gon-tetragon pair configuration B and an m-gon-triangle pair configuration C.

3. The proof

The proof will be performed by constructing a planar map containing all required k-gons for $3 \leqslant k \neq 6$. Using the STEINITZ theorem (cf. [2], [4]) we shall find that this map is also realizable as the boundary of a convex polyhedron. The construction is divided into two parts. In the first part all k-gons with $k \geqslant 7$ are constructed and in addition p_4' tetragons, p_3' triangles and p_5' pentagons, where

a/ if $p_5 \equiv 0 \pmod 3$, then

$$p_4' = p_4,\ p_3' = p_3 + \left[\frac{p_5}{3}\right],\ p_5' = 0$$

b/ if $p_5 \equiv 1 \pmod 3$, then

$$p_4' = p_4,\ p_3' = p_3 + \left[\frac{p_5}{3}\right],\ p_5' = 1$$

c/ if $p_5 \equiv 2 \pmod 3$, then p'

$$p_4' = p_4 + 1,\ p_3' = p_3 + \left[\frac{p_5}{3}\right],\ p_5' = 0$$

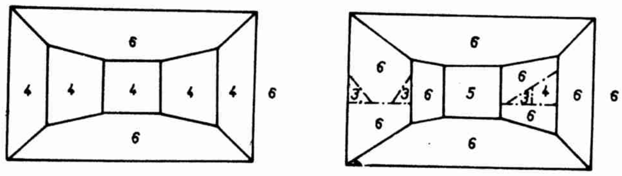
In the second part, $p_3' - p_3$ triangles and $p_4' - p_4$ tetragons are used to form the necessary pentagons.

In the course of the construction p_4' tetragons are constructed first together with part of the required k-gons $k \geqslant 7$, and then p_3' triangles together with the remaining necessary k-gons, $k \geqslant 7$.

If $p_4' \geqslant 5$, we perform the construction starting from a configuration B of a six-sided prism (see fig. 3a).

If $p_4' < 5$, we start from one of configurations A, B or C obtained by a simple modification of the six-sided prism together with p_4' tetragons and part of the p_3' triangles. Thus e.g. fig. 3b shows a starting configuration with a configuration A for the case $p_4' = 1$,

$p_3' \geqslant 4$ (the dot-anddash lines denote the edges added to the graph of the sixsided prism).

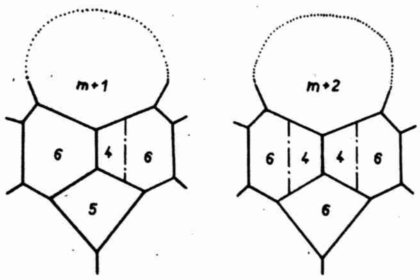


a) b)

ι) The necessary number p_4' of tetragons is constructed as follows: if $p_4' < 5$, they are in the starting configuration (see fig. 3b). If $p_4' \geqslant 5$, we start with a configuration B of the six-sided prism and at the same time we construct the required k-gons, $k \geqslant 7$, in the following way:

ι_1) If a configuration B is available, we proceed as follows: operation μ is performed on the map containing configuration B. In this way we obtain a configuration in which an m-gon and a tetragon will be separated by a pair of adjacent hexagons. Now we have to consider two possibilities:

ι_{11}) If we need an (m+1)-gon, we add an edge to the last mentioned configuration (see fig. 4a here and elsewhere dot-and-dash lines denote the added adges). Thus we obtain the necessary (m+1)-gon together with one tetragon and a configuration A with a hexagon (see fig. 4a).



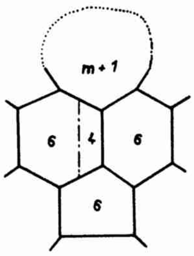
a) b)

α_{12}) If an n-gon is necessary with $n \geqslant m + 2$, we add two edges into the last mentioned configuration B together with an $(m+2)$-gon or with a hexagon if $n = m + 2$ (see fig. 4b).

The resulting configuration A or B is used to construct another tetragon or triangle (if all necessary tetragons have been constructed) and at the same time a k-gon with $k > m + 2$ or another required j-gon with $j \geqslant 7$.

α_2) If a configuration A is available, operation μ is performed on the map in which it is contained. In the resulting configuration, an m-gon and a pentagon are separated by a pair of adjacent hexagons. Adding one edge to this configuration leads to a configuration B with an $(m+1)$-gon or a hexagon (see fig. 5) according as an $(m+1)$-gon is necessary or not. The resulting configuration B can be used to construct another tetragon (if necessary) or triangle, giving at the same time a k-gon with $k \geqslant m + 2$ or another necessary j-gon, $j \geqslant 7$.

Using the above procedure, we start with a configuration B of a six-sided prism and obtain p_4' tetragons, some of the necessary k-gons ($k \geqslant 7$) and one of the configurations A, B (configuration C does not occur in the construction of tetragons). If $p_3' = 0$, this procedure gives us all k-gons with $k \geqslant 7$, p_4' tetragons and one pentagon in case b.

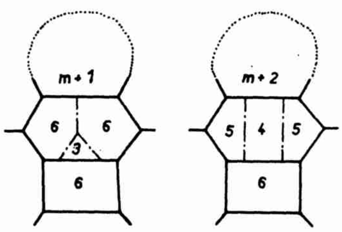


β) The configuration A, B or C (if $p_4' < 5$) is used to construct the required number p_3' of triangles, and the remaining k-gons with $k \geqslant 7$, as follows:

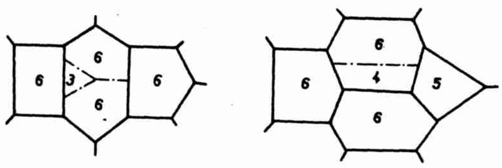
β_1) If a configuration A is available, we proceed as in case α_2).

If a configuration B is available, we perform operation ε on the map in which it is contained, obtaining a configuration in which a tetragon and an m-gon are separated by a hexagon. Again we have to consider two possibilities:

β_{21}) If an $(m+1)$-gon is needed, it is obtained by adding two edges, together with a configuration C with a hexagon (see fig. 6a).



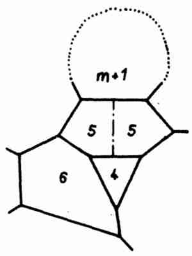
a)　　b)



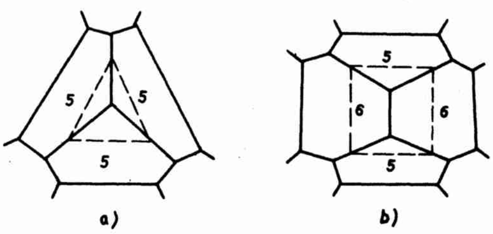
a)　　b)

β_{22}) If an n-gon is needed with $n \geqslant m + 2$, we add two edges the configuration. We obtain a configu ration A with an $(m+2)$-gon or a hexagon (if $n = m+2$) and an adjacent tetragonpentagon pair (see fig. 6b). After the next operation $\mathcal{E}$ is performed on this configuration, we form a triangle by adding three edges to the hexagon separating this pair (see fig. 7a). If, however, operation μ is used first, we join the tetragon-pentagon pair (which is now separated by two adjacent hexagons) by an edge added to one of these hexagons (see fig. 7b). The resulting configuration is used to form another triangle simultaneously with an n-gon $(n > m + 2)$ or another j-gon with $j \geqslant 7$.

β_3) If configuration C is available, we perform operation $\mathcal{E}$ on the map in which it is contained to obtain a configuration in which an m-gon and a triangle are separated by a hexagon and two more hexagons are adjacent to the triangle. By joining the triangle to the m-gon by a new edge across the hexagon separating then we obtain configuration A with an (m+1)-gon or a hexagon (if we need the (m+1)-gon) and an adjacent tetragon-pentagon pair (see fig. 8) which is used to form a triangle as described sub β_{22}). The resulting configuration is used to construct other required triangles and k-gons with $k \geqslant 7$.



In this way we obtain a map M_0 containing all required k-gons with $k \geqslant 7$, p_4' tetragons, p_3' triangles and p_5' pentagons. Performing operation $\mathcal{E}$ on this map, we obtain a map M_1 in which every triangle and tetragon are surrounded by hexagons. New the transformation shown in fig. 9 is applied to each of $p_3' - p_3$ triangles and $p_4' - p_4$ tetragons (the broken lines show the original triangle or tetragon). This gives us the map M which contains all required k-gons with $3 \leqslant k \neq 6$.



According to the STEINITZ' S theorem (which states that a graph G is a graph of a convex polyhedron iff it is planar and at least 3-connected [2], [4]) the map thus obtained is realizable as the boundary of a convex polyhedron.

This completes the proof.

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Author's address:

Prírodovedecká fakulta UPJŠ
Komenského 14
Košice

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Nový dôkaz Eberhardovej vety

STANISLAV JENDROĽ

Resumé

V práci autor uvádza nový dôkaz Eberhardovej vety (viď [1], [2], [3]):

Veta: Pre každú postupnosť $(p_k : 3 \leq k \neq 6)$ nezáporných celých čísel, ktorá vyhovuje podmienke

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{k \geq 7} (k - 6)\, p_k$$

existujú hodnoty p_6 také, že postupnosť $(p_k : k \geq 3)$ existuje 3- valentný konvexný mnohosten, ktorý obsahuje p_k k-uholníkových stien

Новое доказательство теоремы Эбергарда

СТАНИСЛАВ ЭНДРОЛЬ

Р е з ю м е

В статьи дано новое доказательство теоремы В. Эбергарда (см. [1], [2], [3]):

Т е о р е м а : Для любой последовательности (p_k : $3 \leqslant k \neq 6$) неотрицательных целых чисел, которая удовлетворяет условию

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{k \geqslant 7} (k - 6)\, p_k$$

существуют такие значения p_6, что для последовательности (p_k : k 3) существует 3-валентный многограник, который содержит p_k k-угольниковых граней.

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ON THE PROPERTIES OF SOLUTIONS OF NON-LINEAR DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER WITH DELAY

JÁN FUTÁK, Žilina

In paper [1] there are introduced sufficient conditions for solutions of linear differential equation of the 4^{th} order with delay to be non-oscillatory. In papers [2] and [3] there are investigated certain properties of the solutions of ordinary linear differential equation of the 4^{th} order.

In this paper we shell prove some properties of solutions of non-linear differential equation of the 4^{th} order with delay of the form:

$$(1)\quad y^{(4)}(t) + p(t)y'(t) + r(t)y(t) + y(t)\sum_{i=1}^{n} q_i(t)F_i(y[h_i(t)]) = g(t),$$

where the functions $f(t)$, $q_i(t)$, $i = 1,2,\dots,n$, $r(t)$, $g(t)$ belong to the class $C_0(j)$ /$j \equiv \langle t_0, \infty)$/ and n is a positive integer.

In further investigation we shell suppose that $\inf\,[t-h_i(t)] \geqslant d > 0$, $\lim h_i(t) = +\infty$ $i = 1,2,\dots,n$ and $F_i(z) \in C_0(-\infty, \infty)$, $F_i(z) \geqslant 0$, $i = 1,2,\dots,n$.

Let $\Phi(t)$ be a continuous function defined on the initial set $E_{t_0} = \bigcup_{i=1}^{n} E^i_{t_0}$ /$E^i_{t_0} = \{x;\ x = h_i(t) < t_0,\ t \in j\} \cup \{t_0\}$ / and let $y_0^{(k)}$, $k = 1,2,3$ be arbitrary real numbers. Then we define the fundamental initial problem /further only initial problem/ for differential equation (1) as follows: We search the solution $y(t)$ of (1) in the interval j which fulfils the initial conditions:

$$(2)\quad y(t_0) = \Phi(t_0) = y_0,\ y^{(k)}(t_0 + 0) = y_0^{(k)},\ k = 1,2,3,$$

$$y[h_i(t)] \equiv \Phi\,[h_i(t)] \quad \text{for } h_i(t) < t_0,\ i = 1,2,\dots,n,$$

On existence and uniqueness of the solution $y(t)$ of initial problem (1), (2) the following theorem says.

Theorem 1. Let the functions $p(t)$, $r(t)$, $q_i(t)$, $i = 1,2,\dots,n$, $g(t)$ belong to the class $C_o(j)$ and let the function $\Phi(t)$ be defined, continuous and bounded on the initial set E_{t_o}. Then the initial problem (1), (2) has the only solution on the interval j.

Proof. From the assumption $\lim_{t\to\infty} h_i(t) = +\infty$, $i = 1,2,\dots,n$ and from properties of $\Phi(t)$ it follows that for finding of a solution of the initial problem (1), (2) one can use the method of steps, therefore the further process of proving this theorem is analogical as in a case of linear differential equation.

The solution $y(t)$ of the initial problem (1), (2) will be called oscilatory, if it has infinite number of zero points on every interval (α,∞) $(t_o < \alpha)$.

The solution $y(t)$ of the initial problem (1), (2) will be called non-oscillatory, if there exists such a number $\alpha(> t_o)$, that the function $y(t)$ has not any zero point for any $t > \alpha$.

Lemma 1. Let $y(t)$ be a solution of the initial problem (1), (2). Then $y(t)$ fulfils the following identities:

$$(3)\quad y'''(t)y(t) - \int_{t_o}^{t} y'''(s)y'(s)ds + \int_{t_o}^{t} p(s)y'(s)y(s)ds + \int_{t_o}^{t} r(s)y^2(s)ds + \sum_{i=1}^{n}\int_{t_o}^{t} q_i(s)y^2(s)F_i(y[h_i(s)])\,ds = y_o'''y_o + \int_{t_o}^{t} g(s)y(s)ds,$$

$$(4)\qquad y'''(t)y'(t) - \frac{1}{2}y''^{2}(t) + \int_{t_o}^{t} p(s)y'^{2}(s)ds +$$

$$+ \int_{t_o}^{t} r(s)y(s)y'(s)ds + \sum_{i=1}^{n}\int_{t_o}^{t} q_i(s)y'(s)y(s)F_i(y[h_i(s)])ds =$$

$$= y_o'''y_o' - \frac{1}{2}y_o''^{2} + \int_{t_o}^{t} g(s)y'(s)ds.$$

P r o o f. After multiplying differential equation (1) by $y(t)$ and integrating from t_o to t from $t \in j$, we obtain the identity (3). After multiplying (1) by $y'(t)$ and integrating on the interval $\langle t_o, t)$ for $t \in j$ we get the identity (4).

T h e o r e m 2. Let the functions $p(t)(\leqslant 0)$, $r(t)(\leqslant 0)$, $q_i(t)(\leqslant 0)$, $i = 1,2,\dots,n$, $g(t)$ belong to the class $C_o(j)$ and let for any $t \in j$ a/ $g(t) \geqslant 0$, b/ $g(t) \leqslant 0$ hold, whereby let $p^2(t) + r^2(t) + \left[\sum_{i=1}^{n} q_i(t)\right]^2 + g^2(t) \equiv 0$ do not hold on any subinterval of the interval j. If $y(t)$ is a solution of the initial problem (1), (2), for which there holds

$$(5)\qquad \text{a/}\quad y_o^{(k)} \geqslant 0,\quad y_o''' > 0,\quad k = 0,1,2$$

$$\text{b/}\quad y_o^{(k)} \leqslant 0,\quad y_o''' < 0,\quad k = 0,1,2$$

and

$$y_o'''y_o' - \frac{1}{2}y_o''^{2} \geqslant 0,$$

then $y(t)$, $y'(t)$, $y''(t)$, $y'''(t)$ have no zero points on the interval (t_o, ∞).

P r o o f will be done in indirect way for the case a/ $y_o^{(k)} \geqslant 0$, $k = 0,1,2$, $y_o''' > 0$, $g(t) > 0$. Let the assumptions of theorem hold. Suppose that $y'''(t)$ has zero points on j. Let us denote $\xi \in (t_o, \infty)$ the first zero point of $y'''(t)$ on the right-hand side from t_o. With regard to a/ it holds that $y'''(t) > 0$ for $t \in \langle t_o, \xi)$ and $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$ for $t \in (t_o, \xi)$ as well. Then from identity (4) for $t = \xi$ we obtain contrafiction because the left side is negative and the right one is non-negative. Therefore it must hold that $y'''(t) > 0$ for $t \in j$ and also $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$ for $t \in (t_o, \infty)$.

The case b/ can be proved analogically.

In a similar way we can prove

T h e o r e m 3. Let the assumptions of Theorem 2 hold and let $r(t) \in C_1(j)$, $r'(t) \geqslant 0$, $r(t) \leqslant 0$, whereby let $p^2(t) + r^2(t) + r'^2(t) + \left[\sum_{i=1}^{n} q_i(t)\right]^2 + g^2(t) \equiv 0$ do not hold on any subinterval of the interval j. If $y(t)$ is a solution of the initial problem (1), (2), fulfilled (5) and

$$y_o''' y_o' - \frac{1}{2} y_o''^2 - \frac{1}{2} r(t_o) y_o^2 \geqslant 0,$$

then $y(t)$, $y'(t)$, $y''(t)$, $y'''(t)$ have no zero points on the interval (t_o, ∞).

In further investigation we shell suppose that $\int_{t_o}^{\infty} g^2(s)ds < \infty$.

T h e o r e m 4. Let the functions $p(t)$, $g(t)$, $r(t)\ (\leqslant 0)$, $q_i(t) (\leqslant 0)$, $i = 1,2,\dots,n$ belong to the class $C_o(j)$ and let $2p(t) + 1 \leqslant 0$ hold for $t \in j$, whareby let $[2p(t) + 1]^2 + r^2(t) + [\sum_{i=1}^{n} q_i(t)]^2 \equiv 0$ do not hold on any subinterval of the interval j. If $y(t)$ is solution of the initial problem (1), (2) for which (5) holds and

$$y_o''' y_o' - \frac{1}{2} y_o''^2 \geqslant \frac{1}{2} \int_{t_o}^{\infty} g^2(s)ds,$$

then $y(t)$, $y'(t)$, $y''(t)$, and $y'''(t)$ have not any zero points on the interval (t_0, ∞).

P r o o f. Let the assumptions of theorem hold. From the identity (4) with regard to inequality

$$(6) \qquad 2\int_{t_0}^{t} a(s)b(s)ds \leqslant \int_{t_0}^{t} a^2(s)ds + \int_{t_0}^{t} b^2(s)ds,$$

we obtain

$$(7) \qquad y'''(t)y'(t) - \frac{1}{2}y''^2(t) + \int_{t_0}^{t}\left[p(s) + \frac{1}{2}\right] y'^2(s)ds +$$

$$\int_{t_0}^{t} r(s)y(s)y'(s)ds + \sum_{i=1}^{n}\int_{t_0}^{t} q_i(s)y'(s)y(s)F_i(y[h_i(s)])ds \geqslant$$

$$\geqslant y_0'''y_0' - \frac{1}{2}y_0''^2 - \frac{1}{2}\int_{t_0}^{t} g^2(s)ds.$$

If $\xi \in (t_0, \infty)$ is the zero point of $y'''(t)$, then with regard to the initial conditions (2) and (5) a/, it holds $y'''(t) > 0$ for $t \in \langle t_0, \xi)$. From the inequality (7) for $t = \xi$ we again obtain the contradiction. Therefore $y'''(t) > 0$ for $t \in j$ and also $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$ for $t \in (t_0, \infty)$.

The case (5) b/ may be proved in analogical way.

T h e o r e m 5. Let functions $p(t)(\leqslant 0)$, $r(t)$, $q_i(t)(\leqslant 0)$, $i = 1,2,\ldots,n$, $g(t)$ belong to the class $C_0(j)$ and let $2r(t) + 1 \leqslant 0$ hold, whereby let the relation $p^2(t) + [\sum_{i=1}^{n} q_i(t)]^2 + [2r(t) + 1]^2 \equiv$

$\equiv 0$ do not hold on any subinterval of the interval j. If y(t) is a solution of the initial problem (1), (2) for which (5) hold and

$$y_o''' y_o - \frac{1}{2}\int_{t_o}^{\infty} g^2(s)ds > 0,$$

then y(t), y'(t), y''(t), y'''(t) have not any zero points on the interval (t_o, ∞).

Proof. Let the assumptions of theorem hold. From the identity (3) with regard to the inequality (6) we get:

$$y'''(t)y(t) - \int_{t_o}^{t} y'''(s)y'(s)ds + \int_{t_o}^{t} p(s)y'(s)y(s)ds +$$

$$+ \int_{t_o}^{t} \left[r(s) + \frac{1}{2}\right] y^2(s)ds + \sum_{i=1}^{n} \int_{t_o}^{t} q_i(s)y^2(s)F_i(y[h_i(s)])\, ds \geqslant$$

$$\geqslant y_o''' y_o - \frac{1}{2}\int_{t_o}^{t} g^2(s)ds.$$

The further process of proving is analogical like in the theorem 4.

Similarly, by means of the identity (3) we can prove the following theorem.

Theorem 6. Let the functions r(t), g(t), $q_i(t) (\leqslant 0)$ $i = 1,2,\ldots,n$ belong to the class $C_o(j)$, $p(t) \in C_1(j)$, $p(t) \leqslant 0$

and let $2r(t) - p'(t) + 1 \leqslant 0$, hold, whereby let $p^2(t) + \left[\sum_{i=1}^{n} q_i(t)\right]^2 + [2r(t) - p'(t) + 1]^2 \equiv 0$ do not hold on any subinterval of the interval j. If a solution y(t) of the initial problem (1), (2) fulfils the conditions (5) and

$$y_o''' y_o + \frac{1}{2} p(t_o) y_o^2 - \frac{1}{2} \int_{t_o}^{\infty} g^2(s) ds > 0,$$

then $y(t)$, $y'(t)$, $y''(t)$, $y'''(t)$ have not any zero points on the interval (t_o, ∞).

If we shall suppose in further considerations that the conditions (5) do not hold for the solution y(t) of the initial problem (1),(2) we are able to prove the following theorems.

Theorem 7. Let the functions $r(t)$, $q_i(t) (\geqslant 0)$, $i = 1, 2, \ldots, n$, $g(t)$ belong to the class $C_o(j)$, $p(t) \in C_1(j)$ and let $2r(t) - p'(t) - 1 \geqslant 0$ hold for any $t \in j$, whereby let the expression $\left[\sum_{i=1}^{n} q_i(t)\right]^2 + [2r(t) - p'(t) - 1]^2 \equiv 0$ do pot hold on any subinterval of the interval j. Then for any solution y(t) of the initial problem (1), (2) for which

$$(8) \qquad y_o''' y_o - y_o'' y_o' + \frac{1}{2} p(t_o) y_o^2 + \frac{1}{2} \int_{t_o}^{\infty} g^2(s) ds \leqslant 0,$$

holds there does not exist the point $\xi \in (t_o, \infty)$ such that

a/ $\quad y(\xi) = y'(\xi) = 0$,

b/ $\quad y(\xi) = y''(\xi) = 0$.

Proof. Proof will be done in indirect way for the case a/ by using the identity (3). Let there exists such point $\xi > t_o$ that $y(\xi) = y'(\xi) = 0$ hold. With regard to the inequality (6) the identity (3) will be of the form:

$$(9)\qquad y'''(t)y(t) - y'(t)y''(t) + \frac{1}{2}\,p(t)y^2(t) +$$

$$+ \int_{t_o}^{t} \left\{ y''^2(s) + \left[r(s) - \frac{1}{2}\,p'(s) - \frac{1}{2} \right] y^2(s) \right\} ds +$$

$$+ \sum_{i=1}^{n} \int_{t_o}^{t} q_i(s)y^2(s)F_i(y[h_i(s)])\,ds \leqslant y'''_o y_o - y'_o y''_o +$$

$$+ \frac{1}{2}\,p(t_o)y_o^2 + \frac{1}{2}\int_{t_o}^{t} g^2(s)\,ds.$$

If $t = \xi\ (\xi > t_o)$ in the inequality (9), we obtain:

$$\int_{t_o}^{\xi} \left\{ y''^2(t) + \left[r(t) - \frac{1}{2}\,p'(t) - \frac{1}{2} \right] y^2(t) \right\} dt +$$

$$+ \sum_{i=1}^{n} \int_{t_o}^{\xi} q_i(t)y^2(t)F_i(y[h_i(t)])\,dt \leqslant y'''_o y_o - y'_o y''_o +$$

$$+ \frac{1}{2}\,p(t_o)y_o^2 + \frac{1}{2}\int_{t_o}^{\xi} g^2(t)\,dt \leqslant 0,$$

what is contradiction, because the left side is non-negative. The assertion b/ one can prove in analogical way.

T h e o r e m 8. Let for differential equation (1) the assumptions of theorem 7 hold and let $p(t) \geqslant 0$ for $t \in j$, whereby let $p^2(t) +$

$$+ \left[\sum_{i=1}^{n} q_i(t)\right]^2 + \left[2r(t) - p'(t) - 1\right]^2 \equiv 0$$ do not hold on any subinterval of the interval j. Then for any solution $y(t)$ of the initial problem (1), (2) for which (8) is fulfilled, there does not exist a point $\xi \in (t_0, \infty)$ such that

$$\text{a/} \quad y''(\xi) = y'''(\xi) = 0,$$
$$\text{b/} \quad y'(\xi) = y'''(\xi) = 0.$$

The proof of the assertion of theorem can be carried on by means of the inequality (9), similarly as in theorem 7.

L e m m a 2. Let the functions, $r(t)$, $g(t)$, $q_i(t) (\geqslant 0)$, $i=1,2,\dots,n$ belong to the class $C_0(j)$, $p(t)$ $C_1(j)$, $p(t) \geqslant 0$ and let $2r(t) - p'(t) - 1 \geqslant 0$ be fulfilled for any $t \in j$, whereby let

$$+ \left[\sum_{i=1}^{n} q_i(t)\right]^2 + \left[2r(t) - p'(t) - 1\right]^2 \equiv 0$$ do not hold on any subinterval of the interval j. If $y(t)$ is a solution of the initial problem (1), (2) for which

$$(10) \qquad y_0 y_0''' - y_0' y_0'' + \frac{1}{2} p(t_0) y_0^2 + \frac{1}{2} \int_{t_0}^{\infty} g^2(s)ds \leqslant m \leqslant 0$$

holds, where m is constant value, then zero points of $y(t)$ are separated by zero points of $y''(t)$ and couversely zero points of $y''(t)$ are separated by zero points of $y(t)$.

P r o o f. Lemma will be prove in indirect way. Let $t_1 < t_2$ be two neighbouring zero points of $y''(t)$ and let $y(t) \neq 0$ for $t \in \langle t_1, t_2 \rangle$, $t_1 > t_0$. From the identity (3) and inequality (9) it follows that

$$(11)\quad \left(\frac{y''(t)}{y(t)}\right)' = \frac{y'''(t)y(t) - y'(t)y''(t)}{y^2(t)} \leqslant -\frac{p(t)}{2} -$$

$$-\frac{1}{y^2(t)}\int_{t_o}^{t}\left\{y''^2(s) + \left[r(s) - \frac{1}{2}\, p'(s) - \frac{1}{2}\right]y^2(s)\right\} ds -$$

$$-\frac{1}{y^2(t)}\sum_{i=1}^{n}\int_{t_o}^{t} q_i(s)y^2(s)F_i\left(y[h_i(s)]\right)ds +$$

$$+\frac{1}{y^2(t)}\left[y_o y_o''' - y_o'' y_o' + \frac{1}{2}\, p(t_o)y_o^2 + \frac{1}{2}\int_{t_o}^{t} g^2(s)ds\right].$$

After integrating (11) from t_1 to t_2 we get:

$$0 \leqslant \int_{t_1}^{t_2}\left\{-\frac{p(t)}{2} - \frac{1}{y^2(t)}\int_{t_o}^{t}\left[y''^2(s) + \left(r(s) - \frac{1}{2}\, p'(s) - \frac{1}{2}\right)y^2(s)\right] ds -\right.$$

$$-\frac{1}{y^2(t)}\sum_{i=1}^{n}\int_{t_o}^{t} q_i(s)y^2(s)F_i\left(y[h_i(s)]\right)ds +$$

$$\left.+\frac{1}{y^2(t)}\left(y_o y_o''' - y_o'' y_o' + \frac{1}{2}\, p(t_o)y_o^2 + \frac{1}{2}\int_{t_o}^{t} g^2(s)ds\right)\right\} dt < \frac{m}{y^2(t)} \leqslant 0.$$

This contradiction proves the correctness of the assertion of lemma. It is still to prove that between two neighbouring zero points of solution $y(t)$ there lies the only zero point of $y''(t)$. The proof is analogical.

<u>Theorem 9</u>. Let the functions $r(t)(\geq 0)$, $q_i(t)$, $i = 1,2,\ldots n$ belong to the class $C_o(j)$, $p(t) \in C_1(j)$ and let $p(t) \geq 0$ hold for any $t \in j$, $2r(t) - p'(t) - 1 \geq m_1 > 0$, $q_i(t) \geq m_2 > 0$, $i = 1,2,\ldots,n$,

$$\left|\int_{t_o}^{\infty} g(s)\,ds\right| \leq M < \infty$$

and let more the functions $F_i(z)$, $i = 1,2,\ldots,n$ be increasing. Then each solution $y(t)$ of the initial problem (1), (2) fulfilled (10), is either oscillatory of there holds $\lim\limits_{t\to\infty} y(t) = \lim\limits_{t\to\infty} y'(t) = 0$.

<u>Proof</u>. The solution $y(t)$ of the initial problem (1), (2) fulfilled (10) is either a/ oscillatory or b/ non-oscillatory. Therefore the case a/ is complete. Cousider the case b/. Without loss of universality suppose that $y(t) > 0$ for $t \in \langle t_1, \infty\rangle$ where $t_1 \geq t_o$. For $y(t)$ and $y'(t)$ there may be happen the following cases: 1. $y(t) > 0$, $y'(t) \geq 0$; 2. $y(t) > 0$, $y'(t)$ is oscillatory; 3. $y(t) > 0$, $y'(t) \leq 0$; for $t \in \langle t_1, \infty)$, where $t_1 \geq t_o$. We shall show at first that the cases 1., 2., under assumptions of theorem can never happen.

1. $y(t) > 0$, $y'(t) \geq 0$ for $t \in \langle t_1, \infty)$, $t_1 \geq t_o$. With regard to that $F_i(z) \geq 0$, $F_i(z)$ are increasing, $p(t) \geq 0$, $r(t) \geq 0$ and $q_i(t) \geq m_2 > 0$, $i = 1,2,\ldots,n$, from differential equation (1) it follows that

$$y^{(4)}(t) \leq g(t) - m_2 y(t_1) \sum_{i=1}^{n} F_i\left(y[h_i(t_1)]\right). \tag{12}$$

If we integrate (12) from t_1 to t, where $t_1 \geq t_o$, we obtain:

$$(13) \qquad y'''(t) \leq y'''(t_1) + \int_{t_1}^{t} g(s)ds -$$

$$- m_2 y(t_1) \sum_{i=1}^{n} F_i\left(y[h_i(t_1)]\right) \int_{t_1}^{t} ds \leq$$

$$\leq m_3 - m_2 y(t_1) \sum_{i=1}^{n} F_i\left(y[h_i(t_1)]\right) (t - t_1),$$

where $m_3 = y'''(t_1) + M$. From (13) it follows that $\lim_{t \to \infty} y'''(t) = -\infty$, from this we have that $y'(t) < 0$ for sufficiently large t, what is in contradiction with the assumption $y'(t) \geq 0$.

2. $y(t) > 0$, $y'(t)$ is oscillatrory, $t \in \langle t_1, \infty)$, $t_1 \geq t_o$. This case cannot happen because if $y'(t)$ is oscillatory then $y''(t)$ must be also oscillatory and from this it follows by lemma 2 that $y(t)$ is oscillatrory, too. It is in contradiction with $y(t) > 0$.

3. $y(t) > 0$, $y'(t) \leq 0$ for $t \in \langle t_1, \infty)$, $t_1 \geq t_o$. For proving this case we use inequality (9) from which it follows that

$$y(t)y'''(t) - y'(t)y''(t) \leq -\frac{1}{2} p(t)y^2(t) -$$

$$- \int_{t_o}^{t} \left\{ y''^2(s) + \left[r(s) - \frac{1}{2} p'(s) - \frac{1}{2} \right] y^2(s) \right\} ds -$$

$$- \sum_{i=1}^{n} \int_{t_o}^{t} q_i(s)y^2(s)F_i\Big(y\big[h_i(s)\big]\Big)\,ds + y_o'''y_o - y_o'y_o'' +$$

$$+ \frac{1}{2}\, p(t_o)y_o^2 + \frac{1}{2}\int_{t_o}^{t} g^2(s)ds.$$

With regard to assumptions (2) and (10) after arrangement we get;

$$y(t)y'''(t) - y'(t)y''(t) \leqslant m - \frac{1}{2}\, p(t)y^2(t) -$$

$$- \int_{t_o}^{t} \Big\{ y''^2(s) + \Big[r(s) - \frac{1}{2}\, p'(s) - \frac{1}{2}\Big]y^2(s) +$$

$$+ \sum_{i=1}^{n} q_i(s)y^2(s)F_i\Big(y\big[h_i(s)\big]\Big)\Big\}\,ds.$$

We shall show that

$$\text{(14)} \qquad \int_{t_o}^{\infty} \Big\{ y''^2(s) + \Big[r(s) - \frac{1}{2}\, p'(s) - \frac{1}{2}\Big]y^2(s) +$$

$$+ \sum_{i=1}^{n} q_i(s)y^2(s)F_i\Big(y\big[h_i(s)\big]\Big)\Big\}\,ds < \infty .$$

If integral (14) vould be divergent, then it would be fulfilled that $\lim\limits_{t\to\infty} [y(t)y'''(t) - y'(t)y''(t)] = -\infty$. With regard to assumptions it means that $\lim\limits_{t\to\infty} y''(t) = -\infty$ and so $\lim\limits_{t\to\infty} y(t) = -\infty$; what is in contradiction with $y(t) > 0$ for $t \in \langle t_1, \infty)$. Therefore (14) must be fulfilled. From (14) there holds that integrals

$$\int_{t_o}^{\infty} y''^2(t)dt, \quad \int_{t_o}^{\infty} y^2(t)dt, \quad \int_{t_o}^{\infty} \sum_{i=1}^{n} y^2(t)F_i(y[h_i(t)])\, dt$$

are also convergent and therefore $\int_{t_o}^{t} y'^2(t)dt < \infty$ as well. Then as to the monotonicity of functions $y^2(t)$, $y'^2(t)$ and convergency of corresponding integrals it holds that $\lim\limits_{t\to\infty} y(t) = \lim\limits_{t\to\infty} y'(t) = 0$ and, theorem is being proved.

R e m a r k. We can analogically prove the cases: 1'. $y(t) < 0$, $y'(t) \leqslant 0$; 2'. $y(t) < 0$, $y'(t)$ is oscillatory; 3'. $y(t) < 0$, $y'(t) \geqslant 0$; for $t \in \langle t_1, \infty)$, $t_1 \geqslant t_o$.

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Author's address:

Katedra matematiky VŠD
Žilina
Marxa-Engelsa 25

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Resume

O vlastnostiach riešení nelineárnej diferenciálnej rovnice 4. rádu s oneskoreným argumentom

JÁN FUTÁK

Vo vetách 2, 3, 4, 5, 6 sú postačujúce podmienky na to, aby riešenia $y(t)$ diferenciálnej rovnice (1) a ich derivácie do 3. rádu boli monotónne pri daných začiatočných podmienkach. Vo vede 9 sú postačujúce podmienky, pri ktorých riešenie $y(t)$ rovnice (1) osciluje, alebo platí: $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0$.

Резюме

О свойствах решений нелинейного дифференциального уравнения 4-ого порядка с запаздывающим аргументом

ЯН ФУТАК

В теоремах 2, 3, 4, 5, 6 приведены достаточные условия для того, чтобы решения $y(t)$ дифференциального уравнения (1) и их производные до 3-го порядка были монотонны, при заданных начальных условиях. В теореме 9 приведены достаточные услобия, при которых решение $y(t)$ уравнения (1) колебляется или: $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0$.

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ABOUT CONGRUENCE RELATIONS OF SEMILATTICES

JUHANI NIEMINEN, Heinola

The main subject of this work is to consider the congruence relations on a join semilattice. In what follows, L will mean a join semilattice with the exceptions separately mentioned.

1. Modular semilattices.

In this paragraph we consider the concept of modularity of a semilattice L. The distributivity of a semilattice L has been defined as follows:

Definition 1. A semilattice L is distributive if and only if for every three elements $a, b, c \in L$, $a \leq b \cup c$, there are elements $b_1 \leq b$, $c_1 \leq c$ in L such that $b_1 \cup c_1 = a$.

Definition 1 suggests to define the modularity of a semilattice L as follows:

Definition 2. A semilattice L is modular if and only if for every three elements $a, b, c \in L$, $a \leq b \cup c$, $b \leq a$, there are elements $b_1 \leq b$, $c_1 \leq c$ in L such that $b_1 \cup c_1 = a$.

The distributivity of L implies the modularity of L according to Definition 1. Further, the modularity of L implies the following lemma:

Lemma 1. Let L be a modular semilattice, then for every two distinct elements $a, b \in L$ there is a common lower bound $k \in L$, $k \leq a, b$.

Proof. If there are two elements $a, b \in L$ without a common lower bound, then $a \cup b = c > a$, $a \leq a$, and there are no elements $a_1, b_1 \in L$, $a_1 \leq a$ and $b_1 \leq b$ such that $b_1 \cup a_1 = a$.

It has been mentioned in [6, Lemma 2] that a semilattice L is distributive if and only if the corresponding lattice $\mathcal{I}(L)$ of the ideals of L is a distributive lattice. We can prove a similar theorem concerning the modularity of a semilattice. At first we prove:

Lemma 2. If a semilattice L is modular, then the ideals of L form a lattice.

Proof. The definition for the join operation for two ideals I_1 and I_2 of L is the same as that in the case of lattices and obviously there is no need for the proof. For the meet operation we define: $I_1 \cap I_2 = I_1 \wedge I_2$, where I_1, I_2 are two nonempty ideals of L, and $\wedge$ is the settheoretical intersection. Let $a \in I_1$ and $b \in I_2$, then according to the modularity of L, a and b have a common lower bound $k \in L$, $k < a, b$. The definition of the ideal of L implies that $k \in I_1, I_2$ and thus $I_1 \cap I_2$ s nonempty. Obviously $I_1 \cap I_2$ is an ideal of L.

Theorem 1. semilattice L is modular if and only if the lattice $\mathcal{I}(L)$ of the ideals of L is a modular lattice

Proof. 1° Let $\mathcal{I}(L)$ be a modular lattice. Then there is for every element $a \in L$ a principal ideal (a). Let a, b, c be three elements of L such that $a \cup b > c$ and $b \leqslant c$. According to the definition of the join on $\mathcal{I}(L)$, $a \cup b \in (a] \cup (b]$, and then $c \in (a] \cup (b]$, whence $(c] \cap ((a] \cup (b]) = (c]$. As $c > b$, then $(c] \geqslant (b]$, and now, according to the modularity of $\mathcal{I}(L)$, we obtain $(b] \cup ((a] \cap (c]) = ((b] \cup (a]) \cap (c]$. But $c \in (b] \cup ((a] \cap (c])$, and thus the join operation on $\mathcal{I}(L)$ implies $c \leqslant a_1 \cup b_1$, where $b_1 \in (b]$, $a_1 \in (a] \cap (c]$. Hence, $b_1 \leqslant b$ and $a \leqslant a$. On the other hand, $b_1, a_1 \in (b] \cup ((a] \cap (c]) = (c]$, which implies $c \geqslant a_1 \cup b_1$. Hence, $c = a_1 \cup b_1$, where $a_1 \leqslant a$, $b_1 \leqslant b$.

2° Let L be a modular semilattice and A, B, C three arbitrary nonempty ideals of L such that $B \leqslant A \leqslant B \cup C$. In order to prove the modularity of $\mathcal{I}(L)$, it is sufficient to show that $B \cup (A \cap C) = A$.

Trivially $B \cup (A \cap C) < A$. If $B \cup (A \cap C) < A$, then there is an element $a \in A$ such that $a \notin B \cup (A \cap C)$. But as $A < B \cup C$, then $a \in B \cup C$ and the join of $\mathcal{I}(L)$ implies that there are elements $b \in B$ and $c \in C$ such that $a < b \cup c$. If we put $a_1 = b \cup a$, then $a_1 \notin B \cup (A \cap C)$. Now $a_1 = b \cup a < b \cup c$, where $a_1 < b$ and according to the modularity of L, there is an element c_1 in L such that $c_1 < c$ and $b \cup c_1 = a \in B \cup (A \cap C)$, which is a contradiction. Hence, $B \cup (A \cap C) = A$.

2. On the congruence relations on a semilattice

At first we prove a necessary and sufficient condition for a binary relation Θ to be a congruence relation on a semilattice L.

Lemma 3. A binary relation Θ is a congruence relation on a semilattice L if and only if the following conditions are valid.

1^o $x\Theta x$ for every $x \in L$.

2^o $x\Theta y$ if and only if $x\Theta x\cup y$ and $y\Theta x\cup y$ for any pair x, y of elements of L.

3^o $x \leq y$ and $x\Theta y$ imply $x\cup t\Theta y\cup t$ for every $t \in L$.

4^o $x \geq y$, $x \geq z$, and $x\Theta y$, $x\Theta z$ imply $y\Theta z$.

5^o $x \geq y$, $y \geq z$, and $x\Theta y$, $y\Theta z$ imply $x\Theta z$.

Proof. If Θ is a congruence relation on L, then it obviously satisfies the conditions 1^o - 5^o.

Suppose that Θ satisfies the conditions 1^o - 5^o. According to 1^o, Θ is reflexive and according to 2^o, Θ is symmetric, since 2^o implies: $x\Theta y \Leftrightarrow x\Theta x\cup y$ and $y\Theta x\cup y \Leftrightarrow y\Theta x$.

Consider the substitution law. Assume that $x\Theta y$, $x, y \in L$. According to 2^o $x\Theta y \Leftrightarrow x\Theta x\cup y$ and $y\Theta x\cup y$. By applying 3^o we obtain: for every $t \in L$, $x\cup y\cup t\Theta x\cup t$ and $x\cup y\cup t\Theta y\cup t$, which imply $(x\cup t)\cup(y\cup t)\ \Theta x\cup t$ and $(x\cup t)\cup(y\cup t)\Theta y\cup t$. According to 2^o the last two relations give $x\cup t\Theta y\cup t$ for every $t \in L$.

Consider the transitivity law and assume that $x\Theta y$ and $x\Theta z$, where $x, y, z \in L$. According to the substitution law we obtain $x\cup z\Theta y\cup z$, $x\cup y\Theta z\cup y$, and $x\cup z\Theta y\cup z\cup x$, $x\cup y\Theta y\cup z\cup x$. Besides, $x\cup z\Theta z$ and $x\cup y\Theta y$. By applying 5^o we obtain $z\cup x\cup y\Theta z$ and $z\cup x\cup y\Theta y$, which, according to 4^o, show that $z\Theta y$. This completes the proof.

Following J. Varlet [8] we shall use the following notations. By a part $\langle a,b\rangle$ of a semilattice L, $a, b \in L$, we mean the settheoretical union of the elements of two closed intervals $[a, a\cup b]$ and $[b, a\cup b]$ of L. If $a \leq b$, then obviously $\langle a,b\rangle = [a,b]$. The part $\langle a\cup x, b\cup x\rangle = \langle a,b\rangle_x$ is called a supertransposition of the part $\langle a,b\rangle$ for $x \in L$.

We shall characterize the minimal congruence relation Θ_{ab} on L generated by two elements $a, b \in L$.

Lemma 4. Let L be a semilattice and a, b two elements of L, $a \neq b$. Let Θ_{ab} be a binary relation on L such that $x\Theta_{ab}y$ if and only if 1^o alone is valid, or 2^o and 3^o together are valid, where

1° $x = y$;
2° $a \cup b \cup x = a \cup b \cup x \cup y$ and $a \cup b \cup y = a \cup b \cup x \cup y$;
3° $a \cup x = x$ or $b \cup x = x$ and $a \cup y = y$ or $b \cup y = y$.
Then Θ_{ab} is a congruence relation on L and it is a minimal congruence relation on L collapsing the elements a and b.

P r o o f. At first we show by using Lemma 3 that Θ_{ab} is a congruence relation on L. According to 1° $x\Theta_{ab}x$ for every $x \in L$. We show the validity of the condition 2° in Lemma 3; the proof for the other conditions are similar and we omit them.

$$x\Theta_{ab}y \iff \begin{array}{l} a \cup b \cup x = a \cup b \cup x \cup y \\ a \cup b \cup y = a \cup b \cup x \cup y \\ a \cup x = x \text{ or } b \cup x = x \\ a \cup y = y \text{ or } b \cup y = y \end{array}$$

$$\iff \begin{array}{ll} a \cup b \cup x \cup y = a \cup b \cup x \cup y \cup x & \\ a \cup b \cup x = a \cup b \cup x \cup y \cup x & \iff x\Theta_{ab}x \cup y \\ a \cup x = x \text{ or } b \cup x = x & \\ a \cup x \cup y = x \cup y \text{ or } b \cup x \cup y = x \cup y & \\ a \cup b \cup x \cup y = a \cup b \cup x \cup y \cup y & \\ a \cup b \cup y = a \cup b \cup x \cup y \cup y & \iff y\Theta_{ab}x \cup y \\ a \cup y = y \text{ or } b \cup y = y & \\ a \cup x \cup y = x \cup y \text{ or } b \cup x \cup y = x \cup y & \end{array}$$

It remains to show that Θ_{ab} is a minimal congruence relation on L collapsing $a, b \in L$. According to 2° and 3° of Lemma 3, if Θ_{ab} collapses a and b, it collapses the part $\langle a, b\rangle$ of L. Let Θ be a congruence relation on L such that $a\Theta b$ and let $x, y \in L$ be two elements of L, $x \neq y$, such that $x\Theta_{ab}y$. Assume that $a \cup x = x$ and $a \cup y = y$; the other cases are similar. Now, $a\Theta b$ implies $x\Theta b \cup x$ and $y\Theta b \cup y$. According to the definition of Θ_{ab}, $y\Theta_{ab}a \cup b \cup y = a \cup b \cup x \cup y$ and $x\Theta_{ab}a \cup b \cup x = a \cup b \cup x \cup y$. Now the transitivity readily shows that $x\Theta y$.

C o r o l l a r y 1. The congruence class of Θ_{ab} containing a and b is the part $\langle a, b\rangle$ of L.

P r o o f. Let $a\Theta_{ab}x$, then according to Lemma 4, $a \leqslant x$ or $b \leqslant x$. Further, $a \cup b = a \cup b \cup x$ implies $x \leqslant a \cup b$, which implies that $x \in \langle a, b\rangle$.

C o r o l l a r y 2. $\Theta_{ab} = \Theta_{a\, a \cup b} \cup \Theta_{b\, a \cup b}$.
The proof is obvious.

Finally we consider a congruence relation on L generated by an element $d \in L$, which congruence relation we shall need later.

Lemma 5. Let L be a semilattice and let $d \in L$. Then the equivalence Θ_d on L satisfying the condition

$a\Theta_d b$ if and only if $a = b$ or $a, b \geqslant d$ is a congruence relatin on L.

The proof is obvious.

3. On the minimal congruence relations generated by an ideal I of a semilattice L.

Consider a congruence relation on a semilattice L generated by an ideal I of L. We define the relation differently from the defintion proposed by Krishnan (see [1] p 26).

Lemma 6. Let I be an ideal of semilattice L and Θ_I a binary relation on L relating to I and satisfying the condition: $x\Theta_I y$ if and only if there exists in L an element $i \in I$ such that $x \cup i = x \cup y = y \cup i$, $x, y, i \in L$. Then Θ_I is a congruence relation on L if and only if L is modular; further, for $t \in I$, $x\Theta_I t$ if and only if $x \in I$.

Proof. Let L be a modular semilattice. According to Lemma 3 we shall show that Θ_I is a congruence relation on L.

1^{oo} Let i be an element of I, $x \in L$, and x and i noncomparable in L. According to the modularity of L there is in L an element $i_1 \leqslant i, x$ and thus $x \cup i_1 = x \cup x = x \cup i_1$, whence $x\Theta_I x$. The cases $i < x$, $i \geqslant x$ are trivial.

The proofs of 2^o, 3^o and 5^o of Lemma 3 are obvious and we omit them.

4^{oo} Let $x \geqslant y$, $x \geqslant z$ and $x\Theta_I y$, $x\Theta_I z$. Then $x \cup i_1 = x = y \cup i_1$ and $z \cup i_2 = x = x \cup i_2 \Rightarrow y \cup i_1 = x \cup y \cup z = z \cup i_2$, where $y \leqslant z \cup y$ and $z \leqslant z \cup y$. According to the modularity of L there are two elements $y' \leqslant y$, $i_1' \leqslant i_1$ in L such that $y' \cup i_1' = z \cup y \Rightarrow y \cup i_1' = z \cup y$. Similarly $z \cup i_2' = z \cup y$, and combining the last results, $y \cup (i_1' \cup i_2') = z \cup y = z \cup (i_1' \cup i_2')$, we obtain $z\Theta_I y$.

2^o Suppose that there exists on L for every ideal I of L a congruence relation Θ_I satisfying the condition of the lemma. Then for every two elements $a, b \in L$ there is in L a common lower bound $k \leqslant a, b$, $k \in L$. Indeed, if there would not be any common lower bound, then $b\Theta_{(a]} b$ implies $b \cup i = b \cup b = b \Rightarrow i \leqslant b$ and $i \in (a]$, which is a contradiction.

Assume that $a \leqslant b \cup c$, $b < a$. If $a = b \cup c$, $b = a$, or c and b are comparable, then there is no need for the proof; thus assume that $a < b \cup c$, $b < a$ and b and c are noncomparable. Let L be nonmodular, then $b_1 \cup c_1 < a$ for every $b_1 \leqslant b$ and $c_1 < c$, $c_1 < a$. In particular, $b \cup c_1 < a$ for every $c_1 < c, a$. Obviously for every c_1, $b \cup c_1 \Theta_{(c]} b \cup c$, which implies that $b \cup c_1 \Theta_{(c]} a$. According to the property of $\Theta_{(c]}$, $b \cup c_1 \cup i = b \cup c_1 \cup a = a \cup i$, $i \in (c]$, which is a contradiction, since $a, c > i$.

The proof of the last part of the lemma is obvious.

The congruence relation Θ_I characterizes the modularity of a semilattice L. Besides this property, Θ_I characterizes a minimal congruence relation generated by the ideal I of L. Let the notation $\Theta[I]$ mean the minimal congruence relation generated by I.

Theorem 2. Let I be an ideal of a modular semilattice L and $\Theta[I]$ the least congruence relation on L having I as a congruence class; then $a\Theta[I]b$ if and only if $a \cup b = i \cup b = i \cup a$, $a, b \in L$, $i \in I$.

Proof. 1^o Let $\Theta[I]$ have the minimality property of the theorem and suppose that Θ is a congruence relation on L such that $a\Theta b$ if and only if there is an element $i \in I$ such that $a \cup i = a \cup b = b \cup i$. If $x, y \in I$, then $x \cup y \in I$ and $x \cup (x \cup y) = x \cup y = y \cup (x \cup y)$, whence $x\Theta y$. Thus I is contained by a congruence class of Θ. According to the minimality of $\Theta[I]$, $\Theta[I] < \Theta$, which implies: if $a\Theta[I]b$, then there is $j \in I$ such that $a \cup i = b \cup i = b \cup a$.

2^o Assume that $a, b \in L$ are two elements such that there is an element $i \in I$ having the property $a \cup i = a \cup b = b \cup i$. According to Lemma 6, L is modular. Then $a = a \cup i_a \Theta[I] a \cup i$ and $b \cup i \Theta[I] b \cup i_b = b$, where i_a is a common lower bound of a and i, and i_b that of b and i. These relations imply $a\Theta[I]b$, since $b \cup i = a \cup i$. This completes the proof.

Now we can prove a theorem which is analogous to the results concerning lattice congruences proved by Grätzer and Schmidt in [3] § 3.

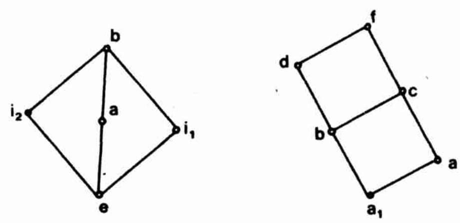
Theorem 3. The congruence relations of the type $\Theta[I]$ on a distributive semilattice L, where I is an ideal of L, form a distributive sublattice $\Theta(I, L)$ of the lattice $\Theta(L)$ of congruence relations on L.

Proof. We show in 1^o, $\Theta[I_1] \cup \Theta[I_2] = \Theta[I_1 \cup I_2]$, in 2^o, $\Theta[I_1] \cap \Theta[I_2] = \Theta[I_1 \cap I_2]$, and in 3^o that $\Theta(I, L)$ is distributive. 1^o and 2^o show that $\Theta(I, L)$ is a sublattice of $\Theta(L)$.

1^{o} Without the distributivity of L we can prove the relation $\Theta[I_1] \cup \Theta[I_2] = \Theta[I_1 \cup I_2]$. If $x(\Theta[I_1] \cup \Theta[I_2])\, y$ then there is a sequence $x = u_0, u_1, \ldots, u_n = y$ of elements of L such that of L such that $u_j \Theta[I_1] u_{j-1}$ or $u_j \Theta[I_2] u_{j-1}$ for every $j = 1, \ldots, n$. Then $u_j \cup i_1 = u_{j-1} \cup i_1 = u_j \cup u_{j-1}$ or $u_j \cup i_2 = u_{j-1} \cup i_2 = u_{j-1} \cup u_j$, where $i_1 \in I_1$ and $i_2 \in I_2$, whence $i_1, i_2 \in I_1 \cup I_2$. In particular, the relation $i_1, i_2 \in I_1 \cup I_2$ implies $u_j \Theta[I_1 \cup I_2] u_{j-1}$ for every j and thus, according to the transitivity of Θ, we obtain $x\Theta[I_1 \cup I_2]y$. The relation $\Theta[I_1 \cup I_2] \leqslant \Theta[I_1] \cup \Theta[I_2]$ is obvious. Thus $\Theta[I_1 \cup I_2] = \Theta[I_1] \cup \Theta[I_2]$.

2^{o} Clearly $\Theta[I_1 \cap I_2] \leqslant \Theta[I_1] \cap \Theta[I_2]$. Let $x(\Theta[I_1] \cap O[I_2])y$ which implies $x \cup i_1 = y \cup i_1 = x \cup y$ and $x \cup i_2 = y \cup i_2 = x \cup y$, where $i_1 \in I_1$ and $i_2 \in I_2$. The lattice $\mathcal{I}(L)$ of ideals of L is distributive, whence $(x] \cup ((i_1] \cap (i_2]) = ((x] \cup (i_1]) \cap ((x] \cup (i_2]) = (x \cup y]$. Accordingly it follows that there is an element $j \in (i_1] \cap (i_2]$ such that $x \cup j = x \cup y$. Similarly there exists $\bar{j} \in (i_1] \cap (i_2]$: $y \cup \bar{j} = x \cup y$. Hence, $j \cup \bar{j} \in I_1 \cap I_2$ and $x \cup j \cup \bar{j} = y \cup j \cup \bar{j} = x \cup y$. Therefore, $x\Theta[I_1]y$ and $x\Theta[I_2]y$, and the relation $\Theta[I_1 \cap I_2] = \Theta[I_1] \cap \Theta[I_2]$ follows.

3^{o} $\Theta[I_1] \cup (\Theta[I_2] \cap \Theta[I_3]) = \Theta[I_1] \cup \Theta[I_2 \cap I_3] = \Theta[I_1 \cup (I_2 \cap I_3)] = \Theta[(I_1 \cup I_2) \cap (I_1 \cup I_3)] = \Theta[I_1 \cup I_2] \cap \Theta[I_1 \cup I_3] = (\Theta[I_1] \cup \Theta[I_2]) \cap (\Theta[I_1] \cup \Theta[I_3])$, which shows the distributivity of $\Theta(I, L)$.



The modularity of a semilattice is a too weak property to imply the proof of 2^{o} above. We can see this by considering semilattice L of Figure 1 A, where L obviously is modular but nondistributive and where $a\Theta[(i_1]]\,b$ and $a\Theta[(i_2]]\,b$. But there is in L no element $i \in (i_1] \cap (i_2]$ such that $i \cup a = a \cup b = b \cup i$.

Let Θ_I^o be the maximal congruence relation on L generated by the ideal I of L, where $x\Theta_I^o y$ if and only if $x, y \in I$ or $x, y \notin I$.

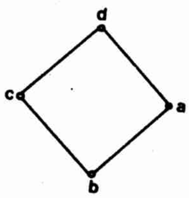
The distributive semilattice L of Figure 1 B shows that $\Theta[I]$ is not a neutral element of $\Theta(L)$. It is valid in L that $b\Theta[(a]]c$, since $a \cup b = b \cup c = a \cup c$, and $b(\Theta_{bd} \cup \Theta^{o}_{(b]})c$, since $b\Theta_{bd}d$ and $d, c \notin (b]$, whence $b(\Theta[(a]] \cap (\Theta_{bd} \cup \Theta^{o}_{(b]}))c$. But $\Theta[(a]] \cap \Theta_{bd}$ is the identical congruence relation and therefore $(\Theta[(a]] \cap \Theta_{bd}) \cup$ $\cup\, (\Theta[(a]] \cap \Theta^{o}_{(b]}) = \Theta[(a]] \cap \Theta^{o}_{(b]}$. Evidently $b(\Theta[(a]] \cap \Theta^{o}_{(b]})c$ is not true. So $\Theta[(a]]$ is not a neutral element of $\Theta(L)$.

We shall say that a congruence class modulo Θ, $\Theta \in \Theta(L)$, is trivial if the class consists of one element only. The following theorem is obvious.

<u>Theorem 4</u>. Let L be a modular semilattice and let I be an ideal of L containing at least two elements. Then there exists exactly one nontrivial congruence class modulo $\Theta[I]$ if and only if L is a chain.

4. On the distributivity of Θ L.

D. PAPERT has defined a necessary and sufficient condition for the distributivity of $\Theta(L)$ ([5] Thm. 7). In this paragraph we shall consider some properties of a semilattice L caused by the distributivity of $\Theta(L)$.



Consider the semilattice L of Figure 2. The congruence relation $\Theta_{ad} \cup \Theta^{o}_{(a]} = 1$ according to the maximality of $\Theta^{o}_{(a]}$, and thus $b(\Theta_{ad} \cup \Theta^{o}_{(a]})c$. However, $b\not\Theta_{ad}{}^{o}$ and $b\not\Theta^{o}_{(a]}c$, which imply the non-existence of a chain $b = x_0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n = c$, where $x_i\Theta_{ad}x_{i-1}$ or $x_i\Theta^{o}_{(a]}x_{i-1}$ for every $i = 1, \ldots, n$. This is a way to define the join on $\Theta(L^*)$ for a lattice L^*. In the case of a semilattice L we can prove:

<u>Lemma 7</u>. Let L be a semilattice and $x, y \in L$. Θ_{xy} is a neutral element in $\Theta(L)$ if and only if $\Theta(L)$ has the following property: Let $\varphi = \bigcup_{i \in A} \Theta_i$, where A is a finite set of indices and $\Theta_i \in \Theta(L)$ for every $i \in A$. If $x\varphi y$, then there are in L two finite chains $x = u_0 \leqslant u_1 \leqslant \ldots \leqslant u_n = x \cup y$ and $y = v_0 \leqslant v_1 \leqslant \ldots \leqslant v_m = x \cup y$ such that $u_j\Theta_i u_{j-1}$ and $v_k\Theta_p v_{k-1}$ for some $i, p \in A$ and $j = 1, \ldots, n$, $k = 1, \ldots, m$.

Proof. 1^o Let $\Theta(L)$ have the property of the lemma. According to Grätzer and Schmidt ([4], p.41) Θ_{xy} is neutral in $\Theta(L)$, if

I: $X \cap (\Theta_{xy} \cup Y) = (X \cap \Theta_{xy}) \cup (X \cap Y)$ and

II: $X \cup (\Theta_{xy} \cap Y) = (X \cup \Theta_{xy}) \cap (X \cup Y)$ for two elements $X, Y \in \Theta(L)$.

I: Let a, b be two elements of L such that $a(X \cap (\Theta_{xy} \cup Y))b$.

$\Longleftrightarrow a(X \cap (\Theta_{xy} \cup Y))b \cup a$ and $b(X \cap (\Theta_{xy} \cup Y))b \cup a$. In the following we shall only consider the relation $a(X \cap (\Theta_{xy} \cup Y))b \cup a$. The proof for the latter relation is similar. We shall use the sign "$\Longleftrightarrow$" althought the notations are not complete.

$\Longleftrightarrow aXa \cup b$ and $a(\Theta_{xy} \cup Y)a \cup b$.

$\Longleftrightarrow aXa \cup b$ and there is in L a chain $a = u_0 \leqslant u_1 \leqslant \ldots \leqslant u_n = a \cup b$ such that $u_i \Theta_{xy} u_{i-1}$ or $u_i Y u_{i-1}$ for $i = 1, \ldots, n$.

$\Longleftrightarrow u_{i-1}(X \cap \Theta_{xy})u_i$ or $u_{i-1}(X \cap Y)u_i$ for $i = 1, \ldots, n$.

$\Longleftrightarrow a((X \cap \Theta_{xy}) \cup (X \cap Y))a \cup b$.

Similarly $b(X \cap (\Theta_{xy} \cup Y)a \cup b \Longleftrightarrow b((X \cap \Theta_{xy}) \cup (X \cap Y))a \cup b$.

$\Longleftrightarrow a((X \cap \Theta_{xy}) \cup (X \cap Y))b$.

II: The relation $A \cup (B \cap C) \leqslant (A \cup B) \cap (A \cup C)$ is true in each lattice L^*, $A, B, C \in L^*$. Therefore, it remains to show that $X \cup (\Theta_{xy} \cap Y) \geqslant ((X \cup \Theta_{xy}) \cap (X \cup Y))$.

Let a, b be two elements of L such that $a((X \cup \Theta_{xy}) \cap (X \cup Y))b$.

$\Longleftrightarrow a((X \cup \Theta_{xy}) \cap (X \cup Y))a \cup b$ and $b((X \cup \Theta_{xy}) \cap (X \cup Y)a \cup b$. We consider merely the notation relating to the former relation.

$\Longleftrightarrow$ There is in L a chain $a = u_0 \leqslant u_1 \leqslant \ldots \leqslant u_n = a \cup b$ such that $u_{i-1} X u_i$ or $u_{i-1} \Theta_{xy} u_i$ for $i = 1, \ldots, n$, and $u_{i-1}(X \cup Y)u_i$ for every i. Now we can apply the properties of $(X \cup Y)$ to every interval $[u_{i-1}, u_i]$ and after the application we obtain a a chain by which we obtain $a(X \cup (\Theta_{xy} \cap Y))a \cup b$. Similarly $b(X \cup (\Theta_{xy} \cap Y))a \cup b$.

$\Longleftrightarrow a(X \cup (\Theta_{xy} \cap Y))b$, which completes the first part of the proof.

2° Suppose that Θ_{xy} is a neutral element of $\Theta(L)$ for any pair $x, y \in L$, and $a(X \cup Y)b$ for two elements $X, Y \in \Theta(L)$. Then $a(X \cup Y)a \cup b$ and $b(Y \cup X)a \cup b$. We consider merely the formen relation; the proof for the latter is similar. Since $\Theta_{a, a \cup b}$ is neutral in $\Theta(L)$, $a(\Theta_{a, a \cup b} \cap (X \cup Y))a \cup b \Longleftrightarrow a((\Theta_{a, a \cup b} \cap Y) \cup (\Theta_{a, a \cup b} \cap X))a \cup b$. The general definition of the meet on $\Theta(L)$ implies the existence of a sequence $a = u_0, u_1, \ldots, u_n = a \cup b$ of elements of L such that $u_i(\Theta_{a, a \cup b} \cap Y)u_{i-1}$ or $u_i(\Theta_{a, a \cup b} \cap X)u_{i-1}$ for every $i = 1, \ldots, n$. As $\Theta_{a, a \cup b}$ is the minimal element of $\Theta(L)$ collapsing the interval $[a, a \cup b]$, and as $u_0 \Theta_{a, a \cup b} u_1$, then according to Corollary 1 of Lemma 4, $a \leqslant u_1$

$\leqslant a \cup b$. Similarly, $a = u_o \Theta_{a, a \cup b} u_i$ for every i, and thus for every i, $a \leqslant u_i \leqslant a \cup b$. By applying Lemma 3 : 2^o we obtain by the aid of the elements u_i the desired chain $a = u_o \leqslant u_o \cup u_1 \leqslant u_o \cup u_1 \cup u_2 \leqslant \ldots \leqslant u_o \cup u_1 \cup u_2 \cup \ldots \cup u_n = a \cup b$, for which we use the abbreviation

$$a = w_o \leqslant w_1 \leqslant w_2 \leqslant \ldots \leqslant w_n = a \cup b. \qquad (1)$$

Thus, if $a(X \cup Y)a \cup b$, then there is in L a chain (1) such that for every $i = 1, \ldots, n$ either $w_i Y w_{i-1}$ or $w_i Y w_{i-1}$.

By the induction we see that the lemma is true for every finite A which completes the proof.

Now we can prove a theorem concerning the distributivity of $\Theta(L)$.

Theorem 5. Let L be a semilattice. $\Theta(L)$ is a distributive lattice if and only if Θ_{xy} is a neutral element in $\Theta(L)$ for any two elements $x, y \in L$.

Proof. If $\Theta(L)$ is distributive, the proof is trivial. Let Θ_{xy} be a neutral element of $\Theta(L)$ for every two $x, y \in L$, and Θ, Φ, Y three elements of $\Theta(L)$. For the distributivity of $\Theta(L)$ it is sufficient to show that $\Theta \cap (\Phi \cup Y) \leqslant (\Theta \cap \Phi) \cup (\Theta \cap Y)$. If $a(\Theta \cap (\Phi \cup Y))b \iff a(\Theta \cap (\Phi \cup Y))a \cup b$ and $b(\Theta \cap (\Phi \cup Y))a \cup b$; we consider only the former relation. According to the neutrality of Θ_{xy}, there is in L a chain $a = u_o < u_1 \leqslant \ldots \leqslant u_n = a \cup b$ such that for every $i = 1, \ldots, n$ either $u_i \Phi u_{i-1}$ or $u_i Y u_{i-1}$. As $a \Theta a \cup b$, then $u_i(\Theta \cap \Phi)u_{i-1}$ or $u_i(Y \cap \Theta)u_{i-1}$ for every i and so $a((\Theta \cap \Phi) \cup (\Theta \cap Y))a \cup b$. Similarly $b((\Theta \cap \Phi) \cup (\Theta \cap Y))a \cup b$ and the theorem follows.

Corollary. Let L be a semilattice. $\Theta(L)$ is distributive lattice if and only if for every congruence relation $\Phi = \bigcup_{i \in A} \Theta_i$, $a \Phi b$, there is in L two chains $a = u_o \leqslant u_1 \leqslant \ldots \leqslant u_n = a \cup b$ and $b = v_o \leqslant v_1 \leqslant \ldots \leqslant v_m = a \cup b$ such that $u_j \Theta_i u_{j-1}$ and $v_k \Theta_p v_{k-1}$ for some $i, p \in P \leqslant A$, where P is finite, and for $j = 1, \ldots, n$, $k = 1, \ldots, m$.

Corollary is an other formulation of Theorem 5.

We shall call a semilattice L, whose $\Theta(L)$ is a distributive lattice, quasidistributive. In the following we consider some criteria and properties concerning a quasidistributive semilattice L.

Theorem 6. A semilattice L is quasidistributive if and only if the only nontrivial congruence class of the congruence relation Θ_{ab} is the part $\langle a,b\rangle$ of L.

Proof. 1° Let L be quasidistributive and $c\Theta_{ab}d$, $c,d \notin \langle a,b\rangle$, $a \neq b$ and $c \neq d$, and $a,b,c,d \in L$. According to the definition of Θ_{ab} only three cases arise: (i) $c \cup d > a \cup b$, (ii) $c \cup d < a \cup b$ and (iii) $c \cup d$ and $a \cup b$ are noncomparable.

(i) $c\Theta_{ab}d \Leftrightarrow c\Theta_{ab}c \cup d$ and $d\Theta_{ab}c \cup d$. Thus $a \cup c \cup d = c \cup d = b \cup c \cup d$. But if c (or d) is noncomparable with $a \cup b$, then $a \cup c \neq c$ and $b \cup c \neq c$ ($a \cup d \neq d$ and $b \cup d \neq d$), since $a \cup b$ and c(d) have not a common lower bound in L (see [5, Thm. 7]). If for c (or d), $c > a \cup b$, then $c \cup a \cup b \neq a \cup b \cup c \cup d$ (or $d \cup a \cup b \neq a \cup b \cup c \cup d$, since $d \neq c$. Hence $c \not\Theta_{ab} d$.

(ii) If $c \cup d < a \cup b$, then $c \cup a \neq c$ and $c \cup b \neq c$, since if $c \cup a = c$ or $c \cup b = c$ then $c \in \langle a,b\rangle$, which is a contradiction.

(iii) $a \cup c \neq c$ and $b \cup c \neq c$, since the noncomparable elements have not a common lower bound in L.

2° Let the only nontrivial congruence class modulo Θ_{ab} be the part $\langle a,b\rangle$ of L for every two elements $a,b \in L$. Assume that two noncomparable elements c and d of L have a common lower bound k in L (see [5, Thm. 7]) and let us consider the congruence relation Θ_{kc}. $d\Theta_{kc}c \cup d$, since $k \cup d = d$, $c \cup d \cup c = c \cup d$, $d \cup k \cup c = d \cup c \cup k \cup c$. But $d \in \langle k,c\rangle = [k,c]$, since d and c are noncomparable, and $d \cup c \in [k,c]$, since $c < d \cup c$. Thus $d\Theta_{kc}c \cup d$ implies a contradiction.

Theorem 7. Let L be a quasidistributive semilattice. For every pair a,b, $a \neq b$, of elements of L Θ_{ab} has a complement in $\Theta(L)$.

Proof. Consider the congruence relation $\bigvee_{x \in A} \Theta^{o}_{(x]} = X$, where $A = \langle a,b\rangle - a \cup b$. The congruence relation exists, since $\Theta(L)$ is a complete lattice. If $z(\Theta_{ab} \cap X)u$ ($z \neq u$, $z,u \in L$), then $z\Theta_{ab}u$ and according to Theorem 6, $z,u \in \langle a,b\rangle$. This implies $\Theta^{o}_{(z]} \in \{\Theta^{o}_{(x]} : x \in A\}$ for which $z \not\Theta^{o}_{(z]} z \cup u$, which is a contradiction. Hence $\Theta_{ab} \cap X = O$.

Consider $\Theta_{ab} \cup X$. Let $z \neq u$ be two elements of L. We shall show that $u(\Theta_{ab} \cup X)\, z \cup u$, which implies $\Theta_{ab} \cup X = 1$. The proof contains three cases: (i) $u \geqslant a \cup b$, (ii) u and $a \cup b$ are noncomparable, and (iii) $u < a \cup b$.

(i) If $u \geqslant a \cup b$, then $u \cup z \geqslant a \cup b$ and $u\, \Theta^{o}_{(x]} z \cup u$ for every $x \in A$.

(ii) If u and a b are noncomparable, then $z \cup u \nleqslant a \cup b$, since $u \leqslant a \cup b$ and thus $z \cup u \in \langle a,b \rangle$. Then $u\Theta^{o}_{(x]} z \cup u$ for every $x \in A$.

(iii) If $u < a \cup b$, then (1) $u \in \langle a,b \rangle$, or (2) $u < a$ (or $u < b$), or (3) $u < a \cup b$ and u is noncomparable with a and b. (1) If $u, z \cup u \in \langle a,b \rangle$, then $u\Theta_{ab} z \cup u$, and if $z \cup u \notin \langle a,b \rangle$, then $z \cup u > a \cup b$, since two noncomparable elements in L have not a common lower bound (see [5, Thm. 7]). Hence $u\Theta_{ab} a \cup b$ and $a \cup b\Theta^{o}_{(x]} z \cup u$ for every $x \in A$. (2) If $u < a$, then $u\Theta^{o}_{(x]} a$ for every $x \in A$, for $u \in (x]$ if and only if $a \in (x]$, since two noncomparable elements of L have not a common lower bound in L. The last part of the proof is similar to that of (1). (3) If $u < a \cup b$ and u is noncomparable with a and b, then $u \notin \langle a,b \rangle$. Thus $u\Theta^{o}_{(x]} u \cup b$ or $u\Theta^{o}_{(x]} u \cup a$ for every $x \in A$ and further, $u \cup b\Theta_{ab} a \cup b$ (or $u \cup a\Theta_{ab} a \cup b$). After this we can continue as in the case (1). Hence X is the complement of Θ_{ab} in $\Theta(L)$.

According to Theorem 6 we can define a minimal congruence relation generated by an ideal I of a quasidistributive semilattice L.

Lemma 8. Let L be a quasidistributive semilattice and I an ideal of L. Then $x\Theta[I]y$ for $x \neq y$ if and only if $x, y \in I$.

The proof follows trivially from Theorem 6.

Theorem 8. Let L be a semilattice and I an ideal of L. L is quasidistributive if and only if $x\Theta[I]y$ for $x \neq y$ implies $x, y \in I$.

Proof. 1° Let $\Theta[I]$ have the property of the theorem and a, b be a pair of elements of L such that a and b are noncomparable and have a common lower bound k in L. Consider the ideal (a] and the congruence relation $\Theta[(a]]$. According to the definition of $\Theta[(a]]$, $\Theta_{ka} \leqslant \Theta[(a]]$, where $a > k$ and $\Theta_{ak} \neq 0$, since a and b are noncomparable. Now $b\Theta_{ak} a \cup b$ and thus $b\Theta[(a]] a \cup b$, $b \neq a \cup b$, and $b, a \cup b \in (a]$, which is a contradiction. The converse proof is obvious according to Lemma 8.

We end this paragraph by proving a theorem concerning the complement of a congruence relation $\Theta[(a]] \in \Theta L$, where L is a quasidistributive semilattice.

Theorem 9. Let L be a quasidistributive semilattice, then every congruence relation $\Theta[(a]]$, $a \in L$, has a complement in $\Theta(L)$.

Proof. Consider the meet $\bigcap_{\substack{x \in (a] \\ x \neq a}} \Theta^{o}_{(x]} = X$, which exists since $\Theta(L)$ is complete. Theorem 8 implies directly $X \cap \Theta[(a]] = 0$. We shall show that $X \cup \Theta[(a]] = 1$. If z, y are two elements of L, then three cases arise: (i) $z, y \in (a]$, or (ii) $z, y \notin (a]$, or (iii) $z \in (a]$ and $y \notin (a]$ ($y \in (a]$ and $z \in (a]$ has a similar proof).

(i) $z, y \in (a] \Rightarrow z\Theta[(a]]y$. (ii) $z, y \in (a]$ implies for every $x \in (a]$, $x \neq a$, $z\Theta^{o}{}_{x}\, y$ and thus zXy. iii If z a , y a , then $z\Theta$ a a and $a\Theta^{o}{}_{(x]}y$ for every $x \in (a]$, $x \neq a$. Hence $\Theta[(a]] \cup X = 1$.

5. On the weak projectivity of a semilattice

L. Varlet's definition of a supertransposition ([8, p. 235]) suggests to define the concept of the weak projectivity of a semilattice L. A supertransposition of a part $\langle a, b\rangle$ of L relating to an element $x \in L$ is the part $\langle a \cup x, b \cup x\rangle = \langle a, b\rangle_x$ of L.

<u>Definition 3</u>. Let L be a semilattice and $a, b, c, d \in L$. A pair a, b of elements of L is weakly projective into a pair c, d of elements, in symbols $\overline{a, b} \to \overline{c, d}$, if there are in L three finite sequences $x_1, x_2, \ldots, x_n$, $a_o, a_1, \ldots, a_{n-1}$, and $b_o, b_1, \ldots, b_{n-1}$ of elements of L such that $\langle a, b\rangle \supseteq \langle a_o, b_o\rangle$, $\langle a_o, b_o\rangle_{x_1} \supseteq \langle a_1, b_1\rangle$, $\langle a_1, b_1\rangle_{x_2} \supseteq \langle a_2, b_2\rangle, \ldots, \langle a_{n-1}, b_{n-1}\rangle_{x_n} \supseteq \langle c, d\rangle$.

<u>Remark 1</u>. The relation "$\longrightarrow$" is transitive. Therefore, $\overline{a, b} \longrightarrow \overline{c \cup t, d \cup t}$ follows from the relations $\overline{a, b} \longrightarrow \overline{c, d}$ and $\overline{c, d}$ — $\overline{c \cup t, d \cup t}$.

<u>Remark 2</u>. The substitution property of a congruence relation Θ of L implies: If $\overline{a, b} \longrightarrow \overline{c, d}$ and $a\Theta b$, then $c\Theta d$.

Introduced by the definition of a minimal lattice congmence relation we define by the aid of the weak projectivity a minimal congruence relation Θ_{abw} generated by a part $\langle a, b\rangle$ of a semilattice L.

<u>Lemma 9</u>. Let L be a semilattice and $a, b \in L$. Let for $c, d \in L$ be $c\Theta_{abw}d$ if and only if $c = d$ or there exists a finite sequence $c = y_o, y_1, \ldots, y_n = d$ such that $\overline{a, b} \longrightarrow \overline{y_i, y_{i+1}}$ for every $i = 0, \ldots, n-1$. Then Θ_{abw} is a congruence relation on L and $\Theta_{abw} = \Theta_{ab}$ is true.

<u>Proof</u>. Let Θ_{abw} be the binary relation defined above; we shall show that Θ_{abw} is a congruence relation on L. $u\Theta_{abw}u$ is trivial and from the relation $\langle u, v\rangle = \langle v, u\rangle$ it follows that

$u\Theta_{abw}v$ if and only if $v\Theta_{abw}u$. Let $u\Theta_{abw}v$ and $v\Theta_{abw}q$, then there are in L two sequences $u = y_o, y_1, \ldots, y_n = v = z_o, z_1, \ldots, z_m = q$ such that $\overline{a,b} \longrightarrow \overline{y_i, y_{i+1}}$ and $\overline{a,b} \longrightarrow \overline{z_j, z_{j+1}}$ for every i and j, $i=0,\ldots,n-1$, $j=0,\ldots, m-1$. Then immediately $u\Theta_{abw}q$. Remark 1 implies the substitution property.

Since $\overline{a,b} \longrightarrow \overline{a,b}$, $a\Theta_{abw}b$, and according to the minimality of Θ_{ab}, $\Theta_{ab} < \Theta_{abw}$. Further, the relation $\Theta_{abw} \leqslant \Theta_{ab}$ follows immediately from Remark 2, and hence $\Theta_{ab} = O_{abw}$. This completes the proof.

The lemma above implies a corollary for Theorem 6.

<u>Corollary</u>. Let L be a quasidistributive semilattice, then for every part $<a,b>$ of L, $\overline{a,b} \longrightarrow \overline{c,d}$ if and only if $c,d \in <a,b>$.

We define a minimal part $<a,b>$ of a semilattice L.

Definition 4. A part $<a,b>$, $a \neq b$, of a semilattice L is a minimal part of L if and only if it follows from the relation $\overline{a,b} \rightarrow \overline{c,d}$ ($c \neq d$): there is in L a finite sequence $a = z_o, z_1, \ldots, z_n = b$ of elements of L, $z_i \in <a,b>$, such that $\overline{c,d} \longrightarrow \overline{z_i, z_{i+1}}$ for every $i = 0,\ldots,n-1$.

We can now derive results on a semilattice L similar to those proposed by P. Crawley, [2], in the case of lattices.

<u>Lemma 10</u>. A part $<a,b>$ of a semilattice L generates an atom of the lattice $\Theta(L)$ if and only if $<a,b>$ is a minimal part of L.

<u>Proof</u>. Assume that $\Theta \in \Theta(L)$ is an atom of $\Theta(L)$ and $<a,b>$ is the part of L generating Θ. Let $\overline{a,b} \longrightarrow \overline{c,d}$, where $<c,d>$ is an arbitrary but proper part of L, i.e. $c \neq d$. Then $c\Theta d$, and thus $\Theta_{cd} \leqslant \Theta$, and since Θ is an atom, $\Theta = \Theta_{cd}$. In particular, $a\Theta_{cd}b$, and according to Lemma 9, there is in L a finite sequence $a = y_o, y_1, \ldots, y_n = b$ of elements such that $\overline{c,d} \rightarrow \overline{y_i, y_{i+1}}$ for every $i = 0,\ldots,n-1$. Since $c\Theta d$, then $a\Theta y_1$ and further, $a\Theta_{ab}y_1$. By applying the transitivity we see that $y_i\Theta a \Rightarrow y_iO_{ab}a$ for every i. According to Corollary 1 of Lemma 4, $y_i\Theta_{ab}\, a$ implies $y_i \in <a,b>$ for every i.

2° Let $<a,b>$ be a minimal part of L, then $a\Theta_{ab}b$ and thus $\Theta_{ab} > O$. Let Θ be a congruence relation of L such that there is in L some proper part $<c,d>$ for which $c\Theta d$ and $c\Theta_{ab}d$. We show that $\Theta_{ab} \leqslant \Theta$, which proves that Θ_{ab} is an atom of $\Theta(L)$. As $c\Theta_{ab}d$, then, according to Lemma 9, there is in L a sequence $c = z_o, z_1, \ldots, z_n = d$ of elements of L such that $a,b \longrightarrow z_i, z_{i+1}$ for every $i = 0,\ldots,n-1$. Since $<c,d>$ is proper, then at least one part $<z_i, z_{i+1}>$ is proper, say $<z_k, z_{k+1}>$. According to the definition of a minimal part, there is in L a sequence $a = u_o, u_1, \ldots, u_n = b$ such that $\overline{z_k, z_{k-1}} \longrightarrow \overline{u_j, u_{j+1}}$ for every $j = 1, \ldots, m-1$. Now $c\Theta d$ implies $z_k \Theta z_{k+1}$, and further, according to the weak projectivity, $u_j \Theta u_{j+1}$ for every j. It follows from the transitivity, $a\Theta b$, and since Θ_{ab} is a minimal congruence relation collapsing the part $\langle a, b \rangle$, $\Theta_{ab} \leqslant \Theta$. Hence the lemma.

According to the lemma above we can prove a theorem concerning the boolean property of the lattice $\Theta(L)$. The proof is a direct copy of that proposed by P. Crawley in the case of lattices ([2, Thm. 4.1]).

Theorem 10. Let L be a semilattice. The lattice $\Theta(L)$ is boolean if and only if for each proper part $\langle a, b \rangle$, $a < b$, of L there is in L a finite sequence $a = z_o < z_1 < z_2 < \ldots < z_n = b$ such that each part $\langle z_i, z_{i+1} \rangle$, $i = 0, \ldots, n-1$, is a minimal part of L.

In the following we consider the structure of L when $\Theta(L)$ is atomic.

Theorem 11. Let L be a quasidistributive lattice. $\Theta(L)$ is atomic if and only if for every proper part $\langle x, y \rangle$ of L there is in L a minimal part $\langle p, s \rangle$ such that $\overline{x, y} \longrightarrow \overline{p, s}$.

Proof. 1^o Let a part $\langle x, y \rangle$ have the property defined above and let Θ be an element of $\Theta(L)$, $\Theta \neq 0$. Then there is at least one proper part $\langle x, y \rangle$ of L such that $x\Theta y$. According to the assumption, $\overline{x, y} \longrightarrow \overline{p, s}$, where $\langle p, s \rangle$ is a minimal part of L, and thus $\Theta \geqslant \Theta_{xy} \geqslant \Theta_{ps}$, where Θ_{ps} is an atom of $\Theta(L)$.

2^o Let $\Theta(L)$ be atomic, and assume that $\langle x, y \rangle$ is a proper part of L such that the following relation is valid for no minimal part $\langle p, s \rangle$ of L: $\overline{x, y} \longrightarrow \overline{p, s}$. Since $\langle x, y \rangle$ is proper, then $\Theta_{xy} \neq 0$, and since $\Theta(L)$ is atomic there is an atom $\Theta_{ps} \leqslant \Theta_{xy}$. Thus $p\Theta_{xy}s \Rightarrow p\Theta_{xy} p \cup s$ and $s\Theta_{xy} p \cup s$. Consider the former relation and assume that $p \neq p \cup s$. According to Lemma 9 and the corollary after it, there is in L a sequence $p = z_o, z_1, \ldots, z_n = p \cup s$ of elements such that $x, y \longrightarrow z_i, z_{i+1}$ for every $i = 0, \ldots, n-1$. Since $\langle p, p \cup s \rangle$ is proper, then at least one part $\langle z_i, z_{i+1} \rangle$ is proper, say $\langle z_k, z_{k+1} \rangle$. We show that $\langle z_k, z_{k+1} \rangle$ is a minimal part of L. According to the corollary of Theorem 5, $z_k, z_{k+1} \in \langle p, p \cup s \rangle$ and then $\overline{p, s} \longrightarrow \overline{z_k, z_{k+1}}$, which implies $z_k \Theta_{ps} z_{k+1}$. Since $\langle z_k, z_{k+1} \rangle$ is proper, $\Theta_{z_k z+1} \neq 0$. $z_k \Theta_{ps} z_{k+1} \Rightarrow O_{z_k z_{k+1}} \leqslant \Theta_{ps}$ and since Θ_{ps} is an atom, $\Theta_{ps} = \Theta_{z_k z_{k+1}}$. But then $\langle z_k, z_{k+1} \rangle$ is a minimal part and $\overline{x, y} \longrightarrow \overline{z_k, z_{k+1}}$, which is a contradiction.

According to the corollary after Lemma 9, we obtain the following corollary:

Corollary. Let L be quasidistributive. Then $\Theta(L)$ is atomic if and only if every proper part of L contains a minimal part of L.

Finally, we construct by the aid of the weak projectivity a theorem concerning pseudocomplements in $\Theta(L)$, L is a semilattice.

Theorem 12. Let Θ be a congruence relation on a semilattice L, and Θ^+ a binary relation on L satisfying the following condition: $x\Theta^+y$ if and only if there exists in L no proper part $\langle u,z\rangle$ such that $\overline{x,y} \to \overline{u,z}$ and $u\Theta z$. Then Θ^+ is a congruence relation on L and it is a pseudocomplement of Θ.

Proof. By D. Papert ([5, Thm. 2]) there exists in $\Theta(L)$ a pseudocomplement Θ^+ for any $\Theta \in \Theta(L)$. Let $A = \{\langle x,y\rangle : x,y \in L, \overline{x,y} \to \overline{y,v}$ and $u\Theta v \Rightarrow u = v\}$ and put $\phi = \bigcup_{\langle x,y\rangle \in A} \Theta_{xy}$. Since $\Theta \cap \Theta_{xy} = 0$ for $\langle x,y\rangle \in A$, then $\phi \leqslant \Theta^+$. For $p \neq v$ such that $p\Theta^+v$, and $\overline{p,v} \to \overline{u,z}$, and $u\Theta z$, we obtain $u = v$. Thus $\langle p,v\rangle \in A$. Therefore $\Theta^+ \leqslant \phi$. Hence $\phi = \Theta^+$ which completes the proof.

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Juhani Nieminen
Tampere university of technology
Pyynikintie 2
33230 Tampere 23
Finland

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Súhrn

O reláciách kongruencií na polozväzoch

JUHANI NIEMINEN

V práci autor skúma relácie kongruencie na polozväzoch. V niektorých prípadoch dokazuje vety, ktoré sú zovšeobecnením analogických tvrdení z teórie zväzov.

Содержание

Об отношениях конгруэнтностей на полуструктурах

ЮХАНИ НИЭМИНЕН

В работе изучаются отношения конгруэнтностей на полуструктурах. Для специальных классов получаются результаты, которые являются обобщением аналогических теорем известных из теории структур.

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ON THE K-THIN ARITHMETICAL SETS

EVA NYULASSYOVÁ, Trnava

This paper is divided into three parts. In the first part we shall give some estimations and eventually precise values of the function f_k defined in the sequel.

The second part contains some generalizations of certain P. Erdös' results on the asymptotic densities of k-thin sets.

The third part of this paper is devoted to the solution of analogous questions for the multiplicative k-thinness.

1.

In this part we shall give a generalization of a result of P. Turán /see [1] /.

Definition 1,1. Let $k \geqslant 2$ be an integer. A set M of natural numbers is called k-thin, if the sum of no k distinct elements of M belongs to M.

For natural n denote by $f_k(n)$ the greatest natural number $m > n$ such that the set $\{n, n-1, \dots, m\}$ can be decomposed into two disjoint k-thin subsets. In the paper [1] a proof of the following theorem according to P. TURÁN is given.

Theorem A. The set $\{n, n+1, \dots, 5n+3\}$ of natural numbers cannot be decomposed into two disjoint 2-thin subsets.

We shall prove:

Theorem 1,1.

$$f_k(n) \geqslant n(k^2+k-1) + \frac{1}{2}(k-1)(k^2+2k-2) - 1.$$

Proof. Let

$$\alpha = n+(n+1)+\ldots+(n+k-1) = nk + \frac{1}{2}k(k-1),$$
$$\beta = \alpha+(\alpha+1)+\ldots+(\alpha+k-1) = nk^2 + \frac{1}{2}k^2(k-1)+\frac{1}{2}k(k-1),$$
$$\gamma = n+(n+1)+\ldots+(n+k-2)+\beta = n(k^2+k-1) + \frac{1}{2}(k-1)(k^2+2k-2).$$

Consider the following decomposition of the set $\{n, n+1, \ldots, \gamma-1\}$ into two subsets A, B.

$$A = \{n, n+1, \ldots, \gamma-1\} \cup \{\beta, \beta+1, \ldots, \gamma-1\},$$
$$B = \{\alpha, \alpha+1, \ldots, \beta-1\}.$$

It can be easily checked that both A and B are k-thin. Hence $\gamma-1 \leqslant f_k(n)$.

Theorem 1,2. $f_3(n) = 11n+12$.

Remark. Theorem A and Theorem 1,2 show that in the case k=2 and k=3 in Theorem 1,1 the relation " $\geqslant$ " can be replaced by " = ".

Proof of Theorem 1,2. Consider the decompositions of the set $\{n, n+1, \ldots, m\}$ $(m > 11n+12)$ into two disjoint subsets A and B. We shall show that no decomposition exists so that both A and B are 3-thin. We shall test all 16 possibilities to place the five elements $n, n+1, n+2, n+3$ and $n+4$ into A or B.

I. If $\{n, n+1, n+2, n+3\} \subset A$, $n+4 \in B$, then $\{3n+3, 3n+4, n+4\} \subset B$, hence $\{7n+11, n+2, n+3\} \subset A$. It follows $\{5n+6, 3n+3, n+4\} \subset B$, hence $9n+13 \in A$. But $9n+13$ is a sum of tree elements of A: $n, n+2, 7n+11$. Hence A is not 3-thin. The description of this situation will be shortened in the following.

II. If $\{n, n+1, n+2, n+4\} \subset A$, $n+3 \in B$, then $\{3n+3, 3n+5, n+3\} \subset B$, hence $\{7n+11, n+1, n+4\} \subset A$, it follows $\{5n+6, 3n+3, n+3\} \subset B$, hence $9n+12 \in A$, but $9n+12 = n+(n+1)+(7n+11)$.

III. If $\{n, n+1, n+3, n+4\} \subset A$, $n+2 \in B$, then $\{3n+5, 3n+4, n+2\} \subset B$, hence $\{7n+11, n+1, n+4\} \subset A$, it follows $\{5n+6, 3n+4, n+2\} \subset B$, hence $9n+12 \in A$, but $9n+12 = n+(n+1)+(7n+11)$.

IV. If $\{n, n+2, n+3, n+4\} \subset A$, $n+1 \in B$, then $\{3n+5, 3n+6, n+1\} \subset B$, hence $\{7n+12, n, n+4\} \subset A$, it follows $\{5n+8, 3n+5, n+1\} \subset B$, hence $9n+14 \in A$, but $9n+14 = n+(n+2)+(7n+12)$.

V. If $n \in A$, $\{n+1, n+2, n+3, n+4\} \subset B$, then $\{3n+6, 3n+7, n\} \subset A$, hence $\{7n+13, n+3, n+4\} \subset B$, it follows $\{5n+6, 3n+6, n\} \subset A$, hence $\{9n+12, n+1, n+2\} \subset B$, then $\{7n+9, 3n+6, n\} \subset A$, it follows

$\{3n+3, n+1, n+3\} \subset B$, hence $\{5n+7, 3n+6, n\} \subset A$, then $\{9n+13, n+1, n+2\} \subset B$, it follows $\{7n+10, 3n+6, n\} \subset A$, hence $\{3n+4, n+1, n+2\} \subset B$, then $\{5n+7, 3n+7, n\} \subset A$, it follows $\{9n+14, n+1, n+2\} \subset B$, hence $\{7n+11, 3n+6, n\} \subset A$, then $\{3n+5, 3n+4, 3n+3\} \subset$ $\subset B$, hence $9n+12 = (5n+6)+(3n+6)+ n$.

VI. If $\{n, n+3, n+4\} \subset A$, $\{n+1, n+2\} \subset B$, then $\{3n+7, n+1, n+2\} \subset B$, hence $\{5n+10, n, n+3\} \subset A$, it follows $\{7n+13, 3n+7, n+1\} \subset B$, hence $\{3n+5, n, n+4\} \subset A$, then $\{5n+9, n+1, n+2\} \subset B$, hence $\{3n+6, 3n+5, n\} \subset A$, it follows $\{7n+11, n+1, n+2\} \subset B$, then $\{5n+8, n, 3n+6\} \subset A$, hence $9n+14 \in B$, but $9n+14 = (n+1) +$ $+ (n+2)+(7n+11)$.

VII. If $\{n, n+1, n+4\} \subset A$, $\{n+2, n+3\} \subset B$, then $\{3n+5, n+2, n+3\} \subset B$, hence $\{5n+10, n, n+4\} \subset A$, it follows $\{3n+6, 3n+5, n+$ $+3\} \subset B$, hence $7n+14 \in A$, but $7n+14 = n + (n+4) + 5n+10$.

VIII. If $\{n, n+1, n+2\} \subset A$, $\{n+3, n+4\}$ B, then $\{5n+10, n, n+1\} \subset A$, hence $\{7n+11, 3n+3, n+4\} \subset B$, it follows $\{3n+4, n+1, n+2\} \subset A$, then $\{5n+7, 3n+3, n+4\} \subset B$, hence $9n+14 \in A$, but $9n+14 = n + (3n+4) + (5n+10)$.

IX. If $\{n, n+1\} \subset A$, $\{n+2, n+3, n+4\} \subset B$, then $\{3n+9, n, n+1\} \subset A$, hence $\{5n+10, n+2, n+3\} \subset B$, it follows $\{3n+5, 3n+9, n+1\} \subset A$, hence $7n+15 \in B$, but $7n+15 = (n+2) + (n+3) + (5n+10)$.

X. If $\{n, n+2\} \subset A$, $\{n+1, n+3, n+4\} \subset B$, then $\{3n+8, n, n+2\} \subset A$, hence $\{5n+10, n+1, n+4\} \subset B$, it follows $\{3n+5, n, n+2\} \subset A$, hence $\{5n+7, n+1, n+3\} \subset B$, then $\{3n+3, 3n+8, n\} \subset A$, hence $(7n+11) \in B$, but $7n+11 = (n+1) + (n+3) + (5n+7)$.

XI. If $\{n, n+3\} \subset A$, $\{n+1, n+2, n+4\} \subset B$, then $\{3n+7, n, n+3\} \subset$ $\subset A$, hence $\{5n+10, n+1, n+4\} \subset B$, it follows $\{3n+5, 3n+7, n+3\} \subset$ $\subset A$, hence $7n+15 \in B$, but $7n+15 = (n+1) + (n+4) + (5n+10)$.

XII. If $\{n, n+1, n+3\} \subset A$, $\{n+2, n+4\} \subset B$, then $\{3n+4, n+2, n+4\} \subset$ $\subset B$, hence $\{5n+10, n+1, n+3\} \subset A$, it follows $\{3n+6, 3n+4, n+4\} \subset$ $\subset B$, hence $7n+14 \in A$, but $7n+14 = (n+1) + (n+3) + (5n+10)$.

XIII. If $\{n, n+4\} \subset A$, $\{n+1, n+2, n+3\} \subset B$, then $\{3n+6, n, n+4\} \subset$ $\subset A$, hence $\{5n+10, n+1, n+2\} \subset B$, it follows $\{3n+7, 3n+6, n\} \subset A$, hence $7n+13 \in B$, but $7n+13 = (n+1) + (n+2) + (5n+10)$.

XIV. If $\{n, n+2, n+3\} \subset A$, $\{n+1, n+4\} \subset B$, then $\{3n+5, n+1, n+4\} \subset B$, hence $\{5n+10, n, n+3\} \subset A$, it follows $\{3n+7, 3n+5, n+1\} \subset B$, hence $7n+13 \in A$, but $7n+13 = n+(n+3)+(5n+10)$.

In the case XV. we shall consider two possibilities of the localisation of the element $2n+1$. We suppose that $2n+1 > n+4$, i.e. $n > 4$. The cases for $n=1$, $n=2$ and $n=3$ will be investigated separately.

XV. a/ If $\{n, n+2, n+4\} \subset A$, $\{2n+1, n+1, n+3\} \subset B$, then $\{2n+1, n+1, n+3, 3n+6\} \subset B$, hence $\{6n+10, \dots, n, n+2, n+4\} \subset A$, it follows $\{2n+1, 4n+4, 4n+8, 3n+4, \dots, n+1, n+3\} \subset B$, then $\{5n+8, 4n+5, n\} \subset A$, hence $10n+13 \in B$, but $10n+13 = (4n+4) + (4n+8) + (2n+1)$.

XV. b/ If $\{n, n+2, n+4, 2n+1\} \subset A$, $\{n+1, n+3\} \subset B$, then $\{3n+6, 4n+3, 4n+5, \dots, n+1, n+3\} \subset B$, it follows $\{5n+10, 6n+7, 6n+9, \dots, n, n+2, n+4, 2n+1\} \subset A$, then $\{3n+2, 3n+4, 3n+6\} \subset B$, hence $9n+12 \in A$, but $9n+12 = (6n+9) + (2n+1) + (n+2)$.

For $n=1$: If $\{1, 3, 5\} \subset A$, $\{2, 4\} \subset B$, then $\{9, 2, 4\} \subset B$, it follows $\{15, 1, 3\} \subset A$, then $\{11, 2, 4\} \subset B$, hence $\{17, 1, 3\} \subset A$, then $\{13, 2, 4\} \subset B$, hence $19 \in A$, but $19 = 15 + 1 + 3$.

For $n=2$: If $\{2, 4, 6\} \subset A$, $\{3, 5\} \subset B$, then $\{12, 3, 5\} \subset B$, it follows $\{20, 2, 4\} \subset A$, then $\{14, 3, 5\} \subset B$, hence $\{22, 2, 4\} \subset A$, then $\{16, 3, 5\} \subset B$, hence $\{24, 2, 4\} \subset A$, it follows $\{18, 3, 5\} \subset B$, hence $26 \in A$, but $26 = 20 + 2 + 4$.

For $n=3$: If $\{3, 5, 7\} \subset A$, $\{4, 6\} \subset B$, then $\{15, 4, 6\} \subset B$, it follows $\{25, 3, 5\} \subset A$, hence $\{17, 4, 6\} \subset B$, then $\{27, 3, 5\} \subset A$, hence $\{19, 4, 6\} \subset B$, it follows $\{29, 3, 5\} \subset A$, hence $\{21, 4, 6\} \subset B$, then $\{31, 3, 5\} \subset A$, it follows $\{23, 4, 6\} \subset B$, hence $33 \in A$, but $33 = 25 + 3 + 5$.

In the case XVI. we will investigate two possibilities of the localisation of the element $2n$. We suppose that $2n > n+4$, i.e. $n > 5$. The cases for $n=1, n=2, n=3$ and $n=4$ will be considered separately.

XVI. a/ If $\{n, n+1, n+2, n+3, n+4, 2n\} \subset A$, then $\{3n+3, 3n+4, 3n+5, \dots, 4n+3, 4n+7\} \subset B$, it follows $\{9n+12, n, n+1\} \subset A$, hence $11n+13 \in B$, but $11n+13 = (3n+3) + (4n+3) + (4n+7)$.

XVI. b/ If $\{n, n+1, n+2, n+3, n+4\} \subset A$, $2n \in B$, then $\{3n+3, 3n+4, 3n+5, 3n+7, 2n\} \subset B$, it follows $\{9n+12, n, n+1, 8n+7\} \subset A$, then $\{6n+6, 3n+7, 2n\} \subset B$, hence $11n+13 \in A$, but $11n+13 = n + (n+1) + (9n+12)$.

For $n=1$: If $\{1, 2, 3, 4, 5\} \subset A$, then $\{6, 7, 8, \dots, 11\} \subset B$, it follows $\{21, 1, 2\} \subset A$, hence $24 \in B$, but $24 = 6 + 7 + 11$.

For $n=2$: If $\{2, 3, 4, 5, 6\} \subset A$, then $\{9, 10, 11, \dots, 14\} \subset B$, it follows $\{30, 2, 3\} \subset A$, hence $35 \in B$, but $35 = 10 + 11 + 14$.

For $n=3$: If $\{3, 4, 5, 6, 7\} \subset A$, then $\{12, 13, 14, \dots, 16, 17, 18\} \subset B$, hence $\{39, 3, 4\} \subset A$, then $46 \in B$, but $46 = 12 + 16 + 18$.

For n=4: If $\{4,5,6,7,8\} \subset A$, then $\{15,16,17,18,19,20,21\} \subset$ $\subset B$, it follows $\{48,4,5\} \subset A$, hence $57 \in B$, but $57 = 18 + 19 +$ $+ 20$.

By inspecting all 16 possibilities quoted above we see that $f_3(n) \leqslant 11n+12$. Now to finish the proof it suffices to use Theorem 1,1.

2.

Further we shall deal with the existence of k-thin sets for various k. It is easy to see that there is an infinite set of natural numbers, which is k-thin for every $k > 1$ as is shown by the following

<u>Example 2,1</u>. Let $M = \{1, 2, 2^2, \ldots, 2^k, \ldots\}$. The set M is k-thin for each $k > 1$. This follows from the fact, that the number $2^{\alpha_1} + 2^{\alpha_2} + \ldots + 2^{\alpha_k}$, is not a power of the number 2 for any $k > 1$, if $0 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_k$. The next theorem gives a generalization of Example 2,1.

<u>Theorem 2, 1</u>. Let A be a non-empty set of natural numbers. Then there exists an infinite set B of natural numbers, which is not k-thin for any $k \in A$, and is k-thin for each $k \notin A$, $(k > 1)$.

In the proof of Theorem 2,1 we shall use the following lemma:

<u>Lemma 2,1</u>. Let m, n be natural numbers, $n > 1$, $k > 1$. Then the set $B_n^m = \{2^m, 2^{m+1}, \ldots, 2^{m+n-1}, 2^m(2^n-1)\}$ is not n-this, but is k-thin for every $k \neq n$.

The proof of the lemma is easy so we omit it.

<u>Proof of Theorem 2,1</u>. Let us suppose, that A is an infinite set. Put $A = \{1 < k_1 < k_2 < \ldots\}$. We construct by induction the sequence $\{A_n\}_{n=1}^{\infty}$ in the following way: Put $A_1 = B_{k_1}^0$. Then the set A_1 is not k_1-thin, but is k-thin for each $k \neq k_1$. Let us suppose now that it has been constructed (a finite) set A_n, which is not k-thin for $k = k_1, k_2, \ldots, k_n$ and is k-thin for any other $k > 1$. Let σ_n be the sum of all numbers of the set A_n and let m be such natural number, that

$$(2) \qquad \sigma_n < 2^m$$

is true.

Let us put $A_{n+1} = A_n \cup B^m_{k_{n+1}}$. It is clear from the construction of the set A_{n+1}, that A_{n+1} is not k-thin for $k=k_1, k_2, \dots k_n, k_{n+1}$.

Now we will show by contradiction that the set A_{n+1} is k-thin for any other $k > 1$. Assume that there exist some $k_o \neq k_i$ $(i=1,2,\dots,n+1)$ and numbers $a_1, a_2, \dots, a_{k_o} \in A_{n+1}$ such that $a_1 + a_2 + \dots + a_k = a \in A_{n+1}$. Denote the set of those a_j which belong to A_n by the symbol C_1 and the set of those a_j which belong to $A_{n+1} - A_n = B^m_{k_{n+1}}$ by the symbol C_2. Both sets C_1 and C_2 are non-empty. If . fact, if $C_2 = \emptyset$, then $a < \sigma_n < 2^m$ and the set A_n. Let us put

$$c_1 = \sum_{d \in C_1} d, \quad c_2 = \sum_{d \in C_2} d$$

Any member of the set $B^m_{k_{n+1}}$ is divisible by 2^m, thus also the numbers a, c_2 are divisible by 2^m, and so the number $c_1 = a - c_2$ is divisible by 2^m. But $c_1 \leqslant \sigma_n < 2^m$ and it is in contradiction with $C_1 \neq \emptyset$.

So we have constructed by induction a sequence of sets $A_1 \subset A_2 \subset \subset \dots \subset A_n \subset \dots$. Let us put $B = \bigcup_{n=1}^{\infty} A_n$. The set B is clearly infinite and is not k-thin for $k=k_1, k_2, \dots$. On the other hand B is k-thin for any other $k > 1$. In fact, if there exist some $k_o \geqslant 2$ and numbers $b_1, b_2, \dots, b_{k_o} \in B$ so that $b_1 + \dots + b_{k_o} = b \in B$, then there exists an index n such that $b_1, \dots, b_{k_o} \in A_n$ and with respect to (2) b belongs also to A_n. This means, that b_o is one of the numbers $k_1, k_2, \dots, k_n$, and that is not possible.

We have proved the statement of Theorem 2,1 for the case when A is infinite. If a set A is finite and contains elements $s_1 < \dots < < s_t$, in the given proof we may replace the sequence $k_1, k_2, \dots, k_n, \dots$ by the sequence $s_1, s_2, \dots, s_{t-1}, s_t, s_t, \dots$. By the same procedure we construct an infinite set B with the required properties. The proof of the theorem 2,1 is finished.

In the next part of the paper we shall show some relationships between the k-thinness of the set and its asymptotic density.

Let A be a set of natural numbers. Let us put $A(n) = \sum_{a \leq n, a \in A} 1$. Further let

$$\delta_1(A) = \liminf_{n \to \infty} \frac{A(n)}{n}, \quad \delta_2(A) = \limsup_{n \to \infty} \frac{A(n)}{n}.$$

The numbers $\delta_1(A)$, $\delta_2(A)$ are called the lower and the upper asymptotic density of the set A, respectively. If there exists $\delta(A) = \lim_{n\to\infty} \frac{A(n)}{n}$, then $\delta(A)$ is called the asymptotic density of the set A. If a set B has the zero asymptotic density, it does not follow from it, that the set B is k-thin for some $k \geq 2$: The following example shows that there is a set with the zero asymptotic density, which is not k-thin for any k.

<u>Example 2.2.</u> Let us put for any natural number $k > 1$ $A_k = \{2^{2^k} + 1, 2^{2^k} + 2, \dots, 2^{2^k} + k, \sum_{i=1}^{k} (2^{2^k} + i)\}$, and let $B = \bigcup_{k=2}^{\infty} A_k$. It is clear that B is not k-thin for any $k \geq 2$. Let n be a natural number. Let us choose k_o in such a way that

$$2^{2^{k_o}} \leqslant n < 2^{2^{k_o+1}}$$

Let us estimate $\frac{B(n)}{n}$. Let $C = \{x \in B;\ x \leqslant 2^{2^{k_o+1}}\}$. It is easy to see, that $C = \bigcup_{i=2}^{k_o} A_i$; thus the number of the elements of the set C is

$$\sum_{i=2}^{k_o} (i+1) = \frac{(k_o-1)(k_o+4)}{2}$$

and this is not less than B(n). Therefore

$$\frac{B(n)}{n} \quad \frac{B(2^{2^{k_o+1}})}{2^{2^{k_o}}} = \frac{\frac{(k_o-1)(k_o-4)}{2}}{2^{2^{k_o}}} = d_{k_o} .$$

It is clear that $\lim_{k\to\infty} d_{k_o} = 0$, hence $\delta(B) = 0$.

In the next theorem we will show that the k-thinness under some assumptions implies the zero asymptotic density of the set (see [2]).

L e m m a 2.2. Let $k_1 < k_2 < \dots$ be a sequence of natural numbers and let $A = \{a_1, a_2, \dots\}$ be a set of natural numbers. Let $A_0 = A$, and let $A_i (i=1,2,\dots)$ denote the set of all numbers of the form $a_1 + a_2 + \dots + a_{k_i} + a_s$, where s is running through all numbers $\geq k_i + 1$. If the sets $A_i, i=1,2,\dots$ are mutually disjoint, then $\delta(A) = 0$.

P r o o f. Let r be an arbitrary natural number and let $x \geq r$. Let us denote by $A_i(x)$ the number of all elements of the set A_i, which are not greater than x. Then it is obvious (with respect to the construction of the set A_i) that

$$(3) \qquad x \geq \sum_{i=0}^{r} A_i(x) \geq (r+1) \, . \, A_r(x)$$

holds. $A_r(x)$ denotes the number of those indices $s \geq k_r + 1$, for which $a_1 + a_2 + \dots + a_s \leq x$. Hence

$$(4) \qquad A_r(x) = A\left(x - \sum_{j=1}^{k_r} a_j\right) - k_r \geq A(x) - \sum_{j=1}^{k_r} a_j - k_r .$$

From (3) and (4) we obtain

$$(r+1) \left[A(x) - \sum_{j=1}^{k_r} a_j - k_r \right] \leq x .$$

The last inequality implies

$$\delta_2(A) = \limsup_{x \to \infty} \frac{A(x)}{x} \leq \frac{1}{r-1} .$$

Since the last statement holds for any natural r, we get $\delta_2(A) = 0$, thus $\delta(A) = 0$.

R e m a r k. The foregoing lemma generalizes the result of the paper [2] according to which $\delta(A) = 0$ if A is k-thin for each $k = 2, 3, \dots$. It suffices to choose for $\{k_i\}$ the set of all natural numbers greater than 1.

T h e o r e m 2,2. Let $k_1 < k_2 < \dots$ be a sequence of natural numbers greater than 1. Suppose that the set $A = \{a_1, a_2, \dots\}$ is a k-thin for any $k = k_j - k_i + 1$, where i, j are arbitrary natural numbers, $1 \leqslant i < j$. Then $\delta(A) = 0$.

P r o o f. This Theorem will be proved by using the lemma 2,2. Let the sets A_i, $i=0,1,2,\dots$, have the same meaning as in Lemma 2,2. We will show by contradiction that they are pairwise disjoint. Let $i < j$, and let $A_i \cap A_j \neq \emptyset$. Then there exists an $a \in A_i \cap A_j$. From the definition of the sets A_i and A_j we have $a_1 + a_2 + \dots + a_{k_i} + a_s = a = a_1 + a_2 + \dots + a_{k_j} + a_{s'}$, $s \geqslant k_i + 1$, $s' \geqslant k_j + 1$. Hence $a_s = a_{k_i+1} + \dots + a_{k_j} + a_{s'}$. Since the number of members on the right side of the last equation is $k_j - k_i + 1$, we have a contradiction with $k_j - k_i + 1$-thinness of the set A. The proof of the theorem 2,2 is finished.

C o r o l l a r y 2,1. Let p be a natural number and let a set A be k-thin for any $k = jp + 1$, where $j = 1, 2, \dots$. Then $\delta(A) = 0$.

P r o o f. It suffices to verify that for the sequence $\{k_j\}_{j=1}^{\infty}$, $k_j = jp + 1$, $(j = 1, 2, \dots)$ conditions of Theorem 2,2 are fulfilled. In fact, if $1 \leqslant i < j$, then

$$k_j - k_i + 1 = (j-i)p + 1$$

is again an element of the sequence $\{k_j\}_{j=1}^{\infty}$.

3.

In this part we shall deal with the multiplicative version of k-thinness.

D e f i n i t i o n 3,1. Let k be a natural number, $k > 1$. The set M of natural numbers is called a multiplicatively k-thin set, if no product of arbitrary k distinct numbers from M belongs to M.

Similarly, as at the beginning of this paper, let $F_k(n)$ be the greatest natural $m > n$ with the following property: The set $\{n, n+1, \dots, m\}$ can be decomposed into two disjoint multiplicative k-thin sets.

We shall give a certain upper estimation for $F_2(n)$.

Theorem 3.1. For each natural n we have

$$F_2(n) \leqslant n^2(n+1)^2(n+2) - 1.$$

Remark. For n=1 we have $F_2(1) = 5$.

Proof of Theorem 3.1. We shall show that the set $\{n, n+1, \dots, n^2(n+1)^2(n+2)\}$ of natural numbers cannot be decomposed into two disjoint multiplicatively 2-thin subsets A and B. We can arbitrarily localise the elements n, n+1, n+2 and n+3 into sets A and B. By the localisation of the mentioned elements we shall have the next eight possibilities:

I. Let $\{n, n+1, n+2, n+3\} \subset A$. Then $\{n(n+1), n(n+2)\} \subset B$, and so $n^2(n+1)(n+2) \in A$. If $n(n+1)(n+2) \in A$, then, since $n \in A$, we should get, that $n^2(n+1)(n+2) \in B$. Hence $n(n+1)(n+2) \in B$. Since $n(n+1) \in B$ we have

$$(5) \qquad n^2(n+1)^2(n+2) \in A.$$

Further from $n+1 \in A$ and $n^2(n+1)(n+2) \in A$ it follows $n^2(n+1)^2(n+2) \in B$. It is in contradiction with (5).

The description of this situation will be shortened as follows.

II. If $\{n, n+1, n+2\} \subset A$, $n+3 \in B$, then $\{n(n+1), (n+3)\} \subset B$, hence $\{n(n+1)(n+3), n+1\} \subset A$. Since $\{n(n+3), n(n+1)\} \subset B$, it follows

$$(6) \qquad n^2(n+1)(n+3) \in A.$$

But the number $n^2(n+1)(n+3)$ is a product of elements $n \in A$ and $n(n+1)(n+3) \in A$, thus it belongs to B. It is in contradiction with (6).

III. If $\{n, n+1, n+3\} \subset A$, $n+2 \in B$, then $\{n(n+1), (n+2)\} \subset B$, hence $\{n(n+1)(n+2), n+1\} \subset A$. Since $\{n(n+2), n(n+1)\} \subset B$, it follows

$$(7) \qquad n^2(n+1)(n+2) \in A.$$

But the number $n^2(n+1)(n+2)$ is a product of elements $n \in A$ and $n(n+1)(n+2) \in A$, thus it belongs to B. It is in contradiction with (7).

IV. If $\{n, n+2, n+3\} \subset A$, $n+1 \in B$, then $\{n(n+2), n+1\} \subset B$, hence $\{n(n+1)(n+2), n+2\} \subset A$. Since $\{n(n+1), n(n+2)\} \subset B$, it follows

$$(8) \qquad n^2(n+1)(n+2) \in A.$$

But the number $n^2(n+1)(n+2)$ is a product of elements $n \in A$ and $n(n+1)(n+2) \in A$, thus it belongs to B. It is in contradiction with (8).

V. If $\{n+1, n+2, n+3\} \subset A$, $n \in B$, then $\{(n+1)(n+2), n\} \subset B$, hence $\{n(n+1)(n+2), n+2\} \subset A$. Since $\{n(n+1), (n+1)(n+2)\} \subset B$, it follows

$$(9) \qquad n(n+1)^2 (n+2) \in A.$$

But the number $n(n+1)^2(n+2)$ is a product of elements $n+1 \in A$, and $n(n+1)(n+2) \in A$, thus it belongs to B. It is in contradiction with (9).

VI. If $\{n, n+1\} \subset A$, $\{n+2, n+3\} \subset B$, then $\{n(n+1), n+3\} \subset B$, hence $\{n(n+1)(n+3), n+1\} \subset A$. Since $\{n(n+3), n(n+1)\} \subset B$, it follows

$$(10) \qquad n^2(n+1)(n+3) \in A.$$

But the number $n^2(n+1)(n+3)$ is a product of elements $n \in A$ and $n(n+1)(n+3) \in A$, thus it belongs to B. It is in contradiction with (10).

VII. If $\{n, n+2\} \subset A$, $\{n+1, n+3\} \subset B$, then $\{n(n+2), n+1\} \subset$ $\subset B$, hence $\{n(n+1)(n+2), n+2\} \subset A$. Since $\{n(n+1), n(n+2)\} \subset$ $\subset B$, it follows

$$n^2(n+1)(n+2) \in A. \tag{11}$$

But the number $n^2(n+1)(n+2)$ is a product of elements $n \in A$ and $n(n+1)(n+2) \in A$, thus it belongs to B. It is in contradiction with (11).

VIII. If $\{n, n+3\} \subset A$, $\{n+1, n+2\} \subset B$, then $\{n(n+3), n+1\} \subset B$, hence $\{n(n+1)(n+3), n+3\} \subset A$. Since $\{n(n+3), n(n+1)\} \subset B$, it follows

$$n^2(n+1)(n+3) \in A. \tag{12}$$

But the number $n^2(n+1)(n+3)$ is a product of elements $n \in A$ and $n(n+1)(n+3) \in A$, thus it belongs to B. It is in contradiction with (12).

As it follows from the above stated,

$$F_2(n) < n^2(n+1)^2(n+2),$$

The proof of Theorem 3,1 is finished.

It is easy to prove an analogous result to the theorem 2,1.

At first we will prove the following lemma.

Lemma 3,1. Let $s > 1$ be a natural number, p a prime. Then the set

$$M_s^p = \{p, p^2, p^{2^2}, p^{2^3}, \ldots, p^{2^{s-1}}, p^{2^s-1}\}$$

is not multiplicatively s-thin and it is multiplicatively k-thin for any $k \neq s$, $(k > 1)$.

The proof of the lemma may be omitted.

Theorem 3,2. Let A be a set of natural numbers greater than 1. Then there exists an infinite set M of natural numbers, which is not multiplicatively k-thin for any $k \in A$, and is multiplicatively k-thin for any other $k > 1$.

Proof. Let $A = \{k_1 \leqslant k_2 \leqslant \dots\}$. We can suppose, that the set A is infinite. Let $p_1, p_2, \dots$ be increasing sequence of primes. Let us put

$$B = \bigcup_{n=1}^{\infty} M_{k_n}^{p_n}$$

It is clear, that the set B has all the required properties.

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Súhrn

O k-riedkych aritmetických množinách

EVA NYULASSYOVÁ

V práci autorka zavádza a skúma pojmy aditívne ako aj multiplikatívne k-riedkej množiny prirodzených čísel. Pomocou aditívnej k-riedkosti uvádza postačujúcu podmienku na to, aby množina prirodzených čísel mala nulovú asymptotickú hustotu.

Author's address:
Katedra matematiky Pedagogickej fakulty UK
Trnava

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Содержание

<u>О к-редких арифметических множествах</u>

ЭВА НЮЛАШИОВА

В работе вводятся и рассматриваются понятия как аддитивно так и мултипликативно к-редких множеств. При помощи аддитивной к-редкости дается достаточное условие для того, чтобы множество натуральных чисел имело нолевую асимптотическую плотность.

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SUR LA STRUCTURE ALGÉBRIQUE DE LA THÉORIE DES TRANSFORMATIONS DIFFÉRENTIELLES LINÉAIRES DU DEUXIÈME ORDRE

O. BORŮVKA, Brno

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I. Préliminaires

1. Dans la présente conférence je me propose de donner un aperçu de la structure algébrique de la théorie des transformations des équations différentielles ordinaires linéaires et homogènes du deuxième ordre dans le champ réel et de développer un modèle algébrique abstrait de cette théorie. Nous considérons les équations en question dans la forme jacobienne

$$(q) \qquad y'' = q(t)y$$

et nous supposons que le coefficient q, appelé par occasion le porteur de l'équation (q), est une fonction continue dans un intervalle considéré, $j : q \in C^0_j$. Qu'il me soit permis d'indiquer et de souligner dès le commencement que, pour de bonnes raisons dont je parlerai plus tard, nous nous bornons aux équations (q) définies et oscillatoires dans l'intervalle $j = (-\infty, \infty)$. On appelle une équation (q) oscillatoire dans l'intervalle j si toute intégrale de (q) admet infiniment beaucoup de zéros vers les deux extrémités de j. Un prototype de ces équations oscillatoires est évidemment l'équation (-1) $y'' = -y$ dans l'intervalle $j = (-\infty, \infty)$.

A la base de notre théorie figurent deux notions importantes, à savoir celles des (premières) phases et des dispersions centrales (de la première espèce). Nous allons d'abord rappeler leurs définitions et quelques propriétés en tant qu'elles nous seront utiles dans la suite.

Pour abréger des notations ultérieures introduissons dès maintenant les systèmes denombrables suivants

$$\mathfrak{Z} = \{\dots t-2\pi, t-\pi, t, t+\pi, t+2\pi, \dots\}, \quad \mathfrak{Z}_0 = \{\dots, t-2\pi, t, t+2\pi, \dots\}$$

et désignons, pour $n = 0, \pm 1, \pm 2, \dots, \quad t \in j$:

$$c_n(t) = t + n\pi .$$

On a, évidemment, $\mathfrak{Z} = \mathfrak{Z}_0 \vee Z_1; \quad \mathfrak{Z}_0 = \{c_{2n}\}, \quad Z_1 = \{c_{2n+1}\}.$

2. Phases. Soit (q) une équation jacobienne et $B = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ une base de (q), les composantes u, v étant, par conséquent, des intégrales linéairement indépendantes de (q). Le wronskien w ($= uv' - u'v$) est alors une constante non nulle. On appelle phase de B toute fonction α qui est continue dans j et qui satisfait dans cet intervalle j, en dehors des zéros de v, à l'équation $\operatorname{tg}\alpha(t) = u(t) : v(t)$. On démontre que les phases de B forment un système dénombrable (α) qui est précisément $(\alpha) = \mathfrak{Z}\alpha$; α designe une phase quelconque de B et on applique la notation $\mathfrak{Z}\alpha = \{\dots$ $\dots, \alpha - 2\pi, \alpha - \pi, \alpha, \alpha + \pi, \alpha + 2\pi, \dots\}$.

Il est bien connu que chaque phase α de B jouit dans j des propriétés suivantes:

$$(1) \quad 1.\ \alpha \in C_j^3; \qquad 2.\ \alpha' \neq 0; \qquad 3.\ \lim_{t \to \sigma\infty} \alpha(t) = \sigma\infty \operatorname{sgn} \alpha' \quad (\sigma = \pm 1)$$

dont la 3-ième caractérise les équations (q) oscillatoires. On a en outre la formule

$$(2) \qquad q(t) = -\{\alpha, t\} - \alpha'^2(t) \qquad (t \in j)$$

$\{\alpha, t\}$ étant la dérivée schwarzienne de α au point t.

Introduissons, pour simplifier les notations, la base de l'équation (-1) : $I(t) = \begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$.

A l'aide de toute phase $\alpha \in (\alpha)$ la base B s' exprime en coordonnées polaires de la façon suivante

$$(3) \qquad B = \sigma\sqrt{|w|}\,\frac{1}{\sqrt{|\alpha'|}}\, I\alpha \qquad (\sigma = \pm 1)$$

La phase α s'appelle propre ou bien impropre suivant que $\sigma = 1$ ou bien $\sigma = -1$. Les phases propres, par exemple, forment un sous-système de (α), $(\alpha)_0$; α étant une phase propre on a : $(\alpha)_0 = \mathfrak{Z}_0\alpha$.

A coté de la notion des phases d'une base de (q) on introduit la notion des phases de l'équation (q) elle-même: Sous une phase de l'équation (q) on entend une phase d'une base quelconque de (q).

Pour aller plus loin appelons fonction-phase toute fonction α définie dans l'intervalle j et jouissante dans cet intervalle des propriétés (1). Par la fonction-phase α se trouve bien déterminnée, d'après (2), une équation oscillatoire (q). On vérifie que α est une phase de (q). Les bases de (q) admettant la phase α forment un systeme des bases proportionnelles, $\bar{B} = \{\mu\, I\alpha : \sqrt{|\alpha'|}\}$, const $= \mu \neq 0$.

La fonction-phase α s'appelle élementaire si elle vérifie, dans j, la relation suivante

$$(4) \qquad \alpha(t+\pi) = \alpha(t) + \pi . \operatorname{sgn} \alpha'.$$

Cette formule entraîne la relation plus générale

$$(5) \qquad \alpha c_n(t) = c_{n.\operatorname{sgn}\alpha'}\alpha(t) \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

Subsistent, par conséquent, les formules

(6) $$\mathfrak{z}\alpha = \alpha \mathfrak{z} \quad ; \quad \mathfrak{z}_0\alpha = \alpha \mathfrak{z}_0$$

3. <u>Dispersions centrales</u>. On entend sous la dispersion centrale d'indice n $(=0, \pm 1, \pm 2, \ldots)$ de l'équation (q) la fonction φ_n définie dans j de manière que, pour chaque nombre $t \in j$, la valeur $\varphi_n(t)$ est le n-ième nombre conjugué avec t et situé à droite ou à gauche de t suivant que $n > 0$ ou $n < 0$. En particulier, la fonction φ_1 s'appelle la dispersion fondamentale. Quant à la fonction φ_0, on pose pour $t \in j$: $\varphi_0(t) = t$.

On sait que chaque fonction φ_n est une fonction-phase qui est constamment croissante et satisfait pour $t \in j$ à l'inégalité $\varphi_{n-1}(t) < \varphi_n(t)$.

Rappelons finalement qu'on entend sous la fonction-distance de l'équation (q) la fonction d définie dans j par la formule

$$d(t) = \varphi_1(t) - t.$$

Au sujet de cette fonction d, dont la signification est évidente, contentons-nous de mentionner le résultat suivant: Pour que la fonction-distance d soit périodique avec π, il faut et il suffit que la dispersion fondamentale φ_1 soit élémentaire.

II. Fondaments de la théorie des transformations des équations jacobiennes

1. On sait depuis une quatre-vingtaine d'années que toute transformation biunivoque du plan (T, Y) sur le plan (t, y), telle que $T = X(t)$, $y = f(t, Y)$ et qui transforme les intégrales de chaque équation jacobienne (Q) dans celles d'une autre, (q), a la forme suivante

(X) $$T = X(t), \quad y = \frac{c}{\sqrt{|X'(t)|}}\, Y \qquad (0 \neq c = \text{const.});$$

on suppose, sans parler de certaines propriétés d'inversibilité et de différentiabilité de la fonction f, que X est une fonction-phase.

Dans la suite nous prenons, pour simplifier les formules, $c=1$

2. Transformations des bases. Soient (Q) une équation jacobienne, $B_A(T)$ une base de (Q) admettant la phase propre A et W le wronskien correspondant. Subsiste alors une formule telle que (3) écrite en lettres majuscules ($\sigma=1$). Appliquons la transformation (X) aux composantes de B_A. Nous avons

$$(7) \qquad B_A(T) \xrightarrow{X} \sqrt{|W|}\, \frac{1}{\sqrt{|AX(t)'|}}\, I A X(t) .$$

Or, la fonction $\alpha(t) = AX(t)$ étant une fonction-phase, elle détermine, d'après (2), une équation oscillatoire (q), qui admet la phase α. En faisant le calcul on trouve le porteur correspondant q:

$$(Qq) \qquad -\{X,t\} + Q(X)X'^2(t) = q(t).$$

On voit que le second membre de (7) est une base de (q), B_α, admettant la phase α. Parmi les bases proportionnelles de (q) qui admettent la phase α, la base B_α se trouve bien déterminée par ceci que, son wronskien a la valeur $W.\operatorname{sgn} X'$ et la fonction α est une phase propre de B_α. Remarquons que la base transformée B_α ne dépend point du choix de la phase $A \in (A)_0$.

L'équation (Qq) s'appelle l'équation de Kummer. Les fonctions Q, q étant données, les solutions X de cette équation sont précisément les fonctions-phases qui réalisent les transformations (X) des bases de (Q) dans celles de (q). Ces solutions s'appelent les dispersions générales des équations (Q), (q) et leur ensemble est nommé le complexe de Kummer, $K_{Q,q}$. Dans le cas $Q=q$ on parle des dispersions et du complexe de Kummer de l'équation (q), K_q.

Envisageons la transformation considérée en tant qu'une opération faisant associer à toute base B_A de (Q) et à toute fonction-phase X, la base B_α de (q); désignons cette opération par o de sorte que $B_\alpha = B_A \circ X$. On trouve illustré le processus menant des données B_A, X au résultat $B_A \circ X$ de l'opération en question dans le schème a la p. ; il y est marqué par les flèches. On a désigné par L_Q, L_q les espaces vectoriels linéaires formés des intégrales des équations (Q), (q) et par $\mathfrak{G}$ l'ensemble de toutes les fonctions-phases.

3. Transformations des intégrales. Il reste à dire quelques mots au sujet des transformations (X) de différentes intégrales de (Q) dans celles de (q). Y(T) étant une intégrale de (Q) on a, évidemment,

$$Y(T) = (\gamma_1, \gamma_2)\, B_{\mathcal{A}}\,(T), \tag{8}$$

γ_1, γ_2 (= const.) étant les coordonnées de Y par rapport a la base $B_{\mathcal{A}}$. L'intégrale transformée, y(t), est alors

$$y(t) = (\gamma_1, \gamma_2)\,(B_{\mathcal{A}} \circ X(t)). \tag{9}$$

On démontre que l'intégrale y de (q) ne dépend pas du choix de la base $B_{\mathcal{A}}$.

III. L'Algèbre des phases

1. Le groupe des phases, $\mathfrak{G}$. Revenons à la notation ci-dessus et désignons par $\mathfrak{G}$ l'ensemble formé par les fonctions-phases. Il est clair que les fonctions inverses et de même les fonctions composées des fonctions-phases sont encore des fonctions-phases. Par conséquent, si l'on introduit dans $\mathfrak{G}$ l'opération binaire, dite la multiplication et consistant en composition des fonctions, $\mathfrak{G}$ devient un groupe. Nous appelons $\mathfrak{G}$ le groupe des phases et ses éléments, occasionnellement, les phases. L'élement neutre de $\mathfrak{G}$ est, évidemment, la fonction $\varphi_0(t) = t$.

Remarquons à cette ocasion que, notre supposition faite dès le commencement, à savoir qu'il s'agisse des équations (q) définies et oscillatoires dans l'intervalle $j = (-\infty, \infty)$, est tout-à-fait esentielle pour nos raisonnements. En effet, les équations (q) étant oscillatoires, les valeurs des élements du groupe $\mathfrak{G}$ recouvrent l'intervalle j qui est précisément le domaine de définition de ces élements. Cela entraîne que le groupe $\mathfrak{G}$ est bien déterminé.

Les fonctions-phases croissantes forment un sous-groupe de $\mathcal{G}$, $\mathcal{G}_0$, qui est invariant dans $\mathcal{G}$ et dont l'indice égale à 2. L'autre élément du groupe-facteur $\mathcal{G}/\mathcal{G}_0$, G_1, consiste en fonctions-phases décroissantes.

2. Le sous-groupe fondamental $\mathfrak{F}$. Soit $\mathfrak{F}$ l'ensemble des phases de l'équation $(-1) : y'' = -y$. On démontre, en se servant de (2), que l'ensemble $\mathfrak{F}$ consiste précisément en solutions de l'équation de Kummer $(-1,-1)$ et qu'il forme un sous-groupe de $\mathcal{G}$: $\mathfrak{F} \subset \mathcal{G}$. Nous appelons $\mathfrak{F}$ le sous-groupe fondamental de $\mathcal{G}$, plus brièvement: le groupe fondamental.

Les systèmes $\mathfrak{Z}$ et $\mathfrak{Z}_0$ sont, évidemment, des sous-groupes monogènes du groupe $\mathfrak{F}_0$, aux générateurs c_1, c_{-1} et c_2, c_{-2} respectivement. On pose $\mathfrak{F}_0 = \mathfrak{F} \cap \mathcal{G}_0$, $E_1 = \mathfrak{F} \cap G_1$.

On démontre que toute phase $\mathcal{E} \in \mathfrak{F}$ est élémentaire. Subsiste, par conséquent, une relation telle que (5) $(\alpha \equiv \mathcal{E})$; donc, la phase $\mathcal{E}$ est échangeable ou bien inversement échangeable avec chaque élement $c_n \in \mathfrak{Z}$, suivant que $\mathcal{E} \in \mathfrak{F}_0$ ou bien $\mathcal{E} \in E_1$. Il en résulte que le groupe $\mathfrak{Z}$ fait partie du centre du groupe $\mathfrak{F}_0$ et on démontre que $\mathfrak{Z}$ est le centre même de $\mathfrak{F}_0$. D'apres (6) les groupes $\mathfrak{Z}$, $\mathfrak{Z}_0$ sont invariants dans $\mathfrak{F}$. Les groupes-facteurs $\mathfrak{F}/\mathfrak{Z}$, $\mathfrak{F}/\mathfrak{Z}_0$ vérifient, évidemment, la relation $\mathfrak{F}/\mathfrak{Z} \geqslant \mathfrak{F}/\mathfrak{Z}_0$ et on démontre que, tout élément de $\mathfrak{F}/\mathfrak{Z}$ est la réunion exactement de deux élements de $\mathfrak{F}/\mathfrak{Z}_0$.

Une autre propriété importante de groupe $\mathfrak{F}$ consiste en ceci, que ce groupe $\mathfrak{F}$ coincide avec le complexe de Kummer de l'équation (-1), K_{-1}. Donc, $\mathcal{E} \in \mathfrak{F}$ étant un élément arbitraire, la transformation $(\mathcal{E})$ fait passer la base $I(t)$ de l'équation (-1) dans une base de la même équation (-1). On a, par conséquent,

$$\frac{1}{\sqrt{|\mathcal{E}'(t)|}} I\mathcal{E}(t) = \mathcal{H}_{\mathcal{E}} I(t)$$

$\mathcal{H}_{\mathcal{E}}$ étant une matrice régulière 2x2 sur le corps des nombres réels, R. On démontre que la correspondance $\mathcal{H} : \mathcal{E} \to \mathcal{H}_{\mathcal{E}}$ est un homomorphisme du groupe $\mathfrak{F}$ sur le groupe $\mathfrak{U}$ formé des matrices unimodulaires 2x2 sur R.

Indiquons finalement la suivante propriété du groupe $\mathfrak{F}$: Le normalisateur de $\mathfrak{F}$ dans $\mathcal{G}$, $\mathfrak{N}_{\mathfrak{F}}$, se confond avec $\mathfrak{F}$: $\mathfrak{N}_{\mathfrak{F}} =$

3. <u>La décomposition</u> $\mathfrak{G}/_d\mathfrak{F}$. L'importance du groupe $\mathfrak{F}$ pour la théorie qui nous occupe tient à ceci: Les systèmes des phases de différentes équations (q) sont précisément les éléments de la décomposition du groupe $\mathfrak{G}$ en classes latérales à droite, $\mathfrak{G}/_d\mathfrak{F}$.

A tout élément $\bar{a} \in \mathfrak{G}/_d\mathfrak{F}$ se trouvent biunivoquement associés:

D'abord, une équation (q) dont les phases forment précisément l'élément $\bar{a}$. Cette équation (q) est bien déterminée par une formule telle que (2), α étant un arbitraire élément de $\bar{a}$;

puis, le complexe de Kummer de l'équation (q), K_q. Ce complexe K_q est bien déterminé par la formule

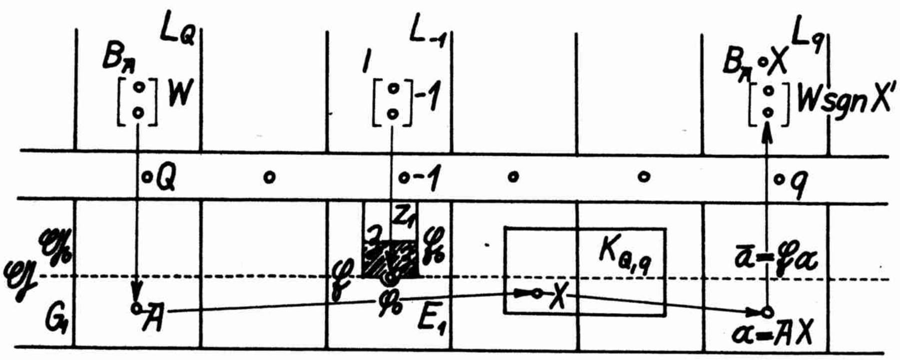
$$K_q = \alpha^{-1}\mathfrak{F}\alpha,$$

α étant encore un arbitraire élément de $\bar{a}$. Par conséquent, K_q est le groupe conjugué avec $\mathfrak{F}$ par rapport aux éléments de $\bar{a}$;

finalement, l'espace vectoriel linéaire formé par les intégrales de l'équation (q), L_q.

Et voici encore une caractérisation algébrique du complexe de Kummer, $K_{Q,q}$: Le complexe $K_{Q,q}$ consiste précisément en fonctions-phases X qui transforment le complexe K_Q sur K_q suivant la formule:

$$K^{-1}K_QX = K_q.$$



4. Remarque. Le temps prévu pour ma conférence me ne permet pas d'insister sur la nature algébrique de certaines notions importantes, d'origine analytique, qui interviennent dans la théorie considérée. Cela concerne surtout les dispersions centrales des équations (q), les fonctions-phases élémentaires, des équations (q), ($\bar{q}$) mutuellement inverses et les équations (q) aux fonctions-distances périodiques. Au sujet des équations mutuellement inverses et celles dont les fonctions-distances sont périodiques on a récemment trouvé des résultats, me paraît-il, bien intéressants. Figurent parmi ces dernières équations les équations (q) aux porteurs q périodiques. Remarquons que, le porteur q d'une équation jacobienne résulte périodique avec π, si et seulement si le groupe K_q contient le centre $\mathfrak{Z}$: $K_q \supset \mathfrak{Z}$. V. à ce sujet mon article sous presse qui va paraître dans le Volume dédié à M. A. Kawaguchi pour son $70^{\text{-ième}}$ anniversaire.

IV. Le modèle algébrique abstrait de la théorie en question

Nous voilà arrivés à la présentation d'un modèle algébrique abstrait de la théorie considérée. L'exposé précédent ayant été conçu de manière à faire apparaître, en grandes lignes, la structure logique de ce modèle, nous sommes en mésure d'aborder la question rapidement et sans aucune espèce de difficultés.

Nous désignons encore par $\mathfrak{U}$ le groupe des matrices unimodulaires 2x2 sur le corps R et nous posons $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Le modèle en question consiste en éléments suivants:

1. Le groupe des phases, $\mathfrak{G}$;
2. les espaces vectoriels linéaires $L_{\bar{a}}$ associés aux élements $\bar{a} \in \mathfrak{G}/_d \mathfrak{f}$;
3. les quasinormes des bases des espaces $L_{\bar{a}}$;
4. les transformations kummeriennes des bases et des éléments des espaces $L_{\bar{a}}$.

1. Le groupe des phases, $\mathfrak{G}$, est un groupe abstrait satisfaisant aux propositions suivantes:

A. $\mathfrak{G}$ contient un sous-groupe d'indice 2, $\mathfrak{G}_0$.

Cela entraîne que le groupe-facteur $\mathfrak{G}/\mathfrak{G}_0$ consiste en deux éléments $\mathfrak{G}_0$, G_1. Pour $a \in \mathfrak{G}$ nous posons sgn $a' = 1$ ou bien sgn $a' = -1$ suivant que $a \in \mathfrak{G}_0$ ou bien $a \in G_1$. Subsiste alors, pour $a, b \in \mathfrak{G}$, la formule: sgn $(ab)'$ = sgn a'. sgn b'.

B. $\mathfrak{G}$ contient un sous-groupe $\mathfrak{F}$, appelé le sous-groupe fondamental, jouissant des propriétés suivantes:

a/ Le centre $\mathfrak{Z}$ de $\mathfrak{F} \cap \mathfrak{G}_0$ est un groupe monogène $\mathfrak{Z} = \{c^n\}$ dont les éléments vérifient pour $e \in \mathfrak{F}$ la relation $ee^n = c^{n.\operatorname{sgn} e'}e$ $(n=0, \pm 1, \pm 2, \ldots;\ c^0 = 1)$.

b/ Il existe un homomorphisme $\mathcal{H}$ de $\mathfrak{F}$ sur $\mathfrak{U}$ tel que (i) $\det \mathcal{H}e = \operatorname{sgn} e'$ pour $e \in \mathfrak{F}$; (ii) $\mathcal{H}c^n = (-1)^n E$, les c^{2n} étant les seuls éléments de $\mathfrak{F}$ qui se représent sur E.

c/ Le normalisateur de $\mathfrak{F}$ dans $\mathfrak{G}$ se confond avec $\mathfrak{F}$: $\mathfrak{N}_{\mathfrak{F}} = \mathfrak{F}$.

D'après a., les groupes $\mathfrak{Z} = \{c^n\}$, $\mathfrak{Z}_0 = \{c^{2n}\}$ sont invariants dans $\mathfrak{F}$. Tout élément de $\mathfrak{F}/\mathfrak{Z}$ est la réunion de deux éléments de $\mathfrak{F}/\mathfrak{Z}_0$. D'après b., le noyau de $\mathcal{H}$ est $\mathfrak{Z}_0$. Donc, d'après le premier théorème d'isomorphisme pour les groupes, il existe un isomorphisme $\bar{\mathcal{H}}$ de $\mathfrak{F}/\mathfrak{Z}_0$ sur $\mathfrak{U}$. Cet isomorphisme est de sorte que, les images de deux éléments $\bar{e}_1, \bar{e}_2 \in \mathfrak{F}/\mathfrak{Z}_0$ dont la réunion est un élément de $\mathfrak{F}/\mathfrak{Z}$ diffèrent exactemant en signe: $\bar{\mathcal{H}}\bar{e}_1 = -\bar{\mathcal{H}}\bar{e}_2$.

2. Les espaces vectoriels linéaires $L\bar{a}$ associée aux élements $\bar{a} \in \mathfrak{G}/_d\mathfrak{F}$. On suppose qu'à tout élément $\bar{a} \in \mathfrak{G}/_d\mathfrak{F}$ se trouve biunivoquement associé un espace vectoriel linéaire à deux dimensions, $L\bar{a}$, dont les bases sont dans certaines relations avec les éléments de $\bar{a}$.

Soit $a \in \bar{a}$ un élément arbitraire. On a alors $\bar{a} = \mathfrak{F}a$. Les groupes $\mathfrak{F}/\mathfrak{Z}$, $\mathfrak{F}/\mathfrak{Z}_0$ induisent les décompositions de $\bar{a}$, $\bar{a}/_d\mathfrak{Z} = (\mathfrak{F}/\mathfrak{Z})\,a$, $\bar{a}/_d\mathfrak{Z}_0 = (\mathfrak{F}/\mathfrak{Z}_0)\,a$. Celles-ci ne dépendent pas du choix de $a \in \bar{a}$. Tout élément de $\bar{a}/_d\mathfrak{Z}$ est la réunion de deux éléments de $\bar{a}/_d\mathfrak{Z}_0$.

Désignons par $\bar{B}$ le système des bases de $L\bar{a}$ provenant de la base B par la multiplication de B par les nombres positifs.

Les relations en question sont les suivantes:

a/ A tout élément $x \in \bar{a}$ correspond un système $\bar{B}_x$ bien déterminé.

b/ $\bar{B}_{ex} = \mathcal{H}e.\bar{B}_x$, pour $x \in \bar{a}$, $e \in \mathfrak{F}$.

c/ $\bar{B}_y = \bar{B}_x$; $x, y \in \bar{a}$, entraîne $y = c^{2n}x$, n étant un entier convenable.

D'après b., c. il existe une correspondance biunivoque entre l'ensemble des systèmes $\bar{B}$ et la décomposition $\bar{a}/_d\mathfrak{Z}_0$: $\bar{B} \to \bar{c}$ entraîne $\bar{B} = \bar{B}_x$ pour $x \in \bar{c}$ $(\in \bar{a}/_d\mathfrak{Z}_0)$. Toute base B de $L\bar{a}$ se trouve contenue, évidemment, dans un système $\bar{B}$ bien déterminé et l'on a $\bar{B} \to \bar{c}_1$, $-\bar{B} \to \bar{c}_2$, $\bar{c}_1 \vee \bar{c}_2 \in \bar{a}/_d\mathfrak{Z}$. Les éléments de l'ensemble $\bar{c}_1$ resp. $\bar{c}_2$ s'appelent les phases propres resp. impropres de la base B.

3. Les quasinormes des bases des espaces $L\bar{a}$. On suppose qu'à toute base B de $L\bar{a}$ se trouve associé un nombre réel non nul, $\|B\|$, appelé la quasinorme (wronskien) de B, de manière que

a/ $\operatorname{sgn} \|B\| = - \operatorname{sgn} a$; pour $B \in \bar{B}_a$.
b/ $\|MB\| = \det M . \|B\|$, pour toute matrice régulière 2x2 sur R.

4. Les transformations kummeriennes des bases et des éléments des espaces $L\bar{a}$.

Une transformation kummerienne des bases fait correspondre à toute base B de $L\bar{a}$ et tout élément $x \in \mathcal{G}$ la base $B \circ x$ de l'espace $L\bar{b}$, d'après la formule

$$B \circ x = \sqrt{\frac{\operatorname{abs} \|B\|}{\operatorname{abs} \|B_b\|}}\ \mathcal{H}e . B_b ;$$

ici la signification de différents symboles est la suivante: $ax \in \bar{b} \in \mathcal{G}/_d \mathcal{f}$, les éléments $b \in \bar{b}$ et $B_b \in \bar{B}_b$ ont éte choisis arbitrairement, et, finalement, $e \in \mathcal{f}$ est la solution de l'équation $ax = eb$.

Une transformation kummerienne des vecteurs fait correspondre à tout vecteur $Y \in L\bar{a}$ et tout élément $x \in \mathcal{G}$ le vecteur $Y \circ x$ tel que

$$Y = (\gamma_1, \gamma_2)\, B, \quad Y \circ x = (\gamma_1, \gamma_2)\,(B \circ x) \quad (\gamma_1, \gamma_2 = \text{const.});$$

B désigne une base de $L\bar{a}$ choisie arbitrairement.

Voilà le modèle algébrique abstrait de la théorie des transformations des équations jacobiennes oscillatoires dans l'intervalle $j = (-\infty, \infty)$. Tous les éléments de la théorie en question, tels que le complexe de Kummer, $K_{Q,q}$, celui K_q, les dispersions centrales, les phases élémentaires, etc., admettent des analogues abstraits bien évidents. Ainsi, par exemple, le complexe de Kummer, K_q, se manifeste en tant que le groupe conjugué avec le groupe $\mathcal{f}$ par rapport aux éléments d'une classe $\bar{a} \in \mathcal{G}/_d \mathcal{f}$, etc.

V. Problèmes ouverts

Je me permets de terminer ma conférence par indiquer quelques problèmes ouverts qui s'attachent à l'exposé précédent.

1. On appelle deux équations (q), $(\bar{q})$ mutuellement inverses si elles admettent des phases α, $\bar{\alpha}$ qui sont des fonctions inverses: $\bar{\alpha} = \alpha^{-1}$. Il s'agit d'étudier la géométrie centro-affine des courbes intégrales des équations (q), $(\bar{q})$ mutuellement inverses.

2. Soient (Q), (q) des équations dont les porteurs diffèrent l'un de l'autre d'une constante $-\lambda^2$, $Q = -\lambda^2 + q$, et dont les dispersions centrales Φ_m, φ_n résultent confondues : $\Phi_m = \varphi_n$. On sait que, dans ces conditions, les fonctions Q, q sont périodiques avec $\omega (> 0)$ et les dispersions en question sont de la forme: $\Phi_m(t) = \varphi_n(t) = t + \omega$. Il s'agit de déterminer toutes les équations (Q), (q) jouissant de ces propriétés.

3. Appliquer la théorie des transformations considérée aux équations (q) aux porteurs périodiques. Etudier la théorie de Floquet en relation avec le modèle algébrique abstrait de ci-dessus.

4. Développer la composante numérique de la théorie considérée. Il s'agit surtout des méthodes effectives du calcul des phases et des dispersions centrales. Payer attention aux équations spéciales intervenantes dans les applications techniques.

5. Récemment, M. F. Neuman a développé une théorie géométrique des transformations des équations différentielles ordinaires linéaires et homogènes du $n^{\text{-ième}}$ ordre (sous presse). Il s'agit d'étudier cette théorie du point de vue algébrique.

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L'adresse de l'auteur:
Otakar Borůvka
Institut Mathématique de l'Académie Tchécoslovaque des Sciences
Janáčkovo náměstí 2a
Brno
Tchécoslovaquie

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O. BORŮVKA

Algebraická struktura teorie lineárních diferenciálních transformací druhého řádu

Resumé

Předmětem tohoto článku je obsah plenární přednášky proslovené dne 28. srpna 1972 na konferenci EQUADIFF III v Brně. Přednáška obsahovala popis postupu vedoucího od prvních začátku analytické teorie Kummerových transformací oscilatorických diferenciálních rovnic $y'' = q(t).y$, $q(t) \in C^0_{j=(-\infty,\infty)}$, k abstraktnímu, na algebraických pojmech axiomaticky založenému modelu této teorie.

Jednotlivé kapitoly předloženého článku mají tento obsah: I. Úvod, II. Základy teorie transformací jacobiovských rovnic, III. Algebra fází, IV. Abstraktní algebraický model teorie transformací, V. Otevřené problémy.

Резюме

Алгебраическая структура теории линейных дифференциальных трансформаций второго порядка

О. БОРУВКА

Предметом этой статьи является содержание пленарной лекции прочтенной 28-го августа 1972 г. на конференции EQUADIFF III в городе Брно. В лекции описан процесс ведущий от первых начал аналитической теории трансформаций Куммера колеблющихся дифференциальных уравнений $y'' = q(t)y$, $q(t) \in C^0_{j=(-\infty,\infty)}$, к абстрактной, на алгебраических понятиях аксиоматическо основанной, модели этой теории.

Название отдельных глав статьи: I. Введение, II. Основы теории трансформаций уравнений Якоби III. Алгебра фаз, IV. Абстрактная алгебраическая модель теории трансформаций. V. Открытые проблемы.

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EINIGE INTEGRALBEDINGUNGEN FÜR DAS NICHTOSZILLIEREN DER LINEAREN DIFFERENTIALGLEICHUNG DRITTER ORDNUNG

MILAN GERA, Bratislava

Vorläufige Erwägungen

Begriffe und Bezeichnungen. Wir sagen, dass die lineare Differentialgleichung n-ter Ordnung

$$(L_n) \qquad L_n x \equiv x^{(n)} + a_1(t)x^{(n-1)} + \dots + a_n(t)x = 0$$

mit stetigen Koeffizienten im Intervall J in J nichtoszillatorisch /disconjugate/ ist, wenn jede ihre nichttriviale Lösung im Intervall J höchstens n-1 Nullstellen, die Vielfachheit inbegriffen hat.

In dieser Arbeit wollen wir die Integralbedingungen eines anderen Typs wie in der Arbeiten [1], [2], [7], [13], [15] für das Nichtoszillieren der linearen Differentialgleichung dritter Ordnung

$$Lx \equiv x''' + a_1(t)x'' + a_2(t)x' + a_3(t)x = 0$$

im Intervall J ableiten. Von den Koeffizienten $a_1(t)$, $a_2(t)$, $a_3(t)$ setzen wir voraus, dass diese zusammen mit ihren Ableitungen stetige Funktionen im Intervall J sind, welche in unseren Erwägungen vorkommen. Im weiteren wird bedeuten:

I - einen kompakten oder halboffenen Intervall, welcher ein Teil des Intervalls J ist;

t_0 - einen Endpunkt des Intervalls I, welcher zu I gehört;

I^o - einen Intervall, welcher t_o nicht enthält, d.h. $I^o = I - \{t_o\}$;
$C^3_+(I)$ - eine Menge positiver Funktionen aus $C^3(I)$.

Es sei $w = w(t) \in C^3_+(I)$. Durch die Transformation geht die

$$x = wY$$

geht die Differentialgleichung $Lx = 0$ in die Differentialgleichung

$$\mathcal{L}Y \equiv H(Y', w) + YLw = 0,$$

über, wo

$$H(u,w) \equiv wu'' + [3w' + a_1(t)w]u' + [3w'' + 2a_1(t)w' + a_2(t)w]\, u$$

ist.
Die Differentialgleichung $\mathcal{L}Y = 0$ geht durch die Substitution

$$Y = \frac{y}{w}\, e^{-\frac{1}{3}\int_{t_o}^{t} a_1(\tau)\, d\tau}$$

in die Differentialgleichung dritter Ordnung der Form

$$Ky \equiv y''' + a(t)y' + b(t)y = 0$$

über, wo

$$a(t) = a_2(t) - \frac{1}{3} a_1^2(t) - a_1'(t),$$

$$b(t) = a_3(t) - \frac{1}{3} a_1(t)a_2(t) + \frac{2}{27} a_1^3(t) - \frac{1}{3} a_1''(t)$$

ist.

Unter der zur Differentialgleichung $Lx = 0$ bzw. $Ky = 0$ adjungierten Differentialgleichung verstehen wir die Differentialgleichung der Form

$$L^*z \equiv z''' - a_1(t)z'' + [a_2(t) - 2a_1'(t)]z' + [a_2'(t) - a_1''(t) - a_3(t)]z = 0$$

bzw.

$$K^*v \equiv v''' + a(t)v' + [a'(t) - b(t)]v = 0.$$

Für die Lösungen der Differentialgleichung $Ky = 0$ im Intervall I gilt die folgende Integralidentität [3]

$$(1) \quad y(t)y''(t) - \frac{1}{2}y'^2(t) + \frac{1}{2}a(t)y^2(t) + \int_{t_0}^{t}\left[b(\tau) - \frac{1}{2}a'(\tau)\right]y^2(\tau)d\tau = \text{Konst.}$$

Es sei $y_1 = y_1(t)$ die Lösung der Differentialgleichung $Ky = 0$, welche in der Zahl $t = t_0$ den Anfangsbedingungen

$$y_1(t_0) = y_1'(t_0) = 0, \quad y_1''(t_0) = 1$$

entspricht. In I gilt dann

$$(2) \qquad y_1v'' - y_1'v' + [y_1'' + a(t)y_1]v = 0,$$

wo

$$v = \begin{vmatrix} y & y_1 \\ y' & y_1' \end{vmatrix}$$

und y die Lösung der Differentialgleichung $Ky = 0$ ist. Wenn $y_1(t) > 0$ in I^o, dann geht die Differentialgleichung (2) durch die Substitution $v = u\sqrt{y_1}$ in die Differentialgleichung

$$(3) \qquad u'' + \left[a(t) + \frac{3}{2}\,\frac{y_1''}{y_1} - \frac{3}{4}\,\frac{y_1'^2}{y_1^2}\right]u = 0, \qquad t \in I^o$$

über.

Weiter haben wir aus der Integralidentität (1) für den Koeffizienten bei u in I^o

$$\frac{3}{2}\,\frac{y_1''}{y_1} - \frac{3}{4}\,\frac{y_1'^2}{y_1^2} + a(t) =$$

$$= \frac{1}{4}\,a(t) - \frac{3}{2}\,\frac{1}{y_1^2(t)}\int_{t_o}^{t}\left[b(\tau) - \frac{1}{2}\,a'(\tau)\right]\,y_1^2(\tau)\,d\tau\,.$$

Es ist dann also möglich die Differentialgleichung (3) in der Form

$$(4) \qquad u'' + \left\{\frac{1}{4}\,a(t) - \frac{3}{2}\,\frac{1}{y_1^2(t)}\int_{t_o}^{t}\left[b(\tau) - \frac{1}{2}\,a'(\tau)\right]y_1^2(\tau)\,d\tau\right\}u = 0$$

$$(t \in I^o)$$

zu schreiben.

Definieren wir die Funktion $Q(t;y_1)$ in I, wo $y_1 = y_1(t) > 0$ im Intervall I^o ist, auf folgende Weise

$$Q(t;y_1) = \frac{1}{y_1^2(t)}\int_{t_o}^{t}\left[b(\tau) - \frac{1}{2}\,a'(\tau)\right]y_1^2(\tau)\,d\tau \quad \text{für } t \in I^o$$

und

$$\left[Q(t;\, y_1)\right]_{t=t_o} = 0.$$

Die Funktion $Q(t; y_1)$ ist im Intervall I ersichtlich stetig.

Im weiteren werden wir über die Funktionen $f(t) \in C^n(I)$, wo I ein kompakter Intervall auf der Zahlenachse ist, voraussetzen, dass sie zusammen mit ihrer n-ten Ableitung definiert und stetig auf irgendeinem offenen Intervall $J^o \supset I$ sind. (Es ist immer möglich die Funktion $f(t) \in C^n(I)$ auf dem Intervall J^o derart stetig fortzusetzen, dass $f(t) \in C^n(J^o)$ ist.)

Die Tatsache, dass die Funktion $g(t) \in C^n(J)$ in der Zahl $t_o \in J$ den Bedingungen

$$g(t_o) = c_o, \quad g'(t_o) = c_1, \ldots, g^{(n-1)}(t_o) = c_{n-1}$$

entspricht, wo $c_o, c_1, \ldots, c_{n-1}$ gegebene Zahlen sind werden wir folgenderweise aufzeichnen

$$g(t) = g(t;\, t_o, c_o, c_1, \ldots, c_{n-1}).$$

Hilfssatz 1. I sei ein halboffener Intervall. Die Differentialgleichung $L_n x = 0$ ist dann und nur dann im Intervall I nichtoszillatorisch, wenn sie in I^o nichtoszillatorisch ist ([4], [5]).

Hilfssatz 2. Die Differentialgleichung $L_n x = 0$ ist im kompakten Intervall I dann und nur dann nichtoszillatorisch, wenn sie in irgendeinem halboffenen /offenen/ Intervall $J^o \supset I$ nichtoszillatorisch ist ([4], [5]).

F sei eine im halboffenen Intervall J definierte Menge von Funktionen $f(t)$. Diese Menge F habe eine solche Eigenschaft (V) im kompakten Intervall $I \subset J$, dass ein solcher halboffener Intervall $J^o \supset I$ $(J^o \subset J)$ existiert, dass die Funktionen $f(t) \in F$ die Eigenschaft (V) auch in J^o haben sollen. Aus dem Hilfssatz 2 folgt dann:

Die Differentialgleichung $L_n x = 0$ sei im halboffenen Intervall I^o nichtoszillatorisch unter folgenden Bedingungen:

(P_1) - eine gewisse Menge von Funktionen $f(t) \in F$ hat die Eigenschaft (V) in I^o,

(P_2) - die linearen Differentialgleichungen der Form (L_m) $(2 \leq m \leq n)$ sind in I^o nichtoszillatorisch.

Dann ist die Differentialgleichung $L_n x = 0$ im kompakten Intervall I unter den Voraussetzungen (P_1), (P_2) im Intervall I nichtoszillatorisch.

Aus diesem Grunde führen wir die bekannten Behauptungen an, welche unter den Voraussetzungen des Typs (P_1), (P_2) im halboffenen Intervall I das Nichtoszillieren der Differentialgleichung $L_n x = 0$ in I sichern, auch in dem Falle, dass I ein kompakter Intervall ist.

Bemerkung 1. Die Differentialgleichung $L_n x = 0$ ist im offenen Intervall J dann und nur dann nichtoszillatorisch, wenn sie in einem beliebigen kompakten /halboffenen bzw. offenen/ Intervall $j \subset J$ nichtoszillatorisch ist.

Hilfssatz 3 [6]. Wenn die Differentialgleichung $Lx = 0$ im Intervall $j \subset J$ nichtoszillatorisch ist, dann sind die Differentialgleichungen $\mathcal{L}Y = 0$, $Ky = 0$, $L^* z = 0$, $K^* v = 0$ nichtoszillatorisch im Intervall j. Ungekehrt, wenn eine der Differentialgleichungen $\mathcal{L}Y = 0$, $Ky = 0$, $L^* z = 0$, $K^* v = 0$ im Intervall j nichtoszillatorisch ist, dann ist auch die Differentialgleichung $Lx = 0$ in j nichtoszillatorisch.

Nichtoszillatorische Differentialgleichungen

1. Satz 1.1 [2]. Die Lösung $y_1 = y_1(t;\ t_o, 0, 0, 1)$ der Differentialgleichung $Ky = 0$ sei positiv im Intervall I^o. Die Differentialgleichung $Ky = 0$ ist im Intervall I dann und nur dann nichtoszillatorisch, wenn die Differentialgleichung

$$(1.1) \qquad u'' + \left[\frac{1}{4}\, a(t) - \frac{3}{2}\, Q(t;y_1)\right] u = 0$$

im Intervall I nichtoszillatorisch ist.

Satz 1.2. Es gelte für die Lösung $y_1(t) = y_1(t; t_o, 0, 0, 1)$ im Intervall I^o $y_1(t) > 0$, $(t-t_o)y_1'(t) \geq 0$. Weiter sei die Differentialgleichung zweiter Ordnung

$$(1.2)\qquad lu \equiv u'' + \left\{\frac{1}{4} a(t) - \frac{3}{2} \inf_{\xi \in I_1} \int_{\xi}^{t} \left[b(\tau) - \frac{1}{2} a'(\tau)\right] d\tau \right\} u = 0,$$

wo I_1 ein Intervall mit den Endpunkten t_o, t ist, im Intervall I nichtoszillatorisch. Dann ist die Differentialgleichung $Ky = 0$ im Intervall I nichtoszillatorisch.

B e w e i s. Da $y_1(t) > 0$, $(t-t_o)y_1'(t) \geq 0$ in I^o ist, existiert auf Grund des zweiten Satzes über den Mittelwert der Integralrechnung eine solche Zahl $\xi \in I_1$, dass

$$Q(t; y_1) = \int_{\xi}^{t} \left[b(\tau) - \frac{1}{2} a'(\tau)\right] d\tau$$

für $t \in I^o$. Mit Rücksicht darauf, dass

$$Q(t; y_1) \geq \inf_{\xi \in I_1} \int_{\xi}^{t} \left[b(\tau) - \frac{1}{2} a'(\tau)\right] d\tau \qquad (t \in I)$$

ist der Koeffizient bei u in der Differentialgleichung (1.1) kleiner oder gleich dem Koeffizienten bei u in der Differentialgleichung (1.2). Da die Differentialgleichung (1.2) in I nichtoszillatorisch ist, ist auf Grund des Sturmschen Satzes auch die Differentialgleichung (1.1) in I nichtoszillatorisch. Laut Satz 1.1 ist deshalb die Differentialgleichung $Ky = 0$ in I nichtoszillatorisch

S a t z 1.3 [7]. Es existiere die Lösung $\bar{x}(t)$ der Differentialgleichung $Lx = 0$ mit der Eigenschaft $\bar{x}(t) > 0$, $(t-t_o)\bar{x}'(t) > 0$ für $t \in I^o$ und dabei sei $(t-t_o)\,a_3(t) \geq 0$ in I. Dann ist die Differentialgleichung $Lx = 0$ im Intervall I nichtoszillatorisch (siehe auch [8]).

H i l f s s a t z 1.1. Es existiere eine solche Funktion $w(t) \in C_+^3(I)$, dass $(t-t_o)\,Lw \leq 0$ für $t \in I^o$ ist und die Differentialgleichung zweiter Ordnung

$$H(u,w) = 0$$

ist im Intervall I nichtoszillatorisch. Für die Lösung $Y_1(t) = Y_1(t; t_o, 0, 0, 1)$ der Differentialgleichung $\mathcal{L}Y = 0$ im Intervall I^o gilt dann $Y_1(t) > 0$, $(t-t_o)Y_1'(t) > 0$. Wenn ausserdem $(t-t_o)\left[3w'(t) + a_1(t)w(t)\right] \geqq 0$ $\left((t-t_o)w'(t) \geqq 0\right)$ für $t \in I^o$ ist, dann gilt für die Lösung $y_1(t) = y_1(t; t_o, 0, 0, 1)$ $\left(x_1(t) = x_1(t; t_o, 0, 0, 1)\right)$ der Differentialgleichung $Ky = 0$ $(Lx = 0)$ im Intervall I^o $y_1(t) > 0$, $(t-t_o)y_1'(t) > 0$ $\left(x_1(t) > 0,\ (t-t_o)x_1'(t) > 0\right)$.

<u>Beweis</u>. Es sei $u_1(t) = u_1(t; t_o, 0, 1)$ die Lösung der Differentialgleichung $H(u,w) = 0$ und $C(t,\tau)$ sei ihre Cauchysche Funktion. Dann ist die Lösung $Y_1(t)$ der Differentialgleichung $\mathcal{L}Y = 0$ auch die Lösung der Integralgleichung

$$(1.3) \qquad Y_1(t) = \int_{t_o}^{t} u_1(\tau)\,d\tau + \int_{t_o}^{t} R(t;\tau)\,Y_1(\tau)\,d\tau,$$

wo

$$R(t;\tau) = -Lw(\tau)\int_{\tau}^{t} C(\eta,\tau)\,d\eta$$

ist.

Da die Differentialgleichung $H(u,w) = 0$ im Intervall I nichtoszillatorisch ist, ist $(t-t_o)u_1(t) > 0$ für $t \in I^o$ und $C(t,\tau) \geqq 0$ in $G_o = \{(t,\tau) \in I^o x I^o \mid (t-t_o)(t-\tau) \geqq 0\}$. Wegen dieser Tatsachen und weil $(t-t_o)\,Lw \leqq 0$ in I^o ist, gilt für den Kern $R(t,\tau)$ der Integralgleichung (1.3) in G_o $(t-\tau)R(t,\tau) \geqslant 0$. Auf Grund des Hilfssatzes 1 [7] ist deshalb

$$Y_1(t) \geq \int_{t_o}^{t} u_1(\tau)\,d\tau > 0$$

für $t \in I^o$.

Aus der Integralgleichung (1.3) folgt weiter

$$Y_1'(t) = u_1(t) - \int_{t_o}^{t} Lw(\tau)\, C(t,\tau)\, Y_1(\tau)\, d\tau .$$

Mit Rücksicht auf die Eigenschaften der Funktionen $u_1(t)$, Lw, $C(t,\tau)$, $Y_1(t)$ erhalten wir, dass $(t-t_o)Y_1'(t) > 0$ für $t \in I^o$ ist. Im Falle, dass $(t-t_o)\,[3w'(t) + a_1(t)\; w(t)] \geq 0$ $((t-t_o)\,w'(t) \geq 0)$ für $t \in I^o$ ist, folgt dann aus dem Zusammenhang zwischen den Lösungen der Differentialgleichungen $\mathcal{L}\,Y = 0$, $Ky = 0$ $(Lx=0)$ die Behauptung des Hilfssatzes für die Lösung $y_1(t)$ $(x_1(t))$ der Differentialgleichung $Ky = 0$ $(Lx=0)$. Aus dem Hilfssatz 1.1 folgt im Falle, dass $w=1$ ist, die

Folgerung 1. Es sei $(t-t_o)a_3(t) \leq 0$ für $t \in I^o$ und die Differentialgleichung zweiter Ordnung

$$l_1 u \equiv u'' + a_1(t)u' + a_2(t)u = 0$$

ist im Intervall I nichtoszillatorisch. Für die Lösung $x_1(t) = x_1(t;t_o,0,0,1)$ der Differentialgleichung $Lx = 0$ im Intervall I^o gilt dann $x_1(t) > 0$, $(t-t_o)x_1'(t) > 0$. Wenn ausserdem $(t-t_o)a_1(t) \geq 0$ für $t \in I^o$ ist, dann gilt für die Lösung $y_1(t) = y_1(t;t_o,0,0,1)$ der Differentialgleichung $Ky = 0$ im Intervall I^o $y_1(t) > 0$, $(t-t_o)y_1'(t) > 0$.

Folgerung 1a. Es sei $(t-t_o)b(t) \leq 0$ für $t \in I^o$ und die Differentialgleichung zweiter Ordnung

$$l_2 u \equiv u'' + a(t)u = 0$$

sei im Intervall I nichtoszillatorisch. Für die Lösung $y_1(t) = y_1(t;\, t_o,0,0,1)$ der Differentialgleichung $Ky = 0$ im Intervall I^o gilt dann $y_1(t) > 0$, $(t-t_o)\, y_1'(t) > 0$. Wenn ausserdem

$(t-t_o)\, a_1(t) \leqq 0$ $\left((t-t_o)^3 \left[3w'(t) + a_1(t)w(t)\right] \leqq 0\right.$ für irgendeine Funktion $w(t) \in C^3_+(I)$) in I^o, dann gilt für die Lösung $x_1(t) = x_1(t;\, t_o, 0, 0, 1)$ $\left(Y_1(t) = Y_1(t,\, t_o,\, 0,\, 0,\, 1)\right)$ der Differentialgleichung $Lx=0$ $(\mathcal{L}Y=0)$ im Intervall I^o $x_1(t) > 0$, $(t-t_o)x_1'(t) > 0$ $(Y_1(t) > 0,\, (t-t_o)\, Y_1'(t) > 0)$.

Auf Grund des Satzes 1.3, des Hilfssatzes 3, des Hilfssatzes 1.1 und dessen Folgerungen erhalten wir folgende Sätze:

Satz 1.4. Es sei $(t-t_o)\, a_3(t) \geqq 0$ $((t-t_o)\, b(t) \geqq 0)$ für $t \in I^o$ und es existiere eine solche Funktion $w(t) \in C^3_+(I)$, dass $(t-t_o)Lw \leqq 0$, $(t-t_o)w'(t) \geqq 0$ $((t-t_o)\; Lw \leqq 0$, $(t-t_o)\times$ $\left[3w'(t) + a_1(t)w(t)\right] \geqq 0)$ für $t \in I^o$ und die Differentialgleichung zweiter Ordnung

$$H(u,w) = 0$$

sei im Intervall I nichtoszillatorisch. Dann sind die Differentialgleichungen $Lx=0$, $L^*z = 0$, $Ky = 0$, $K^*v = 0$ im Intervall I nichtoszillatorisch.

Satz 1.5. Es sei $(t-t_o)\, a_3(t) \geqq 0$ $(\leqq 0)$, $(t-t_o)\, b(t) \leqq 0$ $(\geqq 0)$, $(t-t_o)\, a_1(t) \leqq 0$ $(\geqq 0)$ für $t \in I^o$ und die Differentialgleichung zweiter Ordnung $l_2 u = 0$ $(l_1 u = 0)$ sei nichtoszillatorisch im Intervall I. Dann sind die Differentialgleichungen $Lx = 0$, $L^*z = 0$, $Ky = 0$, $K^*v = 0$ im Intervall I nichtoszillatorisch.

Hilfssatz 1.1'. Es sei I ein kompakter Intervall. Es existiere eine solche Funktion $w(t) \in C^3_+(I)$, dass $(t-t_o)\, Lw \leqq 0$ für $t \in I^o$ ist und die Differentialgleichung zweiter Ordnung

$$H(u,w) = 0$$

ist im Intervall I^o nichtoszillatorisch. Für die Lösung $Y_1(t) = Y_1(t; t_o, 0, 0, 1)$ der Differentialgleichung $\mathcal{L}Y = 0$ im Intervall I^o gilt dann $Y_1(t) > 0$, $(t-t_o)\, Y_1'(t) \geqq 0$. Wenn dabei $Lw \not\equiv 0$ ist, dann ist $(t-t_o)\, Y_1'(t) > 0$ für $t \in I^o$. Ausserdem sei $(t-t_o)\left[3w'(t) + a_1(t)w(t)\right] \geqq 0$ $((t-t_o)w'(t) \geqq 0)$ für $t \in I^o$, dann gilt für die Lösung $y_1(t) = y_1(t; t_o, 0, 0, 1)$ $(x_1(t) = x_1(t;$ $t_o, 0, 0, 1))$ der Differentialgleichung $Ky = 0$ $(Lx = 0)$ im Intervall I^o $y_1(t) > 0$, $(t-t_o)y_1'(t) \geqq 0$. $(x_1(t) > 0$, $(t-t_o)\times$

$x_1'(t) \geq 0$). Wenn dabei $Lw \not\equiv 0$ in I^o oder die Funktion $3w'(t) + a_1(t)w(t)$ $(w'(t))$ ist im Endpunkt des Intervalls I^o, welcher zu I^o gehört, verschieden von Null, dann ist $(t-t_o)\, y_1'(t) > 0$ $((t-t_o)\, x_1'(t) > 0)$ für $t \in I^o$.

Beweis. Dieser kann auf dieselbe Weise wie der Beweis des Hilfssatzes 1.1 durchgeführt werden.

Folgerung 1'. I sei ein kompakter Intervall. Es sei $(t-t_o) \times a_3(t) \leq 0$ für $t \in I^o$ und die Differentialgleichung zweiter Ordnung $l_1 u = 0$ sei im Intervall I^o nichtoszillatorisch. Für die Lösung $x_1(t) = x_1(t;\, t_o, 0, 0, 1)$ der Differentialgleichung $Lx = 0$ im Intervall I^o gilt dann $x_1(t) > 0$, $(t-t_o)\, x_1'(t) \geq 0$. Wenn dabei $a_3(t) \not\equiv 0$ in I^o ist, dann ist $(t-t_o)\, x_1'(t) > 0$ für $t \in I^o$. Wenn ausserdem $(t-t_o)\, a_1(t) \geq 0$ fur $t \in I^o$ ist, dann gilt für die Lösung $y_1(t) = y_1(t;\, t_o, 0, 0, 1)$ der Differentialgleichung $Ky = 0$ im Intervall I^o $y_1(t) > 0$, $(t-t_o)\, y_1'(t) \geq 0$. Wenn dabei $a_3(t) \not\equiv 0$ in I^o oder die Funktion $a_1(t)$ ist im Endpunkt des Intervalls I^o, welcher zu I^o gehört verschieden von Null, dann ist $(t-t_o)\, y_1'(t) > 0$ für $t \in I^o$.

Folgerung 1a'. Es sei I ein kompakter Intervall. Es sei $(t-t_o)\, b(t) \leq 0$ für $t \in I^o$ und die Differentialgleichung zweiter Ordnung $l_2 u = 0$ sei im Intervall I^o nichtoszillatorisch. Für die Lösung $y_1(t) = y_1(t; t_o, 0, 0, 1)$ der Differentialgleichung $Ky = 0$ im Intervall I^o gilt dann $y_1(t) > 0$, $(t-t_o)\, y_1'(t) \geq 0$. Wenn dabei $b(t) \not\equiv 0$ in I^o, dann ist $(t-t_o)\, y_1'(t) > 0$ für $t \in I^o$. Ausserdem sei $(t-t_o)\, a_1(t) \leq 0$ $((t-t_o)\, [3w'(t) + a_1(t)w(t)] \leq 0$ für irgendeine Funktion $w(t) \in C_+^3(I))$ in I^o, dann gilt für die Lösung $x_1(t) = x_1(t; t_o, 0, 0, 1)$ $(Y_1(t) = Y_1(t; t_o; 0, 0, 1))$ der Differentialgleichung $Lx = 0$ $(\mathcal{L}Y = 0)$ im Intervall I^o $x_1(t) > 0$, $(t-t_o)\, x_1'(t) \geq 0$ $(Y_1(t) > 0, (t-t_o)\, Y_1'(t) \geq 0)$.
Wenn dabei $b(t) \not\equiv 0$ in I^o ist, oder die Funktion $a_1(t)$ $(3w'(t) + a_1(t)w(t))$ ist im Endpunkt des Intervalls I^o, welcher zu I^o gehört verschieden von Null, dann ist $(t-t_o)\, x_1'(t) > 0$ $((t-t_o) Y_1'(t) > 0)$ für $t \in I^o$.

Satz 1.4'. Es sei I ein kompakter Intervall. Es sei $(t-t_o)\, a_3(t) \geq 0$ $((t-t_o)\, b(t) \geq 0)$ für $t \in I^o$ und es existiere eine solche Funktion $w(t) \in C_+^3(I)$, dass $(t-t_o)\, Lw \leq 0$, $(t-t_o) w'(t) \geq 0$ $((t-t_o)\, Lw \leq 0, (t-t_o)\, [3w'(t) + a_1(t)w(t)] \geq 0)$ für $t \in I^o$ und die Differentialgleichung zweiter Ordnung

$$H(u, w) = 0$$

ist im Intervall I^o nichtoszillatorisch. Wenn dabei $Lw \not\equiv 0$ in I^o oder die Funktion $w(t)\left(3w'(t) + a_1(t)w(t)\right)$ im Endpunkt des Intervalles I^o, welcher zu I^o gehört, verschieden von Null ist, dann sind die Differentialgleichungen $Lx = 0$, $L^*z = 0$, $Ky = 0$, $K^*v = 0$ im Intervall I nichtoszillatorisch.

Satz 1.5'. Es sei I ein kompakter Intervall. Es sei $(t-t_o) \times a_3(t) \geqq 0$ $(\leqq 0)$, $(t-t_o)\,b(t) \leqq 0$ $(\geqq 0)$, $(t-t_o)a_1(t) \leqq 0$ $(\geqq 0)$ für $t \in I^o$ und die Differentialgleichung zweiter Ordnung $l_2 u = 0$ $(l_1 u = 0)$ sei nichtoszillatorisch im Intervall I^o. Wenn dabei $b(t) \not\equiv 0$ $(a_3(t) \not\equiv 0)$ in I^o oder die Funktion $a_1(t)$ ist im Endpunkt des Intervalls I^o, welcher zu I^o gehört, verschieden von Null, dann sind die Differentialgleichungen $Lx = 0$, $L^*z = 0$, $Ky = 0$, $K^*v = 0$ im Intervall I nichtoszillatorisch.

Die Sätze 1.4' und 1.5' folgen aus dem Satz 1.3, dem Hilfssatz 3 und dem Hilfsatz 1.1' und deren Folgerungen.

Satz 1.6. Die Differentialgleichung zweiter Ordnung $lu = 0$ sei nichtoszillatorisch im Intervall I und es existiere eine solche Funktion $w(t) \in C_+^3(I)$, dass $(t-t_o)\ Lw \leqq 0$, $(t-t_o)\left[3w'(t) + a_1(t)w(t)\right] \geqq 0$ für $t \in I^o$ und die Differentialgleichung zweiter Ordnung $H(u,w) = 0$ ist im Intervall I^o nichtoszillatorisch. Dann sind die Differentialgleichungen $Lx = 0$, $L^*z = 0$, $Ky = 0$, $K^*v = 0$ nichtoszillatorisch im Intervall I.

Dieser Satz folgt aus dem Satz 1.2, dem Hilfssatz 1.1, 1.1' und dem Hilfssatz 3.

Folgerung 1. Es sei $(t-t_o)a_3(t) \leqq 0$, $(t-t_o)\,a_1(t) \geqq 0$ für $t \in I^o$ und die Differentialgleichung zweiter Ordnung $l_1 u = 0$ sei im Intervall I^o nichtoszillatorisch, die Differentialgleichung $lu = 0$ nichtoszillatorisch in I. Dann sind die Differentialgleichungen $Lx = 0$, $L^*z = 0$, $Ky = 0$, $K^*v = 0$ nichtoszillatorisch im Intervall I.

Folgerung 1a. Es sei $(t-t_o)b(t) \leqq 0$ für $t \in I^o$ und die Differentialgleichung zweiter Ordnung $lu = 0$ sei nichtoszillatorisch im Intervall I und die Differentialgleichung $l_2 u = 0$ nichtoszillatorisch in I^o. Dann sind die Differentialgleichungen $Ky = 0$, $K^*v = 0$ nichtoszillatorisch im Intervall I.

Wenn wir in den vorhergegangenen Sätzen die Aufgaben der Differentialgleichungen $Lx = 0$, $L^*z = 0$ bzw. $Ky = 0$, $K^*v = 0$ vertauschen, erhalten wir neue hinreichende Bedingungen für das Nichtoszillieren der Differentialgleichungen $Lx = 0$, $L^*z = 0$, $Ky = 0$, $K^*y = 0$ im Intervall I /Hilfssatz 3/. Zum Beispiel aus der Folgerung 1a des Satzes 1.6 erhalten wir:

Es sei $(t-t_0)\ [a'(t) - b(t)] \leqq 0$ für $t \in I^0$ und die Differentialgleichung zweiter Ordnung $l_2 u = 0$ sei nichtoszillatorisch im Intervall I^0 und die Differentialgleichung

$$u'' + \left\{\frac{1}{4} a(t) + \frac{3}{2} \sup_{\xi \in I_1} \int_{\xi}^{t} [b(\tau) - \frac{1}{2} a'(\tau)]\, d\tau\right\} u = 0,$$

wo I_1 ein Intervall mit den Endpunkten t_0 und t ist, ist nichtoszillatorisch im Intervall I. Dann sind die Differentialgleichungen $Ky = 0$, $K^* v = 0$ nichtoszillatorisch im Intervall I.

2. Es sei α der linke und β der rechte Endpunkt des Intervalles J und $J_\alpha = J - \{\alpha\}$, $J_\beta = J - \{\beta\}$.

<u>Satz 2.1</u>. Es sei $b(t) \leqq 0 \quad (\geqq 0)$ für $t \in J$ und die Funktion

$$\int_{\xi}^{t} \left[b(\tau) - \frac{1}{2} a'(\tau)\right] d\tau\,.$$

bei jedem festem $t \in J$, sei mit Rücksicht auf $\xi \in (\alpha, t]\ (\xi \in [t, \beta))$ von unten begrenzt. Weiter sei die Differentialgleichung zweiter Ordnung $l_2 u = 0$ nichtoszillatorisch im Intervall $J_\alpha \quad (J_\beta)$ und die Differentialgleichung

$$(2.1) \qquad u'' + \left\{\frac{1}{4} a(t) - \frac{3}{2} \inf_{\xi \in (\alpha, t]} \int_{\xi}^{t} \left[b(\tau) - \frac{1}{2} a'(\tau)\right] d\tau\right\} u = 0$$

$$(2.1') \qquad \left(u'' + \left\{\frac{1}{4} a(t) - \frac{3}{2} \inf_{\xi \in [t, \beta)} \int_{\xi}^{t} \left[b(\tau) - \frac{1}{2} a'(\tau)\right] d\tau\right\} u = 0\right)$$

ist nichtoszillatorisch im Intervall J. Dann sind die Differentialgleichungen $Ky = 0$, $K^*v = 0$ nichtoszillatorisch im Intervall J.

Beweis. Um zu zeigen, dass die Differentialgleichungen $Ky = 0$, $K^*v = 0$ im Intervall J nichtoszillatorisch sind, genügt es zu zeigen, dass sie in einem beliebigen kompakten Subintervall I des Intervalls J nichtoszillatorisch sind.

Es sei t_0 der linke (rechte) Endpunkt des Intervalls I und es sei $b(t) \leqq 0$ $(\geqq 0)$ für $t \in J$. Wir zeigen, dass unter den gegebenen Voraussetzungen die Differentialgleichung $lu = 0$ in I und die Differentialgleichung $l_2u = 0$ in I^o nichtoszillatorisch ist. Auf Grund der Folgerung. 1a des Satzes 1.6 wird damit gezeigt sein, dass die Differentialgleichungen $Ky = 0$, $K^*v = 0$ im Intervall I nichtoszillatorisch sind. Weil die Differentialgleichung $l_2u = 0$ in J_α (J_β) nichtoszillatorisch ist, ist sie es auch in I^o. Jetzt wollen wir zeigen, dass die Differentialgleichung $lu = 0$ in I nichtoszillatorisch ist. Es sei $I_1 = (t_0, t]$ $(I_1 = [t, t_0))$ und die Differentialgleichung 2.1 ((2.1′)) sei nichtoszillatorisch in J. Mit Rücksicht darauf, dass

$$-\inf_{\xi \in (\alpha, t]} \int_\xi^t \left[b(\tau) - \frac{1}{2}a'(\tau)\right] d\tau \geqq -\inf_{\xi \in I_1} \int_\xi^t \left[b(\tau) - \frac{1}{2}a'(\tau)\right] d\tau$$

$$\left(-\inf_{\xi \in [t, \beta)} \int_\xi^t \left[b(\tau) - \frac{1}{2}a'(\tau)\right] d\tau \geqq -\inf_{\xi \in I_1} \int_\xi^t \left[b(\tau) - \frac{1}{2}a'(\tau)\right] d\tau\right)$$

für $t \in I$ ist, folgt aus dem Vergleich der Differentialgleichungen 2.1 ((2.1′)), $lu = 0$, dass auch die Differentialgleichung $lu = 0$ in I nichtoszillatorisch ist. Damit ist der Beweis des Satzes beendet.

Folgerung. Es sei $a(t) \geqq 0$, $b(t) \leqq 0$ $(\geqq 0)$, $b(t) - \frac{1}{2}a'(t) \geqq 0$ $(\leqq 0)$ für $t \in J$ und die Differentialgleichung $l_2u = 0$ sei im Intervall J nichtoszillatorisch. Dann sind die Differentialgleichungen $Ky = 0$, $K^*v = 0$ im Intervall J nichtoszillatorisch.

Satz 2.2. Es sei $b(t) \leqq 0$ $(\geqq 0)$, $b(t)-\frac{1}{2}a'(t) \leqq 0$ $(\geqq 0)$ für $t \in J$ und es sei

$$\lim_{t\to\alpha^+} \frac{3}{4}a(t) = A \in (-\infty, \infty), \quad \int_c^{\alpha} b(\tau)\,d\tau < +\infty, \quad c \in J_\alpha$$

$$\left(\lim_{t\to\beta^-} -\frac{3}{4}a(t) = B \in (-\infty, \infty), \quad \int_c^{\beta} b(\tau)\,d\tau < +\infty, \quad c \in J_\beta\right).$$

Weiter sei

1. die Differentialgleichung $l_2u = 0$ nichtoszillatorisch im Intervall J, wenn

$$A \geqq -\frac{3}{2}\int_{\alpha}^{\beta} b(\tau)\,d\tau \qquad \left(B \geqq \frac{3}{2}\int_{\alpha}^{\beta} b(\tau)\,d\tau\right)$$

2. die Differentialgleichung zweiter Ordnung

$$\text{(A)} \qquad u'' + \left[a(t) - A - \frac{3}{2}\int_{\alpha}^{t} b(\tau)\,d\tau\right]u = 0$$

$$\text{(B)} \qquad \left(u'' + \left[a(t) - B + \frac{3}{2}\int_{t}^{\beta} b(\tau)\,d\tau\right]u = 0\right)$$

nichtoszillatorisch im Intervall J, wenn

$$A < -\frac{3}{2}\int_{\alpha}^{\beta} b(\tau)\,d\tau \qquad \left(B < \frac{3}{2}\int_{\alpha}^{\beta} b(\tau)\,d\tau\right)$$

Dann sind die Differentialgleichungen $Ky = 0$, $K^*v = 0$ im Intervall J nichtoszillatorisch.

B e w e i s . Es sei, $b(t) \leqq 0$ für $t \in J$. Um zu zeigen, dass die Differentialgleichungen $Ky = 0$, $K^*v = 0$, unter gegebenen Voraussetzungen in J nichtoszillatorisch sind, genügt es auf Grund des Satzes 2.1 zu zeigen, dass die Differentialgleichungen (2.1) und $l_2 u = 0$ in J nichtoszillatorisch sind.
Es sei $b(t) \leqq \frac{1}{2} a'(t)$ für $t \in J$.

1. Die Differentialgleichung $l_2 u = 0$ sei nichtoszillaterisch in J und es sei

$$A \geqq -\frac{3}{2}\int_{\alpha}^{\beta} b(\tau)\,d\tau ,$$

für den Koeffizienten bei u in der Differentialgleichung (2.1) in J gilt dann

$$\frac{1}{4}a(t) - \frac{3}{2}\inf_{\xi \in (\alpha, t]} \int_{\xi}^{t} \left[b(\tau) - \frac{1}{2}a'(\tau)\right] d\tau = \frac{1}{4}a(t) -$$

$$-\frac{3}{2}\int_{\alpha}^{t} \left[b(\tau) - \frac{1}{2}a'(\tau)\right] d\tau = a(t) - A - \frac{3}{2}\int_{\alpha}^{t} b(\tau)\,d\tau \leqq a(t).$$

Aus dieser Ungleichheit und daraus, dass die Differentialgleichung $l_2 u = 0$ in J nichtoszillatorisch ist, erhalten wir auf Grund des Vergleichungsatzes von Sturm, dass auch die Differentialgleichung (2.1) in J nichtoszillatorisch ist.

2. Die Differentialgleichung (A) sei nichtoszillatorisch in J und es sei

$$A < -\frac{3}{2}\int_{\alpha}^{\beta} b(\tau)\,d\tau ,$$

dann haben wir für $t \in J$

$$a(t) < a(t) - A - \frac{3}{2}\int_{\alpha}^{t} b(\tau)\,d\tau = \frac{1}{4}a(t) - \frac{3}{2}\inf_{\xi\in(\alpha,t]}\int_{\xi}^{t}\Big[b(\tau) - \frac{1}{2}a'(\tau)\Big]d\tau .$$

Mit Rücksicht darauf, dass die Differentialgleichung (A) in J nichtoszillatorisch ist erhalten wir daraus, dass auch die Differentialgleichungen (2.1) und $l_2 u = 0$ in J nichtoszillatorisch sind. Im zweiten Fall wird der Beweis auf ähnliche Weise durchgeführt.

Bemerkung 2.1. Aus dem Satz 2.2 folgt der Satz 3.5 [9], welcher von A. C. Lazer bewiesen wurde.

Bemerkung 2.2. Da die Differentialgleichung $Ky = 0$ dann und nur dann im Intervall J nichtoszillatorisch ist, wenn die adjungierte Differentialgleichung $K^* v = 0$ im Intervall J nichtoszillatorisch ist /Hilfsatz 3/, ist es auf Grund des Satzes 2.1 bzw. des Satzes 2.2 möglich durch Umformulierung der gegebenen Voraussetzungen an die Koeffizienten $a(t)$, $a'(t) - b(t)$ der Differentialgleichung $K^* v = 0$ für das Nichtoszillieren der Differentialgleichungen $Ky = 0$, $K^* v = 0$ neue hinreichende Bedingungen zu erhalten.

Hilfssatz 2.1 [10]. Es seien $b_1(t)$, $b_2(t)$ stetige Funktionen im Intervall J und die Differentialgleichungen

$$y''' + a(t)y' + b_1(t)y = 0, \qquad y''' + a(t)y' + b_2(t)y = 0$$

seien im Intervall J nichtoszillatorisch. Es sei weiter

$$b_1(t) \leqq b(t) \leqq b_2(t)$$

für $t \in J$. Dann ist auch die Differentialgleichung $Ky = 0$ im Intervall J nichtoszillatorisch.
Bezeichnen wir

$$b_+(t) = \frac{b(t) + |b(t)|}{2}, \qquad b_-(t) = \frac{b(t) - |b(t)|}{2},$$

dann ist $b_-(t) \leqq 0$, $b_+(t) \geqq 0$ für $t \in J$.

Satz 2.3. Die Differentialgleichung zweiter Ordnung $l_2 u = 0$ sei in den Intervallen J_α und J_β[1] nichtoszillatorisch. Es sei bei jedem festen $t \in J$ die Funktion

$$\int_\xi^t \left[b_-(\tau) - \frac{1}{2} a'(\tau)\right] d\tau$$

mit Rücksicht auf $\xi \in (\alpha; t]$ begrenzt von unten und die Funktion

[1]Wenn die Differentialgleichung $l_2 u = 0$ in J_α (J_β) nichtoszillatorisch ist, dann ist sie auch in J_β (J_α) nichtoszillatorisch [4].

$$\int_{\xi}^{t} \left[b_+(\tau) - \frac{1}{2}a'(\tau)\right] d\tau$$

sei mit Rücksicht auf $\xi \in [t, \beta)$ ebenfalls von unten begrenzt. Weiter seien die Differentialgleichungen zweiter Ordnung

$$u'' + \left\{\frac{1}{4}a(t) - \frac{3}{2}\inf_{\xi\in(\alpha,t]} \int_{\xi}^{t} \left[b_-(\tau) - \frac{1}{2}a'(\tau)\right] d\tau\right\} u = 0,$$

$$u'' + \left\{\frac{1}{4}a(t) - \frac{3}{2}\inf_{\xi\in[t,\beta)} \int_{\xi}^{t} \left[b_+(\tau) - \frac{1}{2}a'(\tau)\right] d\tau\right\} u = 0$$

im Intervall J nichtoszillatorisch. Dann sind die Differentialgleichungen $Ky = 0$, $K^*v = 0$ im Intervall J nichtoszillatorisch.

Beweis. Aus den gegebenen Voraussetzungen an die Funktion $a(t)$, $b_-(t)$, $b_+(t)$ und aus dem Satz 2.1, folgt, dass die Differentialgleichungen

$$y''' + a(t)y' + b_-(t)y = 0, \qquad y''' + a(t)y' + b_+(t)y = 0$$

im Intervall J nichtoszillatorisch sind. Weil

$$b_-(t) \leqq b(t) \leqq b_+(t)$$

für $t \in J$ ist, ist gemäss dem Hilfssatz 2.1 auch die Differentialgleichung $Ky = 0$ im Intervall J nichtoszillatorisch und daher ist auch die Differentialgleichung $K^*v = 0$ im Intervall J nichtoszillatorisch. /Hilfssatz 3/.

Auf ähnliche Weise erhalten wir aus dem Satz 2.2 und aus dem Hilfssatz 2.1 den

Satz 2.4. Es sei $\lim_{t\to\alpha^+} \frac{3}{4} a(t) = A \in (-\infty, \infty$, $\lim_{t\to\beta^-} \frac{3}{4} a(t) = B \quad (-\infty, \infty)$,

$$\int_\alpha^c b_-(\tau)\,d\tau > -\infty, \quad \int_c^\beta b_+(\tau)\,d\tau < +\infty, \qquad c \in (\alpha,\beta)$$

und es sei $b_-(t) \leqq \frac{1}{2} a'(t) \leqq b_+(t)$ für $t \in J$.

Weiter sei

1. die Differentialgleichung $l_2 u = 0$ nichtoszillatorisch im Intervall J, wenn

$$A \geqq -\frac{3}{2}\int_\alpha^\beta b_-(\tau)\,d\tau\,;$$

2. die Differentialgleichung zweiter Ordnung

$$u'' + \left[a(t) - A - \frac{3}{2}\int_\alpha^t b_-(\tau)\,d\tau\right] u = 0$$

im Intervall J nichtoszillatorisch, wenn

$$A < -\frac{3}{2}\int_\alpha^\beta b_-(\tau)\,d\tau\,;$$

3. die Differentialgleichung $l_2 u = 0$ nichtoszillatorisch im Intervall J, wenn

$$B \geq \frac{3}{2} \int_{\alpha}^{\beta} b_+(\tau) d\tau ;$$

4. die Differentialgleichung zweiter Ordnung

$$u'' + \left[a(t) - B + \frac{3}{2} \int_{t}^{\beta} b_+(\tau) d\tau\right] u = 0$$

nichtoszillatorisch im Intervall J, wenn

$$B < \frac{3}{2} \int_{\alpha}^{\beta} b_+(\tau) d\tau .$$

Dann sind die Differentialgleichungen $Ky = 0$, $K^*v = 0$, im Intervall J nichtoszillatorisch.

Über eine Integralbedingung

3. Es sei $\bar{I} = [T_0, T_1] \subset J$. R.M. Mathsen [12] bewies folgende Sätze:

Satz 3.1. Es seien $a_1(t)$, $a_2(t)$, $a_3(t)$ nichtpositive Funktionen im Intervall $\bar{I}$ und dabei solche, dass

$$(3.1) \qquad 2a_1(t) \leq 3 \int_{T_0}^{t} a_2(\tau) e^{\int_t^{\tau} a_1(s) ds} d\tau ,$$

$$(3.2)\qquad a_2(t) \leqq 3\int_{T_o}^{t} a_3(\tau)\, e^{\int_t^{\tau} a_1(s)ds}\, d\tau$$

für $t \in \bar{I}$. Dann ist die lineare Differentialgleichung $Lx = 0$ im Intervall $\bar{I}$ nichtoszillatorisch.

Satz 3.1′. Es sei $a_1(t) \geqq 0$, $a_2(t) \leqq 0$, $a_3(t) \geqq 0$ für $t \in \bar{I}$ und es sei

$$(3.1')\qquad 2a_1(t) \geqq 3\int_{T_1}^{t} a_2(\tau)\, e^{\int_t^{\tau} a_1(s)ds}\, d\tau,$$

$$(3.2')\qquad a_2'(t) \leqq 3\int_{T_1}^{t} a_3(\tau)\, e^{\int_t^{\tau} a_1(s)ds}\, d\tau$$

für $t \in \bar{I}$. Dann ist die lineare Differentialgleichung $Lx = 0$ im Intervall $\bar{I}$ nichtoszillatorisch.

Beweisen wir den folgenden Satz, aus welchen die angeführten Sätze 3. 1 und 3.1′ hervorgehen.

Satz 3.2. c sei irgendeine Zahl aus J und es sei

$$(3.3)\qquad a_2(t)e^{\int_c^{t} a_1(s)ds} \leqq K + \int_c^{t} a_3(\tau)\, e^{\int_c^{\tau} a_1(s)ds}\, d\tau \leqq 0\,.$$

für $t \in J$, wo K eine nichtpositive Konstante ist. Dann ist die lineare Differentialgleichung $Lx = 0$ im Intervall J nichtoszillatorisch.

Bemerkung 3.1. Aus der Bedingung 3.3 folgt, dass die Bedingungen (3.1) und $a_1(t) \leqq 0$ ((3.1′) und $a_1(t) \geqq 0$) im Satz 3.1 (im Satz 3.1′) für das Nichtoszillieren die Differentialgleichung $Lx = 0$ im Intervall $\bar{I}$ nicht notwendig sind. Dabei ist die Bedingung (3.3) besser als die Bedingungen (3.2), $a_3(t) \leqq 0$ ((3.2′), $a_3(t) \geqq 0$) im Intervall $\bar{I}$.

Bemerkung 3.2. Aus dem Satz 3.2 geht auch die Folgerung 3 des Satzes 5 [13] hervor.

Bevor wir den Beweis des Satzes 3.2 antreten, wollen wir einen Hilfssatz anführen, welchen wir bei diesen Beweis anwenden werden.

Hilfssatz 3.1 [14]. Es seien $y_1(t) = y_1(t; T_o, 0, 0, 1)$, $y_2(t) = y_2(t; T_o, 0, 1, 0)$ Lösungen der linearen Differentialgleichung $Lx = 0$ und es sei

$$v_1(t) = \begin{vmatrix} y_2(t) & y_1(t) \\ y_2'(t) & y_1'(t) \end{vmatrix} .$$

Dann ist die lineare Differentialgleichung $Lx = 0$ im Intervall $\bar{I}$ nichtoszillatorisch dann und nur dann, wenn $y_1(t) > 0$ und $v_1(t) > 0$ für $t \in (T_o, T_1]$.

Bemerkung 3.3. Die Funktion $v_1(t)$ aus dem Hilfssatz 3.1 ist die Lösung der Differentialgleichung

$$Mv \equiv \left[(v'' + a_1(t)v' + a_2(t)v)\, e^{\int_c^t a_1(s)ds} \right]' - a_3(t) e^{\int_c^t a_1(s)ds} v = 0^{1)} \qquad (c \in J)$$

1) Über $a_1(t)$, $a_2(t)$, $a_3(t)$ wird hier nur ihre Stetigkeit im Intervall J vorausgesetzt.

mit der Eigenschaft $v_1(T_o) = v_1'(T_o) = 0$, $v_1''(T_o) = 1$ ([6]).

<u>Beweis des Satzes 3.2.</u> Es sei $I = [t_o, T]$ ein beliebiger aber fester Subintervall des Intervalls J. Es sei $y_1(t) = y_1(t; t_o, 0, 0, 1)$ die Lösung der Differentialgleichung $Lx = 0$ und $v_1(t) = v_1(t; t_o, 0, 0, 1)$ sei die Lösung der Differentialgleichung $Mv = 0$. Dann ist $y_1''(t)$ die Lösung der Integralgleichung

$$(3.4) \qquad y_1''(t) = e^{-\int_{t_o}^{t} a_1(s)ds} + \int_{t_o}^{t} A_2(t, \tau) y_1''(\tau) d\tau ,$$

wo

$$A_2(t, \tau) = -\int_{\tau}^{t} \left[a_2(\xi) + (\xi - \tau) a_3(\xi)\right] e^{\int_{t}^{\xi} a_1(s)ds} d\xi$$

und $v_1'(t)$ ist die Lösung der Integralgleichung

$$(3.5) \qquad v_1'(t) = (t - t_o) e^{-\int_{t_o}^{t} a_1(s)ds} + \int_{t_o}^{t} C_1(t, \tau) v_1'(\tau) d\tau ,$$

wo

$$C_1(t, \tau) = \int_{\tau}^{t} \left[(t - \xi) a_3(\xi) - a_2(\xi)\right] e^{\int_{t}^{\xi} a_1(s)ds} d\xi$$

ist (siehe [13]).

Wir zeigen, dass $A_2(t,\tau) \geqq 0$, $C_1(t,\tau) \geqq 0$ in $G_o = \{(t,\tau) \in \in I^o \times I^o \mid t \geqq \tau\}$, wo $I^o = (t_o, T]$. Tatsächlich, aus der Ungleichheit (3.3) für $A_2(t,\tau)$ in G_o erhalten wir

$$A_2(t,\tau) \geqq -\int_\tau^t \left[Ke^{\int_\xi^c a_1(s)ds} + \int_c^\xi a_3(\tau)e^{\int_\xi^\tau a_1(s)ds} d\tau + (\xi-\tau)a_3(\xi)\right] \times$$

$$e^{\int_t^\xi a_1(s)ds} d\xi = -e^{\int_t^c a_1(s)ds} \left[K(t-\tau) + \int_\tau^t (\xi-\tau)\, a_3(\xi) e^{\int_c^\xi a_1(s)ds} d\xi + \right.$$

$$\left. + \int_\tau^t \left(\int_c^\xi a_3(\tau) e^{\int_c^\tau a_1(s)ds} d\tau \right) d\xi \right].$$

Weil

$$\int_\tau^t \left(\int_c^\xi a_3(\tau)\, e^{\int_c^\tau a_1(s)ds} d\tau \right) d\xi =$$

$$= (t-\tau)\int_c^t a_3(\tau)e^{\int_c^\tau a_1(s)ds} d\tau - \int_\tau^t (\xi-\tau)\, a_3(\xi)e^{\int_c^\xi a_1(s)ds} d\xi ,$$

ist

$$A_2(t,\tau) \geqq -(t-\tau)e^{\int_t^c a_1(s)ds}\left[K + \int_c^t a_3(\eta)e^{\int_c^\eta a_1(s)ds} \cdot d\eta\right] \geqq 0,$$

$$(t,\tau) \in G_o.$$

Ähnlich

$$C_1(t,\tau) \geqq - e^{\int_t^c a_1(s)ds}\left[K(t-\tau) - \int_\tau^t (t-\tau)\, a_3(\xi)e^{\int_c^\xi a_1(s)ds} d\xi \, + \right.$$

$$\left. + \int_\tau^t \left(\int_c^\xi a_3(\eta)e^{\int_c^\eta a_1(s)ds} d\eta \right) d\xi \right] =$$

$$= - e^{\int_t^c a_1(s)ds}\left[K(t-\tau) - \int_\tau^t (t-\xi)\, a_3(\xi)e^{\int_c^\xi a_1(s)ds} d\xi \, + \right.$$

$$+ (t-\tau)\int_c^\tau a_3(\eta)e^{\int_c^\eta a_1(s)ds} d\eta + \int_\tau^t (t-\xi)a_3(\xi)e^{\int_c^\xi a_1(s)ds} d\xi =$$

$$= -(t-\tau)e^{\int_t^c a_1(s)ds}\left[K + \int_c^{\tau} a_3(\tau)e^{\int_c^{\tau} a_1(s)ds}\,d\tau\right] \geqq 0$$

für $(t, \tau) \in G_o$.
Also haben wir auf Grund des Hilfssatzes 1 [7] für die Lösungen $y_1''(t)$, $v_1'(t)$ der Integralgleichungen (3.4) ,(3.5) in I

$$y_1''(t) \geqq e^{-\int_{t_o}^{t} a_1(s)ds}, \qquad v_1'(t) \geqq (t-t_o)\, e^{-\int_{t_o}^{t} a_1(s)ds}$$

Daraus erhalten wir, dass

$$y_1(t) \geqq \int_{t_o}^{t} (t-\xi)\, e^{-\int_{t_o}^{\xi} a_1(s)ds}\, d\xi > 0,$$

$$v_1(t) \geqq \int_{t_o}^{t} (\xi - t_o) e^{-\int_{t_o}^{\xi} a_1(s)ds}\, d\xi > 0$$

für $t \in I^o$. Gemäss dem Hilfssatz 3.1 (siehe Bemerkung 3.3) ist deshalb die Differentialgleichung $Lx = 0$ im Intervall I nichtoszillatorisch. Da I ein beliebiger kompakter Subintervall des Intervalls J ist, ist die Differentialgleichung $Lx = 0$ auch in J nichtoszillatorisch.

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Súhrn

Niektoré integrálne podmienky na neosciláciu lineárnej diferenciálnej rovnice tretieho rádu

MILAN GERA

V práci sú odvodené integrálne podmienky pre neosciláciu riešení diferenciálnej rovnice tretieho rádu

$$x''' + a_1(t)x'' + a_2(t)x' + a_3(t)x = 0. \tag{1}$$

Niektoré z týchto podmienok sú zovšeobecnením istých výsledkov A.C. LAZERA [9], R.M. MATHSENA [12] a autora [13]. Napríklad tvrdenie, ktoré zovšeobecňuje výsledok R.M. MATHSENA, je takéto:

Ak c je nejaké číslo z intervalu J, $a_i(t) \in C(J)$, $i = 1,2,3$ a

$$a_2(t)\, e^{\int_c^t a_1(s)ds} \leq K + \int_c^t a_3(\tau)\, e^{\int_c^\tau a_1(s)ds}\, d\tau \leq 0$$

pre $t \in J$, kde K je nekladná konštanta, potom diferenciálna rovnica (1) je neoscilatorická v intervale J.

Резюме

Некоторые интегральные условия для неосцилляции линейного дифференциального уравнения третьего порядка

МИЛАН ГЕРА

В работе приведены интегральные условия для неосцилляции решений дифференциального уравнения третьего порядка

(1) $$x''' + a_1(t)x'' + a_2(t)x' + a_3(t)x = 0.$$

Некоторые из этих условий являются обобщением некоторых результатов А. Ц. ЛЕЙЗЕРА (A.C.LAZER)[9] , Р. М. МЭТСЕНА (R.M.MATHSEN)[12] и самого автора [13].

Например, утверждение обобщающее результат Р. М. МЭТСЕНА:

Если c некоторая точка из интервала J, $a_i(t) \in C(J)$, $i = 1,2,3$ и

$$a_2(t)e^{\int_c^t a_1(s)ds} \leq K + \int_c^t a_3(\tau)e^{\int_c^\tau a_1(s)ds} d\tau \leq 0$$

для $t \in J$, где K неположительная постоянная, то дифференциальное уравнение (1) неосцилляционное в интервале J.

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BEMERKUNG ZUR OSZILLATIONSFÄHIGKEIT DER LÖSUNGEN DER GLEICHUNG $y^{(IV)} + A(x)y' + B(x)y = 0$

JURAJ MAMRILLA, Bratislava

In der Arbeit [1] wurde folgender Satz (Satz 1) bewiesen: Es sei $A', B \in C(\langle x_0, \infty)), x_0 > 0$ und es sei

$$(*) \qquad \int^{\infty} \frac{\int_{x_0}^{x} A v^2 \, dt}{v^2} \, dx = \infty,$$

wo $v = v(x)$ die Lösung der Gleichung

$$(**) \qquad 4v''' - Av = 0$$

ist, welche in x_0 den Anfangsbedingungen $v(x_0) = v'(x_0) = 0$, $v''(x_0) = 1$ entspricht. Ausserdem sei

$$(***) \qquad B - \frac{3}{4} \frac{(Av)'}{v} \geq \frac{2}{x^3} - \frac{4}{x^2} \frac{v'}{v} + \frac{6}{x} \frac{v''}{v}.$$

Dann sind alle Lösungen der Gleichung

$$(****) \qquad y^{(IV)} + Ay' + By = 0$$

oszillatorisch für $x > x_0$.

Es entsteht die Frage, für welches $A = A(x) \geq 0$ hat die Differentialgleichung (**) eine Lösung $v = v(x)$ mit doppelter Nullstelle in x_0, dass die Bedingung (*) erfüllt wird.

Wir beweisen folgenden

Satz 1. Es sei $A \in C(\langle x_0, \infty))$ und $A \geq (x \ln x \ldots \ln_n x)^{-1}$, wo $\ln_n x = \ln(\ln_{n-1} x) > 0$ für $x > x_0$ ist. Dann gilt (*).

Beweis. Wir beweisen diese Behauptung indirekt. Die Voraussetzung unseres Satzes sei erfullt und (*) sei ungültig. Mit Rücksicht darauf, dass $v^{-2}\int_{x_0}^{x} Av^2 dt > 0$ für $x > x_0$, wenn dann (*) nicht gilt, muss $\int^{\infty}(v^{-2}\int_{x_0}^{x} Av^2 dt)\, dx < \infty$ sein. Bezeichnen wir $v^{-1}v' = z$ und auch

$$v^{-2}\int_{x_0}^{x} Av^2 dt \equiv (4vv'' - 2v'^2)v^{-2} \equiv 4z' + 2z^2 = \varphi(x).$$

Die Funktionen $z = z(x)$, $\varphi = \varphi(x)$ sind positiv für $x > x_0$ und $\int^{\infty} \varphi \, dx < \infty$.

Erwägen wir jetzt die Differentialgleichung

$$(1) \qquad 4Z' + 2Z^2 = \varphi(x)$$

wo $\varphi(x)$ gerade die oben difinierte Funktion ist. Überprüfen wir die Existenz positiver Lösungen der Gleichung (1). Die Menge solcher Lösungen ist unleer, weil $Z = z$ eine solche Lösung ist. Aus (1) folgt, dass wenn $Z > 0$, $x > x_1 \geq x_0$, dann ist $\lim_{x\to\infty} Z = 0$, $\int^{\infty} Z^2 dx < \infty$. Also auch $\lim_{x\to\infty} z = 0$, $\int^{\infty} z^2 dx < \infty$.

Auf der anderen Seite rechnen wir

$$\lim_{x\to\infty} \frac{\left[x\ln x \ldots \ln_n x\right] \int_{x_0}^{x} A v^2\, dt}{v^2} =$$

$$= \lim_{x\to\infty} \frac{\left[x\ln x \ldots \ln_n x\right]' \int_{x_0}^{x} A v^2 dt + \left[x\ln x \ldots \ln_n x\right] A v^2}{2 v v'} \geqslant$$

$$\geqslant \lim_{x\to\infty} \frac{\left(x\ln x \ldots \ln_n x\right) A}{2} \frac{v}{v'} \geqslant \lim_{x\to\infty} \frac{1}{2 z(x)} = \infty .$$

Also existiert die Konstante $L > 0$ derart, dass für $x > x_3 \geqslant x_2$

$$v^{-2} \int_{x_0}^{x} A v^2 dt \geqslant L \left(x\ln x \ldots \ln_n x\right)^{-1}$$

gilt. Es folgt daraus, dass $\int^{\infty} \left(v^{-2} \int_{x_0}^{x} A v^2 dt\right) dx = \infty$, was ein Widerspruch mit der Voraussetzung ist. Damit ist der Satz bewiesen.

Die Voraussetzung (*) in der Arbeit [1] wird nur dazu benutzt, damit gezeigt werden kann, dass die Gleichung

$$(3) \qquad v y'' - 2 v' y' + 3 v'' y = z,$$

wo $v = v(x)$ die Lösung der Gleichung (**) mit doppelter Nullstelle in x_0 ist und $z = z(x)$ eine negative und konkave Funktion für $x > x_0$ ist, keine positive Lösung $y = y(x)$ für $x > x_1 \geqslant x_0$ hat. Nach Einführung der Substitution $y = uv$ in die Gleichung (3) erhalten wir

$$(4) \qquad u'' + \frac{4vv'' - 2v'^2}{v^2}\,u = \frac{z}{v^2}, \quad x > x_1 \geq x_o .$$

Bezeichnen wir $A_1 = (4vv'' - 2v'^2)v^{-2}$, $f = zv^{-2}$. Es ist leicht ersichtlich, dass wenn $A_1 \geq 0$, $\int_\infty A_1 dx = \infty$, $f \leq 0$ ist, dann hat die Gleichung (4) und auch die Gleichung (3) auf (x_1, ∞) keine positive Lösung. Diese Behauptung behält ihre Gültigkeit auch dann, wenn wir voraussetzen werden, dass die Gleichung

$$(5) \qquad u'' + A_1 u = 0, \quad A_1 \geq 0$$

oszillatorisch ist (alle ihre Lösungen sind oszillatorisch auf $< x_1, \infty)$) und $f \leq 0$ ist. Tatsächlich, wenn eine positive Lösung $u = u_1(x) > 0$, $x > x_1$ der Gleichung (4) existierte, dann würde

$$u_1'' + \left(A_1 - \frac{f}{u_1}\right)u_1 = 0$$

gelten, d.h. die Gleichung

$$(6) \qquad V'' + A_2 V = 0, \qquad A_2 = A_1 - \frac{f}{u_1} \geq A_1$$

hat eine nichtoszillatorische Lösung $V = u_1 > 0$. Wie aber aus dem Vergleichssatz hervorgeht, ist dies nicht möglich.

Die Gleichung (5) ist oszillatorisch, wenn $\int x^{1-\alpha} A_1 dx = \infty$, $A_1 \geq 0$ wo $\alpha =$ konst. > 0. Wenn $\alpha = 0$ ist, dann kann die Gleichung (5) bekanntlich schon nicht oszillatorisch sein und die Gleichung (4) könnte auch eine positive Lösung haben, obwohl $f \leq 0$ ist. Diese Bedingung für das Oszillieren geht aus der notwendigen Bedingung für das Nichtoszillieren der Gleichung (5) hervor, welche

$$\int^{\infty} x^{1-\alpha} A_1 dx < \infty \qquad \text{ist} \quad ([2], \text{Seite } 434).$$

Beweisen wir jetzt

Satz 2. Es sei $A \in C\ (<x_o,\infty))$ und $A \geqslant (x^{3-\beta} \ln x \ldots \ldots \ln_n x)^{-1} > 0$, wo $\beta =$ konst. > 0 ist, dann ist $\int^{\infty} x^{1-\alpha} A_1 dx = \infty$ wo $1 > \alpha$ = konst. > 0 der Ungleichheit $\beta > 2\alpha$ entspricht,

Beweis. Verfahren wir analogisch, wie im vorhergehenden Satz. Bezeichnen wir $v^{-1}v' = z$ und

$$x^{1-\alpha} v^{-2}(4vv'' - 2v'^2) = x^{1-\alpha} v^{-2} \int_{x_o}^{x} Av^2 dt = x^{1-\alpha}(4z' + 2z^2) = \mathscr{C}_1(x).$$

Es sei $\int^{\infty} \mathscr{C}_1 dx < \infty$. Nach der Integration der Gleichung $x^{1-\alpha}(4z' + 2z^2) = \mathscr{C}_1$ in $<x_1, x>$ haben wir dann

$$4x^{1-\alpha} z + 2\int_{x_1}^{x} t^{-\alpha} z\,[tz - 2(1-\alpha)]\,dt = 4x_1^{1-\alpha} z(x_1) + \int_{x_1}^{x} \mathscr{C}_1 dx,$$

$$x_1 > x_o.$$

Inwiefern

$$xz = x\,\frac{v'}{v} = x\,\frac{(x-x_o) + \int_{x_o}^{x} (x-t)\frac{A}{4}\,vdt}{\frac{(x-x_o)^2}{2} + \int_{x_o}^{x} \frac{(x-t)^2}{2}\,\frac{A}{4}\,vdt} \geqslant$$

$$> \frac{(x-x_o)^2 + \int_{x_o}^{x} (x-t)^2 \frac{A}{4} v dt}{\frac{(x-x_o)^2}{2} + \int_{x_o}^{x} \frac{(x-t)^2}{2} \frac{A}{4} v dt} = 2 > 2(1-\alpha)$$

ist, konvergiert mit Rücksicht auf $z > 0$ das Integral $\int_{x_o}^{x} t^{-\alpha} z\, [tz - 2(1-\alpha)] dt$ für $x \to \infty$ und auch $\lim x^{1-\alpha} z = a > 0$. Weiter folgt daraus, dass $z = v^{-1} v' < bx^{\alpha-1}$, $a < b$, $b =$ konst > 0 und also $v/v' > b^{-1} x^{1-\alpha}$.

Andererseits

$$\lim_{x \to \infty} (x^{2-\alpha} \ln x \ldots \ln_n x) v^{-2} \int_{x_o}^{x} Av^2 dt \geqq \frac{1}{2} \lim_{x \to \infty} (Ax^{2-\alpha} \ln x \ldots \ln_n x) \frac{v}{v'} >$$

$$> \frac{1}{2b} \lim_{x \to \infty} (A x^{3-2\alpha} \ln x \ldots \ln_n x) \geqq \frac{1}{2b} \lim_{x \to \infty} x^{3-2\alpha} = \infty .$$

Die Konstante $L > 0$ existiert also derart, dass fur $x > x_2 \geqq x_1$

$$\varphi_1 = x^{1-\alpha} v^{-2} \int_{x_o}^{x} Av^2 dt \geqq L(x \ln x \ldots \ln_n x)^{-1} > 0 .$$

gilt. Daraus folgt allerdings, dass $\int^{\infty} \varphi_1 \, dx = \infty$, was mit der Voraussetzung im Widerspruch steht. Damit ist der Satz bewiesen.

Jetzt wollen wir versuchen $v^{-1}v'$, $v^{-1}v''$ abzuschätzen, damit wir die Bedingung (***) nur mit Hilfe von A, A', B ausdrücken können. Aus der Gleichung (**) folgt

$$(7) \qquad v'' = 1 - \int_{x_o}^{x} \frac{A}{4} v\,dt, \quad v' = x - x_o + \int_{x_o}^{x} (x-t)\frac{A}{4} v\,dt,$$

$$v = \frac{(x-x_o)^2}{2} + \int_{x_o}^{x} \frac{(x-t)^2}{2} \frac{A}{4} v\,dt, \quad vv'' - \frac{1}{2} v'^2 = \int_{x_o}^{x} \frac{A}{4} v^2\,dt$$

und mit Rücksicht darauf, dass $v = v(x)$ eine wachsende Funktion ist haben wir

$$\frac{v''}{v} \leqslant \frac{1}{v} + \int_{x_o}^{x} \frac{A}{4}\,dt \leqslant \frac{2}{(x-x_o)^2} + \int_{x_o}^{x} \frac{A}{4}\,dt, \quad x > x_o.$$

Auf der anderen Seite

$$(x-x_o)^2 \frac{v''}{v} = (x-x_o)^2 \frac{1 + \int_{x_o}^{x} \frac{A}{4} v\,dt}{\frac{(x-x_o)^2}{2} + \int_{x_o}^{x} \frac{(x-t)^2}{2}\frac{A}{4} v\,dt} \geqslant 2.$$

Als Ergebnis erhalten wir

$$(8)\quad \frac{2}{(x-x_o)^2} \leqslant \frac{v''}{v} \leqslant \frac{2}{(x-x_o)^2} + \int_{x_o}^{x} \frac{A}{4}\,dt, \qquad x > x_o .$$

Aus der letzten Identität (7) folgt auch

$$0 < \frac{v''}{v} - \frac{1}{2}\left(\frac{v'}{v}\right)^2 \leqslant \int_{x_o}^{x} \frac{A}{4}\,dt$$

und mit Berücksichtigung von (8) haben wir für $v^{-1}v'$ die Abschätzung von oben

$$(9)\quad \frac{2}{x-x_o} < \frac{v'}{v} \leqslant \sqrt{2}\left[\frac{2}{(x-x_o)^2} + \int_{x_o}^{x} \frac{A}{4}\,dt\right]^{1/2} .$$

Die Abschätzung von unten wurde analog wie die Abschätzung von unten für $v^{-1}v''$ gewonnen.

Aufgrund von (8) und (9) können wir die Bedingung (***) durch die Bedingung

$$(10)\qquad B(x) \geqslant B_1(x)$$

ersetzen, wo

$$B_1(x) = \frac{3}{4} A' + \frac{3}{4} A \sqrt{2} \left[\frac{2}{(x-x_0)^2} + \int_{x_0}^{x} \frac{A}{4} dt \right]^{1/2} +$$

$$+ \frac{2}{x^3} - \frac{4}{x^2} \frac{2}{x-x_0} + \frac{6}{x} \left[\frac{2}{(x-x_0)^2} + \int_{x_0}^{x} \frac{A}{4} dt \right]$$

ist.

Nun kann der folgende Satz formuliert werden:

Satz 3'. Es sei $A', B \in C(\langle x_0, \infty))$. Die Funktion $A = A(x)$ erfülle die Voraussetzung des Satzes 2. Die Funktion $B = B(x)$ erfülle die Voraussetzung (10). Dann oszillieren alle Lösungen der Gleichung (****) auf $\langle x_0, \infty)$.

Beweis. Die Gleichung (****) multiplizieren wir mit $v(x)$, wo $v = v(x)$ die Lösung der Gleichung (**) mit doppelter Nullstelle in x_0 ist. Für $x > x_0$ haben wir dann

$$(vy'' - 2v'y' - 3v''y)'' + \left(B - \frac{3}{4} \frac{(Av)'}{v}\right) vy = 0$$

oder auch

$$(3) \qquad vy'' - 2v'y' + 3v''y = z$$

$$(11) \qquad z'' = - \left[B - \frac{3}{4} \frac{(Av)'}{v}\right] vy .$$

Um unsere Behauptung zu beweisen, genügt es zu zeigen, dass die Voraussetzung der Existenz irgendeiner nichtoszillatorischen Lösung $y = \bar{y}(x)$ der Gleichung (****) zu einem Widerspruch führt. Es sei $y = \bar{y}(x) > 0$ für $x > x_1 \geqslant x_o$. Da $\frac{2}{x^3} - \frac{4}{x^2}\frac{v'}{v} + \frac{6}{x}\frac{v''}{v} > 0$ für jedes $A \geqslant 0$, $x > 0$ [1], ist dann die Funktion $\bar{z}(x) = v\bar{y}'' - 2v'\bar{y}' + 3v''\bar{y}$ konkav und ab eines gewissen $x_2 \geqslant x_1$ ist diese positiv oder negativ.

Erwägen wir die Differentialgleichung

$$vy'' - 2v'y' + 3v''y = \bar{z}(x). \tag{3'}$$

Aus dem Satz 2 folgt, dass die Gleichung (3') keine positive Lösung hat, wenn $\bar{z}(x) < 0$ ist. Also mit Rücksicht darauf, dass $y = \bar{y}(x)$ die Lösung von (3') ist, muss für $\bar{z}(x) > k_o > 0$, $x > x_2$, k_o = konst gelten.

Überprüfen wir die Eigenschaften der positiven Lösungen der Gleichung (3') wobei $\bar{z}(x) > k_o > 0$, $x > x_2$.

Wenn wir die Substitution $y = \frac{u}{v}$ in (3') einführen und die so gewonnene Gleichung auf x^{-1} multiplizieren, erhalten wir nach ihrer Integration in $\langle x_2, x\rangle$

$$\frac{u'}{x} - \left(\frac{4}{x}\frac{v'}{v} - \frac{1}{x^2}\right)u + \int_{x_2}^{x}\left(\frac{2}{t^3} - \frac{4}{t^2}\frac{v'}{v} + \frac{6}{t}\frac{v''}{v}\right)u\,dt =$$

$$= k_2 + \int_{x_2}^{x}\frac{\bar{z}}{t}\,dt,$$

wo k_2 = konst ist. Daraus, dass $\int^{\infty}\frac{\bar{z}}{t}dt = \infty$ und $\left(\frac{2}{t^3} - \frac{4}{t^2}\frac{v'}{v} + \frac{6}{t}\frac{v''}{v}\right)u > 0$ für $A(x) \geqslant 0$ ist, folgt $\int^{\infty}\left(\frac{2}{t^3} - \frac{4}{t^2}\frac{v'}{v} + \right.$

$+ \frac{6}{t}\frac{v''}{v}\Big)udt = \infty$. Aus der Gleichung (11) erhalten wir für jede positive Lösung $y(x)$ der Gleichung (3')

$$\bar{z}'(x) = \bar{z}'(x_2) - \int_{x_2}^{x} \Big[B - \frac{3}{4}\frac{(Av)'}{v}\Big] yvdt \leqslant \bar{z}'(x_2) -$$

$$- \int_{x_2}^{x} \Big(\frac{2}{t^3} - \frac{4}{t^2}\frac{v'}{v} + \frac{6}{t}\frac{v''}{v}\Big)udt$$

voraus $\lim_{x\to\infty} \bar{z}'(x) = -\infty$ folgt, was ein Widerspruch mit der Voraussetzung $\bar{z}(x) > k_0$, $x > x_2$ ist.

Um den Beweis zu beenden, genügt es einzusehen, dass die Erfüllung der Bedingung (10) die Erfüllung der Bedinung (***) impliziert.
Es gilt auch folgender

Satz 4. Es sei $\int^{\infty} Ax^2\,dx < \infty$, $A \geqslant 0$ und die Bedingung (10) sei erfüllt, dann sind alle Lösungen der Gleichung (****) oszillatorisch auf $\langle x_0, \infty)$.
Da die Bedingung (10) die Voraussetzung (***) impliziert, ist der Satz 4 eine Folgerung des in der Arbeit [1] bewiesenen Satzes 3. Wir führen den Satz 4 deshalb an, weil dieser in gewissem Sinne den Satz 3 im Hinblick auf die Funktion $A(x)$ ergänzt, d.h. im Satz 3 werten die Voraussetzung $A(x) \geqslant (x^{3-\beta}\ln x \ldots \ln_n x)^{-1}$ aus und also auch $\int^{\infty} Ax^2 dx = \infty$ und im Satz 4 wieder die Voraussetzung $\int^{\infty} Ax^2 dx < \infty$. Die zweite Vorausetzung in beiden Sätzen ist dieselbe - Bedingung (10).

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Adresse des Verfassers:
Katedra matematickej analýzy PFUK
816 31 Bratislava
Mlynská dolina
pavilón matematiky

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Súhrn

Poznámka k oscilatoričnosti riešení rovnice $y^{(IV)} + A(x)y' + B(x)y = 0$

JURAJ MAMRILLA

Táto poznámka nadväzuje na prácu [1], ktorá sa zaoberá postačujúcimi podmienkami oscilatoričnosti všetkých riešení diferenciálnej rovnice (****). Hlavný výsledok je obsiahnutý vo vete 3, ktorej predpoklad o koeficiente A(x) je slabší a predpoklad o koeficiente B(x) je konkrétnejší ako vo vete 1 [1].

Резюме

Заметка к колеблемости решений уравнения $y^{(IV)} + A(x)y' + B(x)y = 0$

Ю. МАМРИЛЛА

Эта заметка является продолжением работы [1] касающейся колеблемости всех решений уравнения (****). Главный результат дан теоремой 3, в которой условие налагаемое на коэффициент A(x) более слабое и на коэффициент B(x) более конкретное чем в теореме 1 работы [1].

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REMARKS ON OPEN EVERYWHERE DISCONTINUOUS FUNCTIONS

A. NEUBRUNNOVÁ and T. ŠALÁT, Bratislava

In the paper [1] S. Marcus formulated the following problems:

1. Does the exist an open function without Darboux property?
2. Does there exist an open function without Darboux property in any interval?
3. Does there exist an open everywhere discontinuous function?
4. Does there exist an open everywhere discontinuous function of the second Baire class?

All his questions were answered in an affirmative way. The positive answers can be found in [2], [3], [4].

The present note contains a construction of functions, which are defined by means of subseries and which are open everywhere discontinuous and of the third Baire class. A simple modification of them gives Borel functions which are open everywhere discontinuous and have Darboux property in no interval. Thus a new solution of the questions 2. and 3. of S. Marcus is obtained. Further it is shown that every continuous function can be represented as a uniform limit of open everywhere discontinuous functions.

Under a series of the type (γ) (see [5]) we mean a series $\sum_{k=1}^{\infty} a_k$ with real terms such that

$$\sum_{k;\, a_k < 0} a_k = -\infty, \quad \sum_{k;\, a_k \geq 0} a_k = +\infty .$$

Denote by $C(\sum_1^\infty a_n)$ and $D(\sum_1^\infty a_n)$ the set of all $x \in (0,1>$, $x = \sum_{k=1}^\infty \varepsilon_k(x) \cdot 2^{-k}$ (the dyadic expansion of x, $\varepsilon_k(x) = 0$ or 1, $\varepsilon_k(x) = 1$ for infinitely many k' s) for which the series $\sum_{k=1} \varepsilon_k(x) a_k$ converges or diverges, respectively.

It was shown in [5] that $C(\sum_1 a_n)$ is a $F_{\sigma\delta}$ - set and $D(\sum_1^\infty a_n)$ a $G_{\delta\sigma}$ -set in $(0,1>$.

Lemma. Let $\sum_{n=1}^\infty a_n$ be a series of the type (γ) and let $a_n \to 0$. Then for each real number c the set of all $x \in (0,1>$, for which $\sum_{k=1}^\infty \varepsilon_k(x) a_k = c$, is dense in $(0,1>$.

Proof. Let $x_0 \in (0,1>$ be arbitrarily chosen. Let $\delta > 0$. Choose m, r, $0 \leq r \leq 2^m - 1$ such that $x_0 \in \left(\frac{r}{2^m}, \frac{r+1}{2^m}\right> \subset (x_0 - \delta, x_0 - \delta)$. Let $c_m = \sum_{k=1}^m \varepsilon_k(x_0) a_k$. The series $\sum_{k=m+1}^\infty a_k$ is again of the type (γ). Using Theorem 1,2 from [5] a sequence

$$\eta_{m+1}, \quad \eta_{m+2}, \ldots, \eta_{m+n}, \ldots$$

of numbers 0,1 (with infinitely many η_j's such that $\eta_j = 1$ can be obtained such that $\sum_{k=m+1}^\infty \eta_k a_k = c - c_m$. Choose $y = \sum_{k=1} \varepsilon_k(y) 2^{-k}$ where $\varepsilon_k(y) = \varepsilon_k(x_0)$ $(k = 1, 2, \ldots, m)$ and $\varepsilon_k(y) = \eta_k$ for $k > m$. We have $y \in \left(\frac{r}{2^m}, \frac{r+1}{2^m}\right> \subset (x_0 - \delta, x_0 - \delta)$ and $\sum_{k=1} \varepsilon_k(y) a_k = c$.

Theorem 1. There exists an open Borel everywhere discontinuous on $E_1 = (-\infty, -\infty)$ without Darboux property in any interval.

P r o o f. Put $f(x) = 0$ if the series $\sum_{k=1}^{\infty} \varepsilon_k(x)a_k$ is divergent and $f(x) = \sum_{k=1}^{\infty} \varepsilon_k(x)a_k$ if it is convergent. From Lemma immediately follows that $f(I) = E_1$ for any non-degenerate interval $I \subset (0,1 \rangle$. Hence f is an open function. The discontinuity of f is also an immediate consequence of Lemma.

Let us consider the sets

$$M^a = \{x \in (0,1\rangle \ ; \ f(x) < a\} \text{ and } M_a = \{x \in (0,1\rangle \ ; \ f(x) > a\}.$$

For $a \leq 0$ we have $M^a \subset C\left(\sum_1^{\infty} a_n\right)$, $M^a = \left(M^a \cap X\right) \cup \left(M^a \cap R\right)$, where X and R are the sets of all irrational and rational numbers in $(0,1\rangle$, respectively.

Putting $f_n(x) = \sum_{k=1}^{n} \varepsilon_k(x)a_k$ $(n = 1,2,\dots)$, we get $M^a \cap X = \bigcup_{k=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{n=p}^{\infty} B(k,n)$, where $B(k,n) = \{x \in C\left(\sum_1^{\infty} a_n\right) \cap X;\ f_n(x) \leq a - \frac{1}{k}\}$. Since f_n is continuous in $(0,1\rangle \cap X$, the set $B(k,n)$ is closed in $C\left(\sum a_n\right) \cap X$. Hence $M^a \cap X$ is of the type F_σ in $C \cap X$ $\left(C^1 = C\left(\sum_1^{\infty} a_n\right)\right)$. Since C is of the type $F_{\sigma\delta}$ the set M^a is an $F_{\sigma\delta}$ - set (in $0,1\rangle$), too.

For $a > 0$ we have $M^a = D\left(\sum_1^{\infty} a_n\right) \cup \{x \in C;\ f(x) < a\}$. The set $D\left(\sum a_n\right)$ is a $G_{\delta\sigma}$ -set (in $(0,1\rangle$) -see [5], the set $\{x \in C^1;\ f(x) < a\}$ is a $F_{\sigma\delta}$ - set, hence M^a is a $F_{\sigma\delta\sigma}$ - set (in $(0,1\rangle$).

In an analogous way it can be proved that for each real a the set M_a is a $F_{\sigma\delta\sigma}$ - set(in $(0,1\rangle$).

Hence f is a Baire function (of the third class). To obtain the open everywhere discontinuous function on $(-\infty, +\infty)$ with the property $f(I) = E_1$. for any non-degenerate interval I, it is sufficient to consider the periodic extension of f. We shall use the notation f also for the extension.

Now we define g on $(-\infty, +\infty)$ as follows (see also [4]): $g(x) = f(x)$ if $f(x) \neq 0$ and $g(x) = 1$ if $f(x) = 0$. The function g is open. In fact $g(I) = E_1 - \{0\}$ for any nondegenerate interval I. It is a Borel function as a composition of two Borel functions and is evidently everywhere discontinous. It has not the Darboux property in any interval I because $g(I) = E_1 - \{0\}$. This ends the proof.

Theorem 2. For any continuous function h defined on $(-\infty, +\infty)$ and any $\varepsilon > 0$ there exists an open everywhere discontinuous Borel function $\varphi = \varphi_\varepsilon$ defined on $(-\infty, +\infty)$ such that $|h(x) - \varphi(x)| < \varepsilon$ for each $x \in (-\infty, +\infty)$.

Proof. Let h be continuous on $(-\infty, +\infty)$. Let f be open everywhere discontinuous on $(-\infty, +\infty)$ and such that $f(I) = E_1$ for any non-degenerate interval I. Such a function exists (see the proof of Theorem 1). Define $\varphi = \varphi_\varepsilon$ as follows:

$\varphi(x) = f(x)$ if $f(x) \in (h(x) - \varepsilon, h(x) + \varepsilon)$ and $\varphi(x) = h(x)$ if $f(x) \notin (h(x) - \varepsilon, h(x) + \varepsilon)$.

It follows from the properties of the function f and from the definition of φ that the oscillation of φ is positive in any point x, hence φ is discontinuous in any point.

We prove that φ is an open function. Let $I \subset (-\infty, -\infty)$ be any open interval and let

$$\alpha = \inf_{x \in I} (h(x) - \varepsilon), \qquad \beta = \sup_{x \in I} (h(x) + \varepsilon).$$

From the definition of φ it follows that

$$(1) \qquad \varphi(I) \subset (\alpha, \beta).$$

Now let $y \in (\alpha, \beta)$. We shall prove that $y \in \varphi(I)$. Consider at first the case $y \geq h(x)$ for each $x \in I$. Since $y < \beta$ there exists $x_0 \in I$ such that $h(x_0) + \mathcal{E} > y$, hence $h(x_0) + \mathcal{E} > y \geq h(x_0) > h(x_0) - \mathcal{E}$. The continuity of the functions $h(x) + \mathcal{E}$ and $h(x) - \mathcal{E}$ at x_0 implies that a neighbourhood U of the point x_0 exists such that

$$(2) \qquad h(x) + \mathcal{E} > y > h(x) - \mathcal{E}.$$

for any $x \in U$. Since $I \cap U$ is a non-degenerate interval, the function f takes in $I \cap U$ all the real values hence $y = f(x_1)$ for a point $x_1 \in I \cap U$. Thus according to (2) and the definition of φ we get $\varphi(x_1) = y$. Hence $y \in \varphi(I)$.

The case $y \leq h(x)$ for all $x \in I$ is analogous.

If the function h takes in I both values less and greater than y then there exists a point $x' \in I$ for which $h(x') = y$. We have

$$(3) \qquad h(x') - \mathcal{E} < h(x') = y < h(x') + \mathcal{E}.$$

The continuity of the functions $h(x) - \mathcal{E}$, $h(x) + \mathcal{E}$ guarantees that (3) is valid when x' is substituted by x, where x belongs to a suitably chosen neighbourhood V of x'. Since $I \cap V$ is a non-degenerate interval there exists $x_2 \in I \cap V$ such that $y = f(x_2)$. Using (3) for $x' = x_2$ and the definition of φ we have $y = \varphi(x_2)$. Hence $y \in \varphi(I)$. So $(\alpha, \beta) \subset \varphi(I)$ and this toghether with (1) gives $\varphi(I) = (\alpha, \beta)$.

The definition of φ gives immediately $|h(x) - \varphi(x)| < \mathcal{E}$ for any $x \in E_1$. The only property to be proved is that φ is Borelian. Choosing any open set $G \subset E_1$ we have

$$\varphi^{-1}(G) = \Big[\{x;\ f(x) \geq h(x) + \mathcal{E}\} \cup \{x;\ f(x) \leq h(x) - \mathcal{E}\} \cup \{x;\ h(x) - \mathcal{E} < f(x) < h(x) + \mathcal{E}\}\Big] \cap \varphi^{-1}(G) = \{x; f(x) \geq h(x) + \mathcal{E}\} \cap h^{-1}(G) \cup \{x;\ f(x) \leq h(x) + \mathcal{E}\} \cap h^{-1}(G) \cup \{x;\ h(x) - \mathcal{E} < f(x) < h(x) + \mathcal{E}\} \cap f^{-1}(G).$$

We can see from what is written above that $\varphi^{-1}(G)$ is a finite union of Borel sets. Hence φ is a Borel function. The proof is complete.

Corollary. Any continuous function on $(-\infty, +\infty)$ is a uniform limit of a sequence of open everywhere discontinuous Borel functions.

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Autorova adresa:
Katedra matematickej analýzy PFUK
816 31 Bratislava,
Mlynská dolina
pavilón matematiky

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Súhrn

Poznámky o otvorených všade nespojitých funkciách

A. NEUBRUNNOVÁ a T. ŠALÁT

Pomocou istých výsledkov o čiastočných radoch v práci autori konštruujú príklady všade nespojitých otvorených borelovských funkcií, ktoré nemajú Darbouxovú vlastnosť v žiadnom intervale. Ďalej ukazujú, že každá spojitá reálna funkcia na $(-\infty, +\infty)$ je rovnomernou limitou postupnosti otvorených všade nespojitých borelovských funkcií.

Р е з ю м е

Заметки о открытых всюду разрывных функциях

А. НОЙБРУННОВА, Т. ШАЛАТ

На основе некоторых результатов о частичных рядах в работе строятся примеры всюду разрывных открытых бореловских функций, которые не обладают свойством Дарбу в никаком интервале. Дальше доказывается, что любая непрерывная вещественная функция на $(-\infty, \infty)$ является равномерным пределом последовательности открытых всюду разрывных функций

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ON BILATERAL SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH LAGS

K. SMÍTALOVÁ, Bratislava

Consider the differential equation

$$\dot{x}(t) = f(t, x(t), x(t - \tau(t))) \tag{1}$$

where $\tau(t)$ is a non-negative function. The usual (Cauchy) problem connected with this equation is, roughly speaking, to find a function $x(t)$ which is a solution of (1) for $t \geqslant t_0$ and such that $x(t) = \varphi(t)$ for $t < t_0$, where $\varphi(t)$ is some initial function. Some sufficient conditions for the existence of such solutions can be found e.g. in the monograph [1].

But there is another problem, rather difficult, to find a function $x(t)$ which is, for all t, a solution of (1) and such that $x(t_0) = x_0$. Such $x(t)$ is called to be a bilateral solution of (1). It is easy to find an example of a function $y(t)$, which is a solution of (1) for $t \geqslant t_0$, $y(t) = \varphi(t)$ for $t < t_0$ and such that $y(t)$ cannot be extended to a solution $z(t)$ of (1) defined for all t, see e.g. [1].

Recently has been shown in [1], [2], [3], [4], [5], [6] that in the case when (1) is a linear differential equation with continuous coefficients bounded by a constant M then through each point goes at least one bilateral solution of (1) provided $\tau(t)$ is a continuous function bounded by a small constant depending on M.

Such a condition is not necessary since each linear differential equation with constant lag and constant coefficients has bilateral solutions.

In the present paper we show that bilateral solutions exist for homogeneous linear differential equations with non-negative continuous coefficients and with arbitrary non-negative continuous lags.

In the sequel, a point a_0 will be called a singular point of (1) provided $\inf\{t - \tau(t) \leqslant a_0;\ t \geqslant a_0\} = a_0$.

Now we are able to prove the following

Theorem 1. Let a, b be two successive singular points of the differential equation

$$\dot{x}(t) = M(t)x(t-\tau(t)) \tag{2}$$

(i.e. there is no singular point of (2) between a and b). Assume that $M(t)$, $\tau(t)$ are continuous real functions defined in (a, b). Let $M(t)$ be non-negative and $\tau(t)$ positive, for each $t \in (a,b)$. If (t_0, x_0) is a point of the plane with $t_0 \in (a, b)$. If (t_0, x_0) is a point of the plane with $t_0 \in (a, b)$, then there exists at least one solution x of (2) defined in (a, b) such that $x(t_0) = x_0$.

Proof: If $x_0 = 0$ then $x(t) \equiv 0$ is a desired solution. If a solution $x(t)$ goes through (t_0, x_0) then $-x(t)$ goes through $(t_0, -x_0)$. So we may assume $x_0 > 0$.

Define a sequence $\{t_k\}_{k=1}^{\infty}$ of points from (a, b) as follows: Choose some $t_1 \in (a,b)$, $t_1 < t_0$, put $r_2 = \min\{t - \tau(t);\ t \in [t_1, t_0]\}$. and let $t_2 = (a + r_2)/2$. Clearly $a < t_2 < t_1$. If $t_1 > t_2 > \dots > t_k$ has been chosen let $r_{k+1} = \min\{t-\tau(t);\ t \in [t_k, t_0]\}$ and put $t_{k+1} = (a + r_{k+1})/2$. We have $a < t_{k+1} < t_k$, and $\lim\limits_{k\to\infty} t_k = a$.

For each k, let φ_k be a positive continuous function defined in $[t_{k+1}, t_k]$. Then there exists a solution y_k of the Cauchy problem (2) in $[t_k, t_0]$ with initial function φ_k. It is easy to verify that y_k is positive in $[t_k, t_0]$. Put

$$x_k(t) = \begin{cases} \dfrac{x_0}{y_k(t_0)}\, y_k(t) & \text{for} \quad t \geqslant t_k, \\[2ex] x_k(t_k) & \text{for} \quad t < t_k. \end{cases}$$

Clearly $x_k(t) > 0$ is a solution of (2) in $[t_k, t_0]$ and $x_k(t_0) = x_0$. Moreover, x_k is non-decreasing in $(a, t_0]$ since $\dot{x}_k(t) = M(t)x_k(t - \tau(t)) \geq 0$ for $t \in [t_k, t_0]$. So the functions $\{x_k\}_{k=1}^{\infty}$ are uniformly bounded in $(a, t_0]$ by the constant $\max\{x_k(t);\ t \in (a, t_0]\} = x_0$. The functions $\{x_k\}_{k=1}^{\infty}$ are also equicontinuous on each compact $X \subset (a, t_0]$. To see this it suffices to show that $\{x_k\}_{k=1}^{\infty}$ are equicontinuous on each interval $[t_n, t_0]$. But for $k > n$ and for each $t \in [t_n, t_0]$, $0 \leq \dot{x}_k(t) = M(t)x_k(t - \tau(t)) \leq x_0 . M_n$, where $M_n = \max M(t)$ for $t \in [t_n, t_0]$. So $\{x_k\}_{k=n+1}^{\infty}$ are equicontinuous on $[t_n, t_0]$. Since $x_1, \ldots, x_n$ are continuous, they are also uniformly continuous and consequently, equicontinuous in $[t_n, t_0]$. Thus $\{x_k\}_{k=1}^{\infty}$ are equicontinuous in $[t_n, t_0]$.

Hence there exists a subsequence $\{x_{n_k}(t)\}_{k=1}^{\infty}$ which converges almost uniformly in $(a, t_0]$ (i.e. uniformly on each compact) to a function $z(t)$. Clearly $z(t_0) = x_0$. We show that $z(t)$ is a solution of (2) in $(a, t_0]$. For each $i, j \geq k + 1$ and for each $t \in [t_k, t_0]$ we have

$$|\dot{x}_{n_i}(t) - \dot{x}_{n_j}(t)| = |M(t)||x_{n_i}(t - \tau(t)) - x_{n_j}(t - \tau(t))|$$

and the right-hand side of the equality tends to 0 whenever $i, j \to \infty$ ($M(t)$ is bounded in $[t_k, t_0]$). So $\{x_{n_i}\}_{i=1}^{\infty}$ converges uniformly in $[t_k, t_0]$ to the function $x(t)$. Now in the inequality $|\dot{x}(t) - M(t)x(t - \tau(t))| = |\dot{x}(t) - \dot{x}_{n_i}(t) + M(t)x_{n_i}(t - \tau(t)) - M(t)x(t - \tau(t))| \leq |\dot{x}(t) - \dot{x}_{n_i}(t)| + |M(t)||x_{n_i}(t - \tau(t)) - x(t - \tau(t))|$ the right-hand side tends to 0 whenever $i \to \infty$ which shows that $x(t)$ is a solution of (2) in $(a, t_0]$. Now by a well-known theorem, $x(t)$ can be uniquely extended to the whole (a, b), q.e.d.

It is easy to verify, that the function $x(t)$ which has been constructed in the proof of Theorem 1, is positive in (a, b). Indeed, if $x(v) = 0$ for some $v \in (a, b)$, then $x(t) = 0$ for each $t \in (a, v]$ and hence, by the uniqueness theorem, $x(t) = 0$ for each $t \in (a, b)$. So $x(t)$ is in (a, b) a solution of the differential equation

$$\dot{x}(t) = A(t)x(t), \tag{3}$$

where $A(t) = \dot{x}(t)/x(t)$ is a positive continuous function. Thus we have proved the following

Theorem 2. Let all assumptions of the Theorem 1 be satisfied. Then there exists a positive continuous function $A(t)$ defined in (a,b) such that each solution $x(t)$ of the differential equation (3) is a solution of (2).

The results obtained in the preceeding theorems can be generalised to more general linear differential equations as is shown in the following theorem. The proof of this theorem is omitted since it is similar to the proofs of the preceeding theorems.

Theorem 3. Let $N(t)$, $M_i(t)$, resp. $\tau_i(t)$ be continuous functions defined in (a,b), for $i = 1,2, \dots, k$. Let $N(t) \geq 0$, $M_i(t) \geq 0$, $\tau_i(t) > 0$, and let $t - \tau_i(t) \in (a,b)$, for each $t \in (a,b)$. Let x_0 be a real number and let $t_0 \in (a,b)$. Then there exists a bilateral solution $x(t)$ of the differential equation

$$(4) \qquad \dot{x}(t) = \sum_{i=1}^{k} M_i(t)x(t - \tau_i(t)) + N(t)x(t)$$

defined in (a,b) such that $x(t_0) = x_0$.

Moreover, there exists a positive continuous function $A(t)$ such that each solution of (3) in (a,b) is a solution of (4).

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Autorova adresa:
Katedra matematickej analýzy PFUK
816 31 Bratislava,
Mlynská dolina
pavilón matematiky

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Resume

O obojstranných riešeniach lineárnych diferenciálnych rovníc s oneskoreným argumentom

K. SMÍTALOVÁ

V článku autorka dokazuje existenciu riešenia problému $\dot{x}(t) = M(t)x(t - \tau(t))$, $x(t_0) = x_0$.

Funkcie $M(t) \geq 0$, $\tau(t) \geq 0$ sú spojité na intervale (a, b), kde a, b sú dva susedné singulárne body skúmanej rovnice. Ohraničenosť funkcií $M(t)$, $\tau(t)$ sa nepredpokladá.

Р е з ю м е

О двусторонних решениях линейных дифференциальных уравнений с запаздывающим аргументом

К. СМИТАЛОВА

В статье доказывается существование решения задачи

$$\dot{x}(t) = M(t)x(t - \tau(t)), \; x(t_0) = x_0.$$

Функции $M(t) \geq 0$, $\tau(t) \geq 0$ непрерывные в промежутке (a,b), где a,b - две соседние особые точки рассматриваемого уравнения. Ограниченность функций $M(t)$, $\tau(t)$ не предполагается.

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