ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE

MATHEMATICA XXIX

1974

SLOVENSKÉ PEDAGOGICKÉ NAKLADATEĽSTVO
BRATISLAVA

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA

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Zborník Acta Facultatis rerum naturalium Universitatis Comenianae. Vydáva Slovenské pedagogické nakladateľstvo, Bratislava, Sasinkova 5, č. tf. 645-51. Povolilo Povereníctvo kultúry číslom 2265/56-IV/1.

(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXIX, 1974)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXIX - 1974

ON CERTAIN TYPE OF GENERALIZED RANDOM VARIABLES

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1. Introduction

Generalized random variables which are studied in problems appearing in quantum mechanics are usualy called observables. This note concerns so called simultaneously observable. The definitions will be given in what follows. They are given also in [4], [5]. The obtained results are completion of some results of [4] and [5], concerning the existence of Boolean sub-algebras containing a given set $A \subset L$ and contained in the logic L. An aplication to simultaneous observables is given.

2. Notations and notions

A partially ordered set L with the first element O and the last element 1 will be considered. L is supposed to be complemented i. e. to any $x \in L$ there exist an uniquelly determined element x'.
If $a \leqq b'$ we say that a and b are disjoint or ortogonal. In this case we write $a \perp b$. $a \vee b$ and $a \wedge b$ stand for sup $\{a, b\}$, inf $\{a, b\}$, if the mentioned exist. If $a \perp b$ then $a + b$ instead of $a \vee b$ is used.
L is said to be a σ - logic (cf [4] or [5]) if the following axioms are satisfied.
(1) $(a')' = a$ for any $a \in L$,
(2) $a \leqq b$ implies $b' \leqq a'$,
(3) $a \vee a' = 1$, $a \wedge a' = 0$.
(4) if $a \leqq b$, $a, b \in L$ then there is $c \in L$ such that $a + c = b$,

(5) if $a_1, a_2, \dots a_n \dots$ belong to L and $a_i \perp a_k$ whenever $i \neq k$ then $a_1 + a_2 + \dots + a_n + \dots$ exists.

L is said to be a s-logic if (5) is substituted by (5') $a + b \in L$ for any $a, b \in L$ such that $a \perp b$.

If nothing else is said then the word logic will mean a s-logic.

An observable or L - valued measure x is a σ-homomorphism from the Borel sets on the real line into the σ-logic L, i. e.

(1) $E \cap F = \emptyset$ implies $x(E) \perp x(F)$,

(2) if $E_1, E_2, \dots$ is a sequence of pairwise disjoint elements then $x(E_1 \cup E_2 \cup \dots) = x(E_1) + x(E_2) + \dots$,

(3) $x(\emptyset) = 0$ and $x((-\infty, \infty)) = 1$.

The compatibility of two elements $a, b \in L$ (notation $a \leftrightarrow b$) is defined in the usual way, i. e. the elements $a, b \in L$ are said to be compatible if there exist u, v, z mutually disjoint such that $a = u + z$, $b = v + z$. If $A \subset L$ and $a \leftrightarrow b$ for any $a, b \in A$ then A is said to be compatible. If moreover for any $a, b \in A$, $a \leftrightarrow b$, the elements u, v, z belong to a given $L_0 \subset L$ then a, b are said to be compatible in L_0 (notation $a \overset{L_0}{\longleftrightarrow} b$).

The compatibility of $A \subset L$ in a set $L_0 \subset L$ means that $a \overset{L_0}{\longleftrightarrow} b$ for any $a, b \in A$.

A special type of s-logic and σ-logic are s-class and σ-class respectively (see [1], [2], [3]). Recall that a s-class S is a collection of subsets of a given set Ω which is closed under forming of the union of any two disjoint sets and under the complementations, while the σ-class is a s-class which is closed under forming of countable unions of pairwise disjoint sets. The examples of s- classes (σ-classes) which are not Boolean algebras (σ-algebras) are well known.

3. Generated logics and σ-logics

The problem whether for a set $A \subset L$ where L is a logic (σ-logic) there exists a Boolean algebra (σ-algebra) E such that $A \subset E \subset L$ was many times studied and has applications. In general the answer is not positive. In what follows we shall give under certain assumptions a positive answer to this question and we shall give an application of our solution to so called simultaneous observables. The difference between our results and those given in [4] or [5] can be shortly explained as follows. We use neither the condition that L is a lattice which should be used in [5], nor the condition $a \leftrightarrow b$, $a \leftrightarrow c$ implies $a \leftrightarrow b \vee c$ which is usually assumed.

There are counterexamples given by POOL (see also RAMSAY [4]) showing that without asuming the condition $a \leftrightarrow b$, $a \leftrightarrow c \Rightarrow$ implies $a \leftrightarrow b \vee c$, the positive answer to our problems may not be true even if $A \subset L$ is supposed to be compatible. But we show that the answer is positive when the condition for a set A to be compatible in L is substituded by a slight stronger condition. In case of so called simultaneous observables a result is obtaied using a slightly diferent notion of simultaneous observability (see below).

Theorem 1. Let L be a s-logic (σ-logic). Let $A \subset L$ and L_0 be the s-logic (σ-logic) generated by A. A necessary and sufficient condition for L_0 to be a Boolean sub-algebra (sub σ-algebra) is the compatibility of A in L_0.

Note that the above theorem is an analogy of a theorem appearing in quantum probability spaces (see [3]). There is a reason to formulate and prove such a theorem for logics, since a logic need not be isomorphic with a s- class as easily follows from a result obtained in [2].

Proof of Theorem 1. The necessity is trivial. Let us prove the sufficiency.

Suppose that L and L_0 are s-logics. The proof for σ-logic is analogical. Let $a \in A$ be a fixed element. Denote $S(a)$ the set of all those $b \in L_0$ for which $a \overset{L_0}{\longleftrightarrow} b$.

We shall prove that $S(a)$ is a s-logic.

If $b \in S(a)$ then $a = u + z$, $b = v + z$ where $u, v, z \in L_0$ and u, v, z are mutually ortogonal. Hence $u \leqq v'$, $u \leqq z'$, consequently $u \leqq v' \wedge z' = (v + z)' = b'$. So $b' = u + c$ (according to (4)). It can be easily seen that $c = b' \wedge u'$ (cf [5] proposition 3.2). Thus $c = b' \wedge u' = (b + u)' \in L_0$. So $a = z + u$ $b' = c + u$ where u, z, c belong to L_0 and $z \perp u$, $c \perp u$. Note that $z \perp c$. In fact, $z \leqq b$, hence $b' \leqq z'$ which implies $c = b' \wedge u' \leqq z'$. We have $a \overset{L_0}{\longleftrightarrow} b'$ which means $b' \in S(a)$.

Now let $b_1, b_2 \in S(a)$, $b_1 \perp b_2$. We have to prove $a \overset{L_0}{\longleftrightarrow} b_1 + b_2$. Under the assumption $a = u_1 + z_1$, $a = u_2 + z_2$, $b_1 = v_1 + z_1$, $b_2 = v_2 + z_2$ where u_1, v_1, z_1, are mutually ortogonal and the same holds for u_2, v_2, z_2. Moreover the elements u_i, v_i, z_i $(i = 1, 2)$ belong to L_0.

The elements z_1, z_2 are mutually ortogonal because $z_1 \leqq b_1 \leqq b_2' \leqq z_2'$, hence $z_1 + z_2$ exists.

Evidently $z_1 + z_2 \leqq a$. Thus

$$a = (z_1 + z_2) + a \wedge (z_1 + z_2). \qquad (i)$$

Further the sum $(b_1 \wedge z_1') + (b_2 \wedge z_2')$ exists because of the fact $(b_1 \wedge z_1') \leqq b_1 \leqq b_2' \leqq (b_2 \wedge z_2')'$.

Now we shall prove that

$$[(b_1 \wedge z_1') + (b_2 \wedge z_2')] \perp (z_1 + z_2). \qquad \text{(ii)}$$

In fact, $(b_1 \wedge z_1') \leqq b_1 \leqq b_2' \leqq z_2'$ and evidently $(b_1 \wedge z_1') \leqq \leqq z_1'$. Hence $(b_1 \wedge z_1') \leqq z_1' \wedge z_2' = (z_1 \vee z_2)'$. Analogicaly $(b_2 \wedge z_2') \leqq (z_1 \wedge z_2)'$, so (ii) is true.

Evidently

$$b_1 + b_2 = [(b_1 \wedge z_1') + (b_2 \wedge z_2')] + (z_1 + z_2). \qquad \text{(iii)}$$

Now we shall prove that

$$[(b_1 \wedge z_1') + (b_2 \wedge z_2')] \perp a_1 \wedge (z_1 + z_2)'. \qquad \text{(iv)}$$

We have

$$a \wedge (z_1 \vee z_2)' \leqq a \wedge z_1' = u_1 \leqq v_1' = (b_1 \wedge z_1')'$$

and analogicaly

$$a \wedge (z_1 + z_2)' \leqq (b_2 \wedge z_2')'.$$

Hence $a \wedge (z_1 + z_2)' \leqq [(b_1 \wedge z_1') + (b_2 \wedge z_2')]'$ and (iv) is proved.

From (i) - (iv) it follows

$a \overset{L_o}{\longleftrightarrow} b_1 + b_2$. (Note the important fact that

$z_1 + z_2 \in L_o$, $a \wedge (z_1 + z_2)' \in L_o$

and $[(b_1 \wedge z_1') + (b_2 \quad z_2')] \in L_o$).

Thus $S(a)$ is a s-logic containing A. Hence $S(a) = L_o$. Now let $C \in L_o$ be arbitrary element. Consider the set $S(c)$ of all those $b \in L_o$ for which $b \overset{L_o}{\longleftrightarrow} c$. It can be seen in the same way as above that $S(c)$ is a s-logic. From what was proved it follows that $S(c) = L_o$. Thus for any two a, b, $\in L_o$ we have $b \in S(a)$ hence $a \overset{L_o}{\longleftrightarrow} b$. But $a \wedge b$ exists for any two a, b such that $a \leftrightarrow b$ (see [5], proposition 3. 7). Moreover the result of [5] asserts that $a \wedge b = z$ where $a = u + z$, $b = v + z$ is the representation of a and b following from the compatibility. Since our assumption is the compatibility int L_o, we have $z = a \wedge b \in L_o$.

The existence of $a \vee b \in L_o$ for any a, $b \in L_o$ easily follows from the fact that $a \vee b = (a' \wedge b')'$. Thus L_o is a lattice. The distributivity of L_o easily follows from [4] Lemma 4. Hence L_o is a Boolean algebra.

4. Applications to simultaneous observability

If L is a σ-logic, x, y two observables then x, y are called simultaneously observable if $x(E) \perp y(F)$ for any two Borel sets E, F. A collection $\{x_t\}$ $(t \in T)$ is called a collection of simultaneous observables if any two of x_t $(t \in T)$ are simultaneous observables. The following problem is of interest. Given a collection x_t of simultaneous observables, does there exists an observable x and a collection f_t of Borel measurable functions such that $x_t = f_t(x)$, $(t \in T)$. ($f(x)$ is defined as $f(x)(E) = x[f^{-1}(E)]$ for any Borel set E.)

The answer is positive in case of two observables (see [4][5]). In case of arbitrary collection of observables the proof given by VARADARAJAN works only when an additional condition is given (see [4][5]). In general the answer is not positive (see [4]). We shall show that the VARADARAJAN'S proof works when a stronger definition of simultaneous observability is given. Neither the condition that L is a lattice nor another additional condition will be used.

The observables $x_t(t \in T)$ will be called strongly simultaneously observable if for any $t_1 \neq t_2$ and any Borel sets E, F $x_{t_1}(E) \overset{L_0}{\longleftrightarrow} x_{t_2}(F)$ where L_0 is the σ-logic generated by $\{x_t(E)\}$ $(t \in T)$, E any Borel set.

Theorem 2. Let x_n $n = 1, 2 \ldots$ be a countable collection of strongly simultaneous observables. Then there exists an observable x and a sequence f_n of Borel measurable functions with $x_n(E) = x(f_n^{-1}(E))$ for every Borel set E.

Under our assumptions the idea of VARADARAJAN'S proof may be followed without the assumption of L being a lattice or any other assumption. The proof uses Theorem 1 and the following two lemmas (see [5] Propositions 3. B and 3.15.)

Lemma 1. Let $\mathcal{S}$ be a countably generated Boolean σ-algebra of subsets of a given set S. Then $\mathcal{S}$ is a σ-homomorphic image of the σ-algebra $\mathcal{B}$ of all Borel sets on the real line.

Lemma 2. Let $\mathcal{E}$ be a σ-algebra of subsets of a given set E and h a σ-homomorphism of $\mathcal{E}$ on to a Boolean σ-algebra $\mathcal{F}$ If x is any σ-homomorphism of the system $\mathcal{B}$ of Borel sets into $\mathcal{F}$, then there exists a real valued function f on E such that f is $\mathcal{E}$-measurable and $x(B) = h(f^{-1}(B))$ for all Borel sets B on the real line.

Proof of Theorem 2. Let X_n be the range of x_n $(n = 1, 2, \ldots)$. The X_n's are homomorphic images of $\mathcal{B}$, hence they are σ-subalgebras of L. Under our assumptions $\bigcup_{n=1}^{\infty} X_n$

is compatible in the generated σ-logic L_o, hence according Theorem 1, L_o is a Boolean σ-algebra. L_o is evidently countably generated. By Lemma 1 there exists a σ-homomorphism x from $\mathcal{B}$ on to L_o. By Lemma 2 for any n = 1, 2, ... there exists a Borel measurable function f_n such that $x_n(B) = x(f_n^{-1}(B)$ for any Borel set B.

Added in proofs: While in proofs a paper [6] appeared in which a method is given for proving results similar to the theorem contained in [3].

REFERENCES

[1] Gudder S. P., Quantum probality spaces, Proc. Amer. Math. Soc. 21, 1969, 296-302

[2] Katriňák T., Neubrunn T., On certain generalized probality domains, Mat. časopis 23, 1973, 209-215

[3] Neubrunn T., A note on quantum probalitiy spaces, Proc. Amer. Math. Soc., 25, 1970, 672-675

[4] Ramsay A., A theorem on two comuting observables, J. Math. Mech. 15, 1966, 227-234

[5] Varadarajan V. S., Probability in physic and a theorem on simultaneous observability, Comm. Pure Appl. Math., 15 1962, 189-217, correction loc. cit. 18, 1965

[6] Pettis B. J., On some theorems of Sierpiński on subalgebras of boolen σ-rings, Bul. de la Acad. Pol. des Sciencies, XIX (1971), 563-568

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Received: November 11, 1971, at Publishers May 28, 1973.

(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXIX, 1974)

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXIX – 1974

ON THE ISOMORPHISM OF REGULAR TOURNAMENTS

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In this paper we consider some properties of the regular tournaments. On the basis of these properties it is possible to propose a method of a construction of regular tournaments with m vertices. At the same time the question of their numbers is also solved. The use of this method is laborious, it can be done however with the help of a computer.

Let us consider a quadratic matrix $R = (r_{ij})$ with m rows where $m = 2n+1$, n is a natural number. The elements r_{ij} have the following properties:

1. r_{ij} are not negative integers for $i, j = 1, 2, \ldots, m$
2. $r_{ii} = 0$ for $i = 1, 2, \ldots, m$
3. $r_{ij} + r_{ji} = 1$ for $i, j = 1, 2, \ldots, m$
4. $r_{i1} + r_{i2} + \ldots + r_{im} = n$ for $i = 1, 2, \ldots, m$
5. $r_{1j} + r_{2j} + \ldots + r_{mj} = n$ for $j = 1, 2, \ldots, m$

We will call such a matrix a ρ-matrix.

Let us have an arbitrary regular tournament /the outdegree and indegree of every vertex v is equal to n/ with m vertices, where $m = 2n + 1$, and n is a natural number. The adjacency matrix R of this tournament is a $m \times m$ matrix (r_{ij}) with $r_{ij} = 1$ if $v_i\ v_j$ is an arc of the tournaments and 0 otherwise. It is clear, that between the set of all regular tournaments with m vertices and the set of all ρ-matrices with m rows there exist an one to one correspondence. Theorefore we can examine in the following parts the regular tournaments only with the help of the corresponding ρ-matrices.

Definition 1. By an E-arrangement of an arbitrary matrix we understand the exchange of two arbitrary rows and of the same two columns.

Note 1. By the E-arrangement the properties of the $\wp$ -matrices are preserved.

Definition 2. Two matrices are isomorphic, if one of them after the finite number of E-arrangements gives the other.

Definition 3. Two regular tournaments are isomorphic, if their correspondent $\wp$ -matrices are isomorphic.

Definition 4. We will call the matrix $H_o = R\tilde{R}$, where $\tilde{R}$ is the transposed matrix R, the characteristical matrix of the order of zero of the matrix R, briefly $\mathcal{X}_o$-matrix of the matrix R.

Note 2. If R is a $\wp$ -matrix, then the element h^o_{ij} of its $\mathcal{X}_o$-matrix H_o gives the number of units, contained in the i-th row and the j-th row of the-matrix R in the common columns. We indicate here some properties of the $\mathcal{X}_o$-matrices:

1. The $\mathcal{X}_o$-matrix is a quadratic symmetric matrix with a not negative integer elements where

$$h^o_{ii} = n \qquad \text{for } i = 1, 2, \ldots, m,$$
$$h^o_{ij} < n \qquad \text{for } i \neq j.$$

2. For the elements of the matrix H_o holds

$$h^o_{1j} + h^o_{2j} + \ldots + h^o_{mj} = n^2 \text{ for } j = 1, 2, \ldots, m,$$
$$h^o_{i1} + h^o_{i2} + \ldots + h^o_{im} = n^2 \text{ for } i = 1, 2, \ldots, m.$$

3. Let us consider some $\mathcal{X}$ -matrix R and its corresponding $\mathcal{X}_o$-matrix H_o. After an E-arrangement of the matrix R we obtain a matrix R'. Then we obtain the $\mathcal{X}_o$-matrix $H'_o = R'\tilde{R}'$ from the matrix H_o by the accomplishment of the same E-arrangement.

Definition 5. We will call the matrix $H_i = H_{i-1} H_{i-1}$ where $H_o = R\tilde{R}$, R is a $\wp$ -matrix, the characteristical matrix of the order i, briefly the $\mathcal{X}_i$-matrix of the matrix R.

Note 3. The property 3. of the $\mathcal{X}_o$-matrices have also the $\mathcal{X}_i$-matrices of the higher orders in their relation to the $\mathcal{X}_i$-matrices of a degree smaller.

Definition 6. a/ We call the row-structure of an arbitrary $\wp$ -matrix with m rows a vector

$$p = (p_1 \, p_2 \ldots p_m),$$

where

$$p_k = r_{k1} + r_{k2} + \ldots + r_{km} \text{ for } k = 1, 2, \ldots, m$$

b/ We call the columnial-structure of an arbitrary $\wp$ -matrix with m rows a vector

$$q = (q_1 \; q_2 \ldots q_m)\cdot$$

where $\qquad q_k = r_{1k} + r_{2k} + \ldots + r_{mk}$ for $k = 1, 2, \ldots, m.$

<u>Definition 7.</u> We will call the row-structure of an arbitrary $\mathcal{X}_i$-matrix the matrix

$$A^i = \left\| \begin{matrix} a^i_{10} & a^i_{11} & a^i_{12} \ldots & a^i_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ a^i_{mo} & a^i_{m1} & a^i_{m2} \ldots & a^i_{mn} \end{matrix} \right\|$$

where a^i_{jk} is the equal number to the number of k in the j-th row of the $\mathcal{X}_i$-matrix.

<u>Note 4.</u> The equation $\sum_{j=1}^{n} k \; a^i_{jk} = n^2$ for $i = 1, 2, \ldots, m.$

<u>Definition 8.</u> Two $\mathcal{X}_i$-matrices have the same row-structures, if the row-structure of one of them after some permutation of the rows given the row-structure of the other.

Let us consider now the set of all $\mathcal{X}$ -matrices for some fixed m. Let us decompose this set into a system of disjuncted subsets:

$$M_m = M_{m_1} \cup M_{m_2} \cup \ldots \cup M_{m_k}$$

so that an arbitrary matrix, belonging to some of the subsets is isomorphic with every matrix belonging to some other subset.

Such a decomposition is posssible and unambiguous because the isomorphism of $\wp$ -matrices is a property symmetric and transitive, which can be seen from the definition 2.

<u>Definition 9.</u> We will call the decomposition of the set M_m into a system of subsets $M_{m1}, M_{m2}, \ldots, M_{mk}$ an ν -decomposition if it has the following properties:

1. Two arbitrary matrices from M_m, belonging to the same subset are isomorphic.

2. Two arbitrary matrices from M_m, not belonging to the same subset are not isomorphic.

<u>Definition 10.</u> We will call the ν -basis of the set M_m such a set of $\wp$ -matrices $R_1, R_2, \ldots, R_k$ any two of which do not

belong to the same subset of the ν -decomposition of all the matrices M_m. The number k is the number of all the not isomorphic ρ -matrices for the given m.

Definition 11. We will say, that the ρ -matrix $R = (r_{ij})$ is in normal form when for its elements holds:

1. $r_{21} = r_{31} = \dots = r_{n+1\,1} = 1,$

2. $$\sum_{j=n+2}^{m} r_{2j} \geq \sum_{j=n+2}^{m} r_{3j} \geq \dots \geq \sum_{j=n+2}^{m} r_{n+1\,j},$$

3. $$\sum_{i=2}^{n+1} r_{i\,n+2} \geq \sum_{i=2}^{n+1} r_{i\,n+3} \geq \dots \geq \sum_{i=2}^{n+1} r_{im}.$$

Theorem 1. The set of all ρ -matrices in normal form contains at least one ν -basis of the set of all ρ -matrices M_m.

Proof. It is sufficient to show that the arbitrary ρ -matrix can be transformed to its normal form with the help of E-arrangements. In this case it is clear that the arbitrary ρ -matrix is isomorphic with at least one ρ -matrix in normal form.

a/ We can obtain the property 1/ of ρ -matrices in normal form after the most n E-arrangements, which cause the change of some of the elements $r_{21}, r_{31}, \dots, r_{n+1\,1}$ from zero to one and the change of some of the elements $r_{n+2\,1}, r_{n+3\,1}, \dots, r_{2n+1\,1}$ from one to zero.

b/ Let us consider a subset S of a given ρ -matrix, which is given as follows:

$$S = (r_{ij}) \text{ for } i = 2, 3, \dots, n+1; \; j = n+2, n+3, \dots, m.$$

The row-structure of this submatrix is

$$p = (p_1\, p_2 \dots p_n)$$

where $$p_i = \sum_{j=n+2}^{m} r_{i+1\,j} \quad \text{for } i = 1, 2, \dots, n.$$

The columnial - structure of this submatrix is

$$q = (q_1\, q_2 \dots q_n)$$

where $$q_i = \sum_{j=2}^{n+1} r_{j\,n+i+1} \quad \text{for } i = 1, 2, \dots, n.$$

From the definition of the -matrices in normal form follows that must be fulfilled

$$p_1 \geq p_2 \geq \dots \geq p_n \qquad q_1 \geq q_2 \geq \dots \geq q_n$$

It can be obtained with E-arrangements of the matrix R, which cause either only the exchange of rows, or only the exchange of the columns of the submatrix S. By it the property 1., obtained before, is conserved, because the E-arrangements which cause the permutation of the rows or columns of S, will cause in the first row of the matrix R only the exchange of zeros and of ones between them.

Theorem 2. If two $\wp$-matrices are isomorphic, their corresponding $\mathcal{X}_i$-matrices have equal structures and are isomorphic.

Proof. It follows from the property 3/ of $\mathcal{X}_0$-matrices and from t he note 3., on the $\mathcal{X}_i$-matrices.

Let us have now two regular matrices H_i, H_i', where i is a not negative integer. We choose two m-dimensional vectors b, b' so that for its elements holds:

1. $b_j = b_k$ if and only if the j-th row and the k-th row H_i has an equal structure.

2. $b_j' = b_k'$ if and only if the j-th row and the k-th row H_i' has an equal structure.

3. $b_j = b_k'$ if and only if the j-th row H_i and the k-th row H_i' has an equal structure.

On these suppositions the following theorem is valid:

Theorem 3. Let be the vectors $x = (x_1\ x_2\ \dots\ x_m)$ resp. $x' = (x_1'\ x_2'\ \dots\ x_m')$ a solution of the systems $H_i\ x = b$ resp. $H_i'\ x' = b'$ of linear algebraic equations for the elements of which holds:

1. $x_j \neq x_k$ for $j, k = 1, 2, \dots, m,$

2. $x_j' \neq x_k'$ for $j, k = 1, 2, \dots, m,$

3. for every $j = 1, 2, \dots, m$ there is only one k, such that $x_j = x_k'$. Therefore there is a permutation of the elements of the vector x which gives the vector x'.

Thus if the matrix H_i gives after the same permutation of rows and columns the matrix H_i', then H_i and H_i' must be isomorphic and never in elsewhere.

Proof. a/ Let be H_i, H_i' isomorphic. Therefore after the final number of E-arrangements, executed on H_i we obtain H_i'. The

E-arrangements necessary for it, can be expressed by the permutation of rows and columns. If we take the H_i as a matrix of the system of algebraic equations, then the permutation of rows together with the permutation of the vector of the right sides corresponds to permutation of rows, which does not influence the solution of the system, and the permutation of the columns corresponds to the permutation of the elements of the vector of the solutions of the system. In the case the matrices of two systems and the vectors of right sides are equal, the solutions must be equal also.

b/ Let by a certain permutation of the elements the vector x gives x'. Let after the execution of the same permutation on the rows and columns of the matrix H_i we do not obtain H_i'. Then it is impossible for these matrices to be equal by any other permutation, because the vectors of the solutions x, x' can not be equal by any other permutation and we have a system of linear algebraic equations with a regular matrix.

REFERENCES

[1] Harary F., Graph theory, Addison-Wesley Publishing Company 1969

[2] Kotzig A., Les cycles dans les tournois, Théorie des graphes, Actes des Journées Internationales de l'I.C.C., Roma 1969

[3] Kotzig A., Cycles in a complete graph oriented in equilibrium, Mat. fyz. časopis 16 /1966/, 175-182

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Received: November 29, 1971, at Publishers May 28, 1973

(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXIX, 1974)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXIX – 1974

CERTAIN PROPERTIES OF SOLUTIONS OF THE NONLINEAR DIFFERENTIAL EQUATION $y''' + q(x)y' + r(x)g(y) = f(x)$

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The properties of solutions of a linear differential equation of order 3 without a right hand number have been treated by many authors. The present paper investigates certain properties of solutions of the nonlinear differential equation

$$(1) \qquad y''' + q(x)y' + r(x)g(y) = f(x)$$

with $q(x)$, $r(x)$, $f(x)$ and $g(y)$ continuous functions for all $x \in \langle x_0, \infty)$ and $y \in (-\infty, \infty)$ where $x_0 \in (-\infty, \infty)$. Sufficient conditions are given for the existence of solutions with no zeroes or double zeroes, and some results concerning asymptotic properties of solutions are given. Some analogous results for the solutions of a third-order linear differential equation have been presented in [1]; the papers [2] and [3] contain analogous result for a fourth-order equation.

It will be assumed throughout that for $y \neq 0$ $g(y)$ sgn $y > 0$. Let $G(y) = \int_0^y g(s)\,ds$ and let S be the set of all solutions $y(x)$ of 1 which exist on $\langle x_0, \infty)$. Suppose further that

$$F(x) = y(x)y''(x) - \frac{1}{2} y'^2(x) + \frac{1}{2} q(x)y^2(x)$$

where $y(x)$ is a solution of (1).

Theorem 1. Let $r(x) \in C^1 \langle x_0, \infty)$ and suppose further that for all $x \in \langle x_0, \infty)$

$$r(x) \leq 0, \; r'(x) \geq 0, \; 2q(x) + |f(x)| \leq 0.$$

If $y(x)$ is a solution of (1) such that

(2) $y(x_o) \geqq 0, \quad y'(x_o) \geqq 0, \quad y''(x_o) > 0$

or

(2') $y(x_o) \leqq 0, \quad y'(x_o) \leqq 0, \quad y''(x_o) < 0$

and if

(3) $$y'(x_o)y''(x_o) + r(x_o)G(y(x_o)) - \frac{1}{2}\int_{x_o}^{\infty} |f(t)|dt \geqq 0,$$

then $y(x)$, $y'(x)$ and $y''(x)$ have no zeroes to the right of x_o.

Proof. Let $y(x)$ be a solution of (1) satisfying e.g. (2) and (3); let $x_1 > x_o$ be a point such that $y''(x_1) = 0$, $y''(x) \neq 0$ for $x \in \langle x_o, x_1)$. Multiply (1) by $y'(x)$ and integrate over $\langle x_o, x\rangle$ with $x \in \langle x_o, \bar{x})$ where $\langle x_o, \bar{x})$ is the domain of $y(x)$. This yields

(4) $$y'(x)y''(x) - \int_{x_o}^{x} y''^2(t)dt + \int_{x_o}^{x} q(t)y'^2(t)dt + r(x)G(y(x)) =$$

$$= y'(x_o)y''(x_o) + r(x_o)G(y(x_o)) + \int_{x_o}^{x} r'(t)G(y(t))dt +$$

$$+ \int_{x_o}^{x} f(t)y'(t)dt,$$

and using the hypotheses of the theorem we get

$$\int_{x_o}^{x_1} \left[q(t) + \frac{1}{2}|f(t)|\right] y'^2(t)dt > y'(x_o)y''(x_o) + r(x_o)G(y(x_o)) -$$

$$- \frac{1}{2}\int_{x_o}^{x_1} |f(t)|\, dt,$$

which leads to a contradiction. Hence for all $x \in \langle x_o, \bar{x})$ $y''(x) > 0$ and thus also $y(x) > 0$, $y'(x) > 0$ for $x \in (x_o, \bar{x})$.

The method of proof for $y(x)$ satisfying (2') and (3) is analogous.

Theorem 2. Let $q(x) \in C_1^1 < x_o, \infty)$ and further suppose that for all $x \in < x_o, \infty)$

$$q(x) \leqq 0, \quad r(x) \leqq 0, \quad q'(x) - |f(x)| \geqq 0.$$

If $y(x)$ is a solution of (1) satisfying (2) or (2') and

$$(5) \qquad F(x_o) - \frac{1}{2} \int_{x_o}^{\infty} |f(t)|\, dt \geqq 0$$

then $y(x)$, $y'(x)$ and $y''(x)$ have no zeroes to the right of x_o.

Proof. Again let $y(x)$ be a solution of (1) satisfying (5) and e.g. (2); let $y''(x_1) = 0$, $y''(x) \neq 0$ for $x \in < x_o, x_1)$. Multiplying (1) by $y(x)$ and integration from x_o to $x \in < x_o, \bar{x})$ where $< x_o, \bar{x})$ is the domain of $y(x)$ yields - after some manipulating

$$(6) \qquad F(x) + \int_{x_o}^{x} r(t) g(y(t)) y(t)\, dt \geqq F(x_o) - \frac{1}{2} \int_{x_o}^{x} |f(t)| dt +$$

$$+ \frac{1}{2} \int_{x_o}^{x} \left[q'(t) - |f(t)| \right] y^2(t)\, dt.$$

Hence, because of the hypotheses

$$0 > F(x_o) - \frac{1}{2} \int_{x_o}^{x} |f(t)| dt$$

and this contradicts (5). Thus necessarily $y''(x) > 0$ for all $x \in < x_o, \bar{x})$ and also $y(x) > 0$, $y'(x) > 0$ for $x \in (x_o, \bar{x})$.

The second part of the theorem is proved analogously.

Theorem 3. Let $q(x) \in C^1 < x_o, \infty)$; suppose further that for all $x \in < x_o, \infty)$ and $y \in (-\infty, +\infty)$, $y \neq 0$,

$$(7) \qquad q(x) \leqq 0, \quad \frac{2r(x) g(y)}{y} - q'(x) + |f(x)| \leqq 0.$$

If $y(x)$ is a solution of (1) satisfying (2) or (2') with $y(x_o) > 0$ or $y(x_o) < 0$, and if (5) holds, then $y(x)$ $y'(x)$ and $y''(x)$ have no zeroes to the right of x_o.

Proof. From (1) we obtain

$$(8) \qquad F(x) + \int_{x_o}^{x} \left[\frac{r(t) g(y(t))}{y(t)} - \frac{1}{2} q'(t) + \frac{1}{2} |f(t)| \right] y^2\, t\, dt \geqq$$

$$\geq F(x_o) - \frac{1}{2} \int_{x_o}^{x} |f(t)|\, dt,$$

supposing that $x \in \langle x_o, x_1 \rangle$ with $y''(x_1) = 0$, $y''(x) \neq 0$ for $x \in \langle x_o, x_1)$, hence also $y(x) \neq 0$ for $x \in \langle x_o, x_1 \rangle$. Using (8) for $x = x_1$, we get a contradiction, hence $y''(x) \neq 0$ for $x \in \langle x_o, \bar{x})$ and likewise $y(x) > 0$, $y'(x) > 0$ for $x \in (x_o, \bar{x})$, provided (2) holds and $y(x_o) > 0$.

Theorem 4. Let $q(x) \in C^1 \langle x_o, \infty)$ and suppose that for all $x \in \langle x_o, \infty)$,

$$r(x) \leq 0, \quad q'(x) + |f(x)| \geq 0,$$

Moreover, suppose that in no subinterval of $\langle x_o, \infty)$ $r(x) \equiv 0$ and $q'(x) - |f(x)| \equiv 0$ at the same time. Then a solution $y(x)$ of (1) such that

$$F(x_o) - \frac{1}{2} \int_{x_o}^{\infty} |f(t)|\, dt > 0,$$

has no zero to the right of x_o.

Proof. Let $y(x_1) = 0$, $y(x) \neq 0$, $y(x) \neq 0$ for $x \in (x_o, x_1)$. Using (6) we get for $x = x_1$.

$$-\frac{1}{2} y'^2(x_1) + \int_{x_o}^{x_1} r(t) g(y(t)) y(t)\, dt \geq F(x_o) - \frac{1}{2} \int_{x_o}^{x_1} |f(t)|\, dt +$$

$$+ \frac{1}{2} \int_{x_o}^{x_1} \left[q'(t) - |f(t)| \right] y^2(t)\, dt,$$

which leads to a contradiction if the hypotheses are considered.

Theorem 5. Let $q(x) \in C^1 \langle x_o, \infty)$; suppose further that for all $x \in \langle x_o, \infty)$,

$$r(x) \geq 0, \quad q'(x) + |f(x)| \leq 0,$$

and that in no subinterval of $\langle x_o, \infty)$ $r(x) \equiv 0$ and $q'(x) + |f(x)| \equiv 0$ at the same time. Then a solution $y(x)$ of (1) such that

$$F(x_o) + \frac{1}{2}\int_{x_o}^{\infty} |f(t)|dt < 0,$$

has no double zeroes to the right of x_o.

Proof. For some $x_1 > x_o$ let $y(x_1) = y'(x_1) = 0$. Then (1) yields

$$F(x) + \int_{x_o}^{x} r(t)\, g(y(t))y(t)dt \leqq F(x_o) + \frac{1}{2}\int_{x_o}^{x} |f(t)|dt + \frac{1}{2}\int_{x_o}^{x} \left[q'(t) + |f(t)|\right] y^2(t)\,dt,$$

which is a contradiction for $x = x_1$, as $F(x_1) = 0$.

Remark 1. In the Theorems 1-5 we have implicitly supposed the convergence of $\int_{x_o}^{\infty} |f(t)|dt$. If we suppose the convergence of $\int_{x_o}^{\infty} f^2(t)\,dt$, then, evidently, we can formulate analogous theorems. We have, e.g.

Theorem 3'. In the hypotheses of Theorem 3, change (7) to

$$q(x) \leqq 0, \qquad \frac{2r(x)\,g(y)}{y} - q'(x) + 1 \leqq 0$$

and (5) to

$$F(x_o) - \frac{1}{2}\int_{x_o}^{\infty} f^2(t)\,dt \geqq 0.$$

Then $y(x)$, $y'(x)$ and $y''(x)$ have no zeroes to the right of x_o.

A further consequence of Theorems 1, 2 and 3 is.

Corollary 1. Under the assumptions of Theorem 1, 2 or 3 we have for all $x \in (x_o, \bar{x})$,

$$\operatorname{sgn} y(x) = \operatorname{sgn} y'(x) = \operatorname{sgn} y''(x),$$

and if $y(x) \in S$, furthermore

$$\lim_{x \to \infty} y(x) = \lim_{x \to \infty} y'(x) = \pm\infty.$$

The unboundedness of $y(x)$ and $y'(x)$ can be proved under assumptions weaker than those of Theorem 1. We have.

T h e o r e m 6. Let $r(x) \in C^1 \langle x_0, \infty)$ and for all $x \in \langle x_0, \infty)$ $r(x) \leqq 0$, $r'(x) \geqq 0$, $2q^0(x) + |f(x)| \leqq 0$.

Then any $y(x) \in S$ such that

$$y'(x_0)y''(x_0) + r(x_0)\, G(y(x_0)) - \frac{1}{2}\int_{x_0}^{\infty} |f(t)|dt \geqq$$

$$\geqq K_0 > 0,$$

is unbounded together with its first derivative as $x \to \infty$.

P r o o f. Let $y(x) \in S$ and let the hypotheses be valid. Then (4) yields

$$y'(x)y''(x) \geqq y'(x_0)\, y''(x_0) + r(x_0)\, G(y(x_0)) -$$

$$- \frac{1}{2}\int_{x_0}^{x} |f(t)|\, dt,$$

so that for all $x \geqq x_0$

$$\frac{d}{dx}\, y'^2(x) \geqq 2\, K_0,$$

or

$$(9) \qquad y'^2(x) \geqq y'^2(x_0) + 2\, K_0 (x - x_0),$$

which means that $y'(x) \to \pm \infty$ as $x \to \infty$ and therefore so does $y(x)$.

From (9) we see that $y(x)$ lies either over the curve

$$y = p_1(x) = y(x_0) - \frac{1}{3K_0}\, |y'(x_0)|^3 + \frac{1}{3K_0}$$

$$\left[y'^2(x_0) + 2K_0 (x - x_0)\right]^{\frac{3}{2}}.$$

or under the curve

$$y = p_2(x) = y(x_0) + \frac{1}{3K_0}\, |y'(x_0)|^3 -$$

$$- \frac{1}{3K_o} \left[y'^2 (x_o) + 2K_o (x - x_o) \right]^{\frac{3}{2}}$$

Theorem 7. Let $q(x) \in C^1 \langle x_o, \infty)$ and suppose that for all $x \in \langle x_o, \infty)$

$$q(x) \leqq 0, \quad r(x) \leqq 0, \quad q'(x) - |f(x)| \geqq 0.$$

Suppose further that for $y(x) \in S$

$$F(x_o) - \frac{1}{2} \int_{x_o}^{\infty} |f(t)| \, dt \geqq K_o^* = a \geqq 0,$$

while for $a = 0$

$$b = 2 \left[y(x_o) \, y'(x_o) - K_o^* \, x_o \right] > 0.$$

Then $y(x)$ is unbounded as $x \to \infty$, while for all $x \in \langle x_o, \infty)$ $y(x) \geqq (ax^2 + bx + c)^{\frac{1}{2}}$,

with

$$c = K_o^* x_o^2 + y^2(x_o) - 2y(x_o) \, y'(x_o) x_o.$$

Proof. Using the hypotheses and (6) we get

$$F(x) \geqq F(x_o) - \frac{1}{2} \int_{x_o}^{x} |f(t)| \, dt,$$

so that for all $x \in \langle x_o, \infty)$

$$y(x) y''(x) \geqq K_o^*.$$

By integrating this we get

$$\frac{d}{dx} y^2(x) \geqq 2 K_o^* (x - x_o) + 2y(x_o) \, y'(x_o)$$

which yields the required proof.

Theorem 8. Let $q(x) \in C^1 \langle x_o, \infty)$ and suppose that for all $x \in \langle x_o, \infty)$ and $y \in (-\infty, \infty)$

$$q(x) \geqq k_1 > 0, \quad |g(y)| \leqq k_2 < \infty.$$

If

$$\int_{x_0}^{\infty} |f(t)|dt \leqq A < \infty, \quad \int_{x_0}^{\infty} |r(t)|dt \leqq B < \infty,$$

$$\int_{x_0}^{\infty} \{q'(t)\}_+ dt \leqq C < \infty,$$

then for any solution $y(x)$ of (1) $y'(x)$ and $y''(x)$ are bounded and there exists $k \geqq 0$ such that $y(x)$ lies between the lines

$$y = p_1(x) = k(x - x_0) + y(x_0)$$

and

$$y = p_2(x) = -k(x - x_0) + y(x_0)$$

($\{q'(x)\}_+$ - is the positive part of $q'(x)$).

<u>Proof.</u> Let $y(x)$ be a solution of (1) defined on $\langle x_0, \bar{x})$, $\bar{x} \leq +\infty$. If $y(x)$ is a linear function, then the theorem holds. Now let $y''(x) \neq 0$. Multiply (1) be $y''(x)$ and simplify to obtain

$$(10) \quad y''^2(x) + q(x)y'^2(x) + 2\int_{x_0}^{x} r(t)g(y)y''(t)dt \leqq$$
$$\leqq y''^2(x_0) + q(x_0)y'^2(x_0) + \int_{x_0}^{x} |f(t)|dt +$$
$$+ \int_{x_0}^{x} |f(t)|y''^2(t)dt + \int_{x_0}^{x} q'(t)y'^2(t)dt.$$

Hence

$$y''^2(x) + k_1 y'^2(x) \leq y''^2(x_0) + q(x_0)y'^2(x_0) + A +$$
$$+ k_2B + \int_{x_0}^{x} \left[|f(t)| + k_2|r(t)| + \{q'(t)\}_+\right] \cdot \left[y''^2(t) +\right.$$
$$\left. + y'^2(t)\right]dt.$$

Putting $a_o = \min\{1, k_1\}$, we can use Bellman's lemma (cf. [4]) and the last relation to prove

$$y''^2(x) + y'^2(x) \leqq K \exp\left\{\frac{1}{a_o} \int_{x_o}^{x} \left[|f(t)| + k_2 |r(t)| + \right.\right.$$

$$\left.\left. + \{q'(t)\}_+\right] dt\right\},$$

with K an arbitrary positive constant such that

$$K \geqq \frac{1}{a_o}\left[y''^2(x_o) + q(x_o)\, y'^2(x_o) + A + k_2 B\right].$$

This proves that $y'(x)$ and $y''(x)$ are bounded for $x \in \langle x_o, \bar{x})$. Hence there exists a constant $k \geqq 0$ such that $|y'(x)| \leqq k$ which proves the rest of the theorem.

Corollary 2. Under the assumptions of Theorem 8, $y(x) \in S$.

Proof. From the boundedness of $y'(x)$ and $y''(x)$ and the fact that $y(x)$ lies between two lines passing through the same point $(x_o, y(x_o))$ we deduce that $\bar{x} = +\infty$; thus $y(x) \in S$.

Theorem 9. Under the assumptions of Theorem 8 with the addition that for $x \in \langle x_o, \infty)$

$$r(x) \geqq 0, \qquad q(x) \geqq k_1 > 1,$$

with k_1 a constant, every solution $y(x)$ of (1) is bounded together with its first two derivatives on $\langle x_o, \infty)$.

Proof. It is sufficient to prove the boundedness of $y(x)$ as $x \to \infty$. Multiply (1) by $y(x)$ and integrate to obtain

$$y(x)y''(x) - \frac{1}{2} y'^2(x) + \frac{1}{2} q(x) y^2(x) \leqq F(x_o) +$$

$$+ \frac{1}{2} \int_{x_o}^{x} |f(t)\, dt| + \frac{1}{2} \int_{x_o}^{x} \left[|f(t)| + q'(t)\, y^2(t)\right] dt.$$

Since $y'(x)$ and $y''(x)$ are bounded, this yields

$$[q(x) - 1]\, y^2(x) \leqq K + \int_{x_o}^{x} \left[|f(t)| + q'(t)\right] y^2(t)\, dt$$

with $K > 0$ a constant. Hence $y^2(x)$ is bounded. Since $y(x)$, $y'(x)$ and $y''(x)$ are bounded, evidently $y(x) \in S$. This completes the proof.

Theorem 10. Let $q(x) \in C^1\langle x_0, \infty)$, $r(x) \in C^1\langle x_0, \infty)$, $g(y) \in C^1(-\infty, \infty)$ and suppose that for all $x \in \langle x_0, \infty)$ and $y \in (-\infty, +\infty)$.

$$(11) \quad |r(x)| \leqq k_1 < \infty, \; |g(y)| \leqq k_2 < \infty, \; q(x) - k_1 k_2 \geqq a > 0,$$
$$2r(x) g'(y) + q'(x) \leqq 0.$$

If

$$\int_{x_0}^{\infty} |f(t)| dt \leqq A < \infty, \quad \int_{x_0}^{\infty} |r'(t)| dt \leqq B < \infty,$$

then $y(x) \in S$, $y'(x)$ and $y''(x)$ are bounded on $\langle x_0, \infty)$ and $y(x)$ lies between two lines passing through the point $(x_0, y(x_0))$.

If in addition

$$(12) \quad r(x) \geqq 0, \; q(x) \geqq k_3 > 1, \; \int_{x_0}^{\infty} \{q'(t)\}_+ dt \leqq C < \infty,$$

then $y(x)$ is also bounded on $\langle x_0, \infty)$.

Proof. Consider a solution $y(x)$ of (1) defined on $\langle x_0, \bar{x})$. We shall prove that $y'(x)$ and $y''(x)$ are bounded for $x \in \langle x_0, \bar{x})$, hence $y(x) \in S$.

Since

$$(13) \quad \int_{x_0}^{x} r(t) g(y) y''(t) dt = r(x) g(y) y'(x) - r(x_0) g(y(x_0))$$
$$y'(x_0) - \int_{x_0}^{x} r'(t) g(y) y'(t) dt - \int_{x_0}^{x} r(t) \frac{\partial g}{\partial y} y'^2(t) dt,$$

we can use the hypothese and (10) to deduce

$$y''^2(x) + a y'^2(x) \leqq K + \int_{x_0}^{x} |r'(t) g(y)| dt +$$
$$+ \int_{x_0}^{x} |f(t)| y''^2(t) dt + \int_{x_0}^{x} \{ 2r(t) \frac{\partial g(y)}{\partial y} + q'(t) +$$
$$+ r'(t) g(y)| \} y'^2(t) dt.$$

Hence

$$y''^2(x) + y'^2(x) \leqq \frac{K_1}{a_o} \exp\left[\frac{1}{a_o}\int_{x_o}^{x}\{|f(t)| + k_2|r'(t)|\}\,dt\right],$$

with $a_o = \min\{1, a\}$, $K_1 > 0$, $K_1 \geqq K$. This proves that $y'(x)$ and $y''(x)$ are also bounded. The proof of boundedness of $y(x)$ follows the same line of reasoning as the corresponding part of the proof of Theorem 9.

Theorem 11. Let $q(x) \in C^1\langle x_o, \infty)$, $r(x) \in C^2\langle x_o, \infty)$, $g(y) \in C^1(-\infty, \infty)$ and suppose that for $x \in \langle x_o, \infty)$, $y \in (-\infty, \infty)$ the following relation, as well as (11), holds:

$$|r'(x)| \leqq k_3 < \infty, \quad G(y) \leqq k_4 < \infty.$$

If

$$\int_{x_o}^{\infty}|f(t)|\,dt \leqq A < \infty, \quad \int_{x_o}^{\infty}\{-r''(t)\}_+\,dt \leqq B < \infty.$$

then $y(x) \in S$, $y'(x)$ and $y''(x)$ are bounded on $\langle x_o, \infty)$ and $y(x)$ lies between two lines passing through $(x_o, y(x_o))$. If in addition (12) is satisfied, then $y(x)$ is also bounded on $\langle x_o, \infty)$.

Proof. It is sufficient to prove that $y'(x)$ and $y''(x)$ are bounded. We use (10) and (13) to deduce

$$y''^2(x) + ay'^2(x) - 2r'(x)G(y(x)) + 2\int_{x_o}^{x} r''(t)\, G(y(t))\,dt \leqq K + \int_{x_o}^{x}|f(t)|y''^2(t)\,dt + \int_{x_o}^{x}\left\{2r(t)\frac{\partial g(y)}{\partial y} + q'(t)\right\}y'^2(t)\,dt,$$

and since $r'(x)$ and $G(y)$ anre bounded, this yields

$$(14)\quad y''^2(x) + ay'^2(x) \leq K_1 + \int_{x_0}^{x} |f(t)\, y''^2(t)\, dt + k_4 \int_{x_0}^{x} \{-2r''(t)\}_+ dt,$$

or

$$y''^2(x) \leq K_2 \exp\left[\int_{x_0}^{x} |f(t)|\, dt\right] \leq K_3 < \infty .$$

Now (14) yields

$$y'^2(x) \leq \frac{1}{a}(K_1 + AK_3 + B k_4),$$

hence $y'(x)$ and $y''(x)$ are bounded.

The boundedness of $y(x)$ is proved analogously as in the proof of Theorem 9.

The following Theorem deals with the oscillatory properties of solutions of (1). We start by proving the following

Lemma. Let $q(x) \in C^1 < x_0, \infty)$ and suppose that for $x \in < x_0, \infty)$ and $y \in (-\infty, \infty)$, $y \neq 0$

$$q(x) \geq 0, \quad q'(x) + |f(x)| - \frac{2r(x)\, g(y)}{y} \leq 0$$

with the understanding that in both cases equality is not identically attained on any subinterval of (x_0, ∞) for any $y \neq 0$.

If $y(x) \in S$ and

$$F(x_0) + \frac{1}{2} \int_{x_0}^{\infty} |f(t)|\, dt = K_0 \leq 0,$$

Then the zeroes of $y(x)$ and $y'(x)$ are separated on (x_0, ∞).

Proof. Evidently it is sufficient to prove that if $y'(x_1) = y'(x_2) = 0$, $y'(x) \neq 0$ for $x \in (x_1, x_2)$, then there exists a point $\xi \in < x_1, x_2 >$ such that $y(\xi) = 0$. For $x \in < x_1, x_2 >$ let $y(x) \neq 0$. Then (1) yields

$$y(x)y''(x) - \frac{1}{2} y'^2(x) \leqq K_o - \frac{1}{2} q(x)y^2(x) +$$
$$+ \frac{1}{2} \int_{x_o}^{x} \left[q'(t) + |f(t)| \right] y^2(t)dt - \int_{x_o}^{x} r(t)g(y)y(t)dt,$$

hence for $x \in \langle x_1, x_2 \rangle$

$$y(x)y''(x) - y'^2(x) \leqq y(x)y''(x) - \frac{1}{2} y'^2(x) \leqq K_o -$$
$$- \frac{1}{2} q(x)y^2(x) + \frac{1}{2} \int_{x_1}^{x} \left[q'(t) + |f(t)| - \frac{2r(t)g(y(t))}{y(t)} \right]$$
$$y^2(t)dt,$$

and therefore

$$\left[\frac{d}{dx} \left[\frac{y'(x)}{y(x)} \right] \leqq \frac{K_o}{y^2(x)} - \frac{1}{2} q(x) + \frac{1}{2y^2(x)} \int_{x_1}^{x} q'(t) + \right.$$
$$+ |f(t)| + \frac{2r(t)g(y(t))}{y(t)} \quad y^2(t)dt.$$

Integrate from x_1 to x_2 to obtain a contradiction. This proves the existence of at least one $\xi \in \langle x_1, x_2 \rangle$ such that $y(\xi) = 0$. Evidently there is exactly one point ξ with this property.

Theorem 12. Suppose that, in addition to the hypotheses of the Lemma, we have

$$q'(x) \leqq 0, \quad r(x) \geqq 0, \quad \int_{x_o}^{\infty} r(t)dt = +\infty,$$

and let $g(y)$ be a nondecreasing function for $y \in (-\infty, \infty)$. Then either $y(x) \in S$ on $\langle x_o, \infty)$ is oscillatory or $\lim_{x \to \infty} y(x) = 0$.

Proof. Suppose that $y(x) \in S$, the hypotheses hold and $y(x)$ is not oscilatory. We shall prove that in that case $\lim_{x \to \infty} y(x) = 0$. Five cases will have to be considered:

I. $y'(x)$ is oscillatory on $\langle x_o, \infty)$, i.e. for every $a \geqq x_o$ there exists $\xi \in (a, \infty)$ such that $y'(\xi) = 0$. According to the lemma, $y(x)$ is likewise oscillatory.

II. $y(x) > 0$, $y'(x) \geqq 0$ for $x \in \langle x_1, \infty)$, $x_1 \geqq x_o$. From (1) we deduce

(15) $y''' = -q(x)y'(x) - r(x)g(y(x)) + f(x),$

hence

$$y'''(x) \leqq - r(x)g(y(x_1)) + f(x).$$

Since $\int_{x_1}^{\infty} |f(t)|\,dt$ is convergent, this yields

$$y''(x) \leqq y''(x_1) + A - g(y(x_1)) \int_{x_1}^{x} r(t)\,dt,$$

with A a constant. Thus $y'(x) \longrightarrow -\infty$ as $x \longrightarrow \infty$ which is a contradiction.

III. $y(x) < 0,\ y'(x) \leqq 0$ for $x \in \langle x_1, \infty),\ x_1 \geqq x_o.$

We use (15) to prove that

$$y'''(x) \geqq - r(x)g(y(x)) + f(x),$$

hence

$$y''(x) \geqq y''(x_1) - A - g(y(x_1)) \int_{x_1}^{x} r(t)\,dt$$

so that $y'(x) \longrightarrow +\infty$ for $x \longrightarrow \infty.$

IV. $y(x) > 0,\ y'(x) \leqq 0$ for $x \in \langle x_1, \infty),\ x_1 \geqq x_o.$ Let $\lim_{x \to \infty} y(x) = C$ We shall prove that $c = 0$. For supposing that $c > 0$, we can use (15) to prove

$$y'''(x) \leqq - q(x)y'(x) - r(x)g(c) + f(x)$$

and therefore

$$y''(x) \leqq y''(x_1) + q(x_1)y(x_1) + A - g(c) \int_{x_1}^{x} r(t)\,dt$$

and considering that $g(c) > 0$, this means that $y(x) \longrightarrow -\infty$ for $x \longrightarrow \infty$ and therefore likewise $y(x) \longrightarrow -\infty$ as $x \longrightarrow \infty$ - a contradiction. Hence $\lim y(x) = 0.$

V. $y(x) < 0,\ y'(x) \geqq 0$ for $x \in \langle x_1, \infty),\ x_1 \geqq x_o.$ If $\lim_{x \to \infty} y(x) = c$, we follow the same line of reasoning as before to show that $c < 0.$

We have proved that either $y(x)$ is oscillatory of $\lim_{x \to \infty} y(x) = 0$ thus completing the proof.

Remark 2. It is again clear from the proofs that analogous theorems can be formulated if $\int_{x_0}^{\infty} f^2(t)dt$ is convergent.

REFERENCES

[1] Greguš, M., Über die lineare homogene Differentialgleichung dritter Ordnung, Wiss. Z. Univ. Halle, Math. - Nat XII/3, S. 265-286, März 1963

[2] Mamrilla J., O niektorých vlastnostiach riešení lineárnej diferenciálnej rovnice $y^{4} + 2Ay' + [A' + b]y = 0$, Acta F. R. N. Univ. Comen., X., 3., Mathem. 1965

[3] Šoltés P., O niektorých vlastnostiach riešení diferenciálnej rovnice 4. rádu, Spisy Přírodov. fak. Univ. J.E. Purkyně v Brně, č. 518 1970, 429--444

[4] Bellman R., Teorija ustojčivosti rešenij diferencialnych uravnenij, Moskva, 1954

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Recived: December 20, 1971, ad Publishers May 28, 1973

(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXIX, 1974)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXIX - 1974

STRONGLY GEODETIC DIRECTED GRAPHS

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In the paper [1] the so-called strongly geodetic graphs having the following property were studied: any two vertices are connected by exactly one path of the length less than or equal to the diameter.

Our paper is devoted to the study of directed graphs (digraphs) with the mentioned property and to an extension of this problem. The notions used here are in the sense of [3].

I

Definition. Let d be natural. A digraph of diameter d is said to be strongly geodetic if arbitrary two vertices are connected by exactly one directed path of the length not greater than d.

Theorem 1. Any strongly geodetic digraph is regular.

Proof. Let $G = (V_G, E_G)$ be a strongly geodetic digraph of diameter d. Our proof contains four steps.

1. G does not contain any cycle of the length f (f-cycle) for $f \leq d$ (this follows directly from the definition).

2. Every edge of G is contained exactly in one $(d+1)$ - cycle.

Take an edge $ab \in E_G$. Obviously the distance $d(b, a) = d$. The only path of the length d from b to a form with the edge ab a $(d+1)$-cycle including the edge ab. It can be easily seen (from the definition) that this is the only $(d+1)$-cycle containing ab.

3. If $ab \in E_G$ then the outdegree of b $(s_1(b))$ is equal to indegree of a $(s_2(a))$.

Denote $U = \{x \mid d(b, x) = 1\}$. There exists exactly one path P_i of the length $\leq$ d from every $u_i \in U$ to the vertex a. Owing to the definition two different paths P_i and P_j are vertex-disjoint (and hence also edge-disjoint). Afterwards, the indegree of a is not smaller than the outdegree of b. The inverse inequality can be proved analogously.

4. For any $a \in V_G$ the outdegree of a is equal to its indegree.

Any edge ax is contained in exactly one $(d+1)$-cycle C_x (see 2.). Owing to the definition two different cycles C_x and C_y are edge--disjoint. Hence the indegree of a cannot be smaller than its otudegree. The inverse can be proved similarly.

Now, consider two arbitrary vertices $a, b \in V_G$. Since the diameter of G is finite, there exists a path $ac_1c_2 \dots b$. By 3. and 4. we get: $s_1(a) = s_2(a) = s_1(c_1) = \dots = s_1(b) = s_2(b)$.

The proof of our theorem is finished.

Now, we say that a strongly geodetic digraph G is of the type (d, c) if its diameter is d and for any $a \in V_G$ we have: outdegree of a = indegree of a = c.

Theorem 2. For any natural d and infinite cardinal c there exists a strongly geodetic digraph of the type (d, c).

Proof. (It is very similar to that of theorem 4 of [1].) For d = 1 obviously the complete digraph with c vertices is of the type (1, c). Suppose $d \geqslant 2$. Consider the digraph G_0 consisting of c isolated vertices. Connect every pair of vertices of G_0 the distance between which is greater that d (hence all pairs in this case) by a single directed path of the length d consisting of new vertices and edges in such a way that no two new paths have a common inner vertex. The so constructed graph G_1 cannot be strongly geodetic, because it is not regular.

Now, make the same operation on G_1, and so on. In such way we get a sequence of digraphs $G_0, G_1, G_2, \dots$ Denote $G = \bigcup_{i=0}^{\infty} G_i$.

By similar considerations as in the proof of theorem 4 of [1] we get that G is a strongly geodetic digraph of the type (d, c).

The finite strongly geodetic digraphs will be studied in the second part.

II

Let r and d be integers with $0 \leqslant r < d$; by the symbol $G_{d,r}$ denote any finite digraph with the following properties:

a) the diameter of $G_{d,r}$ is d,

b) for every two vertices $x, y \in V_{G_{d,r}}$ there exists exactly one directed walk from x to y of the length not smaller than r and not greater than d,

c) $G_{d,r}$ contains no k-cycle for $1 \leqslant k \leqslant d - r$.

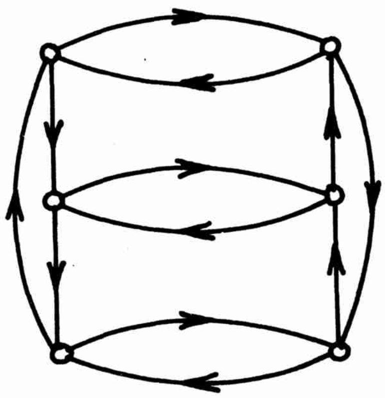
Remark. The notion of $G_{d,0}$ is equivalent with that of finite strongly geodetic digraph.

Theorem 3. 1. any $G_{1,0}$ is a complete digraph without loops ,

2. if $d \geqslant 2$, then $G_{d,0}$ is a $(d+1)$-cycle,

3. $G_{d,r}$ does not exist if $r > 0$, $d \geqslant r+2$.

Remark. We do not study the case $r = d - 1 > 0$. We cannot construct all the graphs of this kind. However, it is easy to show that the number of vertices of the graph $G_{d,d-1}$ is of the form $m^{d-1}(m+1)$, where m is natural. In the Fig. 1 it is shown an example of $G_{2,1}$.



Proof of Theorem 3. The proof of 1. is evident. Hence we can suppose $d \geqslant r+2$. Let A be the adjacency matrix of the graph $G_{d,r}$ with n vertices. Then we have

$$(1) \qquad A^d + A^{d-1} + \dots + A^r = J,$$

where J consists exclusively of 1's. The eigenvalues of J are $\lambda_0 = n$, $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = 0$. Thus if μ is an eigenvalue of A, then we have either

(2) $\mu^d + \mu^{d-1} + \ldots + \mu^r = n,$ or

(3) $\mu^d + \mu^{d-1} + \ldots + \mu^r = 0.$

The matrix A has hence one eigenvalue μ_o, which is a root of (2) and n - 1 eigenvalues satisfying (3). Hence the eigenvalues of A are $\mu_o, \omega_1, \omega_2, \ldots, \omega_{d-r}, 0.$

(4) (where $\omega_p = e^{i\frac{p\pi}{d-r+1}}$, $p = 1, 2, \ldots, d - r$)

with the multiplicities $1, a_1, \ldots, a_{d-r}, a_{d-r+1}$, respectively, whereby a_i are non-negative integers with

(5) $1 + a_1 + \ldots + a_{d-r} + a_{d-r+1} = n.$

Since $G_{d,r}$ contains no k -cycle for $k = 1, \ldots, d - r$, thus $\operatorname{tr}(A^q)$ (i.e. the sum of diagonal elements) is 0 for $q = 1, 2, \ldots, d - r$. Thus we have

(6) $\mu_o^q + a_1\omega_1^q + \ldots + a_{d-r}\omega_{d-r}^q = 0$, for $q = 1, \ldots, d - r$.

Let us consider (6) as a system of d - r linear equations for $a_1, \ldots, a_{d-r}$. The determinant of (6) is (Vandermonde)

(7) $$D = \left(\prod_{p=1}^{d-r} \omega_p\right) \prod_{1 \leq q < p \leq d-r} (\omega_p - \omega_q).$$

Obviously $D \neq 0$ and so

(8) $$a_p = -\frac{D_p}{D} \quad \text{for } p = 1, \ldots, d - r,$$

where D_p arises from D in the known manner.

Since for $n > 0$ the number μ_o cannot be a root of (3), using (2), (3), (7) and (8) we get

(9) $$a_p = -\frac{\mu_o}{\omega_p} \prod_{\substack{j=1 \\ j\neq p}}^{d-r} \frac{\mu_o - \omega_j}{\omega_p - \omega_j} = -\frac{\mu_o(u_o^{d-r} + \ldots + \mu_o + 1)}{(\mu_o - \omega_p)\,\omega_p \prod_{\substack{j=1 \\ j\neq p}}^{d-r}(\omega_p - \omega_j)}$$

Suming all equations (6) we obtain

$$(10)\quad \sum_{q=1}^{d-r} \mu_o^q + a_1 \sum_{q=1}^{d-r} \omega_1^q + \dots + a_{d-r} \sum_{q=1}^{d-r} \omega_{d-r}^q = 0.$$

According to (3) for $j = 1, \dots, d-r$ the equality $\sum_{q=1}^{d-r} \omega_j^q = -1$ holds, thus using (5) from (10) it follows

$$(11)\quad \sum_{q=0}^{d-r} \mu_o^q = n - a_{d-r+1} \qquad (\text{put } a_{d+1} = 0).$$

Using geometrical identities we get

$$(12)\quad \omega_p - \omega_q = 2i \sin \frac{(p-q)\pi}{d-r+1}\, e^{i\frac{(p+q)\pi}{d-r+1}}.$$

Thus

$$(13)\quad \omega_p \prod_{\substack{q=1\\ q\neq p}}^{d-r} (\omega_p - \omega_q) = i^{d-r+1}\, e^{i\frac{\pi}{d-r+1}\left(2p + \sum_{q=1}^{d-r}(p+q) - 2p\right)} \prod_{\substack{q=1\\ q\neq p}}^{d-r} 2 \sin \frac{(p-q)\pi}{d-r+1} =$$

$$= i\, e^{-i\frac{p\pi}{d-r+1}}\, t_p,$$

where t_p is a real number. Denote

$$\varepsilon_p = e^{i\frac{p\pi}{d-r+1}} = \cos \frac{p\pi}{d-r+1} + i \sin \frac{p\pi}{d-r+1} = \cos \alpha_p + i \sin \alpha_p.$$

Substituting (11) and (13) into (9) we get

$$a_p = \frac{\mu_o(n - a_{d-r+1})}{i(\varepsilon_p - \mu_o\, \varepsilon_p^{-1})t_p}$$

and from this it follows that

(14) $i(\varepsilon_p - \mu_o \varepsilon_p^{-1})\mu_o^{-1}$ is a real number for all $p = 1, \dots, d - r$. If $\mu_o = a(\cos\varphi + i\sin\varphi)$, then from (14) we get

(15) $(\cos\varphi - a)\cos\alpha_p = \sin\varphi\sin\alpha_p$, for $p = 1, \dots, d - r$.

According to $d - r \geq 2$

$$0 < \alpha_1 < \frac{\pi}{2} \quad \text{and} \quad \alpha_{d-r} = \pi - \alpha_1 \text{ hold.}$$

Putting $p = 1$ and $p = d - r$ in (15) we obtain

(16) $(\cos\varphi - a)\cos\alpha_1 = \sin\varphi\sin\alpha_1$ and

(17) $-(\cos\varphi - a)\cos\alpha_1 = \sin\varphi\sin\alpha_1$, respectively.

This is possible only for $\varphi = 0$ and $a = 1$, what means that $\mu_o = 1$. Substituting into (2)

(18) $n = d - r + 1$.

Now, if $r = 0$, than $n = d + 1$ and for $d \geq 2$ there exists only one graph $G_{d,0}$, namely the $(d + 1)$-cycle (assertion 2.).

If $r > 0$, then $n < d + 1$, but such a graph cannot exist. This completes the proof of theorem 3.

Corollary. G is a finite strongly geodetic digraph if and only if one of the following two cases occurs:

1. G is a complete digraph without loops,
2. G is a cycle.

REFERENCES

[1] Bosák J., Kotzig A., Znám Š., Strongly geodetic graphs, J. Comb. Theory, 5 1968, 170-176

[2] Hoffman A. J., Singleton R. R., On Moore graphs with diameter 2 and 3, IBM J. Res. Develop., 4 1960, 497-504

[3] Harary F., Graph Theory, Addison-Wesley Publishing Company, 1969

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Received: March 16, 1972, at Publishers May 28, 1973

(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXIX, 1974)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXIX – 1974

ON A GENERALIZED DISTRIBUTIVITY IN MODULAR LATTICES

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The aim of this paper is to study, when the generalized distributive identities

(1a) $$a \wedge \bigvee_J a_j = \bigvee_J (a \wedge a_j),$$

(1b) $$a \vee \bigwedge_J a_j = \bigwedge_J (a \vee a_j)$$

hold in a complete modular lattice L. We show that a complete distributive lattice L is Brouwerian if and only if L is meet-continuous. Brouwerian lattice will be characterized by help of sublattices too.

The symbols $\vee$, $\wedge$, $\bigvee_J$, $\bigwedge_J$ denote the lattice-theoretical operations (where the index set J can be finite of infinite) and symbol $||$ denotes uncomparability of elements of a lattice.

For any elements a, a_j $(j \in J)$ of a complete lattice there hold these inequalities $a \wedge \bigvee_J a_j \geqq \bigvee_J (a \wedge a_j)$, $a \vee \bigwedge_J a_j \leqq \bigwedge_J (a \vee a_j)$. It implies, if (1a) does not hold in a complete lattice L, then there exist elements a, $a_j \in L$ $(j \in J)$, such that

(2a) $$a \wedge \bigvee_J a_j > \bigvee_J (a \wedge a_j).$$

If (1b) does not hold in a complete lattice L, then there exist elements a, $a_j \in L$ $(j \in J)$, such that

(2b) $$a \vee \bigwedge_J a_j < \bigwedge_J (a \vee a_j).$$

Lemma 1. Let L be a complete modular lattice and let x, $x_j \in L$, $j \in J_1$. If $x \wedge \bigvee_{J_1} x_j > \bigvee_{J_1} (x \wedge x_j)$ and $J = \{ j \in J_1 : x_j \parallel x \}$, then $J \neq \emptyset$ and

$x \wedge \bigvee_J x_j > \bigvee_J (x \wedge x_j)$.

Proof. If $J = J_1$ the condition is fulfilled. Let $J \neq J_1$ and $J_1 - J = J_2$. Then

$$(3) \qquad x \wedge \bigvee_{J_2} x_j = \bigvee_{J_2} (x \wedge x_j).$$

It implies $J \neq \emptyset$ according to the assumption. With respect to the modularity of the lattice L we get

$$(4) \qquad x \wedge (\bigvee_J x_j \vee \bigvee_{J_2} x_j) = (x \wedge \bigvee_J x_j) \vee (x \wedge \bigvee_{J_2} x_j) >$$

$$> \bigvee_J (x \wedge x_j) \vee \bigvee_{J_2} (x \wedge x_j).$$

Let us denote $a = x \wedge \bigvee_J x_j$, $b = \bigvee_J (x \wedge x_j)$, $c = \bigvee_{J_2} (x \wedge x_j)$. $a = b$ implies $a \vee c = b \vee c$ and by (3) it contradicts (4). Hence $a > b$ because $a \geq b$ obviously holds.

Lemma 2. Let elements x, x_j $(j \in J)$ of a complete lattice L satisfy $x \parallel x_j$ for any $j \in J$. Then

(i) If $x \wedge \bigvee_J x_j > \bigvee_J (x \wedge x_j)$, then $x \wedge \bigvee_J x_j \parallel x_j$ for any $j \in J$.

(ii) If $x \vee \bigwedge_J x_j < \bigwedge_J (x \vee x_j)$, then $x \vee \bigwedge_J x_j \parallel x_j$ for any $j \in J$.

Proof. (i). $x \wedge \bigvee_J x_j \ngeq x_j$ immediately follows from the assumptions. If there exists $j(0) \in J$ such that $x \wedge \bigvee_J x_j < x_{j(0)}$, then $x \wedge \bigvee_J x_j = (x \wedge \bigvee_J x_j) \wedge x_{j(0)} = x \wedge x_{j(0)} \leq \bigvee_J (x \wedge x_j)$ and it contradicts the assumptions.

(ii) It can be proved dually.

Lemma 3. Let elements x, x_j $(j \in J)$ of a complete modular lattice L satisfy $x \| x_j$ for any $j \in J$. Then

(i) If $\bigvee_J (x \wedge x_j) < x$, then $x \| x_j \vee \bigvee_J (x \wedge x_j)$ for any $j \in J$.

(ii) If $\bigwedge_J (x \vee x_j) > x$, then $x \| x_j \wedge \bigwedge_J (x \vee x_j)$ for any $j \in J$.

Proof. (i). $x \not\geq x_j \vee \bigvee_J (x \wedge x_j)$ obviously follows from the assumption. If there exists $j(0) \in J$ such that $x < x_{j(0)} \vee \bigvee_J (x \wedge x_j)$, then $x_{j(0)} \| \bigvee_J (x \wedge x_j)$, $x_{j(0)} \vee x = x_{j(0)} \vee \bigvee_J (x \wedge x_j)$, $x_{j(0)} \wedge x = x_{j(0)} \wedge \bigvee_J (x \wedge x_j)$. Hence the lattice L contains a nonmodular sublattice and it is a contradiction to the supposition.

(ii) It can be proved dually.

Theorem 1. The generalized distributive identity (1a) holds in a complete modular lattice L if and only if L does not contain a set of elements x, x_j $(j \in J)$ such that

(*) for any $j \in J$ $x_j \| x$ and $x \wedge x_i = x \wedge x_j$ for any $i, j \in J$ and $\bigvee_J x_j > x$.

Proof. Let (1a) do not hold in a complete modular lattice L. By Lemma 1, there exist elements y, $y_j \in L$, $j \in J$ such that $y \| y_j$ for any $j \in J$ and $y \wedge \bigvee_J y_j > \bigvee_J (y \wedge y_j)$. Let us denote $y \wedge \bigvee_J y_j = x$. Then $x \| y_j$ for any $j \in J$ by Lemma 2. Further $x > \bigvee_J (y \wedge y_j) = \bigvee_J (x \wedge y_j)$. By Lemma 3, $x \| y_j \vee \bigvee_J (x \wedge y_j)$ for any $j \in J$. Let us denote $y_j \vee \bigvee_J (x \wedge y_j) = x_j$. With respect to the modularity of the lattice L $x \wedge x_j = x \wedge (y_j \vee \bigvee_J (x \wedge y_j)) = (x \wedge y_j) \vee \bigvee_J (x \wedge y_j) = \bigvee_J (x \wedge y_j)$ for any $j \in J$. Further $\bigvee_J x_j = \bigvee_J (y_j \vee \bigvee_J (x \wedge y_j)) = \bigvee_J y_j \vee \bigvee_J (x \wedge y_j) > x$. The converse assertion is obvious.

Lemma 4. If (*) and (2b) holds for elements x, x_j $(j \in J)$, of a complete modular lattice L, then L is not distributive.

Proof. Let us denote $x \vee \bigwedge_J x_j = y$. By Lemma 2, $y \| x_j$ for any $j \in J$. $\bigwedge_J (y \vee x_j) = \bigwedge_J (x \vee x_j) > y$ holds too. Let us denote $x_j \wedge \bigvee_J (y \vee x_j) = y_j$ for any $j \in J$. By Lemma 3, $y_j \| y$ for any $j \in J$. Since $\bigwedge_J x_j \geq x \wedge x_j$ and the lattice L is modular, we get $y \wedge y_j = (x \vee \bigwedge_J x_j) \wedge (x_j \wedge \bigwedge_J (y \vee x_j)) =$

$$= \bigwedge_J x_j \vee (x \wedge x_j \wedge \bigwedge_J (y \vee x_j)) =$$

$$= \bigwedge_J x_j,$$

$$y \vee y_j = (x \vee \bigwedge_J x_j) \vee (x_j \wedge \bigwedge_J (y \vee x_j)) =$$

$$= \bigwedge_J (y \vee x_j) \wedge (x \vee x_j \vee \bigwedge_J x_j) =$$

$$= \bigwedge_J (y \vee x_j).$$

Since $\bigwedge_J y_j = \bigwedge_J x_j < y$ but $y_j \| y$ for any $j \in J$, obviously there exist $j(1), j(2) \in J$ such that $y_{j(1)} \neq y_{j(2)}$. If $y_{j(1)} < y_{j(2)}$, then the elements $\bigwedge_J y_j < y_{j(1)} < y_{j(2)} < \bigwedge_J (y \vee y_j)$, and y form a nonmodular sublattice, which is a contradiction. Hence $y_{j(1)} \| y_{j(2)}$. If $y \wedge (y_{j(1)} \vee y_{j(2)}) = \bigwedge_J x_j$, then the elements $\bigwedge_J x_j < y_{j(1)} < y_{j(1)} \vee y_{j(2)} < \bigwedge_J (y \vee x_j)$ and y form a nonmodular sublattice, which is a contradiction. Therefore $y \wedge (y_{j(1)} \vee y_{j(2)}) > \bigwedge_J x_j = y \wedge y_{j(1)} = y \wedge y_{j(2)} = (y \wedge y_{j(1)}) \vee (y \wedge y_{j(2)})$. This completes the proof.

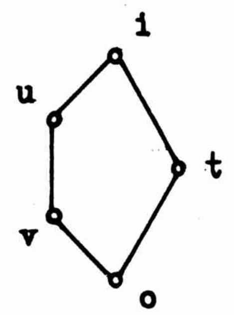
Corollary 1. The identities (1a) and (1b) hold in a complete distributive lattice L if and only if for arbitrary elements x, $x_j \in L (j \in J)$:

$$(5) \quad x \wedge \bigvee_J x_j = \bigvee_J (x \wedge x_j) \iff x \wedge \bigwedge_J x_j = \bigwedge_J (x \vee x_j).$$

Proof. If (1a) does not hold and the condition (5) holds in a complete distributive lattice L, then (by Theorem 1) there exist elements x, $x_j \in L$ $(j \in J)$, satisfying (*) and then with respect to the assumption the condition (2b) holds for these elements too. Hence (by Lemma 4) we get that the lattice L is not distributive. If (1b) does not hold in a complete distributive lattice L, then from dual proposition to Theorem 1 and Lemma 4 it follows that the lattice L is nondistributive. The converse assertion is obvious.

Remark. (1a) holds in a complete lattice L if and only if L is a Brouwerian lattice (cf. [1]) i. e. if for any two elements a, $b \in L$ there exists a relative pseudo-complement b:a (the largest element x with $a \wedge x \leqq b$).

Definition. Let L be a lattice and $a \geqq b$, $a, b \in L$. Let us call the element $x \in L$ such that $a \wedge x = b$ the meet-semicomplement of the element a with respect to b. If exists the greatest meet-semicomplement of a with respect to b, we will denote it by x_{ab}.



In the nonmodular lattice N_5 (its diagram in Figure 1) for any a with $a > b$ there exists the element x_{ab} but for $a = u$, $b = v$, there does not exist the element b:a. It is easy to prove the following proposition:

(6) If for elements $a \geqq b$ of lattice L there exists the element $b{:}a \in L$, then $b{:}a = x_{ab}$.

Lemma 5. Let L be a modular lattice and $a > b$ ($a, b \in L$. Then, there exists the element $x_{ab} \in L$ if and only if there exists the element $b{:}a \in L$ and moreover $x_{ab} = b{:}a$.

Proof. Let the element $x_{ab} \in L$ exist. If $a \wedge x \leqq b$, then with respect to the assumptions we get $a \wedge (b \vee x) = b \vee (a \wedge x) = b$. Hence $x \leqq b \vee x \leqq x_{ab}$. It implies $x_{ab} = b{:}a$. If there exists the element $b{:}a \in L$, then the assertion holds by (6).

It is obvious that $b:a = (a \wedge b) : a$ for any elements a, b of lattice L. By Remark and Lemma 5 it follows:

Corollary 2. The generalized distributive identity (1a) holds in a complete modular lattice L if and only if there exists $x_{ab} \in L$ for any $a, b \in L$ satisfying $a > b$.

A complete lattice L is said to be meet-continuous (cf. [1]) when, for any $x \in L$ and any directed $\{x_d\}$ $(d \in D)$ of elements of L

$$x \wedge \bigvee_D x_d = \bigvee_D (x \wedge x_d).$$

Lemma 6. Let L be a complete distributive lattice. The following conditions are equivalent:

(i) L is a Brouwerian lattice.

(ii) (1a) holds for any $x \in L$ and any chain $\{x_j\}$ $\{j \in J\}$ of elements of L.

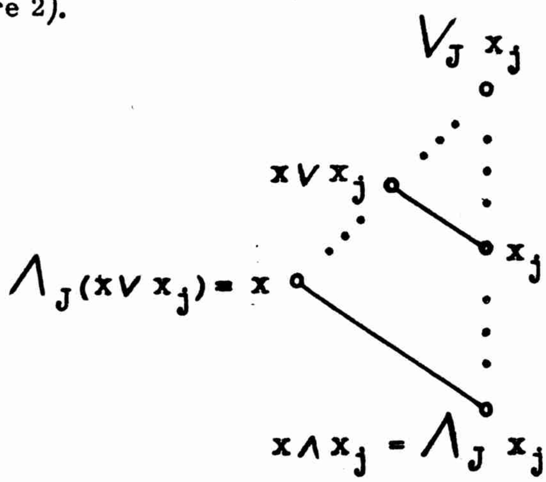
(iii) L is meet-continuous.

Proof. (i) implies (ii) by Remark. O. FRINK proved, that (ii) implies (iii) in any a complete lattice (cf. in [1] a Chap. VIII. Exercises of § § 4-5). (iii) implies (i) by Remark.

Theorem 2. A complete distributive lattice L is Brouwerian if and only if L does not contain an element x and a chain $\{x_j\}$ $(j \in J)$, such that the following condition is fulfilled

$$(7) \quad \begin{array}{l} x \parallel x_j \text{ for any } x_j \neq \bigwedge_J \; (j \in J), \; \bigvee_J x_j > x, \\ x \wedge x_j = \bigwedge_J x_j \text{ for any } j \in J, \; \bigwedge_J (x \vee x_j) = x \end{array}$$

(i. e. if it does not contain a sublattice which is isomorphic to the lattice in Figure 2).



P r o o f. If a complete distributive lattice L is not Brouwerian, then (by Lemma 6) it contains an element $y \in L$ and a chain of elements $y_j \in L$ $(j \in J_1)$ such that

$y \wedge \bigvee_{J_1} y_j > \bigvee_{J_1} (y \wedge y_j)$. Let $J = \{ j \in J_1 : y_j \parallel y \}$.

By Lemma 1 $J \neq \emptyset$ and $y \wedge \bigvee_J y_j > \bigvee_J (y \wedge y_j)$. Let us denote $x_o = y \wedge \bigvee_J y_j$ and for any $j \in J$ $x_j = y_j \vee \bigvee_J (x_o \wedge y_j)$. Obviously the elements x_j $(j \in J)$ form a chain. Likewise as in the proof of Theorem 1 it can be shown

(8) $\quad x_o \parallel x_j$ for any $j \in J$, $x_o \wedge x_j = \bigvee_J (x_o \wedge y_j)$, $\bigvee_J x_j > x_o$ (i. e. for elements x_o, x_j the condition $(*)$ holds).

If $x_o \vee \bigwedge_J x_j < \bigwedge_J (x_o \vee x_j)$, then by Lemma 4, the lattice L is nondistributive which is a contradiction. Therefore $x_o \vee \bigwedge_J x_j = \bigwedge_J x_o \vee x_j$. If $\bigwedge_J x_j < x_o$ we set $x = x_o$. If $\bigwedge_J x_j \parallel x_o$ we set $x = x_o \vee \bigwedge_J x_j$. Then $x \not\leq x_j$ (for any $j \in J$) follows immediate from (8). If there exists $j(0) \in J$ such that $x > x_{j(0)}$, $x_{j(0)} \neq \bigwedge_J x_j$, then the elements $x_o \wedge x_j = x_o \wedge \bigwedge_J x_j < \bigwedge_J x_j < x_{j(0)} < x$, and x_o form a nonmodular sublattice, which is a contradiction. Hence $x \parallel x_j$ for any $x_j \neq \bigwedge_J x_j$ $(j \in J)$. Because the lattice L is modular $x < \bigvee_J x_j$ holds too. Further $x \wedge x_j = (x_o \vee \bigwedge_J x_j) \wedge x_j = (x_o \wedge x_j) \vee \bigwedge_J x_j = \bigwedge_J x_j$, $\bigwedge_J (x \vee x_j) = \bigwedge_J (x_o \vee x_j \vee \bigwedge_J x_j) = \bigwedge_J (x_o \vee x_j) = x_o \vee \bigwedge_J x_j = x$. The converse assertion is obvious.

T h e o r e m 3. A complete lattice L is Brouwerian and dually Brouwerian too if and only if for any element $x \in L$ and any elements $x_j \in L$ $(j \in J)$ the next condition holds:

(9) $\quad \bigvee_J (x \wedge x_j) \vee \bigwedge_J x_j = \bigvee_J x_j \wedge \bigwedge_J (x \vee x_j)$.

P r o o f. Let for elements of a complete lattice L the identity (9) hold. If $J = \{1, 2\}$ the identity (9) has form $(x \wedge x_1) \vee (x \wedge x_2) \vee (x_1 \wedge x_2) = (x_1 \vee x_2) \wedge (x \vee x_1) \wedge (x \vee x_2)$. It implies that according to the assumption is lattice L distributive [1]. If L is not a Brouwerian lattice, then (by Theorem 2) there exist elements x, $x_j \in L$ $(j \in J)$ satisfying (7) and for these elements (9) does not hold. It is a contradiction. If L is not a dually Brouwerian lattice, then with respect to the duality of (9) our assertion holds too. If L is a Brouwerian and dually Brouwerian lattice, then $\bigvee_J (x \wedge x_j) \vee \bigwedge_J x_j = (x \wedge \bigvee_J x_j) \vee \bigwedge_J x_J = \bigvee_J x_j \wedge (x \vee \bigwedge_J x_j) = \bigvee_J x_j \wedge \bigwedge_J (x \vee x_j)$.

REFERENCES

[1] Birkhoff G., Lattice theory, Amer. Math. Soc. Colloq. Publ. vol. 25, New York, N. Y. 1967

[2] Szász G., Einfuhrung in die Verbandstheorie, Akad. Kiadó Verlag der Ungar. Akad. der Wiesenschaften, Budapest 1962

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Received: March 19, 1971, at Publishers May 28, 1973

(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXIX, 1974)

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE

MATHEMATICA XXIX - 1974

ON CERTAIN TRANSFORMATIONS OF SETS IN R_N

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For each element ω belonging to a metric space Ω, certain transformation T_ω of the system $\mathcal{L}$ into $\mathcal{L}$ is associated, where $\mathcal{L}$ denotes the family of all Lebesgue measureable subsets of the set of all real numbers. Let the following conditions be satisfied.

(i) There exists $\omega_0 \in \Omega$ such that for every interval $[a, b]$ and every sequence $\{\omega_n\}$ of elements belonging to Ω and converging to ω_0,

$$\lim_{n\to\infty} \left[\inf T\omega_n ([a, b])\right] = a,$$

$$\lim_{n\to\infty} \left[\sup T\omega_n ([a, b])\right] = b$$

holds.

(ii) If $E, F \in \mathcal{L}$ and $E \subset F$, then for every $\omega \in \Omega$, $T\omega(E) \subset T\omega(F)$.

(iii) If $E \in \mathcal{L}$ and $\omega_n \to \omega_0$ (in Ω), then

$$\lim_{n\to\infty} |T\omega_n(E)| = |T\omega_0(E)| = |E|,$$

where $|A|$ denotes the Lebesgue measure of the set $A \in \mathcal{L}$. Then among other results T. NEUBRUNN and T. ŠALÁT [3] have proved the following

Theorem 1.1. Let Ω and $T\omega\ (\omega \in \Omega)$ have the same meaning as above and let conditions (i), (ii) and (iii) be satisfied. Let $A \in \mathcal{L}$, $|A| > 0$ and $\omega_n \longrightarrow \omega_o$ (in Ω). Then there exists a natural number n_o such that for $n \geqslant n_o$, $A \cap T\omega_n(A) \neq \emptyset$ holds.

The purpose of the present paper is to study some properties of sets in R_N (N-dimensional Euclidean space) under transformation like $T\omega$ which transforms a measurable set in R_N into a measurable set in R_N. In the first part, we introduce three conditions in R_N which are equivalent to the above three conditions when $N = 1$, and extend the result of Theorem 1. 1. of [3]. Here we consider the sets always in R_N. In the second part we extend the idea of universal sets as introduced by Kestelman in [1] and prove some relevatn resulsts.

Now, we explain some notations:

(i) $S[c, \rho]$ stands for the closed sphere with centre c and radius ρ, while $S(c, \rho)$ denotes the open sphere with the same centre and radius. (ii) Difference set $D(A)$ means the set of all vectors $x - y$, where $x, y \in A$, a set in R_N. (iii) $|x|$ denotes the norm of the vector x. (iv) $\bar{A}$ denotes the closure of the set A. (v) A/B denotes the set of all those vectors of the set A which do not belong to the set B. (vi) For $a \in R_N$, $A \subset R_N$ the symbol $\{|a - A|\}$ denotes the set of all numbers $|a - x|$, where $x \in A$.

In this context, we note a well-known result [2] that if $E_1, E_2, E_3, \ldots$ be measurable sets and $E = \bigcap_{k=1}^{\infty} E_k$ such that

$$E_1 \supset E_2 \supset E_3 \supset \ldots, \text{ then } |E| = \lim_{n \to \infty} \left[|E_n| \right]$$

1. An extension of a theorem of T. Neubrunn and T. Šalát

Suppose that to every ω belonging to a metric space there is a certain transformation $T\omega$ which transforms a measurable set in R_N into a measurable set in R_N.

Let the transformations $T\omega$ satisfy the following conditions:

(I) There exists $\omega_o \in \Omega$ such that for every sphere $K = S[a, r] \subset R_N$ and every sequence $\{\omega_n\}$ $(\omega_n \in \Omega)$ converging to ω_o,

$$\lim_{n \to \infty} \left[\sup \{ |a - T\omega_n(K)| \} \right] = r$$

holds.

(II) If E and F be measurable sets in R_N such that $F \subset E$, then for every $\omega \in \Omega$, $T\omega(F) \subset T\omega(E)$.

(III) If E be a measurable set in R_N and $\omega_n \to \omega_o$ (in Ω), then

$$\lim_{n \to \infty} |T\omega_n(E)| = |T\omega_o(E)| = |E|.$$

When $N = 1$, conditions (II) and (III) are the same as (ii) and (iii) respectively of [3]. It is easy to see that the condition (i) of [3] implies our condition (I), but the converse is not true. However, the conditions (I) and (III) together imply the condition (i) of [3] as can be verified.

Therefore, it follows that when $N = 1$ the conditions (i), (ii) and (iii) of [3] together are equivalent to the conditions (I)(II) and (III).

Theorem 1. Let $T\omega_n$ $(\omega_n \in \Omega)$ be the transformations satisfying the conditions (I), (II) and (III), and the sequence $\{\omega_n\}$ converge to ω_o (in Ω). Let A be a set of positive measure in R_N. Then there exists a natural number N_o such that for $n \geq N_o$, $A \cap T\omega_n(A)$ is a set of positive measure.

Proof. Since A is a set of positive measure, there exists a sphere $K_1 = S[a, r]$, such that $|K_1 / A| < \mathcal{E} |K_1|$, where $0 < \mathcal{E} < \frac{1}{2^N + 2}$ and $|K_1| = \delta$, say.

Let $K_2 = S[a, s]$, where $s = \frac{r}{2}$ and

$$\sup \{ |a - T\omega_n(K_2)| \} = d_n .$$

On account of the condition (I), there exists a natural number N_1 such that for $n \geq N_1$, $|d_n - s| < r - s = \frac{r}{2}$. So, for $n \geq N_1$, $T\omega_n(K_2 \cap A) \subset K_1$. Further, according to the condition (III), there exists a natural number N_2 such that for $n \geq N_2$,

$$\Big| |T\omega_n(K_2 \cap A)| - |K_2 \cap A| \Big| < \frac{\delta}{2^{N+1}} .$$

Let $N_o = \max(N_1, N_2)$. Also. let $C_1 = K_1 \cap A$ and $C_2^n = T\omega_{n'}(K_2 \cap A)$.

Now, for $n \geq N_o$, $C_1 \cap C_2^n = K_1 - \left[C_1' \cup C_2'^n \right]$, where dashes denote the complements with respect to K_1.

Then, for $n \geq N_o$,

$$|C_1 \cap C_2^n| \geq |K_1| - \left[|C_1'| + |C_2'^n| \right]$$

$$= |K_1| - \left[|K_1/A| + |K_1| - |T\omega_n(K_2 \cap A)| \right]$$

$$= |T\omega_n(K_2 \cap A)| - |K_1/A|,$$

$$> |K_2 \cap A| - \frac{\delta}{2^{N+1}} - |K_1/A|$$

$$= |K_2| - |K_2/A| - |K_1/A| - \frac{\delta}{2^{N+1}} ,$$

$$\geq |K_2| - |K_1/A| - |K_1/A| - \frac{\delta}{2^{N+1}}$$

$$> |K_2| - 2\varepsilon |K_1| - \frac{\delta}{2^{N+1}}$$

$$= \frac{\delta}{2^N} - 2\,\mathcal{E}\,\delta - \frac{\delta}{2^{N+1}}$$

$$= \left[\frac{1}{2^{N-1}} - 2\mathcal{E} \right] \delta ,$$

$$> 0, \text{ since } 0 < \mathcal{E} < \frac{1}{2^{N+1}} .$$

Thus, for $n \geqslant N_o$, $(K_1 \cap A) \cap T\omega_n\,(K_2 \cap A)$ is a set of positive measure. Hence by applying condition (II) we obtain that $A \cap T\omega_n\,(A)$ is of positive measure for $n \geqslant N_o$.

Corollary 1. If A be a set of positive measure in R_N, then the difference set $D(A)$ contains a sphere $S\,[0, \eta]$, $\eta > 0$.

Proof. Let $\{\omega_n\}$ $(\omega_n \in R_N)$ be a sequence converging to 0. Let $g_{\omega_n}(x) = x + \omega_n$, where $x \in R_N$. Evidently g_{ω_n} satisfies the conditions (I), (II) and (III) in R_N.

So, by Theorem 1 there exists a natural number N_o such that for every $n \geqslant N_o$ there exists a vector x_n for which $x_n + \omega_n \in A$.

Let $|\omega_{N_o}| = \eta$. Then for every $\omega \in S\,[0, \eta]$, there is a vector $x \in A$ such that $x + \omega \in A$.

Thus the difference set $D(A)$ contains a sphere $S\,[0, \eta]$, $\eta > 0$.

Corollary 2. When $N = 1$, the theorem runs as follows:

Let $T\omega_n$ $(\omega_n \in \Omega)$ denote certain transformations satisfying the conditions (i), (ii) and (iii) in R_1.

Let A be a set of positive measure in R_1 and $\{\omega_n\}$ be a sequence beloging to Ω and converging to ω_o (in Ω). Then there exists a natural number N_o such that for $n \geqslant N_o$, $A \cap T\omega_n\,(A)$ is a set of positive measure.

This corollary is a generalisation of Theorem 1. 1 of [3] in the sense that in [3] it has been shown that the set $A \cap T\omega_n\,(A)$ is non-null for $n \geqslant N_o$; but here it follows that under the same conditions $A \cap T\omega_n\,(A)$, for $n \geqslant N_o$, is of positive measure.

Theorem 2. Let the transformations $T\omega_n$ ($\omega_n \in \Omega$ and $\{\omega_n\} \to \omega_o \in \Omega$, $\omega_n \neq \omega_{n+1}$, $n = 1, 2, \ldots$) satisfy the conditions (I), (II) and (III) in R_N. Let A be a set of positive measure in R_N and p be any positive integer. Then there exist mutually distinct $\omega_1, \omega_2, \ldots, \omega_p$ in Ω such that

$$A \cap T\omega_1(A) \cap T\omega_2(A) \cap \ldots \cap T\omega_p(A) \text{ is}$$

a set of positive measure.

Proof. Let $\{\omega_n'\}(\omega_n' \in \Omega)$ be a sequence converging to ω_o (in Ω). Since A is a set of positive measure, by Theorem 1, there exists a natural number N_1 such that for $n \geqslant N_1$, $A \cap T\omega_n'(A)$ is a set of positive measure. Let us choose a natural number $n_1 \geqslant N_1$. Then $A \cap T\omega_{n1}'(A)$ is a set of positive measure. Let $\omega_{n1}' = \omega$, and $A_1 = A \cap T\omega_1(A)$. Then $|A_1| > 0$, $A_1 \subset A$.

Next, let $\{\omega_n''\}$ ($\omega_n'' \in \Omega$) be a sequence converging to ω_o. Since, $|A_1| > o$, by Theorem 1 there is a natural number N_2 such that for $n \geqslant N_2$, $A_1 \cap T\omega_n''(A_1)$ is a set of positive measure. Let n_2 be such that $n_2 \geqslant N_2$ so that $A_1 \cap T\omega_{n2}''(A_1)$ is a set of positive measure. We put $\omega_{n_2}'' = \omega_2$ ($\neq \omega_1$). Let $A_2 = A_1 \cap T\omega_2(A_1)$. Then $|A_2| > o$ and $A_2 \subset A_1 \subset A$.

Continuing this process, we obtain a set A_{p-1} and ω_p ($\omega_1 \neq \omega_2 \neq \ldots \neq \omega_p$) such that $A \supset A_1, \supset \ldots \supset A_{p-1}$; and $A_p = A_{p-1} \cap T\omega_p(A_{p-1})$ is a set of positive measure.

Thus, $A \cap T\omega_1, (A) \cap T\omega_2(A_1) \cap \ldots \cap T\omega_p(A_{p-1})$ is a set of positive measure. Since $A \supset A_1 \supset \ldots \supset A_{p-1}$, it follows that

$$A \cap T\omega_1(A) \cap T\omega_2(A) \cap \ldots \cap T\omega_p(A) \text{ is a set}$$

of positive measure.

This completes the proof.

Now we shall prove a theorem in which the transformations $T\omega_n$ $(\omega_n \in \Omega)$ satisfy one more condition in R_N stated below:

(IV) Let $\{\omega_n\}$ $(\omega_n \in \Omega)$ be a sequence converging to ω_o (in Ω). If A be a closed set, then $T\omega_n(A)$ is closed for sufficiently large n.

Theorem 3. Let the transformations $T\omega_n$ $(\omega_n \in \Omega$ and $\{\omega_n\} \to \omega_o \in \Omega$, $\omega_n \neq \omega_{n+1}$, $n = 1, 2, \ldots)$ satisfy the conditions (I), (II), (III) and (IV) in R_N. Let A be a closed set of positive measure in R_N. Then there exist $\{\omega_k\}$, $\omega_k \in \Omega$ and a vector $\xi \in A$ such that $T\omega_k(\xi) \in A$, $k = 1, 2, \ldots$.

Proof. Let $\{\omega_n'\}$ $(\omega_n' \in \Omega)$ be a sequence converging to ω_o (in Ω). Since $|A| > o$ and $\bar{A} = A$ then by Theorem 1 there exists a natural number n_1 for which $A \cap T\omega_{n1}'(A)$ is a closed set of positive measure. Let $\omega_{n1}' = \omega$, and $A_1 = A \cap T\omega_1(A)$. Then $|A_1| > o$, $A = \bar{A}_1$ and $A_1 \subset A$.

Let $\{\omega_n''\}$ $(\omega_n'' \in \Omega)$ be a sequence converging to ω_o. Since $|A_1| > o$ and $\bar{A}_1 = A_1$, we get a natural number n_2 such that $\omega_{n2}'' \neq \omega_1$ and $A_1 \cap T\omega_{n2}''(A_1)$ is a closed set of positive measure. Put $\omega_{n2}'' = \omega_2$ $(\neq \omega_1)$ and $A_2 = A_1 \cap T\omega_2(A_1)$. Then $|A_2| > 0$, $A_2 = \bar{A}_2$ and $A_2 \subset A_1 \subset A$.

Continuing this process indefinitely we obtain a sequence of closed sets $\{A_n\}$ such that

$$A = A_o \supset A_1 \supset A_2 \supset \ldots \supset A_{n-1} \supset A_n \supset \ldots$$ and

a sequence $\{\omega_k\}$ $(\omega_k \in \Omega)$, $\omega_i \neq \omega_j$ $(i \neq j)$ such that $A_k = A_{k-1} \cap T\omega_k(A_{k-1})$, $|A_k| > o$, $A_k = \bar{A}_k$, $k = 1, 2, \ldots$.

So, there exists a vector $\xi \in A$ such that $T\omega_k(\xi) \in A$, $k = 1, 2, \ldots$.

This completes the proof.

2. A few theorems extending the idea of "Universal sets"

In this part we prove some theorems extending some results on universal sets of [1]. In doing so in place of translation we apply transformation $T\omega$ as defined earlier.

Theorem 4. Let the transformations $T\omega_n$ $(\omega_n \in \Omega)$ satisfy the condition (I) in R_N, where the sequence $\{\omega_n\}$ $(\omega_n \in \Omega)$ converges to ω_o (in Ω) and $S(a, r)$ be a sphere in R_N. If ξ be any vector belonging to the sphere, then $T\omega_n(\xi) \in S(a, r)$ for sufficiently large n.

Proof. Let us consider the sphere $K = S(a, r)$. Let $\xi \in K$. Then if we apply the condition (I) to any closed sphere enclosing ξ and contained in K, then, since K is open, obviously $T\omega_n(\xi) \in K$ for sufficiently large n.
This proves the theorem.

In proving our next theorem we require the following condition:

(V) If $|E| = 0$, then $|T\omega_n(E)| = 0$ where $\{\omega_n\} \to \omega_o$, $\omega_n \in \Omega$, $\omega_o \in \Omega$.

Theorem 5. Let the transformations $T\omega_n$ with their inverses satisfy the conditions (I) and (V). If A be a set (in R_N) which contains almost all the points of a sphere $K = S(a, r)$ then there exists a vector ξ_o in R_N such that $T\omega_n(\xi_o) \in A$ for sufficiently large n.

Proof. Let $X = K \cap \left(\bigcup_{n=1}^{\infty} T^{-1}_{\omega_n}(E)\right)$, where E is a hull subset of K. Then $|X| = 0$ and $A \supset K - E$. Let $\xi_o \in K - X$. So, $\xi_o \bar{\in} X$. In other words, $\xi_o \bar{\in} T^{-1}_{\omega_N}(E)$, $n = 1, 2, \ldots$ i.e. $T\omega_n(\xi_o) \bar{\in} E$, $n = 1, 2, \ldots$. Since K is open, it follows from condition (I) that $T\omega_n(\xi_o) \in K$ for sufficiently large n. So, $T\omega_n(\xi_o) \in K - E$ for sufficiently large n, i.e. $T\omega_n(\xi_o) \in A$ for sufficiently large n.
This proves the theorem.

Corollary. If $\omega_o = 0$ and $g\omega_n = x + \omega_n$, where $x \in R_N$ and $\omega_n \in R_N$ then the conditions of the above theorem are satisfied, and we obtain Theorem 4 of [1].

Theorem 6. Let the transformations $T\omega_n$ with their inverses satisfy the condition (I) and preserve the category, where the sequence $\{\omega_n\}$ $(\omega_n \in \Omega)$ converges to ω_o (in Ω). If A be a set (in R_N) which contains all the points of a sphere $K = S(a, r)$ with the exception of those belonging to a set E of the first category, then there exists a vector $\xi_o \in R_N$ such that $T\omega_n(\xi_o) \in A$ for sufficiently large n.

Proof. The statement "$|X| = 0$" in the proof of Theorem 5 is to be replaced by "X is of the first category". The set K - X is now of the second category and for every ξ_o in K - X we have $T\omega_n(\xi_o) \in A$ for all large n.
This completes the proof.

Corollary. If $\omega_o = 0$ and $g\omega_n = x + \omega_n$, where $x \in R_N$ and $\omega_n \in R_N$ then the conditions of Theorem 6 are satisfied and we obtain Theorem 5 of [1].

Theorem 7. Let the transformations $T\omega_n$ with their inverses satisfy the conditions (I) and (III) in R_N, where the sequence $\{\omega_n\}$ $\omega_n \in \Omega)$ converges to ω_o (in Ω). If A be a set in R_N containing almost all the points of a sphere K = = S (a, r), then there exists a vector ξ_o in R_N and a subsequence $\{\omega_{n_i}\}$ of $\{\omega_n\}$ such that $T\omega_{n_i}(\xi_o) \in A$ for sufficiently large i.

Proof. Let E be a subset of K of measure zero. Let $\eta_1 + \eta_2 + \eta_3 + \dots$ be a convergent series of positive terms. On account of the condition (III)

$$\lim_{n\to\infty} |T^{-1}_{\omega_n}(E)| = |E|.$$

So, for η_1, there exists a natural number n_1 such that

$$\left| |T^{-1}_{\omega_{n1}}(E)| - |E| \right| < \eta_1$$

i. e. $|T^{-1}_{\omega_{n1}}(E)| < \eta_1$.

Similarly for η_2, we can choose a natural number $n_2 > n_1$ such that

$$|T^{-1}_{\omega_{n_2}}(E)| < \eta_2 .$$

In this way, we choose a subsequence $\{\omega_{n_i}\}$ of $\{\omega_n\}$ such that corresponding to η_i

$$\left|T^{-1}_{\omega_{n_i}}(E)\right| < \eta_i ,$$

where $n_1 < n_2 < \cdots < n_i < \cdots$.

Let $X_i = K \cap \{\bigcup_{k=i}^{\infty} T^{-1}_{\omega_{n_k}}(E)\}$, $Q = \bigcap_{i=1}^{\infty} X_i$.

Since, $X_1 \supset X_2 \supset X_3 \supset \ldots$, $|X_i| \to |Q|$.

Now, $|X_i| < \sum_{k=i}^{\infty} \eta_k$ so that $|X_i| \to 0$ and hence $|Q| = 0$.

Let $\xi_0 \in K - Q$. So, $\xi_0 \bar{\in} Q$. Then $\xi_0 \bar{\in} X_{i_0}$ for some natural number i_0. In other words,

$$\xi_0 \bar{\in} T^{-1}_{\omega_{n_i}}(E), \quad \text{for all } i \geqslant i_0 ,$$

$$\text{i. e. } T\omega_{n_i}(\xi_0) \bar{\in} E, \quad \text{for all } i \geqslant i_0 .$$

Since, K is open, on account of the condition (I), it follows that $T\omega_{n_i}(\xi_0) \in K$, for sufficiently large i.

Then $T\omega_{n_i}(\xi_0) \in K - E$, for suffieciently large i and thus the theorem concludes.

Lastly, the author is thankful to Dr. K. C. Ray of the University of Kalyani for his kind help in preparing the paper and to Prof. Dr. T. Šalát for his suggestions towards the improvement of the paper.

REFERENCES

[1] Kestelman H., The convergent sequences belonging to a set, J. Lond. Math. Soc. Vol. XXII (1947), p. 130-135

[2] Natanson I. P., Theory of functions of a real variable, Vol. I Frederick Ungar Publishing Co., New York, (1964), p. 70

[3] Neubrunn T. and Šalát T., Distance sets, Ratio sets and Certain transformations of sets of real numbers, Čas. Pěst. Mat. 94 (1969), p. 381-393

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Received: November 25, 1971, at Publishers May 28, 1973

(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXIX, 1974)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXIX - 1974

ABSTRACT ENTROPY

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In the paper we introduce an abstract scheme including the topological entropy (see [1] and also [2] and [3]) as well as the Kolmogoroff-Sinaj's entropy (see [4], [5]) and also some further invariants. *)

In Theorems 1 - 4 we generalize the corresponding results of [1]. Theorem 5 contains a new result concerning the algebraic entropy introduced in the paper. Finally we present a few problems connected with the notions introduced in the paper.

First recall the definition of the topological entropy. Let X be a topological space, $f : X \longrightarrow X$ a deformation (i. e. a continuous map), P the family of all open finite coverings of X. For $A \in P$ we put

$$H(A) = \log \min \{ \text{card } B;\ B \text{ is a refinement of } A \}.$$

If $A_i \in P$ $(i = 1, 2, \dots, n)$, then denote

$$\bigvee_{i=1}^{n} A_i = \{ \bigcap_{i=1}^{n} E_i;\ E_i \in A_i \}$$

(Of course, $\bigvee_{i=1}^{n} A_i$ is an element of P.) Then for every $A \in P$ there exists the following limit (see [1])

*) Some results of the paper were communicated at the Third Prague Topological Cymposium in September 1971.

$$h(A, f) = \lim \frac{1}{n} H \bigvee_{i=0}^{n-1} f^{-i}(A)).$$

The topological entropy of the deformation f is defined as

$$h(f) = \sup \{ h(A, f); \quad A \in P \}$$

what is a real number.

In our general case we start with with a quasiordered set P (i. e. a set P with a reflexive and transitive relation $\leqq$). On the set P it is defined an associative binary operation $\vee$ such that $A \vee B \geqq A$ and $A \vee B \geqq B$ for every A, B $\in$ P. Finally let $T : P \longrightarrow P$ and $H : P \longrightarrow \langle 0, \infty)$ be any functions satisfying the following conditions:

1. $H\left(\bigvee_{i=0}^{k} T^i(A)\right) \leq H\left(\bigvee_{i=0}^{j} T^i(A)\right) + H\left(\bigvee_{i=j+1}^{k} T^i(A)\right)$

for all $A \in P$.

2. $T(A \vee B) = T(A) \vee T(B)$ for all A, B $\in$ P.

3. $H(T(A)) \leqq H(A)$ for all $A \in P$.

Sometimes we shall need also the condition

4. $A \leqq B \Longrightarrow H(A) \leqq H(B), \quad T(A) \leqq T(B).$

<u>Lemma.</u> Under the assumptions 1 - 3 $\lim \frac{1}{n} H(\bigvee_{i=0}^{n-1} T^i(A))$ esists and $\lim \frac{1}{n} H(\bigvee_{i=0}^{n-1} T^i(A)) \leqq H(A)$.

<u>Proof.</u> Put $H_k = H(\bigvee_{i=0}^{k} T^i(A))$ $(k = 1, 2, \dots)$. Then

$$H_{n+m} = H(\bigvee_{i=0}^{n-1} T^i(A) \vee \bigvee_{i=n}^{n+m-1} T^i(A)) \leqq$$

$$H\Big(\bigvee_{i=0}^{n-1} T^i(A)\Big) + H\Big(T^n\Big(\bigvee_{i=0}^{m-1} T^i(A)\Big)\Big) \leqq$$

$$\leqq H\Big(\bigvee_{i=0}^{n-1} T^i(A)\Big) + H\Big(\bigvee_{i=0}^{m-1} T^i(A)\Big) = H_n + H_m .$$

The existence of $\lim \frac{1}{n} H_n$ follows from the inequality $H_{n+m} \leqq H_n + H_m$. It can be proved as an exercice. E.g. by the following way.

Put $x = \inf \frac{1}{n} H_n$. Let $\mathcal{E} > 0$. There is k such that $x + \mathcal{E}/2 > 1/k\, H_k \geqq x$. Further $H_{km} \leqq m H_k$. If $n \geqq k$, then $n = km + i$, where $0 \leqq i \leqq k-1$ and

$$\frac{H_n}{n} \leqq \frac{H_{km}}{km+i} + \frac{H_i}{km+i} \leqq \frac{H_{km}}{km} + \frac{iH_1}{km+i} \leqq x + \frac{\mathcal{E}}{2} + \frac{iH_1}{km+i} .$$

Choose m_o such that $\frac{iH_1}{km_o + i} < \frac{\mathcal{E}}{2}$ /i=1, ... , k-1/. Then for $n > km_o$ we have

$$x \leqq \frac{H_n}{n} < x + \mathcal{E} ,$$

hence $\lim \frac{1}{n} H_n = x$. From the inequality $H_n \leqq n H_1$ it follows that $\lim \frac{1}{n} H_n \leqq H_1 = H(A)$.

Definition 1. For any given P, T, H and $A \in P$ let us put

$$h(A, T) = \lim \frac{1}{n} H\Big(\bigvee_{i=0}^{n-1} T^i(A)\Big),$$

$$h(T) = \sup \{ h(A, T);\ A \in P \} ;$$

$h(T)$ is called the entropy of then triple (P, T, H). The triple (P, T, H) is called the base of the entropy $h(T)$.

Examples *)

1. Topological entropy

In this case P is the family of all open finite coverings of a topological space X, where $A \leqq B$, A, $B \in P$ iff B is a refinement of A, $A \vee B = \{ E \cap F;\ E \in A,\ F \in B \}$. Further $H(A) = \log \operatorname{card} A$ and $T(A) = f^{-1}(A)$, where f is a deformation of X.

The properties 2 and 4 are evident. Since $\operatorname{card} f^{-1}(A) \leqq \operatorname{card} A$, we have $H(f^{-1}(A)) \leqq H(A)$. Finally, for every A, $B \in P$

$\operatorname{card} A \vee B \leqq \operatorname{card} A \cdot \operatorname{card} B,$

$H(A \vee B) \leqq H(A) + H(B),$

hence

$$H\Big(\bigvee_{i=0}^{k} f^{-i}(A)\Big) \leqq H\Big(\bigvee_{i=0}^{j} f^{-1}(A)\Big) + H\Big(\bigvee_{i=j+1}^{k} f^{-i}(A)\Big).$$

2. Kolmogoroff-Sinaj's entropy

Let (X, S, m) be a probability measure space, $f : X \longrightarrow X$ a measure preserving transformation ((X, S, m, f) is called a dynamical system). Denote P the family of all finite measurable decompotisions A of X such that $A, f^{-1}(A), \ldots, f^{-k}(A)$ are independent i. e.

$$m\Big(\bigcap_{i=1}^{p} f^{-j_i}(E_i)\Big) = \prod_{i=1}^{p} m(f^{-j_i}(E_i)) \quad (0 \leqq j_1 < \ldots < j_p \leqq k).$$

As before, $A \leqq B$ iff B is a refinement of A. Further put

$$H(A) = - \sum_{E \in A} m(E) \log m(E)$$

and $T(A) = f^{-1}(A)$.

Put $B = \bigvee_{i=0}^{j} T^i(A)$, $C = \bigvee_{i=j+1}^{k} T^i(A)$. The decompositions B, C are independent. Therefore

*) The examples 1, 2 and 5 are known; the examples 3, 4, 6 and 7 are new.

$$H(B \vee C) = -\sum_{E \in B \vee C} m(E) \log m(E) =$$

$$= -\sum_{E \in B} \sum_{F \in C} m(E \cap F)$$

$$\log m(E \cap F) =$$

$$= -\sum \sum m(E \cap F) \log m(E) . m(F) =$$

$$= -\sum_{E} \sum_{F} m(E \cap F) \log m(E) -$$

$$- \sum_{F} \sum_{E} m(E \cap F) \log m(F) =$$

$$= -\sum_{E} m(E) \log m(E) - \sum_{F} m(F)$$

$$\log m(F) = H(B) + H(C).$$

The condition 2 is evident, as well as the implication $A \leqq B \Longrightarrow T(A) \leqq T(B)$. Let $A \in P$. Then

$$H(A) = -\sum_{E \in A} m(E) \log m(E) = -\sum_{E \in A} m(T(A))$$

$$\log m(T(A)) \geqq -\sum_{F \in T(A)} m(F) \log m(F) = H(T(A)).$$

Finally, let $A \leqq B$. Then

$$H(A) = -\sum_{E \in A} m(E) \log m(E) =$$

$$= -\sum_{E \in A} \sum_{F \in B} m(E \cap F) \log m(E) =$$

$$= - \sum_{F \in B} \sum_{E \in A} m(E \cap F) \log m(E) \leqq$$

$$\leqq - \sum_{F \in B} \log m(F) \sum_{E \in A} m(E \cap F) =$$

$$= - \sum_{F \in B} m(F) \log m(F) = H(B).$$

3. Entropy of an automorphism of a Boolean algebra

Let B be a Boolean algebra, f an automorphism of B. Let P be the set of all finite decompositions of the greatest element of B. For $A \in P$ put $H(A) = \log \operatorname{card} A$, $T(A) = f(A)$.

4. Another type of topological entropy

Let P be the family of all open coverings of a topological space X having refinements of finite orders, i. e. $P = \{A ;$ there is $B \geqq A$, B is a covering of finite order$\}$. As before T is a functions generated by a deformation, but now $H(A) = \log \min \{\text{order } B;\ B \geqq A\}$.

The properties 2 - 4 are evident. We prove only the condition 1. It is enough to prove that $H(A \vee B) \leqq H(A) + H(B)$ for every $A, B \in P$. Let $H(A) = \log m$, $H(B) = \log n$. Then there are $C \geqq A$, $D \geqq B$ such that order $C = m$ and order $D = n$. Therefore arbitrary $m + 1$ sets of C and arbitrary $n + 1$ sets of D have the empty intersection. Take arbitrary $mn + 1$ sets of $C \vee D$. They have the form $E_i \cap F_j$ where $E_i \in C$, $F_j \in D$. Among $mn + 1$ couples (i, j) there is different either $m + 1$ of the first indices or $n + 1$ of the second indices. Therefore

$$\bigcap (E_i \cap F_j) = \bigcap_i E_i \cap \bigcap_j F_j = \emptyset.$$

Hence order $C \vee D \leqq mn$ and $H(A \vee B) \leqq \log mn = \log m + \log n = H(A) + H(B)$.

5. Entropy of a group endomorphism ([1])

Let G be an Abelian group, P the family of all finite subgroups; $A \leq B$ iff $A \subset B$, T endomorphism, $H(A) =$ log order A.

6. Another type of measure preserving transformation

Let P be a lattice of subsets of a set X (e. g. a ring of sets), H a non-negative subadditive function on P (e. g. a measure), $T : X \longrightarrow X$ such a transformation that $H(T^{-1}(E)) \leq H(E)$ (e. g. a measure preserving transformation).

Of course, we can specify, e. g. P may be the family of all compact sets, H a volume. Another example : H - the Haar measure, $T : x \longmapsto ax$.

If H is a finite measure on an algebra, then $h(T) = 0$. Namely, $0 \leq h(T, E) \leq h(T, X)$ for all measurable sets E. But h $h(T, X) = \lim \frac{1}{n} H(\bigvee_{i=0}^{n-1} T^i(X)) \leq \lim \frac{1}{n} H(X) = 0$. If H is not defined on an algebra, h need not be trivial. E. g. let $X = (-\infty, \infty)$, S be the ring generated by bounded intervals, H be the Lebesgue measure, $T(x) = x + c$. Then $h(T) = c$. An interesting example we obtain if H is k-dimensional Hausdorff measure, P the family of all sets of finite measure, T a contraction i. e. such a transformation that $d(T(x), T(y)) \leq d(x, y)$. E. g. if $k = 1$ and T is a rotation trough α (α irational) and E is a straight line interval with an endpoint in the origine, then $h(T, E)$ equals the length of the interval, hence $h(T) = \infty$.

7. Entropy of an operator

Let P be a linear lattice of functions (ordered as usually), H be a positive linear functional, $T(f) = f - g$, where g is a fixed non-negative fuction. Another possibility (g arbitrary measure preserving transformation, H integral):
$T(f)(x) = f(g(x))$. In this case

$$H(T(f)) = \int_X T(f)\,dm = \int_X fg\,dm = \int_X f\,dm = H(f).$$

Usually, if "two systems are isomorphic" then their entropies are equal. A similar theorem holds also in the general case.

Definition 2. Let (P, T, H), (R, S, G) be two bases of entropy. We shall say that (P, T, H) is immersed in (R, S, G), if there is a mapping $U : P \longrightarrow R$ such that

1. $U(A \vee B) = U(A) \vee U(B)$ for all $A, B \in P$.
2. $U \circ T = S \circ U$.
3. $G(U(A)) = H(A)$ for all $A \in P$.

The bases are equivalent if U is a bijection.

Theorem 1. If (P, T, H) is an immersion in (R, S, G) then $h(T) \leq h(S)$. If (P, T, H) and (R, S, G) are equivalent then $h(T) = h(S)$.

Proof. $U(\bigvee T^i(A)) = \bigvee U(T^i(A)) = \bigvee S^i(U(A))$ hence $H(\bigvee T^i(A)) = G(\bigvee S^i(U(A)))$. Therefore $h(T, A) = h(S, U(A)) \leq h(S)$ for all $A \in P$, hence $h(T) \leq h(S)$. If U is a bijection then to any $B \in R$ there is an $A \in P$ such than $B = U(A)$. Hence $G(\bigvee S^i(B)) = G(\bigvee S^i(U(A))) = G(U(\bigvee T^i(A))) = H(\bigvee T^i(A))$, $h(S, B) = h(T, A) \leq h(T)$ and so $h(S) \leq h(T)$.

In some special cases Theorem 1 describes the situation in factor spaces; U corresponds to the projection. E. g. if (R, S, G) is a base of the topological entropy of a space X and a deformation f, Y is a factor space, g induced deformation and (P, T, H) the base generated by the couple (Y, g) and $p : X \longrightarrow Y$ the projection, then (P, T, H) is immersed in (R, S, G); the corresponding immersion $U : P \longrightarrow R$ can be defined as $U(A) = p^{-1}(A)$. Theorem 1 implies $h(T) \leq h(S)$ i. e. $h(g) \leq h(f)$. (The topological entropy of induced deformation in factor space is not greater than the entropy of the original deformation.) More general example of an immersion we get, if we take any invariant (with respect to S) subset P of the set R. U is in this case the injection, $U : P \longrightarrow R$ and $T = S/P$..

From Theorem 1 it follows that the same entropy have e. g. isomorphic dynamical systems *), homeomorphic topological spaces with deformations commuting corresponding homeomorphism etc. An interesting application of Theorem 1 is presented at the end of the article.

*) Recently D. Ornstein [(6)] proved the opposite theorem for Bernoulli dynamical systems.

Definition 3. Let (P, T, H) be a base of the entropy, $Q \subset P$. We say that Q generates P if to any $A \in P$ there exists $B \in Q$ such that $\bigvee_{i=0}^{k-1} T^i(B) \geqq A$ for some k.

Theorem 2. If Q generates P then $h(T) = \sup \{ h(T,A); A \in Q\}$.

Proof. Let $B \in Q$. Then

$$h(T, \bigvee_{i=0}^{k-1} T^i(B)) = \lim \frac{1}{p} H(\bigvee_{j=0}^{p-1} T^j(\bigvee_{i=0}^{k-1} T^i(B))) =$$

$$= \lim_p \frac{1}{p} H(\bigvee_{u=0}^{k+p-2} T^u(B)) =$$

$$= \lim_p \frac{1}{t+1-k} H(\bigvee_{u=0}^{t-1} T^u(B)) =$$

$$= \lim \frac{t}{t+1-k} \frac{1}{t} H(\bigvee_{u=0}^{t-1} T^u(B)) =$$

$$= h(T, B),$$

hence

$$h(T, A) \leqq h(T, \bigvee_{i=0}^{k-1} T^i(B)) = h(T, B).$$

In Theorem 1 we were interested in factor or quocient spaces. What can we say about product spaces.

Definition 4. Let (P_1, T_1, H_1) and (P_2, T_2, H_2) be bases of the entropy. Their product is the base (P, T, H), where $P = \{ (E, F); E \in P_1, F \in P_2 \}$ $((E_1, E_2) \leqq (F_1, F_2)$ iff $E_1 \leqq F_1$ and $E_2 \leqq F_2$; $(E_1, E_2) \vee (F_1, F_2) = (E_1 \vee F_1, E_2 \vee F_2))$, $T((E, F)) = (T_1(E), T_2(F))$, $H((E, F)) =$

$$= H_1(E) + H_2(F).$$

Theorem 3. $h(T) = h(T_1) + h(T_2)$.

Proof. Clearly $\bigvee T^i(A_1, A_2) = (\bigvee T_1^i(A_1), \bigvee T_2^i(A_2))$.

Hence $h(T, (A_1, A_2)) = h(T_1, A_1) + h(T_2, A_2) \leqq$

$\leqq h(T_1) + h(T_2)$ and therefore $h(T) \leqq h(T_1) +$

$+ h(T_2)$. On the other hand there are A_1, A_2 such that

$$h(T_1) + h(T_2) - \delta < h(T_1, A_1) + h(T_2, A_2) =$$

$$= h(T, (A_1, A_2)) \leqq h(T).$$

Theorem 3 can be applied to the product of topological spaces and topological entropy, since the coverings corresponding to the couples (A, B) (i. e. the coverings $\{E \times F;\ E \in A,\ F \in B\}$) generate the family of all finite coverings in the product space. In the case of dynamical systems the decompositions corresponding to the couples (A, B) need not generate the family of all measurable decompositions in the product space. Therefore we know only that $h(T) \geqq h(T_1) + h(T_2)$.

Theorem 4. Let (P, T, H) be a base of the entropy satisfying the assumptions 1 - 4. Then $h(T^k) = k\, h(T)$ for every positive integer k.

Proof. $h(T^k) \geqq h(T^k, \bigvee_{i=0}^{k-1} T^i(A)) =$

$$= k \lim_n \frac{1}{nk} H(\bigvee_{j=0}^{n-1} T^{kj}(\bigvee_{i=0}^{k-1} T^i(A))) =$$

$$= k \lim_n \frac{1}{nk} H(\bigvee \bigvee T^{kj+i}(A)) =$$

$$= k \lim \frac{1}{nk} H\left(\bigvee_{p=0}^{nk-1} T^p(A)\right) = k\, h(T, A)$$

for all $A \in P$, hence $h(T^k) \geqq k\, h(T)$. On the other hand, according to the condition 4

$$h(T, A) = \lim \frac{1}{nk} H\left(\bigvee_{i=0}^{nk-1} T^i(A)\right) \geqq$$

$$\geqq \lim_n \frac{1}{nk} H\left(\bigvee_{i=0}^{n-1} (T^k)^i (A)\right) =$$

$$= \frac{1}{k} H(T^k, A).$$

Finally we mention two structures on the set X_n of all sequences $\{x_i\}_{i=-\infty}^{\infty}$ of integers 0, 1, ..., n-1. The first structure is the algebra B_n generated by the family C_n of all sets of the type $\{\{x_u\}_{u=-\infty}^{\infty} ;\ x_i = j\}$, where $j = 0, 1, \ldots, n-1$ and $i = 0, \pm 1, \pm 2, \ldots$ Let T_n be the shift i. e. $T_n(\{x_u\}_{u=-\infty}^{\infty}) = \{y_u\}_{u=-\infty}^{\infty}$, where $y_u = x_{u+1}$. Let $h(T_n)$ be the algebraic entropy defined in Example 3.

Theorem 5. $h(T_n) = \log n$.

Proof. Put $K_p = \{E_o^p, \ldots, E_{n-1}^p\}$ where $E_j^p = \{\{x_u\}_{u=-\infty}^{\infty};\ x_p = j\}$ $(j=0, \ldots, n-1)$. Then

$$h(T_n, K_p) = \lim \frac{1}{k} H\left(\bigvee_{i=0}^{k-1} T^i(K_p)\right) = \lim \frac{1}{k} \log n^k = \log n.$$

Since the family $Q = \{K_p\}_{p=-\infty}^{\infty}$ generates P, we have $h(T_n) = \log n$.

Corrolary. Given $n \neq m$ there is no isomorphism $U : B_n \to B_m$ commuting shifts.

Let S_n be the σ-algebra generated by C_n. What can we say about the algebraic entropy $h_o(T_n)$ of T_n in S_n? According to Theorem 1 we have $h(T_n) \leqq h_o(T_n)$, hence $h_o(T_n) \geqq \log n$. It seems to us that $h_o(T_n) = \log n$ but we are not able to prove it. (In [7] it is proved that there is no isomorphism $U : S_n \longrightarrow S_m$ commuting shifts.)

Another natural structure on X_n is the topology U_n generated by C_n. Let $h\tau(T_n)$ be now the topological entropy. Since every open set is measurable here (i. e. $U_n \subset S_n$), our topological base is immersed into the corresponding algebraic base. Therefore according to Theorem 1

$$h\tau(T_n) \leqq h_o(T_n).$$

Since $h\tau(T_n) = \log n$ we get also only $h_o(T_n) \geqq \log n$.

Problems

In what follows h_i denotes the entropy appearing in Example i.

1. Compare the entropy h_1 of a deformation and the entropies h_5 of induced endomorphisms of the group of homology. (This problem was given by A. ARCHANGELSKIJ in Ref. Journal 1967.)

2. What is the connection between the topological dimension and h_4? Evidently $h_1 \geqq h_4$. If X has finite dimension, then $\log(1+\dim X) \geqq h_4(T)$ for all deformations T. Possible hypothesis:

$$\dim X = \sup \left\{ e^{h_4(T)} - 1 \; ; \; T \text{ is a deformation of } X \right\} ;$$

$$\dim X = \left[e^{h_4(T)} \right].$$

3. Let (X, S, m, T) be a dynamical system, (S, d) the pseudometric space where $d(E, F) = m(E \triangle F)$. Compare the entropies h_1, h_2, h_4.

5. Let X be a topological space, B the σ-ring generated e.g. by compact sets. Compare h_1 and h_3.

REFERENCES

[1] Adler R. L., Konheim A. G., McAndrew M. H., Topological entropy, Trans. AMS, 114 (1965) 309-319

[2] Goodwyn L. W., Topological entropy and expansive cascades, PhD dissertation, University of Maryland 1968

[3] England J. W., Martin N. F. G., On the topological entropy of solenoids, J. Math. Mech. 19 (1969), 139-142

[4] Kolmogoroff A. N., A new metric invariant of transitive dynamical systems and automorphisms of Lebesgue spaces, DAN SSSR 119 (1958) 861--864

[5] Sinaj Ja/. G., On the entropy of a metric automorphism, DAN SSSR 124 (1959) 980-983

[6] Ornstein D., Bernoulli shifts with the same entropy are isomorphic, Advances in math., 4 (1970) 337-352

[7] Kluvanek I., Riečan B., Some properties of Bernoulli schemes, Mat.-fyz. časop. 14 (1964) 84-88

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Recived : January 3, 1972, at Publishers May 28, 1973

(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXIX, 1974)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXIX - 1974

EINIGE EIGENSCHAFTEN DER LÖSUNGEN OHNE NULLSTELLEN DER LINEAREN DIFFERENTIALGLEICHUNG DRITTER ORDNUNG

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In der vorliegenden Arbeit wurden hinreichende Bedingungen abgeleitet, unter welchen wenigstens eine Lösung ohne Nullstellen der Differentialgleichung dritter Ordnung existiert. Weiter wurden hinreichende Bedingungen abgeleitet, unter welchen die Lösung mit der doppelten Nullstelle rechts von dieser keine Nullstelle hat.

Erwägen wir die Differentialgleichung dritter Ordnung

$$(1) \qquad y''' + p(x)y'' + 2A(x)y' + [A'(x) + b(x)]\, y = 0,$$

wo

$$b(x),\ p(x) \in C^0(I),\ A(x) \in C^1(I),\ I = (-\infty;\infty).$$

Für eine beliebige Lösung $y(x)$ der Differentialgleichung (1) gilt die Integralidentität

$$(2) \qquad [\,y(x)y''(x) - \tfrac{1}{2}\, y'^2(x) + A(x)y^2(x)]\exp\Big\{\int_{x_0}^{x} p(t)dt\Big\} +$$

$$+ \frac{1}{2}\int_{x_0}^{x} p(t)y'^2(t)\exp\Big\{\int_{x_0}^{t} p(s)\,ds\Big\}\,dt +$$

$$+\int_{x_0}^{x} [\,b(t) - A(t)p(t)]\; y^2(t)\exp\Big\{\int_{x_0}^{t} p(s)\,ds\Big\}\,dt =$$

$$= y(x_0)y''(x_0) - \frac{1}{2}\, y'^2(x_0) + A(x_0)\, y^2(x_0).$$

Gemäß [3] können wir jede Lösung $y(x)$ der Differential - - gleichung (1) mit der Eigenschaft $y(x_0) = 0$, $x_0 \in J$, $J = (-\infty; \infty)$ in der Form $y = c_1 y_1 + c_2 y_2$ schreiben, wo $y_1(x)$, $y_2(x)$ zwei linear unabhängige Lösungen der Differentialgleichung (1) mit der Eigenschaft $y_1(x_0) = y_1'(x_0) = 0$, $y_1''(x_0)$, $y_2(x_0) = y_2''(x_0) = 0$, $y_2'(x_0) \neq 0$ sind.

Die Menge der Lösungen der Differentialgleichung (1) $y = c_1 y_1 + c_2 y_2$ mit der Eigenschaft $y(x_0) = 0$ nennen wir ein Büschel von Lösungen erster Art der Differentialgleichung (1) im Punkte x_0, kurz ein Büschel von Lösungen erster Art im Punkte x_0.

Gemäß [3] entspricht das Büschel von Lösungen erster Art der Differentialgleichung zweiter Ordnung, welche die Form

$$(3) \qquad w y'' - w'y' + [w'' + p(x)w' + 2A(x)w]\, y = 0,$$

hat, wobei $w(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ die Lösung der Differentialgleichung

$$(4) \qquad [w'' + p(x)w']' + p(x)w'' + [p^2(x) + 2A(x)]w' + [A'(x) - b(x) + 2A(x)p(x)]\, w = 0 \quad \text{ist.}$$

Für jede Lösung $w(x)$ der Differentialgleichung (4) gelten folgende Integralidentitäten:

$$(5) \qquad \left[w(x)w''(x) + p(x)w(x)w'(x) + A(x)w^2(x) - \frac{1}{2} w'^2(x)\right] \exp\left\{\int_{x_0}^{x} p(t)dt\right\} =$$

$$= \frac{1}{2}\int_{x_0}^{x} p(t)w'^2(t) \exp\left\{\int_{x_0}^{t} p(s)ds\right\} dt +$$

$$+ \int_{x_0}^{x} [b(t) - A(t)p(t)]\, w^2(t) \exp\left\{\int_{x_0}^{t} p(s)ds\right\} dt +$$

$$+ w(x_0)w''(x_0) + p(x_0)w(x_0)w'(x_0) + A(x_0)w^2(x_0) -$$
$$- \frac{1}{2} w'^2(x_0),$$

$$(6) \quad \left[w''(x) + p(x)w'(x) + 2A(x)w(x)\right] \exp\left\{\int_{x_0}^{x} p(t)dt\right\} =$$
$$= w''(x_0) + p(x_0)w'(x_0) + 2A(x_0)w(x_0) + \int_{x_0}^{x} \left[A'(t) +\right.$$
$$\left.+ b(t)\right] w(t) \exp\left\{\int_{x_0}^{t} p(s)ds\right\} dt .$$

Bemerkung 1. Die Gleichung (4) ist auf Grund der Substitution

$$w - u_1, \quad w' = u_2, \quad w'' + p(x)w' = u_3$$

äquivalent mit dem System linearer Differentialgleichungen

$$u_1' = u_2$$
$$u_2' = -p(x)u_2 + u_3$$
$$u_3' = \left[A'(x) - b(x) + 2A(x)p(x)\right] u_1 + 2\left[A(x) +\right.$$
$$\left.+ p^2(x)\right]u_2 - p(x)u_3,$$

für welche der Satz über Existenz und Eindeutigkeit gilt. Also existiert gerade eine Lösung $w(x)$ der Differentialgleichung (4), welche die Anfangsbedingungen

$$w(x_0) = w_0, \quad w'(x_0) = w_0', \quad w''(x_0) + p(x_0)w'(x_0) = w_0^*$$

erfüllt. Die letzte Bedingung kann durch die Bedingung $w''(x_0) = w_0''$ ersetzt werden.

Satz 1. Es sei $w_1 = w_1(x)$ eine beliebige nichttriviale Lösung der Differentialgleichung (4). Dann existieren solche zwei lineare unabhängige Lösungen $y_1(x)$, $y_2(x)$ der Differentialgleichung (1), dass

$$w_1 = y_1 y_2' - y_1' y_2$$

gilt. Der Beweis wird ähnlich wie in der Arbeit [2] durchgeführt.

Satz 2. Es sei $p(x), b(x) \in C^0(I)$, $A(x) \in C^1(I)$, $I = (-\infty; \infty)$. Es gelte $p(x) \gtreqless 0$, $b(x) - A(x)p(x) \gtreqless 0$ für alle $x \in J$. Die Funktionen $p(x)$, $b(x) - A(x)p(x)$ seien nicht gleichzeitig indentisch gleich Null in keinem Teilintervall. Dann haben die Differentialgleichungen (1) und (4) wenigstens eine Lösung ohne Nullstellen für $x \in J$.

Den Bewei führen wir für die Differentialgleichung (4) durch. Die Methode des Beweises stammt von M. GREGUŠ aus der Arbeit [2]. Es sei $z_1(x)$, $z_2(x)$, $z_3(x)$ ein Fundamentalsystem von Lösungen (4) mit der Eigenschaft

$$z_1(x_o) = z_1'(x_o) = 0, \quad z_1''(x_o) = 1,$$

$$z_2(x_o) = z_2''(x_o) = 0, \quad z_2'(x_o) = 1,$$

$$z_3'(x_o) = z_3''(x_o) = 0, \quad z_3(x_o) = 1.$$

Es sei $x_o > x_1 > x_2 > x_3 > \dots$ eine Folge von Punkten, welche zu $-\infty$ divergiert. Aus der Integralidentität (5) folgt, daß $z_1(x)$ rechts von x_o keine Nullstelle hat.

Bilden wir eine Folge von Lösungen der Differentialgleichung (4)

$$\{u_n\}_{n=1}^{\infty}, \quad u_n = c_{1n} z_1 + c_{2n} z_2 + c_{3n} z_3$$

mit der Eigenschaft

$$u_n(x_n) = u_n'(x_n) = 0, \quad u_n''(x_n) > 0,$$

wobei

$$u_n^2(x_o) + u_n'^2(x_o) + u_n''^2(x_o) = 1.$$

Aus der Integralidentität (5) folgt für $u_n(x)$, dass $u_n(x)$ rechts von x_n keine Nullstelle hat und für $x > x_n$ gilt $u_n > 0$.

Bilden wir die nachstehenden Folgen

$$\{u_n(x_o)\}_{n=1}^{\infty}, \quad \{u_n'(x_o)\}_{n=1}^{\infty}, \quad \{u_n''(x_o)\}_{n=1}^{\infty}$$

Es ist ersichtlich, daß diese Folgen begrenzt sind und deshalb existieren ausgewählte Folgen, welche konvergieren. Der Sicherheit wegen setzen wir voraus, daß dies dieselben Folgen sind. Ihre Limite bezeichnen wir u_o, u_o', u_o''.

Es sei $u(x)$ die Lösung der Differentialgleichung (4) welche den Bedingungen

$$u(x_o) = u_o, \quad u'(x_o) = u_o', \quad u''(x_o) = u_o''$$

entspricht. Die Lösung $u(x)$ ist nichttrivial, da

$$\lim_{n \to \infty} [u_n^2(x_o) + u_n'^2(x_o) + u''^2(x_o)] = u_o^2 + u_o'^2 + u_o''^2 = 1.$$

Die Lösung $u_n(x)$ ist

$$u_n(x) = u_n''(x_o)\, z_1(x) + u_n'(x_o)\, z_2(x) + u_n(x_o)\, z_3(x)$$

und die Lösung $u(x)$ der Form

$$u(x) = u_o''\, z_1(x) + u_o'\, z_2(x) + u_o\, z_3$$

woraus folgt, daß

$$\lim_{n \to \infty} u_n(x) = u(x)$$

für alle $x \in (-\infty ; \infty)$.

Zeigen wir, daß $u(x) \neq 0$ für alle $x \in J$ ist. Die Integralidentität (5) für die Lösung $u_n(x)$ hat die Form

$$\left[u_n(x)u_n''(x) + p(x)u_n(x)u_n'(x) - \frac{1}{2} u_n'^2(x) + A(x)u_n^2(x)\right] \cdot$$

$$\exp\left\{\int_{x_n}^{x} p(t)dt\right\} = \frac{1}{2}\int_{x_n}^{x} p(t)u_n'^2(t)\exp\left\{\int_{x_n}^{t} p(s)ds\right\}$$

$$dt + \int_{x_n}^{x}\left[b(t) - A(t)p(t)\right]u_n^2(t)\exp\left\{\int_{x_n}^{t} p(s)ds\right\} dt.$$

Daher ist für $n \longrightarrow \infty$

$$u(x)u''(x) + p(x)u(x)u'(x) - \frac{1}{2} u'^2(x) + A(x)u^2(x) =$$

$$= \frac{1}{2}\int_{-\infty}^{x} p(t)u'^2(t)\exp\left\{-\int_{t}^{x} p(s)ds\right\} dt + \int_{-\infty}^{x}\left[b(t) -\right.$$

$$\left. - A(t)p(t)\right]u^2(t)\exp\left\{-\int_{t}^{x} p(s)ds\right\} dt.$$

Dies ist die Integralidentität für die Lösung $u(x)$, aus welcher folgt, daß $u(x)$ für $x \in J$ keine Nullstelle hat.

Ähnlich wird die Behauptung des Satzes für die Differentialgleichung (1) bewiesen. Im Beweis wird die Integralidentität (2) und die Folge $\{x_n\}_{n=1}^{\infty}$ welche zu $+\infty$ divergiert verwendet.

Folgerung. Wenn die Voraussetzungen des Satzes 2 erfüllt sind, dann existiert das Büschel von Lösungen $c_1 y_1 + c_2 y_2$ der Differentialgleichung (1) im Punkte $-\infty$ und erfüllt die Differentialgleichung (3).

Satz 3. Es bei $b(x)$, $p(x) \in C^0(I)$, $A(x) \in C^1(I)$, $I = -\infty ; \infty)$ und x_0 sei eine beliebige Zahl. Es gelte für alle $x \in J_1 = (x_0 ; \infty)$ $A(x) > 0$, $p(x) \leqslant 0$, $b(x) - A(x)p(x) \leqslant 0$, $A'(x) - b(x) + 2A(x)p(x) \leqslant 0$ wobei die Funktionen $p(x)$, $b(x) - A(x)\,p(x)$ in keinem Teilintervall gleichzeitig identisch gleich Null sind. Für jede Lösung $y(x)$ der Differentialgleichung (1) mit der Eigenschaft

$$y(x_0) = y'(x_0) = 0, \quad y''(x_0) = k > 0,$$

bzw. $k < 0$

für $x \in J_1$ gilt dann

$$0 < y(x) < \frac{k}{A(x)} \exp\left\{ - \int_{x_0}^{x} p(t)dt \right\},$$

bzw.

$$0 > y(x) > \frac{k}{A(x)} \exp\left\{ - \int_{x_0}^{x} p(t)dt \right\},$$

$$y'^2(x) < \frac{2k^2}{A(x)} \exp\left\{ - 2 \int_{x_0}^{x} p(t)dt \right\}.$$

Beweis. Es sei $y(x)$ die Lösung der Differentialgleichung (1) mit der Eigenschaft

$$y(x_0) = y'(x_0) = 0, \quad y''(x_0) = k > 0, \quad \text{bzw.} \quad k < 0.$$

Aus der Integralidentät (2) folgt dann, daß $y(x) > 0$ bzw. $y(x) < 0$ fur $x \in J_1$ ist. Für die erwogene Lösung $y(x)$ der Differentialgleichung (1) aus der Integralidentität (2) folgt, daß

$$y(x)y''(x) - \frac{1}{2} y'^2(x) + A(x)y^2(x) > 0$$

für $x > x_0$ ist.

Für jede Lösung $y(x)$ der Differentialgleichung (1) gilt auch die Integralidentität

$$[y''(x) + 2A(x)y(x)] \exp\left\{ \int_{x_0}^{x} p(t)dt \right\} - y''(x_0) -$$

$$- 2A(x_0)y(x_0) = \int_{x_0}^{x} [A'(t) - b(t) + 2A(t)p(t)]\, y(t)$$

$$\exp\left\{\int_{x_o}^{t} p(s)ds\right\} dt .$$

Diese Identität kann für die erwogene Lösung $y(x)$ in die Form

$$\left[y(x)y''(x) - \frac{1}{2} y'^2(x) + A(x)y^2(x)\right] \exp\left\{\int_{x_o}^{t} p(t)dt\right\} =$$

$$= y(x)y''(x_o) - \left[A(x)y^2(x) + \frac{1}{2} y'^2(x)\right] \exp\left\{\int_{x_o}^{x} p(t)dt\right\} +$$

$$+ \; y(x) \int_{x_o}^{x} \left[A'(t) - b(t) + 2A(t)p(t)\right] y(t) \exp$$

$$\left\{\int_{x_o}^{t} p(s)ds\right\} dt$$

umgeschrieben werden, woraus für $x > x_o$

$$k\; y(x) - \left[A(x)y^2(x) + \frac{1}{2} y'^2(x)\right. \exp\left\{\int_{x_o}^{x} p(t)dt\right\} > 0$$

folgt.

Aus der letzten Ungleichheit sind die Ungleichheiten

(7) $$k\, y(x) - A(x)y^2(x) \exp\left\{\int_{x_o}^{x} p(t)\, dt\right\} > 0,$$

(8) $$k\; y(x) - \frac{1}{2} y'^2(x) \exp\left\{\int_{x_o}^{x} p(t)\, dt\right\} > 0$$

ersichtlich. Aus der Beziehung (7) für $k > 0$ erhalten wir

(9) $$0 < y(x) < \frac{k}{A(x)} \exp\left\{-\int_{x_o}^{x} p(t)dt\right\}.$$

Aus der Ungleichheit (8) erhalten wir durch Anwendung von (9) schrittweise

$$\frac{k^2}{A(x)} \exp\left\{-\int_{x_0}^{x} p(t)dt\right\} - \frac{1}{2} y'^2(x) \exp$$

$$\left\{\int_{x_0}^{x} p(t)dt\right\} > 0,$$

$$(10) \quad y'^2(x) < \frac{k^2}{A(x)} \exp\left\{-2\int_{x_0}^{x} p(t)dt\right\}.$$

Wenn $k < 0$ ist, erwägen wir die Lösung $-y(x)$ und aus (9) folgt

$$(11) \quad 0 > y(x) > \frac{k}{A(x)} \exp\left\{-\int_{x_0}^{x} p(t)dt\right\}$$

und die Abschätzung (10). Damit ist der Beweis beendet.

Satz 4. Es sei $b(x), p(x) \in C^0(I)$, $A(x) \in C^1(I)$, $I = (-\infty;\infty)$. Es sei x_0 eine reelle Zahl. Für alle $x \in J_1 = (x_0;\infty)$ gelte $A(x) > 0$, $p(x) \geqslant 0$, $b(x) - A(x)p(x) \geqslant 0$, $A'(x) + b(x) \leqslant 0$, wobei die Funktionen $p(x)$, $b(x) - A(x) p(x)$ in keinem Teilintervall gleichzeitig identisch gleich Null sind. Für jede Lösung $z(x)$ der Differentialgleichung (4) mit der Eigenschaft

$$z(x_0) = z'(x_0) = 0, \quad z''(x_0) = k > 0, \qquad \text{bzw. } k < 0$$

für $x \in J_1$ gilt dann

$$\frac{k}{A(x)} \exp\left\{-\int_{x_0}^{x} p(t)dt\right\} > z(x) > 0,$$

bzw.

$$0 > z(x) \frac{k}{A(x)} \exp\left\{-\int_{x_0}^{x} p(t)\,dt\right\}$$

und $$z'^2(x) < \frac{2k^2}{A(x)} \exp\left\{ -2 \int_{x_0}^{x} p(t)dt \right\}.$$

Beweis. Es sei $z(x)$ die Lösung der Differentialgleichung (4) mit der Eigenschaft

$$z(x_0) = z'(x_0) = 0, \quad z''(x_0) = k > 0, \qquad \text{bzw. } k < 0.$$

Gemäß (5) können wir dann leicht feststellen, dass $z(x) > 0$ bzw. $z(x) < 0$ für $x > x_0$ ist.

Gemäß (5) erhalten wir für das erwogene $z(x)$ für $x > x_0$

$$z(x)z''(x) + p(x)z(x)z'(x) - \frac{1}{2} z'^2(x) + A(x)z^2(x) > 0.$$

Für dasselbe $z(x)$ kann man die integrale Identität (6) in der Form

$$\left\{ z(x)[z''(x) + p(x)z'(x)] - \frac{1}{2} z'^2(x) + A(x)z^2(x) \right\} \exp \left\{ \int_{x_0}^{x} p(t)dt \right\} = z(x)z''(x_0) - \left[A(x)z^2(x) + \frac{1}{2} z'^2(x) \right] \exp\left\{ \int_{x_0}^{x} p(t)dt \right\} + z(x) \int_{x_0}^{x} [A'(t) + b(t)] z(t) \exp \left\{ \int_{x_0}^{t} p(s)ds \right\} dt$$

schreiben, aus welcher wir für $x > x_0$

$$k\, z(x) - \left[A(x)z^2(x) + \frac{1}{2} z'^2(x) \right] \exp\left\{ \int_{x_0}^{x} p(t)dt \right\} > 0$$

erhalten. Aus der letzten Ungleichheit für $x > x_0$ folgt

$$(12) \quad k\, z(x) - A(x)z^2(x) \exp\left\{ \int_{x_0}^{x} p(t)dt \right\} > 0,$$

$$(13)\quad k\,z(x) - \frac{1}{2}\,z'^2(x)\exp\left\{\int_{x_0}^{x} p(t)dt\right\} > 0.$$

Wenn $k > 0$, bzw. $k < 0$ ist, dann erhalten wir aus (12)

$$0 < z(x) < \frac{k}{A(x)}\exp\left\{-\int_{x_0}^{x} p(t)dt\right\},$$

bzw.

$$0 > z(x) > \frac{k}{A(x)}\exp\left\{-\int_{x_0}^{x} p(t)dt\right\}.$$

Mit Hilfe der zwei letzten Ungleichheiten erhalten wir aus der Beziehung (13)

$$z'^2(x) < \frac{2k^2}{A(x)}\exp\left\{-2\int_{x_0}^{x} p(t)dt\right\}.$$

<u>Satzt 5.</u> Es sei $b(x)$, $p(x) \in C^0(I)$, $A(x) \in C^1(I)$, $I = (-\infty;\infty)$. Für alle $x \in J$ gelte $p(x) \geqslant 0$, $b(x) - A(x)p(x) \geqslant 0$.
$p(x)$ $b(x) - A(x)\,p(x)$ seien in keinem Teilintervall gleichzeitig identisch gleich Null. Es sei $x_0 \in J$ eine beliebige Zahl. Für $x \in J_1 = (x_0;\infty)$ gelte $A(x) \geqslant 0$, $A'(x) - b(x) + 2A(x)p(x) \leqslant 0$ und dabei ist $A'(x) - b(x) + 2A(x)p(x)$ nicht identisch gleich Null in keinem Teilintervall
Es sei

$$\int_{x_0}^{\infty} p(x)dx < \infty,$$

$$\int_{x_0}^{\infty}\left[b(x) - A(x)p(x)\right]dx = +\infty.$$

Es sei $0 \leqslant f(x) \in C^2 \langle x_0;\infty)$ derart, daß $f''(x) + p(x)f'(x) \in C^1(I)$ und für $x \in J_1$ gilt

$$\left[f''(x) + p(x)f'(x)\right]' + p(x)f''(x) + \left[p^2(x) + 2A(x)\right]f'(x) + \left[A'(x) - b(x) + 2A(x)p(x)\right]f(x) \leqslant 0.$$

Dann existiert für jede Lösung $z(x)$ der Differentialgleichung (4) ohne Nullstellen in J_2, $J_2 = (a; \infty)$ $a \geqslant x_0$ ein solches $\bar{a} \geqslant a$, dass für $x \geqslant \bar{a}$

$$|z(x)| - k\, f(x) > 0$$

gilt, wo k irgendeine positive Konstante ist.

Beweis. Es sei $z_1(x)$ die Lösung der Differentialgleichung (4) mit der Eigenschaft

$$z_1(a) = f(a), \quad z_1'(a) = f'(a), \quad z''(a) = f''(a).$$

Es sei $u(x) = z_1(x) - f(x)$, dann entspricht $u(x)$ der Differentialgleichung

$$(14) \quad [u'' + p(x)u']' + p(x)u'' + [p^2(x) + 2A(x)]u' + $$
$$+ [A'(x) - b(x) + 2A(x)p(x)]\, u = -\{ [f''(x) +$$
$$+ p(x)f'(x)]' + p(x)f''(x) + [p^2(x) + 2A(x)]f'(x) +$$
$$+ [A'(x) - b(x) + 2A(x)p(x)]\, f(x) = B(x).$$

Mit dem Übergang zu dem der Gleichung (4) äquivalentem Differentialsystem und durch Verwendung der Variationsmethode der Konstanten erhalten wir, daß auch für die Lösungen der nichthomogenen Gleichung (4) ähnliche Formeln wie für die Gleichung (1) gelten und also $u(x)$ kann in der Form

$$(15) \quad u(x) = \int_a^x B(t) \frac{W(x, t)}{W(t)}\, dt,$$

geschrieben werden, wo

$$W(x, t) = \begin{vmatrix} u_1(x), & u_2(x), & u_3(x) \\ u_1(t), & u_2(t), & u_3(t) \\ u_1'(t), & u_2'(t), & u_3'(t) \end{vmatrix}$$

wobei $u_1(x)$, $u_2(x)$, $u_3(x)$ ein Fundamentalsystem von Lösungen der Differentialgleichung (4) ist, deren Wronskian $W(x)$

positiv ist. Bei festem t ist $W(x, t)$ die Lösung der Differentialgleichung (4) mit einer doppelten Nullstelle im Punkt t und auf Grund der Identität (5) ist für $x \geqslant t$ keine Nullstelle. Es gilt auch, daß $W(x, t) \geqslant 0$ für $x \geqslant t$ ist, weil $W_x''(t, t) > 0$.

Aus der Beziehung (15) folgt, daß $u(x) \geqslant 0$ für $x > a$ ist und also auch $z_1(x) \geqslant f(x)$ für $x > a$ ist. Es sei $z(x)$ eine solche beliebige Lösung der Differentialgleichung (4) ohne Nullstellen in J_2, daß $z(x) > 0$ für $x \in J_2$ ist. Dann können für $z'(x)$ mehrere Fälle eintreten. Wir zeigen, daß für alle $x \in J_2$ kein $z'(x) < 0$ entstehen kann. Wenn dem so wäre, dann würde aus der Differentialgleichung (4)

$$\left\{ \left[z''(x) + p(x) z'(x) \right]' + p(x) z''(x) + p^2(x) z'(x) \right\} \exp \left\{ \int_a^x p(t)\,dt \right\} \geqslant 0$$

folgen. Die letzte Ungleichheit ist mit der Ungleichheit

$$\left(\left[z''(x) + p(x) z'(x) \right] \exp \left\{ \int_a^x p(t)\,dt \right\} \right)' \geqslant 0$$

äquivalent, und aus dieser folgt, daß die Funktion

$$\left[z''(x) + p(x) z'(x) \right] \exp \left\{ \int_a^x p(t)\,dt \right\} = \left[z'(x) \exp \left\{ \int_a^x p(t)\,dt \right\} \right]'$$

für $x \in J_2$ nicht fallend ist. Deshalb ist

$$\left[z'(x) \exp \left\{ \int_a^x p(t)\,dt \right\} \right]' \longrightarrow k_1$$

fur $x \longrightarrow \infty$. Wenn $k_1 > 0$ ist, dann ist von einem gewissen x beginnend $z'(x) \geqslant 0$, was im Widerspruch mit der Voraussetzung steht.

Wenn $k_1 \leqslant 0$ dann, mit Rücksicht darauf, daß

$$\left[z'(x) \exp \left\{ \int_a^x p(t)dt \right\} \right]'$$

nichtfallend für $x \in J_2$ ist, gilt

$$\left[z'(x) \exp \left\{ \int_a^x p(t) \right\} \right]' \leqslant k_1 .$$

Durch Integration der letzten Ungleichheit von a bis x erhalten wir

$$z'(x) \exp \left\{ \int_a^x p(t) dt \right\} \leqslant k(x-a) + z'(a).$$

Im Falle $k_1 < 0$ ist $z'(x) \longrightarrow -\infty$ für $x \longrightarrow \infty$ und also $z(x) \longrightarrow -\infty$ für $x \longrightarrow \infty$. Dies ergibt abermals einen Widerspruch.

Im Falle $k_1 = 0$ gilt

$$z'(x) \leqslant \frac{z'(a)}{\exp \left\{ \int_a^\infty p(t) dt \right\}}$$

und also, wenn $z'(a) < 0$ geht $z(x)$ nach $-\infty$, wenn $x \longrightarrow \infty$. Wenn $z'(a) = 0$ wählen wir die Zahl $a_1 \in J_2$ so, daß $z'(a_1) < 0$ / ein solches a_1 existiert/ und wir erwägen nocheinmal auf dieselbe Weise mit dem Punkt a_1 /anstatt a/.

Deshalb ist von einem gewissen x beginnend entweder $z'(x) \geqslant 0$ oder $z(x)$ hat in J_2 unendlich viele Minima.

Im ersten Falle ist $z(x) \geqslant c > 0$ von einem gewissen $\bar{x} > a$. Dann ist für $x \longrightarrow \infty$

$$\int_{\bar{x}}^x \left[b(t) - A(t)p(t) \right] z^2(t) \exp \left\{ \int_{\bar{x}}^x p(s)ds \right\} dt \geqslant$$

$$\geqslant c^2 \int_{\bar{x}}^x \left[b(t) - A(t)p(t) \right] \exp \left\{ \int_{\bar{x}}^t p(s)ds \right\} dt .$$

Aus der Integralidentität (5) folgt, daß für alle genügend grossen b

$$(16)\quad z(b)z''(b) + p(b)z(b)z'(b) - \frac{1}{2} z^2(b) + A(b)z^2(b) > 0$$

gilt.

Im zweiten Falle erhält $z(x)$ unendlich oft Minimum. Es sei $\bar{b}$ eines davon. Dann gilt

$$z(\bar{b}) > 0, \quad z'(\bar{b}) = 0, \quad z''(b) \geqslant 0$$

und auch (16) von (5) in $b > \bar{b}$. Bezeichnen wir $z_2(x) = z(x) - kz_1(x)$ und wählen wir $k > 0$ so klein, daß

$$(17)\quad z(b) > 0, \quad z(b)z''(b) + p(b)z(b)z'(b) - \frac{1}{2} z'^2(b) + A(b)z^2(b) > 0$$

gilt. Dies ist in beiden Fällen immer möglich. Es sei $x_1 > b$ die erste Nullstelle $z_2(x)$. Für die Lösung $z_2(x)$ im Punkte x_1 hat die Integralidentität (5) die Form

$$(18)\quad \left[z_2(x_1)z_2''(x_1) + p(x_1)z_2(x_1)z_2'(x_1) - \frac{1}{2} z_2'^2(x_1) + A(x_1)z_2^2(x_1)\right] \exp\left\{\int_{x_1}^{b} p(x)dx\right\} - \frac{1}{2}\int_b^{x_1} p(x)z_2'^2(x) \exp\left\{\int_b^x p(t)dt\right\} dx - \int_b^{x_1} \left[b(x) - A(x)p(x)\right]z_2^2(x) \exp\left\{\int_b^x p(t)dt\right\} dx = z(b)z_2''(b) + p(b)z_2(b)z_2'(b) - \frac{1}{2} z'^2(b) + A(b)z_2^2(b) > 0.$$

Und dies ergibt einen Widerspruch mit der Voraussetzung. Also ist $z_2(x) > 0$ für $x > b$ und also

$$z(x) - k\, z_1(x) > 0, \qquad z(x) > k\, z_1(x) \geqslant k \cdot f(x)$$

für $x > b$. Damit ist der Beweis des Satzes beendet.

LITERATÚRA

[1] Greguš M., Über die asymptotischen Eigenschaften der Lösungen der linearen Differentialgleichung dritter Ordnung, Annali di Matematica pura ed applicata, IV., LXIII, 1963, 1-10

[2] Greguš M., Über die lineare homogene Differentialgleichung dritter Ordnung, Wiss. Zeitschr. Univ. Halle, Math., Nat., XII/3, 1963, 256-286

[3] Moravský L., O niektorých vlastnostiach riešení diferenciálnej rovnice tvaru $y''' + p(x)y'' + 2A(x)y' + [A'(x) + b(x)]y = 0$, Acta, F. R. N. Univ. Comen., Mathematica XIII, 1966, 61-68

[4] Ráb M., Oscilační vlastnosti integrálu difereniální rovnice 3. řádu. Práce Brn. Československ. Akad. Věd 27, (7) 1955

[5] Švec M., Sur une propriéte des intégrales de 1 equation $y^{(n)} + Q(x)y = 0$, $n = 3, 4$, Czechoslovak Math. J. 7/82, 1957, 450-461

[6] Švec M., Neskoľko zamečanij o linejnom diferencijaľnom uravneniji tretjego porjadka, Čech. mat. žurnal, 15 (90) 1965, 42-49

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Eingegangen am 5. Mai 1971, in den Verlag am 28. Mai 1973

(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXIX, 1974)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXIX - 1974

ON SOME CLASSES OF SETS OF NATURAL NUMBERS

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In this paper we shall follow the paper [1], where certain subfamilies of the family U of all infinite subsets of the set N of all natural numbers are considered.

According to [1] let S_1, be the family of all $A \in U$ with the property: The set A contains with every its element all its natural divisors. Futher let S_2 be the family of all $A \in U$ such, that if $a, b \in A$, $a \neq b$, then $a \not| b$ and $b \not| a$.

We shall also consider families $T_i(F)(\subset U)(i = 1, 2)$ introduced (in [1]) by the following way: Let F be any function defined on N with values in U. Then

$$T_1(F) = \{A \in U : \text{if } a \notin A, \text{ then } F_a \cap A = \emptyset\}$$

$$T_2(F) = \{A \in U : \text{if } a \in A, \text{ then } F_a \cap A = \emptyset\},$$

where $F_a = F(a)$.

Let ϱ be a function defined on U, which to each $A \in U$ assignes its dyadic value, i. e. $\varrho(A) = \sum_{k=1}^{\infty} \mathcal{E}_k 2^{-k}$, where $\mathcal{E}_k = 1$ if $k \in A$ and $\mathcal{E}_k = 0$ in the opposite case. Obviously ϱ is a one to one map of U onto $(0, 1\rangle$. It is easy to see, that if n is fixed then the whole interval $(0, 1\rangle$ is the sum of the basic intervals of the order n

$$i_n^{(s)} = (s/2^n, (s+1)/2^n\rangle \qquad (s = 0, 1, \ldots, 2^n - 1).$$

Each of these intervals is determined by the suquence of n elements $\mathcal{E}'_1, \ldots, \mathcal{E}'_n$ of zeros and ones (if $x = \sum_{k=1}^{\infty} \mathcal{E}_k 2^{-k}$ is the infinite dyadic expansion of any number $x \in (s/2^n, (s+1)/2^n\rangle$ then $\mathcal{E}'_k = \mathcal{E}_k$ for $k = 1, \ldots, n$) and different basic intervals correspond to different sequences.

Further we shall investigate sets $\varrho(S_i)$ and $\varrho(T_i(F))$ ($i = 1, 2$) from the point of view of Borel's classification of sets.

Let V be any subfamily of the family $U (V \subset U)$. Let us define for each $n \in N$ a family $V(n) (\subset U)$ in the following way:

$$V(n) = \{A \in U : \exists_{B \in V} A \cap \langle 1, n\rangle = B \cap \langle 1, n \rangle\}$$

We shall say, that the family V is of the type (α) if the following inclusion holds

$$(\alpha) \qquad \bigcap \{V(n) : n \in N\} \subset V.$$

Obviously for every $n \in N$ we have $V \subset V(n)$.

Above introduced family S_1 is of the type (α), because $S_1(n) = \{A \in U : A$ contains with every its element $a \leq n$ all its natural divisors$\}$. If $A \notin S_1$, then there is $a \in A$ such, that some of its natural divisors d ($d \leq a$) does not belong to A, i.e. $A \notin S_1(a)$, hence the inclusion (α) is fulfilled if V is replaced by S_1. Analogously it is possible to persuade that every of introduced families S_2 and $T_i(F)$ ($i = 1, 2$) is of the type (α).

We shall introduce an example of family V, which is not of the type (α). Let $\{a_n\}_{n \in N}$ be a sequence of positive numbers such, that $a_n \to 0$ and $\sum_{n \in N} a_n = +\infty$. Let $V = \{A \in U: \sum_{n \in A} a_n < +\infty\}$. The inclusion (α) is not fulfilled, because $N \in \bigcap \{V(n) : n \in N\} = U$, but $N \notin V$.

T h e o r e m 1. Let a family $V (\subset U)$ be of the type (α). Then $\varrho(V) (\subset (0, 1\rangle)$ is a set of the type G_δ.

Proof. The inclusions $V \subset V(n)$ $(n \in N)$ imply $V \subset \bigcap \{V(n) : n \in N\}$. Together with the inclusion (α) validity of which is quarranteed by the assumption of Theorem 1 $V = \bigcap \{V(n) : n \in N\}$ follows. $\wp(V) = \bigcap \{\wp(V(n)) : n \in N\}$ follows from the properties of the map $\wp$. Hence, it is sufficient to prove, that each of sets $\wp(V(n))$ $(n \in N)$ is of the type G_δ.

Let us define for each $n \in N$ a family h_n of basic intervals of the order n by the following way : $h_n = \{ i_n^{(s)} :$ there is $A \in V(n)$ such, that $i_n^{(s)}$ is determined by the sequence $\mathcal{E}_1, \dots, \mathcal{E}_n$, where $\mathcal{E}_k = 1$ if $k \in A$ and $\mathcal{E}_k = 0$ if $k \notin A \}$. We shall prove $\wp(V(n)) = \bigcup \{ i_n^{(s)} : i_n^{(s)} \in h_n \}$. Let $x \in \wp(V(n))$ ($x = \sum_{k=1}^{\infty} \mathcal{E}_k 2^{-k}$ is the infinite dyadic expansion of x). Then there is $A \in V(n)$ such, that $x = \wp(A)$, i. e. $x \in i_n^{(s)}$, where the interval $i_n^{(s)}$ is determined by the sequence $\mathcal{E}_1, \dots, \mathcal{E}_n$ and obviously belongs to the family h_n - $x \in \bigcup \{ i_n^{(s)} : i_n^{(s)} \in h_n \}$. If $x \notin \wp(V(n))$, then for each $A \in V(n)$ $\wp(A) \neq x$ holds and x does not belong to any interval of the family h_n, $x \in \bigcup \{ i_n^{(s)} : i_n^{(s)} \in h_n \}$. So we proved, that every of sets $\wp(V(n))$ $(n \in N)$ as a finite sum of sets of the type G_δ is G_δ.

Theorem 2. Let each of families S_i and $T_i(F)$ $(i = 1, 2)$ have the introduced meaning. Then sets $\wp(S_i)$ and $\wp(T_i(F))$ $(i = 1, 2)$ are of the type G_δ.

Proof. Families, which are concerned by the statement of Theorem 2, are of the type (α). Hence according to Theorem 1 sets $\wp(S_i)$ and $\wp(T_i(F))$ $(i = 1, 2)$ are of the type G_δ.

Remark. Non-void sets $\wp(S_i)$ $(i = 1, 2)$ belong to no zero Borel´s class. Above mentioned sets are according to Theorem 3 and Theorem 1 of [1] nowhere dense in $(0, 1)$, hence are not open. It is easy to see, that $\wp(S_i)$ $(i = 1, 2)$ are also not closed. Really, let $\{p_n\}_{n \in N}$ be the increasing sequence of all prime numbers, let $A_n = \{1, p_n, p_{n-1}, \dots\}$ $(n \in N)$. Obviously

$A_n \in S_1$. Then $\varrho(A_n) = 2^{-1} + \sum_{k=n}^{\infty} 2^{-p_k} \longrightarrow 2^{-1} = \varrho(A)$, where $A = \{2, 3, \dots\} \notin S_1$. If we put $B_n = \{2, p_n, p_{n+1}, \dots\}$ $(n = 2, 3, \dots)$, then $B_n \in S_2$ and $\varrho(B_n) = 2^{-2} + \sum_{k=n}^{\infty} 2^{-p_k} \longrightarrow 2^{-2} = \varrho(B)$, where $B = \{3, 4, \dots\} \in S_2$. Cosequently sets $\varrho(S_i)$ $(i = 1, 2)$ are not closed.

Similarly, it is possible to check that sets $\varrho(T_i(F))$ $(i = 1, 2)$ in general belong to the no Borel's zero class. Sets $\varrho(T_i(F))$ $(i = 1, 2)$ are open if and only if are void. It follows immediately from Theorem 3 of [1] according to which $\varrho(T_i(F))$ are nowhere dense. If for every $a \in N$ we put $F_a = \{2, 4, 6, \dots\}$ and define sets $A_n = \{1, 2n+1, 2n+3, \dots\} \in T_i(F)$ $(n \in N)$, then $\varrho(A_n) = 2^{-1} + \sum_{k=1}^{\infty} 2^{-2n-2k+1} = 2^{-1} + 3^{-1} . 2^{1-n} \longrightarrow 2^{-1} =$

$= \varrho(A)$, where $A = \{2, 3, \dots\} \notin T_i(F)$ $(i = 1, 2)$. Hence in general the sets $\varrho(T_i(F))$ $(i = 1, 2)$ are not closed.

But there are sequences of elements from $\varrho(V)$ $(V \subset U)$ such, that under certain circumstances ones converge to an element in $\varrho(V)$. It says the following theorem.

Theorem 3. Let the family $V(\subset U)$ be of the type (α) and let $x_m \in \varrho(V)$ $(m \in N)$, $x_m \longrightarrow x \in (0, 1\rangle$, where x is not dyadic rational number. Then $x \in \varrho(V)$.

Proof. Let $x_m = \sum_{k=1}^{\infty} \varepsilon_k^{(m)} 2^{-k} \in \varrho(V)$ and let $x_m \longrightarrow$ $\longrightarrow x \in (0, 1\rangle$ $(x = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k})$. Let $x = \varrho(A)$. We shall show that $A \in V$. The point x determines a sequence of intervals $i_n^{(s_n)}$ $(n \in N)$ tuch, that for each n x is an interior point of the interval $i_n^{(s_n)}$. For every n there is m_n such that, for $m \geq m_n$, $x_m \in i_n^{(s_n)}$ holds, i. e. $\varepsilon_k^{(m)} = \varepsilon_k$ for $k = 1, 2, \dots, n$. Consequently the set $A_{m_n} \in V(x_{m_n} = \wp(A_{m_n}))$ agrees with the set A on the interval $\langle 1, n \rangle$ and hence $A \in V(n)$ holds for each $n \in N$, $A \in V$.

T h e o r e m 4. Let $x_m \in \wp(S_i)$ $(x_m \in \wp(T_i(F)))(m \in N)$, $x_m \longrightarrow x \in (0, 1\rangle$, where x is not dyadic rational number. Then $x \in \wp(S_i)$ $(x \in \wp(T_i(F)))(i = 1, 2)$.

P r o o f. Because families S_i $(T_i(F))$ $(i = 1, 2)$ are of the type (α) the assertion of Theorem 4 is a consequence of Theorem

REFERENCES

[1] Šalát T., On certain classes of sets of natural numbers, Matemat. časop. (to appear)

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Received August 10, 1970, at Publishers May 28, 1973

(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXIX, 1974)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXIX - 1974

A NOTE ON TOPOLOGIES COMPATIBLE WITH THE ORDERING

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In the present paper a theorem concerning topologies on ordered sets is proved. As a corollary we get a positive answer to the problem 4.21 of the paper [1].

1. Basic contepts

By an ordered set we mean a partially ordered set. Given a subset F of an ordered set A we denote $F^{*} = \{ z : z \geqq x \text{ for all } x \in F \}$, $F^{+} = \{ z : z \leqq x \text{ for all } x \in F \}$. $(x] = \{ z : z \leqq x \}$, $[x) = \{ z : z \geqq x \}$.

Consider a set A with an order and with a topology τ. We say the topology τ is strongly compatible with the given ordering [1] if τ is a T_1-topology and if for every pair $a, b \in A$, $a < b$, there exists a neigbourhood $0(a)$ of the point a and a neigbourhood $0(b)$ of the point b such that

$$x \in 0(a), \quad y \in 0(b) \Longrightarrow x < y \text{ or } x \parallel y .$$

Let $\mathfrak{m}$ be an arbitrary infinite cardinal number. A subset I of an ordered set A is called an $\mathfrak{m}$- ideal (dual $\mathfrak{m}$- ideal) if for each non empty subset M of I with card $M < \mathfrak{m}$ the inclusion $M^{*+} \subset I$ ($M^{+*} \subset I$) holds. An $\mathfrak{m}$-ideal I (a dual $\mathfrak{m}$-ideal) is called completely irreducible iff it is not an intersection of $\mathfrak{m}$-ideals (dual $\mathfrak{m}$-ideals) different from I.

By an $\mathfrak{m}$-ideal topology on an ordered set A we mean the topology whose subbasis of open sets is the set of all completely irreducible $\mathfrak{m}$-ideals and dual $\mathfrak{m}$-ideals of A.

In the case $\mathfrak{m} = \aleph_0$ we get the ideal topology in the sense of FRINK[2].

2. The following lemma is obvious

Lemma. Let $\mathfrak{m}$ be an infinite cardinal number. An intersection of $\mathfrak{m}$ - ideals or of dual $\mathfrak{m}$-ideals is an $\mathfrak{m}$-ideal or a dual $\mathfrak{m}$-ideal respectively.

Theorem. Let A be an ordered set, let $\mathcal{T}_{\mathfrak{m}}$ be the $\mathfrak{m}$-ideal topology on a which fulfils the axiom T_2. Then $\mathcal{T}_{\mathfrak{m}}$ is strongly compatible with the ordering.

Proof. Let $a, b \in A$, $a < b$. Suppose first that there exist such neighbourhoods $0(a)$, $0(b)$ that $0(a)$ is an $\mathfrak{m}$-ideal, $0(b)$ a dual $\mathfrak{m}$-ideal and $0(a) \cap 0(b) = \emptyset$. If $x \in 0(a)$, $y \in 0(b)$ then $y \leqslant x$ fails to be true.

In fact, $y \leqslant x$ implies $y \in (x]$, hence $y \in 0(a)$ (since $(x] \subset \subset 0(a)$)- a contradiction.

To complete the proof of the theorem it is therefore sufficient to prove that to any pair of points a, b with $a < b$ there exist disjoint neighbourhoods $0(a)$, $0(b)$ such that $0(a)$ is an $\mathfrak{m}$-ideal and $0(b)$ a dual $\mathfrak{m}$-ideal.

Let $a < b$. By the hypothesis of the Theorem there exist open subsets 0, Q such that $a \in 0$, $b \in Q$, $0 \cap Q = \emptyset$. Without logs of generality we can assume that both 0 and Q are elements of a basis of open sets. For the basis we can set the family of all intersections of finite number of subbasis elements. Thus $0 = = \bigcap\{A_k : 1 \leqq k \leqq n\}$, $Q = \bigcap\{B_k : 1 \leqq k \leqq m\}$, the family $\mathcal{S} = \{A_1, A_2, \dots A_n, B_1, B_2, \dots B_m\}$ being a subfamily of the subbasis. The elements of $\mathcal{S}$ are either $\mathfrak{m}$-ideals or dual $\mathfrak{m}$-ideals.

$$\text{Set } 0(a) = \bigcap\{C : C \in \mathcal{S},\ C \text{ is an } \mathfrak{m}\text{-ideal}\},$$

$0(b) = \bigcap\{C : C \in \mathcal{S}$, C is a dual $\mathfrak{m}$-ideal$\}$. If all elements A_k $k = 1, 2, \dots n$ were dual $\mathfrak{m}$-ideals, then by Lemma, 0 would be a dual $\mathfrak{m}$-ideal too. Since $a \in 0$, it holds $[a) \subset 0$ and therefore $b \in 0$ which gives a contradiction. Hence at least one element of $\mathcal{S}$ is an $\mathfrak{m}$-ideal and $0(a)$ is an intersection of a nonempty family of sets. In a similar way it can be proved that at least one element of $\mathcal{S}$ is a dual $\mathfrak{m}$-ideal and therefore $0(b)$ is an intersection of a non empty family.

Let A_k be an $\mathfrak{m}$-ideal in $\mathcal{S}$. Since $a \in 0$, $a \in A_k$. Let B_k be an $\mathfrak{m}$-ideal in $\mathcal{S}$. Then $b \in Q$ implies $b \in B_k$. B_k being an $\mathfrak{m}$-ideal, $a \in B_k$. It was just shown that every $\mathfrak{m}$-ideal in $\mathcal{S}$ contains a. Thus $a \in 0(a)$. It can be proved in a similar way that $b \in 0(b)$.

Now prove that $0(a) \cap 0(b) = \emptyset$. In fact $0(a) \cap 0(b) = A_1 \cap A_2 \cap \ldots \cap A_n \cap B_1 \cap B_2 \cap \ldots \cap B_m = 0 \cap Q = \emptyset$. By Lemma $0(a)$ is an $\mathfrak{m}$-ideal and $0(b)$ a dual $\mathfrak{m}$-ideal. Thus the sets $0(a)$, $0(b)$ have the described properties.

Remark. Setting $\mathfrak{m} = \aleph_0$ we get the theorem 1 [3].

Problem. Does the $\mathfrak{m}$-ideal topology for $\mathfrak{m} = \aleph_0$ satisfy the axiom T_2?

REFERENCES

[1] Sekanina A., Sekanina M., Topologies compatible with ordering. Archivum Mathematicum (Brno) 2 (1966), 113-120

[2] Frink O., Ideals on a partially ordered sets. Am. Math. Monthly 61 (1954), 223-234

[3] Rosický J., A note on topology compatible with the ordering. Archivum Mathematicum (Brno) 5 (1969), 19-24

[4] Mayer J., Novotný M., On some topologies on products of ordered sets. Archivum Mathematicum (Brno) 1 (1965), 251-257

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Received, April 29, 1971, at Publishers May 28, 1973

(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXIX, 1974)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXIX – 1974

TRANSLATIONS AND ENDOMORPHISMS IN UNIVERSAL ALGEBRAS

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Translations in lattices and semilattices were investigated by G. SZÁSZ [1], [2], SZÁSZ and SZENDREI [3] and M. KOLIBIAR [4]. In the paper [5] SZÁSZ is dealing with translations in groupoids.

In this Note we prove that some results from [5] (Theorems 1 and 2) can be generalized for universal algebras with finitary operations. On the other hand the Proposition 3 [5] does not hold in general for algebras with n-ary operations. SZÁSZ [5] put the question if every translation that is an endomorphism on a groupoid must be idempotent. A simple example shows that the answer is negative. We also prove that in an idempotent algebra every translation is an endomorphism.

We shall be concerned with an algebra A with one n-ary operation (i. e., in the whole paper, "algebra" means an algebra with one n-ary operation ω); the case of more operations is analogous. We need the following definitions:

Definition 1. A subset I of an algebra A is said to be an ideal of the algebra A if $\omega(a_1, \ldots, a_i, \ldots, a_n) \in I$ whenever at least one of the elements $a_1, \ldots, a_n$ belongs to I.

Definition 2. A mapping φ of the set A into A is called a translation of the algebra A, if for each n-tuple $(a_1, \ldots, a_i, \ldots, a_n)$ $(a_i \in A)$ and each i $(1 \leq i \leq n)$.

$$\varphi[\omega(a_1, \ldots, a_i, \ldots, a_n)] = \omega(a_1, \ldots, \varphi(a_i), \ldots, a_n)$$

is true.

Definition 3. An endomorphism of an algebra A is a mapping $\varphi : A \longrightarrow A$ with the property

$$\varphi[\omega(a_1, \ldots, a_i, \ldots, a_n)] = \omega(\varphi(a_1), \ldots, \varphi(a_i), \ldots, \varphi(a_n))$$

for any n-tuple $(a_1, \ldots, a_i, \ldots, a_n)$ $(a_i \in A)$.

Definition 4. The mapping φ with the property $\varphi(a) = \varphi[\varphi(a)]$ for each $a \in A$ is called idempotent (we shall write $\varphi^2(a)$ instead of $\varphi[\varphi(a)]$).

Definition 5. The algebra is factorisable if for each $a \in A$ there exists an n-tuple $(a_1, \ldots, a_i, \ldots, a_n)$ such that $\omega(a_1, \ldots, a_i, \ldots a_n) = a$.

Definition 6. An algebra A will be called idempotent if $\omega(a, \ldots, a, \ldots, a) = a$ for each $a \in A$.

The following Theorems 1 and 2 generalize Theorems 1 and 2 from the paper [5].

Theorem 1. Let φ be a translation of an algebra A. Then $\varphi(A)$ is an ideal in A. If φ is an idempotent mapping, then φ is an endomorphism.

Proof. First we prove the second assertion of the Theorem. Let φ be an idempotent translation of the algebra A. Cleary $\varphi = \varphi^n$. Then for any n-tuple $(a_1, \ldots, a_i, \ldots, a_n)$ $(a_i \in A)$ $\varphi[\omega(a_1, \ldots, a_i, \ldots, a_n)] = \varphi^n[\omega(a_1, \ldots, a_i, \ldots, a_n)] = \varphi^{n-1}[\varphi(\omega(a_1, \ldots, a_i, \ldots, a_n))] = \varphi^{n-1}[\omega(\varphi(a_1, a_2, \ldots, a_i, \ldots, a_n))] =$

$= \ldots = \varphi[\omega(\varphi(a_1), \ldots, \varphi(a_i), \ldots, \varphi(a_{n-1}), a_n)] =$

$= \omega(\varphi(a_1), \ldots, \varphi(a_i), \ldots, \varphi(a_n))$;

thus φ is an endomorphism in A.

Next we prove that $\varphi(A)$ is an ideal in A where φ is a translation of the algebra A. Consider the n-tuple $(a_1, \ldots, a_i, \ldots a_n)$ such that $a_1 \in \varphi(A)$. Then there exists $x \in A$ with $\varphi(x) = a_1$, and $\omega(a_1, \ldots, a_i, \ldots a_n) = \omega(\varphi(x), a_2, \ldots, a_i, \ldots, a_n) = \varphi[\omega(x, a_2, \ldots, a_i, \ldots, a_n)] \in \varphi(A)$ is valid. The proof for $a_k \in \varphi(A)$, $k = 2, \ldots, n$ is similar.

Theorem 2. Let φ be an idempotent endomorphism of the algebra A such that $\varphi(A)$ is an ideal in A. Then φ is a translation of the algebra A .

Proof. Choose an arbitrary n-tuple $(a_1, \ldots, a_i, \ldots, a_n)$. Then $\omega(a_1, \ldots, \varphi(a_i), \ldots, a_n) \in \varphi(A)$ for each $i = 1, \ldots, n$. Hence there exists $z \in A$ with the property $\varphi(z) = \omega(a_1, \ldots, \varphi(a_i), \ldots, a_n)$.

Further we have

$$\omega(a_1, \ldots, \varphi(a_1), \ldots, an) = \varphi(z) = \varphi^2(z) = \varphi[\varphi(z)] = \varphi[\omega(a_1, \ldots, \varphi(a_i), \ldots, a_n)] = \omega(\varphi(a_1), \ldots, \varphi^2(a_i), \ldots, \varphi(a_n)) = \varphi[\omega(a_1, \ldots, a_i, \ldots, a_n)].$$

Thus we have shown that φ is a translation of the algebra A.

In the Theorem 2, the assumption of the idempotency of the mapping φ cannot be omitted. Example: Take the open interval $(0; \infty) = K$ with a ternary operation ω defined as follows:

$$\omega(x, y, z) = x y z \quad \text{for each} \quad x, y, z \in K.$$

Put $\varphi(x) = \frac{1}{x}$ for each $x \in K$. Then φ is an endomorphism and $\varphi(K)$ is an ideal in K because $\varphi(K) = K$. Obviously φ φ is not a translation.

Let G be a groupoid and assume that $\varphi : G \to G$ is a mapping such that φ is an endomorphism and a translation. SZÁSZ [5] put the question whether φ must be idempotent (he proved that the answer is positive in the case when G is factorisable). The following simple example shows that for general groupoids the answer is negative.

Let $G = \{a, b, c\}$ be a set with three elements and define $x.y = b$ for each $x, y \in G$. Put $\varphi(a) = \varphi(b) = b$, $\varphi(c) = a$ Then φ satisfies the conditions

$$(1^o) \qquad \varphi(x.y) = \varphi(b) = b,$$

(2°) $\qquad x \,.\, \varphi(y) = \varphi(x).\, y = b,$

(3°) $\qquad \varphi(x).\ \varphi(y) = b,$

for each $x, y \in G$. (1°), (2°) imply that φ is a translation. From (1°) and (3°) it follows that φ is an endomorphism. The mapping φ is not idempotent because of $\varphi^2(c) = b \neq a = \varphi(c)$.

Theorem 3. Let φ be a translation of an algebra A. Assume that φ is also an endomorphism of A and that the ideal $\varphi(A)$ is factorisable. Then $\varphi = \varphi^n$.

Proof. Let $a \in A$. Since $\varphi(A)$ is factorisable and $\varphi(a) \in \varphi(A)$, there are elements $a_1, \ldots, a_n \in A$ such that

$$\varphi(a) = \omega(\varphi(a_1), \ldots, \varphi(a_i), \ldots, \varphi(a_n)).$$

Then we have

$$\varphi(a) = \omega(\varphi(a_1), \ldots, \varphi(a_i), \ldots, \varphi(a_n)) = \varphi[\omega(a_1, \varphi(a_2), \ldots, \varphi(a_i), \ldots, \varphi(a_n))] = \ldots = \varphi^{n-1}[\omega(a_1, \ldots, a_i, \ldots, a_{n-1}\varphi(a_n))] = \varphi^{n-1}[\varphi(\omega(a_1, \ldots, a_i, \ldots, a_n))] = \varphi^{n-1}[\omega(\varphi(a_1), \ldots, \varphi(a_i), \ldots, \varphi(a_n))] =$$

$$\varphi^{n-1}[\varphi(a)] = \varphi^n(a).$$

Let us remark that under the same ussumptions as in Theorem 3 the mapping φ need not be idempotent. Let us consider the closed interval $A = \langle -1; 1\rangle$ with the ternary operation ω defined by the rule $\omega(x, y, z) = xyz$ for each $x, y, z \in A$. Evidently A is factorisable. Define the mapping φ in A by putting $\varphi(x) = -x$. Then φ is an endomorphism and a translation, too, but $\varphi(x) = -x$ $\varphi^2(x) = -(-x) = x$, thus φ is not idempotent.

Theorem 4. Every translation of an idempotent algebra is an endomorphism.

Proof. If suffices to show that $\varphi = \varphi^n$ and then in view of the proof of the Theorem 1 the proof will be complete. Let φ be a translation of an idempotent algebra A, let $a \in A$ be an arbitrary element. Then $a = \omega(a, \ldots, a, \ldots, a)$ holds. From this we get $\varphi(a) = \omega(\varphi(a), \ldots, \varphi(a), \ldots, \varphi(a)) = \varphi[\omega(a, \varphi(a), \ldots, \varphi(a), \ldots, \varphi(a))] = \ldots = \varphi^n[\omega(a, \ldots, a, \ldots, a)] = \varphi^n(a)$.

Using the example given to the question of SZÁSZ it is easy to construct an example of the algebra with an n-ary operation for each $n > 2$ to show that the assumption $\varphi = \varphi^n$ is not necessary for the translation of an algebra to be an endomorphism.

The following theorem gives a necessary condition.

Theorem 5. Let φ be a translation of an algebra A such that φ is an endomorphism on A. Then for each $a \in A$ we have

$$(1)\ \omega(\varphi(a), \ldots, \varphi(a), \ldots, \varphi(a)) = \omega(\varphi^n(a), \ldots, \varphi^n(a), \ldots, \varphi^n(a)).$$

Proof. $\omega(\varphi(a), \ldots, \varphi(a), \ldots, \varphi(a)) = \varphi[\omega(a, \varphi(a), \ldots, \varphi(a), \ldots, \varphi(a))] = \ldots = \varphi^n[\omega(a, \ldots, a, \ldots, a)] = \varphi^{n-1}[\omega(\varphi(a), \ldots, \varphi(a), \ldots, \varphi(a))] = \ldots = \omega(\varphi^n(a), \ldots, \varphi^n(a), \ldots, \varphi^n(a))$.

Remark. If φ is a translation of an algebra A such that for each $a \in A$ the identity (1) is true then φ need not be an endomorphism. Let us consider the following example:

In the groupoid $G = \langle -1; 1 \rangle$ with the operation of the ordinary multiplication the mapping φ is defined by the rule $\varphi(x) = -x$. The identity (1) from the Theorem 5 says that $[\varphi(x)]^2 = [\varphi^2(x)]^2$. φ is a translation in G. $[\varphi(x)]^2 = (-x).(-x) = x^2$ and $[\varphi^2(x)]^2 = x.x = x^2$. Clearly φ is not an endomorphism.

REFERENCES

[1] Szász G., Translationen der Verbände. Acta fac. rer. nat. Univ. Comenianae. Mathematica. 5 (1961), 449-453

[2] Szász G., Die Translationen der Halbverbände. Acta Sci. Math., 17 (1956), 165-169

[3] Szász G. und Szendrei J., Über die Translationen der Halbverbände Acta Sci. Math., 18 (1957), 44-47

[4] Kolibiar M., Bemerkungen über Translationen der Verbände. Acta fac. rer. nat. Univ. Comen. Math., 5 (1961), 455-458

[5] Szász G., Gruppoid transzlácioi és endomorfizmusai. Acta Acad. Pedagogicae Nyiregyháziensis Tom 2 (1968), 143-146

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Received April 5, 1971, at Publishers May 28, 1973

(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXIX, 1974)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXIX - 1974

Одна оценка функции $S(t)$ в теории дзета-функции Римана

ЯН МОЗЕР, Братислава

В этой заметке приводится одно простое свойство функции $S(t)$. Прежде чем сформулировать соответствующее утверждение, напомним (а) определение функции $S(t)$:

Если t неявляется ординатой нуля $\zeta(s)$, то

$$S(t) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + it\right).$$

где $\arg \zeta(\frac{1}{2} + it)$ получается непрерывным продолжением $\arg \zeta(s)$ вдоль ломаной линии, начинающей в точке $s = 2$, ($\arg \zeta(2) = 0$), илущей к точке $s = 2 + it$ и затем к точке $\frac{1}{2} + it$. Если же t-ордината нуля $\zeta(s)$, то мы положим $S(t) = S(t+0)$, см. [1], гл. IX, п.3., стр. 209.

(б) одну конструкцию:

Кратному нулю $\beta + i\gamma$ функции $\zeta(s)$ ставится в соответствие его γ-ордината.

Конечному числу нулей $\beta + i\gamma$ с одинаковой γ-ординатой ставится в соответствие эта общая γ-ордината.

Таким образом определена возрастающая последовательность γ-ординат нулей функции $\zeta(s)$ и также придан смысл термину "соседние γ-ординаты".

Известно, что независимо от какой бы то ни было гипотезы, имеет место оценка

$$S(t) = O(\ln t).$$

Здесь узнано, что на некторой бесконечной системе промежутков имеет место несколько лучшая оценка. Именно, имеет место

Т е о р е м а. Независимо от гипотезы Римана, существует бесконечная система промежутков $<\gamma', \gamma'')$ такого рода, что

$$S(t) = O\left(\frac{\ln t}{\ln \ln \ln t}\right), \quad t \in <\gamma', \gamma''),$$

в частности

$$S(\gamma''-0) = O\left(\frac{\ln \gamma''}{\ln \ln \ln \gamma''}\right),$$

где γ', γ'' – соседние γ-ординаты.

Доказательство существенно опирается на одну теорему Титчмарша и на одну теорему Литтлвуда.

Сначала перечислим вспомогательные утверждения. Известно, см. [2], стр. 174, Пример (III), что

$$\int_0^{\infty} \frac{[u] - u + \frac{1}{2}}{u + z}\, du = \frac{1}{12\, z} + O\left(\frac{1}{|z|^2}\right).$$

Если принять во внимание, что

$$\text{(a)} \qquad \varphi(u) = \int_0^u \left([v] - v + \frac{1}{2}\right) dv ,$$

$$\varphi(u) = \sum_{\nu=1}^{\infty} \frac{1-\cos 2\pi\nu u}{2\pi^2 \nu^2},$$

$$\int_0^{\infty} \frac{[u] - u + \frac{1}{2}}{u + z}\, du = \int_0^{\infty} \frac{\varphi(u)}{(u+z)^2}\, du,$$

(б) изменение порядка интегрирования и суммирования оправдано, то, интегрируя по частям, нетрудно получить члены обозначенные символом $O\left(\frac{1}{|z|^2}\right)$ в явном виде. Именно, имеет место

Л е м м а 1. Если z комплексное число с положительной действительной частью, то

$$\int_0^{\infty} \frac{[u] - u + \frac{1}{2}}{u + z}\, du =$$

$$= \frac{1}{12z} - \frac{1}{4\pi^4} \sum_{\nu=1}^{\infty} \frac{1}{\nu^4 z^3} + \frac{3}{4\pi^4} \sum_{\nu=1}^{\infty} \frac{1}{\nu^4} \int_0^{\infty} \frac{\cos 2\pi\nu u}{(u+z)^4}\, du .$$

Полагая $z = \frac{1}{4} + i\frac{T}{2}$, и отделяя мнимую часть, получается

Л е м м а 2.

$$\mathrm{Im}\left\{ \int_0^{\infty} \frac{u - u + \frac{1}{2}}{u + \frac{1}{4} + i\frac{T}{2}}\, du \right\} =$$

$$= -8 A_1 \frac{T}{1+4T^2} - 128 A_2 \frac{3T + 4T^3}{(1+4T^2)^3} -$$

$$- 2048\, A_3\, T \sum_{\nu=1}^{\infty} \frac{1}{\nu^4} \int_0^{\infty} \frac{(4u+1)-4T^2}{\left[(4u+1)^2+4T^2\right]^4}\,(4u+1)$$

$\cos(2\pi\nu u)\,du$,

где $A_1 = \frac{1}{12}$, $A_2 = -\frac{1}{4\pi^4}\sum_{\nu=1}^{\infty}\frac{1}{\nu^4}$, $A_3 = \frac{3}{4\pi^4}$.

Просматривая последнее соотношение 3.п., гл. IX, [1], стр. 210, и, используя формулу

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \int_0^{\infty}\frac{[u]-u+\frac{1}{2}}{u+z}\,du$$

при $z = \frac{1}{2} + i\,\frac{T}{2}$, нетрудно получить в явном виде члены, обозначенные там как $O\left(\frac{1}{T}\right)$.

Если $f(T)$ обозначает эти члены, то имеет место

Л е м м а 3.

$$f(T) = \frac{3}{16}\,\frac{1}{T} + \frac{1}{4}\sum_{n=2}^{\infty}(-1)^{n-1}\frac{4n-1}{4n^2-2n}\left(\frac{1}{2T}\right)^{2n-1} +$$

$$+ \operatorname{Im}\left\{\int_0^{\infty}\frac{[u]-u+\frac{1}{2}}{u+\frac{1}{4}+i\frac{T}{2}}\,du\right\},$$

т.е. (см. Л. 2.),

$$f(T) = \frac{3}{16}\,\frac{1}{T} - 8A_1\frac{T}{1+4T^2} - 128A_2\frac{3T+4T^3}{(1+4T^2)^3} +$$

$$+ \frac{1}{4}\sum_{n=2}^{\infty}(-1)^{n-1}\frac{4n-1}{4n^2-2}\left(\frac{1}{2T}\right)^{2n-1} -$$

$$- 2048\, A_3 T \sum_{\nu=1}^{\infty} \frac{1}{\nu^4} \int_0^{\infty} \frac{(4u+1)^2 - 4T^2}{\left[(4u+1)^2 + 4T^2\right]^4} (4u+1)$$

$\cos\,(2\pi\nu u)\, du$.

Используя Л. 3., теорему Лагранжа о среднем значении, и простой алгебраический прием, получается

<u>Лемма 4.</u> Если (для определенности) $t' < t''$, то

$$f(t'') - f(t') = -\frac{3}{16}\,\frac{t''-t'}{T_1^2} - 8\, A_1 \frac{1-4T_2^2}{(1+4T_2)^2}\,(t''-t') -$$

$$- 128\, A_2 \frac{3-48T_3^2-48T_3^4}{(1+4T_3^2)^4}\,(t''-t') +$$

$$+ \sum_{n=2}^{\infty} (-1)^n \left(1-\frac{1}{4n}\right)\left(\frac{1}{2\, Tn}\right)^{2n} (t''-t') -$$

$$- 2048\, A_3 \sum_{\nu=1}^{\infty} \frac{1}{\nu^4} \left\{ \int_0^{\infty} \frac{(4u+1)^2 - 4t''^2}{\left[(4u+1)^2 + 4t''^2\right]^4} (4u+1) \right.$$

$$\cos\,(2\pi\nu u)\, du + 8 t' T_4 \int_0^{\infty} \frac{12T_4^2 - 5(4u+1)^2}{\left[(u+1)^2 + 4T_4^2\right]^5} (4u+1)$$

$$\left. \cos\,(2\pi\nu u)\, du \right\} (t''-t'),$$

где $t' < T_k < t''$, $k = 1, 2, 3, 4,$

$t' < \tilde{T}_n < t''$, $n = 2, 3, \ldots$.

Порядок членов в Л. 4., кроме первых двух, не выше $O\left(\frac{1}{t'^3}\right)$, так что имеет место

<u>Лемма 5.</u> $f(t'') - f(t') = \left[-\frac{1}{48}\,\frac{1}{t'^2} + O\left(\frac{1}{t'^3}\right)\right]$ $(t''-t')$.

Пусть $L(T) = \frac{1}{2\pi} T \ln T - \frac{1+\ln 2\pi}{2\pi} T + \frac{7}{8}$.

Имеет место очевидная

Лемма 6. $L(t'') - L(t') = \frac{1}{2\pi} \left[1 + O(1)\right] (t'' - t') \ln t'$.

Пусть $s = \sigma + it$. Если, как обычно, $N(T)$ обозначает число нулей функции $\zeta(s)$ на прямоугольнике $0 < \sigma < 1$, $0 < t \leqq T$, то

(1) $$N(T) = L(T) + S(T) + f(T),$$

см. [1], стр. 209, соотношение (2).
Имеет место

Лемма 7. Пусть γ', γ'' - соседние γ-ординаты, и $\gamma' \leqq t' < t'' < \gamma''$.

Тогда

$$S(t') - S(t'') = \frac{1}{2\pi} \left[1 + O(1)\right] (t'' - t') \ln t'.$$

Доказательство следует ис Л. 5., Л. 6., и формулы (1), если принять во внимание, что $N(t'') = N(t')$.

Отметим следующие очевидные свойства функции $S(t)$, вытекающие из Л. 7.

Следствие 1. $S(t)$ - убывающая функция на промежутке $\langle \gamma', \gamma'')$, т.е. имеет, самое большее, один нуль на этом промежутке.

С л е д с т в и е 2 . Если $S(t)$ имеет нуль на промежутке (γ', γ''), то кратность этого нуля - нечетное число.

П р и м е ч а н и е 1 . Недоказано, существует-ли бесконечно много нулей функции $S(t)$.

Положим теперь $R(t) = S(t) + f(t)$.

Л е м м а 8 . (Титчмарш) Существует такая последовательность значений $\tau \longrightarrow \infty$, что $R(\tau) = 0$. Доказательство см. [3], стр. 253-254.

Л е м м а 9 . (Литтлвуд) Для разности соседних γ - ординат, $\gamma' < \gamma''$, имеет место оценка

$$\gamma'' - \gamma' < \frac{A_4}{\ln \ln \ln \gamma'} .$$

Доказательство, см. например, [1], гл. IX, п. 12., стр. 223-225.

Теперь нетрудно дать

Доказательство теоремы

Прежде всего, существует такая подпоследовательность значений $\bar{\tau} \longrightarrow \infty$, последовательности упоминавшейся в Л. 8., что всякому $\bar{\tau}$ соответствуют такие соседние γ - ординаты, $\bar{\gamma}' < \bar{\gamma}''$, что $\bar{\tau} \in (\bar{\gamma}', \bar{\gamma}'')$. [Напомним, что значения $\bar{\tau}$ - это те значения, на которых осуществляется переход от положительных к отрицательным значениям функции $R(t)$. С одной стороны, таких переходов должно быть бесконечно много, с другой стороны, соотношение

$$R(\gamma) - R(\gamma - 0) \geqq 1$$

"запрещает" переход такого рода через точку γ. См. упоминавшееся выше доказательство Титчмарша.]

Пусть, дальше, $t \in \langle \bar{\gamma}', \bar{\gamma}'')$.

Тогда:

(а) в силу Л. 7.,

$$S(\tau) - S(t) = \frac{1}{2\pi}\left[1 + O(1)\right](t - \bar{\tau}) \ln \bar{\tau},$$

(б) в силу Л. 8., $S(\bar{\tau}) = - f(\bar{\tau})$,

(в) в силу Л. 3., $|f(\bar{\tau})| < \frac{A_5}{\bar{\tau}}$.

Так что, наконец, в силу (а), (б), (в), и Л. 9.,

$$|S(t)| < \frac{1}{2\pi}\left[1 + |O(1)|\right] |t - \bar{\tau}| \ln \bar{\tau} + \frac{A_5}{\bar{\tau}}$$

$$< \frac{A_4}{2\pi}\left[1 + |O(1)|\right] \frac{\ln \bar{\tau}}{\ln \ln \ln \bar{\gamma}'} + \frac{A_5}{\bar{\tau}}$$

$$< A_6 \frac{\ln \bar{\tau}}{\ln \ln \ln \bar{\gamma}'}$$

$$< A_7 \frac{\ln t}{\ln \ln \ln t}.$$

Приведем еще некоторые примечания к предшествующей заметке [4].

П р и м е ч а н и е 2. В заметке [4], Лемма 5., н названа Ω- теоремой Титчмарша. Однако, ее следует назвать Ω- теоремой Литтлвуда-Титчмарша, так как сначала это утверждение доказал Литтлвул, в предположении справедливости гипотезы Римана, и, после того, Титчмарш, независимо от этой гипотезы.

Известно, что в предположении справедливости гипотезы Римана, имеет место оценка

$$(2) \qquad S(t) = O\left(\frac{\ln t}{\ln \ln t}\right),$$

см., например, [1], гл. XIV, п. 13., стр. 346-349, или гл. XIV, п. 21., стр. 360-362.

Писть теперь $\gamma' < \gamma''$ - соседние γ - ординаты, и

$$t' = \gamma' + \frac{1}{3}(\gamma'' - \gamma'),$$
$$t'' = \gamma' + \frac{2}{3}(\gamma'' - \gamma').$$

Так как $N(t') = N(t'')$, то, используя (1) получается

$$L(t'') - L(t') + S(t'') - S(t') + f(t'') - f(t') = $$
$$= 0 .$$

Из последнего соотношения, используя Л. 5., Л. 6., и (2), нетрудно получить оценку

$$\gamma'' - \gamma' < \frac{A_8}{\ln \ln \gamma'},$$

т.е., имеет место следующее очевидное

П р и м е ч а н и е 3 . В предположении справедливости гипотезы Римана для разности соседних γ- ординат, $\gamma' < \gamma''$, имеет место оценка

$$\gamma'' - \gamma' = O\left(\frac{1}{\ln \ln \gamma'}\right).$$

Из оценки (2) и формулы (1) следует также очевидное

Примечание 4. В предположении справедливости гипотезы Римана, имеет место оценка

$$n(\gamma) = O\left(\frac{\ln \gamma}{\ln \ln \gamma}\right),$$

где $n(\gamma)$ обозначает кратность нуля $\varrho = \frac{1}{2} + i\gamma$ функции $\zeta(s)$.

ЛИТЕРАТУРА

[1] Титчмарш Е. К., Теория дзета-функции Римана, И Л, 1953

[2] Титчмарш Е. К., Теория функций, Москва, Ленинград, 1951

[3] Titschmarsh E. C., The zeros of the Riemann zeta-function, Proc. Royal Soc. A, 151 1935, 234-255

[4] Мозер Ян, О вертикальном распределении нулей дзета-функции Римана, Acta F. R. N. Univ. Comen.-Mathematica (в печати).

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13.IX.1971, в издательство 28.мая 1973

(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXIX, 1974)
ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXIX - 1974

Некоторые следствия из гипотезы Римана

ЯН МОЗЕР, Братислава

> "Мы увидим, что на основе этой гипотезы может быть построена весьма стройная теория, и это лишний раз подтверждает ее справедливость."
> Е.К. ТИТЧМАРШ, 1 , стр. 331.

Пусть, как обычно, $\rho = \beta + i\gamma$ обозначает нетривиальный нуль функции $\zeta(s)$, $s = \sigma + it$. Г. Дэвенпорт, (см. [2], стр. 92, соотношение (11)) постоянную

(1) $$B = -\frac{1}{2}C - 1 + \frac{1}{2}\ln 4\pi,$$

(C - постоянная Эйлера), истолковал как число полученное суммированием ряда

(2) $$-\sum_{\rho}\frac{1}{\rho},$$

таким способом, что сгруппируются члены с ρ и $\bar{\rho} = \beta - i\gamma$. Этим получен следующий результат:

(3) $$B = -2\sum_{\gamma>0}\frac{\beta}{\beta^2+\gamma^2}.$$

Напомним, что ряд

$$\sum_{\rho} \frac{1}{|\rho|}$$

расходится, т.е., соотношение (3) получено сгруппированием членов в ряде, несходящемся абсолютно.

Теперь возможно думать так.

Полагая $\beta = \frac{1}{2}$, (т.е. предполагая справедливость гипотезы Римана) и, используя (1) и (3), получается формула

(4) $$\frac{1}{2} \ln \frac{e^{c+2}}{4\pi} = \sum_{\gamma>0} \frac{1}{\frac{1}{4}+\gamma^2} .$$

Так что, формула (4) получена в следующих предположениях:

(а) суммируемость ряда (2) упоминавшимся способом,

(б) справедливость гипотезы Римана.

Целью этой заметки является получение формулы (4) предполагая только (б).

<u>Примечание 1.</u> Хочу, однако, предупредить, что я касаюсь формулы (4) не потому, что считаю ее полезной (полезность этой формулы, вероятно, заключается просто в том, что она существует) а потому, что считаю ее действительно красивой. Она красива в следующем смысле. С одной стороны, нам неизвестен точный закон вертикального распределения нулей дзета-функции Римана. С другой стороны, она выражает совершенно точно сумму бесконечного ряда чисел обратных квадратам модулей нетривиальных хулей функции $\zeta(s)$, посредством основных постоянных анализа e, π, C.

Прежде чем сформулировать утверждение, напомним следующее.

Пусть

$$\chi(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})},$$

$$\vartheta(t) = \frac{i}{2} \ln \chi(\tfrac{1}{2} + it).$$

Тогда обозначим ([1], стр. 94, (2))

(5) $$Z(t) = e^{i\vartheta(t)} \zeta(\tfrac{1}{2} + it).$$

Буквою τ обозначен член последовательности действительных значений такого рода, что

(6) $$Z'(\tau) = 0, \quad \tau \neq \gamma, \quad \tau \to \infty.$$

Существование последовательности такого рода следует, например, из доказательства Титчмарша ([1], стр. 260--262) теоремы Харди о бесконечности числа нулей $\frac{1}{2} + i\gamma$ функции $\zeta(s)$.

Имеет место

Т е о р е м а . Если справедлива гипотеза Римана, то

(7) $$\frac{1}{2} \ln \frac{e^{c+2}}{4\pi} = \sum_{\gamma>0} \frac{1}{\frac{1}{4}+\gamma^2},$$

т.е., например,

(8) $$\sum_{\gamma>0} \frac{1}{\frac{1}{4}+\gamma^2} = 0{,}023\ 095\ 708\ 966\ 121,$$

и, дальше

(9) $$\frac{1}{8}\pi = \lim_{\tau\to\infty} \sum_{\gamma>0} \frac{\tau}{\gamma^2-\tau^2}.$$

Приступим к перечислению вспомогательных утверждений. Так как

$$(10)\quad -\frac{1}{2}\ln\left(\frac{5}{4}+i\frac{t}{2}\right) = -\frac{1}{2}\ln t + \frac{1}{2}\ln 2 - i\frac{\pi}{4} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k}\left(\frac{5}{2t}\right)^{2k} + \frac{i}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\left(\frac{5}{2t}\right)^{2k-1} = -\frac{1}{2}\ln t + \frac{1}{2}\ln 2 - i\frac{\pi}{4} + 0\left(\frac{1}{t}\right),$$

$$|t| > \frac{5}{4},$$

$$(11)\quad \int_0^{\infty}\frac{u\,du}{\left[u^2+\left(\frac{5}{4}+i\frac{t}{2}\right)^2\right]\left(e^{2\pi u}-1\right)} = 16\int_0^{\infty}\frac{16u^2-4t^2+25}{(16u^2-4t^2+25)^2+400t^2}\,\frac{u}{e^{2\pi u}-1}\,du - i\,320\int_0^{\infty}\frac{t}{(16u^2-4t^2+25)^2+400t^2}\,\frac{u}{e^{2\pi u}-1}\,du = 0\left(\frac{1}{t}\right),$$

$$(12)\quad \frac{1}{5+i2t} = \frac{5}{25+4t^2} - i\frac{2}{25+4t^2} = 0\left(\frac{1}{t}\right),$$

то, полагая $z = \frac{5}{4} + i\frac{t}{2}$ в соотношений ([3], стр. 37)

$$\frac{\Gamma'(z)}{\Gamma(z)} = \ln z - \frac{1}{2} - 2\int_0^{\infty}\frac{u\,du}{(u^2+z^2)\,(e^{2\pi u}-1)},$$

и, используя (10), (11), (12), получается

Лемма 1.

$$-\frac{1}{2}\ \frac{\Gamma'(\frac{5}{4}+i\frac{t}{2})}{\Gamma(\frac{5}{4}+i\frac{t}{2})} = -\frac{1}{2}\ln t+\frac{1}{2}\ln 2+O\left(\frac{1}{t}\right)+i\left[-\frac{1}{4}\pi+\right.$$

$$\left.+\ O\left(\frac{1}{t}\right)\right].$$

Так как ряд

$$\sum_{\rho}\frac{s}{\rho(s-\rho)}$$

авсолютно сходится, (т.е., сгруппирование его членов позводително) то

$$\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)\equiv\sum_{\substack{\rho\\(\gamma>0)}}\left[\frac{1}{(s-\rho)\rho}+\frac{1}{(s-\bar\rho)\bar\rho}\right].$$

Из этого соотношения получается

Л е м м а 2 . Если справедлива гипотеза Римана, то при $s=\frac{1}{2}+it$, $t\neq\gamma$ имеет место

$$\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)=\sum_{\gamma>0}\frac{1}{\frac{1}{4}+\gamma^2}+i\sum_{\gamma>0}\frac{2t}{\gamma^2-t^2}.$$

Так как

$$(13)\qquad -\frac{1}{\frac{1}{2}+it}=-\frac{1}{\frac{1}{4}+t^2}=i\frac{t}{\frac{1}{4}+t^2}\ O\left(\frac{1}{t}\right),$$

то, полагая $s=\frac{1}{2}+it$, $t\neq\gamma$ в формуле

$$\frac{\zeta'(s)}{\zeta(s)}=\ln 2\pi-1-\frac{1}{2}C-\frac{1}{s-1}-\frac{1}{2}\ \frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)}\ +$$

$$+\ \sum\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right),$$

([1], стр. 41), и, используя (13), Л. 1., Л. 2., получается

Л е м м а 3 . Если справедлива гипотеза Римана, то при $t \neq \gamma$ имеет место

$$\frac{\zeta'(\frac{1}{2}+it)}{\zeta(\frac{1}{2}+it)} = -\frac{1}{2}\ln t+\frac{1}{2}\ln 2+\ln 2\pi-1-\frac{1}{2}C+$$

$$+\sum_{\gamma>0}\frac{1}{\frac{1}{4}+\gamma^2}+O\left(\frac{1}{t}\right)+$$

$$+\ i\left[-\frac{1}{4}\pi+\sum_{\gamma>0}\frac{2t}{\gamma^2-t^2}+O\left(\frac{1}{t}\right)\right].$$

И, еще наконец, ([1], стр. 260)

Л е м м а 4 . $\vartheta'(t) = \frac{1}{2}\ln t - \frac{1}{2}\ln 2\pi + O(\frac{1}{t})$.

Тепер мы в состоянии дать

Доказательство Теоремы

Используя (5) и (6), получается

$$\frac{\zeta'(\frac{1}{2}+i\tau)}{\zeta(\frac{1}{2}+i\tau)} + \vartheta'(\tau) = 0. \tag{14}$$

(a) Отделяя действительную часть соотношения (14), используя Л. 3., Л. 4., получается

$$\frac{1}{2}\ln 2 + \frac{1}{2}\ln 2\pi - 1 - \frac{1}{2}C + \sum_{\gamma>0} \frac{1}{\frac{1}{4}+\gamma^2} = o\left(\frac{1}{\tau}\right),$$

и, переходя к пределу при $\tau \to \infty$:

$$C = \ln\frac{4\pi}{e^2} + \sum_{\gamma>0}\frac{1}{\frac{1}{4}+\gamma^2},$$

т.е. (7).

(б) Так как

$\ln 2 = 0,693\ 147\ 180\ 559\ 945\ \ldots$, [4], 106,

$\ln \pi = 0,144\ 729\ 885\ 849\ 400\ \ldots$, [4], 156,

$C = 0,577\ 215\ 664\ 901\ 532\ \ldots$, [5], 1094,

то

$$\frac{1}{2}\left(C - \ln\frac{4\pi}{e^2}\right) \doteq 0,023\ 095\ 708\ 966\ 121\ .$$

Из этого и (7) получается (8).

(в) Отделяя мнимую часть соотношения (14), используя Л. 3., Л. 4., получается

$$-\frac{1}{8}\pi + \sum_{\gamma>0}\frac{\tau}{\gamma^2-\tau^2} = o\left(\frac{1}{\tau}\right),$$

и, переходя к пределу при $\tau \to \infty$, получается (9).

<u>П р и м е ч а н и е 2 . Соотношение</u>

$$C = \ln\frac{4\pi}{e^2} + 2\sum_{\gamma>0}\frac{1}{\frac{1}{4}+\gamma^2}$$

показывает, что даже на фоне вопросов о арифметической и алгебраической природе Эйлеровой постоянной C, выделяется гипотеза Римана.

ЛИТЕРАТУРА

[1] Титчмарш Е. К., Теория дзета-функции Римана, ИЛ, Москва 1953

[2] Девенпорт Г., Мультипликативная теория чисел, "Наука" Москва 1971

[3] Уиттекер Э. Т., Ватсон Дж. Н., Курс современного анализа, т. II, ГИФМЛ, Москва 1963

[4] Эйлер Л., Введение в анализ бесконечных, т. I, ГИФМЛ, Москва 1961

[5] Градштейн И. С., Рыжик И. М., Таблицы интегралов сумм, рядов и произведений, "Наука" Москва 1971

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23.IX.1971, в издательство 28 мая 1973