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GENERALIZED SYMMETRIC MEANS OF A THREE - - WAY ARRAY

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1. Introduction

In this paper we consider a threeway array of numbers $\|x_{ijk}\|$, $(i = i,2,\ldots,r;\ j = 1,2,\ldots,s;\ k = 1,2,\ldots,t)$ which may be regarded as a trisample from an array $\|X_{IJK}\|$, $(I = 1,2,\ldots,R;\ J = 1,2,\ldots,S;\ k = 1,2,\ldots,T)$, a trisample being chosen from a population matrix by taking those elements which are at the intersections of a selected set of r of the R planes and s of the S planes and t of the T planes, (we denote these planes by (i), (j), (k)). We will consider the symmetric polynomial functions of such matrices.

The polynomials of the present paper, called "generalized symmetric means" are symmetric functions in the sense that they are invariant under permutations of planes (i), (j), (k). This paper defines the generalized symmetric means, shows that they are "inherited on the average", develops the formulas for use in random pairing.

The denotation will be analogical to that introduced by Tukey in [1], [2] and by Hook in [3].

2. Trisamples and generalized symetric means

We suppose a population matrix $\|X_{IJK}\|$, $(I = 1,2,\ldots,R;\ J = 1,2,\ldots,S;\ K = 1,2,\ldots,T)$ from which a trisample $\|x_{ijk}\|$, $(i = 1,2,\ldots,r;\ j = 1,2,\ldots,s;\ k = 1,2,\ldots,t)$ is selected. Any polynomial symmetric in the x_{ijk} (in the sense defined in section 1) is a linear combination of the type

$$\sum_{\neq} x^{a}_{i_1 j_1 k_1} x^{b}_{i_2 j_2 k_2} \cdots x^{d}_{i_u j_w k_v}$$

where the symbol $\sum_{\neq}$, for three-way arrays, will mean summation over all subsequent subscripts, with the restriction that plane (i) subscripts represented by different letters must remain different throughout the summation, and the same holds for plane (j) subscripts, and the same holds for plane (k) subscripts. We define generalized symmetric means (g.s.m.).

Definition 1. A generalized symmetric mean is a polynomial

$$\frac{1}{M} \sum_{\neq} x^{a}_{i_1 j_1 k_1} x^{b}_{i_2 j_2 k_2} \cdots x^{d}_{i_u j_w k_v}$$

where the subscripts are summed from 1 to r, 1 to s and 1 to t (for trisamples) or from 1 to R, 1 to S and 1 to T (for populations), the exponents are positive integers and M is the number of terms in the summation.

When the trisample (or population) size is given, the g.s.m. is specified by the exponents, together with information telling us which ones correspond to elements that lie in the same plane (i), and which ones correspond to elements that lie in the same plane (j), and which ones correspond to elements that lie in the same plane (k). A convenient notation for g.s.m.'s is thus provided by placing the exponents in a matrix within brackets in such a way, that exponents with affect elements in the same plane (i) of the matrix $\|x_{ijk}\|$ are entered in the same plane (i), and so on for the remaining planes (j), (k).

For example

$$\left[\begin{array}{cc|cc} p & - & q & - \\ - & - & - & u \end{array}\right] = \frac{1}{r(r-1)s(s-1)t(t-1)} \sum_{\neq} x^{p}_{i_1 j_1 k_1} x^{q}_{i_1 j_1 k_2} x^{u}_{i_2 j_2 k_2}$$

Definition 2. Two g.s.m.'s are identical if the matrix of entries of one can be obtained from the other by permuting planes (i), (or planes (j), or planes (k)).

The different g.s.m.'s of degrees 1 and 2 are these:

Degree 1:

$$\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right] = \frac{1}{rst} \sum_{\neq} x_{ijk}$$

Degree 2:

$$\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \end{array}\right] = \frac{1}{r(r-1)s(s-1)t(t-1)} \sum_{\neq} x_{ijk}\, x_{uvw}$$

$$\left[\begin{array}{cc|cc} 1 & - & - & 1 \\ - & - & - & - \end{array}\right] = \frac{1}{r(r-1)st(t-1)} \sum_{\neq} x_{ijk}\, x_{ujv}$$

$$\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \end{array}\right] = \frac{1}{rs(s-1)t(t-1)} \sum_{\neq} x_{ijk}\, x_{iuv}$$

$$\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \end{array}\right] = \frac{1}{r(r-1)s(s-1)t} \sum_{\neq} x_{ijk}\, x_{uvk}$$

$$\left[\begin{array}{cc|cc} 1 & - & 1 & - \\ - & - & - & - \end{array}\right] = \frac{1}{rst(t-1)} \sum_{\neq} x_{ijk}\, x_{iju}$$

$$\left[\begin{array}{cc|cc} 1 & - & - & - \\ 1 & - & - & - \end{array}\right] = \frac{1}{rs(s-1)t} \sum_{\neq} x_{ijk}\, x_{iuk}$$

$$\left[\begin{array}{cc|cc} 1 & 1 & - & - \\ - & - & - & - \end{array}\right] = \frac{1}{rs(r-1)t} \sum_{\neq} x_{ijk}\, x_{ujk}$$

$$\left[\begin{array}{cc|cc} 2 & - & - & - \\ - & - & - & - \end{array}\right] = \frac{1}{rst} \sum_{\neq} x_{ijk}^{2}$$

We shall now define the partitions of the integer n. The general term

$$x_{i_1 j_1 k_1}^{a}\, x_{i_2 j_2 k_2}^{b} \cdots x_{i_u j_v k_w}^{d}$$

of degree n contains n-factors. To each of these factors we assign a different symbol, and the resulting set of symbols may be partitioned in three ways- once by planes (i), and once by planes (j), and once by planes (k).

D e f i n i t i o n 3. The entries a,b,...,d of a generalized symmetric mean of degree n form a partitions of the integer n. It is convenient to represent every such a partition in terms of n distinct symbols

$$\{p_1 \ldots p_{a_1}, q_1 \ldots q_{b_1}, r_1 \ldots r_{d_1} / p_1 \ldots p_{a_2}, q_1 \ldots q_{b_2}, \ldots, r_1 \ldots$$
$$\ldots r_{d_2} / p_1 \ldots p_{a_3}, q_1 \ldots q_{b_3}, \ldots, r_1 \ldots r_{d_3}\}$$

where commas are used to separate the parts of the partition of the plane (i), and similarly for the remaining partitions of the plane (j), (k), skew lines to separate the different partitions, and the lenghts of the parts are the positive integers, whose sum is n.

Greek letters will be used to represent arbitrary partitions. The secondary definition for the g.s.m.:

D e f i n i t i o n 4. A generalized symmetric means will be then an ordered triple $\langle \alpha / \beta / \gamma \rangle$ of partitions α, β, and γ, each on the same set of symbols. Each part of α will consist of those symbols which correspond to factors having a particular plane (i) subscript, and the parts of β, γ are similarly determined by plane (j) and plane (k) subscripts.

For example

$$\left[\begin{array}{cc|cc} 2 & - & 2 & - \\ - & - & - & 1 \end{array}\right] = \frac{1}{r(r-1)s(s-1)t(t-1)} \sum_{\neq} x^2_{i_1 j_1 k_1} x^2_{i_1 j_1 k_2} x_{i_2 j_2 k_2}$$

becomes, in the secondary notation,

$$\langle \alpha / \beta / \gamma \rangle = \langle \text{pqrv, u/pqrv, u/pq, rvu} \rangle .$$

D e f i n i t i o n 5. A g.s.m. $\langle \alpha / \beta / \gamma \rangle$ is inherited on the average, in the sense that

$$\text{ave} \{\langle \alpha / \beta / \gamma \rangle\} = \langle \alpha / \beta / \gamma \rangle_o$$

where $\langle \alpha / \beta / \gamma \rangle_o = \frac{1}{M} \sum_{\neq} x^a_{i_1 j_1 k_1} x^b_{i_2 j_2 k_2} \ldots x^d_{i_u j_w k_v}$

represent any g.s.m. for an RxSxT population and where "ave" denotes the expectation or average over all possible trisamples from the population.

D e f i n i t i o n 6. A dichotomy of a partition α is an ordered set $\{\alpha_1, \alpha_2\}$ of two partitions α_1 and α_2, such that α_1 consists of some of the symbols comprising α, and α_2 of the remaining ones, and such, that any two symbols which both occur in α_1 or both in α_2 belong to the same part if an only if they belonged to the same part of α.

D e f i n i t i o n 7. The null partition will be denoted by $\emptyset$, so that $\{\emptyset, \alpha\}$ and $\{\alpha, \emptyset\}$ are dichotomies α.

R e m a r k. The idea of random pairing for trisamples is a straightforward extension of that described by Tukey [2]. This means taking two trisamples, $\|x_{ijk}\|$ and $\|y_{ijk}\|$, the order within each having been independently randomized, and adding the two to obtain a new trisample $\|z_{ijk}\|$. For symmetric functions of the z's one wants the average value (where the average is taken with respect both to sampling and to randomization of order within trisamples) expressed, by means of a "pairing formula", in terms of symmetric functions of the two original populations. Using "aver" means average over all possible permutations of the x 's and y 's. Thus "aver" means average over permutations of plans (i) and of plans (j) and plans (k).

T h e o r e m. Let α, β, γ be arbitrary partitions of the same set of the distinct symbols, if $\{\alpha_1, \alpha_2\}$, $\{\beta_1, \beta_2\}$, and $\{\gamma_1, \gamma_2\}$ be all distinct dichotomies of the partitions α, β, γ, if α_1, β_1, γ_1 consist of the same symbols.

Then it holds:

$$\text{aveaver}\ \{\langle \alpha / \beta / \gamma \rangle\} = \sum_{\neq} \langle \alpha_1 / \beta_1 / \gamma_1 \rangle_0 \langle \alpha_2 / \beta_2 / \gamma_2 \rangle_{00} ,$$

where the summation extends over all distinc dichotomies of the partitions α, β, γ.

P r o o f. If α, β, γ are arbitrary partitions of the form (see Definition 3) and if we recall that $\langle \alpha/\beta/\gamma \rangle$ is a g.s.m. (see Definition 4) for a sample of numbers of the form

$(x_{ijk} + y_{\tau(i)\tau(j)\tau(k)})$, we have

$$\langle \alpha/\beta/\gamma \rangle = \frac{1}{V_r^m\, V_s^p\, V_t^u} \sum \Big[\big(x_{i_1 j_1 k_1} + y_{\tau(i_1)\tau(j_1)\tau(k_1)}\big) \cdots$$

$$\big(x_{i_1 j_1 k_1} + y_{\tau(i_1)\tau(j_1)\tau(k_1)}\big)\Big] \cdots \Big[\big(x_{i_u j_v k_w} + y_{\tau(i_u)\tau(j_v)\tau(k_w)}\big)$$

$$\cdots \big(x_{i_u j_v k_w} + y_{\tau(i_u)\tau(j_v)\tau(k_w)}\big)\Big]$$

where in the first square bracket exist c of equal factors and in the second square bracket exists d equal factors a in the last is h equal factors, and V_r^m, V_s^p, V_t^u are variations of the m-th resp. the p, n-th class with r resp. s,t elements. After multiplication of the square brackets, we have $2^{c+d+ \ldots +h}$ terms form

$$x^{c'}_{i_1 j_1 k_1}\, x^{d'}_{i_2 j_2 k_2} \cdots x^{h'}_{i_u j_v k_w} \cdot y^{c-c'}_{\tau(i_1)\tau(j_1)\tau(k_1)}\, y^{d-d'}_{\tau(i_2)\tau(j_2)\tau(k_2)} \cdots y^{h-h'}_{\tau(i_u)\tau(j_v)\tau(k_w)}$$

where $c' + d' \ldots + h' + c-c' + d-d' + \ldots + h-h' = n$, and n is whole degree of the g.s.m. .

Then

$$\langle \alpha/\beta/\gamma \rangle = \frac{1}{V_r^m\, V_s^p\, V_t^u} \sum_{\substack{i_1 \neq i_2 \neq \\ j_1 \neq j_2 \neq \\ k_1 \neq k_2 \neq}} \sum_{2^{c+d+\ldots}} x^{c'}_{i_1 j_1 k_1} \cdots x^{h'}_{i_u j_v k_w} \cdot$$

$$\cdot\, y^{c-c'}_{\tau(i_1)\tau(j_1)\tau(k_1)} \cdots y^{h-h'}_{\tau(i_u)\tau(j_v)\tau(k_w)}$$

$$\text{aver } \langle \alpha / \beta / \gamma \rangle = \sum_{2^{c+d+..}} \frac{1}{V_r^m \, V_s^p \, V_t^u} \cdot \frac{1}{V_r^m!} \sum_{(V_r^m!)} \frac{1}{V_s^p!} \sum_{(V_r^m!)} \frac{1}{V_t^u} \sum_{(V_t^m!)_o}$$

$$\sum_{\substack{i_1 \neq i_2 \neq \dots \\ j_1 \neq j_2 \neq \dots \\ k_1 \neq k_2 \neq \dots}} x^{c'}_{i_1 j_1 k_1} \dots x^{h'}_{i_u j_v k_w} \cdot y^{c-c'}_{\tau(i_1)\tau(j_1)\tau(k_1)} \dots$$

$$\dots y^{h-h'}_{\tau(i_u)\tau(j_v)\tau(k_w)} = \sum_{2^{c+d+..}} \frac{1}{V_r^m \, V_s^p \, V_t^u} \cdot \frac{(V_r^m - 1)!}{V_r^m!} \cdot$$

$$\cdot \frac{(V_s^p - 1)!}{V_s^u \, !} \cdot \frac{(V_t^u - 1)!}{V_t^u \, !} \cdot \sum_{\substack{i_1 \neq i_2 \neq \dots \\ j_1 \neq j_2 \neq \dots \\ k_1 \neq k_2 \neq \dots}} x^{c'}_{i_1 j_1 k_1} \dots$$

$$\dots x^{h'}_{i_u j_v k_w} \cdot y^{c-c'}_{i_1 j_1 k_1} \dots y^{h-h'}_{i_u j_v k_w} = \sum_{2^{c+d+..}} \frac{1}{V_r \, V_s \, V_t} \cdot$$

$$\cdot \sum_{\substack{i_1 \neq i_2 \neq \dots \\ j_1 \neq j_2 \neq \dots}} x^{c'}_{i_1 j_1 k_1} \dots x^{h'}_{i_u j_v k_w} \frac{1}{V_r \, V_s \, V_t} \sum_{\substack{i_1 \neq i_2 \neq .. \\ j_1 \neq j_2 \neq ..}} y^{c-c'}_{i_1 j_1 k_1} \dots$$

$$\dots y^{h-h'}_{i_u j_v k_w} = \sum_{2^{c+d+..}} \langle p_1 p_2 \dots p_{a_1}, \; q_1 q_2 \dots q_{b_1}, \dots, \; r_1 r_2 \dots$$

$$\dots r_{d_1} / p_1 p_2 \dots p_{a_2}, \; q_1 q_2 \dots q_{b_2}, \dots, r_1 r_2 \dots r_{d_2} / p_p p_2 \dots p_{a_3},$$

$$q_1 q_2 \dots q_{b_3}, \dots, r_1 r_2 \dots r_{d_3} \rangle_{*} \; \times \; \langle p_1 p_2 \dots p_{f_1}, q_1 q_2 \dots q_{g_1}, \dots$$

$$\dots, r_1 r_2 \dots r_{h_1} / p_1 p_2 \dots p_{f_2}, q_1 q_2 \dots q_{g_2}, \dots, r_1 r_2 \dots r_{h_2} / p_1 p_2 \dots$$

$$\dots p_{f_3}, \; q_1 q_2 \dots q_{g_3}, \dots, r_1 r_2 \dots r_{h_3} \rangle_{**} .$$

$$\text{aver } \{ \langle \alpha / \beta / \gamma \rangle \} = \sum_{2^{c+d+}} \langle \alpha_1 / \beta_1 / \gamma_1 \rangle_{*} \cdot \langle \alpha_2 / \beta_2 / \gamma_2 \rangle_{**}$$

The final step is to average over all trisamples. Since the samples are chosen independently from different populations, and since the symmetric means are inherited on the average we can put:

$$\text{ave aver } \{\langle\alpha/\beta/\gamma\rangle\} = \sum \langle \alpha_1 / \beta_1 / \gamma_1 \rangle_o \langle \alpha_2 / \beta_2 / \gamma_2 \rangle_{oo}$$

We have the theorem proved. Equations (1) is a pairing formula.

<u>Example.</u> Consider the partition $\{p, qt/p, qt/p, q, t\}$. By the preceding Theorem this becomes in the primary notation:

$$\text{ave aver } \{\langle\alpha/\beta/\gamma\rangle\} = \left[\begin{array}{cc|cc|cc} 1 & - & - & - & - & - \\ - & - & - & 1 & - & 1 \end{array}\right]_o + \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right]_o \cdot \left[\begin{array}{cc|cc} 1 & - & 1 & - \\ - & - & - & - \end{array}\right]_{oo} +$$

$$+ \left[\begin{array}{cc|cc} 1 & - & 1 & - \\ - & - & - & - \end{array}\right]_o \cdot \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right]_{oo} + 2\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right]_o \cdot \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right]_{oo} +$$

$$+ 2\left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & - \end{array}\right]_o \cdot \left[\begin{array}{cc|cc} 1 & - & 1 & - \\ - & - & - & 1 \end{array}\right]_{oo} + \left[\begin{array}{cc|cc|cc} 1 & - & - & - & - & - \\ - & - & - & 1 & - & 1 \end{array}\right]_{oo} .$$

In the secondary notation:

$$\text{ave aver } \{\langle\alpha/\beta/\gamma\rangle\} = \langle p,qt/p,qt/p,q,t\rangle_o + \langle p/p/p\rangle_o \langle qt/qt/q,t\rangle_{oo} +$$

$$\langle qt/qt/q,t\rangle_o \cdot \langle p/p/p\rangle_{oo} + \langle p,q/p,q/p,q\rangle_o \cdot \langle t/t/t\rangle_{oo} +$$

$$\langle t/t/t\rangle_o \cdot \langle p,q/p,q/p,q\rangle_{oo} + \langle p,t/p,t/p,t\rangle_o \langle q/q/q\rangle_{oo} +$$

$$\langle q/q/q\rangle_o \cdot \langle p,t/p,t/p,t\rangle_{oo} + \langle p,qt/p,qt/p,q,t\rangle_{oo} .$$

3. The distinct g.s.m.'s of degree 3

Those will be denoted by u's with subscripts.
Thus

$$u_1 = \left[\begin{array}{ccc|ccc|ccc} 1 & - & - & - & - & - & - & - & - \\ - & - & - & - & 1 & - & - & - & - \\ - & - & - & - & - & - & - & - & 1 \end{array}\right], \quad u_2 = \left[\begin{array}{cc|cc|cc} 1 & - & - & - & - & - \\ - & - & - & 1 & - & - \\ - & - & - & - & - & 1 \end{array}\right],$$

$$u_3 = \left[\begin{array}{ccc|ccc|ccc} 1 & - & - & - & - & - & - & - & - \\ - & - & - & - & 1 & - & - & - & 1 \end{array}\right], \quad u_4 = \left[\begin{array}{ccc|ccc} 1 & - & - & - & - & - \\ - & - & - & - & 1 & - \\ - & - & - & - & - & 1 \end{array}\right],$$

$$u_5 = \left[\begin{array}{cc|cc|cc} - & - & - & - & - & 1 \\ 1 & - & - & 1 & - & - \end{array}\right], \qquad u_6 = \left[\begin{array}{cc|cc} - & - & - & 1 \\ - & 1 & - & - \\ - & - & 1 & - \end{array}\right],$$

$$u_7 = \left[\begin{array}{ccc|ccc} - & - & - & 1 & - & - \\ - & - & 1 & - & 1 & - \end{array}\right], \qquad u_8 = \left[\begin{array}{cc|cc|cc} 1 & - & - & - & - & - \\ - & - & - & 1 & - & 1 \end{array}\right],$$

$$u_9 = \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 1 \\ - & - & - & 1 \end{array}\right], \qquad u_{10} = \left[\begin{array}{ccc|ccc} 1 & - & - & - & - & - \\ - & - & - & - & 1 & 1 \end{array}\right],$$

$$u_{11} = \left[\begin{array}{cc|cc} - & - & - & 1 \\ - & 1 & 1 & - \end{array}\right], \qquad u_{12} = \left[\begin{array}{cc|cc} - & - & 1 & - \\ - & 1 & - & 1 \end{array}\right], \qquad u_{13} = \left[\begin{array}{cc|cc} - & - & - & 1 \\ 1 & - & - & 1 \end{array}\right],$$

$$u_{14} = \left[\begin{array}{cc|cc} - & 1 & - & - \\ - & - & 1 & 1 \end{array}\right], \qquad u_{15} = \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & - & 2 \end{array}\right],$$

$$u_{16} = \left[\begin{array}{cc|cc|cc} 1 & - & - & - & - & - \\ - & - & 1 & - & - & - \\ - & - & - & - & 1 & - \end{array}\right], \qquad u_{17} = \left[\begin{array}{ccc|ccc|ccc} 1 & - & - & - & 1 & - & - & - & 1 \\ - & - & - & - & - & - & - & - & - \end{array}\right]$$

$$u_{18} = \left[\begin{array}{ccc|ccc} 1 & - & - & - & - & - \\ - & 1 & - & - & - & - \\ - & - & 1 & - & - & - \end{array}\right], \qquad u_{19} = \left[\begin{array}{cc|cc|cc} 1 & - & - & - & - & - \\ - & - & 1 & - & 1 & - \end{array}\right],$$

$$u_{20} = \left[\begin{array}{cc|cc|cc} 1 & - & - & 1 & - & 1 \\ - & - & - & - & - & - \end{array}\right], \qquad u_{21} = \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 1 & - \\ - & - & 1 & - \end{array}\right], \qquad u_{22} = \left[\begin{array}{ccc|ccc} 1 & - & - & - & 1 & 1 \\ - & - & - & - & - & - \end{array}\right],$$

$$u_{23} = \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & 1 & - & - \\ - & 1 & - & - \end{array}\right], \qquad u_{24} = \left[\begin{array}{ccc|ccc} 1 & - & - & - & - & - \\ - & 1 & 1 & - & - & - \end{array}\right], \qquad u_{25} = \left[\begin{array}{cc|cc} - & - & 1 & - \\ 1 & - & 1 & - \end{array}\right],$$

$$u_{26} = \left[\begin{array}{cc|cc} - & 1 & 1 & 1 \\ - & - & - & - \end{array}\right], \qquad u_{27} = \left[\begin{array}{cc|cc} - & 1 & - & - \\ 1 & 1 & - & - \end{array}\right], \qquad u_{28} = \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & - & 2 & - \end{array}\right]$$

$$u_{29} = \left[\begin{array}{cc|cc} 1 & - & - & 2 \\ - & - & - & - \end{array}\right], \qquad u_{30} = \left[\begin{array}{cc|cc} 1 & - & - & - \\ - & 2 & - & - \end{array}\right], \qquad u_{31} = \left[\begin{array}{cc|cc|cc} 1 & - & 1 & - & 1 & - \\ - & - & - & - & - & - \end{array}\right],$$

$$u_{32} = \left[\begin{array}{cc|cc} 1 & - & - & - \\ 1 & - & - & - \\ 1 & - & - & - \end{array}\right], \quad u_{33} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & - & - & - \\ - & - & - & - & - & - \end{array}\right], \quad u_{34} = \left[\begin{array}{cc|cc} 1 & - & 2 & - \\ - & - & - & - \end{array}\right],$$

$$u_{35} = \left[\begin{array}{cc|cc} 1 & - & - & - \\ 2 & - & - & - \end{array}\right], \quad u_{36} = \left[\begin{array}{cc|cc} 1 & 2 & - & - \\ - & - & - & - \end{array}\right], \quad u_{37} = \left[\begin{array}{cc|cc} 3 & - & - & - \\ - & - & - & - \end{array}\right].$$

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ON THE MEASURABILITY OF FUNCTIONS OF TWO VARIABLES

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Several sufficient conditions have been published for a function defined on a product of two measurable spaces to be measurable. This paper recalls and generalizes a sufficient condition for the measurability of a real-valued function on a product of two measure metric spaces. The terminology and notation of [1] are used throughout with one exception: a measurable function is to be understood in a more usual sense as such a mapping from a measurable space M into a topological space Z that the inverse image of any Borel subset of Z is a measurable set in M.

J. H. MICHAEL and B. C. RENNIE have proved the following theorem ([3], Theorem 2) concerning Lebesgue measurability of a real-valued function.

Theorem 1. Suppose that $f(x,y)$ is defined on a measurable plane set E and is 0 outside E and that $f(x,y)$ is a continuous function of x on E relatively to E for almost all y and that it is a measurable function of y for almost all x; then $f(x,y)$ is plane measurable.

An analogous proposition for functions on a product of two measurable metric spaces will be given in Theorem 2. To simplify the proof as much as possible we shall prove first Lemmas 1 to 5.

Definition 1. $(X, \varphi; \rho)$ is called a measurable metric space if (X, φ) is a measurable space and ρ a metric on X satisfying the condition that every open set in the metric space (X, ρ) is φ-measurable. If $(X, \varphi; \rho)$ is a measurable metric space and μ a measure on (X, φ) then $(X, \varphi, \mu; \rho)$ will be called a measure metric space.

Definition 2. Let $(X, \varphi, \mu ; \rho)$ be a finite-measure metric space. Measure μ is said to be weakly inner regular if for any φ-measurable $E \subset X$ and any $\varepsilon > 0$ there is a ρ-closed set $F \subset E$ with $\mu(F) \geq \mu(E) - \varepsilon$.

It should be noted here that if $(X, \varphi; \rho_1)$ and $(Y, \tau; \rho_2)$ are separable measurable metric spaces i.e. such measurable metric spaces that (X, ρ_1) and (Y, ρ_2) are separable then their product $(X \times Y, \varphi \times \tau; \rho_1 \times \rho_2)$ is a (separable) measurable metric space too. In fact, any open set G in $(X \times Y, \rho_1 \times \rho_2)$ is a union of some rectangles $A_s \times B_t$ where $\{A_s\}_{s=1}^{\infty}$ and $\{B_t\}_{t=1}^{\infty}$ are countable bases in (X, ρ_1) and (Y, ρ_2) respectively and so G is a countable union of measurable sets and hence is itself $\varphi \times \tau$-measurable.

Let us now consider a non-negative real-valued function f defined on the product of separable finite-measure metric spaces $(X, \varphi, \mu ; \rho_1)$ and $(Y, \tau, \nu; \rho_2)$ and having all its x-sections τ-measurable. We shall further suppose that $f = 0$ outside a closed set $E \subset X \times Y$ and that all the y-sections of f are relatively continuous on E^y.

Since (X, ρ_1) is a separable metric space, we can use the method of T.NEUBRUNN [4] to construct a cellection of its measurable subsets Ω_i^n, $n = 1,2,\ldots$, $i = 1,2,\ldots, i_n \leqslant \infty$ with diameter $d(\Omega_i^n) < \frac{1}{n}$ and such that for any fixed n the sets Ω_i^n are pairwise disjoint and $\bigcup_{i=1}^{i_n} \Omega_i^n = X$. Due to the assumed finiteness of ν, the least upper bound $\sup \{\nu(E_x); x \in \Omega_i^n\}$ exists for every i, n, and so for any positive integer n and any integer i with $1 \leqslant i \leqslant i_n$ we can find a point $a_{ni} \in \Omega_i^n$ satisfying

(1) $$\nu(E_{a_{ni}}) \geqslant \sup_{x \in \Omega_i^n} \nu(E_x) - \frac{1}{n \cdot \mu(X)}$$

Now put

(2) $$A_{ni} = \{(x,y) \in X \times Y;\ x \in \Omega_i^n,\ (a_{ni}, y) \in E\} = \Omega_i^n \times E_{a_{ni}}$$

(3) $$A_n = \bigcup_{k=1}^{i_n} A_{nk}$$

(4) $$B_n = \bigcup_{k=n}^{\infty} A_k$$

(5) $$B = \bigcap_{k=1}^{\infty} B_k = \limsup A_n$$

All the above sets are $\varphi \times \tau$-measurable since in the measurable metric space $(X=Y, \varphi \times \tau; \rho_1 \times \rho_2)$ the closed set E is $\varphi \times \tau$-measurable.

Applying Fubini's theorem we have

(6) $$\mu \times \nu(E) = \int_X \nu(E_x) d\mu = \sum_{i=1}^{i_n} \int_{\Omega_i^n} \nu(E_x)\, d\mu \leq$$

$$\leq \sum_{i=1}^{i_n} \int_{\Omega_i^n} \left(\nu(E_{a_{ni}}) + \frac{1}{n \cdot \mu(X)}\right) d\mu =$$

$$= \sum_{i=1}^{i_n} \mu \times \nu(\Omega_i^n \times E_{a_{ni}}) + \sum_{i=1}^{i_n} \frac{\mu(\Omega_i^n)}{n \cdot \mu(X)} =$$

$$= \mu \times \nu\left(\bigcup_{i=1}^{i_n} A_{ni}\right) + \frac{1}{n}$$

for every n and hence

(7) $$\mu \times \nu(A_n) \geq \mu \times \nu(E) - \frac{1}{n}$$

L e m m a 1. $B \subset E$ and $\mu \times \nu(E - B) = 0$.

P r o o f. For $n = 1,2,\ldots$ put

(8) $$C_n = \left\{(x,y) \in X \times Y;\ (\rho_1 \times \rho_2)\big((x,y), E\big) \leq \frac{1}{n}\right\}$$

Since E is closed, it holds

(9) $$E = \bigcap_{n=1}^{\infty} C_n$$

By (2), $n \leq m$ implies $A_{mi} \subset C_n$, hence $A_m \subset C_n$ which by (4) yields

(10) $\quad B_n \subset C_n$

As a consequence of (5), (9) and (10) we get

(11) $\quad B \subset E$

On the other hand, by (4) and (7) we have

(12) $$\mu \times \nu (B_n) \geq \mu \times \nu (E) - \frac{1}{n}$$

Due to the finiteness of $\mu \times \nu$, (5) and (12) imply

(13) $$\mu \times \nu (B) = \lim_{n \to \infty} \mu \times \nu (B_n) \geq \mu \times \nu (E)$$

Now (11) and (13) prove the lemma.

For all $n = 1,2,\dots$ and $(x,y) \in X \times Y$ define

(14) $$f_n(x,y) = f(a_{ni},y) \text{ if } x \in \Omega_i^n, \; i = 1,2,\dots$$

L e m m a 2. The functions f_n, $n = 1,2,\dots$ are $\varphi \times \tau$-measurable.

P r o o f. For any fixed n and all $i = 1,2,\dots i_n$ put

$$J_i = \Omega_i^n \times Y$$

By (14), it holds

$$f_n(x,y) = f(a_{ni},y) \text{ if } (x,y) \in J_i$$

Let P be a Borel set. Then

$$\overset{-1}{f_n}(P) = \bigcup_{k=1}^{i_n} \left[J_k \; \overset{-1}{f_n}(P) \right] = \bigcup_{k=1}^{i_n} \left[\Omega_k^n \times \overset{-1}{f_{a_{nk}}}(P) \right] \in \varphi \times \tau$$

L e m m a 3. For any point $(x,y) \in B$, $f(x,y) = \limsup f_n(x,y)$.

P r o o f. Consider a fixed point $(x,y) \in B$. To each k we can find an i with $x \in \Omega_i^k$. By (5), $K = \{ k; (x,y) \in A_k \}$ is an infinite set. Let $\varepsilon > 0$. Since the y-sections of f are relatively continuous on E^y we can find a $\delta > 0$ such that for all ξ satisfying

(15) $(\xi, y) \in E$ and $\rho_1(x, \xi) < \delta$

it holds

$$|f(\xi, y) - f(x,y)| < \varepsilon$$

With $k > \frac{1}{\delta}$, $k \in K$, (15) holds true for $\xi = a_{ki}$ since

$$d(\Omega_i^k) < \frac{1}{k}$$

Therefore

$$|f(a_{ki}, y) - f(x,y)| < \varepsilon$$

that is

$$|f_k(x,y) - f(x,y)| < \varepsilon$$

This shows that the subsequence $\{f_k(x,y)\}_{k \in K}$ of $\{f_n(x,y)\}_{n=1}^{\infty}$ converges to $f(x,y)$. Now take $k \notin K$. Then $(x,y) \notin A_k$, hence by (2) and (3) we get $(a_{ki}, y) \notin E$ and therefore

$$f_k(x,y) = f(a_{ki}, y) = 0$$

Recalling $f \geq 0$ we see that the upper limit of $\{f_n\}_{n=1}^{\infty}$ is equal to $\lim_{k \in K} f_k(x,y) = f(x,y)$ what was to be proved.

<u>L e m m a 4.</u> The function f is $\overline{\mu \times \nu}$ - measurable where $\overline{\mu \times \nu}$ is the completion of $\mu \times \nu$.

<u>P r o o f.</u> By Lemmas 2 and 3 the $\mu \times \nu$ -measurability (i.e. the $\varphi \times \tau$-measurability) of f on the set B is implied. Since $\overline{\mu \times \nu}(E-B) = 0$ by Lemma 1, f is $\overline{\mu \times \nu}$-measurable on E. The $\overline{\mu \times \nu}$ -measurability of f on $X \times Y$ follows from E being a measurable set.

<u>L e m m a 5.</u> Let $(X, \varphi, \mu; \rho_1)$ and $(Y, \tau, \nu; \rho_2)$ be separable finite-measure metric spaces. Let a non-negative real-valued function f on $X \times Y$ and a set $E \in \varphi \times \tau$ have the following properties

(i) for every $x \in X$, the section f_c is a τ-measurable function
(ii) for all $y \in Y$, f^y is relatively continuous on E^y
(iii) $f(x,y) = 0$ outside E

Then f is $\overline{\mu \times \nu}$-measurable on E.

P r o o f. Measures μ and ν being finite, $\mu \times \nu$ is also finite and hence by [5, Theorem 1.2, page 27] it is weakly inner regular. Therefore to any integer $n > 0$ we can find a closed set $F_n \subset E$ with $\mu \times \nu(E - F_n) < \frac{1}{n}$. Put

$$E_n = \bigcup_{i=1}^{n} F_i$$

and define

$$g_n(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in E_n \\ 0 & \text{if } (x,y) \in (X \times Y) - E_n \end{cases}$$

It can readily be seen that for g_n and E_n, $n = 1,2,\ldots$ the initial assumptions hold and therefore the g_n' s are $\overline{\mu \times \nu}$ - measurable.Enough to show that $\{g_n\}_{n=1}$ converges to f pointwise $\overline{\mu \times \nu}$-almost everywhere. Denote the set $E - \bigcup_{n=1}^{\infty} E_n$ by N. Evidently

$$\mu \times \nu(E - \bigcup_{n=1}^{\infty} E_n) \leq \mu \times \nu(E - F_n) < \frac{1}{n}$$

for any n and hence $\mu \times \nu(N) = 0$. Now for $(x,y) \notin N$ we prove $g_n(x,y) = f(x,y)$ for sufficiently great n. If $(x,y) \notin E$ we have $g_n(x,y) = 0 = f(x,y)$ for all n. If $(x,y) \in E$ and under the assumption $(x,y) \notin N$ there can be found an index m such that $(x,y) \in E_m$. Then for $n > m$ we have $g_n(x,y) = f(x,y)$ which completes the proof.

T h e o r e m 2. Suppose $(X, \varphi, \mu; \rho_1)$ and $(Y, \tau, \nu; \rho_2)$ are separable σ-finite-measure metric spaces. Let f be a real-valued function defined on an $\varphi \times \tau$-measurable set $E \subset X \times Y$ with the following properties

(I) for μ-almost every $x \in X$, the section f_x is τ-measurable on E_x

(II) for ν-almost every $y \in Y$, the function f^y is relatively continuous on E^y.

Then f is $\overline{\mu \times \nu}$-measurable.

P r o o f. We can assume $f(x,y) \geq 0$ because in general a real-valued function f can be obtained as a difference of two non-negative functions the measurability of which implies the measurability of f. Let M be the set of those points x that f_x is not τ-measurable on E_x and N the set of all y such that f^y is not relatively continuous on E^y. Then $Z = (M\times Y) \cup (X\times N)$ has $\mu \times \nu(Z) = 0$ and on $(X\times Y) - Z$, (I) and (II) hold true for all $x \in X - M$ and $y \in Y - N$. Thus we can suppose that (I) and (II) hold everywhere. Due to the σ-finiteness of μ and ν we can write

$$X = \bigcup_{i=1}^{\infty} X_i, \qquad Y = \bigcup_{j=1}^{\infty} Y_j$$

where $\mu(X_i) < \infty$, $\nu(Y_j) < \infty$ for all i,j and besides, for $i \neq j$ it holds $X_i \cap X_j = Y_i \cap Y_j = \emptyset$. Consequently Lemma 5 can be applied in any of the spaces $X_i \times Y_j$ yielding $\overline{\mu \times \nu}$-measurability of f on each of the pairwise disjoint sets $E \cap (X_i \times Y_j)$ which finally implies that f is $\overline{\mu \times \nu}$-merasurable on $E = \bigcup_{i,j} E \cap (X_i \times Y_j)$.

Applying a result of E. MARCZEWSKI and R. SIKORSKI [2], the separability of (X, ρ_1) and (Y, ρ_2) can be replaced by a weaker condition, as done in Theorem 3.

D e f i n i t i o n 3. A cardinal number m is said to have measure zero if every finite measure, defined for all subsets of any set of power m and vanishing for all one-point sets, vanishes identically.

T h e o r e m 3. Let $(X, \varphi, \mu; \rho_1)$ and $(Y, \tau, \nu; \rho_2)$ be σ-finite-measure metric spaces, each of them having a base whose power has measure zero. Suppose f is a real-valued function defined on an $\varphi \times \tau$-measurable set $E \subset X \times Y$ with the properties (I) and (II) from Theorem 2. Then f is $\overline{\mu \times \nu}$-measurable.

P r o o f. By Theorem III of [2], the existence in (X, ρ_1) and (Y, ρ_2) of bases having measure zero implies $X = N_1 \cup S_1$ and $Y = N_2 \cup S_2$ where S_1 and S_2 are separable spaces and $\mu(N_1) =$

$= \nu(N_2) = 0$. Our Theorem 2 holds in $S_1 \times S_2$ and then for any Borel set B we get $f^{-1}(B) = \left[f^{-1}(B) \cap (S_1 \times S_2)\right] \cup \left[f^{-1}(B) \cap (N_1 \times Y)\right] \cup \cup \left[f^{-1}(B) \cap (X \times N_2)\right] \in \overline{\varphi \times \tau}$ since $(X \times Y, \overline{\varphi \times \tau}, \overline{\mu \times \nu})$ is the completion of the product space $X \times Y$. Therefore f is $\overline{\varphi \times \tau}$-measurable in all its domain.

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PERIODIC SOLUTION OF A SECOND-ORDER NON-LINEAR DIFFERENTIAL EQUATION

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A. LAZER [1] has proved that the differential equation

$$x'' + g(x) = f(t)$$

has a periodic solution with period T, provided f, g are continuous functions, $f(t + T) = f(t)$, $\lim_{|x| \to \infty} g(x)/x = 0$, and $x.g(x) \geqslant 0$ for $|x| \geqslant b$.

In the present paper a more general result is given. It is shown that the differential equation

$$x'' + Ax' + Bx + g(x) = f(t)$$

where A, B are real constants and f, g continuous function, has a T-periodic solution provided $f(t + T) = f(t)$ for each t, and $\lim_{|x| \to \infty} g(x)/x = 0$. Here all functions are real-valued.

In the sequel, all functions are assumed to be defined on the real line. The norm $\| \ \|$ for functions is defined as usually by $\|f\| = \sup_t |f(t)|$. In the proof of the theorem we use the complex-valued functions. Let P (resp. P^*) denote the set of all real-valued (resp. complex-valued) periodic continuous functions with period T.

It is known ([2]) that the equation

$$(1) \qquad x'(t) + k.x(t) = f(t)$$

where $f \in P^*$, and $k \neq 0$ is a complex constant, has a T-periodic solution

$$x(t) = \frac{e^{-kt}}{e^{kT} - 1} \int_t^{t+T} e^{ks} f(s)ds$$

For each complex number $k \neq 0$, and for each $f \in P^*$ let $I_k(f)$ be the function

$$I_k(f) = \frac{e^{-kt}}{e^{kT} - 1} \int_t^{t+T} e^{ks} f(s)ds$$

Clearly I_k is a linear mapping. The next lemma shows that I_k is a linear operator.

<u>Lemma 1.</u> Let k be a non-zero complex number. Then there is a real number C_k such that I_k is a linear operator from P^* to P^* with norm $\leq C_k$, i.e. for each $f \in P^*$

(2) $\|I_k(f)\| \leq C_k \|f\|$

and for each $f \in P^*$, $I_k(f)$ is a solution of the differential equation (1).

<u>Proof:</u> $\|I_k(f)\| \leqslant |1/(e^{kT}-1)| \left\| \int_t^{t+T} e^{k(s-t)} f(s)ds \right\| \leqslant$

$$\leqslant |1/(e^{kT}-1)| \, \|f\| \int_t^{t+T} |e^{k(s-t)}| \, ds \leqslant |1/(e^{kT}-1)| \, \|f\| \, T \max(|e^{kT}|, 1) =$$

$= C_k \|f\|$.

Now let k_1, k_2 be non-zero complex numbers. Define a mapping $I_{k_1} \circ I_{k_2}$ as follows: $I_{k_1} \circ I_{k_2}(f) = I_{k_2}(I_{k_1}(f))$, for each $f \in P^*$. From the Lemma 1 it follows that, for each $f \in P^*$, $I_{k_1} \circ I_{k_2}(f) \in P^*$ and $\|I_{k_1} \circ I_{k_2}(f)\| \leqslant C_{k_1} C_{k_2} \|f\|$ (see (2)). Moreover, $I_{k_1} \circ I_{k_2}(f)$ is a solution of the differential equation

(3) $x'' + (k_1 + k_2)x' + k_1 k_2 . x = f$

Clearly, we have

$$\left(\frac{d}{dt} + k_2\right) I_{k_2}\left(I_{k_1}(f)\right) = I_{k_1}(f)$$

and

$$\left(\frac{d}{dt} + k_1\right)\left(\frac{d}{dt} + k_2\right) I_{k_2}\left(I_{k_1}(f)\right) = \left(\frac{d}{dt} + k_1\right) I_{k_1}(f) = f$$

Thus we have proved the

Lemma 2. Let k_1, k_2 be non-zero complex numbers. Then $I_{k_1} \circ I_{k_2} = I_{k_2}(I_{k_1})$ is a linear operator from P^* to P^*. For each $f \in P^*$, $I_{k_1} \circ I_{k_2}(f)$ is a solution of the differential equation (3) and

$$\| I_{k_1} \circ I_{k_2}(f) \| \leq C_{k_1} C_{k_2} \| f \|$$

Now we are able to prove the next

Theorem. If f, g are real-valued continuous functions, $f \in P$, and if $\lim_{|x| \to \infty} \frac{g(x)}{x} = 0$, then the differential equation

$$x'' + Ax' + Bx + g(x) = f(t)$$

where A, B are real constants, $B \neq 0$, has a real T - periodic solution.

Proof: We show that the assumptions of the Schauder's fixed-point theorem are satisfied.

Let k_1, k_2 be the zeros of the equation $k^2 - Ak + B = 0$. Clearly $k_1, k_2 \neq 0$. For each $\vartheta \in P$ define a function ϑ^* as follows:

$$\vartheta^* = \frac{1}{2}\left\{ I_{k_2}\left[I_{k_1}(f(t) - g(\vartheta(t))) \right] + I_{k_1}\left[I_{k_2}(f(t) - g(\vartheta(t))) \right] \right\}.$$

We show that $\vartheta^* \in P$. This is clear if k_1, k_2 are reals. If k_1, k_2 are non-real roots, then k_1 is the complex conjugate of k_2 $(k_1 = \bar{k}_2)$. Thus $\overline{I_{k_2}\left[I_{k_1}(f(t) - g(\vartheta(t))) \right]} =$

$= I_{\bar{k}_2}\left[I_{\bar{k}_1}(f(t) - g(\vartheta(t)))\right] = I_{k_1}\left[I_{k_2}(f(t) - g(\vartheta(t)))\right]$, and hence ϑ^* is the real part of the function $I_{k_1}\left[I_{k_2}(f(t)-g(\vartheta(t)))\right]$. From this and from the Lemma 2 we have $\vartheta^* \in P$.

Thus the mapping $\mathcal{A}$ defined on P by $\mathcal{A}(\vartheta) = \vartheta^*$ is a mapping from P to P. Now we show that $\mathcal{A}$ is a continuous mapping. Using the Lemma 2 we have $\|\mathcal{A}(\vartheta_1) - \mathcal{A}(\vartheta_2)\| = \sup_t |\mathcal{A}(\vartheta_1(t)) - \mathcal{A}(\vartheta_2(t))| =$

$= \sup_t \left|\frac{1}{2}\left\{I_{k_2}\left[I_{k_1}(g(\vartheta_1(t))-g(\vartheta_2(t)))\right] + I_{k_1}\left[I_{k_2}(g(\vartheta_1(t))-g(\vartheta_2(t)))\right]\right\}\right| \leqslant$

$\leqslant C_{k_1}C_{k_2} \sup_t |g(\vartheta_1(t)) - g(\vartheta_2(t))|$; if $\|\vartheta_1 - \vartheta_2\| \to 0$ then by the continuity of g we have $\sup_t |g(\vartheta_1(t)) - g(\vartheta_2(t))| = \|g(\vartheta_1) - g(\vartheta_2)\| \to 0$ and hence consequently $\|\mathcal{A}(\vartheta_1) - \mathcal{A}(\vartheta_2)\| \to 0$.

It is easy to verify that

$$\frac{d^2\mathcal{A}(\vartheta)}{dt^2} + (k_1+k_2)\frac{d\mathcal{A}(\vartheta)}{dt} + k_1k_2\mathcal{A}(\vartheta) = f - g(\vartheta)$$

If $\vartheta = \mathcal{A}(\vartheta)$ is a fixed point of $\mathcal{A}$, then

$$\frac{d^2\vartheta}{dt^2} + (k_1 + k_2)\frac{d\vartheta}{dt} + k_1k_2\vartheta = f - g(\vartheta),$$

i.e.

$$\frac{d^2\vartheta}{dt^2} + A\frac{d\vartheta}{dt} + B\vartheta = f - g(\vartheta)$$

We have $\lim_{|x|\to\infty}\frac{g(x)}{x} = 0$ hence to each $\varepsilon > 0$ there is some $r(\varepsilon)$ such that $|g(x)| < \varepsilon|x|$ whenever $|x| \geqslant r(\varepsilon)$. If $M(\varepsilon) = \max\{|g(x)|; |x| \leqslant r(\varepsilon)\}$ then

$$|g(x)| \leqslant \begin{cases} M(\varepsilon) & \text{if } |x| \leqslant r(\varepsilon) \\ \varepsilon|x| & \text{if } |x| \geqslant r(\varepsilon) \end{cases}$$

and hence

(4) $\quad |g(x)| \leqslant \varepsilon . D \qquad$ if $\quad |x| \leqslant D$

where D is a number such that

(5) $$D \geqslant \frac{M(\varepsilon)}{\varepsilon}$$

Let $\varepsilon > 0$ be such that

(6) $$1 - \varepsilon C_{k_1} C_{k_2} > 0$$

Choose a $D = D(\varepsilon)$ such that (5) is satisfied and that

(7) $$D \geqslant C_{k_1} C_{k_2} \|f\| / (1 - \varepsilon C_{k_1} C_{k_2})$$

Now consider the set $K = \{\vartheta \in P; \|\vartheta\| \leqslant D\}$. K is a closed convex subset of P. If we show that

1° $\mathcal{Q}(K) \subset K$

and

2° the closure of $\mathcal{Q}(K)$ is compact

then by applying the Schauder's fixed point theorem the proof will be finished.

The property 1° follows from the Lemma 2 and from relations (4), (6), (7):

$$\|\mathcal{Q}(\vartheta)\| \leqslant \frac{1}{2} (C_{k_1} C_{k_2} \|f - g(\vartheta)\| + C_{k_1} C_{k_2} \|f - g(\vartheta)\|) \leqslant$$

$$\leqslant C_{k_1} C_{k_2} (\|f\| + \|g(\vartheta)\|) \leqslant C_{k_1} C_{k_2} (\|f\| + \varepsilon D) \leqslant D$$

The property 2°: Let $\{\vartheta^*_n\}_{n=1}^{\infty}$ be a sequence of elements from the set $\mathcal{Q}(K)$. From the property 1° it follows that this sequence is bounded. We have also (see the Lemmas 1 and 2)

$$\left\| \frac{d\vartheta^*_n}{dt} \right\| = \frac{1}{2} \| -k_2 \{ I_{k_2} [I_{k_1} (f - g(\vartheta_n))] \} - k_1 \{ I_{k_1} [I_{k_2} (f - g(\vartheta_n))] \} +$$

$$+ I_{k_1} (f - g(\vartheta_n)) + I_{k_2} (f - g(\vartheta_n)) \| \leqslant \frac{1}{2} ((k_1 + k_2) C_{k_1} C_{k_2} +$$

$$+ C_{k_1} + C_{k_2})(\|f\| + \|g(\vartheta_n)\|) < \text{const.},$$

(here ϑ_n is the inverse of ϑ_n^*) hence the functions ϑ_n^* are equicontinuous. Thus the Ascoli's theorem can be applied: There is a subsequence $\{\vartheta_{n_k}^*\}_{k=1}^{\infty}$ of $\{\vartheta_n^*\}_{n=1}^{\infty}$ which converges uniformly, i.e. relative the norm $\| \ \|$ to a function $\vartheta^* \in K$. Hence the closure of $\mathcal{A}(K)$ is a compact, q.e.d.

R e m a r k 1. From the proof of the theorem it follows that the condition $\lim_{|x| \to \infty} \frac{g(x)}{x} = 0$ can be replaced by this weaker condition: There is some $\varepsilon > 0$, and a number $r(\varepsilon) > 0$ such that (6) is satisfied and such that, for each $x \geqslant r(\varepsilon)$, $|g(x)| < \varepsilon |x|$.

R e m a r k 2. The theorem is true also for $B = 0$ if all other assumptions of the theorem are satisfied and if $\int_0^T f(s)ds = 0$. Indeed, let P_0 denote the set of all functions $f \in P$ such that $\int_0^T f(s)ds = 0$ and for each $f \in P_0$ let $I_0(f)$ be the function $I_0(f) = \int_0^t f(s)ds$. It is easy to verify that the mapping $\mathcal{B}$ defined by $\mathcal{B}(f) = I_A(I_0(f))$ is a linear continuos operator from P_0 to P with norm $\|\mathcal{B}\| \leqslant C_A T$, and that for each $f \in P_0$, $\mathcal{B}(f)$ is a solution of the differential equation $x'' + Ax' = f$. Now the proof is similar to the proof of the above cited theorem of A. LAZER [1]. We here give an outline of this proof:

For each $\vartheta \in P$ let $\hat{g}(\vartheta) = g[\vartheta(t)] - N(\vartheta)$, where $N(\vartheta) = \frac{1}{T}\int_0^T g(\vartheta(s))\, ds$; hence for each $\vartheta \in P$ we have $\int_0^T \hat{g}(\vartheta)s))\, ds = 0$.

Let R be the set of real numbers and let $Q = P \times R$. Define

$$|(\vartheta, a)| = \|\vartheta\| + |a|,$$

$$c_1(\vartheta_1, a_1) + c_2(\vartheta_2, a_2) = (c_1 \vartheta_1 + c_2\vartheta_2, c_1 a_1 + c_2 a_2),$$

for each (ϑ, a), (ϑ_1, a_1), $(\vartheta_2, a_2) \in Q$, and $c_1, c_2 \in R$. Clearly B is a linear normed space. On the set Q define an operator φ as follows: $\varphi\left[(\vartheta, a)\right] = (\vartheta^*, a^*)$, where

$$\vartheta^* = a + \mathcal{B}(f - \hat{g}(\vartheta)),$$
$$a^* = a - N(\vartheta^*)$$

φ is a continuous mapping from Q to Q and

$$\frac{d^2\vartheta^*}{dt^2} + A\frac{d\vartheta^*}{dt} = f(t) - \hat{g}(\vartheta(t)) = f(t) - g(\vartheta(t)) + N(\vartheta).$$ Hence if (ϑ, a) is a fixed point of φ then $a = a^*$, $\vartheta = \vartheta^*$, $N(\vartheta^*) = N(\vartheta) = 0$ and consequently $\frac{d^2\vartheta}{dt^2} + A\frac{d\vartheta}{dt} + g(\vartheta) = f$.

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О ВЕРТИКАЛЬНОМ РАСПРЕДЕЛЕНИИ НУЛЕЙ ДЗЕТА-ФУНКЦИИ РИМАНА

Ян МОЗЕР, Братислава

Пусть $s=\sigma+it$ - комплексное переменное, $\zeta(s)$ - дзета-функция Римана. Гипотеза Римана состоит в том, что все нетривиальные нули функции $\zeta(s)$, лежат на критической прямой $\sigma=\frac{1}{2}$.

Пусть $\rho=\frac{1}{2}+i\gamma$ - нуль функции $\zeta(s)$, лежащий на критической прямой. Так как $\overline{\zeta(\frac{1}{2}+it)}=\zeta(\frac{1}{2}-it)$, достаточно предположить, что $\gamma>0$.

Пусть, дальше:

(γ) - обозначает строго возрастающую последовательность ординат нулей ρ, (ордината кратного нуля считается один раз в этой последовательности),

(γ,γ') - обозначает соседние ординаты, т.е. члены последовательности (γ), такого рода, что $\gamma<\gamma'$, и, γ' - наименьший из следующих после γ членов последовательности (γ),

$\{\gamma\}$ - обозначает последовательность пар (γ,γ'), где γ пробегает последовательность (γ),

$n(\gamma)$ - обозначает кратность нуля ρ.

Обозначения O, Ω - используются в обычном, в теории $\zeta(s)$, смысле. $A, A_1, \ldots$ - постоянные, $A(\alpha)$ - постоянная, зависящая от параметра.

Напомним следующее:

(а) существует $\gamma_1>0$, такого рода, что при $\gamma>\gamma_1$, имеет место

$$1\leqq n(\gamma)\leqq[A\ln\gamma]$$

(см.[1], стр. 209, конец п.2), где квадратные скобки (только здесь в этой заметке!) обозначают целую часть,

(б) функция Мебиуса $\mu(n)$, определяется соотношением

$$\sum_{n=1}^{+\infty}\frac{\mu(n)}{n^s}=\prod_{(p)}\left(1-\frac{1}{p^s}\right),\quad \sigma>1$$

где p - пробегает все простые числа,

(в) ослабленная гипотеза Мертенса состоит в том, что

(1) $$\int_1^x \left\{\frac{M(t)}{t}\right\}^2 dt = O(\ell n\, x)$$

где

$$M(x) = \sum_{n \le x} \mu(n)$$

Из соотношения (1) следует:

(1а) гипотеза Римана,

(1б) $\zeta'(\rho) \neq 0$, (см. [1], стр. 374),

(1b) $$\frac{A}{\gamma \exp\left(A \frac{\ell n}{\ell n \ell n \gamma}\right)} < \gamma' - \gamma$$

начиная с некоторого члена последовательности $\{\gamma\}$, (см. [1], стр. 379).

В этой заметке доказывается

Т е о р е м а. Если справедлива гипотеза Римана, то для любого $\alpha \in (0, \frac{1}{2})$, существует последовательность $\{\tilde{\gamma}(\alpha)\}$, последовательности $\{\gamma\}$, такого рода, что если $(\tilde{\gamma}', \tilde{\gamma}'')$ - член этой подпоследовательности, то

$$\psi(\tilde{\gamma}';\alpha) = \frac{\exp\left[\frac{(\ell n\, \tilde{\gamma}')^{\alpha}}{n(\tilde{\gamma}')}\right]}{2\,\ell n\, \tilde{\gamma}' \exp\left[\frac{A\, \ell n\, \tilde{\gamma}'}{n(\tilde{\gamma}')\, \ell n\, \ell n\, \tilde{\gamma}'}\right]} < \tilde{\gamma}'' - \tilde{\gamma}'$$

Из этой Теоремы следует

А л ь т е р н а т и в а. Если справедлива гипотеза Римана, то или

(а) соотношение

$$\psi(\gamma;\alpha) < \gamma' - \gamma,$$

имеет место для всякого члена последовательности $\{\gamma\}$, начиная с некоторого, (т.е., существует $\gamma_2(\alpha) > 0$, такого рода, что это соотношение имеет место для всех (γ, γ'), для которых $\gamma > \gamma_2(\alpha)$),

или

(б) существует подпоследовательность $\{\tilde{\tilde{\gamma}}(\alpha)\}$, последовательности $\{\gamma\}$, такого рода, что если $(\tilde{\tilde{\gamma}}', \tilde{\tilde{\gamma}}'')$ - член этой подпоследовательности, то

$$\tilde{\tilde{\gamma}}'' - \tilde{\tilde{\gamma}}' \leqq \psi(\tilde{\tilde{\gamma}}; \alpha).$$

Если осуществляется возможность (а) , то имеется уточнение Теоремы, т.е., соотношение (1b) улучшено, только в предположений справедливости гипотезы Римана.

Если осуществляется возможность (б) , то

(ба) существует подпоследовательность $\{\tilde{\gamma}(\alpha)\}$, последовательности $\{\gamma\}$, такого рода, что если $(\tilde{\gamma}', \tilde{\gamma}'')$ - член этой подпоследовательности, то

$$\tilde{\gamma}' < \tilde{\gamma}' + \psi(\tilde{\gamma}'; \alpha) < \tilde{\gamma}'' \tag{2}$$

и

(бб) существует подпоследовательность $\{\tilde{\tilde{\gamma}}(\alpha)\}$, последовательности $\{\gamma\}$, такого рода, что если $(\tilde{\tilde{\gamma}}', \tilde{\tilde{\gamma}}'')$ - член этой подпоследовательности, то

$$\tilde{\tilde{\gamma}}' < \tilde{\tilde{\gamma}}'' = \tilde{\tilde{\gamma}}' + \psi(\tilde{\tilde{\gamma}}'; \alpha) \tag{3}$$

т.е., в случае (б) , в последовательности $\{\gamma\}$ осуществляется бесконечно много раз возможность (2) , и, бесконечно много раз возможность (3). Это обстоятельство (возможность (б)), я назову: осциляцией последовательности $\{\gamma\}$, относительно последовательности

$$(\gamma + \psi(\gamma; \alpha))$$

В доказательстве Теоремы существенно следующее: оценка модуля функции $\zeta^{(4)}$ в окрестности нуля ρ, и, Ω - теорема Титчмарша.

Прежде чем приступить к доказательству, приведем некоторые вспомогательные утверждения.

Л е м м а 1. Если справедлива гипотеза Римана, то

$$\zeta(s) = O\left\{\exp\left(A\frac{\ln t}{\ln\ln t}\right)\right\},\ \frac{1}{2} \leqq \sigma \leqq \frac{2}{\ln t} + \frac{1}{2}.$$

Д о к а з а т е л ь с т в о. См. [1] , стр. 350, доказательство Теоремы 1.

Л е м м а 2. Если справедлива гипотеза Римана, то

$$\zeta(s) = O\left\{\exp\left(A\frac{\ln t}{\ln\ln t}\right)\right\},\ \frac{1}{2} - \frac{2}{\ln t} \leqq \sigma < \frac{1}{2}.$$

Д о к а з а т е л ь с т в о. Как обычно, из функционального уравнения Римана, и асимптотической формулы Стирлинга, получается соотношение

$$|\zeta(s)| < A_1\, t^{\frac{1}{2}-\sigma} |\zeta(1-s)|$$

Из этого, на основе Леммы 1., для $\frac{1}{2} - \frac{2}{\ln t} \leqq \sigma < \frac{1}{2}$, получается, что

$$|\zeta(s)| < A_2\, t^{\frac{1}{2}-\sigma} \exp\left(A\frac{\ln t}{\ln\ln t}\right) \leqq$$

$$\leqq A_2\, t^{\frac{2}{\ln t}} \exp\left(A\frac{\ln t}{\ln\ln t}\right) =$$

$$= A_3 \exp\left(A\frac{\ln t}{\ln\ln t}\right)$$

Л е м м а 3. Если справедлива гипотеза Римана, то

$$\frac{\zeta^{(k)}(\varrho)}{k!} = O\left\{(\ln\gamma)^k \exp\left(A\,\frac{\ln\gamma}{\ln\ln\gamma}\right)\right\},\quad k = 1,2,\ldots,$$

где

$$\zeta^{(k)}(\varrho) = \left[\frac{d^k\zeta(s)}{d\,s^k}\right]_{s=\varrho}$$

(постоянная в этих O - соотношениях, независит от k).

Д о к а з а т е л ь с т в о. На основе интегральной формулы Коши, используя Лемму 1., Лемму 2., и, полагая $r = \frac{1}{\ln\gamma}$, получается:

$$\frac{\zeta^{(k)}(\varrho)}{k!} = \frac{1}{2\pi r^k}\int_0^{2\pi}\frac{\zeta\left(\frac{1}{2}+i\gamma+r\,e^{i\varphi}\right)}{e^{ik\varphi}}\,d\varphi\ + $$

$$= O\left\{\frac{1}{r^k}\exp\left[A\,\frac{\ln(\gamma+r)}{\ln\ln(\gamma+r)}\right]\right\} =$$

$$= O\left\{(\ln\gamma)^k \exp\left(A\,\frac{\ln\gamma}{\ln\ln}\right)\right\}$$

Л е м м а 4. Если справедлива гипотеза Римана, то для любого $\alpha \in (0, \frac{1}{2})$, существует $\gamma_5(\alpha) > 0$, такого рода, что при $\gamma > \gamma_5(\alpha)$, и тех t, для которых

$$|t-\gamma| \leqq \psi(\gamma;\alpha)$$

имеет место соотношение

$$\left|\zeta\left(\tfrac{1}{2}+it\right)\right| < A_5 \exp\left[(\ln\gamma)^{\alpha}\right].$$

Д о к а з а т е л ь с т в о. Прежде всего, Лемма 3. утверждает, что

$$(5)\qquad \left|\frac{\zeta^{(k)}(\rho)}{k!}\right| < A_4(\ln\gamma)^k \exp\left(A\,\frac{\ln\gamma}{\ln\ln\gamma}\right),\quad \gamma>\gamma_3>0,\ k=1,2,\dots,$$

(γ_3 - независит от k), и, в промежутке $|t-\gamma|<\sqrt{\gamma^2+\frac{1}{4}}$, имеет место разложение Тейлора

$$(6)\qquad \zeta\left(\frac{1}{2}+it\right) = \sum_{k=n(\gamma)}^{+\infty} \frac{\zeta^{(k)}(\rho)}{k!}\left[i(t-\gamma)\right]^k$$

Если t удовлетворяет неравенству (4), то, используя

(а) формулу (6) и неравенства (5),

(б) обозначение $q(\gamma;\alpha)=\ln\gamma\,\psi(\gamma;\alpha)$, (ясно, что существует $\gamma_4(\alpha)>0$, такого рода, что $0<q(\gamma;\alpha)\leqq\frac{1}{2}$, при $\gamma>\gamma_4(\alpha)$),

в $\dfrac{1}{1-q(\gamma;\alpha)}\leqq 2$, при $\gamma>\gamma_4(\alpha)$,

получим при $\gamma>\gamma_5(\alpha)=\max\{\gamma_3,\gamma_4(\alpha)\}$:

$$\left|\sum_{k=m(\gamma)}^{+\infty}\frac{\zeta^{(k)}(\rho)}{k!}\left[i(t-\gamma)\right]^k\right| \leqq \sum_{k=n(\gamma)}^{+\infty}\left|\frac{\zeta^{(k)}(\rho)}{k!}\right| |t-\gamma|^k <$$

$$< A_4\exp\left(A\,\frac{\ln\gamma}{\ln\ln\gamma}\right)\sum_{k=n(\gamma)}^{+\infty}\left[q(\gamma;\alpha)\right]^k =$$

$$= A_4\,\frac{\left[q(\gamma;\alpha)\right]^{n(\gamma)}}{1-q(\gamma;\alpha)}\exp\left(A\,\frac{\ln\gamma}{\ln\ln\gamma}\right)\leqq 2A_4\exp\left[(\ln\gamma)^{\alpha}\right] =$$

$$= A_5\exp\left[(\ln\gamma)^{\alpha}\right].$$

Л е м м а 5. Независимо от какой бы то ни было гипотезы, для любого $\alpha\in(0,\frac{1}{2})$, имеет место

$$\zeta\left(\frac{1}{2}+it\right)=\Omega\left\{\exp\left[(\ln t)^{\alpha}\right]\right\}.$$

(Постоянная в этой Ω - теореме равна единице.)

Д о к а з а т е л ь с т в о. См. [1] . стр. 201-206

Д о к а з а т е л ь с т в о Т е о р е м ы. Если утверждение Теоремы неверно, то существуют, $\alpha_0 \in (0,\frac{1}{2})$, и, $\gamma_6(\alpha_0) > 0$, такого рода, что для всякого члена (γ', γ'') последовательности $\{\gamma\}$, начиная с того, для которого $\gamma' > \gamma_6(\alpha_0)$, имеет место

(7) $$\gamma'' - \gamma' \leqq \psi(\gamma'; \alpha_0)$$

Положим $\gamma_7(\alpha_0) = \max\{\gamma_5(\alpha_0), \gamma_6(\alpha_0)\}$. Используя Лемму 4., получим

(8) $$\left|\zeta\left(\frac{1}{2} + it\right)\right| < A_5 \exp\left[(\ln\gamma')^{\alpha_0}\right], \quad t \in \langle \gamma', \gamma'' \rangle .$$

Если $t = \gamma'$, то

(9) $$\left|\zeta\left(\frac{1}{2} + it\right)\right| < A_5 \exp\left[(\ln t)^{\alpha_0}\right].$$

Пусть, дальше, $\gamma' < t \leqq \gamma''$. Обозначим

$$K(\gamma'; \alpha_0, t) = \frac{\exp\left[(\ln\gamma')^{\alpha_0}\right] - \exp\left[(\ln t)^{\alpha_0}\right]}{\exp\left[(\ln t)^{\alpha_0}\right]} .$$

В этом случае получается

$$A_5 \exp\left[(\ln\gamma')^{\alpha_0}\right] = A_5 \exp\left[(\ln t)^{\alpha_0}\right]\left\{1 + K(\gamma'; \alpha_0, t)\right\} <$$
$$< 3 A_5 \exp\left[(\ln t)^{\alpha_0}\right] =$$
$$= A_6 \exp\left[(\ln t)^{\alpha_0}\right]$$

так как $|K(\gamma'; \alpha_0, t)| < 2$. Т.е.

(10) $$\left|\zeta\left(\frac{1}{2} + it\right)\right| < A_6 \exp\left[(\ln t)^{\alpha_0}\right], \quad t \in (\gamma', \gamma''\rangle .$$

Тогда, из (9) и (10), получается

$$(11)\qquad \left|\zeta\left(\frac{1}{2}+it\right)\right| < A_6 \exp\left[(\ln t)^{\alpha_0}\right], \quad t\in\langle\gamma',\gamma''\rangle.$$

Так как (11) имеет место для всякого $\gamma'>\gamma_7(\alpha_0)$, то, наконец, получается оценка

$$(12)\qquad \zeta\left(\frac{1}{2}+it\right) = O\left\{\exp\left[(\ln t)^{\alpha_0}\right]\right\}.$$

Соотношение (12), однако, противоречит при $\alpha\in(\alpha_0,\frac{1}{2})$, для достаточно больших t, Лемме 5.

Примечание. Напомним, что в предложении справедливости ослабленной гипотезы Мертенса, для последовательности $\{\gamma\}$, начиная с некоторого члена, имеет место

$$\frac{A}{\gamma\exp\left(A\dfrac{\ln\gamma}{\ln\ln\gamma}\right)} < \gamma'-\gamma < \frac{A}{\ln\ln\ln\gamma}$$

[Оценка снизу-см. (1b). Оценка сверху-простое следствие известной теоремы Литтльвуда, имеющей место независимо от какой бы то ни было гипотезы, (доказательство теоремы Литтльвуда см., например, [1], п. 12., стр. 223-225).]

Так как возможности Альтернативы касаются частей промежутков

$$\left\langle \frac{A}{\gamma\exp\left(A\dfrac{\ln\gamma}{\ln\ln\gamma}\right)}, \frac{A}{\ln\ln\ln\gamma}\right\rangle$$

то получить метод, позволяющий судить, которая возможность Альтернативы, для некоторого $\alpha\in(0,\frac{1}{2})$ осуществляется, вероятно, трудная задача, даже в предположении справедливости ослабленной гипотезы Мертенса.

Л И Т Е Р А Т У Р А

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A GENERALIZATION OF L_1 COMPLETNESS THEOREM

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It is well known that some theorems of measure theory can be formulated and proved by the help of only certain properties of the systems N_n of all measurable sets of measure less than 1/n (see [5], [6], [7]). E. g. we proved in [7] such a generalization of the completness of the space $S(\tau)$ of all measurable sets of finite measure with the pseudometric $d(E, F) = (E \Delta F)$.

We think that it is natural to try to formulate and prove by a similar way some theorems of the integration theory. Of course, we must study here the systems F_n of all integrable functions with integral less than 1/n. In this paper we shall generalise by this way the L_1 completness theorem.

Since we actually prove a more general proposition (part 1), we obtain besides the generalized L_1 completness theorem (part 2) also the generalized $S(\tau)$ completness theorem from [7] as well as an Alfsen's theorem ([1], Theorem 3) about the completness of the metric space induced by his full integral (part 4).

1

We shall assume that there are given a σ-complete lattice M, a sublattice L of M and a sequence $\{R_n\}$ of relations defined on L fulfilling the following axioms:

1. R_n is symmetric and reflexive for any n.
2. If $(x_i, x_{i+1}) \in R_i$, $i=n,\dots,n+r-1$, then $(x_n, x_{n+r}) \in R_{n-1}$.
3. If $x_n \nearrow x$ or $x_n \searrow x$, $x_n \in L$ and $(x_n, x_{n+1}) \in R_n$ for $n > N$, then $x \in L$ and to any m there is n_0 such that $(x, x_n) \in R_m$ for any $n \geqq n_0$.

4. If $(x,y) \in R_n$, then $(z \cap x,\ z \cap x \cap y) \in R_n$ for any $z \in L$.

5. $R_{n+1} \subset R_n$ for any n.

Let us give a very simple but instructive example. Let M be the set of all measurable functions defined on a measure space, L be the set of all integrable functions, and $(x,y) \in R_n$ if and only if $\int |x - y| < 2^{-n}$.

Note that from 2 and 5 it follows

2'. To any n there exists m such that $(x,z) \in R_m$, $(z,y) \in R_m$ implies $(x,y) \in R_n$.

<u>Theorem 1.</u> If $\{R_n\}$ satisfies 1,2' and 5, then $\{R_n\}$ is a base of a uniformity for the space L, hence L is a uniform, pseudometrizable space. If $\{R_n\}$ satisfies 1 - 5, then L is complete.

<u>Proof.</u> It follows from 1 that R_n is symmetric and that R_n contains the diagonal (for n = 1,2,...). 2' states that to any n there exists m such that $R_m \circ R_m = \{(x,y) : \exists z \in M,\ (x,z) \in R_m,\ (z,y) \in R_m\} \subset R_n$. Finally from 5 it follows that the intersection of two relations R_m, R_n is also a relation from this sequence (either R_m or R_n). Hence $\{R_n\}$ is a base for a uniformity, L is a uniform and pseudometrizable space ([4], chap. VI, Th. 2, Th. 13). In order to prove the completness of L it suffices to prove that any Cauchy sequence is convergent ([4], chap. VI., Th. 24).

Now let $\{x_n\}$ be a Cauchy sequence. Then there is a subsequence $\{x_{t_i}\}$ such that $(x_{t_i}, x_{t_{i+1}}) \in R_i$. Put $y_i = x_{t_i}$, $z_n^k = \bigcap_{i=n}^{n+k} y_i$.

Evidently $z_n^k \searrow \bigcap_{i=n}^{\infty} y_i \ (k \to \infty)$. Further $(z_n^k, z_n^{k+1}) =$ $= (z_n^{k-1} \cap y_{n+k},\ z_n^{k-1} \cap y_{n+k} \cap y_{n+k+1}) \in R_{n+1}$ because of $(y_{n+k}, y_{n+k+1}) \in R_{n+k}$ and 4. From 3 it follows that $\bigcap_{i=n}^{\infty} y_i \in L$ and there exists n_0 such that for every $k > n_0$ we have $(z_n^k, \bigcap_{i=n}^{\infty} y_i) \in R_n$. Now fix n and put $z_{n+k} = z_n^k$ (k = 0,1,...) i.e. $z_i = z_n^{i-n}$ (i=n, n+1, ...).

We have $(z_i, z_{i+1}) \in R_i$ $(i = n, n+1, \ldots)$. According to the axiom 2 we have $(y_n, z_n^k) = (z_n^o, z_n^k) = (z_n, z_{n+k}) \in R_{n-1}$. Take $k > n_o$. Then from the relations $(y_n, z_n^k) \in R_{n-1}$, $(z_n^k, \bigcap_{i=n}^{\infty} y_i) \in R_n$ and from 2 it follows $(y_n, \bigcap_{i=n}^{\infty} y_i) \in R_{n-2}$ for any $n > 2$.

Now put $u_n = \bigcap_{i=n}^{\infty} y_i$. By preceding $u_n \in L$ and $(y_n, u_n) \in R_{n-2}$. Moreover, if we put $x = \bigcup_{n=1}^{\infty} u_n$, then $u_n \nearrow x$. Hence, we have $(u_n, y_n) \in R_{n-2} \subset R_{n-3}$, $(y_n, y_{n+1}) = (x_{t_n}, x_{t_{n+1}}) \in R_{n-2}$, $(y_{n+1}, u_{n+1}) \in R_{n-1}$, therefore $(u_n, u_{n+1}) \in R_{n-3}$ by 2. Applying again 3 we get $x \in L$ and $\{u_n\}$ converges to x with respect to the uniform tepology. Also $\{y_n\}$ converges to x. Indeed, for any m and sufficiently large n we have $(u_n, x) \in R_{m+2}$, $(y_n, u_n) \in R_{n-2} \subset R_{m+1}$, hence $(y_n, x) \in R_m$.

We have the following result: there are an element $x \in L$ and a subsequence $\{x_{t_i}\}$ of the sequence $\{x_n\}$ such that $\{x_{t_i}\}$ converges to x. But because $\{x_n\}$ is Cauchy, then the whole sequence $\{x_n\}$ converges to x too.

2

Now we shall formulate and prove the generalized L_1 completness theorem. We shall assume that there are given a measurable space and a sequence $\{F_n\}_{n=o}^{\infty}$ of systems of measurable functions satisfying the following conditions:

2.1 $0 \in F_n$ for any n; $f \in F_n \Longleftrightarrow -f \in F_n$.

2.2 If $f_i \in F_i$, $i = n, n+1, \ldots, n+k$, then $\sum_{i=n}^{n+k} f_i \in F_{n-1}$.

2.3 If $f_n \nearrow f$ or $f_n \searrow f$ and $f_n \in F_o$, $f_n - f_{n+1} \in F_n (n=1,2,\ldots)$, then $f \in F_o$ and to any m there is n such that $f - f_n \in F_m$.

2.4 If f is a measurable function, $g \in F_n$ and $|f| \leqq |g|$, then $f \in F_n$.

2.5 $F_{n+1} \subset F_n$ for every n.

Evidently from 2.2 and 2,5 it follows

2.2´ To any n there exists m such that $f \in F_m$, $g \in F_m$ implies $f + g \in F_n$.

T h e o r e m 2. If $\{F_n\}$ satisfies 2.1, 2.2´ and 2.5 then the system $\{\{(f,g) : f,g \text{ measurable}, f - g \in F_n\} : n = 1,2,\dots\}$ is a base of a uniformity for F_o; F_o is then a uniform, pseudometrizable space. If $\{F_n\}$ satisfies 2.1 - 2.5 then F_o is complete.

P r o o f. Let M be the system of all measurable functions and $L = F_o$. Further define R_n as follows: $(f,g) \in R_n$ $(n = 1,2,\dots)$ if and only if $f - g \in F_n$. The proof of properties 1 - 5 is almost trivial and Theorem 2 follows from Theorem 1.

If we take the system of all integrable functions for F_o and put $F_n = \{f \text{ measurable} : \int |f| < 2^{-n}\}$, then we get the usual L_1 completness theorem.

3

Now we prove as a corollary of Theorem 1 the generalized $S(\tau)$ completness theorem from [7]. (Actually we give another formulation of this theorem.) We assume that there are given a measurable space X with a σ-ring S of subsets of X and a sequence $\{N_n\}_{n=0}^{\infty}$ of subsystems of S satisfying the following conditions:

3.0 $E,F \in N_o \Longrightarrow E \cup F \in N_o$.

3.1 $\emptyset \in N_n$ for every n.

3.2 If $E_i \in N_i$ $(i = n, n+1, \dots)$, then $\bigcup_{i=n}^{\infty} E_i \in N_{n-1}$.

3.3 If $\{E_i\}_{i=1}^{\infty}$ is any non-increasing sequence of sets from N_o, and $\bigcap_{i=1}^{\infty} E_i = \emptyset$, then to any m there exists n such that $E_n \in N_m$.

3.4 If $E \subset F$, $E \in S$, $F \in N_n$, then $E \in N_n$.

3.5 $N_{n+1} \subset N_n$ for every n.

It is evident that if (X,S,τ) is a measure space, $N_0 = \{E \in S: \tau(E) < \infty\}$, $N_n = \{E \in S : \tau(E) < 2^{-n}\}$, then the sequence $\{N_n\}_{n=0}^{\infty}$ satisfies all the assumptions 3.1 - 3.5. (In [7] we assumed that 3.2 holds only for a subsequence $\{k_n\}$. But the difference is formal, since we should put $N_n' = N_{k_n}$ $(n=1,2,\ldots)$.)

As before we hint at the fact that now from 3.2 and 3.5 it follows

3.2′ To any n there exists m such that $E \in N_m$, $F \in N_m$ implies $E \cup F \in N_n$.

<u>Theorem 3.</u>[x)] If $\{N_n\}_{n=0}^{\infty}$ statisfies 3.1, 3.2′, 3.4 and 3.5 then the system $\{\{(E,F): E,F \in S, E \Delta F \in N_n\} : n = 1,2,\ldots\}$ is a base of a uniformity for N_0; N_0 is then uniform and pseudometrizable. If $\{N_n\}_{n=0}^{\infty}$ satisfies 3.0 - 3.5, then N_0 is complete.

<u>Proof.</u> Put $M = S$, $L = N_0$, $R_n = \{(E,F) : E \Delta F \in N_n\}$. Then 1 follows from 3.1, 5 from 3.5, 2 from 3.2 and 3.4 ($E_n \Delta E_{n+k} \subset \bigcup_{i=n}^{n+k-1} (E_i \Delta E_{i+1})$), 4 from 3.4. If $E_n \nearrow E$, $E_n \in N_0$, then $E = E_1 \cup \bigcup_{n=2}^{\infty} (E_n - E_{n-1})$, $E_1 \in N_0$, $\bigcup_{n=2}^{\infty} (E_n - E_{n-1}) \in N_1 \subset N_0$, therefore $E \in N_0$ because of 3.0. If $E_n \searrow E$, then $E \in N_0$ because of 3.4 and 3.5. In the both cases to any m there is n such that $(E,E_n) \in R_m$. Actually $(E,E_k) \in R_m$ for every $k \geqq n$, since $E - E_k \subset E - E_n = E \Delta E_n \in N_m$ resp. $E_k - E \subset E_n - E = E \Delta E_n \in N_m$. Hence all the assumptions of Theorem 1 are satisfied and we see that Theorem 3 follows from Theorem 1.

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x) See [7], Theorem 2

It seems that $S(\tau)$ completness theorem can be generalized by another form, similarly as L_1 completness theorem. Actually, we can formulate the following theorem (the second assumption is replaced by a weaker, the third by a stronger assumption is replaced by a weaker, the third by a stronger assumption):

Theorem 3'. N_i is a complete uniform pseudometrizable space, if $\{N_n\}_{n=0}^{\infty}$ satisfies 3.1, 3.4, 3.5 and the following two conditions:

I. If $E_i \in N_i$ $(i = n, n+1, \dots, n+k)$, then $\bigcup_{i=n}^{n+k} E_i \in N_{n-1}$.

II. If $E_n \nearrow E$, or $E_n \searrow E$ and $E_n \in N_0$ $(n = 1,2,\dots)$, $E_n \triangle E_{n+1} \in N_n$, then $E \in N_0$ and to any m there is n such that $E \triangle E_n \in N_m$.

The proof of this theorem is clear and can be omited.

4

Let M be a σ-continuous lattice i.e. M is σ-complete and $x_n \nearrow x, y_n \nearrow y$ (resp. $x_n \searrow x$, $y_n \searrow y) \Rightarrow x_n \cap y_n \nearrow x \cap y$ (resp. $x_n \cup y_n \searrow x \cup y$). Let L be a sublattice of M and I be an increasing valuation (i.e. such real function, that $I(x) + I(y) = I(x \cup y) + I(x \cap y)$, and $x \leqq y \Rightarrow I(x) \leqq I(y)$) satisfying the following requirement:

$x_n \nearrow x \in M$, $x_n \in L$ $(n = 1,2,\dots)$, $\sup_n I(x_n) < \infty \Rightarrow x \in L$ and $I(x_n) \nearrow I(x)$; and dually.

Then we say that I is a full integral ([1], § 2). L is a pseudometric space with the pseudometric function $d(x,y) = I(x \cup y) - I(x \cap y)$ ([2], p. 77). Moreover, from Theorem 1 we get the following theorem:

Theorem 4. ([1], Theorem 3). L is a complete pseudometric space.

P r o o f. Define $R_n = \{(x,y) : d(x,y) = I(x \cup y) - I(x \cap y) < 2^{-n}\}$. Then 1 and 5 are evident, 2 follows from the triangle inequality, 3 from Beppo Levi requirement. Finally 4 follows from the inequality

$$I(x \cap z) + I(x \cap y) = I((x \cap z) \cup (x \cap y)) + I((x \cap z) \cap (x \cap y)) \leqq$$

$$\leqq I(x \cup y) + I(x \cap y \cap z)$$

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ON COVERING OF RINGS BY THEIR RESIDUE CLASSES

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The disjoint covering of the set of all integers by residue classes is studied in many articles (see for example [1]). In [6] an above estimation for the number of residue classes in a disjoint covering system is shown. In [3] is this result generalized for the disjoint covering of some Abelian groups by their cosets.

Our article contains a generalization of mentioned problem in another direction. We shall study the disjoint covering of principal ideal domains by residue classes and show the mentioned estimation for the number of ideals. Further, we prove our result to be the best possible in some sense.

I.

A commutative integrity domain R with unit is called a principal ideal domain if every ideal of R is principal. That means if I is an ideal of R then there exists an element $a \in R$ such that $I = aR = \{ah : h \in R\}$. Such an ideal I is said to be generated by the element a.

Let R be a ring. Define the function f on R (the image of which is some set of cardinal numbers) in following way

$$a \in R : \qquad f(a) = \text{card } R/aR$$

where R/aR is the factor-ring related with ideal aR.

A residue class of R is a set of the form

$$a + nR = \{a + nh : h \in R\}$$

where $a, n \in R$.

A system of residues related with the element a is a set R_a (of elements of the ring R) with

(i) $0 \in R_a$ (0 is the zero of R),

(ii) if $x, y \in R_a$ and $x \neq y$ then $(x + aR) \cap (y + aR) = \emptyset$,

(iii) $\bigcup_{x \in R_a} (x + aR) = R$.

Later on we shall need the following properties:

1. Any principal ideal domain is a unique factorization domain, i.e. every element of R which does not divide the unit of R is expressible as a product of irreducible elements, and, except for the order of the factors and for unit factors, this representation is unique ([5], chapter IV, § 15, theorem 32).

2. If R is principal ideal domain and $a, b \in R$ then there exists a g.c.d. (a,b) of elements a,b in R.

3. Let p be an irreducible element in the principal ideal domain R and $a, b \in R$ so that $ab \neq 0$ and $p|ab$. Then either $p|a$ or $p|b$.

These two properties follows from the property 1.

4. Let R be a principal ideal domain and let $a, b, c \in R$. Then the diophantine equation $ax + by = c$ is solvable for $x, y \in R$ if and only if $(a,b)|c$.

It is easy to show the necessity of this condition. The sufficiency follows from the fact that the set $\{ax + by : x, y \in R\}$ is an ideal in R generated by g.c.d. (a,b) of elements a,b.

5. Let R_p be a system of residues related with irreducible element p. If $x \in R_p$ and $x \neq 0$ then $p \nmid X$.

In opposite case it holds

$(0 + pR) \cap (x + pR) \neq \emptyset$

And this is a contradiction.

We shall investigate such principal ideal domains for which f(a) is a finite cardinal number for every element $a \in R$, $a \neq 0$.

II

A system of residue classes

$$a_i + n_iR, \quad a_i, n_i \in R, \quad i \in T, \quad |T| > 1 \tag{1}$$

is said to be disjoint covering if

(j) $(a_i+n_iR) \cap (a_j+n_jR) = \emptyset$ for $i \neq j$

(jj) $\bigcup_{i \in T} (a_i+n_iR) = R$

For this system some properties of disjoint covering systems of the set of rational integers can be generalized (see [1], [3] and [6]).

1. $(n_i,n_j) \nmid 1$ for any $i \neq 1$.

Suppose $(n_i,n_j)|1$ for some i,j. Then the diophantine equation $a_i-a_j = xn_j-yn_i$ is solvable and hence the classes a_i+n_iR and a_j+n_jR are not disjoint.

2. Let T be a finite set. Then we have

$$\sum_{i \in T} \frac{1}{f(n_i)} = 1 \quad \text{for (1).}$$

We can easy prove that

$$R_{ab} = \{x+a : x \in R_a, y \in R_b\}$$

which implies $f(ab) = f(a)f(b)$. According to this result the residue class $x + aR$ can be disjointly covered by $f(b) = \frac{f(ab)}{f(a)}$ classes of the form $x+ya+abR$ for $y \in R_b$.

Let $n = \prod_{i \in T} n_i$ with n_i from (1). Replace every residue class a_i+n_iR $(i \in T)$ by a system a_i+yn_i+nR $(y \in R_{n/n_i})$ of $\frac{f(n)}{f(n_i)}$ disjoint classes. Thus we get the following system [from (1)]

$$y + nR, \quad y \in R_n$$

So we have

$$f(n) = \sum_{i \in T} \frac{f(n)}{f(n_i)}$$

and our statement is proved.

3. Let

$$n_{i_o} = \prod_{t=1}^{r} p_t^{\lambda t}, \quad i_o \in T \tag{2}$$

be a decomposition into the irreducible elements. Then

$$\text{card } T \geqslant 1 + \sum_{t=1}^{r} \lambda_t[f(p_t) - 1]$$

To prove this assertion we shall need the following theorem.

T h e o r e m 1. If (1) is a disjoint covering system and (2) holds then the elements

$$a_{i_o} + c_t q_t p_t^{\alpha t} \tag{3}$$

(where $t = 1,2,\ldots,r$; c_t runs over all elements of R_{p_t} except of 0; $q_t = n_{i_o}/p_t^{\lambda t}$; $\alpha_t = 0,1,\ldots, \lambda_t-1$) belong to the distinct classes of the system (1).

To the proof we shall use the following

L e m m a. If

$$a_{i_o} + c_t q_t p_t^{\alpha t} \in a_j + n_j R \tag{4}$$

then $p_t^{\beta t} \mid n_j$ with $\beta_t > \alpha_t$.

P r o o f. From (4) it follows

$$(q_t p_t^{\alpha t}, n_j) \mid a_{i_o} - a_j \tag{5}$$

However, the classes $a_{i_0} + n_{i_0}R$ and $a_j + n_jR$ are disjoint, hence the diophantine equation

$$a_{i_0} + n_{i_0}x = a_j + n_jy \tag{6}$$

is not solvable and so we have $(n_{i_0}, n_j) \nmid a_{i_0} - a_j$. From (5) and (6) it follows that n_j is divisible by $p_t^{\beta_t}$ with $\beta_t > \alpha_t$.

Proof of Theorem 1. The elements of (3) are pairwise distinct. We shall prove it indirectly. Suppose that for some t and t' we have

$$c_t q_t p_t^{\alpha_t} = c_{t'} q_{t'} p_{t'}^{\alpha_{t'}}$$

If $p_t \neq p_{t'}$ then the exponent of p_t on the left-hand side is smaller than that on the right-hand side. Let $p_t = p_{t'}$ and $\alpha_t \neq \alpha_{t'}$. Obviously $\alpha_t < \alpha_{t'}$ may be supposed. Hence we get

$$c_t q_t = c_{t'} q_{t'} p_t^{\alpha_{t'} - \alpha_t}.$$

However $p_t \nmid c_t q_t$ which is a contradiction. If $p_t = p_{t'}$ and $\alpha_t = \alpha_{t'}$, then from $(c_t - c_{t'})\, q_t p_t^{\alpha_t} = 0$ we get $c_t = c_{t'}$.

The remaining part of proof is the same as in the case of rational integers ([6]).

Proof of property 3. The number of elements in (4) is exactly $\sum_{t=1}^{r} \lambda_t [f(p_t) - 1]$ since $\operatorname{card}\{c_t : c_t \in R_{p_t},\ c_t \neq 0\} = f(p_t) - 1$ and any of them does not belong to the class $a_{i_0} + n_{i_0}R$.

By the similar considerations as in [6] the following assertion can be proved.

<u>T h e o r e m 2.</u> If R is a principal ideal domain and nR is its arbitrary ideal generated by n with

$$n = \prod_{t=1}^{r} p_t^{\lambda_t}$$

then there exists a disjoint covering system of the form (1) containing the class $0 + nR$ and consisting of $1 + \sum_{t=1}^{r} \lambda_t[f(p_t) - 1]$ classes.

Remark 1. Using the axiom of choice we can show that the property II.3 also holds for principal ideal domains in which $f(a)$ is any cardinal number.

Remark 2. If R is equal to Z (rational integers), then $f(p_t) = p_t$ and hence property II.3 is a generalization of Mycielski's conjecture from [4].

Remark 3. If R is a ring in which the unique factorization theorem is false then the property II.3 does not hold for arbitrary decomposition into irreducible elements. For example: Let $R = \{a + b\sqrt{-5}: a, b \in Z\}$. The ring R is not unique factorization domain and we have $6 = 2.3 = (1 + \sqrt{-5}).(1 - \sqrt{-5})$, where $2,3,1 + \sqrt{-5}, 1- \sqrt{-5}$ are irreducible in R (see [2], p. 211).

The system

$$k + (1 + \sqrt{-5})R \quad \text{for} \quad k = 0,1,\dots,5$$

disjointly covers the ring R. Similarly the system

$$k(1 + \sqrt{-5}) + 6R \quad \text{for} \quad k = 0,1,\dots,5$$

is disjoint and covers the residue class $0 + (1+ \sqrt{-5})R$. Hence the system

$$0 + 6R$$

$$\left.\begin{array}{l} k + (1 + \sqrt{-5})R \\ k(1 + \sqrt{-5}) + 6R \end{array}\right. \quad k = 1,2,\dots, 5$$

is disjoint covering on R. (The number of classes is 11.)

On the other hand we can easy show that $f(2) = 4$, $f(3) = 9$, (because $R_2 = \{0,1,\sqrt{-5},1+\sqrt{-5}\}$ and $R_3 = \{0,1,2\sqrt{-5},2\sqrt{-5},1 +$

$\sqrt{-5}$, $1+2\sqrt{-5}.2+\sqrt{-5},2+2\sqrt{-5}\}$) and from the factorization 6=2.3 we should get

$$\text{card } T \geq 1 + (4-1) + (9-1) = 12.$$

However, this question is open for unique factorization domains which are not principal ideal domains (or more precisely for rings in with the property I.4 is false).

Remark 4. Our estimation for covering of rings by residue classes is different from like one for covering of groups by their cosets. For example: consider the ring $G = \{a + bi: a, b \in Z\}$ of all gaussian integers. The set $H = 3G = \{3(a + bi): a, b \in Z\}$ form an ideal of ring G and a subgroup of additive group of G.

Now, it is easy to check that the cosets

$$\{1 + 3a + bi: a,b \in Z\} = 1 + \{3a + bi: a, b \in Z\}$$
$$\{2 + 3a + bi: a,b \in Z\} = 2 + \{3a + bi: a, b \in Z\}$$
$$\{a + (1+3b)i: a, b \in Z\} = i + \{a+ 3bi: a, b \in Z\}$$
$$\{a + (2+3b)i: a, b \in Z\} = 2i + \{a + 3bi: a, b \in Z\}$$

together with F form a disjoint covering of G as a group (hence 5 cosets).

From Theorem 1 it follows that the minimal number of residue classes of any disjoint covering of G (as a ring) containing H is 9. Namely, the element 3 is irreducible in G and $R_3 = \{0,1,2,i,2i,1 + i,1+2i,2+i,2+2i\}$, hence $f(3) = 9$, thus card $T \geq 1 + [f(3)-1] = 9$.

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AN EXPLICIT WAY OF FINDING ABSOLUTE MAXIMUM OF STRICTLY CONCAVE PARAMETRIC QUADRATIC FUNCTION WITH LINEAR PARAMETRIC DEPENDENT CONSTRAINTS

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Abstract

A method for solving a maximization problem of strictly concave quadratic parametric function with linear parametric dependent constraints is presented. The described algorithm partitions a given interval of parameter into a finite number of partial intervals in such a way that parametric dependent system of linear equations assigned to each partial interval expresses explicitly an absolute maximum of strictly concave quadratic parametric function. This maximum is identical with the optimal quadratic programming solution.

1. Introduction

One of the importatnt points in mathematical programming problems is to know how an optimal solution behaves when objective function and constraints depend on a parameter which varies in a given interval.

Gass [2]has developed an algorithm for solving problems with linear parametric function and linear constraints. The algorithm is based on simplex method. A given interval is partitioned into a number of partial intervals in such manner that the same optimal solution corresponds to each value from one partial interval.

Saaty [8] has found an algorithm for a general case. Coefficients of constraints and objective function are linear parametric functions.

Wolfe [9] has analyzed a special case of quadratic programming with the linear part of objective function multiplied by parameter.

Ritter [6] has found an algorithm for a case of quadratic programming when objective function coefficients as well as constraint coefficients are linear parametric functions but optimal solution may get negative values.

The presented algorithm for solving maximization problems of strictly concave quadratic parametric function with parametric dependent linear constraints enables to find a nonnegative solution identical with the quadratic programming solution.

2. Problem formulation

Let $c = (c_j)$, $A = (a_{ij})$, $b = (b_i)$, $e = (e_i)$

$d = (d_j)$, $B = (b_{ij})$ for $i = 1,\dots,m$, $j = 1,\dots,n$

and $C = (c_{ij})$, $D = (d_{ij})$ for $i,j = 1,\dots,n$

For each value of parameter from the interval $t_o \leqq t \leqq t_k$ there should be found an absolute maximum, i.e. a vector with the components $x_j \geqq 0$ satisfying the condition:

$$\max \left\{ Q(x,t) \mid (A + tB)\, x \leqq b + te, \quad x \geqq 0 \right\}$$
$$\text{where } Q(x,t) = (c + td)'x - \frac{1}{2} x'(C + tD)\, x \qquad (2.1)$$

Using a shortened form of notation we obtain:

$$\max \left\{ Q(x,t) \mid x \in K \right\}$$
$$\text{where } K = \left\{ x \mid x \geqq 0;\ f_i(x,t) \leqq b_i + te_i, \text{ for } i=1,\dots,m \right\} \qquad (2.2)$$

Let us assume the function $Q(x,t)$ to be strictly concave and the functions $f_i(x,t)$ convex.

At the same time there should be chosen nxn symmetric matrices C,D and the bounds of interval of the parameter t such that the matrix (C + tD) is symmetric and positively definite for all considered values of parameter.

Let x be a solution of problem (2.2) for $t = t_r$. Let the subset of indices corresponding to x be denoted as $M \subset \{1,2,\dots,m\}$ for which

$$f_i\ (x,t_r) = b_i + t_r e_i \qquad \text{for } i \in M$$
$$f_i\ (x,t_r) < b_i + t_r e_i \qquad \text{for } i \notin M \tag{2.3}$$

T h e o r e m 1. Let x be an optimal solution of the problem (2.2). Optimality of x is saved if the constraints $f_i\ (x,t)$ for which x fulfills strict inequalities, are excluded from (2.2).

P r o o f. Let us assume the vector x for $t = t_r$ to be an optimal solution of (2.2). Further let vector x^1 exists for $t = t_r$ such that:

$$x^1_j \geqq 0 \quad \text{for} \quad j = 1,2,\dots,n$$

$$f_i(x^1,t) \leqq b_i + te_i, \qquad \text{for } i \in M$$

$$Q\ (x^1,t) > Q\ (x,t)$$

Then for a sufficient small $\lambda > 0$

$$f_i\{[x + \lambda(x^1,x)],t\} \leqq b_i + te_i, \quad \text{for } i=1,2,\dots,m$$

$$x_j + (x^1_j,x_j) \geqq 0, \qquad \text{for } j=1,2,\dots,n$$

It follows from strict concavity of $Q\ (x,t)$ that

$$Q\ \{[\lambda x^1 + (1-\lambda)x],\ t\} > Q(x^1,t)+(1-\lambda)Q(x,t)$$

i.e.

$$Q\{[x + \lambda\,(x^1-x)]\ ,\ t\} > Q(x,t)$$

As a concave function has at most one absolute maximum the vector x is not optimal solution of (2.2). This is in contradiction with our assumption therefore the theorem 1 is valid.

It follows from the theorem 1 that x is an optimal solution of

$$\max\{Q(x,t) \mid x_j \geqq 0, \qquad \text{for} \quad j = 1,\dots,n$$
$$f_i(x,t) \leqq b_i + te_i \quad \text{for} \quad i \in M\} \tag{2.4}$$

The condition for the components of the vector x to be non-negative may be rewritten as a set of inequalities of constraints without generality being influenced.

The subset of indices will be denoted by $M' \subset \{1,2,\dots,m+n\}$.

The problem (2.4) may be written in the form:

$$\max \{ Q(x,t) \mid f_i(x,t) \leqq b_i + te_i, \quad \text{for} \quad i \in M' \} \tag{2.5}$$

The matrix form of (2.5) is

$$\max \{ (x+td)'x - \frac{1}{2} x'(C+tD)x \mid (A_1+tB_1)\, x \leqq b_1+te_1 \} \tag{2.6}$$

where $A_1 = (a_{ij})$, $B_1 = (b_{ij})$ for $i \in M'$, $j = 1,2,\dots,n$

$b_1 = (b_i)$, $e_1 = (e_i)$ for $i \in M'$.

Consequently the problem (2.1) will be solved as:

$$(c + td)'x - \frac{1}{2} x'(C + tD)\, x \Rightarrow \max$$

with the constraints

$$(A_1 + tB_1)\, x \leqq B_1 + te_1 \tag{2.7}$$

and conditions of feasibility

$$(A_2 + tB_2)\, x \leqq b_2 + te_2 \tag{2.8}$$

With respect to concavity of the function $Q(x,t)$ the Lagrangian function for (2.6) has the form

$$\Phi(x,u) = - Q(x,t) + u'[(A_1 + tB_1)\, x - (b_1 + te_1)] =$$

$$= - (c+td)'x + \frac{1}{2} x'(C+tD)x + u'[(A_1+tB_1)\, x - (b_1+te_1)]$$

Vector x is an optimal solution of (2.6) if according to the Kuhn-Tucker theorem [3], [5] is valid

$$\frac{\partial \Phi(x,u)}{\partial x} = 0, \quad \frac{\partial \Phi(x,u)}{\partial u} = 0, \quad u \geqq 0, \quad \text{i.e.}$$

$$\begin{aligned} (A_1 + tB_1)'u + (C + tD)\, x &= c + td \\ (A_1 + tB_1)\, x &= b_1 + te_1 \\ u &\geqq 0 \end{aligned} \tag{2.9}$$

The conditions (2.9) are necessary and sufficient for the vector x to be an optimal solution of (2.6) for $t = t_r$.

The solution is being found in two steps.

Step 1 of the algorithm finds an initial solution and set it to be a parametric function.

The second step of algorithm checks the following two assumptions:

a) x is optimal at most for the parametric values satisfying the (2.9)

b) x must satisfy the inequalities (2.8) which are the conditions of feasibility

3. Step 1

It may happen that the maximization problem (2.1) for $t = t_o$ has no optimal solution. Then let us find a minimal parametric value $t > t_o$ for which the problem has an optimal solution. Let as assume the given maximization problem has a solution for $t = t_o$ and the optimal solution x_o is found by the Wolfe method (9).

The constraints (2.9) corresponding to the optimal soluiton x_o for the parametric value $t = t_o$ are necessary and sufficient.

The components of the vectors u and x_o can be determined from the system of linear equations (2.9) as functions of parameter t if the matrix

$$M(t) = \begin{pmatrix} (A_1 + tB_1), & C + tD \\ 0, & A_1 + tB_1 \end{pmatrix} \qquad (3.1)$$

is not singular, 0 being $m_o x m_o$ zero matrix.

Singularity of $M_{(t)}$ depends on the sumbatrix $A_1 + tB_1$ only.

<u>Theorem 2.</u> The matrix M(t) defined by (3.1) is not singular if and only if the submatrix $(A_1 + tB_1)$ contains no linearly dependent row. (Proof see Ritter [6]).

Assume the rows of the matrix $(A_1 + tB_1)$ being linearly independent. The solution will be obtained from the system of equations (2.9) as:

$$\begin{pmatrix} u \\ \\ x_o \end{pmatrix} \quad M^{-1}(t) \quad \begin{pmatrix} c + td \\ \\ b_1 + te_1 \end{pmatrix}$$

If (2.9) is solved by means of determinants, the components of the vectors are:

$$u_i = \frac{|M_i(t)|}{|M(t)|} \qquad \text{for } i = 1,2,\ldots,m$$

$$x_j = \frac{|M_j(t)|}{|M(t)|} \qquad \text{for } j = 1,2,\ldots,n \qquad (3.2)$$

The determinants $|M_i(t)|$, $|M_j(t)|$ and $|M(t)|$ are polynomials of m+n order maximally. Consequently if a determinant of the system $|M(t)|$, i.e. a polynom, has a zero point in the considered partial interval, the shown way of calculation cannot be used. For the given value of parameter maximization problem is solved by the Wolfe method.

Another case is that of matrix $(A_1 + tB_1)$ containing linearly dependent rows, which leads to a

$$(A_1 + tB_1)\, x = b_1 + te_1 \qquad (3.2)$$

containing linearly dependent equations.

The system (2.9) can be always reduced to a partial system containing linearly independent equations only. However, it is not sufficient to find out maximal number of linearly independent rows in the matrix $(A_1 + tB_1)$ and take the corresponding equations only. The equation determining the position of optimum x_o may happen to be among the equations left out of (3.2).

To avoid difficulties encountered when choosing the right system of linearly independent equations the maximization problem is solved for $t_1 = t_o + \varepsilon$, where $\varepsilon > 0$ may be arbitrary small.

The following three cases are then considered:

1. An optimal solution x_1 is obtained for t_1. The equations (2.9) corresponding to the solution are linearly independent.

2. The equations (2.9) corresponding to the optimal solution x_1 for t_1 are again linearly dependent.

3. There exists no optimal soluiton for t_1.

In the first case the matrix $M(t)$ for $t = t_1$ is not singular. The system (2.9) can be solved explicitly according to x and u.

If the second case occurs, the maximal number of linearly independent rows of the matrix $(A_1 + t_1 B_1)$ can be found by means of linear maximization problem so that (2.9) holds.

In the third case the method described in the case one is used to find the minimal value of the parameter $t_1 > t_o$, for which the maximization problem has on optimal solution.

4. Step 2

After completing the first step of solution a parametric dependent solution $u_i(t)$, $x_j(t)$ is obtained.

During the second step there is determined the upper bound of a partial interval of parameter t, in which the solution u(t) and x(t) is optimal.

Substituting $u_i(t)$ into the (2.9) for vector $u_i \geqq 0$ and $x_j(t)$ into the conditions of feasibility (2.8) for vector x we get the following system of inequalities for the parameter t:

$$t \leqq p_i \tag{4.1}$$

$$t \geqq p_i \tag{4.2}$$

The upper bound of the partial interval of the parameter t for (4.1) is given by:

$$\min p_i = t_r$$

The solution $u(t)$ and $x(t)$ is optimal for each value of parameter t from the interval (t_{t-1}, t_r).

To determine the optimal solution for $t > t_r$ we use the following procedure:

1. If index i of the value p_i determining the upper bound is $i \in M'$, the system of equations (2.9) can be reduced by excluding the corresponding variable u_i from the system (2.9). Then we start with the first step of solution with the new system.

2. If index i of the value p_i determining the upper bound is $i \notin M' \subset \{1,\dots,n+m\}$ i.e. the matrix $(A_1 + tB_1)$ can be extended by the corresponding row vector from (2.8) and the solution of the new system is started by the first step.

Existence of a unique solution and corvergence follows from strict concavity of the function $Q(x,t)$ and from the theorem about solution of linear equations system when the determinant of system $D \neq 0$.

5. Example

Let us solve the following maximization problem. Find the optimal solution of a quadratic parametric programming problem for each t from the interval $0 \leq t \leq 23$ maximizing the strictly concave objective function given in the form:

$$Q(x,t) = (20-11t)x_1 + (17+6t)x_2 + (23-15t)x_3 + (19-20)x_4 - \\ -1/2(x_1^2+3x_2^2+9x_3^2+19x_4^2-2x_1x_2+4x_1x_3+2x_1x_4-6x_2x_4-12x_3x_4) \quad (5.1)$$

with the following constraints:

$$\begin{aligned} &a) \quad 3x_1 - 5x_2 + 8x_3 + x_4 \leq 30 - 3t \\ &b) \quad -2x_1 + 4x_3 - 10x_4 \leq -13 - 8t \\ &c) \quad x_1 + 4x_2 - x_3 - x_4 \leq 17 + 10t \\ &d) \quad 3x_1 - 9x_2 - 6x_4 \leq 25 + 5t \end{aligned} \quad (5.2)$$

e) $x_1 + 3x_2 + 7x_3 - 8x_4 \leqq 41 - 7t$

f) $-x_1 \leqq 0$

g) $-x_2 \leqq 0$

h) $-x_3 \leqq 0$

i) $-x_4 \leqq 0$

Using the Wolfe method on the computer IBM 7040 we obtain the optimal solution for $t = 0$:

$$
\begin{array}{ll}
x_1 & 8,763597 \\
x_2 & 3,098964 \\
x_3 & 2,149225 \\
x_4 & 2,010227
\end{array}
\qquad (5.3)
$$

It can be seen after substituting the optimal solution in (5.2) that equality holds for the (5.2/a) and (5.2/c). Consequently the optimal solution may be expressed as a function of t if the following system of equations is used:

$$
\begin{array}{rcl}
u_1 + 3u_2 + x_1 - x_2 + 2x_3 + x_4 & = & 20 - 11t \\
4u_1 - 5u_2 - x_1 + 3x_2 - 3x_4 & = & 17 + 6t \\
-u_1 + 8u_2 + 2x_1 + 9x_3 - 6x_4 & = & 23 - 15t \\
-u_1 + u_2 + x_1 - 3x_2 - 6x_3 + 19x_4 & = & 19 - 20t \\
x_1 + 4x_2 - x_3 - x_4 & = & 17 + 10t \\
3x_1 - 5x_2 + 8x_3 + x_4 & = & 30 - 3t
\end{array}
\qquad (5.4)
$$

The solution of the system (5.4) is the following:

$$
\begin{array}{ll}
x_1 = 8,763 + 5,144\ t & x_2 = 3,099 + 0,276\ t \\
x_3 = 2,149 - 1,900\ t & x_4 = 2,010 - 1,852\ t \\
u_1 = 6,331 - 2,169\ t & u_2 = 0,565 - 2,684\ t
\end{array}
\qquad (5.5)
$$

Demand $u_i \geqq 0$ gives $t \leqq 2,924$, $t \geqq 0,210$.

It follows from the check of feasibility

b) $t \geqq 1,84$
d) $t \leqq 2,04$
e) $t \leqq 1,65$
f) $t \geqq -1,70$
g) $t \geqq -11,22$
h) $t \leqq 1,13$
i) $t \leqq 1,08$

Then, according to the above given inequalities, the (5.5) is an optimal solution for $0 \leqq t \leqq 0,210$.

Proceeding further in solution we get the optimal solution

$x_1 = 10,076 - 1,088\ t$
$x_2 = 2,799 + 1,700\ t$
$x_3 = 2,308 - 2,654\ t$
$x_4 = 1,964 - 1,632\ t$
$u_1 = 6,143 - 1,271\ t$

for $0,210 \leqq t \leqq 0,86$.

Similarly

$x_1 = 13,0511 - 4,5089\ t$
$x_2 = 1,2089 - 3,5289\ t$
$x_3 = 0$
$x_4 = 0,8867 - 0,3933\ t$
$u_1 = 7,2711 - 2,5609\ t$
$u_2 = 9,4891 + 10,9029\ t$

is the optimal solution for $0,86 \leqq t \leqq 1,09$

$x_1 = 10,4372 - 1,9626\ t$
$x_2 = 1,4438 + 3,2914\ t$
$x_3 = 0$
$x_4 = -0,7874 + 1,2032\ t$
$u_1 = 5,1858 - 0,6204\ t$
$u_2 = -15,8030 + 17,1340\ t$
$u_3 = -3,3040 + 3,1645\ t$

is the optimal solution for $1,09 \leqq t \leqq 3,77$ and finally the optimal solution for $3,77 \leqq t \leqq 4,33$ looks like:

$x_1 = 22,16177 - 5,11763\ t$
$x_2 = -2,07353 + 4,23529\ t$
$x_3 = 0$
$x_4 = -3,13235 + 1,82353\ t$
$u_1 = 18,66910 - 3,14704\ t$
$u_2 = -55,69845 + 18,32349\ t$
$u_3 = 3,43751 + 1,36766\ t$
$u_4 = 12,89703 + 3,41175\ t$

The solution:

$x_1 = 0$ $\qquad$ $u_2 = -87{,}7130 + 35{,}92\ t$

$x_2 = 17{,}133333 - 0{,}20\ t$ $\qquad$ $u_3 = 3{,}5633 + 1{,}18\ t$

$x_3 = 0$ $\qquad$ $u_4 = -10{,}1666 + 3{,}00\ t$

$x_4 = 1{,}3000 + 0{,}8\ t$ $\qquad$ $u_5 = -53{,}1264 + 12{,}64\ t$

is optimal for $5 \leq t \leq 23{,}826$, what is the solution of our problem.

The table bellow compares the results for given values of parameter t obtained by means of the Wolfe method on the IBM System 7040 with those obtained by the method described. Both solutions are identical.

Value of parameter t	Variable	Solution obtained by:	
		substituting into analytical solution	Wolfe method on the digital computer
4	x_1 x_2 x_3 x_4	1,69125 14,86763 0 4,16177	1,691176 14,867647 0 4,161765
8	x_1 x_2 x_3 x_4	0 15,5333 0 7,7000	0 15,53333 0 7,70000
12	x_1 x_2 x_3 x_4	0 14,7333 0 10,9000	0 14,73333 0 10,90000
20	x_1 x_2 x_3 x_4	0 13,1333 0 17,3000	0 13,13333 0 17,30000
23	x_1 x_2 x_3 x_4	0 12,5333 0 19,7000	0 12,53333 0 19,700000

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ON SUMS OF PRIME POWERS IN ARITHMETIC PROGRESSION

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In [2] T. Šalát and S. Znám obtain an asymptotic formula for the sum $\sum_{p\le x} p^a$ for $a>0$. In this paper we extend this result to primes in arithmetic progression. Also, our result holds for negative powers as well as positive ones.

Let

$$\lambda(n, k, a) = \begin{cases} \log p & \text{if } n = p^{\alpha} \text{ and } n \equiv a \pmod k \\ 0 & \text{if } n \text{ is not a prime power} \end{cases}$$

$$\psi(x, k, a) = \sum_{n=1}^{x} \lambda(n, k, a)$$

and let φ be the Euler φ-function. One version of the prime number theorem tells us

<u>Lemma:</u> If $(a, k) = 1$ then there is a constant $c_1 > 0$, such that

$$\psi(x,k,a) = \frac{x}{\varphi(k)} + O(x \exp(-c_1 \log^{1/2} x)). \tag{1}$$

See for example [1, page 136].

We now consider the sum

$$S = S_r(x,k,a) = \sum_{\substack{p\le r \\ p\equiv a \pmod k}} p^r \tag{2}$$

If $(a,k) \neq 1$ or if $r < -1$, S converges to a constant. If $r = -1$, $S \sim \frac{1}{\varphi(k)} \log\log x$ for $(a,k) = 1$. We are now ready to prove:

Theorem: If $(a,k) = 1$ and $r > -1$, then there is a constant $c_2 > 0$ such that

$$(3) \qquad \sum_{\substack{p \leq x \\ p \equiv a \pmod k}} p^r = \frac{1}{\varphi(k)} \int_2^x \frac{t^r}{\log t}\,dt + O(x^{r+1} \exp(-c_2 \log^{1/2} x))$$

Proof: If $r = 0$ we have the prime number theorem for primes in arithmetic progressions. Hence, assume $r \neq 0$. Let

$$f(t) = -\frac{d}{dt}(t^r/\log t) = t^{r-1}(1-r\log t)/\log^2 t$$

Then

$$S = \sum_{\substack{p \leq x \\ p \equiv a \pmod k}} p^r = \sum_{2 \leq n \leq x} \frac{n^r \lambda(n,k,a)}{\log n} - \sum_{\substack{n=p^\alpha \\ \alpha \geq 2 \\ n \leq x}} \frac{n^r \lambda(n,k,a)}{\log n}$$

$$= \sum_{2 \leq n \leq x} \lambda(n,k,a) \int_n^x f(t)\,dt + \frac{x^r}{\log x} \sum_{2 \leq n \leq x} \lambda(n,k,a)$$

$$+ O(x^{r+1/2}) + O(\log x),$$

since

$$\sum_{\substack{n=p^\alpha \\ \alpha \geq 2 \\ n \leq x}} \frac{n^r \lambda(n,k,a)}{\log n} = \sum_{\alpha=2}^{\left[\frac{\log x}{\log 2}\right]} \frac{1}{\alpha} \sum_{\substack{n=p^\alpha \\ n \leq x}} n^r$$

$$= \sum_{\alpha=2}^{\left[\frac{\log x}{\log 2}\right]} \frac{1}{\alpha} O(1 + \int_1^{x^{1/\alpha}} t^{\alpha r}\,dt) = \sum_{\alpha=2}^{\left|\frac{\log x}{\log 2}\right|} \frac{1}{\alpha} O(1 +$$

$$+ x^{r+1/\alpha} + \log x^{1/\alpha}) = O\left(\sum_{\alpha=2}^{\left|\frac{\log x}{\log 2}\right|} \frac{1}{\alpha} + x^{r+1/\alpha} + \frac{1}{\alpha^2}\log x\right) =$$

$$= O\,(x^{r+1/2} + \log x).$$

Therefore,

$$(4) \qquad S = \int_2^x \psi(t,k,a)f(t)dt + \frac{x^r}{\log x}\psi(x,k,a) + O(x^{r+1/2}) + O(\log x)$$

Letting $E(t,k,a) = \psi(t,k,a) - \frac{t}{\varphi(k)}$, we have

$$(5) \qquad \int_2^x \psi(t,k,a)f(t)dt = \int_2^x \frac{t}{\varphi(k)} f(t)dt + \int_2^x E(t,k,a)f(t)dt$$

$$= \frac{1}{\varphi(k)}\left\{\int_2^x \frac{t^r}{\log t}dt - \frac{x^{r+1}}{\log x} + \frac{2^{r+1}}{\log 2}\right\} + \int_2^x E(t,k,a)f(t)dt$$

Also

$$(6) \qquad \frac{x^r}{\log x}\psi(x,k,a) = \frac{x^{r+1}}{\varphi(k)\log x} + \frac{x^r}{\log x}E(x,k,a)$$

By (4), (5) and (6)

$$(7) \qquad S = \frac{1}{\varphi(k)}\int_2^x \frac{t^r}{\log t}dt + \frac{x^r}{\log x}E(x,k,a) + \int_2^x E(t,k,a)f(t)dt$$

$$+ O(x^{r+1/2}) + O(\log x)$$

By (1)

$$(8) \qquad \frac{x^r}{\log x}E(x,k,a) = O(x^{r+1}\exp(-c_1 \log^{1/2} x))$$

Also, by (1) and the fact that k is a constant

$$(9) \qquad \int_2^x E(t,k,a)f(t)dt = O\left(\int_2^x |E(t,k,a)|\,|f(t)|\,dt\right)$$

$$= O\left(\int_2^{e^k} + \int_{e^k}^{x^{1/2}} + \int_{x^{1/2}}^{x} (|E(t,k,a)|\,|f(t)|\right) dt$$

$$= O(1) + O\left(\int_{e^k}^{x^{1/2}} t^r\,dt\right) + O\left(\int_{x^{1/2}}^{x} t^r \exp(-c_1 \log^{1/2} t)dt\right)$$

$$= O(1) + O\left(x^{\frac{1}{2}(r+1)}\right) + O(\exp(-c_2 \log^{1/2} x) \int_{x^{1/2}}^{x} t^r\,dt$$

$$= O(x^{r+1} \exp(-c_2 \log^{1/2} x)),$$

where $0 < c_2 < c_1$.

Combining (7), (8), and (9) we obtain

$$S = \frac{1}{\varphi(k)} \int_2^x \frac{t^r}{\log t}\,dt + O\,(x^{r+1} \exp(-c_2 \log^{1/2} x))$$

which completes the proof.

Note that $\int_2^x \frac{t^r}{\log t}\,dt \sim \frac{x^{r+1}}{(r+1) \log x}$

Also, a better error term is obtainable in (3) simply by using a version of the prime number theorem with the corresponding error term.

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ÜBER EINIGE OSZILLATORISCHE EIGENSCHAFTEN DER LÖSUNGEN DER QUASILINEAREN DIFFERENTIALGLEICHUNG VIERTER ORDNUNG

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In der Arbeit erwägen wir die quasilineare Differentialgleichung 4. Ordnung $(ry''')' + qy = 0$, deren Lösungen die Eigenschaft (E) haben.

(E) Jede nichttriviale Lösung hat höchstens eine zweifache, bzw. dreifache Nullstelle.

Unter dieser Voraussetzung werden einige weitere Eigenschaften der Lösungen dieser Gleichung sowie auch der Satz über die Trennung der Nullstellen der Lösungen, bzw. der Ableitungen der Lösungen, bewiesen.

In der Arbeit werden die Methoden von M. Švec aus [1] angewendet und die erhaltenen Ergebnisse stellen die Verallgemeinerungen seiner Ergebnisse aus dieser Arbeit vor.

I.

Im ersten Teil wird die Existenz der quasilinearen Differentialgleichung 4. Ordnung, deren Lösungen die Eigenschaft (E) erfüllen, gezeigt. Hier, wie auch im Weiteren $-\infty \leq a < b \leq +\infty$.

S a t z 1.1. Am Intervall (a,b) seien $r(x) > 0, r'(x)$, $p_1(x)$, $p_1'(x)$, $p_2(x)$, $q(x) > 0$, $q'(x)$, $q''(x)$ stetige Funktionen und es gelte

$$q''r + \left(r' - p_1 - \frac{2q'r}{q}\right) q' \geq 0, \qquad \left(\frac{p_2}{q}\right)' + 2 > 0$$

Für jede nicht triviale Lösung y der Differentialgleichung

(a) $(ry''')' + p_1 y''' + \frac{1}{2} p_1' y'' + p_2 y' + qy = 0$

existiert dann höchstens ein Punkt $\xi \in (a,b)$, in welchem die Bedingung

(E_5) $y'(\xi) = y''(\xi) = 0$

erfüllt ist.

B e w e i s. Für die Lösung y gilt die folgende Identität:

(1.1)
$$\left[\frac{1}{q} y''(ry''') + \frac{1}{2}\frac{p_1}{q} y''^2 + \frac{1}{2}\frac{p_2}{q} y'^2 + y y' + \frac{q'r}{2q^2} y''^2\right]' =$$
$$= \frac{r}{q} y'''^2 + \frac{1}{2q^2}\left[r y'' + (r' - p_1 - \frac{2rq'}{q}) q'\right] y'' +$$
$$+ \frac{1}{2}\left[\left(\frac{p_2}{q}\right)' + 2\right] y'^2$$

Es sollen zwei solche Punkte $x_1, x_2 \in (a,b)$ esistieren, in welchen (E_5) gilt. Durch die Integration der Gleichheit (1.1) von x_1 bis x_2 erhalten wir einen Widerspruch, da die linke Seite der Gleichheit gleich Null und die rechte Seite verschieden von Null ist.

F o l g e r u n g 1.1. Es seien die Voraussetzungen des Satzes 1.1 erfüllt. Für jede Nichttriviale Lösung der Differentialgleichung

$(\bar{a})$ $(ry''')' + p_1 y''' + \frac{p_1'}{2} y'' + qy = 0$

existiert dann höchstens ein Punkt $\xi \in (a,b)$ in welchem die Bedingung (E_5), oder die Bedingung

(E_1) $y(\xi) = y''(\xi) = 0$

erfüllt ist.

Der Beweis folgt aus der Gleichheit (1.1) auf analoge Weise wie im vorhergehendem Satz.

S a t z 1.2. Es seien $r(x) > 0$, $p_1(x)$, $p_1'(x)$, $q(x) > 0$, $q'(x)$, $q''(x) \leqq 0$ stetige Funktionen für $x \in (a,b)$. Dann existiert für jede nichttriviale Lösung y der Differentialgleichung $(\bar{a})$ höchstens ein Punkt $\xi \in (a,b)$ in welchem die Bedingung (E_1) erfüllt ist.

B e w e i s. Für y gilt die folgende Identität:

$$(1.2) \qquad \left[y''(ry''') + \frac{1}{2} p_1 y''^2 + qyy' - \frac{1}{2} q' y^2 \right]' = ry'''^2 + qy'^2 - \frac{1}{2} q'' y^2 ;$$

daraus folgt ein Widerspruch auf analoge Weise, wie im Beweis des Satzes 1.1.

Im Weiteren werden wir uns mit der quasilinearen Differentialgleichung

$$(a_1) \qquad (r y''')' + qy = 0$$

beschäftigen, wo $r(x) > 0$, $q(x) > 0$, $q'(x)$, $q''(x) \leqslant 0$ stetige Funktionen in (a,b) sind.

S a t z 1.3. Für jede nichttriviale Lösung y der Differentialgleichung (a_1) existiert höchstens ein Punkt $\xi \in (a,b)$, in welchem die Bedingung (E_1) oder die Bedingung

$$(E_2) \qquad y(\xi) = y'''(\xi) = 0$$

erfüllt ist.

B e w e i s. Das Verfahren ist das gleiche wie bei dem Beweis des Satzes 1.1, ausgehend von der Identität (1.2).

B e m e r k u n g 1.1. Unter dem Symbol

$$\begin{vmatrix} y^{(\alpha_1)}(x) \\ y^{(\alpha_2)}(x) \\ y^{(\beta_1)}(x_1) \\ y^{(\beta_2)}(x_1) \end{vmatrix} \quad \text{bzw.} \quad \begin{vmatrix} y^{(\alpha_1)}(x) \\ (ry^{(\alpha_2)})(x) \\ y^{(\beta_1)}(x_1) \\ y^{(\beta_2)}(x_1) \end{vmatrix}$$

wo α_j, $\beta_j \in \{0,1,2,3\}$ und $j = 1,2$, werden wir die Determinante

$$\begin{vmatrix} y_1^{(\alpha_1)}(x) & y_2^{(\alpha_1)}(x) & y_3^{(\alpha_1)}(x) & y_4^{(\alpha_1)}(x) \\ y_1^{(\alpha_2)}(x) & y_2^{(\alpha_2)}(x) & y_3^{(\alpha_2)}(x) & y_4^{(\alpha_2)}(x) \\ y_1^{(\beta_1)}(x_1) & y_2^{(\beta_1)}(x_1) & y_3^{(\beta_1)}(x_1) & y_4^{(\beta_1)}(x_1) \\ y_1^{(\beta_2)}(x_1) & y_2^{(\beta_2)}(x_1) & y_3^{(\beta_2)}(x_1) & y_4^{(\beta_2)}(x_1) \end{vmatrix},$$

bzw. die Determinante

$$\begin{vmatrix} y_1^{(\alpha_1)}(x) & y_2^{(\alpha_1)}(x) & y_3^{(\alpha_1)}(x) & y_4^{(\alpha_1)}(x) \\ (ry_1^{(\alpha_1)})(x) & (ry_2^{(\alpha_2)})(x) & (ry_3^{(\alpha_2)})(x) & (ry_4^{(\alpha_2)})(x) \\ y_1^{(\beta_1)}(x_1) & y_2^{(\beta_1)}(x_1) & y_3^{(\beta_1)}(x_1) & y_4^{(\beta_1)}(x_1) \\ y_1^{(\beta_2)}(x_1) & y_2^{(\beta_2)}(x_1) & y_3^{(\beta_2)}(x_1) & y_4^{(\beta_2)}(x_1) \end{vmatrix}$$

verstehen, wo y_1, y_2, y_3, y_4 ein Fundamentalsystem von Lösungen der Gleichung (a_1) bedeuten und x, x_1 sind aus ihrem Definitionsinterval.

Denn führen wir folgenden Bezeichnungen ein:

$$(1,3)\qquad P^1_{x_1}(x) = \begin{vmatrix} y(x) \\ y'(x) \\ y(x_1) \\ y''(x_1) \end{vmatrix}, \qquad P^2_{x_1}(x) = \begin{vmatrix} y(x) \\ y'(x) \\ y(x_1) \\ y'''(x_1) \end{vmatrix},$$

$$P^3_{x_1}(x) = \begin{vmatrix} y''(x) \\ (ry''')(x) \\ y(x_1) \\ y''(x_1) \end{vmatrix}, \qquad P^4_{x_1}(x) = \begin{vmatrix} y''(x) \\ (ry''')(x) \\ y(x_1) \\ y'''(x_1) \end{vmatrix};$$

$$(1,4)\qquad P^5_{x_1}(x) = \begin{vmatrix} y'(x) \\ y''(x) \\ y(x_1) \\ y''(x_1 \end{vmatrix}, \qquad P^6_{x_1}(x) = \begin{vmatrix} y'(x) \\ y''(x) \\ y(x_1) \\ y'''(x_1) \end{vmatrix},$$

$$P^7_{x_1}(x) = \begin{vmatrix} y'(x) \\ y''(x) \\ y(x_1) \\ y'(x_1) \end{vmatrix}, \qquad P^8_{x_1}(x) = \begin{vmatrix} y'(x) \\ y''(x) \\ y''(x_1) \\ y'''(x_1) \end{vmatrix};$$

$$(1,5) \qquad P^9_{x_1}(x) = \begin{vmatrix} y'(x) \\ (ry''')(x) \\ y(x_1) \\ y''(x_1) \end{vmatrix}, \qquad P^{10}_{x_1}(x) = \begin{vmatrix} y'(x) \\ (ry''')(x) \\ y(x_1) \\ y'''(x_1) \end{vmatrix},$$

$$P^{11}_{x_1}(x) = \begin{vmatrix} y'(x) \\ (ry''')(x) \\ y(x_1) \\ y'(x_1) \end{vmatrix}, \qquad P^{12}_{x_1}(x) = \begin{vmatrix} y'(x) \\ (ry''')(x) \\ y''(x_1) \\ y'''(x_1) \end{vmatrix}.$$

B e m e r k u n g 1.2. Im Weiteren werden

$$(1.6) \qquad y_1,\ y_2,\ y_3,\ y_4$$

ein solches Fundamentalsystem von Lösungen der Gleichung (a_1) bedeuten, dass ihr Wronskian positiv ist.

H i l f s s a t z 1.1. Es sei x_1 ein beliebiger Punkt aus dem Intervall (a,b). Dann haben die Funktionen $P^1_{x_1}(x)$, $P^2_{x_1}(x)$, $P^3_{x_1}(x)$, $P^4_{x_1}(x)$ gerade eine Nullstelle auf (a,b), und zwar x_1.

B e w e i s. Aus der Bemerkung 1.1 folgt, dass $P^i_{x_1}(x_1) = 0$ $(i = 1,2,3,4)$. Setzen wir voraus, dass für irgendeine Zahl i $(i = 1,2,3,4)$ ein solches $\eta_i \in (a,b)$, $\eta_i \neq x_1$ existiert, dass $P^i_{x_1}(\eta_i) = 0$. Dann existiert zwischen x_1 und η_i ein solches ξ_i, dass $(P^i_{x_1})'(\xi_i) = 0$, wobei

$$(P^1_{x_1})'(x) = \begin{vmatrix} y(x) \\ y''(x) \\ y(x_1) \\ y''(x_1) \end{vmatrix}, \qquad (P^2_{x_1})'(x) = \begin{vmatrix} y(x) \\ y''(x) \\ y(x_1) \\ y'''(x_1) \end{vmatrix}$$

$$(P^3_{x_1})'(x) = \begin{vmatrix} (qy)(x) \\ y''(x) \\ y(x_1) \\ y''(x_1 \end{vmatrix}, \qquad (P^4_{x_1})'(x) = \begin{vmatrix} (qy)(x) \\ y''(x) \\ y(x_1) \\ y'''(x_1) \end{vmatrix}$$

Wenn i eine der Zahlen 1,3 ist, folgt aus der Gleichheit $(P^i_{x_1})'(\xi_i) = 0$, dass eine solche nichttriviale Lösung existiert, welche in x_1 und in ξ_i die Eigenschaft (E_1) hat. Ähnlich, wenn i gleich mit einer der Zahlen 2,4 ist, folgt aus der Gleichheit $(P^i_{x_1})'(\xi_i) = 0$, dass eine solche nichttriviale Lösung y existiert, welche in x_1 die Eigenschaft (E_2) und in ξ_i die Eigenschaft (E_1) hat. Beide Ergebnisse sind mit dem Satz 1.3 im Widerspruch.

H i l f s s a t z 1.2. Es sei z ein beliebiger Punkt aus dem Intervall (a,b). Dann haben die Funktionen

(1,7) $$P_z(x) = \begin{vmatrix} y(x) \\ y'(x) \\ y(z) \\ y'(z) \end{vmatrix}, \qquad \bar{P}_z(x) = \begin{vmatrix} y''(x) \\ (ry''')(x) \\ y''(z) \\ (ry''')(z) \end{vmatrix}$$

in (a,b) gerade eine Nullstelle und zwar z.

B e w e i s. Setzen wir voraus, dass $P_z(x)$, $\bar{P}_z(x)$ zwei Nullstellen $z, \eta \in (a,b)$ haben, und zur Gewissheit $z < \eta$. Dann existiert ein solcher Punkt $x_1 \in (z,\eta)$, dass

$$P_z'(x_1) = \begin{vmatrix} y(x_1) \\ y''(x_1) \\ y(z) \\ y'(z) \end{vmatrix} = P^1_{x_1}(z) = 0, \quad \bar{P}_z'(x_1) = \begin{vmatrix} y''(x_1) \\ -(qy)(x_1) \\ y''(z) \\ (ry''')(z) \end{vmatrix} = q(x_1) P^3_{x_1}(z) = 0$$

was mit dem Hilfssatz 1.1 im Widerspruch steht.

S a t z 1.4. Für jede nichttriviale Lösung y der Differentialgleichung (a_1) existiert höchstens ein Punkt $\xi \in (a,b)$, in welchem y irgendeine der Bedingungen (E_1), (E_2),

(E_3) $\quad y(\xi) = y'(\xi) = 0,$

(E_4) $\quad y''(\xi) = y'''(\xi) = 0$

erfüllt.

B e w e i s. a) Setzen wir voraus, dass eine solche nichttriviale Lösung y existiert, welche in x_1 die Bedingung (E_1) bzw. (E_2) und in $x_2 \neq x_1$ die Bedingung (E_3), bzw. (E_4) erfüllt. Dass heisst, dass $P^1_{x_1}(x_2) = 0$, bzw. $P^3_{x_1}(x_2) = 0$, bzw. $P^2_{x_1}(x_2) = 0$, bzw. $P^4_{x_1}(x_2) = 0$, was mit dem Hilfssatz 1.1 im Widerspruch ist.

b) Weiter müssen wir den Fall ausschliessen, dass die Lösung in zwei verschiedenen Punkten $z, \eta \in (a,b)$, $z < \eta$ (der Fall $\eta < z$ schliesst sich analogisch aus) die Eigenschaft (E_3), bzw. (E_4) hat. Aus der Voraussetzung, dass die nichttriviale Lösung mit der Eigenschaft (E_3), bzw. (E_4) existiert, folgt, dass $P_z(\eta) = 0$, bzw. $\bar{P}_z(\eta) = 0$, was mit dem Hilfssatz 1.2 im Widerspruch ist.

c) Setzen wir voraus, dass eine nichttriviale Lösung y existiert, welche in x_3 die Bedingung (E_3) und in x_2 die Bedingung (E_4) erfüllt, wo $x_2, x_3 \in (a,b)$, $x_2 \neq x_3$. Es gilt also

$$R_{x_2}(x_3) = \begin{vmatrix} y(x_3) \\ y'(x_3) \\ y''(x_2) \\ y'''(x_2) \end{vmatrix} = 0$$

Im Hinblick zur Bemerkung 1.2 ist $R_{x_2}(x_2) > 0$; weiter gilt $R'_{x_2}(x_2) = 0$, $R''_{x_2}(x_2) = 0$. $R''_{x_2}(x)$ ist gleich der Summe der Funktionen

$$R''_{x_2}(x) = \begin{vmatrix} y'(x) \\ y''(x) \\ y''(x_2) \\ y'''(x_2) \end{vmatrix} + \begin{vmatrix} y(x) \\ y'''(x) \\ y''(x_2) \\ y'''(x_2) \end{vmatrix} = S_1(x) + \frac{1}{r(x)} S_2(x)$$

wo über die Funktionen $S_1(x)$ und $S_2(x)$ leicht gezeigt werden kann, dass beide in irgendeiner Umgebung O_{x_2} wachsend sind und es gilt

$S_1(x) < 0$, $S_2(x) < 0$ für $x < x_2$, $S_1(x) > 0$, $S_2(x) > 0$ für $x > x_2$. Daraus folgt, dass $R(x)$ für $x < x_2$ eine positive sinkende und für $x > x_2$ eine positive wachsende Funktion in der Umgebung von x_2 ist. Daraus, sowie auch aus $R_{x_2}(x_3) = 0$ folgt, dass ein solches $x_1 \in (x_2, x_3)$ existiert, dass

$$R'_{x_2}(x_1) = \begin{vmatrix} y(x_1) \\ y''(x_1) \\ y''(x_2) \\ y'''(x_2) \end{vmatrix} = \frac{1}{r(x_2)} P^3_{x_1}(x_2) = 0$$

gilt, was mit dem Hilfssatz 1.1 im Widerspruch ist.

<u>Bemerkung 1.3.</u> Aus der Bemerkung 1.2 und aus der weiteren Untersuchung der Ableitungen der angeführten Funktionen folgt, dass $P^i_{x_1}(x)$ $(i = 1,3)$ für $x < x_1$ positiv und für $x > x_1$ negativ sind. $P^i_{x_1}(x)$, $(i = 2,4)$, $P_{x_1}(x)$, $\bar{P}_{x_1}(x)$ sind für alle $x \neq x_1$ und $R_{x_1}(x)$ für alle x positiv.

<u>Hilfssatz 1.3.</u> Es sei x_1 ein beliebiger Punkt aus dem Intervall (a,b). Dann haben die Funktionen $P^5_{x_1}(x)$, $P^6_{x_1}(x)$, $P^7_{x_1}(x)$, $P^8_{x_1}(x)$ höchstens eine Nullstelle auf (a,b) und zwar x_1.

<u>Beweis.</u> Aus (1.4) folgt, dass $P^i_{x_1}(x_1) = 0$ $(i=5,7,8)$. Setzen wir voraus, dass für irgendeine i von den Zahlen $\{5,6,7,8\}$ $\eta_i \in (a,b)$, $\eta_i \neq x_1$, die Nullstelle von $P^i_{x_1}(x)$ $(i = 5,6,7,8)$ ist.

Dann existiert zwischen x_1 und η_i ein solcher Punkt ξ_i, dass $(P^i_{x_1})'(\xi_i) = 0$ $(i = 5,6,7,8)$, was im Falle $i = 5,7,8$ ersichtlich ist und im Falle $P^6_{x_1}(x)$ wird dessen Existenz analogisch wie im Satz 1.4 im Falle c) bewiesen. Da aber $(P^i_{x_1})'(x_1) = 0$ $(i=6,7,8)$ gilt, existiert zwischen x_1 und ξ_i ein solcher τ_i (i ist gleich irgendeiner von den Zahlen 5,6,7,8), dass $(r(P^i_{x_1})')'(\tau_i) = 0$, $(i = 5,6,7,8)$. Die Existenz des Punktes τ_i folgt (für irgendein i von den Zahlen 6,7,8) aus dem Rolleschen Satz und für $i = 5$ wird sie analogisch wie im Satz 1.4 Fall c) bewiesen. Damit erhalten wir jedoch einen Wiederspruch, da mit Rücksicht auf die Bemerkung 1.3 die Funktion $(r(P^5_{x_1})')'(x) = (q\,P^1_{x_1})(x) + P^3_{x_1}(x)$ links von x_1 positiv, rechts negativ ist und eine einzige Nullstelle x_1 hat. Die Funktion $(r(P^6_{x_1})')'(x) = (q\,P^2_{x_1})(x) + (P^4_{x_1})(x)$ ist mit Rücksicht auf die Bemerkung 1.3 für $x \neq x_1$ ebenfalls positiv und hat eine einzige Nullstelle x_1. Die Funktion $(r(P^7_{x_1})')'(x)$ ist gleich der Summe der Funktionen $(r\,\bar{R}_{x_1})(x) + (q\,P_{x_1})(x)$ wo

$$(1.8) \qquad (r\,\bar{R}_{x_1})(x) = \begin{vmatrix} y(x_1) \\ y'(x_1) \\ y''(x) \\ (ry''')(x) \end{vmatrix}$$

Über die Funktion $(r\,\bar{R}_{x_1})(x)$ kann analogisch wie im Beweis des Satzes 1.4 Fall c) gezeigt werden, dass sie keine Nullstellen hat und für alle $x \in (a,b)$ positiv ist. Da nach Bemerkung 1.3 auch $P_{x_1}(x)$ für alle $x \neq x_1$ positiv ist, erhalten wir einen Widerspruch mit der Voraussetzung der Existenz von η_7 und also ξ_7 und damit auch τ_7. Die Funktion $(r\,(P^8_{x_1})')'(x) = \bar{P}_{x_1}(x) + (q\,R_{x_1})(x)$ ist aus ähnlichen Gründen wie die vorhergehenden Funktionen positiv und hat deshalb in τ_8 keine Wurzel. Damit ist der Hilfssatz bewiesen.

<u>H i l f s s a t z 1.4.</u> Es sei x_1 ein beliebiger Punkt aus dem Intervall (a,b). Dann haben die Funktionen $P^9_{x_1}(x)$, $P^{10}_{x_1}(x)$, $P^{11}_{x_1}(x)$, $P^{12}_{x_1}(x)$ auf (a,b) gerade eine Nullstelle und zwar x_1.

<u>B e w e i s.</u> Aus der Bemerkung 1.1 bzw. aus der Bemerkung 1.2 folgt, dass $P^i_{x_1}(x_1) = 0$ $(i = 10,11,12)$, bzw. $P^9_{x_1}(x_1) < 0$ ist. Setzen wir voraus, dass $\eta_i \in (a,b)$, $\eta_i \neq x_1$ die Nullstelle von $P^i_{x_1}(x)$ für irgendein i von den Zahlen $\{9,10,11,12\}$ ist. Zwischen x_1 und η_i existiert dann ein solches ξ_i, dass $(P^i_{x_1})'(\xi_i) = 0$ (i ist gleich irgendeiner von den Zahlen 9,10,11,12). Im Falle $i = 10,11,12$ ist dies ersichtlich, im Falle $P^9_{x_1}(x)$ folgt dies analogisch aus der Untersuchung des Verlaufes der Funktion in der Umgebung x_1 wie im Beweis des Satzes 1.4. Die Existenz des Punktes ξ_i ist jedoch ausgeschlossen, da die Beziehungen

$$(P^9_{x_1})'(x) = P^3_{x_1}(x) + (q\, P^1_{x_1})(x)$$

$$(P^{10}_{x_1})'(x) = P^4_{x_1}(x) + (q\, P^2_{x_1})(x)$$

$$(P^{11}_{x_1})'(x) = \bar{R}_{x_1}(x) + (q\, P_{x_1})(x)$$

$$(P^{12}_{x_1})'(x) = \frac{1}{r(x)}\bar{P}_{x_1}(x) + (q\, R_{x_1})(x)$$

gelten. Aus den angeführten Beziehungen folgt ein Widerspruch, da die Funktionen $(P^9_{x_1})'(x)$, $(P^{10}_{x_1})'(x)$ eine einzige Nullstelle x_1 und die Funktionen $(P^{11}_{x_1})'(x)$, $(P^{12}_{x_1})'(x)$ für $x \in (a,b)$ keine Nullstelle haben. Damit ist der Hilfssatz bewiesen.

<u>B e m e r k u n g 1.4.</u> Aus den Bemerkungen 1.2, 1.3 und aus der weiteren Untersuchung der Ableitungen der angeführten Funktionen geht hervor, dass $P^5_{x_1}(x)$ für $x < x_1$ positiv und für $x > x_1$ negativ ist, $P^6_{x_1}(x)$ ist für alle $x \in (a,b)$ und $P^i_{x_1}(x)$ $(i = 7,8)$

für $x \neq x_1$, $x \in (a,b)$ positiv. $P^9_{x_1}(x)$ ist für alle $x \in (a,b)$ negativ und $P^i_{x_1}(x)$ $(i = 10,11,12)$ sind für $x < x_1$ negativ und für $x > x_1$ positiv.

S a t z 1.5. Für jede nichttriviale Lösung y der Differentialgleichung (a_1) existiert höchstens ein Punkt $\xi \in (a,b)$, in welchen eine der Bedingungen (E_1), (E_2), (E_3), (E_4), (E_5) und (E_6) $y'(\xi) = y'''(\xi) = 0$. erfüllt ist.

B e w e i s. a) Im Satz 1.4 haben wir bewiesen, dass von den ersten vier Bedingungen keine zwei in zwei verschiedenen Punkten erfüllt sein können. Setzen wir voraus, dass eine solche nichttriviale Lösung y existiert, welche in x_1 eine der Bedingungen (E_j) $(j = 1,2,3,4)$ und in $x_2 \neq x_1$ die Bedingung (E_5) bzw. (E_6) erfüllt. Das heisst aber, dass $P^i_{x_1}(x_2) = 0$ $(i = 5,6,7,8)$, bzw. $P^i_{x_1}(x_2) = 0$ $(i = 9,10,11,12)$ und dies ist mit dem Hilfssatz 1.4 im Widerspruch.

b) Weiter müssen wir den Fall ausschliessen, dass die Lösung in zwei verschiedenen Punkten $\xi, \eta \in (a,b)$, $\xi < \eta$ die Eigenschaft (E_5) bzw. (E_6) hat. (Analogisch könnten wir beweisen, dass auch der Fall $\xi > \eta$ nicht vorkommen kann.) Setzen wir voraus, dass eine solche nichttriviale Lösung y existiert, d.h. dass für die Funktionen

$$(1.9) \qquad Q_\xi(x) = \begin{vmatrix} y'(\xi) \\ y''(\xi) \\ y'(x) \\ y''(x) \end{vmatrix} \qquad \text{bzw.} \qquad \bar{Q}_\xi(x) = \begin{vmatrix} y'(\xi) \\ (ry''')(\xi) \\ y'(x) \\ (ry''')(x) \end{vmatrix}$$

$Q_\xi(\eta) = 0$, bzw. $\bar{Q}_\xi(\eta) = 0$ gilt. Dann existiert ein solches $\tau \in (\xi, \eta)$, dass $Q'_\xi(\tau) = 0$. Da $Q'_\xi(\xi) = 0$, existiert ein solches $x_1 \in (\xi, \tau)$, dass $(r\,Q'_\xi)'(x_1) = 0$, dies ist aber ein Widerspruch, weil $(r\,Q'_\xi)'(x_1) = q(x_1)P^7_{x_1}(\xi) + r(x_1)\,P^8_{x_1}(\xi)$ und aus der Bemerkung 1.4 folgt, dass $P^i_{x_1}(x)$ $(i = 7,8)$ positiv für $x \neq x_1$, $x \in (a,b)$ sind. Eine analogische Behauptung beweisen wir für die Funktion $\bar{Q}_\xi(x)$, für welche mit Rücksicht auf die Bemerkung 1.4 aus der Gleichheit

$$\bar{Q}'_{\xi}(x_1) = q(x_1)\, P^{11}_{x_1}(\xi) + r(x_1) P^{12}_{x_1}(\xi) = 0$$

ein Widerspruch hervorgeht.

c) Zum Schluss setzen wir voraus, dass eine nichttriviale Lösung y existiert, welche in x_2 die Bedingung (E_5) und in x_1 die Bedingung (E_6) erfüllt, wobei $x_1 \neq x_2$, $x_1, x_2 \in (a,b)$. Also für

$$(1.10)^{\cdot} \qquad \bar{\bar{R}}_{x_1}(x) = \begin{vmatrix} y'(x) \\ y''(x) \\ y'(x_1) \\ (ry''')(x_1) \end{vmatrix}$$

gilt $\bar{\bar{R}}_{x_1}(x_1) = \bar{\bar{R}}_{x_1}(x_2) = 0$. Dann existiert ein solches $\eta \in (x_1, x_2)$ dass für

$$\bar{\bar{R}}'_{x_1}(x) = \begin{vmatrix} y'(x) \\ y'''(x) \\ y'(x_1) \\ (ry''')(x_1) \end{vmatrix} = \frac{1}{r(x)}\, \bar{Q}_{x_1}(x)$$

$\bar{\bar{R}}'_{x_1}(\eta) = 0$ ist.

Von der Funktion $\bar{Q}_{x_1}(x)$ haben wir gezeigt, dass diese eine einzige Nullstelle x_1 hat und so folgt aus der Gleichheit $\bar{\bar{R}}_{x_1}(\eta) = \frac{1}{r(\eta)}\, \bar{Q}_{x_1}(\eta) = 0$ ein Widerspruch.

II

Im zweiten Teil der vorliegenden Arbeit führen wir einige Eigenschaften der Lösungen der Differentialgleichung (a_1) unter der Voraussetzung an, dass diese die Bedingung (E) erfüllen und dass alle Lösungen oszillatorisch sind. Eine ähnliche Voraussetzung benutzt M. Švec in der Arbeit [1].

Weiter beweisen wir den Satz über die Trennung der Nullstellen der Lösungen und der Ableitungen der Lösungen der Gleichung (a_1).

Bemerkung 2.1. Als oszillatorisch bezeichnen wir eine solche Lösung, welche rechts vom beliebigen Punkt $x_1 \in (a,b)$ unendlich viele Nullstellen hat.

Satz 2.1. Es sei $x_1 \in (a,b)$ ein beliebiger Punkt. Zwei Lösungen u_1, u_2 der Differentialgleichung (a_1), welche in x_1 drei der Werte $u_j^{(m)}(x_1)$ $(m = 0,1,2,3)$, $(j = 1,2)$ derselben Ableitungen gleich Null haben, sind linear abhängig und haben daher alle übrigen Nullstellen gemeinsam.

Der Beweis folgt aus dem Satz über die Eindeutigkeit der Lösung der Cauchyschen Aufgabe.

Unter der Voraussetzung, dass y_1, y_2, y_3, y_4 ein Fundamentalsystem von Lösungen der Gleichung (a_1) ist, können wir die Lösung $\overset{*}{u}_1(x)$, welche in x_1 eine dreifache Nullstelle hat, als Determinante schreiben:

$$(2.1) \qquad \overset{*}{u}_1(x) = \begin{vmatrix} y_1(x_1) & y_2(x_1) & y_3(x_1) & y_4(x_1) \\ y_1'(x_1 & y_2'(x_1) & y_3'(x_1) & y_4'(x_1) \\ y_1''(x_1) & y_2''(x_1) & y_3''(x_1) & y_4''(x_1) \\ y_1(x) & y_2(x) & y_3(x) & y_4(x) \end{vmatrix}$$

oder

$$(2.2) \qquad \overset{*}{u}_1(x) = -D_{41}y_1(x) + D_{42}y_2(x) - D_{43}y_3(x) + D_{44}y_4(x),$$

wo D_{4i} $(i = 1,2,3,4)$ Determinanten dritter Ordnung sind, die zu der Matrix des Systems

$$(2.3) \qquad \sum_{i=1}^{4} c_i\, y_i^{(m)}(x_1) = 0 \qquad (m = 0,1,2)$$

gehören. Wenigstens eins der Elemente D_{4i} $(i = 1,2,3,4)$ ist verschieden von Null. Wenn nämlich alle Determinanten dritter Ordnung D_{4i} $(i = 1,2,3,4)$ gleich Null wären, wäre auch der Wronskian der Lösungen y_1, y_2, y_3, y_4 gleich Null, was mit der Voraussetzung im Widerspruch ist. Im Weiteren werden wir voraussetzen, dass $D_{44} \neq 0$ ist.

Dann können wir die Lösung mit einer dreifachen Nullstelle in x_1 in der Form

$$(2.4) \qquad u_1(x) = \frac{c_4}{D_{44}} u_1^*(x)$$

schreiben, wo c_4 eine beliebige Konstante ist.

S a t z 2.2. Es sei $x_1, x_2 \in (a,b)$, $x_1 \neq x_2$. Es sei $i(j)$ eine der Zahlen 0,1,2,3, (1,2,3,4,5,6). Dann existiert eine nichttriviale Lösung $u_2(x)$ der Gleichung (a_1), welche in x_1 (E_j) und in x_2 die Gleichheit $u_2^{(i)}(x_2) = 0$ erfüllt. Jede zwei Lösungen, welche in x_1 irgendeine der Eigenschaften (E_j) $(1 \leq j \leq 6)$ haben und in x_2 für eine der Zahlen $i = 0,1,2,3$ der Gleichung $u_2^{(i)}(x_2) = 0$ entsprechen, sind linear abhängig und haben daher alle Nullstellen gemeinsam.

B e w e i s. Die Behauptung beweisen wir nur für den Fall $u_2(x_1) = u_2'(x_1) = 0$, $u_2(x_2) = 0$. In den anderen Fällen wird der Beweis analogisch durchgeführt.

Es sei y_1, y_2, y_3, y_4 ein Fundamentalsystem der Gleichung (a_1). Es ist ersichtlich, dass

$$(2.5) \qquad u_2^*(x) = \begin{vmatrix} y_1(x_1) & y_2(x_1) & y_3(x_1 & y_4(x_1) \\ y_1'(x_1) & y_2'(x_1) & y_3'(x_1) & y_4'(x_1) \\ y_1(x_2) & y_2(x_2) & y_3(x_2) & y_4(x_2) \\ y_1(x) & y_2(x) & y_3(x) & y_4(x) \end{vmatrix}$$

die Lösung der Differentialgleichung (a_1) ist, welche in x_1 eine doppelte und in x_2 eine einfache Nullstelle hat. Diese Lösung ist nicht identisch gleich Null, weil dann $0 = u_2^{*\prime}(x_2) = P_{x_1}(x_2)$, was mit dem Hilfssatz 1.2 im Widerspruch steht.

Es sei $u_2(x)$ eine andere Lösung der Differentialgleichung (a_1), welche in x_1 eine doppelte und in x_2 eine einfache Nullstelle hat. Dann sind $u_2^*(x)$ und $u_2(x)$ linear abhängig. Setzen wir voraus, dass dies nicht gilt. Dann ist

$$(2.6) \qquad y(x) = u_2(x) - \frac{u_2'(x_2)}{\overset{*}{u}_2{}'(x_1)} \overset{*}{u}_2(x)$$

die Lösung der Gleichung (a_1), welche in x_1 und auch in x_2 doppelte Nullstellen hat. Das würde die Eigenschaft (E_3) in zwei verschiedenen Punkten bedeuten, was nicht vorkommen kann. Also gilt

$$(2.7) \qquad u_2(x) = k\, \overset{*}{u}_2(x), \quad k \neq 0 \text{ ist eine Konstante.}$$

<u>Bemerkung 2.2.</u> Es sei $x_1 \in (a,b)$ ein beliebiger Punkt. Es ist zu sehen, dass die Funktion (2.1), bzw.

$$(2.8) \qquad \overset{**}{u}_1(x) = \begin{vmatrix} y_1(x_1) & y_2(x_1) & y_3(x_1) & y_4(x_1) \\ y_1'(x_1) & y_2'(x_1) & y_3'(x_1) & y_4'(x_1) \\ y_1'''(x_1) & y_2'''(x_1) & y_3'''(x_1) & y_4'''(x_1) \\ y_1(x) & y_2(x) & y_3(x) & y_4(x) \end{vmatrix}$$

bzw.

$$(2.9) \qquad \overset{***}{u}_1(x) = \begin{vmatrix} y_1(x_1) & y_2(x_1) & y_3(x_1) & y_4(x_1) \\ y_1''(x_1) & y_2''(x_1) & y_3''(x_1) & y_4''(x_1) \\ y_1'''(x_1) & y_2'''(x_1) & y_3'''(x_1) & y_4'''(x_1) \\ y_1(x) & y_2(x) & y_3(x) & y_4(x) \end{vmatrix}$$

die Bedingung

$$(2.10) \qquad \overset{*}{u}_1(x_1) = \overset{*}{u}_1{}'(x_1) = \overset{*}{u}_1{}''(x_1) = 0, \quad \overset{*}{u}_1{}'''(x_1) \neq 0$$

bzw.

$$(2.11) \qquad \overset{**}{u}_1(x_1) = \overset{**}{u}_1{}'(x_1) = \overset{**}{u}_1{}'''(x_1) = 0, \quad \overset{**}{u}_1{}''(x_1) \neq 0$$

bzw.

$$(2.12) \qquad \overset{***}{u}_1(x_1) = \overset{***}{u}_1{}''(x_1) = \overset{***}{u}_1{}'''(x_1) = 0, \quad \overset{***}{u}_1{}'(x_1) \neq 0$$

erfüllt.

<u>Satz 2.3.</u> Es sei $x_1 \in (a,b)$ ein beliebiger Punkt und y_k $(k = 1,2,3,4)$ sei ein Fundamentalsystem der Differentialgleichung

(a_1). y_k $(k = 1,2,3)$ erfülle keine der Bedingungen (2.10), (2.11), (2.12). Dann ist

a)
$$v_4(x) = \begin{vmatrix} y_1(x_1) & y_2(x_1) & y_3(x_1) \\ y_1'(x_1) & y_2'(x_1) & y_3'(x_1) \\ y_1(x) & y_2(x) & y_3(x) \end{vmatrix}$$

die Lösung der Differentialgleichung (a_1), welche in x_1 die Bedingung

(2.13) $\quad v_4(x_1) = v_4'(x_1) = 0, \quad v_4''(x_1) \neq 0$

erfüllt;

b)
$$\bar{v}_4(x) = \begin{vmatrix} y_1(x_1) & y_2(x_1) & y_3(x_1) \\ y_1''(x_1) & y_2''(x_1) & y_3''(x_1) \\ y_1(x) & y_2(x) & y_3(x) \end{vmatrix}$$

ist die Lösung derselben Gleichung, welche in x_1

(2.14) $\quad \bar{v}_4(x_1) = \bar{v}_4''(x_1) = 0, \quad \bar{v}_4'(x_1) \neq 0$

erfüllt;

c)
$$\bar{\bar{v}}_4(x) = \begin{vmatrix} y_1'(x_1) & y_2'(x_1) & y_3'(x_1) \\ y_1''(x_1) & y_2''(x_1) & y_3''(x_1) \\ y_1(x) & y_2(x) & y_3(x) \end{vmatrix}$$

ist wieder eine Lösung von (a_1), welche in x_1 die Bedingung

(2.15) $\quad \bar{\bar{v}}_4'(x_1) = \bar{\bar{v}}_4''(x_1) = 0, \quad \bar{\bar{v}}_4(x_1) \neq 0$

erfüllt;

d)
$$z_4(x) = \begin{vmatrix} y_1(x_1) & y_2(x_1) & y_3(x_1) \\ y_1'''(x_1) & y_2'''(x_1) & y_3'''(x_1) \\ y_1(x) & y_2(x) & y_3(x) \end{vmatrix}$$

ist eine Lösung der Differentialgleichung (a_1), welche in x_1 die Eigenschaft

(2.16) $z_4(x_1) = z_4'''(x_1) = 0, \quad z_4'(x_1) \neq 0$

hat.

e)

$$\bar{z}_4(x) = \begin{vmatrix} y_1'(x_1) & y_2'(x_1) & y_3'(x_1) \\ y_1'''(x_1) & y_2'''(x_1) & y_3'''(x_1) \\ y_1(x) & y_2(x) & y_3(x) \end{vmatrix}$$

ist eine Lösung von (a_1), für welche

(2.17) $\bar{z}_4'(x_1) = \bar{z}_4'''(x_1) = 0, \quad z_4(x_1) \neq 0$

gilt.

f)

$$\bar{\bar{z}}_4(x) = \begin{vmatrix} y_1''(x_1) & y_2''(x_1) & y_3''(x_1) \\ y_1'''(x_1) & y_2'''(x_1) & y_3'''(x_1) \\ y_1(x) & y_2(x_1) & y_3(x) \end{vmatrix}$$

ist eine Lösung von (a_1) welche in x_1 die Bedingung

(2.18) $\bar{\bar{z}}_4''(x_1) = \bar{\bar{z}}_4'''(x_1) = 0, \quad \bar{\bar{z}}_4(x_1) \neq 0$

erfüllt.

Jede weitere Lösung, welche in x_1 eine der Bedingungen von (2.13) bis (2.18) erfüllt, kann als lineare Kombination geschrieben werden und zwar in den Fällen a), bzw. b), bzw. c) der Lösung $u_1^{*}(x)$ und $v_4(x)$, bzw. $\bar{v}_4(x)$, bzw. $\bar{\bar{v}}_4(x)$, in den Fällen d) bezw. e) der Lösung $u_1^{**}(x)$ und $z_4(x)$, bzw. $\bar{z}_4(x)$ und im Falle f) der Lösung $u_1^{***}(x)$ und $\bar{\bar{z}}_4(x)$.

B e w e i s. Diesen führen wir nur für den Fall a) durch, in den anderen Fällen wird der Beweis analogisch durchgeführt.

Suchen wir eine solche Lösung $u_2(x)$ für welche (2.13) gilt. Wir können sie in der Form

(2.19) $$u_2(x) = \sum_{i=1}^{4} c_i y_i(x)$$

schreiben, wo c_i $(i = 1,2,3,4)$ die Bedingungen

$$\sum_{i=1}^{4} c_i y_i(x_1) = 0$$

$$(2.20) \qquad \sum_{i=1}^{4} c_i y_i'(x_1) = 0$$

$$\sum_{i=1}^{4} c_i y_i''(x_1) = \beta \neq 0$$

erfüllen. Wenigstens eine Determinante dritten Grades der Matrix des Systems (2.20) ist verschieden von Null, weil im Gegenteil y_k $(k = 1,2,3,4)$ nicht linear unabhängig wären. Es sei dies D_{44}. Dann erhalten wir für c_i

$$(2.21) \qquad c_1 = -c_4 \frac{D_{41}}{D_{44}} + \beta \frac{\Delta_{23}}{D_{44}}, \qquad c_2 = c_4 \frac{D_{42}}{D_{44}} - \beta \frac{\Delta_{13}}{D_{44}},$$

$$c_3 = -c_4 \frac{D_{43}}{D_{44}} + \beta \frac{\Delta_{12}}{D_{44}},$$

wo

$$\Delta_{ik} = \begin{vmatrix} y_i(x_1) & y_k(x_1) \\ y_i'(x_1) & y_k'(x_1) \end{vmatrix}$$

Nach Einsetzen von (2.21) in (2.19) erhalten wir

$$(2.22) \qquad u_2(x) = \frac{c_4}{D_{44}} \Big[-D_{41} y_1(x) + D_{42} y_2(x) - D_{43} y_3(x) + D_{44} y_4(x) \Big] +$$

$$+ \frac{\beta}{D_{44}} \Big[\Delta_{23} y_1(x) - \Delta_{13} y_2(x) + \Delta_{12} y_3(x) \Big],$$

oder

$$(2.23) \qquad u_2(x) = \frac{c_4}{D_{44}} u_1^{*}(x) + \frac{\beta}{D_{44}} v_4(x)$$

S a t z 2.4. Es sei $x_1 \in (a,b)$ ein beliebiger Punkt. $u_1(x)$ sei die Lösung der Differentialgleichung (a_1), welche in x_1 die Bedingung (2.10) erfüllt und $u_2(x)$ sei eine Lösung, die in x_1 die Bedingung (2.13) erfüllt. Dann:

a) die Lösungen $u_1(x)$ und $u_2(x)$ haben mit Ausnahme von x_1 keine gemeinsamen Nullstellen;

b) zwischen jeden zwei einfachen Nullstellen von $u_2(x)$ liegt wenigstens eine Nullstelle der Lösung $u_1(x)$;

c) zwischen jeden zwei Nullstellen von $u_1(x)$ liegt wenigstens eine Nullstelle der Lösung $u_2(x)$.

B e w e i s. An Stelle der Lösung $u_1(x)$ werden wir im Beweis die Lösung $u_1^*(x)$ erwägen, was mit Rücksicht auf die lineare Abhängigkeit beider Lösungen möglich ist.

a) Die Behauptung a) folgt aus dem Satz 2.2.

b) c_1 und c_2 seien zwei auf einfache Nachbarnullstellen der Lösung $u_2(x)$. Setzen wir voraus, dass in $\langle c_1, c_2 \rangle$ keine Nullstelle der Lösung $u_1^*(x)$ liegt.

Dann ist $\dfrac{u_2(x)}{u_1^*(x)}$ in $\langle c_1, c_2 \rangle$ eine stetige Funktion und hat dort eine Ableitung erster Ordnung. Aus (2.23) folgt

$$\left[\frac{u_2}{u_1^*}\right]' = \left[\frac{c_4}{D_{44}} + \frac{\beta}{D_{44}} \frac{v_4}{u_1^*}\right]' = \frac{\beta}{D_{44}} \frac{v_4' u_1^* - v_4 u_1^{*\prime}}{u_1^{*2}} \tag{2.24}$$

Die Lösung $u_1^*(x)$ können wir durch Entwicklung der Determinante (2.1) nach der dritten Reihe folgend schreiben:

$$u_1^*(x) = y_1''(x_1)v_1(x) - y_2''(x_1)v_2(x) + y_3''(x_1)v_3(x) - y_4''(x_1)v_4(x), \tag{2.25}$$

wo $v_i(x)$ $(i = 1,2,3)$ Lösungen in der Form der Determinanten wie $v_4(x)$ sind, wobei aus (2.1) die i-te Spalte $(i = 1,2,3)$ und die dritte Reihe weggelassen ist.

Dann ist

$$v_4'(x)\, u_1^*(x) - v_4(x)\, u_1^{*\prime}(x) = y_1''(x_1)(v_4'v_1 - v_4v_1')(x) - \tag{2.26}$$

$$- y_2''(x_1)(v_4'v_2 - v_4v_2')(x) + y_3''(x_1)(v_4'v_3 - v_4v_3')(x)$$

wobei wir die Ausdrücke in den Klammern folgend schreiben können:

$$v_4'v_1 - v_4v_1' = \Delta_{23}\Big[- A_{12}\,\Delta_{34} + A_{13}\,\Delta_{24} - A_{14}\,\Delta_{23} -$$

$$- A_{23}\,\Delta_{14} + A_{24}\,\Delta_{13} - A_{34}\,\Delta_{12}\Big],$$

$$v_4'v_2 - v_4v_2' = \Delta_{13}\Big[- A_{12}\,\Delta_{34} + A_{13}\,\Delta_{24} - A_{14}\,\Delta_{23} - \tag{2.27}$$

$$- A_{23}\,\Delta_{24} + A_{24}\,\Delta_{13} - A_{34}\,\Delta_{12}\Big]$$

$$v_4'v_3 - v_4v_3' = \Delta_{12}\Big[- A_{12}\,\Delta_{34} + A_{13}\,\Delta_{24} - A_{14}\,\Delta_{23} -$$

$$- A_{23}\,\Delta_{14} + A_{24}\,\Delta_{13} - A_{34}\,\Delta_{12}\Big]$$

so dass wir nach Einsetzen der Ausdrücke (2.27) in (2.26)

$$v_4'u_1^* - v_4u_1^{*\prime} = -\Big[y_1''(x_1)\,\Delta_{23} - y_2''(x_1)\,\Delta_{13} + \tag{2.28}$$

$$+ y_3''(x_1)\,\Delta_{12}\Big]\,P_{x_1}(x) = -D_{44}\,P_{x_1}(x)$$

erhalten, wo

$$A_{ik} = \begin{vmatrix} y_i(x) & y_k(x) \\ y_i'(x) & y_k'(x) \end{vmatrix}$$

Nach Einsetzen von (2.28) in (2.24) erhalten wir

$$\left[\frac{u_2(x)}{u_1^*(x)}\right]' = -\beta\frac{P_{x_1}(x)}{u_1^{*2}(x)} \tag{2.29}$$

Durch Integration von (2.29) von c_1 bis c_2 erhalten wir

$$\frac{u_2(c_2)}{u_1^*(c_2)} - \frac{u_2(c_1)}{u_1(c_1)} = -\beta\int_{c_1}^{c_2}\frac{P_{x_1}(x)}{u_1^{*2}(x)}\,dx \tag{2.30}$$

Aus dem Hilfssatz 1.2 und aus der Bemerkung 1.3 folgt, dass die rechte Seite der Gleichheit (2.30) positiv, während die linke Seite gleich Null ist; dies ist aber ein Widerspruch.

c) 1. Es seien $d_1, d_2, d_1 < d_2 \neq x_1$, $d_1 \neq x_1$ zwei einfache Nachbarnullstellen der Lösung $u_1^*(x)$. Setzen wir voraus, dass $u_2(x)$ zwischen d_1 und d_2 keine Nullstellen hat. Dann ist die Funktion $\dfrac{u_1^*(x)}{u_2(x)}$ in $\langle d_1, d_2 \rangle$ stetig und hat dort eine Ableitung 1. Ordnung:

$$\left[\frac{u_1^*(x)}{u_2(x)}\right]' = \frac{\beta}{D_{44}} \frac{v_4' u_1^* - v_4 u_1^{*\prime}}{u_2^2} = \beta \frac{P_{x_1}(x)}{u_2^2(x)} \tag{2.31}$$

Analogisch wie im Falle vorher erhalten wir einen Widerspruch. Damit haben wir bewiesen, dass sich einfache Nullstellen gegenseitig abtrennen.

c) 2. Es sei $d_1 = x_1 < d_2$, wo d_1, d_2 zwei Nachbarnullstellen der Lösung $u_1(x)$ sind, zwischen welchen keine Nullstelle der Lösung $u_2(x)$ liegt. Zuerst beweisen wir, dass in irgendeiner Umgebung von rechts $O_{x_1}^+$ des Punktes x_1 $u_2(x) > u_1^*(x)$ ist. Da $u_1^{*\prime\prime\prime}(x_1) > 0$ (dies folgt aus der Positivität des Wronskiens y_1, y_2, y_3, y_4) und $u_2(x)$ wählen wir derart, dass $u_2''(x_1) > 0$, $u_2(x) > 0$ $u_1^*(x) > 0$ für $x > x_1$, $x \in O_{x_1}^+$ sind. Da $u_1^{*\prime\prime}(x_1) = 0 < u_2''(x_1)$ und $u_1^{*\prime}(x_1) = u_2'(x_1) = 0$, ist $(u_2 - u_1^*)''(x_1) > 0$, für alle $x \in O_{x_1}^+$ ist also auch $(u_2 - u_1^*)''(x) > 0$. Daraus folgt, dass $(u_2 - u_1^*)'(x)$ für $x \in O_{x_1}^+$ eine wachsende Funktion ist und da $(u_2 - u_1^*)'(x_1) = 0$ ist, gilt für $x \in O_{x_1}^+$, $x \neq x_1$ die Ungleichheit $(u_2 - u_1^*)'(x) > 0$. Daraus folgt wieder analogisch, dass $(u_2 - u_1^*)(x) > 0$ für $x \in O_{x_1}^+$ und also für irgendeine Umgebung von rechts $O_{x_1}^+$ des Punktes x_1 ist $u_2(x) > u_1^*(x)$. Die Funktionen $u_1^*(x)$, $u_2(x)$ können am Intervall (x_1, d_2) folgende Eigenschaften haben:

(A) $u_1^*(x) < u_2(x)$ für alle $x \in (x_1, d_2)$

oder

(B) $u_1^*(x) < u_2(x)$ nur für $x \in \sigma_{x_1}^+$ und in den anderen Teilen des Intervalls (x_1, d_2) ist $u_1^*(x)$ wechselnd kleiner oder grösser als $u_2(x)$. $u_1^*(x)$ und $u_2(x)$ durchschneiden sich dabei in einer geraden Anzahl von Punkten.

Es existieren also die Punkte ξ_i, $\xi_i \in (x_1, d_2)$, $(i = 1,2,\ldots, 2k)$, für welche folgendes gilt:

$$u_1^*(\xi_i) = u_2(\xi_i)$$

$$(2.32) \qquad u_1^{*\prime}(\xi_i) > u_2'(\xi_i) \qquad (i = 1,3,\ldots 2k-1)$$

und

$$u_1^*(\xi_i) = u_2(\xi_i)$$

$$(2.33) \qquad u_1^{*\prime}(\xi_i) < u_2'(\xi_i) \qquad (i = 2,4,\ldots 2k)$$

Zuerst untersuchen wir den Fall (B). Es sei z.B. $k = 1$, d.h. es seien zwei beliebige Punkte aus dem Intervall (x_1, d_2), für welche

$$(2.34) \qquad 0 < u_1^*(\xi_1) = u_2(\xi_1), \quad u_1^{*\prime}(\xi_1) > u_2'(\xi_1)$$

und

$$(2.35) \qquad 0 < u_1^*(\xi_2) = u_2(\xi_2), \quad u_1^{*\prime}(\xi_2) < u_2'(\xi_2)$$

gilt. Dann ist

$$W(\xi_1) = \begin{vmatrix} u_1^* & u_2 \\ u_1^{*\prime} & u_2' \end{vmatrix} (\xi_1) = \left[u_1^* (u_2' - u_1^{*\prime})\right](\xi_1) = -c_1 < 0$$

und

$$W(\xi_2) = \begin{vmatrix} u_1^* & u_2 \\ u_1^{*\prime} & u_2' \end{vmatrix} (\xi_2) = \left[u_1^* (u_2' - u_1^{*\prime})\right] (\xi_2) = c_2 > 0$$

wo c_1, c_2 positive Zahlen sind. Daraus folgt, dass die Funktion $W(x)$ im Intervall (x_1, d_2) wenigstens eine Nullstelle hat, die wir mit α bezeichner.

Der Fall (A) kann auf den Fall (B) überführt werden, wenn wir $u_1^*(x)$ mit einer derart geeigneten Konstante multiplizieren, dass $u_1^*(x)$ wenigstens zweimal $u_2(x)$ durchschneidet. Daraus, dass der Wronskian im Punkte α gleich Null ist, folgt die Möglichkeit solche Konstanten k_1, k_2 zu finden, für welche

$$
(2.36) \qquad \begin{aligned} k_1 u_1^*(\alpha) + k_2 u_2(\alpha) &= 0 \\ k_1 u_1^{*\prime}(\alpha) + k_2 u_2'(\alpha) &= 0 \end{aligned}
$$

gilt. Das bedeutet, dass eine solche Lösung der Gleichung (a_1)

$$
(2.37) \qquad z(x) = k_1 u_1^*(x) + k_2 u_2(x)
$$

existiert, welche im Punkte x_1 und auch im Punkte α eine doppelte Nullstelle hat. Dies ist aber ein Widerspruch mit der Eigenschaft (E_3). Damit haben wir bewiesen, dass zwischen jeden zwei Nullstellen der Lösung $u_1^*(x)$ wenigstens eine Nullstelle der Lösung $u_2(x)$ liegt.

Zuletzt beweisen wir, dass zwischen x_1 und der ersten rechts liegenden Nullstelle ξ der Lösung $u_2(x)$ keine Nullstelle der Lösung $u_1^*(x)$ liegt. Setzen wir also voraus, dass zwischen x_1 und ξ die Nullstelle η der Lösung $u_1^*(x)$ liegt. Dann liegt gemäss c) 2. zwischen x_1 und η die Nullstelle der Lösung $u_2(x)$ und ξ ist also nicht die erste Nullstelle $u_2(x)$, welche rechts von x_1 liegt, was ein Widerspruch ist. Damit ist der Satz vollkommen bewiesen.

<u>S a t z 2.5.</u> Es seien $u_2(x)$ und $\bar{u}_2(x)$ zwei linear unabhängige Lösungen der Differentialgleichung (a_1), welche in x_1 eine doppelte Nullstelle haben. Dann trennen sich die einfachen Nullstellen beider Lösungen gegenseitig voneinader ab.

<u>B e w e i s.</u> Dieser ist ähnlich dem Beweis des Satzes 7 der Arbeit [1].

H i l f s s a t z 2.1. Es seien $u_1(x)$ und $u_2(x)$ zwei linear unabhängige Lösungen der Differentialgleichung (a_1). Dann sind auch $u_1^{(k)}(x)$ und $u_2^{(k)}(x)$ $(k = 1,2,3)$ linear unabhängig.

Der B e w e i s wird analogisch wie im Satz 9 der Arbeit [1] durchgeführt.

S a t z 2.6. Es sei $u_1(x)$ eine Lösung der Differentialgleichung (a_1), welche in x_1 eine dreifache Nullstelle hat, $u_2(x)$ ist eine Lösung welche in x_1 eine dopplete Nullstelle hat. Dann

a) $u_1'(x)$, $u_2'(x)$ haben keine gemeinsamen Nullstellen, ausser x_1;

b) $u_1^{(k)}(x)$, $u_2^{(k)}(x)$ $(k = 2,3)$ haben keine gemeinsamen Nullstellen;

c) zwischen jeden zwei Nullstellen von $u_2'(x)$, verschieden von x_1, liegt wenigstens eine Nullstelle von $u_1'(x)$ und zwischen jeden zwei Nullstellen von $u_1'(x)$ liegt wenigstens eine Nullstelle von $u_2'(x)$.

d) Die Nullstellen von $u_1^{(k)}(x)$ und $u_2^{(k)}(x)$ trennen sich gegenseitig ab.

B e w e i s. Aus den Voraussetzungen über $u_1(x)$ und $u_2(x)$ ist ersichtlich, dass x_1 eine gemeinsame Nullstelle $u_1^{(k)}(x)$, $u_2^{(k)}(x)$ nur für $k = 0, 1$ ist.

a) b) Setzen wir voraus, dass $u_1^{(k)}(x)$ und $u_2^{(k)}(x)$ (k ist eine der Zahlen 1,2,3) eine gemeinsame Nullstelle $\xi \neq x_1$ haben, d.i.

$$(2.38)\qquad u_1^{(k)}(\xi) = u_2^{(k)}(\xi) = 0$$

Aus dem Satz 2.4 folgt, dass $u_1^2(\xi) + u_2^2(\xi) \neq 0$. Es sei z.B. $u_2(\xi) \neq 0$, dann ist die Funktion

$$(2.39)\qquad y(x) = u_1(x) - \frac{u_1(\xi)}{u_2(\xi)}\, u_2(x)$$

die Lösung der Differentialgleichung (a_1), für welche

$$(2.40)\qquad y(x_1) = y'(x_1) = 0, \qquad y(\xi) = y^{(k)}(\xi) = 0$$

gilt, was mit der Eigenschaft (E_3) für $k = 1$, mit der Eigenschaft (E_1) für $k = 2$ und mit der Eigenschaft (E_2) für $k = 3$ im Widerspruch ist.

Im Weiteren werden wir anstatt der Lösung $u_1(x)$ die Lösung $u_1^*(x)$ erwägen. Analogisch wie wir die Beziehungen (2.27) und (2.28) beim Beweis des Satzes 2.4 erhielten, erhalten wir die Beziehungen:

$$v_4'' \, u_1^{+'} - v_4' \, u_1^{*''} = - D_{44} \, P_{x_1}^{7}(x) \tag{2.41}$$

$$v_4''' \, u_1^{*''} - v_4'' \, u_1^{*'''} = - D_{44} \, \bar{R}_{x_1}(x) \tag{2.42}$$

$$(rv_4''')' . (r\, u_1^{*'''}) - (r\, v_4''').(r\, u_1^{*'''})' = -q(x) D_{44} F_{x_1}(x) \tag{2.43}$$

wo

$$F_{x_1}(x) = \begin{vmatrix} y(x) \\ (ry''')(x) \\ y(x_1) \\ y'(x_1) \end{vmatrix}$$

ist.

Aus dem Beweis des Hilfssatzes 1.3 und aus der Bemerkung 1.4 folgt, dass die Funktionen $\bar{R}_{x_1}(x)$ und $P^7_{x_1}(x)$ für $x \in (a,b)$ nichtnegativ sind.

Über die Funktion $F_{x_1}(x)$ setzen wir voraus, dass sie zwei Nullstellen x_1, x_2, $x_1 < x_2$ hat. Dann existiert ein solcher Punkt ξ, dass $F'_{x_1}(\xi) = 0$ ist. Aus der Gleichheit $F'_{x_1}(\xi) = P^{11}_{x_1}(\xi) = 0$ folgt jedoch ein Widerspruch. Deshalb hat $F_{x_1}(x)$ eine einzige Nullstelle und zwar x_1. Aus der Bemerkung 1.4 folgt weiter, dass $F_{x_1}(x)$ für $x \neq x_1$ positiv ist.

c) 1. Es seien $\eta_1 < \eta_2$ von x_1 verschiedene Nachbarnullstellen von $u_2'(x)$. Setzen wir voraus, dass $u_1^{*'}(x)$ im Intervall $\langle \eta_1, \eta_2 \rangle$ keine Nullstelle hat. Dann ist die Funktion

$\dfrac{u_2'(x)}{u_1^{*'}(x)}$ in $\langle \eta_1, \eta_2 \rangle$ stetig und hat dort eine Ableitung

1. Ordnung:

$$\left[\frac{u_2'}{u_1^{*\prime}}\right]' = \left[\frac{\frac{c_4}{D_{44}} u_1^{*\prime} + \frac{\beta}{D_{44}} v_4'}{u_1^{*\prime}}\right] = \frac{\beta}{D_{44}} \frac{v_4'' u_1^{*\prime} - v_4' u_1^{*\prime\prime}}{u_1^{*\prime 2}} \tag{2.44}$$

Nach Einsetzen von (2.41) in (2.44) erhalten wir

$$\left[\frac{u_2'}{u_1^{*\prime}}\right]' = -\beta \frac{P_{x_1}^{7}}{u_1^{*\prime 2}} \tag{2.45}$$

Wenn wir (2.45) von η_1 bis η_2 integrieren, erhalten wir einen Widerspruch.

c) 2.I. $\xi_1 < \xi_2$ seien zwei von x_1 verschiedene Nachbarnullstellen von $u_1^{*\prime}(x)$. Setzen wir voraus, dass $u_2'(x)$ im Intervall $<\xi_1, \xi_2>$ keine Nullstelle hat. Dann ist die Funktion $\frac{u_1^{*\prime}(x)}{u_2'(x)}$ in $<\xi_1, \xi_2>$ stetig und hat dort eine Ableitung erster Ordnung:

$$\left[\frac{u_1^{*\prime}}{u_2'}\right]' = \frac{\beta}{D_{44}} \frac{u_1^{*\prime\prime} v_4' - u_1^{*\prime} v_4''}{u_2'^{2}} = \beta \frac{P_{x1}^{7}}{u_2'^{2}(x)} \tag{2.46}$$

Durch die Integration von (2.46) von ξ_1 bis ξ_2 erhalten wir einen Widerspruch.

c) 2. II. Es seien $\xi_1 = x_1 < \xi_2$ zwei Nachbarnullstellen von $u_1^{*\prime}(x)$. Aus dem Beweis des Satzes 2.4 ist ersichtlich, dass in irgendeiner Umgebung von rechts $\sigma_{x_1}^{+}$ des Punktes x_1 ist $u_2'(x) > u_1^{*\prime}(x)$. Die Funktionen $u_1^{*\prime}(x)$, $u_2'(x)$ haben im Intervall $<x_1, \xi_2>$ folgende Eigenschaften:

(A) $\qquad u_1^{*\prime}(x) < u_2'(x) \quad$ für alle $\quad x \in (x_1, \xi_2)$

oder

(B) $\qquad u_1^{*\prime}(x) < u_2'(x) \quad$ nur für $\quad x \in \sigma_{x_1}^{+}$

und in den übrigen Teilen des Intervalls (x_1, ξ_2) ist $u_1^{*\prime}(x)$ wechselnd kleiner oder grösser als $u_2'(x)$. Dabei durchschneiden sich die Funktionen $u_1^{*\prime}(x)$ und $u_2'(x)$ in einer geraden Anzahl von Punkten.

Für die Schnittpunkte dieser Funktionen gilt:

$$(2.47) \qquad u_1^{*'}(\eta_i) = u_2'(\eta_i), \quad u_1^{*''}(\eta_i) > u_2''(\eta_i), \qquad (i = 1,3,\ldots 2k-1)$$

und

$$(2.48) \qquad u_1^{*'}(\eta_i) = u_2'(\eta_i), \quad u_1^{*''}(\eta_i) < u_2''(\eta_i), \qquad (i = 2,4,\ldots 2k)$$

wo $\eta_i \in (x_1, \xi_2)$, $(i = 1,2,\ldots, k)$.

Erwägen wir zuerst den Fall (B). Es sei $k = 1$, d.h. η_1, η_2 sind zwei beliebige Punkte aus dem Intervall (x_1, ξ_2) und zwar solche, dass

$$(2.49) \qquad \begin{aligned} 0 < u_1^{*'}(\eta_1) &= u_2'(\eta_1), \qquad u_1^{*''}(\eta_1) > u_2''(\eta_1) \\ 0 < u_1^{*'}(\eta_2) &= u_2'(\eta_2), \qquad u_1^{*''}(\eta_2) < u_2''(\eta_2) \end{aligned}$$

Dann ist

$$(2.50) \qquad \begin{aligned} \bar{W}(\eta_1) &= \begin{vmatrix} u_1^{*'} & u_2' \\ u_1^{*''} & u_2'' \end{vmatrix} (\eta_1) = u_1^{*'}(\eta_1)\left[u_2'' - u_1^{*''}\right](\eta_1) = -\bar{c}_1 < 0 \\ \bar{W}(\eta_2) &= \begin{vmatrix} u_1^{*'} & u_2' \\ u_1^{*''} & u_2'' \end{vmatrix} (\eta_2) = u_1^{*'}(\eta_2)\left[\omega_2'' - u_1^{*''}\right](\eta_2) = \bar{c}_2 > 0 \end{aligned}$$

wo $\bar{c}_1$, $\bar{c}_2$ positive Konstanten sind. Daraus folgt, dass die Funktion $\bar{W}(x)$ im Intervall (x_1, ξ_2) wenigstens eine Nullstelle hat.

Der Fall (A) kann auf den Fall (B) überführt werden wenn wir $u_1^{*'}(x)$ mi einer derart entsprechenden Konstante $\bar{c} > 1$ multiplizieren, dass $u_2'(x)$ wenigstens zweimal von $u_1^{*'}(x)$ durchschnitten wird. Es ist also möglich solche Konstanten $\bar{k}_1$ $\bar{k}_2$ zu finden, welche dem System von Gleichungen:

$$(2.51) \qquad \begin{aligned} \bar{k}_1\, u_1^{*'}(\alpha_1) + \bar{k}_2\, u_2'(\alpha_1) &= 0 \\ \bar{k}_1\, u_1^{*''}(\alpha_1) + \bar{k}_2\, u_2''(\alpha_1) &= 0 \end{aligned}$$

entsprechen. Das bedeutet, dass eine solche Lösung der Gleichung (a_1)

$$(2.52) \quad z(x) = \bar{k}_1 u_1^{*}(x) + \bar{k}_2 u_2(x)$$

existiert, welche im Punkte x_1 eine doppelte Nullstelle hat (d.h. sie hat die Eigenschaft (E_3)) und im Punkte α_1 erfüllt sie (2.15) (d.h. die Eigenschaft (E_5)). Dies ist aber ein Widerspruch mit dem Satz 1.5. Damit wurde bewiesen, das zwischen je zwei Nullstellen der Ableitung der Lösung $u_1^{*\prime}(x)$ wenigstens eine Nullstelle der Ableitung der Lösung $u_2'(x)$ liegt.

Zum Schluss wird bewiesen, dass zwischen x_1 und der ersten rechts liegenden Nullstelle ξ der Ableitung der Lösung $u_2'(x)$ keine Nullstelle der Ableitung der Lösung $u_1^{*\prime}(x)$ liegt. Der Beweis wird analogisch wie der Beweis des entsprechenden Teiles der Behauptung des Satzes 2.4 durchgeführt.

B e m e r k u n g 2.3. Es kann auch folgende Behauptung bewiesen werden: Es seien $u_2(x)$ und $\bar{u}_2(x)$ zwei nichtlineare unabhängige Lösungen der Differentialgleichung (a_1), welche in x_1 eine doppelte Nullstelle haben.

1. $u_2'(x)$ und $\bar{u}_2'(x)$ haben dann mit Ausnahme von x_1 keine gemeinsamen Nullstellen;
2. $u_2^{(k)}(x)$ und $\bar{u}_2^{(k)}(x)$ $(k = 2,3)$ haben keine gemeinsamen Nullstellen;
3. zwischen jeden zwei Nullstellen $u_2'(x)$, verschieden von x_1, liegt eine Nullstelle $\bar{u}_2'(x)$ und umgekehrt;
4. die Nullstellen $u_2^{(k)}(x)$ und $\bar{u}_2^{(k)}(x)$ $(k = 2,3)$ trennen sich gegenseitig ab.

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MATHEMATICA XXVII – 1972

A NOTE ON THE CLASS M_2

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All functions in the sequel are real-valued defined on the real line. If f is a function and λ a number then the set $\{x; f(x) > \lambda\}$ is denoted, for abbreviation, by $[f > \lambda]$, and similarly are defined symbols $[f < \lambda]$, $[\lambda < f < \eta]$. A function f is in the class M_2' of ZAHORSKI [4] provided for each real λ, $[f<\lambda] \cap J \neq \emptyset$ implies $|[f < \lambda] \cap J| > 0$ (by $|A|$ is denoted the Lebesgue measure of A) and $[f > \lambda] \cap J \neq \emptyset$ implies $|[f > \lambda] \cap J| > 0$. A function f has the Denjoy property ($f \in D_1$) provided for each non-trivial closed interval J and for each pair of reals λ_1, λ_2, $[\lambda_1 < f < \lambda_2] \cap J \neq \emptyset$ implies $|[\lambda_1 < f < \lambda_2] \cap J| > 0$.

L. MIŠÍK [2] has shown that $B_1 \cap M_2' = B_1 \cap D_1$ (B_α is the Baire α class of function). In connection with this result L. MIŠÍK has posed the following problem: Does there exist a function $f \in B_2 \cap M_2'$ with the Darboux property such that $f \notin D_1$? The problem was solved by J. S. LIPIŃSKI [1], and T. ŠALÁT [3], and the answer is positive. In the present note there is given another example of such function, which is more simple than the examples in [1], [3].

<u>Theorem.</u> There is a function f with the Darboux property such that $f \in B_2$, $f \in M_2'$, $f \notin D_1$.

<u>Proof:</u> Let $\{J_n\}_{n=1}^{\infty}$ be a basis of open sets. We construct by induction the system $\{P_n^k\}_{n=1}^{\infty}$, k=1,2,3, of pairwise disjoint non-empty nowhere dense perfect sets P_n^k such that, for each n,k, $P_n^k \subset J_n$, $|P_n^1| = 0$, and $|P_n^k| > 0$ for $k > 1$. Let P_1^1,

P_1^2, P_1^3 be pairwise disjoint non-empty nowhere dense perfect sets contained in J_1 such that $|P_1^1| = 0$, and $|P_1^k| > 0$, for $k > 1$. If the sets $\{P_n^k\}_{n=1}^{m}$, $k = 1, 2, 3$ have been constructed let

$P = \bigcup_{n=1}^{m} \bigcup_{k=1}^{3} P_n^k$. Clearly P is a nowhere dense set, hence there is some non empty open interval $J \subset J_{m+1} - P$. Let P_{m+1}^1, P_{m+1}^2, P_{m+1}^3, be pairwise disjoint non-empty nowhere dense perfect sets contained in J such that $|P_{m+1}^1| = 0$, and $|P_{m+1}^k| > 0$ for $k > 1$. Thus the system $\{P_n^k\}_{n=1}^{\infty}$, $k = 1, 2, 3$, has been constructed. Now for each n, let g_n be a continuous function from P_n^1 onto the closed unit interval $\langle 0, 1 \rangle$. Define the function f as follows:

$$f(x) = \begin{cases} g_n(x) & \text{if } x \in P_n^1, \text{ for some } n, \\ 1 & \text{if } x \in P_n^2, \text{ for some } n, \\ 0 & \text{otherwise.} \end{cases}$$

The function f has the Darboux property since each open interval $J \neq \emptyset$ contains some P_n^1 and consequently, $f(J) = \langle 0,1 \rangle$.

There is also $f \in M_2'$. To see it consider the set $[f > \lambda]$. If $\lambda \geqslant 1$ the $[f > \lambda] = \emptyset$. If $\lambda < 1$, then $[f > \lambda] \supset \bigcup_{n=1}^{\infty} P_n^2$ and since each non-empty open interval J contains some P_n^2, J intersects the set $[f > \lambda]$ in a set of positive measure. The proof for the set $[f < \lambda]$ is similar.

To show that $f \in B_2$ it suffices to show that each of the sets $[f > \lambda]$, $[f \geqslant \lambda]$, is of the type F_σ. But $[f > \lambda] = \emptyset$ for $\lambda \geqslant 1$, $[f > \lambda] = (-\infty, +\infty)$ for $\lambda < 0$, and $[f > \lambda] = \bigcup_{n=1}^{\infty} P_n^2 \cup \bigcup_{n=1}^{\infty} [g_n > \lambda]$ if $0 \leqslant \lambda < 1$, and clearly each of the sets P_n^2, $[g_n > \lambda]$ is of the type F_σ. Similarly $[f \geqslant \lambda] = \emptyset$ if $\lambda > 1$, $[f \geqslant \lambda] = (-\infty, +\infty)$ if $\lambda \leqslant 0$, and $[f \geqslant \lambda] = \bigcup_{n=1}^{\infty} P_n^2 \cup \bigcup_{n=1}^{\infty} [g_n \geqslant \lambda]$ if $0 < \lambda \leqslant 1$, where each of the sets $[g_n \geqslant \lambda]$ is of the type F_σ.

Finally we show that $f \notin D_1$. This follows from the fact, that the set $[0 < f < 1]$ is dense on the real line, and that

$$\left|[0 < f < 1]\right| \leq \left|\bigcup_{n=1}^{\infty} P_n^1\right| = 0.$$

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REMARKS ON THE FUNDAMENTAL THEOREM OF ARITHMETIC

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In the paper two new proofs of the fundamental theorem of arithmetic are given.

The following is called the fundamental theorem of arithmetic.

Theorem A. Let a,b,c be real or complex integers, $a \neq 0$. If $a \mid bc$ and $(a,b) = 1$, then $a \mid c$ (cf. [1], p. 14, 424).

In this note we give two proofs of Theorem A for complex integers different from the proof of this theorem given in [1] p. 424. The first of our proofs is based on the same idea as Surányi's proof of Theorem A for real integers (cf. [2]). The second is based as well as the proof in [1] on the following fact:

If $(a,b) = 1$, then there exist complex integers x,y such that $ax + by = 1$.

The metod of proving the last fact is different from that obe used in [1] (p. 422 - 423).

Proof I. Denote K the set of all complex integers. Let $a, b, c \in K$, $a \neq 0$, $a \mid bc$ and $(a,b) = 1$. Put $K_1 = \{x \in K;\ a \mid bx\}$. Obviously $a \in K_1$, the same being true for da $(d \in K)$. It suffices to prove that K_1 consits precisely of the numbers da $(d \in K)$. This will be proved if we prove the following two assertions:

T_1: Let $N(x)$ denote the norm of the number $x \in K$*). Let c_0 denote such an element of the set K_1 that $N(c_0) = \min\limits_{x \in K_1, x \neq 0} N(x)$

*) If $x = x_1 + ix_2$ (x_1, x_2 are real), then $N(x) = x_1^2 + x_2^2$ (cf. [1], p. 417).

(since the norms of elements of K_1 are non-negative integers, the existence of c_o is obvious). Then for each $x \in K_1$ we have $c_o | x$.

T_2: $|c_o| = |a|$.

<u>Proof of T_1.</u> Let $x \in K_1$. It is well-known that there exist such numbers $t, r \in K$ that

(1) $\quad x = t\, c_o + r$

(1′) $\quad N(r) < N(c_o)$

(cf. [1], p. 420). From (1) we get $br = bx - b\,t\,c_o$. Since a / bx, $a | b\,c_o$, the number br is divisible by a. Thus $r \in K_1$. If $N(r) > 0$, then (1′) contradicts the definition of c_o. Hence $N(r) = 0$, $r = 0$ and according to (1) we have c_o / x.

<u>Proof of T_2.</u> Since $a \in K_1$, we have $c_o | a$ according to T_1. Thus there exists a number $d_1 \in K$ such that

(2) $\quad a = d_1\, c_o$

Further $a | b\, c_o$, because c_o belongs to K_1. In view of the last fact the existence of a number $d_2 \in K$ follows such that

(3) $\quad b\, c_o = a\, d_2$.

From (2), (3) we get $b\,c_o = d_1 d_2 c_o$ and since $c_o \neq 0$, we have

(4) $\quad b = d_1 d_2$

It follows from (2) and (4) that $d_1 | a$ and simultaneously $d_1 | b$. Since $(a,b) = 1$, it must be $|d_1| = 1$ and so from (2) we get $|a| = |c_o|$.

At first we prove two auxiliary results.

<u>Lemma 1.</u> Let $a, b \in K$, $a, b \neq 0$. Then

$$N(a + bj) < \max\,(N(a), N(b))$$

for a suitable $j \in \{1, -1, i, -i\}$.

P r o o f. Let e.g.

(*) $N(b) \leqq N(a)$

We shall prove Lemma 1 indirectly. Let

(5) $N(a + b) \geqq N(a)$,

(6) $N(a - b) \geqq N(a)$

(7) $N(a + ib) \geqq N(a)$

(8) $N(a - ib) \geqq N(a)$

We write $a = a_1 + ia_2$, $b = b_1 + ib_2$, where a_k, b_k $(k = 1,2)$ are real integers. From (5), (6) we get

$$-(b_1^2 + b_2^2) = 2(a_1b_1 + a_2b_2) \leqq b_1^2 + b_2^2$$

i.e.

(9) $2|a_1b_1 + a_2b_2| \leqq N(b)$

Analogously from (7), (8) we get

(10) $2|a_1b_2 - a_2b_1| \leqq N(b)$

An easy computation leads from (9), (10) to the inequality $2\,N(a)\,N(b) \leqq N^2(b)$, so $N(b) \geqq 2\,N(a)$. The last together with (*) gives $N(b) \geqq 2\,N(b)$, $1 \geqq 2$.

L e m m a 2. Let $a, b \in K$, $(a,b) = 1$. Then there exist $x, y \in K$ such that $ax + by = 1$.

P r o o f. We shall prove by induction that for each natural number k the following assertion A_k holds:

If $a, b \in K$, $(a, b) = 1$, $N(a) + N(b) < k$, then there exist $x, y \in K$ such that $ax + by = 1$.

First of all the assertion A_1 is true since there are no numbers $a,b \in K$ satisfying the conditions $(a,b) = 1$, $N(a) + N(b) < 1$.

Suppose A_k for each $k \leqq n$ to be true. We prove that A_{n+1} holds. Let $a,b \in K$, $(a,b) = 1$, $N(a) + N(b) < n + 1$. If one of the numbers a, b is 0, then the other must be a unit (accorging to (a, b) = 1). In this case the existence of numbers $x, y \in K$ with $ax + by = 1$ is obvious. Hence let $a, b \neq 0$ and let e.g. $N(b) \leq N(a)$. In view of Lemma 1 there exists a number $j \in \{1, -1, i, -i\}$ such that $N(a + bj) < N(a)$. Hence

$$N(a + bj) + N(b) < N(a) + N(b) < n + 1$$

Thus $N(a + bj) + N(b) < n$. Obviously $(a + bj, b) = 1$ and by the induction hypothesis there are $x_1, y_1 \in K$ such that $(a + bj) x_1 + by_1 = 1$, i.e. $ax_1 + b(jx_1 + y_1) = 1$. Putting $x = x_1$, $y = jx_1 + x_2$, we see that $x, y \in K$ and $ax + by = 1$. Hence A_{n+1} holds.

Proof II of Theorem A. Let $a|bc$, $(a,b) = 1$. Following the Lemma 2 there exist $x, y \in K$ such that $ax + by = 1$. Multiply the last equality by c we get

$$c = acx + bcy$$

From this Theorem A follows immediately.

Remark 1. The fundamental theorem of arithmetic (F.T.A) for the set C of all real integers can be proves by the same method which was used in Proof II. It suffices only to take C instead of K, to substitute $\{1, -1, i, -i\}$ in Lemma 1 by $\{1, -1\}$ and $N(x)$ by $|x|$ (the absolute value of x).

Remark 2. F.T.A. for the set C is also an easy consequence of F.T.A. for the set K. This can be shown in the following way: Let $a|bc$, $(a,b) = 1$, $a,b,c \in C$, $a \neq 0$. Then obviously $a|bc$ also in K. We assert that $(a,b) = 1$ also in K. Let contrary to this assertion $a = a_1 d$, $b = b_1 d$, where d is not a complex unit. Then we have

$$a^2 = N(a) = N(a_1)N(d), \qquad b^2 = (N(b) = N(b_1)N(d), \qquad N(d) > 1.$$

Hence $(a^2, b^2) > 1$ in C and therefore $(a, b) > 1$ in C, too. This is a contradiction to the assumption $(a,b) = 1$ in C. Hence $(a,b) = 1$ in K and so owing to the F.T.A. for K we have $a|c$ in K. Thus there exists a $c_1 \in K$ such that $c = c_1 a$. If $c = 0$ then $a|c$ in C evidently. If $c \neq 0$ then c_1 is real, because a,c are real numbers. Hence $c_1 \in C$ and so $a|c$ in C.

M. Kolibiar called our attention to the possibility of using the methods of Proofs I, II for proving some theorems analogous to the F.T.A. for some integral domains.

Let I be an integral domain. Suppose a function $N:I \to C_0$ to be given, C_0 being the set of all non-negative integers. If $a,b \in I$ and b is divisible by a, then we write $a|b$. Further $a,b \in I$ are called coprime if from $d|a$, $d|b$ it follows that $N(d) = 1$. We shall say that for I the F.T.A. holds if the following assertion is true:

Let $a,b,c \in I$, $a \neq 0$. Let a,b be comprime and $a|bc$. Then $a|c$.

Let us suppose that I is an Euclidean integral domain (cf.[3], p. 91). Hence there exists a function $N:I \to C_0$ such that for each $a, b \in I$, $b \neq 0$ there are some $r, t \in I$ with $a = bt + r$, where $r = 0$ or $N(r) < N(b)$. Let us suppose that the function N satisfies also the following conditions:

(i) $N(x) = 0 \Rightarrow x = 0$

(ii) $N(xy) = N(x)\, N(y)$

By the method used in Proof I the following theorem can be proved.

<u>Theorem B.</u> Let I be an euclidean integral domain in which the function N satisfies the conditions (i), (ii). Then for I the F.T.A. Holds.

Moreover by a little modification of the method used in Proof I the result of Theorem B can be extended for all principal ideal domains.

Analogously by the method used in Proof II the following result can be obtained.

<u>Theorem C.</u> Let I be an integral domain. Suppose a function $N : I \to C_0$ to be given. Suppose moreover the following conditions for N to be satisfied:

a) $N(a) = 0 \Rightarrow a = 0$;

b) If $a,b \in I$, $a,b \neq 0$, a,b are coprime, then there exists an element $j \in I$ such that $N(a + bj) < \max\,(N(a), N(b))$.

Then for I the F.T.A. holds.

Example 1. For the integral domain $T[x]$ of all polynomials in the indeterminate x over a given field T the F.T.A. holds if we define the function $N : T[x] \to C_0$ as follows: $N(0) = 0$ and $N(P) = h(P) + 1$ if $P \neq 0$ and $h(P)$ denotes its degree. The condition a) from theorem C is obviously by the function N satisfied. We have yet to show that also the condition b) is by N satisfied. Let $P, Q \in T[x]$, $P, Q \neq 0$. If $N(P) < N(Q)$, then for $R = 0$ we get

$$N(P + RQ) = N(P) < N(Q) = \max (N(P), N(Q))$$

If $N(P) \geqq N(Q)$, i.e. if $h(P) \geqq h(Q)$, then let $a_0 x^n$ and $b_0 x^m$ be the highest term of the polynomial P and Q, respectively. Hence $N(P) = n + 1$, $N(Q) = m + 1$, $a_0 \neq 0 \neq b_0$. Putting $R = - a_0 b_0^{-1} x^{n-m}$ we get $N(P + RQ) < N(P) = \max (N(P), N(Q))$.

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ON DIFFERENCE SETS OF SETS OF NON - NEGATIVE INTEGERS

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In the sequel N (resp. C) denotes the set of all non-negative integers (resp. of all integers). In the theory of numbers besides the sums and products of sets of non-negative integers also the difference sets of sets of non-negative integers has been studied, (see e.g. [1]; [2], p. 177 - 182).

For $A, B \subset N$ we put $D(A,B) = \{x \in C : x = a-b \mid a \in A;\ b \in B\}$. The set $D(A,B)$ will be called a difference set of the sets A and B. We put $D(A) = D(A,A)$. (see [1], p. 112).

The set $A \subset N$ is called a difference basis for C if for each $c \in C$ there exist $a, b \in A$ such that $c = a - b$. (see [1]; [4]).

For $A \subset N$ we put $A(n) = \sum_{\substack{a \in A \\ 0 < a \leq n}} 1$ and $\delta_2(A) = \limsup_{n \to \infty} \frac{A(n)}{n}$.

$\delta_2(A)$ is called the upper asymptotic density of the set $A \subset N$.

If there exists $\delta(A) = \lim_{n \to \infty} \frac{A(n)}{n}$, then it is called the asymptotic density of the set A.

In the paper [1] the following result is proved: There exists a set $A \subset N$ for which $A(x) = O(\sqrt{x})$ and $D(A) = C$.

It is shown in the paper [4] that there exists a difference basis A for which the function $A(n)$ arbitrarily slowly tends to infinity.

W. Sierpiński [3] has proved that if $\delta_2(A) > \frac{1}{2}$ then $D(A) = C$ and for each $x \in C$ there exist infinitely many pairs of numbers $a,b \in A$ such that $x = a - b$. This condition is only sufficient but not necessary for $D(A,B) = C$, as it follows from the quoted result of [1].

The following theorem converning difference sets of two sets gives a result analogous to the above quoted Theorem of W. Sierpiński.

<u>Theorem 1.</u> Let $A,B \subset N$ and assume that one of the following two conditions

(a) $\delta(A) > \frac{1}{2}$ and $\delta_2(B) > \frac{1}{2}$,

(b) $\delta_2(A) > \frac{1}{2}$ and $\delta(B) > \frac{1}{2}$,

is satisfied. Then to each $c \in C$ there exists an infinite number of pairs $(a,b) \in A \times B$ such that $c = a - b$.

<u>Proof 1.</u> Let for A,B the condition (a) be satisfied. Assume that there is a number $c \in C$ such that for each pair $(a,b) \in A \times B$ we have $c \neq a - b$. Let $A = \{a_1 < a_2 < \ldots\}$; from (a) it follows that the set $A^* = \{a_1-c, a_2-c, \ldots, a_k-c, \ldots\}$ has the asymptotic density greater than $\frac{1}{2}$. Clearly $A^* \cap B = \emptyset$ and hence $\delta_2(B) < \frac{1}{2}$, contrary to the assumption that $\delta_2(B) > \frac{1}{2}$.

Now we assume that there exists a number $c \in C$ such that $c = a - b$ only for a finite number of pairs $(a,b) \in A \times B$. Let (c_i,d_i); $(i = 1,2,\ldots,k)$ be all pairs from $A \times B$ such that $c = c_i - d_i$.

Put $m = \max(c_1,\ldots,c_k; d_1,\ldots,d_k)$. From the condition (a) we get $\delta(A) = \frac{1}{2} + 2\tau$, where $\tau > 0$. Therefore there exists some n_1 such that the inequality

$$\frac{A(n)}{n} > \frac{1}{2} + \tau \qquad (1)$$

holds for each $n > n_1$. Let $c_1 = \min c_i$ and $d_1 = \min d_i$ $(i = 1,2,\dots,k)$, and put $d = \max (c_1,d_1)$, $d' = \min (c_1,d_1)$. Choose a number n_2 such that for $n > n_2$, the inequality $\frac{m + (d - d')}{2n} < \gamma$ is satisfied. Put $n_o = \max (m,n_1,n_2)$. For $n > n_o$ we form a sequence

$$m + 1,\ m + 2, \dots,\ n. \qquad (2)$$

From the definition of the number m it follows that for each pair a', b' of numbers in (2) such that $a' \in A$ and $b' \in B$, we have $a' - b' \neq c$. The numbers $c_1 + j$, $d_1 + j$ belong to (2) if

$$m - d' < j \leqq n - d. \qquad (3)$$

Let j satisfy the condition (3). Since $(c_1 + j)-(d_1 + j) = c_1 - d_1 = c$, it must hold $c_1 + j \notin A$ or $d_1 + j \notin B$. Let $M_1(M_2)$ be the set of all j satisfying (3) for which $(c_1 + j) \notin A$, $((d_1 + j) \notin B)$. If $P(M_i)$ denotes the number of elements of the set $M_i (i = 1,2)$, then clearly

$$P(M_1) + P(M_2) \geqq (n - d) - (m - d'). \qquad (4)$$

It follows from (4) that at least one the numbers $P(M_i),(i = 1,2)$ is greater or equal to $\frac{1}{2}\left[(n-d) - (m-d')\right]$. If $P(M_1) \geqq \frac{1}{2}\left[(n-d) - (m-d')\right]$ then for $A(n)$ we have $A(n) \leqq n - \frac{1}{2}\left[(n-d) - (m-d')\right]$. Hence $\frac{A(n)}{n} \leqq \frac{1}{2} + \frac{m + (d - d')}{2n} < \frac{1}{2} + \gamma$ which is a contradiction to (1).

On the other hand if we have $P(M_2) \geqq \frac{1}{2}\left[(n - d)-(m - d')\right]$ hence $\frac{B(n)}{n} \leqq \frac{1}{2} + \frac{m + (d - d')}{2n}$. Thus $\delta_2(B) \leqq \frac{1}{2}$, which is again a contradiction to the asumption of Theorem.

2. Let for A,B the condition (b) be satisfied. If $c \in C$, then on the basis of the preceeding part of Theorem 1 there exists an infinite

number of pairs $(b,a) \in B \times A$ such that $b-a = -c$. From this it is clear that there exists an infinite number of pairs $(a,b) \in A \times B$ such that $a-b = c$. This finishes the proof of Theorem 1.

<u>Remark.</u> The conditions introduced in the Theorem 1 are only sufficient for the equality $D(A, B) = C$. (see [1], [4]).

The following theorem shows that the condition (a) or (b) in Theorem 1 cannot be replaced by the condition $\delta_2(A) = \delta_2(B) = 1$.

<u>Theorem 2.</u> There exist sets $A, B \subset N$ such that $\delta_2(A) = \delta_2(B) = 1$, and $C - D(A,B)$ is an infinite set.

<u>Proof.</u> Let $A = \bigcup_{m=1}^{\infty} A_m$ where A_m is the set of all natural numbers from the interval $\langle (10^m)^{10^m}, (10^m + 1)^{10^m+1})$, and $B = \bigcup_{m=1}^{\infty} B_m$ where B_m is the set of all natural numbers from the interval $\langle (10^m+1)^{10^m+1}, \frac{1}{2}(10^{m+1})^{10^{m+1}})$, $m = (1,2,\ldots)$. At first we shall show that $\delta_2(A) = \delta_2(B) = 1$.

We have $A((10^m+1)^{10^m+1}) \geqq (10^m+1)^{10^m+1} - (10^m)^{10^m}$ and so

$$\frac{A((10^m+1)^{10^m+1})}{(10^m+1)^{10^m+1}} \geqq 1 - \frac{(10^m)^{10^m}}{(10^m+1)^{10^m}} \cdot \frac{1}{(10^m+1)} \geqq 1 - \frac{1}{10^m+1},$$

for $m = 1,2,\ldots$ Hence $\delta_2(A) = 1$.

Further $B\left(\frac{1}{2}(10^{m+1})^{10^{m+1}}\right) \geqq \frac{1}{2}(10^{m+1})^{10^{m+1}} - (10^m+1)^{10^m+1}$ and so

$$\frac{B\ \frac{1}{2}(10^{m+1})^{10^{m+1}})}{\frac{1}{2}(10^m+1)^{10^{m+1}}} \geqq 1 - \frac{(10^m+1)^{10^m+1}}{\frac{1}{2}(10^{m+1})^{10^{m+1}}} = 1 - \frac{2(10^m+1)^{10^m+1}}{(10^{m+1})^{10^{m+1}-1} \cdot (10^{m+1})} \geqq$$

$\geqq 1 - \dfrac{1}{10^{m+1}}$. Hence $\delta_2(B) = 1$. Now we are going to show that D(A,B) contains no numbers of the form $\frac{1}{2}(10^m)^{10^m}$, $(m = 1,2,\ldots)$.

Let n be arbitrary natural number and let $\frac{1}{2}(10^n)^{10^n} = x - y$ with $x \in A$, $y \in B$. Then there exist indices i,j such that $x \in A_i$, $y \in B_j$. We have clearly $j < i$. Consider the following four cases.

a) $j < i \leqq n$

b) $j < n \leqq i$

c) $j \leqq n < i$

d) $n \leqq j < i$

a) If $j < i = n$, then $x \in A_n$, $y \in B_j$; $j < n$,

$$x-y > (10^n)^{10^n} - \frac{1}{2}(10^{j+1})^{10^{j+1}} \geqq (10^n)^{10^n} - \frac{1}{2}(10^n)^{10^n} = \frac{1}{2}(10^n)^{10^n} .$$

Let further $j < i < n$. Then

$$x - y < (10^i+1)^{10^i+1} - (10^j+1)^{10^j+1} < \frac{1}{2}(10^n)^{10^n}$$

b) In this case we have

$$x - y > (10^i)^{10^i} - \frac{1}{2}(10^{j+1})^{10^{j+1}} \geqq (10^n)^{10^n} - \frac{1}{2}(10^n)^{10^n} = \frac{1}{2}(10^n)^{10^n} .$$

c) A simple estimation gives

$$x - y > (10^i)^{10^i} - \frac{1}{2}(10^{j+1})^{10^{j+1}} \geqq (10^{n+1})^{10^{n+1}} - \frac{1}{2}(10^{n+1})^{10^{n+1}} =$$

$$= \frac{1}{2}(10^{n+1})^{10^{n+1}} > \frac{1}{2}(10^n)^{10^n} .$$

d) If $n = j < i$, then

$$x - y > (10^i)^{10^i} - \frac{1}{2}(10^{n+1})^{10^{n+1}} \geqq (10^{n+1})^{10^{n+1}} - \frac{1}{2}(10^{n+1})^{10^{n+1}} =$$

$$= \frac{1}{2}(10^{n+1})^{10^{n+1}} > \frac{1}{2}(10^n)^{10^n}.$$

For $n < j < i$ we obtain $x - y > (10^i)^{10^i} - \frac{1}{2}(10^{j+1})^{10^{j+1}} \geqq$

$$\geqq (10^{j+1})^{10^{j+1}} - \frac{1}{2}(10^{j+1})^{10^{j+1}} = \frac{1}{2}(10^{j+1})^{10^{j+1}} > \frac{1}{2}(10^n)^{10^n}.$$

The proof of Theorem 2 is finished.

The following result can be found in [4]: To every set $B = \{b_1 < b_2 < \dots\}$ there exists a difference basis A such that $A(n) \leqq 2B(n)$.

We shall say that the set $A \subset N$ is a strong difference basis if for each $c \in C$ there exists an infinite number of pairs $(a,b) \in A \times A$ such that $c = a - b$.

In connection with the above mentioned result of [4] we prove the following

T h e o r e m 3. To every infinite set $B = \{b_1 < b_2 < \dots\}$ there exists a strong difference basis A such that $A(n) \leqq B(n)$.

P r o o f. Put
$$A_1 = \{a_1, a_1+1\} \qquad a_1 \geqq b_2$$
$$A_2 = \{a_2, a_2+1, a_2+2\} \qquad a_2 \geqq b_5$$
$$\vdots$$
$$A_k = \{a_k, a_k+1, \dots, a_k+k\} \qquad a_k \geqq b_{\frac{1}{2}k(k+3)}$$
$$\vdots$$

$$A = \bigcup_{k=1}^{\infty} A_k.$$

At first we are going to show that $A(n) \leqq B(n)$.

Let n be an arbitrary natural number. Let j be the highest index such that $b_{\frac{1}{2}j(j+3)} \leqq n < b_{\frac{1}{2}(j+1)(j+4)}$; for such index j we have $\frac{1}{2}j(j+3) \leqq B(n) < \frac{1}{2}(j+1)(j+4)$. (5)

From the definition of the set A_j it follows that $a_j \geqq b_{\frac{1}{2}j(j+3)}$, and hence clearly

$$A(n) \leqq \frac{1}{2} j(j+3) \tag{6}$$

It follows from (5) and (6) that $A(n) \leqq B(n)$, for arbitrary $n \in N$. The fact that A is a strong difference basis is evident.

In the next section we shall show how the property of the set "to be a difference basis" is related with some other properties of the set.

In [5] the notions of a rare set and a sparse set have been defined only for the subsets of the set of all prime numbers. Now we shall extend this definition for sets $A \subset N$.

<u>Definition 1.</u> The set $A = \{n_1 < n_2 < \dots\} \subset N$ is called rare if $\sum_{k=1}^{\infty} \frac{1}{n_k} < +\infty$, and sparse if $A(x) = o\left(\frac{x}{\log x}\right)$.

Let A be a set of natural numbers; consider the following three statements on the set A:

$P_1(A)$:A is a difference basis for C

$P_2(A)$:A is rare

$P_3(A)$:A is sparse.

<u>Theorem 4.</u> There exist sets $A_1, A_2, A_3, A_4 \subset N$ such that $P_1(A_1)$ and $P_2(A_1)$, $P_1(A_2)$ and non $P_2(A_2)$, non $P_1(A_3)$ and $P_2(A_3)$, non $P_1(A_4)$ and non $P_2(A_4)$.

<u>Proof.</u> a) Let $A_1 = B_1 \cup B_2$, where B_1 and B_2 are the sets of all non-negative integers of the form

$$B_1 = \left\{ \sum_{\ell=0}^{n} \xi_\ell 4^\ell \right\} \quad \text{where} \quad \xi_\ell = 1 \text{ or } 3 \text{ for each } \ell$$

$$B_2 = \left\{ \sum_{\ell=0}^{n} \xi_\ell 4^\ell \right\} \quad \text{where} \quad \xi_\ell = 0 \text{ or } 1 \text{ for each } \ell .$$

Put $A_1 = \{a_1, a_2, \ldots, a_k, \ldots\}$, $a_i < a_{i+1}$.

In [1] it has been shown that A_1 is a difference basis for C and that $A_1(x) = O(\sqrt{x})$. We shall show that $\sum_{i=1}^{\infty} \frac{1}{a_i} < +\infty$. Since $A_1(x) = O(\sqrt{x})$ there exists some $c_1 > 0$ such that for each $x \geqq 1$ we have $A_1(x) \leqq c_1 \sqrt{x}$. Let $x \geqq a_1$. Then there exists a natural number k such that $1 \leqq k = A_1(x) \leqq c_1 \sqrt{x}$. From this we get,

$$a_{k+1} > x > \frac{1}{c_1^2} \cdot k^2, \quad \text{hence} \quad \sum_{k=2}^{\infty} \frac{1}{a_k} < c_1^2 \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty .$$

b) Let $A_2 = \{1,2,3,\ldots\}$. Then A_2 is a difference basis but it is not rare.

c) Let $A_3 = \{2,2^2,2^3,\ldots\}$. Then A_3 is rare but fails to be a difference basis.

d) Let $A_4 = \{2,4,\ldots, 2k,\ldots\}$. Then A_4 fails to be a difference basis and it is not rare.

<u>Theorem 5.</u> There exist sets $A_1,A_2,A_3,A_4 \subset N$ such that $P_1(A_1)$ and $P_3(A_1)$, $P_1(A_2)$ and non $P_3(A_2)$, non $P_1(A_3)$ and $P_3(A_3)$, non $P_1(A_4)$ and non $P_3(A_4)$.

<u>Proof.</u> Let A_1 is the set in the proof of the preceeding Theorem. In [1] it is shown that A_1 is a difference basis and from $A_1(x) = O\left(\frac{x}{\log^2 x}\right) = o\left(\frac{x}{\log x}\right)$ it follows that A_1 is sparse.

b) Let $A_2 = \{1,2,3,\ldots\}$. Then A_2 is a difference basis but it is not sparse since

$$\lim_{x \to \infty} \frac{A_2(x)}{\left(\frac{x}{\log x}\right)} = \lim_{x \to \infty} \log x = +\infty$$

c) Let $A_3 = \{2, 2^2, 2^3, \ldots\}$. Then A_3 is not a difference basis but it is sparse since for an arbitrary $x > 0$ we have

$$A_3(x) = \left[\frac{\log x}{\log 2}\right] \leqq \frac{\log x}{\log 2},$$

and hence $\lim_{x \to \infty} \frac{A_3(x)}{\left(\frac{x}{\log x}\right)} = 0$.

d) Let $A_4 = \{2, 4, \ldots, 2k, \ldots\}$. Then A_4 obviously fails to be a difference basis. Further let x be an arbitrary positive integer.

Then $\frac{x-1}{2} \leqq A_4(x) = \left[\frac{x}{2}\right] \leqq \frac{x}{2}$, and hence clearly

$$\lim_{x \to \infty} \frac{A_4(x)}{\left(\frac{x}{\log x}\right)} = +\infty$$

Thus A_4 is not sparse.

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