(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXV - 1971)

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE

MATHEMATICA

PUBL. XXV

1971

SLOVENSKÉ PEDAGOGICKÉ NAKLADATEĽSTVO
BRATISLAVA

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ON ALTITUDES OF TRIANGLES IN THE HERMITIAN PLANE

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All terms and notation in this paper have been introduced in the opening parts of the paper [2]. We only recall the most important of them.

Let T be a triangle of the Hermitian plane H_2 with the vertices A^o, A^1, A^2. The barycentric coordinates of points and vectors of the plane H_2 will refer to the triangle T. Any non-zero multiples of the barycentric coordinates of a point are called its homogeneous barycentric coordinates. Lines are expressed by homogeneous linear equations the coefficients of which are not equal. A point [vector] incides with the line [its direction] if and only if its barycentric coordinates satisfy the equation of the line. The line which coincides with two distinct points $A = [a_i]$, $B = [b_i]$ is given by:

$$\begin{vmatrix} z_o & z_1 & z_2 \\ a_o & a_1 & a_2 \\ b_o & b_1 & b_2 \end{vmatrix} = 0 \tag{1}$$

In the case $\sum_{i=0}^{2} b_i = 0$, the equation (1) represents a line which is incident with the point A and is parallel to the vector $\beta = [b_i]$.

The vectors $\delta_o, \delta_1, \delta_2$ defined as follows:

a) $\delta_o + \delta_1 + \delta_2 = 0$

b) for all $i \in N = \{0,1,2\}$ the vectors δ_j, δ_k form the biorthogonal basis associated with the basis $\langle \gamma_j = A^j - A^i$, $\gamma_k = A^k - A^i \rangle$, where $j,k \in N$ and $i \neq j \neq k \neq i$;

are called direction vectors of external normals of the triangle T.

By means of the vectors δ_i, $i \in N$ the so-called g-norms of the triangle T are defined

(2) $\quad g_{ij} = (\delta_i, \delta_j)$, where $i, j \in N$.

The numbers:

(3) $$e_{ij} = \frac{1}{3G}\left[3g_{ii} + 3g_{jj} + 4g_{ij} + 2g_{ji}\right],$$

where $G = \begin{vmatrix} g_{ii} & g_{ij} \\ g_{ji} & g_{jj} \end{vmatrix} \neq 0$ (independently on i and j)

and $i,j \in N$, $i \neq j$ are called e-norms of the triangle T. We put $e_{ii} = 0$, $i \in N$.

The scalar product of the vectors $\xi = [x_0, x_1, x_2]$, $\eta = [y_0, y_1, y_2]$ is given by the following relations:

(4) $$(\xi,\eta) = \frac{1}{G}\left[g_{jj}x_k\bar{y}_k - g_{jk}x_j\bar{y}_k - g_{kj}x_k\bar{y}_j + g_{kk}x_j\bar{y}_j\right]; \qquad j,k \in N,\ j \neq k$$

(5) $$(\xi,\eta) = -\frac{1}{2}\,[ex\bar{y}] = -\frac{1}{2}\sum_{i,j=0}^{2} e_{ij}x_i\bar{y}_j$$

Sometimes it is useful to evaluate the scalar product using the formula:

(6) $$(\xi,\tau) = \sum_{i=0}^{2} x_i\bar{t}_i, \quad \text{where} \quad \tau = \sum_{k=0}^{2} t_k\delta_k .$$

Likewise in [2] the coefficient

$$q = -\frac{(g_{ij} - g_{ji})^2}{4G} \qquad \text{(independently on } i,j,\ i \neq j)$$

will play an important part in this paper.

Let us return now to the subject of this paper. We shall begin with the C e v a s Theorem for triangles in the Hermitian plane.

Theorem 1: Let P^i, $i = 0,1,2$ be an arbitrary point of the line $\overline{A^jA^k}$; $j,k \in N$ and $i \neq j \neq k \neq i$, not identical neither with the point A^j nor with the point A^k. The equality:

$$(7) \qquad (A^1A^2P^0)(A^2A^0P^1)(A^0A^1P^2) = -1$$

is a necessary and sufficient condition for the lines $\overline{A^iP^i}$ $i = 0,1,2$ being either intersecting or parallel.

The theorem can be proved in the same manner as in the real Eucliden plane, if we define the affine ratio $\lambda = (ABC)$ of three collinear points A,B,C as follows

$$C - A = \lambda(C - B) .$$

Lemma 1: The numbers $v_i = \sum_{k=0}^{2} g_{ki}\bar{a}_k$, $i = 0,1,2$ are the barycentric coordinates of the normal vector to the line p expressed by the equation $\sum_{i=0}^{2} a_i z_i = 0$.

Proof: Since $h(g_{ij}) = 2$ and for all $i \in N$ $\sum_{k=0}^{2} g_{ki} = 0$, all solutions of the following system of homogeneous linear equations

$$\sum_{k=0}^{2} g_{ki} z_k = 0, \qquad i = 0,1,2$$

are multiples of the solution (1,1,1). From this fact follows that at least one of the numbers v_i is not zero.

We can take the numbers v_i for the barycentric coordinates of a vector if the sum of them is zero:

$$\sum_i v_i = \sum_{i,k} g_{ki}\bar{a}_k = \sum_k \left[\sum_i g_{ki}\right] \bar{a}_k = 0 .$$

It remains to prove now that the vector with barycentric coordinates v_i is perpendicular to every direction vector $\xi = [z_i]$ of the line p, that is $[ev\bar{z}] = 0$.

$$[ev\bar{z}] = \sum_{i,j} e_{ij} v_i \bar{z}_j = \sum_{i,j,k} e_{ij} g_{ki} \bar{a}_k \bar{z}_j = \sum_{j,k} \left\{ \sum_i e_{ij} g_{ki} \right\} \bar{a}_k \bar{z}_j =$$

$$= \sum_{j,k} \left\{ -2\, \delta_{jk} - g_{k3} \right\} \bar{a}_k \bar{z}_j = -2 \overline{\sum_j a_j z_j} - \sum_j \bar{z}_j \sum_k g_{k3} \bar{a}_k = 0$$

Remark: The numbers g_{i3}, $i \in N$ have been introduced in the relations (10) and (11) of the paper [2].

Theorem 2: The altitudes of the triangle T intersect in a unique point if and only if for each pair of indices $i, j \in N$ the number g_{ij} is real.

Proof: On the basis of Lemma 1 the coordinates of the normal vector to the line A^jA^k are $[x_i = g_{ii},\ x_j = g_{ij},\ x_k = g_{ik}]$, thus the altitude v_i of the triangle T which passes through the vertex A^i has an equation:

$$\begin{vmatrix} z_i & z_j & z_k \\ 1 & 0 & 0 \\ g_{ii} & g_{ij} & g_{ik} \end{vmatrix} = 0, \quad \text{i.e.} \quad g_{ik} z_j - g_{ij} z_k = 0 .$$

As the point $P^i = v_i \cap \overline{A^jA^k}$ has the homogeneous barycentric coordinates $y_i = 0$, $y_j = g_{ij}$, $y_k = g_{ik}$, there $(A^iA^kP^i) = -\dfrac{g_{ik}}{g_{ij}}$ holds.

Since the altitudes of a triangle can not be mutually parallel, they have, in accordance with Theorem 1, one common point if and only if the condition (7) is fulfilled. When the above stated results are substituted in (7) we have after reduction the equality:

$$g_{01} g_{12} g_{20} = g_{02} g_{21} g_{10} ,$$

which is equivalent to:

$$(g_{ij} - g_{ji}) \begin{vmatrix} g_{ii} & g_{ij} \\ g_{ji} & g_{jj} \end{vmatrix} = 0, \text{ and } \bar{g}_{ij} = g_{ij} \text{ respectively.}$$

R e m a r k: Since the g-norms of the triangle T are real if and only if its e-norms are real the Theorem 2 remains true if we replace g_{ij} by e_{ij}.

Further we shall take into consideration only the triangles the altitudes of which do not intersect in a unique point, that is, they form the triangle ${}^{(1)}T$ with the vertices: ${}^{(1)}A^0 = v_1 \cap v_2$, ${}^{(1)}A^1 = v_0 \cap v_2$, ${}^{(1)}A^2 = v_0 \cap v_1$. These triangles fulfil the condition: $g_{ji} \neq g_{ij}$ ($g_{ij} \neq \bar{g}_{ij}$ respectively) for each couple of different indices $i,j \in N$. Unless being told otherwise we shall assume further that $i,j,k \in N$ and $i \neq j \neq k \neq i$.

From the calculations in the proof of Theorem 2 it follows that the altitudes v_j, v_k have the equations:

$$g_{jk}z_i - g_{ji}z_k = 0, \qquad g_{kj}z_i - g_{ki}z_j = 0 .$$

By solving them we obtain the homogeneous barycentric coordinates of the point ${}^{(1)}A^i$: $[x_i = g_{ji}g_{ki},\ \ x_j = g_{ji}g_{kj},\ \ x_k = g_{ki}g_{jk}]$. The sum of the coordinates is $g_{ji}(g_{ki} + g_{kj}) + g_{ki}g_{jk} = -g_{ji}g_{kk} + g_{ki}g_{jk} = -(g_{ji}+g_{jk})g_{kk} + (g_{ki}+g_{kk})g_{jk} = g_{jj}g_{kk} - g_{kj}g_{jk} = G$, thus the j-th [i-th] non-homogeneous barycentric coordinate of the point ${}^{(1)}A^i$ is given by the relations:

$$(8) \qquad {}^{(1)}a^i_j = \frac{1}{G} g_{ji}g_{kj} , \qquad \left[{}^{(1)}a^i_i = \frac{1}{G} g_{ji}g_{ki}\right] .$$

L e m m a 2: The g-norms of the triangle ${}^{(1)}T$ are given by the relations:

$$(9) \qquad {}^{(1)}g_{ij} = \frac{1}{h} g_{ji}$$

where $h = 4q$ and i,j is an arbitrary pair of indices from N.

P r o o f: The direction vectors of the external normals of the triangle ${}^{(1)}T$ are defined by the following relations:

$$\Delta_i = \frac{1}{g_{ij} - g_{ji}} \left[g_{ki}\delta_j - g_{ji}\delta_k\right] , \qquad i = 0,1,2$$

since

1) $\sum_{i=0}^{2}\Delta_i = \sum_{i=0}^{2}\delta_i = 0$

2) $$\left({}^{(1)}A^i - {}^{(1)}A^j, \Delta_i\right) = \frac{1}{G} g_{kj}(g_{ji}-g_{ij}) \frac{g_{ik}}{g_{ji}-g_{ij}} - \frac{1}{G}(g_{ki}g_{jk} - g_{kj}g_{ik}) \frac{g_{ij}}{g_{ji}-g_{ij}} = \frac{1}{G}\left[g_{kj}g_{ik} - g_{ij}g_{kk}\right] = 1,$$

3) $$\left({}^{(1)}A^k - {}^{(1)}A^j, \Delta_i\right) = \frac{1}{G} g_{ij}(g_{jk}-g_{kj}) \frac{g_{ik}}{g_{ji}-g_{ij}} - \frac{1}{G} g_{ik}(g_{jk} - g_{ki}) \frac{g_{ij}}{g_{ji}-g_{ij}} = 0.$$

Then

$$ {}^{(1)}g_{ij} = (\Delta_i, \Delta_j) = \frac{1}{(g_{ij}-g_{ji})^2}(g_{ki}\delta_j - g_{ji}\delta_k, g_{kj}\delta_i - g_{ij}\delta_k) = $$

$$ = \frac{g_{ji}}{(g_{ij}-g_{ji})^2}(-g_{jk}g_{ki}+g_{ji}g_{kk}) = -\frac{G}{(g_{ij}-g_{ji})^2} g_{ji} = \frac{1}{4q} g_{ji}. $$

<u>Corollary 1:</u> The e-norms of the triangles T, ${}^{(1)}T$ are related by

(10) $\quad {}^{(1)}e_{ij} = he_{ji}$

for all pairs of indices $i,j \in N$.

<u>Corollary 2:</u>

(11) $$ {}^{(1)}h = 4\,{}^{(1)}q = -\frac{\left[{}^{(1)}g_{ij} - {}^{(1)}g_{ji}\right]^2}{{}^{(1)}G} = -\frac{\left[g_{ij}-g_{ji}\right]^2}{G} = 4q = h, $$

where

$$ {}^{(1)}G = \begin{vmatrix} {}^{(1)}g_{ii} & {}^{(1)}g_{ij} \\ {}^{(1)}g_{ji} & {}^{(1)}g_{jj} \end{vmatrix} . $$

From the relations (9) and Theorem 2 it follows that the altitudes of the triangle ${}^{(1)}T$ form a triangle ${}^{(2)}T$, g-norms of which are given by the relations:

$$ {}^{(2)}g_{ij} = \frac{1}{h^2} g_{ij} . $$

Repeating this consideration for the triangle ${}^{(2)}T$, we obtain a triangle ${}^{(3)}T$ and so on. The n-th term of the sequence $\{{}^{(n)}T\}_{n=0}^{\infty}$ constructed in such a way is formed by the altitudes of the previous term and its norms (e and g) are given by the relations:

$$ (12) \quad \begin{aligned} {}^{(n)}e_{ij} &= h^n e_{ij}, \quad {}^{(n)}g_{ij} = \frac{1}{h^n} g_{ij}, \quad \text{for all n even,} \\ {}^{(n)}e_{ij} &= h^n e_{ji}, \quad {}^{(n)}g_{ij} = \frac{1}{h^n} g_{ji}, \quad \text{for all n odd;} \end{aligned} $$

it follows that ${}^{(n)}G = \dfrac{1}{h^{2n}} G$.

From the relations (4) and (12) we have

$$ (13) \quad \rho^2\left({}^{(n)}A^i, {}^{(n)}A^j\right) = \frac{1}{{}^{(n)}G} {}^{(n)}g_{kk} = h^n \frac{1}{G} g_{kk} = h^n \rho^2(A^i, A^j) $$

for all natural numbers n and

$$ (14) \quad \left({}^{(n)}A^j - {}^{(n)}A^i, {}^{(n)}A^k - {}^{(n)}A^i\right) = -\frac{1}{{}^{(n)}G} {}^{(n)}g_{jk} = h^n \left[-\frac{1}{G} g_{jk}\right] = $$

$$ = h^n (A^j - A^i, A^k - A^i) $$

for all even.

From these relations it follows that the triangles ${}^{(2n)}T$, $n=1,2,3,\ldots$ are similar to the triangle T. One can also easily prove that the terms of the sequence $\left\{{}^{(n)}T\right\}_{n=0}^{\infty}$ with odd indices are mutually similar.

Moreover, from (13) and (14) also follows that if $0<h<1$, then the lenghts of the edges of the triangles ${}^{(n)}T$, $n = 0, 1, 2, \ldots$ converge to zero, and if $h>1$ they tend to ∞. In the case when $h = 1$, the mentioned sequences are stationary and for all n even the triangles ${}^{(n)}T$ are congruent to the triangle T.

In the following consideration we shall express the coordinates of the vertices of the triangle ${}^{(n)}T$ by means of the numbers g_{ij}. Let us remark first that the relation between two systems of coordinates with respect to T and ${}^{(n)}T$ is given by the equations of the form:

$$(15) \qquad x_j = \sum_{\alpha=0}^{2} {}^{(n)}a_j^{\alpha} y_{\alpha}\,, \qquad j = 0, 1, 2$$

where (x_j) denote the barycentric coordinates of a point with respect to T and (y_j) the barycentric coordinates of the same point with respect to ${}^{(n)}T$.

Since ${}^{(n+1)}T$ is formed by the altitudes of the triangle ${}^{(n)}T$, the barycentric coordinates of its vertices with respect to ${}^{(n)}T$ are given by the relations:

$$(16) \qquad {}^{(n+1)}b_j^i = \frac{1}{{}^{(n)}G}\, {}^{(n)}g_{ji}\, {}^{(n)}g_{kj} = \begin{cases} \dfrac{1}{G}\, g_{ji} g_{kj} = {}^{(1)}a_j^i & \text{for all n even,} \\[2ex] \dfrac{1}{G}\, g_{ij} g_{jk} = \overline{{}^{(1)}a_j^i} & \text{for all n odd.} \end{cases}$$

The resulting equality holds obviously for $j = i$, too.

If we put $n = 1$, $y_{\alpha} = {}^{(2)}b_{\alpha}^i = \overline{{}^{(1)}a_{\alpha}^i}$, $\alpha = 0,1,2$ in (15) and express ${}^{(1)}a_j^i$ according to (8), we obtain:

$$ {}^{(2)}a^i_j = \sum_{\alpha=0}^{2} {}^{(1)}a^\alpha_j \; \overline{{}^{(1)}a^i_\alpha} = \frac{1}{G^2}\left[g_{ji}g_{kj}g_{ij}g_{ik} + g_{ij}g_{kj}g_{ij}g_{jk} + \right. $$

$$ \left. + g_{jk}g_{ij}g_{ik}g_{kj}\right] = \frac{1}{G^2} g_{ij}g_{kj}G = {}^{(1)}a^j_j \, , $$

$$ {}^{(2)}a^i_i = \frac{1}{G^2}\left[g_{ji}g_{ki}g_{ij}g_{ik} + g_{ij}g_{ki}g_{ij}g_{jk} + g_{ik}g_{ji}g_{ik}g_{kj}\right] = $$

$$ = \frac{1}{G^2}\left[g_{ii}g_{ij}g_{ji}^2 + 2g_{ij}^2g_{ji}^2 + g_{ii}g_{ij}^2g_{jj} - g_{ij}^3g_{ji} - \right. $$

$$ \left. - g_{ii}^2g_{ji}g_{jj} - 2g_{ii}g_{ij}g_{jj}\right] = \frac{1}{G}\left[g_{ij}^2 - 2g_{ij}g_{ji} - g_{ii}g_{ji}\right] = $$

$$ = \frac{1}{G}(g_{ij}-g_{ji})^2 + \frac{1}{G}\left[-g_{ii}g_{ji} - g_{ji}^2\right] = -h + {}^{(1)}a^i_i \, . $$

The obtained results may be written in the unified form:

$$ (17) \qquad {}^{(2)}a^i_j = -h\delta_{ij} + {}^{(1)}a^j_j $$

where i,j are two arbitrary elements of N and δ_{ij} is the Kronecker delta.

Then the following relations:

$$ (18) \qquad \begin{aligned} {}^{(3)}a^i_j &= \sum_{\alpha=0}^{2} {}^{(2)}a^\alpha_j \; {}^{(3)}b^i_\alpha = \sum_{\alpha=0}^{2}\left[-h\delta_{j\alpha} + {}^{(1)}a^j_j\right]{}^{(1)}a^i_\alpha = -h\,{}^{(1)}a^i_j + {}^{(1)}a^j_j \\ {}^{(4)}a^i_j &= \sum_{\alpha=0}^{2} {}^{(3)}a^\alpha_j \; {}^{(4)}b^i_\alpha = \sum_{\alpha=0}^{2}\left[-h\,{}^{(1)}a^\alpha_j + {}^{(1)}a^j_j\right]\overline{{}^{(1)}a^i_\alpha} = \\ &= -h\,{}^{(2)}a^i_j + {}^{(1)}a^j_j = h^2\delta_{ij} + (1-h)\,{}^{(1)}a^j_j = h\delta_{ij} + (1-h)\,{}^{(2)}a^i_j \end{aligned} $$

hold also for all pairs of indices $i,j \in N$.

L e m m a 3: For any natural number n and an arbitrary pair of indices $i,j \in N$

$$(19) \qquad {}^{(2n-1)}a^i_j = (-h)^{n-1}\,{}^{(1)}a^i_j + \frac{1-(-h)^{n-1}}{1+h}\,{}^{(1)}a^j_j \quad \text{holds.}$$

P r o o f: The validity of (19) is obvious for n = 1,2. Assuming that the relation (19) holds for a natural number n it follows that the barycentric coordinates of vertices of the triangle ${}^{(2n+1)}T$ with respect to ${}^{(2)}T$ are given by the relations:

$${}^{(2n+1)}c^i_j = (-h)^{n-1}\,\frac{1}{{}^{(2)}G}\,{}^{(2)}g_{ji}\,{}^{(2)}g_{kj} + \frac{1-(-h)^{n-1}}{1+h}\cdot\frac{1}{{}^{(2)}G}\cdot{}^{(2)}g_{ij}\,{}^{(2)}g_{kj} =$$

$$= (-h)^{n-1}\,{}^{(1)}a^i_j + \frac{1-(-h)^{n-1}}{1+h}\,{}^{(1)}a^j_j$$

Then we have:

$${}^{(2n+1)}a^i_j = \sum_{\alpha=0}^{2} {}^{(2)}a^\alpha_j \left[(-h)^{n-1}\,{}^{(1)}a^i_\alpha + \frac{1-(-h)^{n-1}}{1+h}\,{}^{(1)}a^\alpha_\alpha\right] =$$

$$= (-h)^{n-1}\,{}^{(3)}a^i_j + \frac{1-(-h)^{n-1}}{1+h}\sum_{\alpha=0}^{2}\left[-h\delta_{j\alpha} + {}^{(1)}a^j_j\right]{}^{(1)}a^\alpha_\alpha =$$

$$= (-h)^{n-1}\,{}^{(3)}a^i_j + \frac{1-(-h)^{n-1}}{1+h}\,{}^{(1)}a^j_j\left[-h + \sum_{\alpha=0}^{2}\,{}^{(1)}a^\alpha_\alpha\right] =$$

$$= (-h)^{n-1}\left[-h\,{}^{(1)}a^i_j + {}^{(1)}a^j_j\right] + \frac{1-(-h)^{n-1}}{1+h}\,{}^{(1)}a^j_j =$$

$$= (-h)^{n}\,{}^{(1)}a^i_j + \frac{1-(-h)^{n}}{1+h}\,{}^{(1)}a^j_j\,.$$

Corollary 1:

$$(20)\qquad {}^{(2n-1)}a^i_j = h\,\frac{1-(-h)^{n-2}}{1+h}\,{}^{(1)}a^i_j + \frac{1-(-h)^{n-1}}{1+h}\,{}^{(3)}a^i_j\,.$$

Corollary 2:

$$(21)\qquad \begin{aligned} {}^{(2n)}a^i_j &= (-h)^{n-1}\,{}^{(2)}a^i_j + \frac{1-(-h)^{n-1}}{1+h}\,{}^{(1)}a^j_j = \\ &= (-h)^n\,\delta_{ij} + \frac{1-(-h)^n}{1+h}\,{}^{(1)}a^j_j \end{aligned}$$

$$(22)\qquad \begin{aligned} {}^{(2n)}a^i_j &= h\,\frac{1-(-h)^{n-2}}{1+h}\,{}^{(2)}a^i_j + \frac{1-(-h)^{n-1}}{1+h}\,{}^{(4)}a^i_j = \\ &= h\,\frac{1-(-h)^{n-1}}{1+h}\,\delta_{ij} + \frac{1-(-h)^n}{1+h}\,{}^{(2)}a^i_j\,. \end{aligned}$$

Proof: It is sufficient to express ${}^{(2n-1)}a^{\alpha}_j$ in ${}^{(2n)}a^i_j = \sum_{\alpha=0}^{2} {}^{(2n-1)}a^{\alpha}_j\,\overline{{}^{(1)}a^i_{\alpha}}$ using (19) and (20) respectively. From the relations introduced above it follows that if $h = 1$ for the triangle T then the sequence $\left\{{}^{(n)}T\right\}_{n=0}^{\infty}$ contains only four different terms. Hence in the Hermitian plane there exist quadruples of triangles which are closed with respect to the operation of forming triangles of the altitudes.

Theorem 3: For each fixed index $i \in N$ all the points ${}^{(2n)}A^i$, $n = 0,1,2,\ldots$ are collinear. The lines through these points have the common point H with the barycentric coordinates:

$$(23)\qquad \left[\frac{g_{10}g_{20}}{(1+h)G}\,;\ \frac{g_{01}g_{21}}{(1+h)G}\,;\ \frac{g_{02}g_{12}}{(1+h)G}\right].$$

The line through the point H and perpendicular to the line $p_i = \overline{HA^i}$ contains all the points: ${}^{(2n-1)}A^i$, $n = 1,2,3,\ldots$

P r o o f: From the definition of H and its barycentric coordinates follows that:

$$H = \sum_{j=0}^{2} \frac{g_{ij}g_{kj}}{(1+h)G} A^j = \frac{1}{1+h} \sum_{j=0}^{2} {}^{(1)}a^j_j \, A^j \, . \tag{24}$$

Then:

$$^{(2n)}A^i = \sum_{j=0}^{2} {}^{(2n)}a^i_j \, A^j = \sum_{j=0}^{2} \left[(-h)^n \delta_{ij} + \frac{1-(-h)^n}{1+h} {}^{(1)}a^j_j\right] A^j =$$

$$= (-h)^n A^i + \left[1-(-h)^n\right] H = H + (-h)^n \left[A^i - H\right].$$

In a similar manner we find that:

$$^{(2n-1)}A^i = H + (-h)^{n-1} \left[{}^{(1)}A^i - H\right] .$$

It remains to prove that vectors $\xi^i = A^i - H$, $\tau^i = {}^{(1)}A^i - H$ are perpendicular.

From relations (4), (8), (23) and (24) it follows that:

$$(\xi^i, \tau^i) = \frac{1}{G}\left\{g_{jj}\, \xi^i_k \overline{\tau^i_k} - g_{jk} \xi^i_j \overline{\tau^i_k} - g_{kj} \xi^i_k \overline{\tau^i_j} + g_{kk}\, \xi^i_j \overline{\tau^i_j}\right\} =$$

$$= \frac{1}{(1+h)G}\left\{-g_{jj}\,{}^{(1)}a^k_k\left[\overline{{}^{(1)}a^i_k} - \frac{1}{1+h}\overline{{}^{(1)}a^k_k}\right] + g_{jk}\,{}^{(1)}a^j_j\left[\overline{{}^{(1)}a^i_k} - \right.\right.$$

$$\left. - \frac{1}{1+h}\overline{{}^{(1)}a^k_k}\right] + g_{kj}\,{}^{(1)}a^k_k\left[\overline{{}^{(1)}a^i_j} - \frac{1}{1+h}\overline{{}^{(1)}a^j_j}\right] -$$

$$\left. - g_{kk}\,{}^{(1)}a^j_j\left[\overline{{}^{(1)}a^i_j} - \frac{1}{1+h}\overline{{}^{(1)}a^j_j}\right]\right\} = \frac{1}{(1+h)G^3} g_{jk}g_{kj}\left\{-g_{jj}g^2_{ik} + \right.$$

$$+ g_{ij}g_{ik}g_{kj} + g_{ik}g_{ij}g_{jk} - g_{kk}g^2_{ij} + \frac{1}{1+h}\left[g_{jj}g_{ik}g_{ki} - \right.$$

$$\left.\left. - g_{ij}g_{ki}g_{kj} - g_{ik}g_{ji}g_{jk} + g_{kk}g_{ij}g_{ji}\right]\right\} = \frac{1}{(1+h)G^3} g_{ii}g_{jk}g_{kj}\left\{-G + \right.$$

$$\left. + \frac{1}{1+h}\left[g_{jj}g_{kk} + g_{jk}g_{kj} - g^2_{jk} - g^2_{kj}\right]\right\} = 0 \, .$$

C o r o l l a r y: If $h<1$ in the triangle T then for each $i\in N$ the sequence of the points $\left\{{}^{(n)}A^i\right\}_{n=0}^{\infty}$ converges to the point H.

T h e o r e m 4: Besides the sequence $\left\{{}^{(n)}T\right\}_{n=0}^{\infty}$ we can associate with a triangle T another sequence $\left\{{}^{(-n)}T\right\}_{n=0}^{\infty}$ of triangles such that the edges of its arbitrary term are the altitudes of the following term; especialy the edges of the triangle $T = {}^{(o)}T$ are the altitudes of the triangle ${}^{(-1)}T$. If in the triangle T is $h>1$, then for each $i\in N$ the sequence of the points $\left\{{}^{(-n)}A^i\right\}_{n=0}^{\infty}$ converges to the point H.

P r o o f: Let us consider the sequence of triangles the vertices of which are defined by the relations:

$$
\begin{aligned}
{}^{(-n)}A^i &= H + (-h)^{-\frac{n}{2}}\left[A^i-H\right] \quad \text{for } i\in N \text{ and all n even} \\
{}^{(-n)}A^i &= H + (-h)^{-\frac{n+1}{2}}\left[{}^{(1)}A^i - H\right] \quad \text{for all n odd.}
\end{aligned} \tag{25}
$$

If in H_2 there exists a triangle ${}^{*}T$ whose altitudes are the edges of the triangle T then, making use of the relations from the proof of the previous theorem, we have that its vertices ${}^{*}A^i$ are related with vertices of the triangle ${}^{(1)}T$ as follows:

$${}^{(1)}A^i = H + (-h)\left[{}^{*}A^i-H\right] \quad \text{and} \quad {}^{*}A^i = H - \frac{1}{h}\left[{}^{(1)}A^i-H\right]$$ respectively. Hence ${}^{*}T = {}^{(-1)}T$.

On the basis of the relations (24) and (25) we can easily verify that for all n even the following relation holds:

$${}^{(-n-1)}A^k - {}^{(-n-1)}A^j = (-h)^{-\frac{n+2}{2}}\left[{}^{(1)}A^k - {}^{(1)}A^j\right],$$

$${}^{(-n)}A^k - {}^{(-n)}A^j = (-h)^{-\frac{n}{2}}\left[A^k - A^j\right],$$

$${}^{(-n-1)}A^i = \frac{1}{h}\left[\left({}^{(1)}a^j_j - {}^{(1)}a^i_j\right){}^{(-n)}A^j + \left({}^{(1)}a^k_k - {}^{(1)}a^i_k\right){}^{(-n)}A^k\right].$$

From the first two equalities it follows that the vectors ${}^{(-n-1)}A^k - {}^{(-n-1)}A^j$, ${}^{(-n)}A^k - {}^{(-n)}A^j$ are mutually perpendicular, from the third we have that the points ${}^{(-n)}A^j$, ${}^{(-n)}A^k$, ${}^{(-n-1)}A^i$ are collinear. This means, indeed, that the altitudes of the triangle ${}^{(-n-1)}T$ form the triangle ${}^{(-n)}T$; especially the altitudes of the triangle ${}^{(-1)}T$ form the triangle T. The validity of this assertion for n odd will be verified by using the relations:

$$ {}^{(-n-1)}A^k - {}^{(-n-1)}A^j = (-h)^{-\frac{n+1}{2}} \left[A^k - A^j\right] , $$

$$ {}^{(-n)}A^k - {}^{(-n)}A^j = (-h)^{-\frac{n+1}{2}} \left[{}^{(1)}A^k - {}^{(1)}A^j\right] , $$

$$ {}^{(-n-1)}A^i = \frac{1}{h} \left[\left(\overline{{}^{(1)}a^j_j} - \overline{{}^{(1)}a^i_j}\right) {}^{(-n)}A^j + \left(\overline{{}^{(1)}a^k_k} - \overline{{}^{(1)}a^i_k}\right) {}^{(-n)}A^k\right] . $$

The validity of the last part of the theorem follows from the relations (25).

The obtained results can be summarized as follows: With every triangle of the Hermitian plane we can associate the point H with the barycentric coordinates (23). If $h = 0$, that is if the g-norms of the triangle T are real, the point H is the intersection point of the altitudes of the triangle T. If $h \neq 0$ the triangle $T = {}^{(o)}T$ determines a sequence of triangles $\left\{{}^{(n)}T\right\}_{-\infty}^{\infty}$ of such a property that its every term is formed by the altitudes of previous term. If $h < 1$ the vertices of the triangles ${}^{(n)}T$ converge to the point H for $n \to \infty$; if $h > 1$, the vertices of ${}^{(n)}T$ converge to H for $n \to -\infty$. In the case $h = 1$ the above mentioned sequence consists of exactly four different terms.

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Received: October 7, 1969

(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXV - 1971)

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXV - 1971

COSET DECOMPOSITION OF THE ABELIAN GROUPS

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Every group in this paper is considered to be Abelian. The additive group of integers is denoted by Z and the subgroup of Z consisting of all multiples of a number $n \in Z$ is denoted by nZ.

Let $(G, +)$ be a group with the subgroup H and let $a \in G$. The coset $\{a + h \mid h \in H\}$ will be denoted by (a, H). The system

$$(*) \qquad \{(a_i, G_i)\} \qquad i = 0, 1, \dots, k$$

is said to be a coset decomposition of the group $(G, +)$ if it is a disjoint covering of G, i.e. if the following hold:

1. $\bigcup_{i=0}^{k} (a_i, G_i) = G$ and

2. $i \neq j \Longrightarrow (a_i, G_i) \cap (a_j, G_j) = \emptyset$.

(Obviously we can suppose that $a_0 = e$ is the neutral element of the group G).

J. Mycielski made the following conjecture:

Let $(*)$ be a coset decomposition of the group G and let card $G/G_0 = n$ be a finite number. If

$$n = \prod_{t=1}^{r} p_t^{\lambda_t}$$

is the canonical form of n, then

$$k \geq \sum_{t=1}^{r} \lambda_t (p_t - 1).$$

In this article Mycielski's conjecture is proved for a particular case.

Let $a \neq b$ be two arbitrary elements of G. By the symbol $T(a, b)$ we shall denote the coset (a, A), where A is the subgroup of G generated by the element $b - a$, i.e.

$$T(a, b) = \{xa + yb \mid x + y = 1,\ x, y \in Z\}.$$

Two elements $a \neq b$ are said to be unrelated with respect to a subgroup H (of the group G) if

1. $T(a, b) \not\subset H$,

2. $T(a, b) \cap H \neq \emptyset$.

<u>Lemma.</u> Let G/H be the factorgroup of G related to the subgroup H and $f\colon G \to Z$ such a homomorphism that $\operatorname{Ker} f \subset H$ and $f(H) = nZ$. Two elements $a \neq b$ from G are unrelated with respect to the subgroup H if $f(a)$, $f(b)$ are unrelated with respect to the subgroup $nZ \subset Z$.

<u>Proof.</u> If $f(a)$, $f(b)$ are unrelated with respect to nZ, then the system of diophantine equations $x f(a) + y f(b) = nZ$, $x + y = 1$ is solvable. Since $f(H) = nZ$, there exists such an element $h \in H$ that $f(h) = n$. Therefore $f(xa + yb - zh) = 0$ and hence $xa + yb - zh \in \operatorname{Ker} f \in H$, i.e. $xa + yb \in H$. QED.

<u>Theorem 1.</u> Let $\operatorname{card} G/H = n = \prod_{t=1}^{r} p_t^{\lambda_t}$ and let $f\colon G \to Z$ be a surjective homomorphism with the properties: $\operatorname{Ker} f \subset H$ and $f(H) = nZ$. Let us denote

$$K = \sum_{t=1}^{r} \lambda_t (p_t - 1).$$

Then there exists a set of $K + 1$ distinct elements of G

$d_0, d_1, \ldots, d_K$

with the property: $d_0 \in H$, $d_1, \ldots, d_k \notin H$ and d_i, d_j are unrelated with respect to H for all $i, j, = 0, 1, \ldots, k$, $i \neq j$.

P r o o f. We shall consider the following set of integers:

$$(2) \qquad c_t q_t p_t^{\alpha_t}, \quad t = 1, \ldots, r, \quad q_t = n/p_t^{\lambda_t}, \quad c_t = 0, 1, \ldots p_t - 1$$

and $\alpha_t = 0, 1, \ldots, \lambda_t - 1$.

It may be shown that the set (2) contains exactly $1 + \sum_{t=1}^{r} \lambda_t(p_t - 1)$ such distinct elements that only one of them - namely zero - belongs to nZ.

First we observe that arbitrary two elements $a \neq b$ from (2) are unrelated with respect to nZ, i.e. the system of diophantine equations $xa + yb = nz$, $x + y = 1$ is solvable. (The proof of this is only a modification on the proof of Theorem I from [2].) Now, the set (1) may be chosen so, that $f(d_0), \ldots, f(d_K)$ is the set (2) and $f(d_0) = 0$. The remainder of the proof follows immediately from the Lemma.

T h e o r e m 2. Let $G = \{(a_i, G_i)\}$ $i = 0, 1, \ldots, k$ be a coset decomposition of the group G. Let H be one of the subgroups $G_0, \ldots, G_k$ and card $G/H = n = \prod_{t=1} p_t^{\lambda_t}$ is a finite integer. If there exists a surjective homomorphism $f: G \to Z$ with the properties Ker $f \subset H$, $f(H) = nZ$, then

$$k \geq \sum_{t=1}^{r} \lambda_t (p_t - 1)$$

P r o o f. First we shall suppose $H = G_0$. According to the Theorem 1, the elements (1) belong to distinct cosets of the decomposition (a_i, G_i). In fact, let us suppose that two elements d and d' from the set (1) belong to the same coset (a_i, G_i); then $d - d' \in G_i$ and hence $T(d, d') \subset (a_i, G_i)$. The last inclusion is contractible with the fact that d and d' are unrelated with respect to the subgroup H ($i = 0$ is contractible with $T(d, d') \not\subset H$ and $i \neq 0$ with $T(d, d') \cap H \neq \emptyset$).

Now, let $H = G_s$, $s \neq 0$ and h be an arbitrary (but fixed) element of H. The low of composition

$$G * G \to G, \quad (u, v) \to u * v = u + v - h$$

turns G into an Abelian group with h as its neutral element and $2h - u$ as the inverse to u. A substet $G' \subset G$ is a subgroup of $(G, +)$ if and only if the set $h + G'$ is a subgroup of $(G, *)$. Therefore the subset $G' \subset G$ is a coset of $(G, +)$ if and only if it is a coset of $(G, *)$. Hence $G = \{(a_i, G_i)\}$ (see p.1) is a coset decomposition of the group $(G, *)$. Obviously $H = G_s$ is a subgroup of $(G, *)$. The mapping $g: G \to Z$, $a \to f(a) - f(h)$ is a surjective homomorphism of the group $(G, *)$ onto Z. Moreover $\operatorname{Ker} g \subset H$ and $g(H) = nZ$.

Making use of the first part of this proof to the objects $(G, *)$, H and g, the Theorem 2 will be completely proved.

<u>E x a m p l e.</u> Let $G = Z \times Z$ be the cartesian product of two groups $(Z, +)$ an let H be a subgroup of G generated by the elements $\bar{u} = (u_1, u_2)$, $\bar{v} = (v_1, v_2)$ with the property

$$[\bar{u}, \bar{v}] = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \neq 0.$$

It is easy to show, that the mapping

$$f : G \to Z, \quad x \to [\bar{u}, \bar{x}]$$

is a group-homomorphism. Let us consider u_1, u_2 to be coprime. Then the diophantine equation $f(x) = u_1 x_2 - u_2 x_1 = z$ is solvable for any $z \in Z$ and hence f is surjective. On the other hand from the condition $(u_1, u_2) = 1$ it follows that every integer solution of the equation $f(\bar{y}) = 0$ may be written in the form $\bar{y} = p\,\bar{u}$, where $p \in Z$. Therefore $\operatorname{Ker} f \subset H$. Denoting the absolute value of $f(\bar{v})$ by n we obtain $f(H) = nZ$ and hence $\operatorname{card} G/H = n$.

Now the Theorem 2 may be applied.

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Received: November 5, 1969

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(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXV - 1971)

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXV - 1971

A NOTE ON HAHN DECOMPOSITION

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The HAHN decomposition theorem is a well known result of the measure theory ([1], Theorem 29A). The purpose of this note is to formulate and prove the decomposition theorem without introducing any set function, in terms of certain subfamilies of a given σ-ring. The idea of this proof is similar to the one used by T. NEUBRUNN in [2]. The HAHN decomposition of a signed measure space is then a simple consequence of the established result which on the other hand is shown to be in a sense more general.

1. In this note we shall consider a measurable space $(X, \mathcal{S})$ and subfamilies $\mathcal{P}$, $\mathcal{N}$, and $\mathcal{Z}$ of the σ-ring $\mathcal{S}$, having the following properties:

(i) $\mathcal{P} \cup \mathcal{N} = \mathcal{S}$

(ii) $\emptyset \in \mathcal{Z} \subset \mathcal{P} \cap \mathcal{N}$ and $\mathcal{Z}$ is closed under the formation of countable unions of pairwise disjoint sets

(iii) $\mathcal{P} - \mathcal{Z}$ and $\mathcal{N} - \mathcal{Z}$ are closed under the formation of countable unions of pairwise disjoint sets

(iv) For any $G \in \mathcal{Z}$, $E \in \mathcal{P}$, $F \in \mathcal{N}$ for which $G \cap E = \emptyset$ and $G \cap F = \emptyset$, it holds $G \cup E \in \mathcal{P}$ and $G \cup F \in \mathcal{N}$

(v) There is no uncountable family of pairwise disjoint sets in $\mathcal{N} - \mathcal{Z}$, and given any $N \in \mathcal{N} - \mathcal{Z}$, the family $\mathcal{P} - \mathcal{Z}$ contains no uncountable family of pairwise disjoint subsets of N.

We define the families
$$\mathcal{S}^+ = \{E \subset X;\ F \cap E \in \mathcal{P} \text{ for each } F \in \mathcal{S}\}$$
$$\mathcal{S}^- = \{E \subset X;\ F \cap E \in \mathcal{N} \text{ for each } F \in \mathcal{S}\}$$

Now the decomposition theorem is formulated as follows:

Theorem. There exist two disjoint sets A and B whose union is X such that $A \in \mathcal{S}^+$ and $B \in \mathcal{S}^-$.

Proof. Consider in $\mathcal{S}^- - \mathcal{Z}$ a maximal family $\mathcal{B} = \{B_\gamma\}_{\gamma \in \Gamma}$ of pairwise disjoint sets such that all $B_\gamma, \gamma \in \Gamma$ are in $\mathcal{S}$ and write $B = \bigcup_{\gamma \in \Gamma} B_\gamma$. We prove first that $B \in \mathcal{S}^-$. In case there is no set contained in both $\mathcal{S}$ and $\mathcal{S}^- - \mathcal{Z}$ we have $B = \emptyset \in \mathcal{S}^-$. Suppose then that $\mathcal{B} \neq \emptyset$. The sets B_γ being in both $\mathcal{S}$ and $\mathcal{S}^- - \mathcal{Z}$, are consequently all in $\mathcal{N} - \mathcal{Z}$ (since by the definition of $\mathcal{S}^-$ we have $B_\gamma \cap B_\gamma \in \mathcal{N}$) and therefore by (v) the family $\mathcal{B} = \{B_\gamma\}_{\gamma \in \Gamma}$ is countable and we can write $\mathcal{B} = \{B_n\}_{n=1}^{\infty}$. Now for any $F \in \mathcal{S}$ the set $F \cap B = F \cap \left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} (F \cap B_n)$ is a countable union of pairwise disjoint sets in $\mathcal{N}$ (since $B_n \in \mathcal{S}^-$) and therefore by (ii), (iii) and (iv) is itself in $\mathcal{N}$. By the definition of $\mathcal{S}^-$ this proves that $B \in \mathcal{S}^-$.

Let us consider now the set $A = X-B$. To complete the proof of the theorem it is sufficient to show that $A \in \mathcal{S}^+$. It is readily seen that for any $E \in \mathcal{S}$ the set $E \cap A = E \cap (X-B) = (E \cap X) - (E \cap B)$ being a difference of two sets in $\mathcal{S}$ is itself in $\mathcal{S}$. We have to prove it is in $\mathcal{P}$. Suppose, for the contrary, that there exists a set $E \in \mathcal{S}$ such that $E_0 = E \cap A \notin \mathcal{P}$ which immediately implies $E_0 \in \mathcal{N} - \mathcal{Z}$. However, from the maximality of the family $\mathcal{B}$ we conclude that $E_0 \notin \mathcal{S}^-$ and therefore there exists a set $F_0 \in \mathcal{S}$ such that $H = E_0 \cap F_0 \in \mathcal{P}\text{-}\mathcal{N}$ which gives at once $H \in \mathcal{P} - \mathcal{Z}$. We can consider now a maximal family $\mathcal{H}$ of pairwise disjoint sets $H_\lambda \subset E_0$ which are all in $\mathcal{P} - \mathcal{Z}$. By (v), since $E_0 \in \mathcal{N} - \mathcal{Z}$, the family $\mathcal{H}$ is countable, $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$ which in its turn implies that by (iii) the set $H = \bigcup_{n=1}^{\infty} H_n$ is in $\mathcal{P} - \mathcal{Z}$, too. Now denote $K = E_0 - H$. Evidently $K \in \mathcal{S}$ and $K \cap H = \emptyset$, so if K were in $\mathcal{P} - \mathcal{Z}$ or $\mathcal{Z}$, then by (iii) or (iv) respectively it would follow that $E_0 = K \cup H \in \mathcal{P}$ which contradicts our assumption, and so necessarily $K \in \mathcal{N} - \mathcal{Z}$. Due to the maximality of the family $\{H_n\}_{n=1}^{\infty}$ no subset of K is in $\mathcal{P} - \mathcal{Z}$. This implies that for any $E \in \mathcal{S}$ the intersection $E \cap K \notin \mathcal{P} - \mathcal{Z}$, nevertheless, K being in $\mathcal{S}$, it holds $E \cap K \in \mathcal{S}$. As a consequence, $E \cap K$ is in $\mathcal{N}$ and that implies $K \in \mathcal{S}^-$.

Bearing in mind that $K \in \mathcal{N} - \mathcal{Z}$ we see $K \in \mathcal{S}^- - \mathcal{Z}$. This contradicts the maximality of $\mathcal{B}$, since clearly $K \subset X - B$ implies $K \notin \mathcal{B}$. The last contradiction completes the proof of the theorem.

<u>Remark.</u> It is evident that the property (v) could have been replaced by the following:

(v′) There is no uncountable family of pairwise disjoint sets in $\mathcal{P} - \mathcal{Z}$, and given any $P \in \mathcal{P} - \mathcal{Z}$, the family $\mathcal{N} - \mathcal{Z}$ contains no uncountable family of pairwise disjoint subsets of P.

Assuming (i) through (iv) and (v′) and the original definitions of $\mathcal{S}^+$, $\mathcal{S}^+$, the theorem holds as well, since just some notations in its proof are to be mutually exchanged: $\mathcal{P}$ for $\mathcal{N}$, $\mathcal{S}^+$ for $\mathcal{S}^-$, A for B and vice versa.

2. The meaning of the last remark is better seen if we define the families $\mathcal{P}$, $\mathcal{N}$, $\mathcal{Z}$ by means of a given signed measure in order to show that the HAHN decomposition is a consequence of our result.

Suppose we are given a measurable space $(X, \mathcal{S})$ with a signed measure μ. It seems natural to denote

$$(1) \qquad \mathcal{P} = \{E \in \mathcal{S};\ \mu(E) \geq 0\}, \qquad \mathcal{N} = \{E \in \mathcal{S};\ \mu(E) \leq 0\},$$

$$\mathcal{Z} = \{E \in \mathcal{S};\ \mu(E) = 0\}$$

The families $\mathcal{P}$, $\mathcal{N}$, and $\mathcal{Z}$ thus defined have the properties (i) through (iv) and also (v) or (v′), and $\mathcal{S}^+$, $\mathcal{S}^-$ coincide with the families of all positive and negative sets respectively. In fact, (i) is trivial as μ is an extended real valued set function; (ii), (iii) and (iv) are implied by the σ-additivity of μ. In the definition of the signed measure (cf. [1], § 28) it is moreover required that μ assume at most one of the values $+\infty$ and $-\infty$. Suppose $\mu > -\infty$, we prove that then (v) is fulfilled. If there were an uncountable family $\mathcal{V}$ of pairwise disjoint sets in $\mathcal{N} - \mathcal{Z}$, i.e. having strictly negative signed measure values, then at least one of the families

$$\mathcal{V}^n = \left\{E \in \mathcal{V};\ \mu(E) < -\frac{1}{n}\right\}$$

defined for $n = 1, 2, \ldots$, say $\mathcal{V}^{n_0}$ would be uncountable and clearly for any sequence $\{V_i\}_{i=1}^{\infty} \subset \mathcal{V}^{n_0}$ we would get $\mu(\bigcup_{i=1}^{\infty} V_i) = -\infty$

which contradicts the assumption. Similarly, if a set $F \in \mathcal{N} - \mathcal{Z}$ (which evidently implies $-\infty < \mu(F) < 0$) had an uncountable family of pairwise disjoint subsets all being in $\mathcal{P} - \mathcal{Z}$, we could find a set $E \subset F$ such that $\mu(E) = +\infty$. This however is a contradiction to the Theorem 28A of [1] and thus (v) is demonstrated. In the case $\mu < +\infty$ the property (v') would be verified in the same way. As a consequence the considered decomposition exists in any signed measure space, or in other words, the HAHN theorem ([1], Theorem 29A) follows from the theorem established in this note.

3. To justify the axiomatic formulation of our hypotheses instead of assuming a signed measure, we give an example where all the properties (i) through (v) are fulfilled (and hence the decomposition theorem holds) and yet no set function can be defined to meet (1). Clearly the property (ii) is more general than the statement $\mathcal{Z} = \mathcal{P} \cap \mathcal{N}$ evidently implied by (1). And so it is sufficient to find such families $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{Z}$ that (i) through (v) hold but $\mathcal{Z} \neq \mathcal{P} \cap \mathcal{N}$. As an example, consider X to be the interval $\langle -1,1 \rangle$, $\mathcal{S}$ the σ-ring of all its subsets, $\mathcal{Z}$ the family of all sets in $\mathcal{S}$ which contain no rational number. Denote $\mathcal{P}'$ to be the family of all sets in $\mathcal{S}$ which contain at least one non-negative rational number and $\mathcal{N}'$ the family of all subsets of X containing at least one non - positive rational number. Let $\mathcal{P} = \mathcal{Z} \cup \mathcal{P}'$, $\mathcal{N} = \mathcal{Z} \cup \mathcal{N}'$, it is then easily seen that $\mathcal{P} - \mathcal{Z} = \mathcal{P}'$, $\mathcal{N} - \mathcal{Z} = \mathcal{N}'$ and the properties (i) through (iv) are readily verified. Given any family $\mathcal{F}$ of pairwise disjoint sets in $\mathcal{P}'$ or $\mathcal{N}'$, it cannot be uncountable as there exists an injection from $\mathcal{F}$ into the set of all rational numbers and therefore the property (v) holds as well. Since clearly $\mathcal{Z} \neq \mathcal{P} \cap \mathcal{N}$, there exists no set function related with $\mathcal{P}, \mathcal{N}, \mathcal{Z}$ by (1) and this shows that the axiomatic hypotheses in this note are more general than considering a signed measure function.

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Received: November, 25, 1969

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(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXV - 1971)

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXV - 1971

PLANARPUNKTE DER HYPERFLÄCHE $\sum c_j x_n^j = 0$

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Die Arbeit handelt von der Konfiguration der Planarpunkte [P-Punkte] der Hyperfläche $\sum c_j x_j^n = 0$ und von gewissen Quadriken, sowie auch von der Konfiguration der P-Punkte und gewissen Hyperebenen.

Im m-dimensionalen projektiven Raum S_m über dem Körper der komplexen Zahlen sei die Hyperfläche n-ten Grades mit der Gleichung

$$(1) \quad \sum_{j=1}^{m+1} c_j x_j^n = 0,$$

gegeben, wo $c_j \neq 0$ Konstanten und x_j homogene Koordinaten von Punkten sind; n ist eine natürliche Zahl, $n \geq 3$.

Mit Hilfe einer regulären linearen Transformation können wir die Gleichung (1) in die Form

$$(2) \quad \sum_{j=1}^{m+1} x_j^n = 0$$

bringen.

Die Hyperfläche (2) hat keine singulären Punkte. Die Schnittpunkte aller $\binom{m+1}{2}$ Kanten des Koordinatensimplexes mit der Hyperfläche (2) sind sog. planare Punkte der Hyperfläche (2). Wir werden diese kurz als P - P u n k t e bezeichnen.

Auf der Kante $O_j\, O_k$ ($j = 1,2, \ldots, m;\quad k = 2,3, \ldots, m+1;$ $j < k$) sind alle Koordinaten der P-Punkte gleich Null ausser x_j und x_k. Für diese Koordinaten gilt

$$(3)\qquad x_j = e^{[2u_{(j,k)}+1]\pi i/n},$$

$$x_k = 1,$$

$$u_{(j,k)} = 0,\ 1,\ \ldots,\ n-1.$$

Die P-Punkte auf der Kante O_jO_k bezeichnen wir $P_{u_{(j,k)}}$.

Wenn die Quadrik den Koordinatenpunkt O_1 nicht enthält, können wir die Gleichung in der Form

$$(4)\qquad \sum_{j=1}^{m+1} a_{jk}x_jx_k = 0, \qquad \text{wobei} \quad j \leq k, \quad a_{11} = 1$$

schreiben. Die Koeffizienten der Quadrik (4) sind ersichtlich unhomogen.

Wenn die Quadrik (4) durch die Punkte $P_{u_{(1,k)}}$ $P_{\bar{u}_{(1,k)}}$ der Kante $O_1\, O_k$ ($k = 2,3,\ldots,m+1;\quad u_{(1,k)} < \bar{u}_{(1,k)}$) gehen soll, dann gilt für den Koeffizienten a_{kk}

$$(5)\qquad a_{kk} = e^{2[u_{(1,k)} + \bar{u}_{(1,k)} + 1]\pi i/n}$$

<u>Bemerkung.</u> Der Koeffizient a_{1k} ist für unsere weiteren Erwägungen nicht notwendig.

Durch die m Paare der P-Punkte: $P_{u_{(1,k)}}$, $P_{\bar{u}_{(1,k)}}$ ist das $\binom{m}{2}$-dimensionale lineare System von Quadriken bestimmt ($k = 2,3, \ldots, m+1$).

Wenn der Punkt $P_{u_{(j,k)}}$ der Kante O_jO_k ($j = 2, 2,\ldots, m;\ k > j$) auf einer Quadrik des erwähnten $\binom{m}{2}$-dimensionalen linearen Systems von Quadriken liegt, dann ist

$$(6)\qquad a_{jk} = -e^{[2u_{(1,j)} + 2\bar{u}_{(1,j)} + 2u_{(j,k)} + 3]\,\pi i/n} - e^{[2u_{(1,k)} + 2\bar{u}_{(1,k)} - 2u_{(j,k)} + 1]\,\pi i/n}.$$

Durch die Punkte $P_{u_{(1,k)}}$, $P_{\bar{u}_{(1,k)}}$ (für $k = 2,\dots, m+1$, $u_{(1,k)} < \bar{u}_{(1,k)}$), $P_{u_{(j,k)}}$ ($j \geqq 2$; $k > j$) ist eine Quadrik bestimmt. Untersuchen wir unter welchen Voraussetzungen auf dieser Quadrik ein weiterer Punkt $P_{u_{(j,k)}}$ liegt, wobei $u_{(j,k)} < \bar{u}_{(j,k)}$, $j = 2,\dots,m$; $k > j$. Nach dem Einsetzen der Koordinaten des Punktes in die die Quadrik bestimmende quadratische Form erhalten wir die Zahl

$$(7)\qquad e^{2[u_{(1,j)} + \bar{u}_{(1,j)} + 2\bar{u}_{(j,k)} + 2]\pi i/n} + e^{2[u_{(1,k)} + \bar{u}_{(1,k)} + 1]\pi i/n} - e^{2[u_{(1,j)} + \bar{u}_{(1,j)} + u_{(j,k)} + \bar{u}_{(j,k)} + 2]\pi i/n} - e^{2[u_{(1,k)} + \bar{u}_{(1,k)} - u_{(j,k)} + \bar{u}_{(j,k)} + 1]\pi i/n}$$

Antwort auf die gegebene Frage gibt

<u>S a t z 1.</u> Die notwendige und hinreichende Bedingung dazu, dass $2.\binom{m+1}{2}$ verschiedene P-Punkte: $P_{u_{(j,k)}}$, $P_{\bar{u}_{(j,k)}}$, je zwei von jeder Kante des Koordinatensimplexes ($j = 1,2,\dots, m$; $k = 2,3,\dots, m+1$; $k > j$, $u_{(j,k)} < \bar{u}_{(j,k)}$) auf einer Quadrik liegen, ist das Erfülltsein der Beziehungen

$$(8)\qquad u_{(1,j)}+\bar{u}_{(1,j)}+u_{(j,k)}+\bar{u}_{(j,k)} \equiv u_{(1,k)} + \bar{u}_{(1,k)} - 1 \pmod{n}$$

für $j = 2,3,\dots, m$; $k = 3, \dots, m+1$, $k > j$.

<u>B e w e i s.</u> 1. Es gelte die Bedingung (8). Es genügt zu zeigen, dass die Zahl (7) gleich Null ist. Nach Einsetzen von (8) in (7) kann dies leicht festgestellt werden.

2. Es liegen die Punkte $P_{u_{(1,k)}}$, $P_{\bar{u}_{(1,j)}}$, $P_{u_{(1,k)}}$, $P_{\bar{u}_{(1,k)}}$, $P_{u_{(j,k)}}$, $P_{\bar{u}_{(j,k)}}$ auf einer Hyperquadrik. ($j = 2, \dots, m$; $k > j$).

Ersichtlich gilt:

$$(9) \qquad u_{(1,j)} + \bar{u}_{(1,j)} + u_{(j,k)} + \bar{u}_{(j,k)} = u_{(j,k)} + \bar{u}_{(1,k)} - 1+nx+r,$$

wo x, r ganze Zahlen sind.

Im Sinne der Voraussetzung ist der Ausdruck (7) gleich Null, deshalb

$$1 - e^{-2r\pi i/n} = 0.$$

Also $r = n \cdot z$, wo z eine ganze Zahl ist. Das heisst jedoch, dass die Beziehung (8) gilt.

Damit ist der Satz 1 bewiesen.

Wir wollen zwei Hilfssätze anführen, die wir zum Beweis weiterer Sätze verwenden werden.

<u>H i l f s s a t z 1.</u> Es sei n eine gerade Zahl ($n \geq 4$). Es existieren $n/2$ Paare von Indexen bei welchen die Summe $u_{(j,k)} + \bar{u}_{(j,k)}$ gerade ist, wobei alle Summen zueinander verschieden sind. Es sind die z.B. (0,2), (0,4), ..., (0,n-2),(1,n-1). Die Anzahl der Indexenpaare $(u_{(j,k)}, \bar{u}_{(j,k)})$, deren Summen nach dem Modul n kongruent sind, wobei die Summen gerade sind, ist $\frac{n-2}{2}$, $(u_{(j,k)} < \bar{u}_{(j,k)})$. Es existieren $n/2$ Indexenpaare bei welchen die Summe ungerade ist, wobei alle Summen zueinander verschieden sind. Die Anzahl der Indexenpaare bei welchen die Summen nach dem Modul n kongruent sind, wobei die Summen ungerade sind, ist $n/2$.

<u>H i l f s s a t z 2.</u> Es sei n ungerade ($n \geq 3$). Es existieren $\frac{n-1}{2}$ Indexenpaare $(u_{(j,k)}, \bar{u}_{(j,k)})$ bei welchen die Summe gerade ist, wobei die Summen zueinander verschieden sind. Es sind z.B.: (0,2), ..., (0, n-1). Es existieren $\frac{n+1}{2}$ Indexenpaare, bei welchen die Summe ungerade ist, wobei die Summen zueinander verschieden sind. Die Anzahl der Indexenpaare, bei welchen die Summen nach dem Modul n kongruent sind, ist $\frac{n-1}{2}$ ohne Rücksicht darauf, ob die Summen gerade oder ungerade sind.

D e f i n i t i o n 1. Eine Quadrik auf welcher $2 \cdot \binom{m+1}{2}$ P-Punkte - je zwei von jeder Kante des Koordinatensimplexes - liegen, heisst P-Quadrik.

S a t z 2. Im Falle des geraden Exponenten n beträgt die Anzahl der P-Quadriken

$$\sum_{r=0}^{m} \binom{m}{r} \cdot \left(\frac{n}{2}\right)^{\binom{m}{2}+r^2 - rm+m+r} \cdot \left(\frac{n-2}{2}\right)^{m-r+rm-r^2}$$

B e w e i s. Von den Summen $u_{(1,k)} + \bar{u}_{(1,k)}$ seien r Summen ungerade, wobei $k = 2,3,\ldots, m+1$; $0 \leq r \leq m$. Die Gesamtzahl der durch die Punkte $P_{u_{(1,k)}}$, $P_{\bar{u}_{(1,k)}}$ bestimmten $\binom{m}{2}$ - dimensionalen linearen Systeme der Quadriken ist

$$\binom{m}{r} \cdot \left(\frac{n}{2} \cdot \frac{n-2}{2}\right)^{m-r} \cdot \left(\frac{n^2}{4}\right)^{r} .$$

In jedem System existieren $\left(\frac{n-2}{2}\right)^{r(m-r)} \cdot \left(\frac{n}{2}\right)^{\binom{m}{2}-r(m-r)}$ P-Quadriken. (Man benützt die Beziehung (8) und den Hilfssatz 1.)

In diesem Falle beträgt die Gesamtzahl der Quadriken

$$\binom{m}{r} \cdot \left(\frac{n}{2}\right)^{\binom{m}{2}+r^2 - rm+m+r} \cdot \left(\frac{n-2}{2}\right)^{m-r+rm-r^2}$$

Damit sind alle Möglichkeiten erschöpft. Es ist ersichtlich, dass die Behauptung des Satzes richtig ist.

S a t z 3. Im Falle des geraden Exponentpn n gehen durch jeden P-Punkt

$$\sum_{r=0}^{m} \binom{m}{r} \cdot \left(\frac{n}{2}\right)^{\binom{m}{2} + r^2-rm+m+r-1} \cdot \left(\frac{n-2}{2}\right)^{m-r+rm-r^2}$$

P-Quadriken hindurch.

B e w e i s. a) Der Index des festen Punktes P trete in jenen Paaren auf, bei welchen die Summe gerade ist.

Die Anzahl der ungeraden Summen $u_{(1,k)} + \bar{u}_{(1,k)}$ sei r, wobei $k = 2,3,\ldots, m+1$, $0 \leqq r \leqq m-1$.

Die Anzahl der durch die Punkte $P_{u_{(1,k)}}$, $P_{\bar{u}_{(1,k)}}$ bestimmten $\binom{m}{2}$ - dimensionalen linearen Systeme von Quadriken ist

$$\binom{m-1}{r} \cdot \left(\frac{n}{2} \cdot \frac{n-2}{2}\right)^{m-1-r} \cdot \left(\frac{n^2}{4}\right)^{r} \cdot \frac{n-2}{2} .$$

In jedem System existieren

$$\left(\frac{n-2}{2}\right)^{r(m-r)} \cdot \left(\frac{n}{2}\right)^{\binom{m}{2}-r(m-r)} \qquad \text{P-Quadriken.}$$

Wenn wir a) voraussetzen, ist die Anzahl der P-Quadriken

$$(10) \qquad \sum_{r=0}^{m-1} \binom{m-1}{r} \cdot \left(\frac{n}{2}\right)^{\binom{m}{2}+r^2-rm+m+r-1} \cdot \left(\frac{n-2}{2}\right)^{m-r+rm-r^2}$$

b) Der Index des festen P-Punktes trete in ungeraden Paaren auf. Die Anzahl der ungeraden Summen $u_{(1,k)} + \bar{u}_{(1,k)}$ sei r, wobei $k = 2,3,\ldots, m+1$, $1 \leqq r \leqq m$.

Die Anzahl der durch die Punkte $P_{u_{(1,k)}}$, $P_{\bar{u}_{(1,k)}}$ bestimmten $\binom{m}{2}$-dimensionalen linearen Systeme von Quadriken ist

$$\binom{m-1}{r-1} \cdot \left(\frac{n}{2} \cdot \frac{n-2}{2}\right)^{m-r} \cdot \left(\frac{n^2}{4}\right)^{r-1} \cdot \frac{n}{2} .$$

In jedem System existieren

$$\left(\frac{n-2}{2}\right)^{r(m-r)} \cdot \left(\frac{n}{2}\right)^{\binom{m}{2}-r(m-r)} \qquad \text{P - Quadriken.}$$

Wenn wir b) voraussetzen, beträgt die Anzahl der P-Quadriken

$$(11)\qquad \sum_{r=1}^{m} \binom{m-1}{r-1}\cdot\binom{n}{2}^{\binom{m}{2}+r^2-rm+m+r-1}\cdot\left(\frac{n-2}{2}\right)^{m-r+rm-r^2}.$$

Durch die Summierung von (10) und (11) erhalten wir die Anzahl der durch einen festen P-Punkt hindurchgehenden P-Quadriken. Damit ist der Satz bewiesen.

Aus den Sätzen 2 und 3 folgt der

S a t z 4. Im Falle eines geraden Exponenten n bilden die P-Punkte und die P-Quadriken folgende Konfiguration:

$$\left\{ \begin{matrix} n.\binom{m+1}{2}\left[\sum\limits_{r=0}^{m}\binom{m}{r}\cdot\binom{n}{2}^{\binom{m}{2}+r^2-rm+m+r-1}\cdot\left(\frac{n-2}{2}\right)^{m-r+rm-r^2}\right], \\ \sum\limits_{r=0}^{m}\binom{m}{r}\cdot\binom{n}{2}^{\binom{m}{2}+r^2-rm+m+r}\cdot\left(\frac{n-2}{2}\right)^{m-r+rm-r^2} \quad 2.\binom{m+1}{2} \end{matrix} \right\}$$

S a t z 5. Im Falle eines ungeraden Exponenten n beträgt die Anzahl der P-Quadriken

$$[n]^m\cdot\left(\frac{n-1}{2}\right)^{m+\binom{m}{2}}.$$

B e w e i s. Die Behauptung des Satzes ist offenbar.

S a t z 6. Im Falle eines ungeraden Exponenten n gehen durch jeden P-Punkt

$$2\cdot[n]^{m-1}\cdot\left(\frac{n-1}{2}\right)^{m+\binom{m}{2}}$$

P-Quadriken hindurch.

B e w e i s. Die Behauptung des Satzes ist offenbar.

Die Folgerung aus den Sätzen 5 und 6 ist der

S a t z 7. Im Falle eines ungeraden Exponenten n bilden die P-Punkte und P-Quadriken folgende Konfiguration:

$$\left\{\begin{matrix} n\cdot\binom{m+1}{2} & , & \left[n^{m}\cdot\left(\frac{n-1}{2}\right)^{m+\binom{m}{2}}\right] \\ \left[2\cdot n^{(m-1)}\cdot\left(\frac{n-1}{2}\right)^{m+\binom{m}{2}}\right] & & 2\cdot\binom{m+1}{2} \end{matrix}\right\}$$

Durch die m P-Punkte $P_{u_{(1,k)}}$ $(k = 2,3,\dots, m+1)$ ist eine Hyperebene bestimmt.

Untersuchen wir unter welchen Bedingungen in dieser Hyperebene irgendein weiterer Punkt $P_{u_{(j,k)}}$ $(j \geq 2,\ k > j)$ liegt.

Wenn der Punkt $P_{u_{(j,k)}}$ in der erwähnten Hyperebene liegt, dann

$$\begin{vmatrix} e^{(2u_{(1,2)}+1)\pi i/n} & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ e^{(2u_{(1,3)}+1)\pi i/n} & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ e^{(2u_{(1,j)}+1)\pi i/n} & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ e^{(2u_{(i,k)}+1)\pi i/n} & 0 & 0 & 0 & 0 & \dots & 1 \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ e^{(2u_{(1,m+1)}+1)\pi i/n} & 0 & 0 & 0 & 0 & \dots & 0 \dots & 1 \\ 0 & 0 & 0 & 0 & e^{[2u_{(j,k)}+1]\pi i/n} & \dots & 1 & 0 \end{vmatrix} = 0$$

Diese Bedingung können wir auch in folgender Form schreiben:

$$2u_{(1,j)} + 2u_{(j,k)} - 2u_{(1,k)} + 1 = n(2x + 1), \tag{12}$$

wo x eine ganze Zahl ist.

D e f i n i t i o n 2. Die Hyperebene in welcher $\binom{m+1}{2}$ P-Punkte liegen - je einer auf jeder Kante des Koordinatensimplexes - heisst eine P-H y p e r e b e n e.

Aus der Bedingung (12) folgt der

S a t z 8. Im Falle eines geraden Exponenten n existieren keine P-Hyperebenen.

Die Folge des Satzes 8 ist der

S a t z 9. Im Falle eines geraden n sind alle P-Quadriken unzerlegbar.

S a t z 10. Im Falle eines ungeraden Exponenten n ist eine notwendige und hinreichende Bedingung dazu, dass $\binom{m+1}{2}$ P-Punkte - je einer von jeder Kante des Koordinatensimplexes - in einer gemeinsamen Hyperebene liegen, die Gültigkeit der Beziehungen

$$2u_{(1,j)} + 2u_{(j,k)} \equiv 2u_{(j,k)} - 1 \pmod{n} \tag{13}$$

für $j = 2, 3, \ldots, m$; $k = 3, \ldots, m+1$; $k > j$.

B e w e i s. 1. Es gelte die Bedingung (13). Es genügt zu zeigen, dass in der durch die Punkte $P_{u_{(1,h)}}$ $(h = 2,3,\ldots,m+1)$, bestimmten Hyperebene auch der Punkt $P_{u_{(j,k)}}$ $j \geqq 2$, $k > j$) liegt. Dies ist aber ersichtlich auf Grund der Beziehung (12).

2. Es liege in der durch die Punkte $P_{u_{(1,h)}}$ $(h = 2,3,\ldots, m+1)$ bestimmten Hyperebene der Punkt $P_{u_{(j,k)}}$ $(j \geqq 2, k > j)$. Es gilt die Beziehung (12); das ist dasselbe wie (13).

Damit ist der Satz bewiesen.

Auch die folgenden Sätze können leicht bewiesen werden:

S a t z 11. Im Falle eines ungeraden n gehen durch einen festen P-Punkt n^{m-1} P-Hyperebenen hindurch.

S a t z 12. Im Falle eines ungeraden n bilden die P-Punkte und die P-Hyperebenen folgende Konfiguration:

$$\left[n \cdot \binom{m+1}{2}_{n^{m-1}} , \; n^m_{\binom{m+1}{2}} \right] .$$

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Eingegangen am: 26 November 1969

(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXV - 1971)

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXV - 1971

ÜBER EINE BIRATIONALE TRANSFORMATION n_3 - GRADES IM n- DIMENSIONALEN PROJEKTIVEN RAUM S_n

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In [1] ist ein für die Euklidische Ebene gültiger Satz angeführt, in [5] wurde dieser Satz für die erweiterte Euklidische Ebene und für den dreidimensionalen Raum verallgemeinert.

In der vorliegenden Arbeit verallgemeinern wir diesen Satz und beweisen dessen Gültigkeit für den n-dimensionalen projektiven Raum S_n über dem Körper der komplexen Zahlen, zugleich führen wir die Transformationsgleichungen zwischen den entsprechenden Punkten, den Grad der Transformation, die Fundamental - und Hauptvarietät an.

Verallgemeinerung des Satzes und die Transformationsgleichungen

Wählen wir im n-dimensionalen projektiven Raum S_n einen Simplex I_n, welcher gleichzeitig der Koordinatensimplex des Raumes S_n ist.

Satz 1. Es sei I_n eine Simplex des Raumes S_n.P sei ein beliebiger Punkt in S_n, welcher in keinem (n-1)-dimensionalen Teilraum des Simplexes I_n liegt. Die Punkte A_{ij} ((ij) sind alle Kombinationen zweiter Klasse aus den Zahlen 0, 1, ... n) sind Projektionspunkte des Punktes P aus den (n-2)-dimensionalen Teilräumen Σ_{ij} des Simplexes auf die gegenüberliegenden Kanten O_iO_j. Die beliebige durch die Punkte A_{ij} hindurchgehende Quadrik Q schneidet noch die Kanten O_iO_j des Simplexes I_n in den Punkten B_{ij}. Die durch die Punkte B_{ij} und durch die gegenüberliegenden (n-2)-dimensionalen Teilräume Σ_{ij} bestimmten Hyperebenen β_{ij} durchschneiden sich ebenfalls in einem Punkte P'.

B e w e i s. Da der Punkt $P \equiv (y)$ in keinem Teilraum des Simplexes I_n liegt, ist $y_i \neq 0$ für $i = 0,1,\ldots, n$. Die projizierende Hyperebene $\alpha \equiv (\Sigma_{ij}, P)(\Sigma_{ij} \equiv (O_0,\ldots \widehat{O}_i, \ldots, \widehat{O}_j, \ldots \ldots, O_n))$ hat die Gleichung

(1) $\quad y_j x_i - y_i x_j = 0$

und durchschneidet die Kante O_iO_j im Punkte A_{ij}, dessen Koordinaten $y_k = 0$, $k = 0, 1, \ldots, i-1, i+1, \ldots, j-1, j+1, \ldots, n)$ aber $y_i \neq 0$, $y_j \neq 0$.

Die durch die Punkte A_{ij}, deren Anzahl $\binom{n+1}{2}$ ist, durchgehende Quadrik Q schneidet noch jede Kante $O_i O_j$ im Punkte B_{ij}. Die Koordinaten des Punktes B_{ij} auf der Kante O_iO_j sind $x_k = 0$, $k = 0,1,\ldots, n$, mit Ausnahme von x_i, x_j. Wenn wir in die Gleichung Q statt $x_k = 0$ $(k = 0,1,\ldots, i-1, i+1,\ldots, j-1, j+1,\ldots, n)$ setzen, erhalten wir nach Berichtigung für die Koordinaten der Schnittpunkte der Quadrik Q mit der Kante O_iO_j die Geichung

(2) $\quad b_i x_i^2 + b_j x_j^2 + c_{ij} x_i x_j = 0$

Hilfssatz 1. Von den Koeffizienten b_i für $i = 0,1,\ldots, n$ kann höchstens einer gleich Null sein.

B e w e i s. Wenn $b_i = 0$, so folgt daraus, dass die Quadrik Q durch den Grundpunkt O_i des Simplexes I_i hindurchgeht. Sollten zwei der Koeffizienten gleich Null sein, dann ginge die Quadrik Q durch zwei Grundpunkte des Simplexes hindurch, z.B. O_i und O_j, $i \neq j$. Die Quadrik Q könnte die Kante $O_i O_j$ in von O_i und O_j verschiedenen Punkten nichtmehr durchschneiden. Der Punkt P müsste also in der Hyperebene des Simplexes liegen, was der Voraussetzung des Satzes widersprechen würde.

Nach dem Hilfssatz 1 können wir voraussetezen, dass entweder b_i oder b_j von Null verschieden ist, dann ist nach Berichtigung (2) (unter der Voraussetzung $b_i \neq 0$)

$$(3)\qquad \left(\frac{x_i}{x_j}\right)^2 + \frac{c_{ij}}{b_i}\cdot\frac{x_i}{x_j} + \frac{b_j}{b_i} = 0$$

Eine Wurzel der quadratischen Gleichung (3) ist das Verhältnis der Koordinaten $y_i : y_j$ des Punktes A_{ij}. Die andere Wurzel ist das Verhältniss $y_i' : y_j'$, wo $y_i', y_j', y_k' = 0$ ($k = o,1,\dots,n$ mit der Ausnahme $k = i$, $k = j$) Koordinaten des zweiten Schnittpunktes der Quadrik Q mit der Kante O_iO_j sind, also Koordinaten des Punktes B_{ij}. Diese Verhältnisse erfüllen die Beziehung

$$(4)\qquad \frac{y_i'}{y_j'}\cdot\frac{y_i}{y_j} = \frac{b_j}{b_i},$$

daraus

$$y_i' : y_j' = b_j\, y_j : b_i y_i .$$

Die Gleichung der Hyperebene β_{ij}, welche die Hülle des Punktes B_{ij} und des (n-2)-dimensionalen Teilraumes Σ_{ij} des Simplexes I_n ist, lautet

$$(5)\qquad b_i y_i x_i - b_j y_j x_j = 0 .$$

Wenn wir alle Kombinationen (ij) aus den Zahlen $0,1,\dots,n$ nehmen, erhalten wir $\binom{n+1}{2}$ Hyperebenen, deren Gleichungen die Form (5) haben. Unter ihnen sind n linear unabhängig und durchschneiden sich in einem Punkt $P'(y_0', y_1', \dots y_n')$, dessen Koordinaten folgende Beziehung erfüllen:

$$(6)\qquad \begin{aligned} y_0' : y_1' :\dots: y_{n-1}' : y_n' &= b_1b_2b_3\dots b_{n-1}b_n y_1y_2y_3\dots y_{n-1}y_n : \\ &: b_0b_2b_3\dots b_n y_0y_2y_3 \dots y_n :\dots: b_0b_1b_2\dots b_{n-2}b_n y_0y_1 \dots \\ &\dots y_{n-2}y_n : b_0b_1b_2 \dots b_{n-1}y_0y_1y_2 \dots y_{n-1}. \end{aligned}$$

<u>B e m e r k u n g 1.</u> Wenn $b_k = 0$, dann entspricht dem Punkt P der Punkt $P' \equiv O_k$.

Aus dem Beweis des Satzes 1 folgt, dass dem Punkte $P \in S_n$ eindeutig der Punkt $P' \in S_n$ (bis auf gewisse Ausnahmen, da über den

Punkt P vorausgesetzt wurde, dass dieser nicht in den Teilräumen des Simplexes I_n liegt).

D e f i n i t i o n 1. Die Zuordnung $P \to P'$ im Raume S_n nennen wir Transformation T.

S a t z 2. Die durch die Gleichung (6) bestimmte Transformation T ist eine eindeutige involutorische Korrespondenz im Raum S_n (eine Ausnahme bilden Punkte gewisser Varietäten).

B e w e i s. Durch Projizieren des Punktes P auf die Kanten des Simplexes erhielten wir $\binom{n+1}{2}$ Punkte A_{ij}; zur eindeutigen Bestimmung der Quadrik Q sind noch n Punkte K_ℓ, $\ell = 1, \ldots, n$, notwendig; diese wählen wir so, dass sie linear unabhängig seien, weiter seien die Hyperebenen (Σ_{ij}, K_ℓ) paarweise verschieden und die Hyperebene $\eta \equiv (K_1, \ldots K_n)$ soll durch keinen Grundpunkt O_1 des Simplexes I_n hindurchgehen.

So ist dem Punkte $P \in S_n$ eindeutig der Punkt $P' \in S_n$ zugeordnet. Wenn wir P' als Ausgangspunkt wählen, welcher wieder die für den Punkt P ausgesprochenen Bedingungen erfüllt, bestimmen wir mit dessen Hilfe eindeutig die Punkte B_{ij} auf den Kanten des Simplexes, mittels der Quadrik Q bekommen wir die Punkte A_{ij} und so erhalten wir den Punkt P.

Die Transformationsgleichungen zwischen den Punkten P und P' haben wieder die Form (6).

S a t z 3. Die Transformation T ist eine birationale Cremona-sche Transformation n^3 Grades.

B e w e i s. Laut dem vorhergegangenen ist die Quadrik Q durch die Punkte K_ℓ, $\ell = 1, \ldots, n$, und A_{ij} bestimmt. Auf Grund der Koordinaten der erwähnten Punkte kann man z.B. die Verhältnisse $\frac{b_i}{b_o}$ der Koeffizienten der Gleichung der Quadrik Q berechnen. Die Determinanten für Unbekannten $\frac{b_i}{b_o}$ bezeichnen wir D_i, $i = 1, \ldots, n$. (Die übrigen Determinanten führen wir nicht an.) Für die Koeffizienten b_i, $i = 0, \ldots, n$, gilt dann:

$$(7) \qquad b_0 : b_1 : b_2 : \ldots : b_n = D_0 : D_1 : D_2 : \ldots : D_n ,$$

wo D_i $(i = 0, \ldots, n)$ Formen $2\binom{n+1}{2} = n(n+1)$ Grades in y_j $(j = 0,\ldots, n)$ sind, weil jedes Determinantenelement in D_i zweiten Grades ist und die Determinante D_i $\frac{n(n+1)}{2}$ Reihen enthält.

In der Determinante D_0 können wir aus der ersten Reihe y_1, aus der zweiten Reihe y_2 usw. aus der n-ten Reihe y_n (was freilich von der Anordnung der Gleichungen abhängt) herausnehmen. Also

$$D_0 = y_1 y_2 \ldots y_n M_0$$

wo M_0 die Form $n(n+1)-n = n^2$ Grades in $y_0, y_1,\ldots,y_n$ ist. Andere Faktoren können aus D_0 nicht herausgenommen werden.

Das gilt analogisch auch für die Determinanten D_i, $i = 1, \ldots, n$, also

$$(8) \qquad D_i = y_0 y_1 \ldots y_{i-1} y_{i+1} \ldots y_n M_i ,$$

wo M_i $(i = 1,\ldots,n)$ Formen des n^2 Grades in $y_0, y_1,\ldots,y_n$ sind.

Die Koeffizienten b_j in der Gleichung der Quadrik sind homogen und deshalb können wir in die Gleichung (7) anstatt D_i die rechte Seite der Gleichung (8) einsetzen; so erhalten wir

$$(9) \qquad b_i = y_0 y_1 \ldots y_{i-1} y_{i+1} \ldots y_n M_i \quad (i = 0,1,\ldots, n).$$

Setzen wir in die Gleichungen (6) von den Gleichungen (9); wir erhalten

$$(10) \qquad \begin{aligned} &\prod_{i=0}^{n} (y_i)^n M_0 M_2 M_3 \ldots M_n : \prod_{i=0}^{n} (y_i)^n M_0 M_1 M_3 \ldots M_n : , \ldots \\ &: \prod_{i=0}^{n} (y_i)^n M_0 M_1 \ldots M_{k-1} M_{k+1} \ldots M_n : \ldots : \prod_{i=0}^{n} (y_i)^n M_0 M_1 \ldots M_{n-1} . \end{aligned}$$

Der Voraussetzung gemäss liegt der Punkt P in keiner der Hyperebenen des Simplexes und deshalb $y_i \neq 0$, $i = 0,\ldots,n$; also können wir mit dem gemeinsamen Faktor $\prod_{i=0}^{n} (y_i)^n$ kürzen und erhalten dann

$$(11) \quad y_0' : y_1' : \ldots : y_k' : \ldots : y_n' = M_1 M_2 \ldots M_n : M_0 M_2 \ldots M_n : \ldots : \\ : M_0 M_1 \ldots M_{k-1} M_{k+1} \ldots M_n : \ldots : M_0 M_1 \ldots M_{n-1} .$$

Jede der Formen M_i $(i = 0, \ldots, n)$ hat den Grad n^2 und jedes Glied des Verhältnisses auf der rechten seite der Gleichheit (11) enthält n Faktoren, also jedes Glied des Verhältnisses ist eine Form des n^3 grades in $y_0, y_1, \ldots y_n$.

Wenn wir y_i' gegen y_i $(i = 0, \ldots, n)$ und umgekehrt in (11) austauschen, erhalten wir Transformationsbeziehungen zwischen P' und P. Daraus ist ersichtlich, dass die Transformation T eine birationale Cremonasche Transformation n^3 - Grades ist.

<u>H i l f s a t z 2.</u> Jede veränderliche y_i in der Form M_i $(i = 0, \ldots, n)$ kann höchstens n-ten Grades sein.

<u>B e w e i s.</u> Die Formen M_i entstanden aus D_i. Die Veränderliche y_i in D_i findet sich in n Reihen im ersten oder zweiten Grad. Alle y_i^2 sind in einer Kolonne; in dieser Kolonne blieb auch y_i, welches nach dem Herausnehmen von y_i aus der Reihe der Determinante D_i entstand. Daraus folgt, dass wenn wir bei der Bildung der Form M_i aus dem Produkt y_i^2 herausnehmen, bleibt y_i nur mehr in (n-2) Reihen, in jeder nur im ersten Grad. Daraus folgt die Behauptung des Hilfsatzes 2.

Im weiteren werden wir von gewissen Punkten beweisen, dass sie fundamentale Punkte der Transformation T sind, d.h. solche, für welche in den Gleichungen (11) $y_i' = 0$ für $i = 0,\ldots, n$ ist.

<u>S a t z 4.</u> Die Punkte der Varietät

$$(12) \quad \bigcup_{\substack{i<j \\ i = 0, \ldots, n-1 \\ j = 1, \ldots, n}} \overline{M}_i \cap \overline{M}_j \; .$$

wo $\overline{M}_i$ eine durch die Gleichung $M_i = 0$ definierte Hyperfläche ist, sind Fundamentalpunkte der Transformation T.

B e w e i s. Für Punkte mit der angeführten Eigenschaft müssen wenigstens zwei Faktoren aus M_i $(i = 0,\dots,n)$ in den Gleichungen (11) verschwinden, weil, wenn nur ein Faktor M_i verschwindet, wir eine der Koordinaten y_i' von Null verschieden erhalten. Solche Punkte, bei welchen zwei Faktoren M_i verschwinden, gehören in den Durchschnitt $M_i \cap M_j$.

Die Menge (12) ist ersichtlich nicht leer. In diese gehören alle Punkte der (n-2) - dimensionalen Teilräume Σ_{ij}, was aus den Eigenschaften der Formen M_i hervorgeht, da jeder Teilraum Σ_{ij} in jeder Varietät M_i enthalten ist.

In den folgenden Erwägungen suchen wir eine weitere Menge von Fundamentalpunkten.

Es sei O_i ein fester Grundpunkt des Simplexes I_n. η sei eine durch die von uns im Beweis des Satzes 2 eingeführten Punkte K_ℓ $(\ell = 1, \dots, n)$ bestimmte Hyperebene. K_{ij} seien Schnittpunkte der Hyperebene η mit den Kanten O_iO_j (i - fest, $j = 0,1, \dots, i-1, i+1, \dots, n$).

Die Anzahl dieser Punkte ist n. n lineare unabhängige Hyperebenen $\xi^{(j)} \equiv (\Sigma_{ij}, K_{ij})$ $(j = 0, \dots, i-1, i+1, \dots, n)$ durchschneiden sich im Punkte P_i.

S a t z 5. Der Punkt P_i ist ein Fundamentalpunkt und in der Transformation T wird auf denselben die Hyperebene ω_i $\left[\omega_i \equiv (O_0, \dots, O_{i-1}, O_{i+1}, \dots O_n)\right]$ als Ganzes abgebildet.

B e w e i s (synthetisch). Aus der Konstruktion des Punktes P_i folgt, dass die Projektionspunkte des Punktes P_i aus Σ_{ij} (i - fest, $j = 0, \dots, i-1, i+1, \dots, n$) auf $O_i\,O_j$ die Punkte K_{ij} sind. Die Projektionspunkte des Punktes P_i aus Σ_{jk} $(j,k = 0, \dots, i-1, i+1, \dots, n)$ sind bestimmte Punkte A_{jk} auf den Kanten $O_j\,O_k$. Ersichtlich liegen die Punkte A_{jk} in der Hyperebene ω_i. Die durch die Punkte K_{ij}, A_{jk}, $(j,k = 1, \dots, n)$ gehende Quadrik Q besteht aus den Hyperebenen ω_i und η (in ω_i sind $\binom{n}{2}$ Punkte A_{jk}, in η sind 2n Punkte der Quadrik - durch diese Punkte ist die Quadrik Q_i eindeutig bestimmt.) Gemeinsame Punkte der Quadrik Q_i mit den Kanten des Simplexes I_n (Punkte im Sinne B_{ij}) verschieden von den Punkten A_{jk}, K_{ij} sind nun:

a_1) auf den Kanten O_iO_j (i - fest) die Punkte O_j,

b_1) auf den Kanten O_jO_k ($j,k = 0,\ldots, i-1, i+1,\ldots, n$), alle Punkte dieser Kanten ausser den Punkten A_{jk}, dies hat aber auf unsere Erwägungen keinen Einfluss.

Das Projizieren der Punkte B_{ij} von den Kanten Σ_{ij} geschieht folgend:

a_2) Der Punkt O_j (als Punkt der Kante $O_i\ O_j$) wird von Σ_{ij} durch die Hyperebene ω_i projiziert;

b_2) Die Kante $O_j\ O_k$ (als Menge der Punkte B_{jk}) wird von Σ_{jk} durch den ganzen Raum S_n projiziert.

Der Durchschnitt aller Mengen von a_2) und b_2) ist die Hyperebene ω_i als Menge von Punkten, welche nach geometrischer Definition dem Punkt P_i entsprechen.

Es ist leicht zu zeigen, dass jeder Punkt P der Hyperebene ω_i (ausser den Punkten Σ_{jk} ($j,k = 0,\ldots, i-1, i+1,\ldots,n$)) gibt als entsprechendes Gebilde den Punkt P_i.

Analogisch gilt die Erwägung für die Punkte P_i, $i = 0,\ldots, n$.

<u>S a t z 6.</u> Auf den Punkt O_i ($i = 0, \ldots, n$) des Simplexes I_n werden die Punkte der Varietät $M_i = 0$ abgebildet.

<u>B e w e i s.</u> Tauschen wir in den Gleichungen (11 y_i für y_i' aus. (Mit Rücksicht auf Satz 3 ist dies möglich). Statt y_i setzen wir die Koordinaten des Punktes O_i und suchen diejenigen Punkte, deren Koordinaten den veränderten Gleichungen (11) genügen. In der Gleichung für y_i fehlt auf der rechten Seite M_i, wobei $y_i \neq 0$, aber in allen Gleichungen sind alle M_j enthalten, $j = 0,\ldots, n$, wobei $j \neq i$. Das heisst, dass $M_j \neq 0$ ($j = 0, \ldots, n$, $j \neq i$). Falls aber irgendein $M_j = 0$, dann ist auch $y_i = 0$.

Aus dem Beweis des Satzes 6 geht hervor, dass die Punkte der Varietät $M_i = 0$ ($i = 0, \ldots, n$) die Hauptpunkte der Transformation T sind. Ersichtlich ist, dass die Hauptvarietät bestimmt $\bigcup_{i=0} M_i$ enthält; aber zu dieser Vereinigung gehören auch die Fundamentalpunkte. Also die Menge jener Punkte dieser Varietät, in welchen die Abbildung definiert, aber nicht (1,1) - deutig ist, ist die Menge

$$\bigcup_{i=0}^{n} M_i - \bigcup M_i \cap M_j$$

$$i < j$$
$$i = 0, \dots, n-1$$
$$j = 1, \dots, n$$

Auf Grund des Satzes 5 gehören in die Hauptvarietät auch die Hyperebenen ω_i' $(i = 0, \dots, n)$.

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Eingegangen am: 26. November 1969

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXV - 1971

CONVERGENCE OF VARIATIONS AND UNIFORM CONVERGENCE

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In the paper [1], there were sequences $\{f_n\}_{n=1}^{\infty}$ of functions of bounded variation on the interval $(-\infty, \infty)$ studied. A theorem asserting that if $\{f_n\}_{n=1}^{\infty}$ is almost uniformly convergent to a function f of bounded variation on $(-\infty, \infty)$ and if $\overset{\infty}{\underset{-\infty}{V}} f_n \to \overset{\infty}{\underset{-\infty}{V}} f$ ($\overset{\infty}{\underset{-\infty}{V}} g$ stands for the variation of g on $(-\infty, \infty)$), then $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent on $(-\infty, \infty)$.

Let us define, in an analogical way as in [2], the variation of f on an arbitrary set $M \subset (-\infty, \infty)$.

<u>Definition 1.</u> A real function f will be termed of bounded variation on a set M if there exists a real number K such that $\sum_i |f(b_i)-f(a_i)| < K$, for any finite sequence $\{\langle a_i, b_i \rangle\}$ of nonoverlapping intervals, the end-points of which belong to M. We shall put $\underset{\emptyset}{V} f = 0$ and $\underset{M}{V} f = \text{lub} \sum_i |f(b_i)-f(a_i)|$ if $M \neq \emptyset$.

In connection with this definition a question arises, if the above cited theorem from [1] is true when the interval $(-\infty, \infty)$ is substitued by any set M $(-\infty, \infty)$. The following example shows that the answer is negative.

Example 1. Let $M = (-1, 0) \cup (0,1)$. Put

$$f_n(x) = \begin{cases} 0 & \text{if } x \in (-1, 0) \\ n x & \text{if } x \in (0, \frac{1}{n}) \\ 1 & \text{if } x \in \langle \frac{1}{n}, 1) \end{cases}$$

Then $\underset{M}{V} f_n = 1$ for $n = 1, 2, 3, \dots$. The sequence is almost uniformly (it means uniformly on every compact subset of M) convergent on M to f which equals zero on $(-1, 0)$ and 1 on $(0,1)$. In fact, if $K \subset M$ is any compact set, then $K = K_1 \cup K_2$, where $K_1 \subset (1,0)$ and $K_2 \subset (0,1)$ are compact sets. The uniform convergence of f_n on K_1 and K_2 implies the uniform convergence on K. Since $\underset{M}{V} f = 1$, we have $\underset{M}{V} f_n \to \underset{M}{V} f$. But the convergence of $\{f_n\}$ on M is not uniform on M.

So an another question arises. The probleme is to characterize those sets $M \subset (-\infty, \infty)$ for which the mentioned theorem remains to be true. The question is solved by the following two theorems.

T h e o r e m 1. Let f_n and f be functions of bounded variation defined on M. Let M have the following property:

(*) If x_0 is such a limit point of M that $x_0 \notin M$, then either $x < x_0$ for every $x \in M$ or $x > x_0$ for every $x \in M$. Then the almost uniform convergence of $\{f_n\}_{n=1}^{\infty}$ to f on the set M and the condition $\underset{M}{V} f_n \to \underset{M}{V} f$ imply the uniform convergence of $\{f_n\}_{n=1}^{\infty}$ on the set M to the function f.

N o t e 1. The variation of f on the set M, defined as obove, is a generalization of that one defined on an interval. In general the variation as a set function need not be additive considered as a set function. It is sufficient to consider the following simple example.

E x a m p l e 2. Let $M = (-1,0) \cup (0,1)$, let $f(x) = \begin{matrix} 0 \text{ if } x \in (-1,0) \\ 1 \text{ if } x \in (0,1) \end{matrix}$. Put $M_1 = (-1,0)$; $M_2 = (0,1)$. We have $\underset{M}{V} f = 1$, $\underset{M_1}{V} f = 0$, $\underset{M_2}{V} f = 0$, hence $\underset{M}{V} f > \underset{M_1}{V} f + \underset{M_2}{V} f$.

But the following simple result concerning the additivity of the variation will be usefull.

L e m m a 1. Let f be of bounded variation on a set M and let $a \in M$, $b \in M$, $a < b$. Put $M_1' = \{x : x \in M, x \leq a\}$, $M_1'' = \{x : x \in M,$

$x \geqq b\}$, $M_2 = \{x : x \in M,\ a \leqq x \leqq b\}$. Then $\mathop{V}\limits_{M} f = \mathop{V}\limits_{M_1'} f + \mathop{V}\limits_{M_1''} f\ \mathop{V}\limits_{M_2} f$.

P r o o f. Let $\langle a_i, b_i\rangle$, $(i = 1, 2, \ldots, n)$ be any collection of nonoverlapping intervals such that $a_i \in M$, $b_i \in M$. Any of the intervals $\langle a_i, b_i\rangle$ which is neither contained in $\langle a, b\rangle$ nor $b_i \leqq a$ or $a_i \geqq b$ holds, if there exists such an interval, can be divided by means of the points a, b into two or three intervals $\langle a_{i_k}, b_{i_k}\rangle$ in such a way that they are contained in $\langle a, b\rangle$ or one of the inequalities $b_{i_k} \leqq a$, $a_{i_k} \geqq b$ holds. For such an interval $\langle a_i, b_i\rangle$.

(1) $$|f(b_i) - f(a_i)| \leqq \sum_k |f(b_{i_k}) - f(a_{i_k})|$$

If instead of the original system $\langle a_i, b_i\rangle$ the new one which is obtained after the mentioned procedure of division is considered, then in view of (1)

$$\sum_i |f(b_i) - f(a_i)| \leqq \sum_j |f(b_j) - f(a_j)| = \sum{}' |f(b_j) - f(a_j)| +$$

$$+ \sum{}'' |f(b_j) - f(a_j)| + \sum{}''' |f(b_j) - f(a_j)|$$

where $\sum$ denotes the sum over those $\langle a_j, b_j\rangle$ for which $b_j \leqq a$, $\sum''$ the sum over those $\langle a_j, b_j\rangle$ for which $a_j \geqq b$ and $\sum'''$ is the sum over those intervals $\langle a_j, b_j\rangle$ for which $\langle a_j, b_j\rangle \subset \langle a, b\rangle$. Then (2) implies

$$\sum_i |f(b_i) - f(a_i)| \leqq \mathop{V}\limits_{M_1'} f + \mathop{V}\limits_{M_1''} f + \mathop{V}\limits_{M_2} f,$$

hence

(3) $$\mathop{V}\limits_{M} f \leqq \mathop{V}\limits_{M_1'} f + \mathop{V}\limits_{M_1''} f + \mathop{V}\limits_{M_2} f$$

;

Now let $\varepsilon > 0$. There exists a finite system $\langle a_k, b_k\rangle$ of nonoverlapping intervals such thet $a_k \in M_1'$, $b_k \in M_1'$, $k = 1, 2, \ldots, m$ and $\sum_k |f(b_k) - f(a_k)| > \mathop{V}\limits_{M_1'} f - \frac{\varepsilon}{3}$. Similarly there exist finite systems $\langle a_\ell, b_\ell\rangle$ and $\langle a_s, b_s\rangle$, $a_\ell \in M_1''$, $b_\ell \in M_1''$, $\ell = m+1,\dots,m+p$; $a_s \in M_2$, $b_s \in M_2$, $s = m+p+1, \dots, m+p+n$ such that

$$\sum_\ell |f(b_\ell) - f(a_\ell)| > \underset{M_1''}{V} f - \frac{\varepsilon}{3}, \qquad \sum_s |f(b_s) - f(a_s)| > \underset{M_2}{V} f - \frac{\varepsilon}{3}$$

So to any $\varepsilon > 0$ the existence of a finite system of nonoverlapping intervals $\langle a_i, b_i\rangle$, $a_i \in M$, $b_i \in M$, is guaranteed such that

$$\sum_i |f(b_i) - f(a_i)| > \underset{M_1'}{V} f + \underset{M_1''}{V} f + \underset{M_2}{V} f - \varepsilon$$

Hence $\underset{M_1}{V} f \geqq \underset{M_1'}{V} f + \underset{M_1''}{V} f + \underset{M_2}{V} f$

The last and (3) prove the lemma.

N o t e 2. In the proof of Theorem 1 also the lower semicontinuity of the variation will be used. It is as follows:

If $\{f_n\}_{n=1}^{\infty}$ is a sequence converging on M to f, then $\underset{M}{V} f \leqq \liminf\limits_{n \to \infty} V f_n$. The proof of the last fact is not different from that one for the case when $M = \langle a,b\rangle$ (see [3]) and will be omitted.

P r o o f o f T h e o r e m 1. Let $\varepsilon > 0$. Choose a finite collection of intervals $\langle a_i, b_i\rangle$ such that

$$\sum_i |f(b_i) - f(a_i)| > \underset{M}{V} f - \varepsilon \tag{4}$$

Put $a = \min\limits_i a_i$, $b = \max\limits_i b_i$. Using the same notation as in the lemma we have

$$\underset{M_1'}{V} f + \underset{M_1''}{V} f \leqq \varepsilon \tag{5}$$

Put $M_2 = \langle a,b\rangle \cap M$. From the lower semicontinuity $\underset{M_2}{V} f \leqq \liminf\limits_{n \to \infty} \underset{M_2}{V} f_n$. Hence there is a number N_1 such that if $n > N_1$ then

$$\underset{M_2}{V} f_n > \underset{M_2}{V} f - \varepsilon \tag{6}$$

Further, by the assumption of the theorem, there exists a number N_2 such that if $n > N_2$

(7) $\qquad |\underset{M}{V} f_n - \underset{M}{V} f| < \varepsilon$

Using the lemma and the relations (5), (6), (7), we have for $n > \max(N_1, N_2)$

(8) $$\underset{M_1'}{V} f_n + \underset{M_1''}{V} f_n = \underset{M}{V} f_n - \underset{M_2}{V} f_n < \underset{M}{V} f_n - \underset{M_2}{V} f + \varepsilon =$$

$$= \underset{M}{V} f_n - (\underset{M}{V} f - \underset{M_1'}{V} f - \underset{M_1''}{V} f) + \varepsilon < 3\varepsilon$$

Now choose any $x_1 \in M_1'$. Since $\lim_{n \to \infty} f_n(x_1) = f(x_1)$, we have for $n > N_3$

(9) $\qquad |f_n(x_1) - f(x_1)| < \varepsilon$

For $n > \max(N_1, N_2, N_3)$ and any $x \in M_1'$, (8) and (9) give

(10) $$|f_n(x) - f(x)| \leq |f_n(x) - f_n(x_1)| + |f_n(x_1) - f(x_1)| + |f(x_1) - f(x)| < \underset{M_1'}{V} f_n +$$

$$+ \varepsilon + \underset{M_1'}{V} f < 5\varepsilon .$$

By the same consideration as for the set M_1' the inequality $|f_n(x) - f(x)| < 5\varepsilon$ can be proved for any $x \in M_1''$ and for all n begining from a suitable N. Now the only thing which remains to be proved is the uniform convergence on M_2. To prove this, it is sufficient to prove that M_2 is a compact set. M_2 is a bounded set. We shall prove that it is closed. Let x_0 be a limit point of M_2. If $x_0 = a$, or $x_0 = b$, then $x_0 \in M_2$. If $a < x_0 < b$, then $x_0 \in M_2$, according to the property (*) of the set M. Hence M_2 is a compact set.

<u>Note 3.</u> As an example of sets having the property (*) we can take intervals of any type or any closed set. Another type of a set with the propertx (*) is e. g. the set $(-1,0) \cup \langle 1,2$. Any set which has the property * is F_σ as can be easily proved. The converse is evidently not true.

Theorem 2. Let M have not the property $*$. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ and a function f such that f_n is almost uniformly convergent to f on the set M, $\bigvee_M f_n \to \bigvee_M f$ but $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly to f on the set M.

Proof. According to the assumption, there exists a point x_0 such that x_0 is a limit point of M, x_0 does not belong to M and the sets $M_1 = \{x : x \in M,\ x < x_0\}$, $M_2 = \{x : x \in M,\ x > x_0\}$ are not empty. One of them is infinite. Suppose that it is M_2. (In the opposite case the proof is analogical.) Let us define the functions $\{f_n\}_{n=1}^{\infty}$ as follows: For $n = 1, 2, 3, \ldots$

$$f_n(x) = \begin{cases} 0 & \text{if } x \in M_1 \\ n(x - x_0) & \text{if } x \in M_2 \cap (x_0, x_0 + \frac{1}{n}) \\ 1 & \text{if } x \geqq x_0 + \frac{1}{n}, \quad x \in M_2 \end{cases}$$

Immediately can be verified that $\bigvee_M f_n = 1$ for $n = 1, 2, 3, \ldots$ The sequence $\{f_n\}_{n=1}^{\infty}$ is convergent to f, which is defined on M as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in M_1 \\ 1 & \text{if } x \in M_2 \end{cases}$$

Since $\bigvee_M f = 1$, we have $\bigvee_M f_n \to \bigvee_M f$. The convergence on M is almost uniform. It may be shown quite analogicaly as in the example 1. But the convergence is not uniform on M because $\text{lub}\,|f_n(x) - f(x)| = 1$ for $n = 1, 2, 3, \ldots$. The proof is finished.

From Note 3 and Theorem 1 we have the following corollary.

Corollary. If $\{f_n\}_{n=1}^{\infty}$ and f have the same meaning as in Theorem 1, then on an interval of any type and on any closed set the almost uniform convergence of $\{f_n\}_{n=1}^{\infty}$ to f and the condition $\bigvee_M f_n \to \bigvee_M f$ imply the uniform convergence of $\{f_n\}_{n=1}^{\infty}$ to f.

The generalization of the cited theorem from [1] may be obtained also in a little different way. Let us introduce the following type of convergence.

<u>D e f i n i t i o n 2.</u> A sequence $\{f_n\}_{n=1}^{\infty}$ of functions defined on a set M is said to be $\mathcal{Y}$-convergent to f on M if f_n is uniformly convergent to f on every set $\langle a,b\rangle \cap M$ where $a \in M$, $b \in M$, $a \leq b$.

<u>N o t e.</u> In general the $\mathcal{Y}$-convergence on M and the almost uniform convergence do not coincide. For the proof it suffices to consider the example 1.

If the almost uniform convergence is substituted by the $\mathcal{Y}$-convergence, then Theorem 1 may be stated for any set $M \subset (-\infty, \infty)$.

<u>T h e o r e m 3.</u> If $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions with bounded variations which is $\mathcal{Y}$-convergent on M to a function f of bounded variation, then under the assumption $\underset{M}{V} f_n \to \underset{M}{V} f$ is the sequence $\{f_n\}_{n=1}^{\infty}$ uniformly convergent on M.

The proof is similar to that of Theorem 1 and therefore will be omitted.

The relation bettween the almost uniform convergence and the convergence is described in Theorem 4. At first a lemma will be proved.

<u>L e m m a 2.</u> Let $\{f_n\}_{n=1}^{\infty}$ be $\mathcal{Y}$-convergent to f on the set M. Then $\{f_n\}_{n=1}^{\infty}$ converges to f almost uniformly on M.

<u>P r o o f.</u> Let $K \subset M$, $K \neq \emptyset$ be any compact. (The ase $K = \emptyset$ is trivial.) Then there exist $a \in K$, $b \in K$ such that $M \cap \langle a,b\rangle = K$. From the $\mathcal{Y}$-convergence the uniform convergence on K follows. The proof is finished.

<u>T h e o r e m 4.</u> The $\mathcal{Y}$-convergence on M and the almost uniform convergence on M are identical if and only if M has the property (*).

<u>P r o o f.</u> Let M has the propety (*). In view of Lemma 2 the almost uniform convergence from the $\mathcal{Y}$-convergence on M follows.

Now let $\{f_n\}_{n=1}^{\infty}$ be almost uniformly convergent on M let $a \in M$, $b \in M$, $a \leqq b$. To prove the uniform convergence on $\langle a,b\rangle \cap M$ it is sufficiend to prove that $\langle a,b\rangle \cap M$ is compact. But the last is evident, because the boundedness is trivial and the fact that $\langle a,b\rangle \cap M$ is closed follows from (*).

Suppose now that M has not the property (*). From theorem 2 the existence of a sequence $\{f_n\}_{n=1}^{\infty}$ follows such that $\{f_n\}_{n=1}^{\infty}$ is almost uniformly cónvergent to f, $\underset{M}{V} f_n \to \underset{M}{V} f$ and $\{f_n\}_{n=1}^{\infty}$ fails to be uniformly convergent. The proof is finished.

N o t e. The $\mathcal{Y}$-convergence is metrizable. The proof of this fact is not essentialy different from that of the metrizability of almost uniform convergence (see [4] p. 146).

N o t e. The relation of theorms 1, 2, 3 with the infinite series is evident from the known fact that to any sequence $\{f_n\}_{n=1}^{\infty}$ a series $\sum_{n=1}^{\infty} g_n$ may be formed such that for the partial sums $s_n = f_n$ holds. Hence in theorems 1,2,3 instead of a sequence $\{f_n\}_{n=1}^{\infty}$ a series $\sum_{n=1}^{\infty} g_n$ can be considered, if the condition $\underset{M}{V} f_n \to \underset{M}{V} f$ is substitued by the condition $\underset{M}{V} s_n \to \underset{M}{V} \sum_{n=1}^{\infty} g_n$. But also the following sufficient condition for the uniform convergence of infinite series holds.

T h e o r e m 5. Let $\sum_{n=1}^{\infty} g_n(x) = g(x)$ for every $x \in M$, where g_n and g are of bounded variations on M. Let the convergence of $\sum_{n=1}^{\infty} g_n(x)$ is almost uniform on M and let M has the property (*). Let $\sum_{n=1}^{\infty} \underset{M}{V} g_n = \underset{M}{V} \sum_{n=1}^{\infty} g_n$. Then $\sum_{n=1}^{\infty} g_n$ is uniformly convergent on M.

P r o o f. In view of the last note it is sufficient to prove $\underset{M}{V} s_n \to \underset{M}{V} s$, where $s_n = \sum_{i=1}^{n} f_i$. The lower semicontinuity gives

(11) $\quad \underset{M}{V}\, s \leq \liminf_{n \to \infty} \underset{M}{V}\, s_n$

Further

$$\underset{M}{V}\, s_n = \underset{M}{V}\,(g_1 + g_2 + \dots + g_n) \leq \sum_{n=1}^{\infty} \underset{M}{V}\, g_n = \underset{M}{V}\, s$$

Hence

(12) $\quad \limsup_{n \to \infty} \underset{M}{V}\, s_n \leq \underset{M}{V}\, s$

The result follows from (11) and (12).

N o t e. The conditions $\sum_{n=1}^{\infty} \underset{M}{V}\, g_n = \underset{M}{V} \sum_{n=1}^{\infty} g_n$ and $\underset{M}{V}\, s_n \to \underset{M}{V}\, s$ are not equivalent, as the following example shows.

E x a m p l e. Put $g_n(x) = (-1)^{n+1} \frac{x^n}{n}$ for $n = 1,2,\dots$ and for $x \in \langle 0,1 \rangle$.

We have $s_n(x) = \sum_{i=1}^{n} (-1)^{i+1} \frac{x^i}{i}$, $\lim_{n \to \infty} s_n(x) = \log(1+x) = s(x)$.

Since $s_n (m = 1,2,\dots)$ and s are increasing on $\langle 0,1 \rangle$, we have

$$\underset{\langle 0,1 \rangle}{V} s_n = s_n(1) - s_n(0) = s_n(1); \quad \underset{\langle 0,1 \rangle}{V} s = \log 2$$

Hence $\lim_{n \to \infty} \underset{\langle 0,1 \rangle}{V} s_n = \lim_{n \to \infty} s_n(1) = \log 2 = \underset{\langle 0,1 \rangle}{V} s$

On the other hand $\sum_{n=1}^{\infty} \underset{M}{V}\, g_n = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

So the equality $\sum_{n=1}^{\infty} \underset{M}{V}\, g_n = \underset{M}{V} \sum_{n=1}^{\infty} g_n$ does not hold.

The pointwise convergence of $\{f_n\}_{n=1}^{\infty}$ and the condition $\underset{M}{V}\, f_n \to \underset{M}{V}\, f$ does not imply in general the uniform convergence, even in the case when M is an interval. It may be shown by a trivial example $\{x^n\}_{n=1}^{\infty}$ on $\langle 0,1 \rangle$. In what follows, it will be proved that the pointwise convergence and the condition $\underset{M}{V}\, f_n \to \underset{M}{V}\, f$ guarante the μ-almost uniform convergence in the sense of the Lebesgue measure μ.

D e f i n i t i o n. A sequence $\{f_n\}_{n=1}^{\infty}$ of measurable functions is said to be μ-almost uniformly convergent to a function f if to any $\varepsilon > 0$ there exists a measurable set E such that $\mu(E) < \varepsilon$ and $\{f_n\}$ is uniformly convergent to f on $M-E$.

T h e o r e m 6. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions with bounded variations which is convergent on the set M to a function f of bounded variation. Let $\underset{M}{V} f_n \to \underset{M}{V} f$. Then $\{f_n\}_{n=1}^{\infty}$ is μ-almost uniformly convergent to f on M.

P r o o f. In case of finite measure of the set M our Theorem is a corollary of Jegoroff's theorem ([4] p. 249). So let $\mu(M) = \infty$. Denote $M_k = M \cap \langle -k,k \rangle$ for $k = 1,2,3, \dots$. The sets M_k are of finite measure. Let $\varepsilon > 0$. From the convergence of $\{f_n\}_{n=1}^{\infty}$ on M_k, the μ-almost uniform convergence follows on M_k. Hence to $\frac{\varepsilon}{2^k}$ a measurable set $E_k \subset M_k$ exists such that $\mu(E_k) < \frac{\varepsilon}{2^k}$ and the convergence of $\{f_n\}_{n=1}^{\infty}$ is uniform on $M_k - E_k$. Put $E = \bigcup_{k=1}^{\infty} E_k$. Then $\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k) < \varepsilon$. We shall prove that the convergence of $\{f_n\}_{n=1}^{\infty}$ is uniform on $M-E$. In fact, let $\eta > 0$. There exist $a \in M$, $b \in M$, $a < b$ such that $\underset{M_1^*}{V} f < \eta$ on the set $M_1^* = M - (a,b)$. From the last, in the same manner as in the proof of theorem 1, the existence of N_1 follows such that for $n \geq N_1$

(13) $|f_n(x) - f(x)| < \eta$ for every $x \in M_1^*$.

Now let k be such chosen that $\langle a,b \rangle \subset \langle -k,k \rangle$, hence $\langle a,b \rangle \cap M \subset M_k$. Since $M_k - E \subset M_k - E_k$ and the convergence is uniform on $M_k - E_k$, there exists N_2 such that $|f_n(x) - f(x)| < \eta$ for $n \geq N_2$ and every $x \in M_k - E$. From (13) and the last inequality we get $|f_n(x) - f(x)| < \eta$ for $n \geq \max(N_1\ N_2)$ and every $x \in M-E$. The proof is finished.

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Received: January 14,1970

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(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXV – 1971)

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXV - 1971

ON THE VECTOR SOLUTION OF THE SYSTEM OF THE DIFFERENTIAL EQUATIONS $\underline{x}' = A(t)\,\underline{x} + \underline{f}(t)$

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§ 1. Introduction

We shall deal with the system of differential equations

(a) $\quad x' = A(t)\,\underline{x} + \underline{f}(t),$

where $A(t)$ denotes an n.n. continuous matrix of period p and $f(t)$ is a continuous periodic vector of n components of the same period p

(1) $\quad A(t + p) = A(t), \qquad \underline{f}(t + p) = \underline{f}(t).$

The corresponding homogeneous system is

(b) $\quad \underline{y}' = A(t)\,\underline{y}$

and the adjoint

(c) $\quad \underline{z}' = - A^T(t)\,\underline{z}$

where A^T denotes the transposed matrix of A . If

$$Y(t) = (\underline{y}_1(t),\ \underline{y}_2(t),\ \dots,\ \underline{y}_n(t))$$

represents a fundamental matrix solution of (b), then the equation (b) can be written in the form

($\bar{b}$) $\quad Y' = A(t)\,Y$

Let $Y(t)$ be a fundamental matrix solution of (b), such that the constant matrix (sec [1], theorems 1 & 2)

(2) $\quad P = Y^{-1}(t)\,Y(t + p)$

has the normal form

(3) $\quad P = e^{Kp}$

with the submatrices $P_\nu = e^{K_\nu p}$ where the matrix K is in the Jordan canonical normal form

(4) $$K = \begin{bmatrix} K_1 & & \\ & K_2 & \\ & & \ddots \\ & & & K_s \end{bmatrix} \qquad K_\nu = \begin{bmatrix} \alpha_\nu & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha_\nu \end{bmatrix} \quad (\nu=1,2,\dots,s).$$

The submatrices K_ν are of order m_ν. The fundamental matrix solution $Y(t)$ can be written in the form

(5) $$Y(t) = \Phi(t)\, e^{Kt}, \quad \Phi = (\varphi_1, \varphi_2, \dots, \varphi_n),$$

where the matrix $\Phi(t)$ is of period p and the constant matrix is in the Jordan normal form (4). The corresponding fundamental matrix solution of (c) is (see [2], § 3.2 or [3] § 1.4)

(6) $$Z(t) = (Y^{-1}(t))^T = (\Phi^{-1}(t))^T . e^{-K^T t} = \psi(t)\, e^{-K^T t}$$

$$\text{with } \psi = (\psi_1, \psi_2, \dots, \psi_n).$$

Referring to [1], theorem 3 there corresponds to each submatrix K_ν of K with the eigenvalue $\alpha_\nu = 0$ exactly one with p periodic solution $y_{(\nu)}(t) = \underline{\varphi}_{(\nu)}(t)$ of (b), and similarly one with p periodic solution $\underline{z}_{[\nu]}(t) = \underline{\psi}_{[\nu]}(t)$ pf (c). The indices (ν) and $[\nu]$ are defined by

(7) $$(\nu) = m_1 + m_2 + \dots + m_{\nu-1} + 1, \quad [\nu] = m_1 + m_2 + \dots + m_\nu .$$

It is comfortable to write the fundamental matrix solution $Y(t)$ in the form

(8) $$Y = (\underline{y}_1, \underline{y}_2, \dots, \underline{y}_n) = (Y_1, Y_2, \dots, Y_s)$$

$$\text{with } Y_\nu = (\underline{y}_{(\nu)}, \underline{y}_{(\nu)+1}, \dots, \underline{y}_{[\nu]})$$

where Y_ν is a rectangular matrix of type $n \times m_\nu$. Similarly for the other matrices $Z(t)$, $\Phi(t)$ and $\Psi(t)$. Thus referring to (5) and (6), we obtain

$$(9) \qquad Y_\nu(t) = \Phi_\nu(t)\, e^{K_\nu t} \ , \qquad Z_\nu(t) = \Psi_\nu(t)\, e^{-K_\nu^T t} \ .$$

Using the method of variation of parameters, the solution of (a) can be written in the form (see [1], (3))

$$\underline{x}(t) = \sum_1^n \underline{x}_\mu(t) = \sum_1^n \underline{y}_\mu(t)\, c_\mu(t) = Y(t)\, \underline{c}(t) =$$

$$= Y(t) \left(\int_0^t Z^T(\tau)\underline{f}(\tau) d\tau + \underline{c}\,(o) \right).$$

It is convenient to subdivide the vector solution $\underline{x}(t)$ as the sum of s vector components

$$(10) \qquad \underline{x}(t) = \sum_1^s {}^\nu\underline{x}(t)$$

with (see[1], (21) and (22))

$$(11) \qquad {}^\nu\underline{x}(t) = \sum_{(\nu)}^{[\nu]} \underline{x}_\mu(t) = \sum_{(\nu)}^{[\nu]} \underline{y}_\mu(t)\, c_\mu(t) = Y_\nu(t)\, \underline{c}_\nu(t) =$$

$$= Y_\nu(t) \left(\int_0^t Z_\nu^T(\tau)\, \underline{f}(\tau)\, d\tau + \underline{c}_\nu(o) \right),$$

where

$$(12) \qquad \underline{c}_\nu^T = (c_{(\nu)}, c_{(\nu)+1}, \ldots, c_{[\nu]}).$$

It should be noticed that the matrix K, which is defined in (4), plays a decisive role in the study of the boundedness or unboundedness of the solution $x(t)$ of the inhomogeneous differential equation (a) for unbounded increasing t, as it will be shown in § 4. However, the general way to obtain the matrix $K = \frac{1}{p} \log P$ (see (3) and [1], lemma 2) is in general laborius.

In this paper we give certain special cases to find this matrix K in a simple way. Further we prove that the vector components ${}^\nu\underline{x}(t)$

of the solution $x(t)$ of (a), which are formed from the vectors $\underline{x}_\mu(t)$ by the relation (11), satisfy the systems of differential equations that can be derived from (a). At the end of the paper we study the asymptotic behaviour of the vector solutions $^\nu\underline{x}\ (t)$ as well as of the total solutions $\underline{x}\ (t)$ of (a).

§ 2. The matrix K

In order to obtain the matrix K, as it is described in [1], theorem 2, we need in general a fundamental matrix solution $Y(t)$ of (a), from which we get the matrix $P = Y^{-1}(t)\ Y(t+p)$ and then by taking the logarithm and multiplying by the factor $\frac{1}{p}$, we obtain the matrix K.

That this goes in certain special cases without trouble, is shown in the following:

Theorem 1. If the periodic matrix $A(t)$ of period p is commutative with its integral from 0 to t, i.e. if the relation

$$(13) \qquad A(t) \cdot \int_0^t A(\tau)\, d\tau = \int_0^t A(\tau)\, d\tau \cdot A(t)$$

holds identically in t, then it follows with the constant matrix

$$(14) \qquad K^* = \frac{1}{p} \int_0^p A(\tau)\, d\tau$$

that

$$(15) \qquad P = e^{K^* p}.$$

To this belongs the fundamental matrix solution $Y(t)$

$$(16) \qquad Y(t) = e^{\int_0^t A(\tau) d\tau} = \Phi^*(t)\ e^{K^* t}$$

with the periodic matrix of period p,

$$(17) \qquad \Phi^*(t) = e^{\int_0^t (A(\tau)-K^*)\, d\tau}$$

P r o o f. In the following the two following formulas, which can be proved by means of the series representation of the exponential function will be used (see e.g. [4], § 8 or [5], § 10.9)

$$(18) \qquad e^{B+C} = e^{B} \cdot e^{C} , \qquad \text{if } \; B\,C = C\,B \; ;$$

$$(19) \qquad (e^{B(t)})' = e^{B(t)} \cdot B'(t) = B'(t) . e^{B(t)}, \quad \text{if } \; B'B = B\,B' .$$

First of all it follows by means of (19), that $Y(t)$ in equation (16) is a fundamental matrix solution of ($\overline{b}$) (Notice that $Y(o) = I$). The constant matrix P is obtained from (2) with $t = 0$

$$(20) \qquad P = Y^{-1}(o)\, Y(p) = e^{\int_0^p A(\tau)\, d\tau} ,$$

from which the relations (15) and (14) follows.

It remains only to prove that, the matrix $\overset{*}{\Phi}(t)$ in (16) can be written in the form (17) (see (5)). Evidently $\overset{*}{\Phi}(t)$ has the period p, since from (14)

$$\int_0^p (A(\tau) - K^*)\, d\tau = 0 .$$

The relations (16) and (17) leads to the identity

$$(21) \qquad e^{\int_0^t A(\tau)d\tau} = e^{\int_0^t (A(\tau) - K^*)d\tau} \cdot e^{K^* t} ,$$

if it is proved, that the matrix K^* is commutative with the matrix $\int_0^t A(\tau)d\tau$; i.e. if the relation

$$(22) \qquad \int_0^p A(\tau)d\tau \cdot \int_0^t A(\sigma)d\sigma = \int_0^t A(\sigma)d\sigma \cdot \int_0^p A(\tau)d\tau$$

holds. Since this relation is true for $t = 0$, then (22) follows by differentiating it w.r.t. t from the commutative law

$$(23) \qquad \int_0^p A(\tau)d\tau \cdot A(t) = A(t) \int_0^p A(\tau)d\tau .$$

To prove the validity of (23), we calculate by considering the periodicity of $A(t)$ from (1)

$$A(t) . \int_0^p A(\tau)d\tau = A(t) \left(\int_0^{t+p} A(\tau)d\tau - \int_p^{t+p} A(\tau)d\tau \right)$$

$$= A(t+p) \int_0^{t+p} A(\tau)d\tau - A(t) . \int_p^{t+p} A(\tau)d\tau$$

and by considering (13)

$$= \int_0^{t+p} A(\tau)d\tau . A(t+p) - \int_p^{t+p} A(\tau)d\tau . A(t)$$

$$= \left(\int_0^{t+p} A(\tau)d\tau - \int_p^{t+p} A(\tau)d\tau \right) A(t) = \int_0^p A(\tau)d\tau . A(t) .$$

Then (22) and (21) are proved.

By means of canonical transformation (see e.g. [4], § 5 or [6], § 6.20), we can bring the matrix (14) to the Jordan canonical normal form, which is expressed in (4) and moreover, such that the succession of the eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_s$ is the same. Thus we can obtain the matrix K^o, which is related to the matrix K in (4) by the relation.

(24) $\quad e^{K^o p} = P = e^{Kp}$.

The corresponding eigenvalues of the matrices (24) are equal. The eigenvalues α_ν of K and K^o can at most differ by $n_\nu . \frac{2\pi i}{p}$ with integral numbers n_ν . Thus

(25) $$K^o = K + \begin{bmatrix} E_1 & & & \\ & E_2 & & \\ & & \ddots & \\ & & & E_s \end{bmatrix} \quad \text{with } E_\nu = n_\nu . \frac{2\pi i}{p} . I_\nu ,$$

where I_ν represents the unit matrix of order m_ν . Hence we have

T h e o r e m 2. By means of canonical transformation, the matrix K^* in (14) can be transformed to the matrix K^o, which is related to K by the relation (25), such that K can be calculated.

T h e o r e m 3. The assumption (13) for the theorems 1 and 2 can be replaced by the most powerful one

(26) $A(t) \cdot A(\tau) = A(\tau) \cdot A(t)$

with arbitrary t and τ .

P r o o f. There is only to show, that the relation (13)(with arbitrary lower limit) follows from (26).

We calculate:

$$A(t) \cdot \int_{t_o}^{t} A(\tau)d\tau = \int_{t_o}^{t} A(t)\, A(\tau)d\tau = \int_{t_o}^{t} A(\tau)\, A(t)d\tau =$$

$$= \int_{t_o}^{t} A(\tau)d\tau \cdot A(t)$$

§ 3. The vector solution ${}^{\nu}\underline{x}(t)$

On the vectors ${}^{\nu}\underline{x}(t)$ (see (11)), we state the following:

T h e o r e m 4. The vectors ${}^{\nu}\underline{x}(t)$, which are defined in (11) satisfy the systems of differential equations

(27) $${}^{\nu}\underline{x}' = A(t)\, {}^{\nu}\underline{x} + {}^{\nu}\underline{f}(t) , \qquad \nu = 1,2, \ldots, s$$

with

(28) $${}^{\nu}\underline{f}(t) = \Phi_{\nu}(t)\, \underline{b}_{\nu}(t) = \Phi_{\nu}(t)\, \Psi_{\nu}^{T}(t)\, \underline{f}(t)$$

We need the following

L e m m a 1. By means of the transformation

(29) $$\underline{x}(t) = \Phi(t)\, \underline{u}(t)$$

with $\Phi(t)$ from (5), the system of differential equations (a) is reduced to the system with constant coefficients

(30) $\underline{u}' = K\,\underline{u} + \underline{b}(t)$

with (see (6))

(31) $\underline{b}(t) = \psi^T(t)\,\underline{f}(t), \qquad \underline{b}(t+p) = \underline{b}(t).$

P r o o f. Substituting from (29) in (a) and using (6), (31), we obtain

$$\underline{u}' = \Phi^{-1}(A\Phi - \Phi')\,\underline{u} + \underline{b}.$$

It remains only to show that the matrix premultiplied by $\underline{u}$ is equal to K, i.e.

(32) $\Phi' = A\Phi - \Phi K.$

Referring to (5) and (b), we calculate

$$\Phi^{-1}(A\Phi - \Phi') = \Phi^{-1}(A\Phi - Y'e^{-Kt} + Y\,e^{-Kt}\,.\,K)$$
$$= \Phi^{-1}(A\Phi - A\Phi e^{Kt}\,e^{-Kt} + \Phi e^{Kt}\,e^{-Kt}\,K) = K.$$

R e m a r k. The system of differential equations (30) can be subdivided into the s independent systems

(33) $\underline{u}_\nu' = K_\nu\,\underline{u}_\nu + b_\nu, \qquad \nu = 1, 2, \dots, s,$

where m_ν is the dimension of each vector. Here

(34) $\underline{u}^T = (\underline{u}_1, \underline{u}_2, \dots, \underline{u}_s)$ with $\underline{u}_\nu^T = (u_{(\nu)}, u_{(\nu)+1}, \dots, u_{[\nu]})$

and (see (29) and (8))

(35) $\underline{b}_\nu = \Psi_\nu^T\,\underline{f}.$

Referring to (29), (11) and (8), we also obtain

(36) ${}^{\nu}\underline{x} = \Phi_\nu\,\underline{u}_\nu.$

P r o o f o f T h e o r e m 4. Differentiating (36) under consideration of (33), we obtain

$${}^{\nu}\underline{x}' = \Phi_\nu'\underline{u}_\nu + \Phi_\nu\underline{u}_\nu' = \Phi_\nu'\underline{u}_\nu + \Phi_\nu(K_\nu\underline{u}_\nu + \underline{b}_\nu)$$
$$= (\Phi_\nu' + \Phi_\nu K_\nu)\,\underline{u}_\nu + \Phi_\nu\underline{b}_\nu.$$

Referring to (32), it can be easily verified, that

$$\Phi_\nu' + \Phi_\nu K_\nu = A\,\Phi_\nu .$$

Thus we have

$$\nu_{\underline{x}}' = A\Phi_\nu \underline{u}_\nu + \Phi_\nu \underline{b}_\nu .$$

By using (36) and (35), it follows that the relations (27) and (28) are true.

§ 4. The asymptotic behaviour of the vector solutions $\nu_{\underline{x}}(t)$

We prove the following lemmas

Lemma 2. Let $v(t)$ be an arbitrary periodic function of period p. Then

(37) $$\int_0^t v(\tau)d\tau = v_1(t) + k.t ,$$

where $v_1(t)$ is periodic of period p, and k is the constant

(38) $$k = \frac{1}{p}\int_0^p v(\tau)d\tau .$$

Proof. We have

$$\int_0^t v(\tau)d\tau = \int_0^t (v(\tau) - k)d\tau + \int_0^t k\,d\tau = v_1(t) + kt ,$$

where

$$v_1(t) = \int_0^t (v(\tau) - k)\,d\tau .$$

By virtue of (38), it follows immediately that

$$v_1(p) = \int_0^p (v(\tau) - k)d\tau = k\,p - k\,p = 0 = v_1(0) .$$

L e m m a 3. Let $\underline{v}_\nu^T(t) = (v_{(\nu)}, v_{(\nu)+1}, \dots, v_{[\nu]})$ be an arbitrary periodic vector of period p having m_ν components, such that its last component satisfies the relation

$$(39) \qquad \int_0^p v_{[\nu]}(\tau)d\tau \neq 0 .$$

And let

$$(40) \qquad D_\nu = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

be a matrix of rank $m_\nu - 1$ with $m_\nu = [\nu] - (\nu) + 1$ (see (7)). Then

$$(41) \qquad \int_0^t e^{D_\nu(t-\tau)} \underline{v}_\nu(\tau)d\tau = \underline{w}_\nu(t) + \begin{bmatrix} r(t) \\ r'(t) \\ \vdots \\ r^{(m_\nu-1)}(t) \end{bmatrix} ,$$

where $\underline{w}_\nu(t)$ is a periodic vector of period p with m_ν components, and $r(t)$ is a polynomial of degree m_ν .

P r o o f. First we notice that $D_\nu^r = 0$ for $r \geq m_\nu$ (see [4], § 5 especially p. 82). Denotig the unit matrix of rank m_ν by I_ν, we get

$$(42) \qquad \int_0^t e^{D(t-\tau)} \underline{v}_\nu(\tau)d\tau = \int_0^t \left\{ I_\nu + \sum_{\sigma=1}^{m_\nu-1} D_\nu^\sigma \frac{(t-\tau)^\sigma}{\sigma!} \right\} \underline{v}_\nu(\tau)\, d\tau =$$

$$= \int_0^t \left\{ \begin{bmatrix} v_{(\nu)} \\ \vdots \\ v_{[\nu]} \end{bmatrix} + \begin{bmatrix} v_{(\nu)+1} \\ \vdots \\ v_{[\nu]} \\ 0 \end{bmatrix} (t-\tau) + \dots + \begin{bmatrix} v_{[\nu]-1} \\ v_{[\nu]} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \frac{(t-\tau)^{m_\nu-2}}{(m_\nu-2)!} + \right.$$

$$\left. + \begin{bmatrix} v_{[\nu]} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \frac{(t-\tau)^{m_\nu-1}}{(m_\nu-1)!} \right\} d\tau .$$

But by virtue of lemma 2, we obtain for $\sigma = 0,1,\dots,m_\nu-1$ and fixed γ

$$\int_0^t v_\gamma^{(\sigma)}(\tau)d\tau = v_\gamma^{(\sigma+1)}(t) + k_\gamma^{(\sigma+1)}\cdot t \qquad \text{with } v_\gamma^{(0)} = v_\gamma \, ,$$

where $v_\gamma^{(\sigma+1)}$ is periodic of period p and

$$k_\gamma^{(\sigma+1)} = \frac{1}{p}\int_0^p v_\gamma^{(\sigma)}(\tau)d\tau \, . \tag{43}$$

Thus by using partial differentiation, the occuring integrals in (42) can be calculated successively for $\sigma = 0, 1, \dots, m_\nu-1$ and fixed γ

$$\int_0^t \frac{(t-\tau)^\sigma}{\sigma!} v_\gamma(\tau)d\tau = v_\gamma^{(\sigma+1)}(t) + \sum_{\mu=1}^{\sigma+1} k_\gamma^{(\mu)} \frac{t^{\sigma+2-\mu}}{(\sigma+2-\mu)!} \, . \tag{44}$$

Substituting from (44) in (42), we get

$$\int_0^t e^{D_\nu(t-\tau)} \underline{v}_\nu(\tau)d\tau = \begin{bmatrix} v_{(\nu)}^{(1)} + k_{(\nu)}^{(1)}t \\ \cdot \\ \cdot \\ \cdot \\ v_{[\nu]}^{(1)} + k_{[\nu]}^{(1)}t \end{bmatrix} + \begin{bmatrix} v_{(\nu)+1}^{(2)} + \sum_{\mu=1}^{2} k_{(\nu)+1}^{(\mu)} \frac{t^{3-\mu}}{(3-\mu)!} \\ \vdots \\ v_{[\nu]}^{(2)} + \sum_{\mu=1}^{2} k_{[\nu]}^{(\mu)} \frac{t^{3-\mu}}{(3-\mu)!} \\ 0 \end{bmatrix} + \dots +$$

$$+ \begin{bmatrix} v_{[\nu]}^{(m_\nu)} + \sum_{\mu=1}^{m_\nu} k_{[\nu]}^{(\mu)} \frac{t^{m_\nu+1-\mu}}{(m_\nu+1-\mu)!} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

This sum of vectors can be arranged, as it follows

$$
\int_0^t e^{D_\nu(t-\tau)} \underline{v}_\nu(\tau)\,d\tau = \begin{bmatrix} \sum\limits_{\mu=0}^{m_\nu-1} v^{(\mu+1)}_{(\nu)+\mu} \\ \sum\limits_{\mu=1}^{m_\nu-1} v^{(\mu)}_{(\nu)+\mu} - \sum\limits_{\mu=0}^{m_\nu-1} k^{(\mu+1)}_{(\nu)+\mu} \\ \sum\limits_{\mu=2}^{m_\nu-1} v^{(\mu)}_{(\nu)+\mu} - \sum\limits_{\mu=1}^{m_\nu-1} k^{(\mu)}_{(\nu)+\mu} \\ \vdots \\ v^{(1)}_{[\nu]} - k^{(1)}_{[\nu]}\, t \end{bmatrix} + \tag{45}
$$

$$
+ \begin{bmatrix} k^{(1)}_{(\nu)} \cdot t \\ k^{(1)}_{(\nu)} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} k^{(1)}_{(\nu)+1} \dfrac{t^2}{2!} + k^{(2)}_{(\nu)+1} \cdot t \\ k^{(1)}_{(\nu)+1} \cdot t + k^{(2)}_{(\nu)+1} \\ k^{(1)}_{(\nu)+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} +
$$

$$
+ \dots + \begin{bmatrix} \sum\limits_{\mu=1}^{m_\nu} k^{(\mu)}_{[\nu]} \dfrac{t^{m_\nu-\mu+1}}{(m_\nu-\mu+1)!} \\ \sum\limits_{\mu=1}^{m_\nu} k^{(\mu)}_{[\nu]} \dfrac{t^{m_\nu-\mu}}{(m_\nu-\mu)!} \\ \sum\limits_{\mu=1}^{m_\nu-1} k^{(\mu)}_{[\nu]} \dfrac{t^{m_\nu-\mu-1}}{(m_\nu-\mu-1)!} \\ \vdots \\ k^{(1)}_{[\nu]} \end{bmatrix} .
$$

In the first summation we collect all periodic terms together, such that the other summations contain only polynomials in t with fixed lower indices. To every polynomial vector, we have to add suitable constants, such that the relation on the R.S. of (41) is satisfied. Adding the additional negative constants to the periodic vector in the forst summation, we obtain the required formula (41). Now in the formula (45), there exists a term of the highest possible power t^{m_ν}, namely the term $k^{(1)}_{[\nu]} \frac{t^{m_\nu}}{m_\nu!}$. But the factor $K^{(1)}_{[\nu]}$ is different from zero because of the assumptions (39) and (43). Consequently, it follows that the polynomial r(t) in (41) is of degree m_ν, which completes the proof.

<u>L e m m a 4.</u> By adding a vector of the form $e^{D_\nu t} \underline{c}_\nu(o)$ to to both sides of (41) with D_ν and $\underline{c}^{(o)}_\nu$ from (40) and (12), we can obtain the special form

$$(46)\quad \int_0^t e^{D_\nu(t-\tau)} \underline{v}_\nu(\tau)d\tau + e^{D_\nu t} \underline{c}^{(o)}_\nu = \underline{w}_\nu(t)+k^{(1)}_{[\nu]} \begin{bmatrix} \frac{t^{m_\nu}}{m_\nu!} \\ \frac{t^{m_\nu-1}}{m_\nu-1!} \\ \vdots \\ t \end{bmatrix}$$

with (see (43))

$$(47)\quad k^{(1)}_{[\nu]} = \frac{1}{p} \int_0^p v_{[\nu]}(\tau)d\tau ..$$

<u>P r o o f.</u> For arbitrary $\underline{c}^{(o)}_\nu$, we get (see also [1], (18))

$$e^{D_\nu t} \underline{c}^{(o)}_\nu = \begin{bmatrix} c^{(o)}_{(\nu)} + t\, c^{(o)}_{(\nu)+1} + \dots + \frac{t^{m_\nu-1}}{(m_\nu-1)!} c^{(o)}_{[\nu]} \\ c^{(o)}_{(\nu)+1} + \dots + \frac{t^{m_\nu-2}}{(m_\nu-2)!} c^{(o)}_{[\nu]} \\ \vdots \\ c^{(o)}_{[\nu]} \end{bmatrix} = \begin{bmatrix} q(t) \\ q'(t) \\ \vdots \\ q^{(m_\nu-1)}(t) \end{bmatrix}$$

where q(t) is an arbitrary polynomial of degree m -1. Referring to (41) and (45), the relation

$$r(t) = k^{(1)}_{[\nu]} \frac{t^{m_\nu}}{m_\nu!} + h(t)$$

holds, where $h(t)$ is a definite polynomial of degree m_ν-1. We need only to determine the constants $c_{(\nu)}^{(o)}, \ldots, c_{[\nu]}^{(o)}$, such that the polynomial $q(t) = -h(t)$.

On the boundedness of the vector solution ${}^{\nu}\underline{x}(t)$ for unbounded increasing t, we state the following

L e m m a 5. Each component of the vector solution ${}^{\nu}\underline{x}(t)$ is bounded for t increasing without limit and periodic with period p, in either one of the following two cases:

i) When the eigenvalue α_ν of the corresponding submatrix k_ν is not equal zero (here ${}^{\nu}\underline{x}$ is uniquely determined);

ii) when the eigenvalue $\alpha_\nu = 0$ and $\int_0^p \underline{z}^T_{[\nu]}(\tau)\, \underline{f}(\tau)d\tau = 0$ simultaneously (here ${}^{\nu}\underline{x}$ is uniquely determined with the exception of one parameter).

(Proof. See [1], theorem 4)

Now we prove the following theorem and further by using elementary methods.

T h e o r e m 5. Let K_ν be a submatrix of K with the eigenvalue $\alpha_\nu = 0$. And let

$$\int_0^p \underline{z}^T_{[\nu]}(\tau)\, \underline{f}(\tau)d\tau \neq 0.$$

Then the vector solution ${}^{\nu}\underline{x}(t) = \sum_{\mu=(\nu)}^{[\nu]} \underline{x}_\mu(t)$ takes - independent of the initial conditions - values of the power order t^{m_ν}, where m_ν is the order of K_ν.

P r o o f. Referring to (11) and (9), we have

$$(48) \qquad {}^{\nu}\underline{x}(t) = Y_{\nu}(t) \left(\int_0^t Z_{\nu}^T(\tau)\, \underline{f}(\tau)\, d\tau + \underline{c}_{\nu}(0) \right)$$

$$= \Phi_{\nu}(t) \left(\int_0^t e^{D_{\nu}(t-\tau)} \Psi_{\nu}^T(\tau)\, \underline{f}(\tau) d\tau + e^{D_{\nu}t}\, \underline{c}_{\nu}(0) \right).$$

Applying lemma 4, we can choose $\underline{c}_{\nu}(o)$ to obtain a particular solution ${}^{\nu}\underline{x}^*(t)$ in the form

$$(49) \qquad {}^{\nu}\underline{x}^*(t) = \Phi_{\nu}(t) \left\{ \underline{w}_{\nu}(t) + k_{[\nu]}^{(1)} \begin{bmatrix} \frac{t^{m_{\nu}}}{m_{\nu}!} \\ \frac{t^{m_{\nu}-1}}{(m_{\nu}-1)!} \\ \vdots \\ t \end{bmatrix} \right\},$$

where the vector $\underline{w}_{\nu}^T(t) = (w_{(\nu)}, \ldots, w_{[\nu]})$ is periodic of period p. Further from (47), (48) and (9) we have

$$(50) \qquad k_{[\nu]}^{(1)} = \frac{1}{p} \int_0^p v_{[\nu]}(t) d\tau = \frac{1}{p} \int_0^p \psi_{[\nu]}^T (\tau) \underline{f}(\tau) d\tau =$$

$$= \frac{1}{p} \int_0^p z_{[\nu]}^T(\tau)\, \underline{f}(\tau) d\tau \neq 0.$$

Every solution ${}^{\nu}\underline{x}(t)$ can be written in the form

$$(51) \qquad {}^{\nu}\underline{x}(t) = {}^{\nu}\underline{x}^*(t) + \Phi_{\nu}(t)\, e^{D_{\nu}t}\, \underline{c}_{\nu}^*(0).$$

Evidently the particular solution (49) takes values of the power order $t^{m_{\nu}}$. This is also valid for every ${}^{\nu}\underline{x}(t)$ from (51) with arbitrary $\underline{c}_{\nu}^*(0)$ because $t^{m_{\nu}-1}$ is the highest power which is inclued in $e^{D_{\nu}t}\, \underline{c}_{\nu}^*(o)$

Corollary 1. If $\int_0^p \underline{z}^T_{[\nu]}(\tau)\, \underline{f}(\tau)d\tau = 0$, then the particular solution (49) is a periodic function of period p see (50). Also the general solution (51) is periodic of period p, when the first component $c^*_{(\nu)}(o)$ of $\underline{c}^*_\nu(o)$ is arbitrary chosen, while the other components of $\underline{c}^*_\nu(o)$ are equal zero. This result coincides certainly with lemma 5.

Corollary 2. If $\alpha_\nu = 0$ and $\int_0^p \underline{z}^T_{[\nu]}(\tau)\, \underline{f}(\tau)d\tau \neq o$, then each vector component $\underline{x}_\mu(t)$ of the sum ${}^\nu\underline{x}(t) = \sum_{\mu=(\nu)}^{[\nu]} \underline{x}_\mu(t)$ (see (11)) takes values of the power order t^{m_ν}.

Proof. Substituting from (9) in (49), we obtain a particular solution

$$
{}^\nu\underline{x}^*(t) = Y_\nu(t)\, e^{-D_\nu t} \left\{ \underline{w}_\nu(t) + K^{(1)}_{[\nu]} \begin{bmatrix} \frac{t^{m_\nu}}{m_\nu!} \\ \frac{t^{m_\nu-1}}{(m_\nu-1)!} \\ \vdots \\ t \end{bmatrix} \right\}
$$

$$
= (\underline{y}_{(\nu)}, \ldots, \underline{y}_{[\nu]}) \cdot \begin{bmatrix} 1, & -t, & \ldots, & \frac{(-t)^{m_\nu-1}}{(m_\nu-1)!} \\ & & & \vdots \\ & & & 1 \end{bmatrix}
$$

$$\cdot \left\{ \underline{w}_\nu(t)+k^{(1)}_{[\nu]} \begin{bmatrix} \dfrac{t^{m_\nu}}{m_\nu!} \\ \dfrac{t^{m_\nu-1}}{(m_\nu-1)!} \\ \vdots \\ t \end{bmatrix} \right\};$$

i.e.

$$(52) \qquad {}^{\nu}\underline{x}^*(t) = \left(\underline{y}_\nu\,,\ -t\,\underline{y}_{(\nu)}+\underline{y}_{(\nu)+1},\dots,\ \underline{y}_{(\nu)}\frac{(-t)^{m_\nu-1}}{(m_\nu-1)!}+\right.$$

$$\left. +\ \underline{y}_{(\nu)+1}\frac{(-t)^{m_\nu-2}}{(m_\nu-2)!}+\dots+y_{[\nu]}\right)\cdot$$

$$\cdot \left\{ \underline{w}_\nu(t)+k^{(1)}_{[\nu]} \begin{bmatrix} \dfrac{t^{m_\nu}}{m_\nu!} \\ \dfrac{t^{m_\nu-1}}{(m_\nu-1)!} \\ \vdots \\ t \end{bmatrix} \right\}.$$

Referring to (52) and (11)', it follows for the vector components $\underline{x}^*_{(\nu)+\mu}$ ($\mu = 0,1,\dots,m-1$) that

$$(53) \quad x^*_{(\nu)+\mu}(t) = \underline{y}_{(\nu)+\mu}(t)\left(-k^{(1)}_{[\nu]}\frac{(-t)^{m_\nu-\mu}}{(m_\nu-)!}+\sum_{\sigma=\mu}^{m_\nu-1} w_{(\nu)+\sigma}\frac{(-t)^{\sigma-\mu}}{(\sigma-\mu)!}\right)$$

But from (9) we have

$$(54)\qquad \underline{y}_{(\nu)+\mu}(t) = \sum_{\gamma=0}^{\mu} \varphi_{(\nu)+\gamma}(t)\,\frac{t^{\mu-\gamma}}{(\mu-\gamma)!} \quad (\text{for } \mu = 0,1,\dots,m_\nu-1).$$

Setting (54) in (53), we get for $\mu = 0,1,\dots, m_\nu-1$

$$(55)\qquad x^*_{(\nu)+\mu}(t)=(-1)^{m_\nu-\mu+1}\,k^{(1)}_{[\nu]}\cdot \sum_{\gamma=0}^{\mu} \underline{\varphi}_{(\nu)+\gamma}(t)\,\frac{t^{m_\nu-\gamma}}{(\mu-\gamma)!\,(m_\nu-\mu)!} +$$

$$+\sum_{\gamma=0}^{\mu} \underline{\varphi}_{(\nu)+\gamma}(t)\left(\sum_{\sigma=\mu}^{m_\nu-1}(-1)^{\sigma-\mu}\,w_{(\nu)+\sigma}\,\frac{t^{\sigma-\gamma}}{(\mu-\gamma)!\,(\sigma-\mu)!}\right).$$

From (55) it is clear that each of the vector components $\underline{x}^*_{(\nu)+\mu}$ of the sum ${}^{\nu}\underline{x}^*(t) = \sum_{\mu=0}^{m_\nu-1} x^*_{(\nu)+\mu}$ takes values of the power order t^{m_ν}. This is also valid for each vector component $\underline{x}_{(\nu)+\mu}$ from (51) (for $\mu = 0,1,\dots,m_\nu-1$) with arbitrary $\underline{c}^*_\nu(o)$.

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Received: November 10, 1969

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(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXV – 1971)

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXV - 1971

ÜBER DIE SCHWACHE HALBSTETIGKEIT VON UNTEN EINES TYPES NICHTQUADRATISCHER FUNKTIONALE

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1. Wir werden uns mit dem System der partiellen Differentialgleichungen für w, u, v der Form

$$(1)\quad \begin{cases} \dfrac{D}{h}\Delta\Delta w = \dfrac{\partial^2 w}{\partial x^2}\sigma_x + \dfrac{\partial^2 w}{\partial y^2}\sigma_y + 2\dfrac{\partial^2 w}{\partial x\partial y}\tau - k_1(x,y)\,\sigma_x - \\ \qquad - k_2(x,y)\,\sigma_y + \dfrac{q}{h} \\ \dfrac{\partial\sigma_x}{\partial x} + \dfrac{\partial\tau}{\partial y} = 0, \\ \dfrac{\partial\tau}{\partial x} + \dfrac{\partial\sigma_y}{\partial y} = 0 \end{cases}$$

befassen, wo

$$\sigma_x = \frac{E}{1-\mu^2}\left[\frac{\partial u}{\partial x} + k_1(x,y)w + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \mu\left(\frac{\partial v}{\partial y} + k_2(x,y)w + \right.\right.$$
$$\left.\left. + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right)\right];$$

$$\sigma_y = \frac{E}{1-\mu^2}\left[\frac{\partial v}{\partial y} + k_2(x,y)\,w + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x}\right) + k_1(x,y)w + \right.$$
$$\left. + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)\Big];$$

$$\tau = \frac{E}{2(1+\mu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

D, h, E, μ sind positive Konstanten.

Wir werden die Gleichungen (1) im begrenzten Gebeit Ω ($\Omega < E_2$) mit der lipschitzschen Grenze $\partial\Omega$ untersuchen, bei beliebiger (entsprechender) Kombination folgender Randbedingungen:

$$(2)\quad \begin{cases} a_1|\quad w|_{\partial\Omega} = \left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = 0; & a_2|\quad u|_{\partial\Omega} = v|_{\partial\Omega} = 0; \\ b_1|\quad \Delta w|_{\partial\Omega} = 0; & b_2|\quad \sigma_x n_x + \tau n_y|_{\partial\Omega} = h_1 \\ & \qquad \tau n_x + \sigma_y n_y|_{\partial\Omega} = h_2 \end{cases}$$

wo n eine äussere Normale zu $\partial\Omega$ ist und n_x, n_y sind deren Komponenten. Setzen wir voraus, dass die Funktionen $k_1(x,y)$, $k_2(x,y)$ genügend glatt sind und dass $q(x,y) \in L_2(\Omega)$; $h_1 \in L_2(\partial\Omega)$, $h_2 \in L_2(\partial\Omega)$. Wir werden die Bezeichnungen aus [1] benutzen. Die schwache Konvergenz werden wir mit $\rightharpoonup$ bezeichnen.

2. Setzen wir $V = \hat{\mathring{W}}_2^{(2)}(\Omega) \times (\mathring{W}_2^{(1)}(\Omega))^2$ und für das Tripel $(w, u, v) \in V$ definieren wir die Norm mit der Vorschrift

$$\|(w,u,v)\|_V^2 = \|w\|^2_{\mathring{W}_2^{(2)}(\Omega)} + \|u\|^2_{\mathring{W}_2^{(1)}(\Omega)} + \|v\|^2_{\mathring{W}_2^{(1)}(\Omega)}$$

Setzen wir weiter $W = \hat{\mathring{W}}_2^{(2)}(\Omega) \times (W_2^{(1)}(\Omega))^2$. Im Raum W definieren wir die Norm ähnlich wie in V.

Wenn in der Kombination der Randbedingungen die Bedingungen $a_2/$ auftreten, dann ist das Tripel $(w, u, v) \in V$ eine schwache Lösung von (1), (2) wenn für jedes Tripel $(\tilde{w}, \tilde{u}, \tilde{v}) \in V$ gilt

$$\frac{D}{h}\int_\Omega \left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial y^2}\right) d\Omega \; -$$

$$(3)\qquad -\int_{\Omega}\left(\frac{\partial^2 w}{\partial x^2}\sigma_x + \frac{\partial^2 w}{\partial y^2}\sigma_y + 2\frac{\partial^2 w}{\partial x\partial y}\tau\right)\tilde{w}\,d\Omega - \frac{1}{h}\int_{\Omega} q\,\tilde{w}\,d\Omega +$$

$$+\int_{\Omega}\left(\sigma_x\frac{\partial\tilde{u}}{\partial x} + \tau\frac{\partial\tilde{u}}{\partial y} + \tau\frac{\partial\tilde{v}}{\partial x} + \sigma_y\frac{\partial\tilde{v}}{\partial y}\right)d\Omega = 0$$

Das zweite Additionsglied in (3) können wir durch Integration per partes in die Form

$$-\int_{\Omega}\left(\frac{\partial w}{\partial x}\frac{\partial\tilde{w}}{\partial x}\sigma_x + \frac{\partial w}{\partial y}\frac{\partial\tilde{w}}{\partial y}\sigma_y + \frac{\partial w}{\partial x}\frac{\partial\tilde{w}}{\partial y}\tau + \frac{\partial w}{\partial y}\frac{\partial\tilde{w}}{\partial x}\tau\right)d\Omega$$

bringen.

Die linke Seite in (3) bezeichnen wir als $A(w,u,v)(\tilde{w},\tilde{u},\tilde{v})$.

Wenn in der Kombination der Randbedingungen die Bedingungen b_2/ auftreten, dann ist $(w,u,v)\in W$ eine schwache Lösung von (1),(2) wenn für jedes Tripel $(\tilde{w},\tilde{u},\tilde{v})\in W$

$$A(w,u,v)\,(\tilde{w},\tilde{u},\tilde{v}) - \int_{\partial\Omega} h_1\,\tilde{u}\,ds - \int_{\partial\Omega} h_2\,\tilde{v}\,ds = 0$$

gilt.

3. Es sei E ein realer Banachscher Raum. Es sei F ein auf E definierter Operator. Wenn im Punkte $\alpha\in E$

$$\lim_{t\to 0}\frac{F(\alpha + t\beta) - F(\alpha)}{t} = VF(\alpha,\beta), \qquad \beta\in E$$

existiert, dann heisst der Operator $VF(\alpha,\beta)$ Gateauxischer oder schwacher Differential des Operators F im Punkte α. Wenn $VF(\alpha,\beta)$ in Bezung auf β linear ist, bezeichnen wir dies mit $DF(\alpha,\beta)$ und nennen wir ein linears Gateauxisches Differential des Operators F im Punkte α.

Es sei f ein Funktional auf E, welches auf der Menge M E ein linears Gateauxisches Differential $Df(\alpha,\beta)$ besitzt. Der durch die Beziehung

$$Df(\alpha,\beta) = F(\alpha)\,\beta, \qquad \beta\in E$$

definierte Operator $F(\alpha)$ wird Gradient des Funktionals $f(\alpha)$ genannt und wir schreiben $F(\alpha) = \operatorname{grad} f(\alpha)$.

D e f i n i t i o n 1. Es sei F ein Operator von E nach E^* (adjungiert zu E). Wir sagen, dass F auf der Menge $M \subset E$ potential ist, wenn ein solches Funktional f existiert, dass für alle $\alpha \in M$, $\operatorname{grad} f(\alpha) = F(\alpha)$ ist.

H i l f s s a t z 1. ([2], § 5). Es sei $F(\alpha)$ ein Operator von E nach E^*. Es gelte

1. $F(\alpha)$ hat das lineare Gateauxische Differential $DF(\alpha,\beta)$ in jedem Punkt α der Kugel $D_r = \{\alpha ; \|\alpha-\alpha_0\|_E < r,\ r > 0\}$.
2. Das Funktional $DF(\alpha,\beta)\gamma$ ist in Bezug auf α stetig in jedem Punkt $\alpha \in D_r$.

Dann ist die notwendige und hinreichende Bedingung dazu, dass $F(\alpha)$ in D_r potential sei

$$DF(\alpha,\beta)\gamma = DF(\alpha,\gamma)\beta$$

Es existiert ein einziges, in α_0 den angegebenen Wert f_0 erhaltendes Funktional $f(\alpha)$, dessen Gradient der Operator $F(\alpha)$ ist. Dieses Funktional hat die Form

$$(4)\qquad f(\alpha) = f_0 + \int_0^1 (F(\alpha_0 + t(\alpha-\alpha_0)),\ (\alpha-\alpha_0))\, dt$$

S a t z 1. Der operator $A(w,u,v)$ ist auf V sowie auch auf W potential.

B e w e i s. Wir zeigen, dass beide Voraussetzungen des Hilfssatzes 1 erfüllt sind.

Wir bezeichnen $\alpha = (w,u,v)$, $\beta = (w_1, u_1, v_1)$, $\gamma = (\tilde{w}, \tilde{u}, \tilde{v})$.

Wir errechnen den Quotient

$$\frac{A(\alpha + t\beta)\gamma - A(\alpha)\gamma}{t} = \frac{D}{h}\int_\Omega \left(\frac{\partial^2 w_1}{\partial x^2}\frac{\partial^2 \tilde{w}}{\partial x^2} + 2\frac{\partial^2 w_1}{\partial x\partial y}\frac{\partial^2 \tilde{w}}{\partial x\partial y} + \right.$$

$$\left. + \frac{\partial^2 w_1}{\partial y^2}\frac{\partial^2 \tilde{w}}{\partial y^2}\right) d\Omega + \int_\Omega \left\{ \left[\frac{\partial w}{\partial x}\sigma_x(\alpha,\beta) + \frac{\partial w_1}{\partial x}\sigma_x + \frac{t}{2}\left(\frac{\partial w}{\partial x}\left[\left(\frac{\partial w_1}{\partial x}\right)^2 + \right.\right.\right.\right.$$

$$+\mu\left(\frac{\partial w_1}{\partial y}\right)^2\Bigg] + \frac{\partial w_1}{\partial x}\sigma_x(\alpha,\beta) + t\frac{\partial w_1}{\partial x}\left[\left(\frac{\partial w_1}{\partial x}\right)^2 + \mu\left(\frac{\partial w_1}{\partial y}\right)^2\right]\Bigg)\Bigg]\frac{\partial w}{\partial x} +$$

$$+\left[\frac{\partial w}{\partial y}\sigma_y(\alpha,\beta) + \frac{\partial w_1}{\partial y}\sigma_y + \frac{t}{2}\left(\frac{\partial w}{\partial y}\left[\left(\frac{\partial w_1}{\partial y}\right)^2 + \mu\left(\frac{\partial w_1}{\partial x}\right)^2\right] + \frac{\partial w_1}{\partial y}\sigma_y(\alpha,\beta) + \right.\right.$$

$$+ t\frac{\partial w_1}{\partial y}\left[\left(\frac{\partial w_1}{\partial y}\right)^2 + \mu\left(\frac{\partial w_1}{\partial x}\right)^2\right]\Bigg)\Bigg]\frac{\partial w}{\partial y} + \left[\frac{\partial w}{\partial x}\tau(\alpha,\beta) + \frac{\partial w_1}{\partial x}\tau + \right.$$

$$+ t\frac{\partial w_1}{\partial x}\left(\frac{\partial w_1}{\partial x}\frac{\partial w}{\partial x} + \tau(\alpha,\beta) + t\frac{\partial w_1}{\partial x}\frac{\partial w_1}{\partial y}\right)\Bigg]\frac{\partial \tilde{w}}{\partial y} +$$

$$+\left[\frac{\partial w}{\partial y}\tau(\alpha,\beta) + \frac{\partial w_1}{\partial y}\tau + t\frac{\partial w_1}{\partial y}\left(\frac{\partial w}{\partial y}\frac{\partial w_1}{\partial x} + \tau(\alpha,\beta) + \right.\right.$$

$$+ t\frac{\partial w_1}{\partial x}\frac{\partial w_1}{\partial y}\Bigg)\Bigg]\frac{\partial \tilde{w}}{\partial x}\Bigg\}d\Omega + \int_\Omega\left[\left(\sigma_x(\alpha,\beta) + t\left[\left(\frac{\partial w_1}{\partial x}\right)^2 + \right.\right.\right.$$

$$+\mu\left(\frac{\partial w_1}{\partial y}\right)^2\Bigg]\Bigg)\frac{\partial \tilde{u}}{\partial x} + \left(\tau(\alpha,\beta) + t\frac{\partial w_1}{\partial x}\frac{\partial w_1}{\partial y}\right)\left(\frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x}\right) +$$

$$+\left(\sigma_y(\alpha,\beta) + t\left[\left(\frac{\partial w_1}{\partial y}\right)^2 + \mu\left(\frac{\partial w_1}{\partial x}\right)^2\right]\right)\frac{\partial \tilde{v}}{\partial y}\Bigg]d\Omega\,,$$

wo

$$\sigma_x(\alpha,\beta) = \frac{E}{1-\mu^2}\left[\frac{\partial u_1}{\partial x} + k_1 w_1 + \frac{\partial w}{\partial x}\frac{\partial w_1}{\partial x} + \mu\left(\frac{\partial v_1}{\partial y} + k_2 w_1 + \right.\right.$$

$$\left.\left. + \frac{\partial w}{\partial y}\frac{\partial w_1}{\partial y}\right)\right],$$

$$\sigma_y(\alpha,\beta) = \frac{E}{1-\mu^2}\left[\frac{\partial v_1}{\partial y} + k_1 w_1 + \frac{\partial w}{\partial y}\frac{\partial w_1}{\partial y} + \mu\left(\frac{\partial u_1}{\partial x} + k_1 w_1 + \right.\right.$$

$$+ \frac{\partial w}{\partial x}\frac{\partial w_1}{\partial x}\Big)\Big]$$

$$\tau(\alpha,\beta) = \frac{E}{(2(1+\mu)}\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} + \frac{\partial w_1}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w_1}{\partial y}\right)$$

Wie zu sehen ist, der Limes dieses Quotienten für $t \to 0$ existiert und es gilt

$$VA(\alpha,\beta)\gamma = \frac{D}{h}\int_\Omega \left(\frac{\partial^2 w_1}{\partial x^2}\frac{\partial^2 \tilde{w}}{\partial x^2} + 2\frac{\partial^2 w_1}{\partial x \partial y}\frac{\partial^2 \tilde{w}}{\partial x \partial y} + \frac{\partial^2 w_1}{\partial y^2}\frac{\partial \tilde{w}}{\partial y^2}\right) d\Omega +$$

$$(5) \quad + \int_\Omega \Big[\Big(\frac{\partial w}{\partial x}\sigma_x(\alpha,\beta) + \frac{\partial w_1}{\partial x}\sigma_x\Big)\frac{\partial \tilde{w}}{\partial x} + \Big(\frac{\partial w}{\partial y}\sigma_y(\alpha,\beta) +$$

$$+ \frac{\partial w_1}{\partial y}\sigma_y\Big)\frac{\partial \tilde{w}}{\partial y} + \Big(\frac{\partial w}{\partial x}\tau(\alpha,\beta) + \frac{\partial w_1}{\partial x}\tau\Big)\frac{\partial \tilde{w}}{\partial y} + \Big(\frac{\partial w}{\partial y}\tau(\alpha,\beta) +$$

$$+ \frac{\partial w_1}{\partial y}\tau\Big)\frac{\partial \tilde{w}}{\partial x}\Big] d\Omega + \int_\Omega \Big[\sigma_x(\alpha,\beta)\frac{\partial \tilde{u}}{\partial x} +$$

$$+ \tau(\alpha,\beta)\left(\frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x}\right) + \sigma_y(\alpha,\beta)\frac{\partial \tilde{v}}{\partial y}\Big] d\Omega$$

$VA(\alpha,\beta)\gamma$ ist bezüglich zu β linear. Also können wir $VA(\alpha,\beta)\gamma = DA(\alpha,\beta)\gamma$ schreiben. Wenn wir $\|\alpha_n - \alpha\|_{V,W} < \varepsilon$ voraussetzen, dann erhalten wir durch Anwendung der Hölderischen Ungleichung die Stetigkeit $DA(\alpha,\beta)\gamma$ bezüglich auf α. Die Gleichheit $DA(\alpha,\beta)\gamma = DA(\alpha,\gamma)\beta$ erhalten wir durch Errechnung unter Anwendung bekannter Beziehungen.

Also ist $A(w,u,v)$ ein potentialer Operator und deshalb existiert das Funktional $f(w,u,v)$ für welches grad $f(w,u,v) = A(w,u,v)$ gilt. Dieses Funktional kann leicht gefunden werden (z.B. durch Anwendung der Formel (4)). Es hat die Form

$$f(w,u,v) = \frac{D}{2h}\int_{\Omega}(\Delta w)^2\, d\Omega + \frac{E}{2(1-\mu^2)}\int_{\Omega}\Bigg\{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2 +$$

$$+ (1-\mu)\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\frac{1-\mu}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\frac{1}{4}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]^2+$$

$$+\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial v}{\partial y}\left(\frac{\partial w}{\partial y}\right)^2+\mu\left[\frac{\partial v}{\partial y}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial y}\right)^2\right]+k_1^2 w^2+$$

$$+2k_1 w\frac{\partial u}{\partial x}+k_1 w\left(\frac{\partial w}{\partial x}\right)^2+k_2^2 w^2+2k_2 w\frac{\partial v}{\partial y}+k_2 w\left(\frac{\partial w}{\partial y}\right)^2+$$

$$+2\mu\left[\frac{\partial u}{\partial x}k_2 w+2k_1k_2w^2+\frac{1}{2}k_2 w\left(\frac{\partial w}{\partial x}\right)^2+k_1 w\frac{\partial v}{\partial y}+\right.$$

$$\left.+\frac{1}{2}k_1 w\left(\frac{\partial w}{\partial y}\right)^2\right]\Bigg\}d\Omega+\frac{E}{2(1+\mu)}\int_{\Omega}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\,d\Omega -$$

$$-\frac{1}{h}\int_{\Omega} q\, w\, d\Omega - \int_{\partial\Omega} h_1 u ds - \int_{\partial} h_2 v ds$$

Wenn in den Randbedingungen die Form $a_1/$ oder das Gebeit Ω ein rechtwinkelig ist, dann können wir das erste Additionsglied in (6) in die Form [3]

$$\frac{D}{2h}\int_{\Omega}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2+2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right]d\Omega$$

bringen.

4. Es sei auf den rellen Banachschen Raum E irgendein Funktional $f(\alpha)$ gegeben.

D e f i n i t i o n 2. Das Funktional $f(\alpha)$ ist schwach halbsteting von unten im Punkte $\alpha_0 \in E$, wenn für jede Folge $\{\alpha_n\} \subset E$ die Implikation

$$(\alpha_n \xrightarrow[n\to\infty]{} \alpha_0) \Longrightarrow (f(\alpha_0) \leq \liminf_{n\to\infty} f(\alpha_n)) \quad \text{gilt.}$$

H i l f s s a t z 2. Das Funktional $f(\alpha)$ habe in der Kugel D_r das erste und zweite lineare Gateauxische Differential. Es sei in D_r das zweite Gateauxische Differential des Funktionals $f(\alpha)$ nichtnegativ. Dann ist $f(\alpha)$ auf D_r schwach halbstetig von unten. Der Beweis des Hilfssatzes 2 ist in [2] § 8 zu finden.

S a t z 2. Das Funktional (6) ist auf V und W schwach halbstetig von unten.

B e w e i s. Bezeichnen wir die Additionsglieder in (6) folgend:

$$f_1 = \int_\Omega (\Delta w)^2 \, d\Omega$$

$$f_2 = \int_\Omega \left\{ \mu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 + (1-\mu) \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \frac{1-\mu}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right\} d\Omega$$

$$f_3 = \int_\Omega w^2 \left[k_1(x,y) + k_2(x,y) \right]^2 d\Omega$$

$$f_4 = \int_\Omega qw \, d\Omega + \int_{\partial\Omega} h_1 u \, ds + \int_{\partial\Omega} h_2 v \, ds$$

$$f_5 = \int_\Omega w \left[\frac{\partial u}{\partial x} (k_1 + \mu k_2) + \frac{\partial v}{\partial y} (k_2 + \mu k_1) \right] d\Omega$$

$$f_6 = \int_\Omega \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \, d\Omega$$

$$f_7 = \int_\Omega \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right]^2 d\Omega$$

$$f_8 = \int_\Omega \left\{ \frac{\partial u}{\partial x} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial v}{\partial y} \left(\frac{\partial w}{\partial y} \right)^2 + \mu \left[\frac{\partial v}{\partial y} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial u}{\partial x} \left(\frac{\partial w}{\partial y} \right)^2 \right] \right\} d\Omega$$

$$f_9 = \int_\Omega w \left[\left(\frac{\partial w}{\partial x} \right)^2 (k_1 + \mu k_2) + \left(\frac{\partial w}{\partial y} \right)^2 (k_2 + \mu k_1) \right] d\Omega$$

Nach dem Hilfssatz 2 sind die Funktionale f_1, f_2, f_3, f_4 in W und V schwach halbstetig von unten.

Es sei $u_n \rightharpoonup u$ in $W_2^{(1)}(\Omega)$ und also auch in $L_2(\Omega)$. Gemäss den bekannten Sätzen über schwache Konvergenz ist dann

$$\forall_n \left(\|u_n\|_{W_2^{(1)}(\Omega)} \leq N , \quad \|u\|_{W_2^{(1)}(\Omega)} \leq N \right).$$

Also ist $\left\| \frac{\partial u}{\partial x} \right\|_{L_2(\Omega)} \leq R, \quad \forall_n \left\| \frac{\partial u_n}{\partial x} \right\|_{L_2(\Omega)} \leq R$.

$L_2(\Omega)$ ein reflexiver Raum, deshalb können wir von jeder in diesen Raum befindlichen begrenzten Folge eine in ihm schwach konvergente Folge entnehmen. Daraus und aus der Definition der verallgemeinerten Ableitung $\frac{\partial u}{\partial x}$ folgt sogleich, dass die ganze Folge $\left\{\frac{\partial u_n}{\partial x}\right\}_1^\infty$ schwach konvergent ist und in $L_2(\Omega)$ den Limes $\frac{\partial u}{\partial x}$ hat. Ähnlich können wir beweisen, dass auch

$$\frac{\partial v_n}{\partial x} \xrightarrow[L_2(\Omega)]{} \frac{\partial v}{\partial x}, \quad \frac{\partial u_n}{\partial x} \xrightarrow[L_2(\Omega)]{} \frac{\partial u}{\partial y}, \quad \frac{\partial v_n}{\partial y} \xrightarrow[L_2(\Omega)]{} \frac{\partial v}{\partial y}$$

unter der Voraussetzung, dass $\{u_n\}$, $\{v_n\}$ in $W_2^{(1)}(\Omega)$ schwach konvergiert. Es sei $w_n \xrightarrow[\overset{\circ}{W}{}_2^{(2)}(\Omega)]{} w$. Den Raum $\hat{\overset{\circ}{W}}{}_2^{(2)}(\Omega)$ können wir stetig in $\overset{\circ}{W}{}_2^{(1)}(\Omega)$ einbetten. Den Raum $\overset{\circ}{W}{}_2^{(1)}(\Omega)$ können wir kompakt in $\overset{\circ}{W}{}_2^{(1)}(\Omega)$ einbetten. Es ist also möglich $\hat{\overset{\circ}{W}}{}_2^{(2)}(\Omega)$ kompakt in $W_4^{(1)}(\Omega)$ einzubetten. Bei kompakter Abbildung geht die schwach konvergente Folge in eine stark konvergente über. Also $w_n \to w$ in $W_4^{(1)}(\Omega)$. Daraus erhalten wir aber gleich, dass $\frac{\partial w_n}{\partial x} \xrightarrow[L_4(\Omega)]{} \frac{\partial w}{\partial x}$. Ähnlich ist der Beweis für $\frac{\partial w_n}{\partial y}$.

Es ist ersichtlich, dass wir mit Hilfe der bekannten Implikation

$$(\alpha_n \xrightarrow[L_2(\Omega)]{} \alpha, \quad \beta_n \xrightarrow[L_2(\Omega)]{} \beta) \Longrightarrow (\alpha_n \beta_n \xrightarrow[L_2(\Omega)]{} \alpha\beta)$$

und mit Anwendung der Definition schwacher Konvergenz der Elemente in $L_2(\Omega)$ erhalten, dass die Funktionale f_5, f_6, f_7, f_8 auf V und auch auf W schwach stetig sind.

F o l g e r u n g. Das Funktional (6) ist auf V und auch auf W stetig.

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Anschrift des Verfassers: Katedra matematickej analýzy PFUK, Bratislava, Šmeralova 2/b

Eingegangen am: 5. März 1970

(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXV – 1971)

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXV - 1971

ON THE VALUES OF EULER'S FUNCTION

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Euler's function φ is defined for all natural numbers as follows: $\varphi(n)$ denotes the number of numbers $0,1,2,\dots,n-1$ coprime with the number n.

It is known that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ is the standard form of the natural number $n > 1$, then

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_k}\right) \quad ;$$

written in another way

$$(1) \qquad \varphi(n) = p_1^{\alpha_1-1} p_2^{\alpha_2-1} \dots p_k^{\alpha_k-1}(p_1-1)(p_2-1)\dots(p_k-1)$$

From the definition of Euler's function and from (1) we may conclude: $\varphi(1) = \varphi(2) = 1$, $\varphi(n)$ is an even number, if $n > 2$ and $\varphi(p) = p-1$ if p is a prime number.

H. J. KANOLD in his paper [1], I. NIVEN and H. S. ZUCKERMAN in [2] (p.256) and S. PILLAI in his paper [3] proved that the set all the values of Euler's function has the asymptotic density 0. So "most" of natural numbers have the property that they are not values of Euler's function.

We are now going to construct such infinite sets, the elements of which are not values of Euler's function. In the book [4] (p.198-199) the following result is proved: The numbers of the form $2(3k+1)$, where $3k+1$ $(k \geq 1)$ is a prime number are not values of Euler's function.

The result may be generalized in the following way.

T h e o r e m 1. The equation

(2) $\varphi(x) = 2q$.

where q is any prime number of the form $(2a+1)k + a$, (a, k are natural numbers) has no solution in natural numbers.

P r o o f. The proof of this theorem is analogical to that of mentioned assertion from [4].

Let the equation (2) have a solution $x = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$, where p_i $(i = 1,2,\dots, s)$ are prime numbers, $2 < p_1 < p_2 \dots < p_s$, α is non-negative integer and α_i $(i = 1,2,\dots, s)$ are natural numbers. Then from (1) and (2) we conclude

(3) $2^{\alpha-1} p_1^{\alpha_1-1} p_2^{\alpha_2-1} \dots p_s^{\alpha_s-1} (p_1-1)(p_2-1)\dots(p_s-1) = 2q,$

if $\alpha > 1$ and

(4) $p_1^{\alpha_1-1} p_2^{\alpha_2-1} \dots p_s^{\alpha_s-1} (p_1-1)(p_2-1) \dots (p_s-1) = 2q$

if $\alpha \leq 1$.

Since the right-hand side in (3) and (4), respectively is divisible by an odd prime number, $s \geq 1$ holds. Let $s > 1$. Then the standard form of the number x contains al least two different odd prime number p_1, p_2 and hence the left-hand side in (3) and (4), respectively is divisible by 4 and the right-hand side is not; that is impossible. So $s = 1$ and then the equalities (3) and (4) (if we put $p_1 = p$ and $\alpha_1 = \beta$) turns into

(5) $2^{\alpha-1} p^{\beta-1} (p-1) = 2q$

if $\alpha > 1$ and

(6) $p^{\beta-1}(p-1) = 2q$

if $\alpha \leq 1$.

If $\alpha > 1$ then the left-hand side in (5) is divisible by 4 and the right-hand side is not, that is impossible. So $\alpha \leq 1$.

Let $\beta > 2$. Then the left-hand side in (6) is divisible by a square of the prime p and the right-hand side is not, that is imposible.

Let $\beta = 2$. Then from (6) we obtain $p(p-1) = 2q$. Hence $p-1 = 2$, $q = p$ and so $q = 3$. But it is impossible, because the prime number q is greater than 3.

Let $\beta = 1$. Then from (6) we obtain $p-1 = 2q$. From this we get $p = 2q + 1 = (2k + 1)(2a + 1)$. But it is impossible, because p is a prime number.

<u>T h e o r e m 2.</u> The equation

$$(7) \qquad \varphi(x) = 2q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_n^{\alpha_n}, \quad n \geqq 1$$

where $q_i (i = 1,2,\ldots,n)$ are prime numbers, $2 < q_1 < q_2 < \ldots < q_n$, α_i $(i = 1,2, \ldots, n)$ are natural numbers, is solvable if and only if one of these cases occurs:

(a) $\quad n = 1, \; q_1 = 3,$

(b) $\quad 2q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_n^{\alpha_n} + 1$ is a prime number,

(c) $\quad 2q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_{n-1}^{\alpha_{n-1}} + 1 = q_n$

<u>P r o o f.</u> Let the equation (7) have a solution $x = 2^{\alpha} p_1^{\beta_1} p_2^{\beta_2} \ldots p_k^{\beta_k}$, where p_i $(i = 1,2,\ldots,k)$ are prime numbers, $2 < p_1 < p_2 < \ldots < p_k$, β_i $(i = 1,2,\ldots,k)$ are natural numbers and α is a non-negative integer. Then according to (1) and (7) the following must hold

$$(8) \qquad 2^{\alpha-1} p_1^{\beta_1-1} p_2^{\beta_2-1} \ldots p_k^{\beta_k-1}(p_1-1)(p_2-1) \ldots (p_k-1) = $$
$$= 2q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_n^{\alpha_n},$$

if $\alpha > 1$ and

$$(9) \qquad p_1^{\beta_1-1} p_2^{\beta_2-1} \ldots p_k^{\beta_k-1}(p_1-1)(p_2-1)\ldots(p_k-1) = 2q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_n^{\alpha_n}$$

if $\alpha \leqq 1$.

Since the right-hand side in (8) and (9), respectively is divisible by an odd prime, we get $k \geq 1$.

We shall show that the standard form of the number x contains just one odd prime. Let $k > 1$. Then in (8) and (9), respectively is the left-hand side and so also the right-hand one divisible by 4. That is impossible, because the numbers q_i $(i = 1, 2, \ldots, n)$ are odd primes and for this reason $k = 1$. Let us put $p_1 = p$ and $\beta_1 = \beta$. So $x = 2^{\alpha} p^{\beta}$ and (8), (9) turns into

$$(10) \qquad 2^{\alpha-1} p^{\beta-1} (p-1) = 2q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_n^{\alpha_n},$$

if $\alpha > 1$ and

$$(11) \qquad p^{\beta-1} (p-1) = 2q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_n^{\alpha_n}$$

if $\alpha \leq 1$.

Let $\alpha > 1$. In the same way as in the case $k > 1$ it may be proved, that (10) cannot occur. For that reason $\alpha \leq 1$ and so (11) holds. In the relation (11) p-1 is an even number. Let $p-1 = 2^s h$, where s is a natural and h is an odd natural. Then $p = 2^s h+1$ and from (11) we obtain

$$(12) \qquad (2^s h+1)^{\beta-1} 2^s h = 2q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_n^{\alpha_n}.$$

Let $s > 1$. Just as in the case $k > 1$ we may prove, that the relation (12) cannot occur. For this reason $s = 1$. Then $p = 2h+1$ and the equation (12) turns into

$$(13) \qquad (2h+1)^{\beta-1} h = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_n^{\alpha_n}$$

The following three possibilities can occur (with regards to the numbers h and β):

(a) $h = 1$ and $\beta \geq 1$

(b) $h > 1$ and $\beta = 1$

(c) $h > 1$ and $\beta > 1$

In the case (a) $p = 2h+1 = 3$ and (13) is correct if and only if $n = 1$, $q_1 = 3$ and $\beta = \alpha_1+1$.

In the case (b) from (13) we get $h = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n}$. So $p = 2h+1 = 2q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n} + 1$. Conversely, if $p = 2q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n} + 1$ is a prime, then we can easily show, that the numbers $x = p, 2p$ are solutions of the equation (7).

In the case (c) from (13) we obtain

(14) $\quad p = 2h + 1 = q_i$

for the suitable i, $1 \leq i \leq n$, $\beta - 1 = \alpha_i$. Thus

$$h = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_{i-1}^{\alpha_{i-1}} q_{i+1}^{\alpha_{i+1}} \dots q_n^{\alpha_n}. \text{So } p = 2q_1^{\alpha_1} q_2^{\alpha_2} \dots$$

$$\dots q_{i-1}^{\alpha_{i-1}} q_{i+1}^{\alpha_{i+1}} \dots q_n^{n} + 1.$$

If $i < n$, then $p \geq q_n > q_i$ but it contradicts the relation (14). Hence we have $i = n$. Do $p = 2q_1^{\alpha_1} q_2^{\alpha_2} \dots q_{n-1}^{\alpha_{n-1}} + 1 = q_n$ and $\beta = \alpha_n + 1$. Conversely, if $q_n = 2q_1^{\alpha_1} q_2^{\alpha_2} \dots q_{n-1}^{\alpha_{n-1}} + 1$, then it can be easily shown that the numbers $x = q_n^{\alpha_n+1}$, $2q_n^{\alpha_n+1}$ are solutions of the equation (7).

<u>Corollary.</u> If the equation (7) is solvable, then the equation

(15) $\quad \varphi(z) = 2^{\gamma} q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n}$

is also solvable for any natural γ.

<u>Proof.</u> According to theorem 2, if the equation (7) is solvable, then one of the following cases occurs:

(a) $\quad n = 1, \quad q_1 = 3,$

(b) $\quad 2q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n} + 1$ is prime

(c) $\quad 2q_1^{\alpha_1} q_2^{\alpha_2} \dots q_{n-1}^{\alpha_{n-1}} + 1 = q_n$

It is easily to verify that in the case (a) the equation (15) has the solution $z = 2^{\gamma} . 3^{\alpha_1+1}$.

Now (15) can be written as follows

$$\varphi(z) = 2^{\gamma-1} . 2q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n} .$$

The equation $\varphi(x) = 2^{\gamma-1}$ has the solution $x = 2^{\gamma}$ and the equation $\varphi(y) = 2q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n}$ has in the case (b) (according to theorem 2) the solution $y = 2q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n} + 1$. Since φ is a multiplicative function and $(x,y) = (2^{\gamma}, 2q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n} + 1) = 1$ we have $\varphi(xy) = \varphi(x)\varphi(y) = \varphi(2^{\gamma})\varphi(2q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n} + 1) = 2^{\gamma} q_1^{\alpha_1} q_2^{\alpha_2} \dots$ $\dots q_n^{\alpha_n}$. Hence (15) has a solution $z = xy = 2^{\gamma}(2q_1^{\alpha_1} q_2^{\alpha_2} \dots q_n^{\alpha_n} + 1)$.

In the similar way it can be shown that in the case (c) (15) has the solution $z = 2^{\gamma} q_n^{\alpha_n+1}$.

With the mentioned problem is closely related the study of the values of the function $\varphi(n)/n$. In the book [4] (p.210) it is proved that the set of all the values of the function $\varphi(n)/n$ is dense in the interval (0,1). Further in [4] (p. 210-211) also the following result is proved: The set of all the numbers of the form $k/2^h$, where k is an odd natural and h is natural >1, is dense in the interval (0,1) and disjoint with the set $\{\varphi(n)/n\}$.

The following theorem generalizes this result.

T h e o r e m 3. The set M of all numbers of the form k/a^h, where a,h,k are naturals, $a > 1$, $h > 1$ and $(a,k) = 1$, is dense in the interval (o,1) and disjoint with the set $Q = \{\varphi(n)/n\}$.

P r o o f. Since the set of all the numbers of the form $k/2^h$, $(k/2) = 1$, $k > 0$, $h > 1$ is dense in the interval (0,1) and is a subset of the set M, it is sufficient to prove, that the sets M and Q are disjoint.

We shall prove it indirectly. Let us suppose that there is a natural n such that

$$\varphi(n)/n = k/a^h$$

where a, h, k are natural numbers, $a > 1$, $h > 1$ and $(a,k) = 1$. From this we obtain

(16) $a^h \varphi(n) = kn$

Let p be aby prime divisor of the number a; let $a = p^{\alpha} a_1$, where α and a_1 are natural numbers and $p \nmid a_1$. Then (16′)

$$p \nmid k$$

and the equation (16) turns into

(17) $p^{\alpha h} a_1^h \varphi(n) = kn$

From this $p \mid n$, since $(a,k) = 1$. Hence $n = p^{\beta} n_1$, where β and n_1 are natural numbers and

(18) $p \nmid n_1$

Euler's function is known to be multiplicative. On the base of this and of (1) we have $\varphi(n) = \varphi(p^{\beta} n_1) = p^{\beta-1}(p-1)\varphi(n_1)$.

Then from (17) we get

$$p^{\alpha h} a_1^h p^{\beta-1}(p-1)\varphi(n_1) = kp^{\beta} n_1$$

This yields

(19) $p^{\alpha h-1} a_1^h (p-1)\varphi(n_1) = kn_1$

where $\alpha h-1 \geqq 1$.

From (19) it follows that p divides the number kn_1. But it contradicts (16′) and (18).

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Received: May 26, 1970

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(ACTA F. R. N. UNIV. COMEN. - MATHEMATICA XXV - 1971)

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XXV - 1971

REMARKS ON RATIO SETS OF SETS OF NATURAL NUMBERS

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If A, B are two sets of natural numbers then the set $R(A, B)$ of all numbers of the form $\frac{a}{b}$ where $a \in A$, and $b \in B$, is called the ratio set of A and B (see [1]).

If A or B is a finite set then clearly $R(A, B)$ is nowhere dense in the interval $\langle 0, \infty)$. But there are also infinite sets A, B such that the set $R(A, B)$ is nowhere dense in $\langle 0, +\infty)$ (for example, $A = B = \{2, 2^2, \dots, 2^n, \dots\}$). T. ŠALÁT [1, Theorem 5] has proved that $R(A, B)$ is dense in $\langle 0, +\infty)$ provided A or B has a positive asymptotic density and both A and B are infinite. This condition is only sufficient but not necessary. Indeed, a simple modification of the proof of Theorem 7 in [1] gives the following result:

<u>Theorem 1.</u> Let $C = \{c_i\}_{i=1}^{\infty}$, $c_i < c_{i+1}$, for each i, be an infinite set of natural numbers. Then there are two disjoint sets A, B of natural numbers such that $A(x) < C(x)$, $B(x) < C(x)$ ($A(x)$ is the cardinality of the set of all numbers $a \in A$ such that $a \leq x$), and the ratio set $R(A, B)$ is the set $R^+ - \{1\}$ of all positive rationals different from 1.

<u>Proof:</u> Let $r_1, r_2, \dots, r_k, \dots$ be a sequence of all numbers in $R^+ - \{1\}$. We shall define by induction a sequence of numbers $a_1, b_1, a_2, b_2, \dots, a_k, b_k, \dots$ as follows: Choose two positive integers a_1, b_1 such that $c_1 < a_1$, $c_1 < b_1$, and $\frac{a_1}{b_1} = r_1$. Clearly $a_1 \neq b_1$. If numbers a_i, b_i, $1 \leq i < n$ has been chosen such that $a_j \neq b_k$, for $j, k < n$, let a_n, b_n be two positive integers such that

$$a_n > \max(c_n, a_1, \dots, a_{n-1}, b_1, \dots, b_{n-1}),$$

$$b_n > \max\,(c_n, a_1, \dots, a_{n-1}, b_1, \dots, b_{n-1}),$$

and

$$\frac{a_n}{b_n} = r_n$$

Now put $A = \{a_1, a_2, \dots, a_n, \dots\}$ and $B = \{b_1, b_2, \dots, b_n, \dots\}$. It is easy to verify that the sets A, B have all desired properties and hence Theorem 1 is proved.

The Theorem 1 gives a non-effective example of two disjoint sets A, B, whose asymptotic density is zero and such that R(A, B) is dense in $\langle 0, +\infty)$. Now we show a simple effective example of such sets:

<u>T h e o r e m 2.</u> Let A be the set of all prime numbers > 2 and let B be the set of all numbers of the form $p + 1$, $p \in A$. Then the ratio set R(A, B) is dense in $\langle 0, +\infty)$, $A \cap B = \emptyset$ and both A and B has the zero asymptotic density.

<u>P r o o f:</u> The sets are clearly disjoint; from the known fact on asymptotic density of the set of prime integers it follows that both A and B has the zero asymptotic density. Hence it suffices to prove that R(A, B) is dense in $\langle 0, +\infty)$. Let a, b be two real numbers such that $0 < a < b$. Choose some c such that $a < c < b$. It is known that each interval contains infinitely many numbers of the form $\frac{p}{q}$ where p and q are primes (see [2], p. 155). So choose prime numbers p, q such that $\frac{p}{q}$ (c, b) and

$$(1) \qquad q > \frac{b}{c - a}$$

We shall show that

$$\frac{p}{q + 1} \in (a, b).$$

Assume, on the contrary that $\dfrac{p}{q + 1} \leq a$.

Hence

$$(2) \qquad c - a < \frac{p}{q} - \frac{p}{q + 1} = \frac{p}{q(q + 1)}$$

Since $\frac{p}{q} \in (c, b)$ there is $p < bq$ and hence from (2) $c - a < \frac{b}{q}$; thus we have shown that $q < \frac{b}{c - a}$, contrary to (1). The Theorem is proved.

In Theorem 2 the sets A, B are given such that $A(x) = O\left(\frac{x}{\log x}\right)$ and $B(x) = O\left(\frac{x}{\log x}\right)$. Next Theorem 3 gives two disjoint sets A, B such that $A(x) = O(\log x)$ and $B(x) = O(\log x)$.

Theorem 3. Let p, q be two relatively prime positive integers. Put $A = \{p, p^2, \ldots, p^n, \ldots\}$, and $B = \{q, q^2, \ldots, q^n, \ldots\}$. Then $A(x) = O(\log x)$, $B(x) = O(\log x)$, $A \cap B = \emptyset$, and $R(A, B)$ is dense in $\langle 0, \infty)$.

The following two lemmas are necessary for proof of the Theorem:

Lemma 1. If p, q are two relatively prime integers, $1 < p < q$, then there exists a sequence

(3) $\quad \{\alpha_k\}_{k=0}^{\infty}$

of numbers such that $\alpha_k = \frac{p^{m_k}}{q^{n_k}}$ (m_k, n_k are positive integers)

$\alpha_k > 1$, for $k = 0, 1, 2, \ldots$, and $\lim_{k \to \infty} \alpha_k = 1$.

Proof: Since $1 < p < q$ there exists the least positive integer m_0 such that $p^{m_0} > q$. Pu $\alpha_0 = \frac{p^{m_0}}{q}$. Clearly $1 < \alpha_0 < q$. If $\alpha_k = \frac{p^{m_k}}{p^{n_k}}$ is constructed let d be the least integer such that $(\alpha_k)^d > q$ and put $\alpha_{k+1} = \frac{(\alpha_k)^d}{q}$. From the assumption $\alpha_k = \frac{p^{m_k}}{q^{n_k}}$ (m_k, n_k are positive integers) it follows

$$\alpha_{k+1} = \frac{p^{dm_k}}{q^{dn_k+1}} = \frac{p^{m_{k+1}}}{q^{n_{k+1}}} \quad (m_{k+1}, n_{k+1} \text{ are positive integers}).$$

It is easy to see that

(4) $\quad \alpha_k > 1, \qquad$ for $k = 0, 1, 2, \ldots$

The sequence (3) is decreasing, i.e.

(5) $\quad \alpha_k > \alpha_{k+1}, \qquad$ for $k = 0, 1, 2, \ldots$

To see it we assume that $\alpha_{k+1} \geqslant \alpha_k$, for some k. From the definition of the number α_{k+1} we know that $d = \frac{m_{k+1}}{m_k}$ is the least integer (positive) such that $\alpha_k^{\,d} > q$. But from the equation $\alpha_{k+1} = \frac{\alpha_k^{\,d}}{q}$ and from the inequality $\alpha_{k+1} \geqslant \alpha_k$ it follows that

(6) $\quad \alpha_k^{\,d-1} \geqslant q.$

Since $\alpha_k = \frac{p^{m_k}}{q^{n_k}}$ and p and q are relatively prime integers, the equality does not hold in (6), so we have $\alpha_k^{\,d-1} > d$ contrary to the fact that $d = \frac{m_{k+1}}{m_k}$ is the least integer such that $\alpha_k^{\,d} > q$. Hence the inequality (5) holds.

Now for each positive integer s, let A(s) be the set of all positive integers $k \geq 0$ such that $\alpha_k^{\,s} < q$, and $\alpha_k^{\,s+1} > q$, for every $k \in A(s)$ (the equality $\alpha_k^{\,s} = q$ is clearly impossible).

We shall show that

(7) $\quad A(s)$ is a finite set, $\qquad$ for $s = 1, 2, \ldots$

To see it we assume that for some fixed s there exists a sequence $\{k_i\}_{i=0}^{\infty}$ of integers such that $0 \leq k_i < k_{i+1}$, $\alpha_{k_i}^{\,s} < q$, and $\alpha_{k_i}^{\,s+1} > q$, for $i = 0, 1, 2, \ldots$ From (5) it follows

(8) $$\frac{\alpha_{k_i}^{\,s}}{q} \leq \frac{\alpha_{k_0}^{\,s}}{q} = 1 - \eta < 1, \text{ where } 0 < \eta < 1, \text{ for } i = 0,1,2,\ldots$$

For arbitrary foxed i let

(9) $\quad \varepsilon_i = \alpha_{k_i} = \alpha_{k_{i+1}}$

Evidently $\alpha_i > 0$ (see (5)). From (8) and (9) we have

$$(1-\eta)\ \varepsilon_i \geq \frac{\alpha_{k_i}^s}{q}\,(\alpha_{k_i} - \alpha_{k_i+1}) = \frac{\alpha_{k_i}^{s+1}}{q} - \frac{\alpha_{k_i}^s}{q}\,\alpha_{k_i+1}$$

Since $\dfrac{\alpha_{k_i}^{s+1}}{q} = \alpha_{k_i+1}$, from (8), (4) it follows

$$(1-\eta)\ \varepsilon_i \geq \alpha_{k_i+1}\Big(1 - \frac{\alpha_{k_i}^s}{q}\Big) \geq \alpha_{k_i+1}\,\eta > \eta\ ,\ \text{so}\ \varepsilon_i > \frac{\eta}{1-\eta}$$

From (5) and (9) we have

$$\alpha_{k_i} - \alpha_{k_{i+1}} \geq \alpha_{k_i} - \alpha_{k_{i+1}} = \varepsilon_i > \frac{\eta}{1-\eta}$$

$$\alpha_{k_0} - \alpha_{k_{n+1}} \geq \varepsilon_0 + \varepsilon_1 + \dots + \varepsilon_n > (n+1)\,\frac{\eta}{1-\eta}$$ and so we

conclude that $\lim_{i\to\infty} \alpha_{k_i} = -\infty$ contrary to (4). Hence (7) holds.

Further property of (3) is as follows: For every positive integer r let $B(r)$ be the set of all integers $k \geq 0$ such that $\alpha_k^r > q$ Then

(10) $B(r)$ is finite, for $r = 1, 2, \dots$

In fact, $\alpha_k < q$ for $k = 0, 1, 2, \dots$ (see (5) and the definition of α_0), so $B(r)$ is finite for $r = 1$. Now assume by induction, that $B(r)$ is finite for $r = r_0$. It is easy to check that $B(r_0+1) = A(r_0) \cup B(r_0)$. Since $B(r_0)$ is finite, and from (7) it follows that $A(r_0)$ is finite, too, we conclude that $B(r)$ is finite whenever $r = r_0 + 1$. Hence (10) holds.

Finally, we choose a subsequence $\{\beta_k\}_{k=0}^{\infty}$ of the sequence (3) such that $\lim_{k\to\infty} \beta_k = 1$. The inequality (5) then will imply that $\lim_{k\to\infty} \alpha_k = 1$, q.e.d.

Put $\beta_0 = \alpha_0$. From the definition of the number α_0 we hawe that $\alpha_0^1 = \alpha_0 < q$. Assume that β_{k-1} is choosen such that $\beta_{k-1} < q$ and, say, $\beta_{k-1} = \alpha_j$. From (10) it follows that there exists an index $j' > j$ such that $\alpha_{j'}^{k+1} < q$. Put $\beta_k = \alpha_{j'}$. So the sequence $\{\beta_k\}_{k=0}^{\infty}$ is constructed by induction and for every $k = 1, 2, \dots,$ we have $1 < \beta_k^{k+1} < q$ so $1 < \beta_k < q^{\frac{1}{k+1}}$ and hence $\lim_{k \to \infty} \beta_k = 1$. The Lemma 1 is proved.

<u>Lemma 2.</u> If p and q are two relatively prime integers, $1 < p < q$, and a is a non-negative real number, then a is a limit-point of the set of all fractions of the form $\frac{p^m}{q^n}$ where m and n are positive integers.

<u>Proof:</u> If $a = 0$ then $\lim_{n \to \infty} \frac{p}{q^n} = 0 = a$. Assume that $a > 0$. To prove our lemma it suffices to define a sequence $\{b_k\}_{k=0}^{\infty}$ of numbers of the form $b_k = \frac{p^{m_k}}{q^{n_k}}$ (m, n are positive integers) such that $b_k < a$, for $k = 0,1,2, \dots,$ and $\lim_{k \to \infty} b_k = a$. Choose a positive integer n_0 such that $\frac{p}{q^{n_0}} < a$ and put $b_0 = \frac{p}{q^{n_0}}$. By the Lemma 1 there exists a sequence $\gamma_1 > \gamma_2 > \dots > \gamma_k > \dots$ of numbers of the form $\frac{p^m}{q^n}$ such that $1 < \gamma_k < 1 + \frac{1}{k}$. If $b_k = \frac{p^{m_k}}{q^{n_k}}$ is already constructed let d be the greatest integer such that $b_k \gamma_{k+1}^d < a$ and put $b_{k+1} = b_k \gamma_{k+1}^d = \frac{p^{m_{k+1}}}{q^{n_{k+1}}}$. Clearly $b_k < a \leq \leq b_k \gamma_k$, so $|a - b_k| \leq |b_k \gamma_k - b_k| < b_k \cdot \frac{1}{k} < a \frac{1}{k}$. Hence $\lim_{k \to \infty} b_k = a$. The Lemma 2 is proved.

<u>Proof of the Theorem 3</u> Is an easy consequence of Lemmas 1 and 2. We may, without loss of generality assume

that $p < q$, since $R(A, B)$ is dense in $\langle 0, +\infty)$ if and only if $R(B, A)$ is. That $A(x) = O(\log x)$, is clear.

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Received: May 26, 1970

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Acta Facultatis
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Mathematica XXV

Vydalo Slovenské pedagogické nakladateľstvo Bratislava - Schválené výmerom SÚKK č. 293/I-1971 - Prvé vydanie - Náklad 874 - Rukopis zadaný 28.X.1970 - Vytlačené v októbri 1971 - Papier 2 122 3, 70 g - Tlačili Nitrianske tlačiarne, n.p., Nitra - Tlačené ofsetom - Písmo strojopis - Technický redaktor Adam Hanák - 03/2 - AH 4,577 - VH 5,084 - 67 - 246 - 71

Celý náklad prevzala Ústredná knižnica PFUK, Bratislava, ul. 29.augusta č. 5.