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MATHEMATICA XXIII – 1969

ZUR THEORIE DER ZASSENHAUSSCHEN VERFEINERUNGEN ZWEIER REIHEN VON ZERLEGUNGEN, II
VERKNÜPFTE UND ANGESCHALTETE VERFEINERUNGEN

VÁCLAV HAVEL, Brno

In [1], S. 65 befindet sich ein Satz über die Existenz gleichbasig halbverketteter Verfeinerungen zweier einander modularer Zerlegungsreihen und weiter in [1] S. 68 zweiter Satz über die Existenz von gleichbasig verketteten Verfeinerungen zweier einander komplementärer Zerlegungsreihen. Man kann aus der Analyse der in [1] gegebenen Beweisverfahrens folgenden Zusatz bekommen: Für zwei einander modulare Zerlegungsreihen übergehen die gleichbasig halbverketteten Verfeinerungen aus dem ersten Satz in gleichbasig verkettete Verfeinerungen dann und nur dann, wenn ursprüngliche Reihen einander komplementär sind.

Wir können uns aber nur auf die üblichen Untersuchung der lokalen Ketten mit fester Basis (im Sinne von [1] S. 60) begrenzen, was nun der Gegenstand unserer weiteren Betrachtung wird. Wir versuchen dabei den mengentheoretischen Kern des Verfahrens von A.W. Goldie aus [2] zu finden, möglicherweise ohne Anwendung der früheren Ergebnisse aus [2] und [3] 1).

Es sei $M \neq \emptyset$ eine feste Menge mit einem ausgezeichneten Element $e \in M$; jede Untermenge $M' \subseteq M, e \in M'$ soll ausgezeichnete Menge heisen. Ist N eine nichtleere Untermenge aus M und $\mathcal{A}$ eine Zerlegung in M, so bedeute $N(\mathcal{A}$ die Vereinigung aller $N \mathrm{I} \mathcal{A}$ - Blöcke. Sind $\mathcal{A}, \mathcal{B}$ Zerlegungen auf M, so bedeute $\mathcal{A} \square \mathcal{B}$, dass jeder $\mathcal{A}$-Block und jeder $\mathcal{B}$-Block aus demselben $\mathcal{A} \smile \mathcal{B}$-Block einander inzidieren. Sind $\mathcal{A}, \mathcal{B}$ Zerlegungen auf M, so bedeute $\mathcal{A} \blacksquare \mathcal{B}$, dass für je ausgezeichnete Blöcke $A \in \mathcal{A}$, $B \in \mathcal{B}$ die

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1) Terminologie nach [1] S. 6-9. Elemente einer Zerlegung $\mathcal{A}$ nennen wir $\mathcal{A}$ -Blöcke. Für mengentheoretische, bzw. verbandstheoretische Operationen des Durchschnittes und der Vereinigung gebrauchen wir die Symbole $\cap, \cup$ bzw. $\frown, \smile$, für Inzidenz das Symbol I. Für die Mengeninklusion wird das Symbol $\subseteq$ gebraucht, für Verfeinerungsrelation zwischen Zerlegungen das Symbol $\leq$..

Beziehung $A(\mathcal{B} = B(\mathcal{A}$ gilt. Offensichtlich ist $\mathcal{A} \square \mathcal{B} \Rightarrow \mathcal{A} \blacksquare \mathcal{B}$

Zuerst bemerken wir das Folgende: Sind $\mathcal{A}, \mathcal{B}$ Zerlegungen auf M mit ausgezeichneten Blöcken $A \in \mathcal{A}$, $B \in \mathcal{B}$ und gilt $\mathcal{A} \boxminus \mathcal{B}$, so ist $A(\mathcal{B} = B(\mathcal{A}$ der ausgezeichnete $\mathcal{A} \smile \mathcal{B}$ -Block.

In der Tat, es gelte $A(\mathcal{B} = B(\mathcal{A}$ und es sei C der ausgezeichnete $\mathcal{A} \smile \mathcal{B}$ -Block. Setzen wir voraus, dass in C ein mit B disjunkter $\mathcal{A}$ -Block existiert. Dann folgt die Existenz eines $A' \in \mathcal{A}$, $A' \subseteq C$, $A' \cap B = \emptyset$ und eines $B' \in \mathcal{B}$, $B \subseteq C$, so dass $A \mathrel{I} B' \mathrel{I} A'$. Das bedeutet aber, dass $B' \subseteq (A(\mathcal{B})$; damit gehört auch jedes Element aus B' irgend einem mit B inzidenten $\mathcal{A}$ -Block. Daraus folgt $A' \mathrel{I} B$, was der verlangte Widerspruch ist. $A(\mathcal{B} = B(\mathcal{A}$ fällt also mit C zusammen. ∎

Die Zerlegungen $\mathcal{A}, \mathcal{B}$ in M heissen verknüpft, wenn jeder $\mathcal{A}$ -Block mit genau einem $\mathcal{B}$ -Block und jeder $\mathcal{B}$ -Block mit genau einem $\mathcal{A}$ -Block inzidiert. Die Zerlegungen $\mathcal{A}, \mathcal{B}$ in M heissen angeschaltet, wenn eine Zerlegung $\mathcal{C}$ in M existiert, so dass $\mathcal{A}, \mathcal{C}$ und auch $\mathcal{B}, \mathcal{C}$ verknüpft sind.

Hilfssatz 1. Es sei N eine nichtleere Untermenge aus M und $\mathcal{A}, \mathcal{B}$ Zerlegung auf M. Dann sind $N \sqsubset \mathcal{A}$, $N \sqcap \mathcal{A}$ und auch $(N(\mathcal{A}) \sqcap (\mathcal{A} \smile \mathcal{B})$, $N \sqcap (\mathcal{A} \smile \mathcal{B}) = N \sqcap ((N \sqcap \mathcal{A}) \smile (N \sqcap \mathcal{B}))$ verknüpft.

Beweis. Die Verknüpfung zwischen $N \sqsubset \mathcal{A}$, $N \sqcap \mathcal{A}$ ist durch die Vorschrift $X \longleftrightarrow N \cap X$ für jedes $X \in \mathcal{A}$, $X \mathrel{I} N$ vermittelt. Weiter sind also auch $(N(\mathcal{A}) \sqsubset (\mathcal{A} \smile \mathcal{B})$, $(N(\mathcal{A}) \sqcap (\mathcal{A} \smile \mathcal{B})$ verknüpft und für jedes $Y \in (N(\mathcal{A}) \sqsubset (\mathcal{A} \smile \mathcal{B})$ werden wir nun jedem $(N(\mathcal{A}) \frown Y \in (N(\mathcal{A}) \sqcap (\mathcal{A} \smile \mathcal{B})$ das entsprechende $N \cap Y \in N \sqcap (\mathcal{A} \smile \mathcal{B})$ zuordnen. Damit haben wir die gewünschte Verknüpfung hergeleitet. ∎

Hilfssatz 2. Es seien M_1, M_2 zwei einenader inzidente Untermengen aus M und $\mathcal{A}, \mathcal{B}$ Zerlegungen auf M. Dann sind $\mathcal{A}' = ((M_1 \frown M_2)(\mathcal{A}) \sqcap (\mathcal{A} \smile (M_2 \sqcap \mathcal{B}))$, $\mathcal{B} = ((M_1 \frown M_2)(\mathcal{B}) \sqcap ((M_1 \sqcap \mathcal{A}) \smile \mathcal{B})$ verknüpft.

B e w e i s . Im Hilfssatz 1 ersetzen wir $N, \mathcal{A}, \mathcal{B}$ durch $M_1 \cap M_2, \mathcal{A}, M_2 \sqcap \mathcal{B}$ und bekommen sogleich die Verknüpfung zwischen $((M_1 \cap M_2) \subset \mathcal{A}) \sqcap (\mathcal{A} \smile (M_2 \sqcap \mathcal{B}))$ und $(M_1 \cap M_2) \sqcap ((M_1 \cap M_2) \sqcap \mathcal{A}) \smile (M_2 \sqcap \mathcal{B})) = (M_1 \cap M_2) \sqcap ((M_1 \sqcap \mathcal{A}) \smile (M_2 \sqcap \mathcal{B}))$. ∎

H i l f s a t z 3. Es seien $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ Zerlegungen auf M, wobei $\mathcal{A}_1 \geq \mathcal{A}_2, \mathcal{B}_1 \geq \mathcal{B}_2, \mathcal{A}_1 \boxdot \mathcal{B}_2, \mathcal{A}_2 \boxdot \mathcal{B}_1$. Es seien $L_0 \in (\mathcal{A}_1 \smile \mathcal{B}_2) \frown (\mathcal{A}_2 \smile \mathcal{B}_1)$, $L_1 \in \mathcal{A}_1 \frown (\mathcal{A}_2 \smile \mathcal{B}_1)$, $L_2 \in \mathcal{B}_1 \frown (\mathcal{B}_2 \smile \mathcal{A}_1)$ ausgezeichnete Blöcke. Dann

a) $$L_1 \subset (\mathcal{A}_2 \smile \mathcal{B}_2) = L_2 \subset (\mathcal{A}_2 \smile \mathcal{B}_2) = L_0$$ und

b) die Zerlegungen $\mathcal{A}^* = L_1 \sqcap (\mathcal{A}_1 \frown (\mathcal{A}_2 \smile \mathcal{B}_2))$, $\mathcal{B}^* = L_2 \sqcap (\mathcal{B}_1 \frown (\mathcal{A}_2 \smile \mathcal{B}_2))$ sind angeschaltet.

B e w e i s . Es seien $A_i \in \mathcal{A}_i, B_i \in \mathcal{B}_i, C_{i,j} \in \mathcal{A}_i \smile \mathcal{B}_j \ (i,j = 1,2)$ ausgezeichnete Blöcke. Dann ist $L_0 = C_{12} \frown C_{21}$, $L_1 = A_1 \frown C_{21}$, $L_2 = B \frown C_{12}$, wobei C_{ij} die Vereinigung aller mit $A_i (B_j)$ bezüglich $\mathcal{B}_j (\mathcal{A}_i)$ verbundenen ([1], S. 12) $\mathcal{A}_i$-Blöcke ($\mathcal{B}_j$-Blöcke); ist; $(i,j) = (1,2)(2,1)$. Daraus folgt $C_{12} \cap C_{21} = (A_1 \cap C_{21}) \subset (\mathcal{A}_2 \smile \mathcal{B}_2) = (B_1 \cap C_{12}) \subset (\mathcal{A}_2 \smile \mathcal{B}_2)$: In der Tat, jedes $x \in C_{12} \cap C_{21}$ liegt im gewissen mit B_2 inzidenten $\mathcal{A}_j$-Block und im gewissen mit A_2 inzidenten B_j-Block; $(i,j) = (1,2), (2,1)$.
Nach $\mathcal{A}_1 \boxdot \mathcal{B}_2, \mathcal{A}_2 \boxdot \mathcal{B}_1$ liegt dann auch x im gewissen mit A_2 inzidenten $\mathcal{B}_2$-Block und im gewissen mit $\mathcal{B}_2$ inzidenten $\mathcal{A}_2$-Block. Daraus und aus umgekehrtem Verfahren folgt schon der Beweis des Teils a).

b) Weiter beweisen wir, dass $\mathcal{A}^* = (A_1 \cap C_{21}) \sqcap (\mathcal{A}_1 \frown (\mathcal{A}_2 \smile \mathcal{B}_2))$, $\mathcal{B}^* = (B_1 \cap C_{12}) \sqcap (\mathcal{B}_1 \frown (\mathcal{B}_2 \smile \mathcal{A}_2))$ angeschaltet sind. Aus $\mathcal{A}_1 \frown (\mathcal{A}_2 \smile \mathcal{B}_2) = (\mathcal{A}_2 \smile \mathcal{B}_2) \frown ((\mathcal{A}_1 \frown (\mathcal{A}_2 \smile \mathcal{B}_1))$ folgt $\mathcal{A}^* = (A \cap C_{21}) \sqcap (\mathcal{A}_2 \smile \mathcal{B}_2)$; ähnlich bekommt man $\mathcal{B}^* = (B_1 \cap C_{12}) \sqcap (\mathcal{A}_2 \smile \mathcal{B}_2)$.

Nach Hilfssatz 1 gibt es eine Verknüpfung zwischen $\mathcal{A}^*$ und $C^* = (A_1 \cap C_{21}) \sqsubset (\mathcal{A}_2 \smile \mathcal{B}_2)$. Nach Teil a) ist aber $(A_1 \cap C_{21}) \subset (\mathcal{A}_2 \smile \mathcal{B}_2) = C_{12} \frown C_{21}$, so dass $C^* = (C_{12} \cap C_{21}) \sqcap$

$\cap(\mathcal{A}_1 \smile \mathcal{B}_2)$. Ähnlich finden wir eine Verknüpfung zwischen $\mathcal{B}^*$ und $\mathcal{C}^*$. Die Zerlegungen $\mathcal{A}^*, \mathcal{B}^*$ sind somit angeschaltet. ■

Satz. Es seien $(\mathcal{A}_i)_{i=1}^{m+1}$, $(\mathcal{B}_j)_{j=1}^{n+1}$ zwei Reihen von Zerlegungen auf M, wobei $\mathcal{A}_1 = \mathcal{B}_1$ und $\mathcal{A}_{m+1} = \mathcal{B}_{n+1}$. Weiter setzen wir voraus, dass $\mathcal{A}_i \boxminus \mathcal{B}_j$ für jedes $i = 1, \dots, m+1$; $j = 1, \dots, n+1$. Dann gibt es Verfeinerungen $(\mathcal{A}'_k)_{k=1}^{r+1}$ $(\mathcal{B}'_k)_{k=1}^{r+1}$ beider gegebenen Reihen, wobei für ausgezeichnete Blöcke $A'_k \in \mathcal{A}'_k$, $B'_k \in \mathcal{B}'_k$ $(k = 1, \dots, r)$ eine solche Permutation f von $(1, \dots, r)$ existiert, so dass $A'_k \cap \mathcal{A}'_{k+1}$ und $B'_{f(k)} \cap \mathcal{B}'_{f(k)+1}$ für jedes $k = 1, \dots, r$ stets verknüpfte, bzw. stets angeschaltete Zerlegungen sind.

Beweis.[2] Es seien $A_i \in \mathcal{A}_i$, $B_j \in \mathcal{B}_j$ ausgezeichnete Blöcke. Wir setzen $\bar{\mathcal{A}}_{i,j} = \mathcal{A}_{i+1} \smile (\mathcal{A}_i \frown \mathcal{B}_j)$, $\bar{\mathcal{B}}_{j,i} = \mathcal{B}_{j+1} \smile (\mathcal{B}_j \frown \mathcal{A}_i)$ und bezeichnen mit $\bar{A}_{i,j}$, $\bar{B}_{j,i}$ entsprechende ausgezeichnete Blöcke. Aus $\mathcal{A}_{i+1} \boxminus \mathcal{B}_j$ folgt dann $\bar{A}_{i,j} = (A_i \cap B_j) \subset \mathcal{A}_{i+1}$; analogische Schlüsse folgen für $\bar{B}_{j,i}$. Es ist bekannt, dass $\bar{\mathcal{A}}_{i,j}$ bzw. $\bar{\mathcal{B}}_{j,i}$ untere Zassenhaussche Verfeinerungen von $(\mathcal{A}_i)$, $(\mathcal{B}_j)$ bilden.[3] Nach Hilfssatz 3 folgt nun die Existenz einer Verknüpfung zwischen $\bar{A}_{i,j} \cap \bar{\mathcal{A}}_{i,j+1}$ $(j \neq n)$ und $\bar{B}_{j,i} \cap \bar{\mathcal{B}}_{j,i+1}$ $(i \neq m)$ und nach Hilfssatz 1 ergibt sich die Existenz einer Verknüpfung zwischen $\bar{A}_{i,n} \cap \mathcal{A}_{i+1}, (A_i \cap B_n) \cap \mathcal{A}_{i+1} = \bar{B}_{n,i} \cap \bar{\mathcal{B}}_{n,i+1}$; ählich für $\bar{B}_{j,m} \cap \bar{\mathcal{B}}_{j,m+1}$. Damit ist der Beweis für den Fall der verknüpften Verfeinerungen beendet.

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2) i, j durchlaufen die Mengen $\{1, \dots, m+1\}$, $\{1, \dots, n+1\}$ bzw. mit Ausnahme von $m+1, n+1$.

3) Zum Begriff der oberen und unteren Zassenhausschen Verfeinerungen und zu der ausgesprochenen Tatsache siehe V. KOŘÍNEK, Věstník král. čes. spol. nauk, tř. mat. - přír. 1941, Schrift XIV, S. 8-9 und 14.

Weiter setzen wir $\breve{\mathcal{A}}_{i,j} = \mathcal{A}_i \frown (\mathcal{A}_{i+1} \smile \mathcal{B}_j)$, $\breve{\mathcal{B}}_{j,i} = \mathcal{B}_j \frown (\mathcal{B}_{j+1} \smile \mathcal{A}_i)$ und bezeichnen $\bar{A}_{i,j}, \bar{B}_{j,i}$ entsprechende ausgezeichnete Blöcke. Es ist gut bekannt, dass $\breve{\mathcal{A}}_{i,j}$ bzw. $\breve{\mathcal{B}}_{j,i}$ die oberen Zassenhausschen Verfeinerungen von $(\mathcal{A}_i), (\mathcal{B}_j)$ bilden. Nach Hilfssatz 3 b sind aber Zerlegungen $\breve{A}_{i,j} \sqcap \breve{\mathcal{A}}_{i,j+1}$, $\breve{B}_{j,i} \sqcap \breve{\mathcal{B}}_{j,i+1}$ stets angeschaltet, woraus schon das Ende des Beweises für den Fall der angeschalteten Verfeinerungen folgt. ∎

<u>B e m e r k u n g 1.</u> Wenn die Voraussetzungen im Satz für jede Wahl von $e \in M$ erfüllt sind, so dehnen sich die Bedingungen $\mathcal{A}_i \boxdot \mathcal{B}_j$ auf $\mathcal{A}_i \square \mathcal{B}_j$ aus, was bedeutet, dass untersuchte Reihen komplementär sind. Wir bekommen damit den Verfeinerungssatz aus [1], S. 68 .

<u>B e m e r k u n g 2.</u> Man kann das obere Verfahren in folgender Weise abändern: Wir wählen eine nichtleere Untermenge $E \subseteq M$ und erklären für ausgezeichnet jede Untermenge $M' \subseteq M$, welche E enthält. Weiter sollen ausgezeichnet solche Zerlegungen in M heissen, welche einen ausgezeichneten Block besitzen. Dann kann man Hilfssätze 1, 2, 3 natürlicherweise bezüglich E ausdehnen. Auch die Formulation des Verfeinerungssatzes bleibt in Geltung, wenn wir zwei Reihen von ausgezeichneten Zerlegungen bezüglich E mit gleichen Anfangs - und Endgliedern und die Relation $\boxdot$ betrachten.

L i t e r a t u r v e r z e i c h n i s :

[1] O. BORŮVKA, Grundlagen der Gruppoid- und Gruppentheorie, Berlin 1960.

[2] A. W. GOLDIE, The Jordan - Hölder theorem for general abstract algebras, Proc.Lond.Math.Soc., 2nd series, 52 (1951), 107-131.

[3] V. HAVEL, Zur Theorie der Zassenhausschen Verfeinerungen zweier Reihen von Zerlegungen, I (Gleichbasig halbverkettete Verfeinerungen).

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ZUR THEORIE DER ZASSENHAUSSCHEN VERFEINERUNGEN ZWEIER REIHEN VON ZERLEGUNGEN, III
KARTESISCHE STRUKTUREN

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Im weiteren gebrauchen wir die Terminologie und Bezeichnungen aus [III] und werden kartesische Strukturen (in [III] "Cartesian systems" bezeichnet) mit einer festen Indexmenge Γ_0 untersuchen. Es sei also F eine feste kartesische Struktur $((M_{\alpha_0})_{\alpha_0}, f)$ mit einer ausgewählten Unterstruktur $E=((\{e_{\alpha_0}\})_{\alpha_0}, f)$, wo $e_{\alpha_0} \in M_{\alpha_0}$ für jedes α_0;[1] die Untermengen M'_{α_0}, $e_{\alpha_0} \in M'_{\alpha_0} \subseteq M_{\alpha_0}$ nennen wir dabei ausgezeichnet. Die Begriffe einer Zerlegung und einer erzeugenden Zerlegung in einer kartesischen Struktur, der Begriff einer Faktorstruktur u.s.w. wurden in [III], die Relationen $\square$ bzw. $\blacksquare$ für Zerlegungen auf einer Menge in [II] eingeführt.

Wir werden zwei Verfeinerungssätze für die Reihen von Zerlegungen auf F herleiten, entsprechend zu [II], Hauptsatz und Bemerkung 1.

H i l f s s a t z . Es seien $\mathcal{A}=(\mathcal{A}_{\alpha_0})_{\alpha_0}$, $\mathcal{B}=(\mathcal{B}_{\alpha_0})_{\alpha_0}$ erzeugende Zerlegungen auf F. Dann ist auch $\mathcal{A}\frown\mathcal{B}$ erzeugend auf F. Ist sogar $\mathcal{A}_{\alpha_0} \square \mathcal{B}_{\alpha_0}$ für jedes α_0, so ist auch $\mathcal{A}\smile\mathcal{B}$ erzeugend auf F.

B e w e i s . Wählt man also $C_\alpha \in \mathcal{A}_\alpha \frown \mathcal{B}_\alpha$ für jedes α, so ist $C_\alpha = A_\alpha \frown B_\alpha$ für gewisse $A_\alpha \in \mathcal{A}_\alpha$, $B_\alpha \in \mathcal{B}_\alpha$ und es gibt $A_0 \in \mathcal{A}_0$, $B_0 \in \mathcal{B}_0$, so dass $f(\prod_\alpha A_\alpha) \subseteq A_0$, $f(\prod_\alpha B_\alpha) \subseteq B_0$. Hieraus folgt $f(\prod_\alpha (A_\alpha \cap B_\alpha)) \subseteq A_0 \cap B_0$, wo freilich $A_0 \cap B_0 \in \mathcal{A}_0 \frown \mathcal{B}_0$. Gilt insbesondere $\mathcal{A}_{\alpha_0} \square \mathcal{B}_{\alpha_0}$ für jedes α_0, so ist jeder $\mathcal{A}_{\alpha_0}\smile\mathcal{B}_{\alpha_0}$-Block der Form $A^*_{\alpha_0}(\mathcal{B}_{\alpha_0}$, wo $A^*_{\alpha_0}$ ein willkürlicher in diesem

1) Die Verengungen von f werden wir in der Bezeichnung nicht unterscheiden.

$\mathcal{A}_{\alpha_0} \smile . \mathcal{B}_{\alpha_0}$ -Block enthaltener $A^*_{\alpha_0}$-Block ist. Wählen wir also solche $A^*_\alpha \in \mathcal{A}_\alpha$, so gibt es $A^*_0 \in \mathcal{A}_0$, so dass $f(\sqcap_\alpha A^*_\alpha) \subseteq A^*_0$. Für jede Wahl von $B^\circ_\alpha \in \mathcal{B}_\alpha, A^*_\alpha \mathbin{I} B^\circ_\alpha$ existiert ein $B^\circ_0 \in \mathcal{B}_0, A^*_0 \mathbin{I} B^\circ_0$, so dass $f(\sqcap_\alpha B^\circ_\alpha) \subseteq B^\circ_0$. Daraus folgt $f(\sqcap_\alpha(A^*_\alpha(\mathcal{B}_\alpha)) \subseteq A^*_0(\mathcal{B}_0 \in \mathcal{A}_0 \smile \mathcal{B}_0$.

<u>Satz 1.</u>[2] Es seien $\mathfrak{A}=(\mathcal{A}^i)_{i=1}^{m+1}, \mathfrak{B}=(\mathcal{B}^j)_{j=1}^{n+1}$ zwei Reihen von Zerlegungen auf F mit übereinstimmenden Anfangs-und Endgliedern. Wir setzen voraus, dass für ausgezeichnete Blöcke $A^i_{\alpha_0} \in \mathcal{A}^i_{\alpha_0}$, $B^j_{\alpha_0} \in \mathcal{B}^j_{\alpha_0}$ die Beziehungen $f(\sqcap_\alpha A^i_\alpha) = A^i_0$, $f(\sqcap_\alpha B^j_\alpha) = B^j_0$, $\mathcal{A}^i_{\alpha_0} \boxdot \mathcal{B}^j_{\alpha_0}$ gelten, so dass also $G^i = ((A^i_{\alpha_0})_{\alpha_0}, f)$, $H^j = ((B^j_{\alpha_0})_{\alpha_0}, f)$ Unterstrukturen in F sind. Weiter setzen wir voraus, dass $G^i \sqcap \mathcal{A}^{i+1}, H^j \sqcap \mathcal{B}^{j+1}$ erzeugende Zerlegungen auf G^i bzw. H^j sind. Für die (unteren, bzw. oberen) Zassenhausschen Verfeinerungen $\mathfrak{A} = (\mathcal{A}^{i,j}), \mathfrak{B} = (\mathcal{B}^{j,i})$ von $\mathfrak{A}, \mathfrak{B}$ folgt dann:

a) Sind $A^{i,j}_{\alpha_0} \in \mathcal{A}^{i,j}_{\alpha_0}$, $B^{j,i}_{\alpha_0} \in \mathcal{B}^{j,i}_{\alpha_0}$ ausgezeichnete Blöcke, dann $f(\sqcap_\alpha A^{i,j}_\alpha) = A^{i,j}_0$, $f(\sqcap_\alpha B^{j,i}_\alpha) = B^{j,i}_0$, so dass also $G^{i,j} = ((A^{i,j}_{\alpha_0})_{\alpha_0}, f)$, $H^{j,i} = ((B^{j,i}_{\alpha_0})_{\alpha_0}, f)$ Unterstrukturen in F sind.

b) Die Zerlegungen $G^{i,j} \sqcap \mathcal{A}^{i,j+1}, H^{j,i} \sqcap \mathcal{B}^{j,i+1}$ sind erzeugend auf $G^{i,j}$ bzw. $H^{j,i}$ und die Faktorstrukturen $G^{i,j} / / G^{i,j} \sqcap \mathcal{A}^{i,j+1}$, $H^{j,i} / H^{j,i} \sqcap \mathcal{B}^{j,i+1}$ sind isomorph.

<u>Beweis.</u> Wir setzen wie früher in [II] $\mathcal{A}^{i,j}_{\alpha_0} = \mathcal{A}^{i+1}_{\alpha_0} \smile (\mathcal{A}^i_{\alpha_0} \frown \mathcal{B}^j_{\alpha_0})$, $\bar{\mathcal{B}}^{j,i}_{\alpha_0} = \mathcal{B}^{j+1}_{\alpha_0} \smile (\mathcal{B}^j_{\alpha_0} \frown \mathcal{A}^i_{\alpha_0})$, $\bar{\mathcal{A}}^{i,j}_{\alpha_0} = \mathcal{A}^i_{\alpha_0} \frown (\mathcal{A}^{i+1}_{\alpha_0} \smile \mathcal{B}^j_{\alpha_0})$, $\mathcal{B}^{j,i}_{\alpha_0} = \mathcal{B}^j_{\alpha_0} \frown (\mathcal{B}^{j+1}_{\alpha_0} \smile \mathcal{B}^i_{\alpha_0})$.

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2) Die betreffenden Werte für i, j, α_0 durchlaufen die Indexmengen $\{1, \ldots, m+1\}$ bzw. $\{1, \ldots m\}$; $\{1 \ldots, n+1\}$ bzw. $\{1, \ldots n\}$ und Γ_0.

Im ersten Falle ist $A^{i,j}_{\alpha_0} = (A^i_{\alpha_0} \cap B^j_{\alpha_0}) \subset \mathcal{A}^{i+1}_{\alpha_0}$, $\hat{B}^{j,i}_{\alpha_0} = (B^j_{\alpha_0} \cap A^i_{\alpha_0}) \subset \mathcal{B}^{j+1}_{\alpha_0}$, so dass $f(\bigsqcup \hat{A}^{i,j}_{\alpha}) = \hat{A}^{i,j}_0$ und $f(\bigsqcup \hat{B}^{j,i}_{\alpha}) = \hat{B}^{j,i}_0$. Jeder $\hat{A}^{i,j}_{\alpha_0} \sqcap \mathcal{A}^{i,j+1}_{\alpha_0}$ -Block ist Durchschnitt eines $\mathcal{A}^{i,j+1}_{\alpha_0}$ -Blocks mit $(A^i_{\alpha_0} \cap B^j_{\alpha_0}) \subset \mathcal{A}^{i+1}_{\alpha_0}$ d.h. fällt mit einem $(A^i_{\alpha_0} \cap B^j_{\alpha_0}) \sqcap \mathcal{A}^{i+1}_{\alpha_0}$ -Block zusammen. Aus $f(\bigsqcup A^i_{\alpha}) = A^i_0$, $f(\bigsqcup B^j_{\alpha}) = B^j_0$, $f(\bigsqcup A^{i+1}_{\alpha}) = A^{i+1}_0$ und $f(\bigsqcup X^{i+1}_{\alpha}) \subseteq X^{i+1}_0$ (wo $(A^i_{\alpha} \cap B^j_{\alpha}) \mathrel{\mathbb{I}} X^{i+1}_{\alpha} \in \mathcal{A}^{i+1}_{\alpha}$) folgt also $f(\bigsqcup(A^i_{\alpha} \cap B^j_{\alpha} \cap X^{i+1}_{\alpha})) \subseteq A^i_0 \cap B^j_0 \cap X^{i+1}_0$ und die Zerlegung $\hat{G}^{i,j} \sqcap \hat{\mathcal{A}}^{i,j+1}$ ist erzeugend auf $\hat{G}^{i,j}$. Ähnlicher Schluss folgt für $\hat{H}^{j,i} \sqcap \hat{\mathcal{B}}^{j,i+1}$. Weil es sich nach dem Hauptsatz aus [II] um verknüpfte Zerlegungen $\hat{A}^{i,j}_{\alpha_0} \sqcap \mathcal{A}^{i,j+1}_{\alpha_0}$, $\hat{B}^{j,i}_{\alpha_0} \sqcap \mathcal{B}^{j,i+1}_{\alpha_0}$ handelt (und eine solche Verknüpfung ist durch gemeinsame Durchschnitte mit $A^i_{\alpha_0} \cap B^j_{\alpha_0}$ vermittelt), folgt aus der vorigen Betrachtung, dass es sich sogar um isomorphe Faktorstrukturen $\hat{G}^{i,j} / \hat{G}^{i,j} \sqcap \hat{\mathcal{A}}^{i,j+1}$, $\hat{H}^{j,i} / \hat{H}^{j,i} \sqcap \hat{\mathcal{B}}^{j,i+1}$ handelt.

Was den zweiten Fall betrifft, ist ählich $\breve{A}^{i,j}_{\alpha_0} = A^i_{\alpha_0} \cap C^{i+1,j}_{\alpha_0}$, wo $C^{i+1,j}_{\alpha_0}$ der ausgezeichnete $(\mathcal{A}^{i+1}_{\alpha_0} \smile \mathcal{B}^j_{\alpha_0})$ -Block ist. Wegen $\mathcal{A}^{i+1}_{\alpha_0} \boxminus \mathcal{B}^j_{\alpha_0}$ ist aber $C^{i+1,j}_{\alpha_0} \supseteq A^{i+1}_{\alpha_0} \subset \mathcal{B}^j_{\alpha_0} = B^j_{\alpha_0} \subset \mathcal{A}^{i+1}_{\alpha_0}$. Daraus folgt schon die Beziehung $f(\bigsqcup \breve{A}^{i,j}_{\alpha}) = \breve{A}^{i,j}_0$; ähnlich folgt $f(\bigsqcup \breve{B}^{j,i}_{\alpha}) = \breve{B}^{j,i}_0$. Jeder $\breve{A}^{i,j}_{\alpha_0} \sqcap \breve{\mathcal{A}}^{i,j+1}_{\alpha_0}$ -Block ist gleichzeitig ein $A^i_{\alpha_0} \sqcap (\mathcal{A}^{i+1}_{\alpha_0} \smile \mathcal{B}^{j+1}_{\alpha_0})$ -Block und daraus schliessen wir auf $f(\bigsqcup(A^i_{\alpha} \cap X^{i+1,j+1}_{\alpha})) \subseteq A^i_0 \cap X^{i+1,j+1}_0$, wo $A^i_{\alpha_0} \mathrel{\mathbb{I}} X^{i+1,j+1}_{\alpha_0} \in \mathcal{A}^{i+1}_{\alpha_0} \smile \mathcal{B}^{j+1}_{\alpha_0}$, $f(\bigsqcup X^{i+1,j+1}_{\alpha}) \subseteq X^{i+1,j+1}_0$.

Nach dem Hauptsatz aus [II] sind $\breve{A}^{i,j}_{\alpha_0} \sqcap \breve{\mathcal{A}}^{i,j+1}_{\alpha_0}$, $\breve{B}^{j,i}_{\alpha_0} \sqcap \breve{\mathcal{B}}^{j,i+1}_{\alpha_0}$ stets angeschaltet (und diese Anschaltung ist durch simultane Inzidenzen mit je einem $(C^{i,j+1}_{\alpha} \cap C^{j,i+1}_{\alpha_0}) \sqcap (\mathcal{A}^{i+1}_{\alpha_0} \smile \mathcal{B}^{j+1}_{\alpha_0})$ -Block vermittelt). Daraus ergibt sich ähnlich wie früher der Isomorphismus zwischen $\bar{G}^{i,j}/\bar{G}^{i,j} \sqcap \bar{\mathcal{A}}^{i,j+1}$, $\bar{H}^{i,j}/\bar{H}^{i,j} \sqcap \bar{\mathcal{B}}^{j,i+1}$. $\blacksquare$

<u>Satz 2.</u> Es seien $\mathcal{A} = (\mathcal{A}^i)_{i=1}^{m+1}$, $\mathcal{B} = (\mathcal{B}^j)_{j=1}^{n+1}$ zwei Reihen von erzeugenden Zerlegungen auf F mit übereinstimmenden Anfangs- und Endgliedern, wobei $\mathcal{A}^i_{\alpha_0} \square \mathcal{B}^j_{\alpha_0}$ für jedes i, j, α_0. Dann folgt für die Zassenhausschen Verfeinerungen $\bar{\mathcal{A}} = (\mathcal{A}^{i,j})$, $\bar{\mathcal{B}} = (\mathcal{B}^{j,i})$ von $\mathcal{A}, \mathcal{B}$: Gilt $f((m_\alpha)_\alpha) = m_0$ für $m_{\alpha_0} \in M_{\alpha_0}$ und ist $m_\alpha \in A^{i,j}_\alpha \in \mathcal{A}^{i,j}_\alpha$, $m_\alpha \in B^{j,i}_\alpha \in \mathcal{B}^{j,i}_\alpha$, $A^{i,j}_0 = f(\prod_\alpha A^{i,j}_\alpha)$

$B^{j,i}_0 = f(\prod_\alpha B^{j,i}_\alpha)$, so sind $((A^{i,j}_{\alpha_0})_{\alpha_0} f) / ((A^{i,j}_{\alpha_0})_{\alpha_0} f) \sqcap \mathcal{A}^{i,j+1}_{\alpha_0}$, $((B^{j,i}_{\alpha_0})_{\alpha_0} f) / ((B^{j,i}_{\alpha_0})_{\alpha_0}, f) \sqcap \mathcal{B}^{j,i+1}_{\alpha_0}$ isomorphe Faktorstrukturen und $\mathcal{A}^{i,j}_{\alpha_0} \square \mathcal{B}^{j,i}_{\alpha_0}$.

<u>Beweis.</u> Aus der Tatsache, dass für zwei Zerlegungen Z_1, Z_2 auf einer Grundmenge $Z_1 \square Z_2 \Rightarrow Z_1 \boxdot Z_2$ gilt (für jede Wahl des ausgezeichneten Elements aus der Grundmenge), ergibt sich der erste Teil der Behauptung als Folgerung des Hilfssatzes und des Satzes 1. Der Rest der Behauptung ergibt sich nach [3], S. 93. $\blacksquare$

Zum Schluss bemerken wir, dass die von O. BORŮVKA stammende Fassung des Verfeinerungssatzes für komplementäre Reihen der Faktoroide ([1] S. 110-114) ergibt sich als ein Sonderfall des Satzes 2 für $\Gamma = \{1, 2\}$, $M_1 = M_2 = M$, $M_3 \subseteq M$ bei der Begrenzung auf regelmässige Zerlegungen auf F ([III] § 2).

Eine andere Anwendung bietet die Theorie der Multigruppoide an. Dabei ist $\Gamma = \{1, 2\}$, $M_1 = M_2 = M$ und M_3 liegt in der Potenzmenge von M. Auch hier muss man sich auf gewisse "regelmässige" Zerlegungen auf F begrenzen.

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BEDINGUNGEN DER NICHTOSZILLATIONSFÄHIGKEIT FÜR DIE LINEARE DIFFERENTIALGLEICHUNG DRITTER ORDNUNG

$y''' + p_1(x)y'' + p_2(x)y' + p_3(x)y = 0$

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Die vorliegende Arbeit knüpft an die Arbeit [1] an und wir befassen uns in dieser mit den Konkretkriterien der Nichtoszillationsfähigkeit für die lineare Differentialgleichung dritter Ordnung

(1) $L[y] \equiv y''' + p_1(x)y'' + p_2(x)y' + p_3(x)y = 0,$

wo $p_i(x) \in C(\mathcal{J})$; $i = 1,2,3$; $\mathcal{J} = \langle x_0, b)$ bzw. $(a, x_0\rangle$, $-\infty \leq a < x_0 < b \leq \infty$.

Wir leiten dabei auch einige Eigenschaften der Lösungen dieser Differentialgleichung im Intervall $\mathcal{J}$ ab, wie Nichtnegativität, Nichtpositivität und Begrenzung von oben, bzw. von unten mit bestimmten Funktionen.

Unter der zu der linearen Differentialgleichung $L[y] = 0$ im Intervall $\mathcal{J}$ adjungierten Differentialgleichung verstehen wir die Differentialgleichung

(2) $M[z] \equiv [(z' - p_1(x)z)' + p_2(x)z]' - p_3(x)z = 0$

(siehe [1]).

Diese Differentialgleichung geht durch die Substitution $z(x) = v(x)\exp\int_{x_0}^{x} p_1(t)dt$ in die Differentialgleichung

(3) $M_1[v] \equiv \left(L_2[v]e^{\int_{x_0}^{x} p_1(t)dt}\right)' - p_3(x)\,v\,e^{\int_{x_0}^{x} p_1(t)dt} = 0$

über, wo $L_2[v] \equiv v'' + p_1(x)v' + p_2(x)v$ ist.

Weiter bezeichnen wir $I = \mathcal{J} - \{x_0\}$.

Wir sagen, dass die Differentialgleichung $L[y] = 0$ im Intervall $\mathcal{J}$ nichtoszillatorisch ist, wenn jede ihre nichttriviale Lösung im Intervall $\mathcal{J}$ höchstens zwei Nullstellen, die Vielfachheit inbegriffen, hat.

<u>Lemma 1.</u> Es sei die Funktion $A(x,t)$ stetig und nichtnegativ für $x_0 \leq t \leq x < b$ [nichtpositiv für

$a < x \leqq t \leqq x_0]$. Wenn $p(x), \mu(x)(\nu(x))$ stetige Funktionen im Intervall $< x_0, b)\ [(a, x_0 >]$ sind und

$$\mu(x) \leqq p(x) + \int_{x_0}^{x} A(x,t)\mu(t)dt \quad \left(\nu(x) \geqq p(x) + \int_{x_0}^{x} A(x,t)\nu(t)dt\right)$$

für $x \in < x_0, b)\ [x \in (a, x_0 >]$, dann gilt für die Lösung $y(x)$ der Integralgleichung

(4) $\quad y(x) = p(x) + \int_{x_0}^{x} A(x,t) y(t)\, dt$

im Intervall $< x_0, b)\ [(a, x_0 >]$

$$y(x) \geqq \mu(x) \quad (y(x) \leqq \nu(x)).$$

Die Behauptung des Lemma folgt daraus, dass die Resolvente der Integralgleichung (4) unter den gegebenen Voraussetzungen eine nichtnegative [nichtpositive] Funktion für $x_0 \leqq t \leqq x < b\ [a < x \leqq t \leqq x_0]$ ist (siehe [8]).

Wenn zu den Voraussetzungen des Lemma 1 die Voraussetzung $p(x) \geqq 0 (\leqq 0)$ für $x \in < x_0, b)\ [x \in (a, x_0 >]$ hinzufügen, dann gilt für die Lösung $y(x)$ der Integralgleichung (4) im Intervall $< x_0, b)\ [(a, x_0 >]$

$$y(x) \geqq p(x) \geqq 0 \quad (y(x) \leqq p(x) \leqq 0).$$

Die Integralgleichung (4) ist äquivalent mit der Integralgleichung

(5) $\quad y(x) = \varphi(x) + \int_{x_0}^{x} \left\{ \int_{t}^{x} A(x,\tau) A(\tau,t)\, d\tau \right\} y(t)\, dt,$

wo $\varphi(x) = p(x) + \int_{x_0}^{x} A(x,t) p(t)\, dt$ ist und $p(x), A(x,t)$ stetige Funktionen für $x_0 \leqq t \leqq x < b\ [a < x \leqq t \leqq x_0]$ sind.

Wenn die Voraussetzungen des Lemma 1 erfüllt sind und die Funktion $\varphi(x)$ im Intervall $< x_0, b)\ [(a, x_0 >]$ nichtnegativ (nichtpositiv) ist, dann gilt für die Lösung $y(x)$ der Integralgleichung (4) im Intervall $< x_0, b)\ [(a, x_0 >]$

$$y(x) \geqq \varphi(x) \geqq 0 \quad (y(x) \leqq \varphi(x) \leqq 0).$$

<u>L e m m a 2.</u> Die Funktion $A(x,t)$ sei stetig und nichtpositiv für $x_0 \leqq t \leqq x < b$ [nichtnegativ für $a < x \leqq t \leqq x_0$] und die Funktionen $p(x), \varphi(x)$ seien stetig und nichtnegativ (nichtpositiv) für $x \in < x_0, b)\ [x \in (a, x_0 >]$. Dann gilt

für die Lösung $y(x)$ der Integralgleichung (5) im Intervall $\langle x_0, b)$ $[(a, x_0\rangle]$

$$0 \leqq \varphi(x) \leqq y(x) \leqq p(x) \quad (p(x) \leqq y(x) \leqq \varphi(x) \leqq 0).$$

B e w e i s . Aus den Voraussetzungen des Lemma folgt, dass der Kern der Integralgleichung (5) für $x_0 \leqq t \leqq x < b$ nichtnegativ [für $a < x \leqq t \leqq x_0$ nichtpositiv] ist. Da die Funktion $\varphi(x)$ im Intervall $\langle x_0, b)$ $[(a, x_0\rangle]$ nichtnegativ (nichtpositiv) ist, erhalten wir Lemma 1 für die Lösung $y(x)$ der Integralgleichung (5), dass $y(x) \geqq \varphi(x) \geqq 0$ $(y(x) \leqq \varphi(x) \leqq 0)$ für $x \in \langle x_0, b)$ $[x \in (a, x_0\rangle]$ ist. Mit Rücksicht darauf, dass die Integralgleichung (5) äquivalent ist mit der Integralgleichung (4) haben wir für die Lösung $y(x)$ im gegebenen Intervall

$$y(x) - p(x) = \int_{x_0}^{x} A(x,t)\, y(t)\, dt \leqq 0 \ (\geqq 0)$$

d.h. $y(x) \leqq p(x)$ $(y(x) \geqq p(x))$ für $x \in \langle x_0, b)$ $[x \in (a, x_0\rangle]$.
Damit ist der Beweis des Lemma beendet.

Die Differentialgleichungen $L[y] = 0;\ M[z] = 0;\ M_1[v] = 0$ im Intervall $\mathcal{J}$ zusammen mit den gegebenen Anfangsbedingungen in der Zahl x_0

$$y(x_0) = y_0, \quad y'(x_0) = y_0', \quad y''(x_0) = y_0'';$$
$$z(x_0) = z_0, \quad z'(x_0) = z_0', \quad (z' - p_1(x) z)'_{x=x_0} = z_0'';$$
$$v(x_0) = v_0, \quad v'(x_0) = v_0', \quad v''(x_0) = v_0''$$

sind äquivalent mit den entsprechenden Volterraschen Integralgleichungen zweiter Art

(6) $$y^{(\ell)}(x) = P_\ell(x, y_0, y_0', y_0'') + \int_{x_0}^{x} A_\ell(x,t)\, y^{(\ell)}(t)\, dt;$$

(7) $$z^{(\ell)}(x) = Q_\ell(x, z_0, z_0', z_0'') + \int_{x_0}^{x} B_\ell(x,t)\, z^{(\ell)}(t)\, dt;$$

(8) $$v^{(\ell)}(x) = S_\ell(x, v_0, v_0', v_0'') + \int_{x_0}^{x} C_\ell(x,t)\, v^{(\ell)}(t)\, dt;$$

welche äquivalent sind mit den entsprechenden Integralgleichungen

(9) $$y^{(\ell)}(x) = \varphi_\ell(x, y_0, y_0', y_0'') + \int_{x_0}^{x} \left\{ \int_{t}^{x} A_\ell(x,\tau) A(\tau,t)\, d\tau \right\} y^{(\ell)}(t)\, dt;$$

(10) $$u^{(l)}(x) = \Psi_l(x, u_0, u_0', u_0'') + \int_{x_0}^{x} \Big\{ \int_{t}^{x} B_l(x,\tau)\, B_l(\tau,t)\, d\tau \Big\} u^{(l)}(t)\, dt;$$

(11) $$v^{(l)}(x) = \chi_l(x, v_0, v_0', v_0'') + \int_{x_0}^{x} \Big\{ \int_{t}^{x} C_l(x,\tau)\, C_l(\tau,t)\, d\tau \Big\} v^{(l)}(t)\, dt,$$

wo

(12) $$\varphi_l(x, y_0, y_0', y_0'') = P_l(x, y_0, y_0', y_0'') + \int_{x_0}^{x} A_l(x,t)\, P_l(t, y_0, y_0', y_0'')\, dt;$$

(13) $$\Psi_l(x, u_0, u_0', u_0'') = Q_l(x, u_0, u_0', u_0'') + \int_{x_0}^{x} B_l(x,t)\, Q_l(t, u_0, u_0', u_0'')\, dt;$$

(14) $$\chi_l(x, v_0, v_0', v_0'') = S_l(x, v_0, v_0', v_0'') + \int_{x_0}^{x} C_l(x,t)\, S_l(t, v_0, v_0', v_0'')\, dt,$$

l ist eine der Zahlen 0, 1, 2, 3 (in weiteren Erwägungen der Funktion $P_l(x, y_0, y_0', y_0''), \varphi_l(x, y_0, y_0', y_0''); Q_l(x, u_0, u_0', u_0''), \Psi_l(x, u_0, u_0', u_0''); S_l(x, v_0, v_0', v_0''), \chi_l(x, v_0, v_0', v_0'')$ diese dieselbe Bedeutung haben) und

$$P_l(x, y_0, y_0', y_0'') = \begin{cases} -y_0 p_3(x) - y_0'\Big(p_2(x) + \frac{x-x_0}{1} p_3(x)\Big) - y_0''\Big(p_1(x) + \frac{x-x_0}{1} p_2(x) + \frac{(x-x_0)^2}{2} p_3(x)\Big), & \text{für } l=3, \\ y_0'' e^{-\int_{x_0}^{x} p_1(\tau)d\tau} - y_0' \int_{x_0}^{x} \Big(p_2(t) + \frac{t-x_0}{1} p_3(t)\Big) e^{-\int_{t}^{x} p_1(\tau)d\tau} dt - y_0 \int_{x_0}^{x} p_3(t) e^{-\int_{t}^{x} p_1(\tau)d\tau} dt & \text{bzw.} \\ y_0'' - y_0' \int_{x_0}^{x} \Big(p_2(t) + \frac{t-x_0}{1} p_3(t)\Big) dt - y_0 \int_{x_0}^{x} p_3(t)\, dt, & \text{für } l=2, \\ y_0' + y_0'' \int_{x_0}^{x} e^{-\int_{x_0}^{t} p_1(\tau)d\tau} dt - y_0 \int_{x_0}^{x} p_3(t) \int_{t}^{x} e^{-\int_{t}^{\xi} p_1(\tau)d\tau} d\xi\, dt & \text{bzw.} \\ y_0' + (y_0'' + p_1(x_0) y_0')(x-x_0) - y_0 \int_{x_0}^{x} (x-t) p_3(t)\, dt, & \text{für } l=1, \\ y_0 + y_0'(x-x_0) + y_0'' \int_{x_0}^{x} (x-t) e^{-\int_{x_0}^{t} p_1(\tau)d\tau} dt + y_0 p_2(x_0) \int_{x_0}^{x} (x-t) e^{-\int_{x_0}^{t} p_1(\tau)d\tau} dt & \text{bzw.} \\ y_0 + (y_0' + p_1(x_0) y_0)(x-x_0) + \Big[y_0'' + p_1(x_0) y_0' + (p_2(x_0) - p_1'(x_0)) y_0\Big] \frac{(x-x_0)^2}{2}, & \text{für } l=0, \end{cases}$$

$$k_l(x,t)=\begin{cases} -p_1(x)-\frac{x-t}{1}p_2(x)-\frac{(x-t)^2}{2}p_3(x), \text{ für } l=3, \\ -\int_t^x [p_2(\xi)+(\xi-t)p_3(\xi)]e^{-\int_\xi^x p_1(\tau)d\tau}d\xi \quad \text{bzw.} \\ -p_1(t)-\int_t^x [p_2(\xi)+(\xi-t)p_3(\xi)]d\xi, \text{ für } l=2, \\ -p_2(t)\int_t^x e^{\int_\xi^t p_1(\tau)d\tau}d\xi-\int_t^x p_3(s)\int_s^x e^{\int_\xi^s p_1(\tau)d\tau}d\xi\, ds \quad \text{bzw.} \\ -p_1(t)-\frac{x-t}{1}\left(p_2(t)-p_1'(t)\right)-\int_t^x (x-\xi)p_3(\xi)d\xi, \text{ für } l=1, \\ +\frac{\partial}{\partial t}\left(p_2(t)\int_t^x (x-s)e^{\int_s^t p_1(\tau)d\tau}ds\right)-p_3(t)\int_t^x (x-s)e^{\int_s^t p_1(\tau)d\tau}ds \quad \text{bzw.} \\ -p_1(t)-\frac{x-t}{1}\left(p_2(t)-2p_1'(t)\right)-\frac{(x-t)^2}{2}\left(p_3(t)-p_2'(t)+p_1''(t)\right), \text{ für } l=0; \end{cases}$$

$$Q_l(x,\kappa_0,\kappa_0',\kappa_0'')=\begin{cases} (\kappa_0''+p_1(x_0)\kappa_0'+p_1'(x_0)\kappa_0)p_1(x)+\left(2p_1'(x)-p_2(x)\right)\left[\kappa_0'+(x-x_0)(\kappa_0''+\right. \\ \left.+p_1(x_0)\kappa_0'+p_1'(x_0)\kappa_0)\right]+\left(p_1''(x)-p_2'(x)+p_3(x)\right)\left[\kappa_0+(x-x_0)\kappa_0'+\right. \\ \left.+\frac{(x-x_0)^2}{2}(\kappa_0''+p_1(x_0)\kappa_0'+p_1'(x_0)\kappa_0)\right], \text{ für } l=3, \\ \kappa_0''+p_1(x_0)\kappa_0'+p_1'(x_0)\kappa_0+[p_2(x_0)-p_1'(x_0)]\kappa_0-p_1(x_0)\kappa_0'+ \\ +[p_1'(x)-p_2(x)][\kappa_0+(x-x_0)\kappa_0']+p_1(x)\kappa_0'+\kappa_0\int_{x_0}^x p_3(t)dt+ \\ +\kappa_0'\int_{x_0}^x (t-x_0)p_3(t)dt, \text{ für } l=2, \\ \kappa_0 p_1(x)+\kappa_0\int_{x_0}^x [(x-t)p_3(t)-p_2(t)]dt+\kappa_0'-p_1(x_0)\kappa_0+[p_2(x_0)\kappa_0+ \\ +\kappa_0''](x-x_0), \text{ für } l=1, \\ \kappa_0+[\kappa_0'-p_1(x_0)\kappa_0](x-x_0)+[p_2(x_0)\kappa_0+\kappa_0'']\frac{(x-x_0)^2}{2} \quad \text{bzw.} \\ \kappa_0 e^{\int_{x_0}^x p_1(\tau)d\tau}+[\kappa_0'-p_1(x_0)\kappa_0]\int_{x_0}^x e^{\int_t^x p_1(\tau)d\tau}dt+[p_2(x_0)\kappa_0+ \\ +\kappa_0'']\int_{x_0}^x (t-x_0)e^{\int_t^x p_1(\tau)d\tau}dt, \text{ für } l=0, \end{cases}$$

$$\mathop{B}_{\ell}(x,t)=\begin{cases} p_1(x)+\frac{x-t}{1}\left(2p_1'(x)-p_2(x)\right)+\frac{(x-t)^2}{2}\left(p_1''(x)-p_2'(x)+p_3(x)\right), & \text{für } \ell=3, \\ p_1(x)+\frac{x-t}{1}\left(p_1'(x)-p_2(x)\right)+\int_t^x(\xi-t)\,p_3(\xi)\,d\xi, & \text{für } \ell=2, \\ p_1(x)+\int_t^x\left[(x-\xi)\,p_3(\xi)-p_2(\xi)\right]d\xi, & \text{für } \ell=1, \\ p_1(t)-\frac{x-t}{1}\,p_2(t)+\frac{(x-t)^2}{2}\,p_3(t) \quad \text{bzw.} & \\ \int_t^x\left[(\xi-t)\,p_3(t)-p_2(t)\right]e^{\int_\xi^x p_1(\zeta)d\zeta}\,d\xi, & \text{für } \ell=0, \end{cases}$$

$$\mathop{f}_{\ell}(x,v_0,v_0',v_0'')=\begin{cases} -2p_1(x)v_0''-\left[p_2(x)-p_1'(x)+p_1^2(x)\right]\left[v_0'+(x-x_0)v_0''\right]+\left[p_3(x)-p_2'(x)-\right. \\ \left.-p_1(x)p_2(x)\right]\left[v_0+(x-x_0)v_0'+\frac{(x-x_0)^2}{2}v_0''\right] \quad \text{bzw.} \\ -p_1(x)e^{-\int_{x_0}^x p_1(\zeta)d\zeta}\left(L_2[v]\right)_{x=x_0}-p_1(x)\left\{v_0''+\int_{x_0}^x\left[v_0+(t-x_0)v_0'+v_0''\frac{(t-x_0)^2}{2}\right]p_3(t)e^{\int_x^t p_1(\zeta)d\zeta}dt\right\}+ \\ +\left[p_3(x)-p_2'(x)\right]\left[v_0+(x-x_0)v_0'+\frac{(x-x_0)^2}{2}v_0''\right]-\left[p_2(x)+p_1'(x)\right]\left[v_0'+(x-x_0)v_0''\right], \quad \text{für } \ell=3, \\ v_0\int_{x_0}^x p_3(t)e^{\int_x^t p_1(\zeta)d\zeta}dt-v_0p_2(x)+v_0'\int_{x_0}^x(t-x_0)\,p_3(t)e^{\int_x^t p_1(\zeta)d\zeta}dt- \\ -v_0'\left[p_1(x)+(x-x_0)\,p_2(x)\right]+e^{-\int_{x_0}^x p_1(\zeta)d\zeta}\left(L_2[v]\right)_{x=x_0} \quad \text{bzw.} \\ v_0''+v_0p_2(x_0)-p_2(x)\left[v_0+(x-x_0)v_0'\right]-v_0'\int_{x_0}^x p_1^2(t)\,dt+ \\ +\int_{x_0}^x\left[p_3(t)-p_1(t)p_2(t)\right]\left[v_0+(t-x_0)v_0'\right]dt, \quad \text{für } \ell=2, \\ v_0'e^{-\int_{x_0}^x p_1(\zeta)d\zeta}+v_0\int_{x_0}^x\left[(x-t)\,p_3(t)-p_2(t)\right]e^{\int_x^t p_1(\zeta)d\zeta}dt+ \\ +(x-x_0)e^{\int_{x_0}^x p_1(\zeta)d\zeta}\left(L_2[v]\right)_{x=x_0} \quad \text{bzw.} \\ v_0'+v_0\int_{x_0}^x\left[p_3(t)\int_t^x e^{\int_\xi^t p_1(\zeta)d\zeta}d\xi-p_2(t)\right]dt+\left(L_2[v]\right)_{x=x_0}\int_{x_0}^x e^{-\int_{x_0}^t p_1(\zeta)d\zeta}dt, \quad \text{für } \ell=1, \\ \left[v_0+(x-x_0)v_0'+\frac{(x-x_0)^2}{2}\left(L_2[v]\right)_{x=x_0}\right]e^{-\int_{x_0}^x p_1(\zeta)d\zeta} \quad \text{bzw.} \\ v_0+v_0'\int_{x_0}^x e^{-\int_{x_0}^t p_1(\zeta)d\zeta}dt+\left(L_2[v]\right)_{x=x_0}\int_{x_0}^x(t-x_0)e^{-\int_{x_0}^t p_1(\zeta)d\zeta}dt, \quad \text{für } \ell=0, \end{cases}$$

$$
\zeta_\ell(x,t)=\begin{cases}
-2p_1(x)-\frac{x-t}{1}\left[p_2(x)+p_1'(x)+p_1^2(x)\right]+\frac{(x-t)^2}{2}\left[p_3(x)-p_2'(x)-\right.\\
\left.-p_1(x)p_2(x)\right] \quad \text{bzw.}\\
-p_1(x)-\frac{x-t}{1}\left[p_2(x)+p_1'(x)\right]+\frac{(x-t)^2}{2}\left[p_3(x)-p_2'(x)\right]-\\
-p_1(x)\int_t^x\frac{(\xi-t)^2}{2}p_3(\xi)e^{\int_x^\xi p_1(\eta)d\eta}d\xi, \quad \text{für } \ell=3,\\
-p_1(x)-\frac{x-t}{1}p_2(x)+\int_t^x(\xi-t)p_3(\xi)e^{\int_x^\xi p_1(\eta)d\eta}d\xi \quad \text{bzw.}\\
-p_1(t)-p_1(x)-(x-t)p_2(x)-\int_t^x p_1^2(\xi)d\xi+\\
+\int_t^x(\xi-t)\left[p_3(\xi)-p_1(\xi)p_2(\xi)\right]d\xi, \quad \text{für } \ell=2,\\
\int_t^x\left[(x-\xi)p_3(\xi)-p_2(\xi)\right]e^{\int_x^\xi p_1(\eta)d\eta}d\xi \quad \text{bzw.}\\
-p_1(t)+\int_t^x\left[p_3(s)\int_s^x e^{\int_\xi^s p_1(\eta)d\eta}d\xi-p_2(s)\right]ds, \quad \text{für } \ell=1,\\
\left[p_1(t)-\frac{x-t}{1}p_2(t)+\frac{(x-t)^2}{2}p_3(t)\right]e^{\int_x^t p_1(\eta)d\eta} \quad \text{bzw.}\\
\int_t^x\left[(\xi-t)p_3(t)-p_2(t)\right]e^{\int_\xi^t p_1(\eta)d\eta}d\xi, \quad \text{für } \ell=0;
\end{cases}
$$

dabei setzen wir in den gegebenen Integralgleichungen, die Stetigkcit der auftretenden Ableitungen der Funktionen $p_1(x)$, $p_2(x)$ im erwogenen Intervall $\mathcal{Y}$ voraus.

D e f i n i t i o n 1. Wir sagen, dass die Differentialgleichung $L[y]=0$ aus der Klasse $A_\ell^+(\langle x_0,b))$ $[A_\ell^+((a,x_0\rangle)]$ ist, wenn der Kern der betreffenden Integralgleichung (6) $A_\ell(x,t)$ eine nichtnegative [nichtpositive] Funktion für $x_0\leq t\leq x<b$ $[a<x\leq t\leq x_0]$ ist.

D e f i n i t i o n 2. Wir sagen, dass die Differentialgleichung $L[y]=0$ aus der Klasse $A_\ell^-(\langle x_0,b))$ $[A_\ell^-((a,x_0\rangle)]$ ist, wenn der Kern der betreffenden Integralgleichung (6) $A_\ell(x,t)$ eine nichtpositive [nichtnegative] Funktion für $x_0\leq t\leq x<b$ $[a<x\leq t\leq x_0]$ ist.

Ähnlich definieren wir die Klassen $\mathcal{B}_2^+(\mathcal{J})$, $\mathcal{B}_2^-(\mathcal{J})$ bzw. $\mathcal{C}_2^+(\mathcal{J})$, $\mathcal{C}_2^-(\mathcal{J})$ für die adjungierte Differentialgleichung $M[u]=0$ bzw. für die Differentialgleichung $M_1[v]=0$.

D e f i n i t i o n 3. Wir sagen, dass die Differentialgleichung $L[y]=0$ aus der Klasse $\mathcal{D}_{ij}(\mathcal{J})$ ist, wenn sie aus der Klasse $\mathcal{A}_i^+(\mathcal{J})$ oder $\mathcal{A}_i^-(\mathcal{J})$ ist und gleichzeitig ist die zu ihr adjungierte Differentialgleichung $M[u]=0$ aus der Klasse $\mathcal{B}_j^+(\mathcal{J})$ oder $\mathcal{B}_j^-(\mathcal{J})$ bzw. die Differentialgleichung $M_1[v]=0$ ist aus der Klasse $\mathcal{C}_j^+(\mathcal{J})$ oder $\mathcal{C}_j^-(\mathcal{J})$ (i,j ist eine der Zahlen 0,1,2,3).

B e m e r k u n g 1. Wenn die Differentialgleichung $L[y]=0$ aus der Klasse $\mathcal{A}_2^+(\mathcal{J})$ oder aus der Klasse $\mathcal{A}_2^-(\mathcal{J})$ ist, im allgemeinen kann die Differentialgleichung $M[u]=0$ bzw. $M_1[v]=0$ nicht aus einer beliebigen der Klasse $\mathcal{B}_j^+(\mathcal{J})$, $\mathcal{B}_j^-(\mathcal{J})$ bzw. $\mathcal{C}_j^+(\mathcal{J})$, $\mathcal{C}_j^-(\mathcal{J})$ sein. Zum Beispiel, wenn die Differentialgleichung $L[y]=0$ aus der Klasse $\mathcal{A}_3^+(\langle x_0,b))$ ist, ist $p_1(x)\leq 0$ für $x\in\langle x_0,b)$ und wenn $p_1(x)\not\equiv 0$, dann kann die Differentialgleichung $M[u]=0$ nicht aus der Klasse $\mathcal{B}_1^+(\langle x_0,b))$ sein.

Unmittelbar aus dem auf die Integralgleichungen (6), (7), (8) (9), (10), (11) aplizierten Lemma 1 und Lemma 2 und daraus, dass für die beliebige Funktion $F(x)\in C^{\ell}(\mathcal{J})$ $(\ell\leq 1)$

$$F^{(k)}(x)=\sum_{j=k}^{\ell-1}F^{(j)}(x_0)\frac{(x-x_0)^{j-k}}{(j-k)!}+\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}F^{(\ell)}(t)\,dt$$

ist, wo $k=0,1,\dots,\ell-1$, folgen diese Sätze:

S a t z 1. Es sei eine der Zahlen 0,1,2,3. Die Differentialgleichung $L[y]=0$ sei aus der Klasse $\mathcal{A}_2^+(\mathcal{J})$ und für irgendeine Gruppe dreier Zahlen y_0, y_0', y_0'' sei $P_2(x,y_0,y_0',y_0'')\geq 0\ (\leq 0)$ $[\varphi_2(x,y_0,y_0',y_0'')\geq 0\ (\leq 0)]$ für $x\in\mathcal{J}$. Dann gilt für die Lösung $y(x)$ von $L[y]=0$, welche durch die Anfagsbedingungen $y(x_0)=y_0$, $y'(x_0)=y_0'$, $y''(x_0)=y_0''$ bestimmt ist im Intervall $\mathcal{J}$

$$y^{(2)}(x)-P_2(x,y_0,y_0',y_0'')\geq 0\ (\leq 0)$$

und

$$(x-x_0)^{\ell-k}\Big\{y^{(k)}(x)-\sum_{j=k}^{\ell-1} y^{(j)}(x_0)\frac{(x-x_0)^{j-k}}{(j-k)!}-\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}P_\ell(t,y_0,y_0',y_0'')\,dt\Big\}\geq 0\,(\leq 0)$$

für $0\leq k\leq \ell-1$, k eine ganze Zahl

$$[\,y^{(\ell)}(x)-\varphi_\ell(x,y,y_0,y_0',y_0''\geq 0\,(\leq 0)$$

und

$$(x-x_0)^{\ell-k}\Big\{y^{(k)}(x)-\sum_{j=k}^{\ell-1} y^{(j)}(x_0)\frac{(x-x_0)^{j-k}}{(j-k)!}-\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\varphi_\ell(t,y_0,y_0',y_0'')\,dt\Big\}\geq 0\,(\leq 0)$$

für $0\leq k\leq \ell-1$, k eine ganze Zahl].

<u>Satz 2.</u> Es sei ℓ eine der Zahlen 0,1,2,3. Die Differentialgleichung $L[y]=0$ sei aus der Klasse $\mathfrak{A}_\ell^-(J)$ und für irgendeine Gruppe dreier Zahlen y_0, y_0', y_0'' sei $P_\ell(x,y_0,y_0',y_0'')\geq 0\,(\leq 0)$, $\varphi_\ell(x,y_0,y_0',y_0'')\geq 0\,(\leq 0)$ für $x\in J$. Für die Lösung $y(x)$ von $L[y]=0$, welche durch die Anfangsbedingungen $y(x_0)=y_0,\ y'(x_0)=y_0',\ y''(x_0)=y_0''$ bestimmt ist, im Intervall J gilt dann

$$\varphi_\ell(x,y_0,y_0',y_0'')\leq y^{(\ell)}(x)\leq P_\ell(x,y_0,y_0',y_0'')$$

und

$$(x-x_0)^{\ell-k}\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\varphi_\ell(t,y_0,y_0',y_0'')\,dt\leq\Big(y^{(k)}(x)-\sum_{j=k}^{\ell-1} y^{(j)}(x_0)\frac{(x-x_0)^{j-k}}{(j-k)!}\Big)(x-x_0)^{\ell-k}\leq(x-x_0)^{\ell-k}\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}P_\ell(t,y_0,y_0',y_0'')\,dt,$$

für $0\leq k\leq \ell-1$, k eine ganze Zahl

$$\big(\,P_\ell(x,y_0,y_0',y_0'')\leq y^{(\ell)}(x)\leq \varphi_\ell(x,y_0,y_0',y_0'')$$

und

$$(x-x_0)^{\ell-k}\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}P_\ell(t,y_0,y_0',y_0'')\,dt\leq\Big(y^{(k)}(x)-\sum_{j=k}^{\ell-1} y^{(j)}(x_0)\frac{(x-x_0)^{j-k}}{(j-k)!}\Big)(x-x_0)^{\ell-k}\leq(x-x_0)^{\ell-k}\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\varphi_\ell(t,y_0,y_0',y_0'')\,dt$$

für $0\leq k\leq \ell-1$, k eine ganze Zahl).

<u>Satz 3.</u> Es sei ℓ eine der Zahlen 0,1,2,3. Die Differentialgleichung $M[u]=0$ bzw. $M_1[v]=0$ sei aus der Klasse $\mathfrak{B}_\ell^+(J)$ bzw. $\mathfrak{C}_\ell^+(J)$ und für irgendeine Gruppe dreier Zahlen u_0, u_0', u_0'' bzw. v_0, v_0', v_0'' sei $Q_\ell(x,u_0,u_0',u_0'')\geq 0\,(\leq 0)$ bzw. $S_\ell(x,v_0,v_0',v_0'')\geq 0\,(\leq 0)$ [$\psi_\ell(x,u_0,u_0',u_0'')\geq 0\,(\leq 0)$ bzw.

$$\chi_\ell\,(x, v_0, v_0', v_0'') \geqq 0 \;(\leqq 0)\;]$$ für $x \in J$.

Dann gilt für die Lösung $u(x)$ bzw. $v(x)$ von $M[u]=0$ bzw. $M_1[v]=0$, welche durch die Anfangsbedingungen $u(x_0)=u_0$, $u'(x_0)=u_0'$, $(u'-p_1(x)u)'_{x=x_0}=u_0''$ bzw. $v(x_0)=v_0$, $v'(x_0)=v_0'$, $v''(x_0)=v_0''$ bestimmt ist im Intervall J

$$u^{(\ell)}(x) - Q_\ell\,(x, u_0, u_0', u_0'') \geqq 0 \;(\leqq 0)$$

und

$$(x-x_0)^{\ell-k}\Big\{u^{(k)}(x) - \sum_{j=k}^{\ell-1} u^{(j)}(x_0)\,\frac{(x-x_0)^{j-k}}{(j-k)!} - \int_{x_0}^{x} \frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\, Q_\ell(t, u_0, u_0', u_0'')\,dt\Big\} \geqq 0 \;(\leqq 0)$$

für $0 \leqq k \leqq \ell-1$, k eine ganze Zahl

bzw.

$$v^{(\ell)}(x) - S_\ell\,(x, v_0, v_0', v_0'') \geqq 0 \;(\leqq 0)$$

und

$$(x-x_0)^{\ell-k}\Big\{v^{(k)}(x) - \sum_{j=k}^{\ell-1} v^{(j)}(x_0)\,\frac{(x-x_0)^{j-k}}{(j-k)!} - \int_{x_0}^{x} \frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\, S_\ell(t, v_0, v_0', v_0'')\,dt\Big\} \geqq 0 \;(\leqq 0)$$

für $0 \leqq k \leqq \ell-1$, k eine ganze Zahl

$$[\,u^{(\ell)}(x) - \Psi_\ell\,(x, u_0, u_0', u_0'') \geqq 0 \;(\leqq 0)$$

und

$$(x-x_0)^{\ell-k}\Big\{u^{(k)}(x) - \sum_{j=k}^{\ell-1} u^{(j)}(x_0)\,\frac{(x-x_0)^{j-k}}{(j-k)!} - \int_{x_0}^{x} \frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\, \Psi_\ell(t, u_0, u_0', u_0'')\,dt\Big\} \geqq 0 \;(\leqq 0)$$

für $0 \leqq k \leqq \ell-1$, k eine ganze Zahl

bzw.

$$v^{(\ell)}(x) - \chi_\ell\,(x, v_0, v_0', v_0'') \geqq 0 \;(\leqq 0)$$

und

$$(x-x_0)^{\ell-k}\Big\{v^{(k)}(x) - \sum_{j=k}^{\ell-1} v^{(j)}(x_0)\,\frac{(x-x_0)^{j-k}}{(j-k)!} - \int_{x_0}^{x} \frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\, \chi_\ell(t, v_0, v_0', v_0'')\,dt\Big\} \geqq 0 \;(\leqq 0)$$

für $0 \leqq k \leqq \ell-1$, k eine ganze Zahl $]$.

Satz 4. Es sei ℓ eine der Zahlen 0,1,2,3. Die Differentialgleichung $M[u]=0$ $[M_1[v]=0]$ sei aus der Klasse $\mathfrak{B}_\ell^-(J)$ $[\mathfrak{C}_\ell^-(J)]$ und für irgendeine Gruppe dreier Zahlen u_0, u_0', u_0'' $[v_0, v_0', v_0'']$ sei $Q_\ell(x, u_0, u_0', u_0'') \geqq 0 \;(\leqq 0)$, $\Psi_\ell(x, u_0, u_0', u_0'') \geqq 0$

$(\leqq 0)\,[\,S_\ell(x, v_0, v_0', v_0'') \geqq 0\,(\leqq 0),\ \ \chi_\ell(x, v_0, v_0', v_0'') \geqq 0\,(\leqq 0)\,]$
für $x \in \mathcal{J}$. Für die Lösung $u(x)\,[v(x)]$ von $M[u]=0\,[M_1[v]=0]$, welche durch die Anfangsbedingungen $u(x_0)=u_0,\ u'(x_0)=u_0'$, $(u'-p_1(x)u)'_{x=x_0}=u_0''\ [v(x_0)=v_0, v'(x_0)=v_0', v''(x_0)=v_0'']$ bestimmt ist, im Intervall $\mathcal{J}$ gilt dann

$$\Psi_\ell(x, u_0, u_0', u_0'') \leqq u^{(\ell)}(x) \leqq Q_\ell(x, u_0, u_0', u_0'')$$

und

$$(x-x_0)^{\ell-k}\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\,\Psi_\ell(t, u_0, u_0', u_0'')\,dt \leqq \Big(u^{(k)}(x)-\sum_{j=k}^{\ell-1}u^{(j)}(x_0)\frac{(x-x_0)^{j-k}}{(j-k)!}\Big)(x-x_0)^{\ell-k} \leqq$$
$$\leqq (x-x_0)^{\ell-k}\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\,Q_\ell(t, u_0, u_0', u_0'')\,dt$$

für $0 \leqq k \leqq \ell-1$, k eine ganze Zahl

$$(\,Q_\ell(x, u_0, u_0', u_0'') \leqq u^{(\ell)}(x) \leqq \Psi_\ell(x, u_0, u_0', u_0'')$$

und

$$(x-x_0)^{\ell-k}\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\,Q_\ell(t, u_0, u_0', u_0'')\,dt \leqq \Big(u^{(k)}(x)-\sum_{j=k}^{\ell-1}u^{(j)}(x_0)\frac{(x-x_0)^{j-k}}{(j-k)!}\Big)(x-x_0)^{\ell-k} \leqq (x-x_0)^{\ell-k}\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\,\Psi_\ell(t, u_0, u_0', u_0'')\,dt$$

für $0 \leqq k \leqq \ell-1$, k eine ganze Zahl

$$[\,\chi_\ell(x, v_0, v_0', v_0'') \leqq v^{(\ell)}(x) \leqq S_\ell(x, v_0, v_0', v_0'')$$

und

$$(x-x_0)^{\ell-k}\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\,\chi_\ell(t, v_0, v_0', v_0'')\,dt \leqq \Big(v^{(k)}(x)-\sum_{j=k}^{\ell-1}v^{(j)}(x_0)\frac{(x-x_0)^{j-k}}{(j-k)!}\Big)(x-x_0) \leqq (x-x_0)^{\ell-k}\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\,\chi_\ell(t, v_0, v_0', v_0'')\,dt$$

für $0 \leqq k \leqq \ell-1$, k eine ganze Zahl.

$$(\,S_\ell(x, v_0, v_0', v_0'') \leqq v^{(\ell)}(x) \leqq \chi(x, v_0, v_0', v_0'')$$

und

$$(x-x_0)^{\ell-k}\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\,S_\ell(t, v_0, v_0', v_0'')\,dt \leqq \Big(v^{(k)}(x)-\sum_{j=k}^{\ell-1}v^{(j)}(x_0)\frac{(x-x_0)^{j-k}}{(j-k)!}\Big)(x-x_0)^{\ell-k} \leqq (x-x_0)^{\ell-k}\int_{x_0}^{x}\frac{(x-t)^{\ell-1-k}}{(\ell-1-k)!}\,\chi_\ell(t, v_0, v_0', v_0'')\,dt$$

für $0 \leqq k \leqq \ell-1$, k eine ganze Zahl $)\,]$.

B e m e r k u n g 2. Auf Grund der Sätze 1,2,3,4 ist möglich im Falle $b=\infty$ bzw. $a=-\infty$ auf einige asymptotisch Eingenschaften der Lösungen der Differentialgleichungen $L[y]=0$, $M[u]=0$, $M_1[v]=0$ in der Umgebund des Punktes ∞ bzw. $-\infty$ zu schliessen.

S a t z 5. Es sei i, j eine der Zahlen 0,1,2,3. Die

Differentialgleichung $L[y]=0$ sei aus der Klasse $\mathcal{D}_{ij}(\mathcal{J})$ und im Falle, dass die Differentialgleichung

a/ $L[y]=0$ aus der Klasse $\mathcal{A}_i^-(\mathcal{J})$ $(i\leq 2)$ ist, sei

$(x-x_0)^i \varphi_i(x,0,0,1)\geq 0;$

b/ $L[y]=0$ aus der Klasse $\mathcal{A}_3^-(\mathcal{J})$ ist, sei

$(x-x_0)^3 \varphi_3(x,0,0,1)\leq 0,\ (x-x_0)^2+\int_{x_0}^{x}(x-t)^2 P_3(t,0,0,1)\,dt\geq 0;$

c/ $M[u]=0$ aus der Klasse $\mathcal{B}_j^-(\mathcal{J})$ $(j\leq 2)$ ist, sei

$(x-x_0)^i \psi_j(x,0,0,1)\geq 0$

bzw. $M_1[v]=0$ aus der Klasse $\mathcal{C}_j^-(\mathcal{J})$ $(j\leq 2)$ ist, sei

$(x-x_0)^i \chi_j(x,0,0,1)\geq 0;$

d/ $M[u]=0$ aus der Klasse $\mathcal{B}_3^-(\mathcal{J})$ ist, sei

$(x-x_0)^3 \psi_3(x,0,0,1)\leq 0,\ (x-x_0)^2+\int_{x_0}^{x}(x-t)^2 Q_3(t,0,0,1)\,dt\geq 0$

bzw. $M_1[v]=0$ aus der Klasse $\mathcal{C}_3^-(\mathcal{J})$ ist, sei

$(x-x_0)^3 \chi_3(x,0,0,1)\leq 0,\ (x-x_0)^2+\int_{x_0}^{x}(x-t)^2 S_3(t,0,0,1)\,dt\geq 0;$

für $x\in\mathcal{J}$ und ausserdem, wenn die Differentialgleichung $L[y]=0$ aus einer der Klassen $\mathcal{A}_0^-(\mathcal{J})$, $\mathcal{A}_3^-(\mathcal{J})$ und zugleich die Differentialgleichung $M[u]=0$ aus der Klassen $\mathcal{B}_0^-(\mathcal{J})$, $\mathcal{B}_3^-(\mathcal{J})$ bzw. die Differentialgleichung $M_1[v]=0$ aus einer der Klassen $\mathcal{C}_0^-(\mathcal{J})$, $\mathcal{C}_3^-(\mathcal{J})$ ist, gelte eine der entsprechenden Ungleichheiten

$$\varphi_0(x,0,0,1)+\psi_0(x,0,0,1)>0,\quad \varphi_0(x,0,0,1)+(x-x_0)^2+\int_{x_0}^{x}(x-t)^2 Q_3(t,0,0,1)\,dt>0,$$

$$(x-x_0)^2+\int_{x_0}^{x}(x-t)^2 P_3(t,0,0,1)\,dt+\psi_0(x,0,0,1)>0,\quad (x-x_0)^2+\int_{x_0}^{x}\frac{(x-t)^2}{2}\{P_3(t,0,0,1)+Q_3(t,0,0,1)\}\,dt>0$$

bzw.

$$\varphi_0(x,0,0,1)+\chi_0(x,0,0,1)>0,\quad \varphi_0(x,0,0,1)+(x-x_0)^2+\int_{x_0}^{x}(x-t)^2 S_3(t,0,0,1)\,dt>0,$$

$$(x-x_0)^2+\int_{x_0}^{x}(x-t)^2 P_3(t,0,0,1)\,dt+\chi_0(x,0,0,1)>0,\quad (x-x_0)^2+\int_{x_0}^{x}\frac{(x-t)^2}{2}\{P_3(t,0,0,1)+S_3(t,0,0,1)\}\,dt>0$$

für $x\in I$. Dann ist die Differentialgleichung $L[y]=0$ im Intervall $\mathcal{J}$ nichtoszillatorisch.

B e w e i s . Um zu beweisen, dass die Differentialgleichung $L[y]=0$ im Intervall $\mathcal{J}$ nichtoszillatorisch ist, genügt es zu zeigen, dass ihre Lösung $y_1(x)$ mit der Eingenschaft $y_1(x_0)=y_1'(x_0)=0, y_1''(x_0)=1$ und gleichzeitig die Lösung $\alpha_1(x)$ der Differentialgleichung $M[\alpha]=0$ mit der Eingenschaft $\alpha_1(x_0)=\alpha_1'(x_0)=0,\ (\alpha_1'-p_1(x)\alpha_1)'_{x=x_0}=1$ bzw. die Lösung $v_1(x)$ der Differentialgleichung $M_1[v]=0$ mit der Eingenschaft $v_1(x_0)=v_1'(x_0)=0, v_1''(x_0)=1$ für $x\in I$ positiv ist (siehe Folgerung 1 des Satzes 1 [1]).

Den Beweis fürhen wir nur für folgende Fähle durch:

I. Den Fall $\mathcal{D}_{30}(<x_0,b))$, wenn die Differentialgleichung $L[y]=0$ aus der Klasse $\mathcal{A}_3^+(<x_0,b))$ und die Differentialgleichung $M[\alpha]=0$ aus der Klasse $\mathcal{B}_0^-(<x_0,b))$ ist.

II. Den Fall $\mathcal{D}_{33}(<x_0,b))$, wenn die Differentialgleichung $L[y]=0$ aus der Klasse $\mathcal{A}_3^+(<x_0,b))$ und die Differentialgleichung $M[\alpha]=0$ aus der Klasse $\mathcal{B}_3^-(<x_0,b))$ ist.

III. Den Fall $\mathcal{D}_{12}((a,x_0>)$, wenn die Differentialgleichung $L[y]=0$ aus der Klasse $\mathcal{A}_1^-((a,x_0>)$ und die Differentialgleichung $M_1[v]=0$ aus der Klasse $\mathcal{C}_2^-((a,x_0>)$ ist.

IV. Den Fall $\mathcal{D}_{00}(<x_0,b))$, wenn die Differentialgleichung $L[y]=0$ aus der Klasse $\mathcal{A}_0^-(x_0,b))$ und die Differentialgleichung $M[\alpha]=0$ aus der Klasse $\mathcal{B}_0^-(<x_0,b))$ ist.

V. Den Fall $\mathcal{D}_{ij}(<x_0,b))$, wenn die Differentialgleichung $L[y]=0$ aus der Klasse $\mathcal{A}_i^+(<x_0,b))$ und die Differentialgleichung $M_1[v]=0$ aus der Klasse $\mathcal{C}_j^+(<x_0,b))$ ist.

In den anderen Fählen ist die Beweisführung ähnlich.

I. Da die Differentialgleichung $L[y]=0$ aus der Klasse $\mathcal{A}_3^+(<x_0,b))$ ist, ist der Kern der Integralgleichung (6) (für $\ell=3$) $A_3(x,t)=-p_1(x)-\frac{x-t}{1}p_2(x)-\frac{(x-t)^2}{2}p_3(x)$ eine nichtnegative Funktion für $x_0\leqq t\leqq x<b$ und daher auch $A_3(x,x_0)=-p_1(x)-\frac{x-x_0}{1}p_2(x)-\frac{(x-x_0)^2}{2}p_3(x)\geqq 0$ für $x\in<x_0,b)$. Daraus, dass $A_3(x,x_0)=P_3(x,0,0,1)$ ist haben wir für die Lösung $y_1(x)$ gemäss Satz 1 $y_1'''\geqq P_3(x,0,0,1)\geqq 0$, woraus $y_1(x)\geqq\frac{(x-x_0)^2}{2}$ für $x\in<x_0,b)$ ist.

Mit Rücksicht darauf, dass die Differentialgleichung $M[\alpha]=0$ aus der Klasse $\mathcal{B}_0^-(<x_0,b))$ ist, ist der Kern der Integralgleichung (7) (für $\ell=0$) $B_0(x,t)$ eine nichtpositive Funktion für $x_0 \leq t \leq x < b$.

Da $Q_0(x,0,0,1) \geq 0$ ist und nach Voraussetzung auch $\Psi_0(x,0,0,1) \geq 0$ für $x \in <x_0,b)$ ist, dann gilt auf Grund des Satzes 4 für die Lösung $\mu_1(x)$ im Intervall $<x_0,b)$

$$0 \leq \Psi_0(x,0,0,1) \leq \mu_1 .$$

Wir zeigen, dass $\mu_1(x) > 0$ für $x \in (x_0,b)$ ist. Indirekt. In der Zahl $\bar{x}_0 \in (x_0,b)$ sei $\mu_1(\bar{x}_0)=0$. Da $\mu_1(x) \in C^1(<x_0,b))$ und $\mu_1(x) \geq 0$ für $x \in <x_0,b)$ ist auch $\mu_1'(\bar{x}_0)=0$. Laut Lemma 2 [1] ist dann auch $J_1(\bar{x}_0)=0$, was nicht möglich ist, weil $J_1(\bar{x}_0) \geq \frac{(\bar{x}_0-x_0)^2}{2} > 0$. Also $\mu_1(x) > 0$ für $x \in (x_0,b)$.

II. Aus dem Beweis des Falles I folgt, dass es genügt die Nichtnegativität der Lösung $\mu_1(x)$ der Differentialgleichung $M[\alpha]=0$ im Intervall $<x_0,b)$ zu zeigen. Da die Differentialgleichung $M[\alpha]=0$ aus der Klasse $\mathcal{B}_3^-(<x_0,b))$ ist, ist der Kern der Integralgleichung (7) (für $\ell=3$)

$$B_3(x,t)=p_1(x)+(2p_1'(x)-p_2(x))\frac{x-t}{1}+\frac{(x-t)^2}{2}\left(p_1''(x)-p_2'(x)+p_3(x)\right)$$

eine nichtpositive Funktion für $x_0 \leq t \leq x < b$ und also ist auch $B_3(x,x_0) \leq 0$ für $x \in <x_0,b)$. Da $B_3(x,x_0)=Q_3(x,0,0,1)$ und nach der Voraussetzung des Satzes ist $\Psi_3(x,0,0,1) \leq 0$, $(x-x_0)^2+\int_{x_0}^{x}(x-t)^2 Q_3(t,0,0,1)\,dt \geq 0$ für $x \in <x_0,b)$ auf Grund des Satzes 4 erhalten wir für die Lösung $\mu_1(x)$ im Intervall $<x_0,b)$

$$\mu_1(x) \geq \frac{(x-x_0)^2}{2}+\int_{x_0}^{x}\frac{(x-t)^2}{2}Q_3(t,0,0,1)\,dt \geq 0$$

d.h. $\mu_1(x) \geq 0$ für $x \in <x_0,b)$

III. In diesem Falle ist der Kern der Integralgleichung (6) (für $\ell=1$) $A_1(x,t)$ eine nichtnegative Funktion für $a < x \leq t \leq x_0$ ebenso ist der Kern der Integralgleichung (8) (für $\ell=2$) $C_2(x,t)$ eine nichtnegative Funktion für $a < x \leq t \leq x_0$. Da

$P_1(x,0,0,1) \leq 0$; $S_2(x,0,0,1) \geq 0$ ist und nach der Voraussetzung ist auch $\varphi_1(x,0,0,1) \leq 0$, $\chi_2(x,0,0,1) \geq 0$ für $x \in (a, x_0>$; aus den Sätzen 2 und 4 haben wir für die Lösungen $y_1(x)$, $v_1(x)$ im Intervall $(a, x_0>$

$$P_1(x,0,0,1) \leq y_1'(x) \leq \varphi_1(x,0,0,1) \leq 0,$$

$$0 \leq \int_{x_0}^{x} \varphi_1(t,0,0,1)\,dt \leq y_1(x) \leq \int_{x_0}^{x} P_1(t,0,0,1)\,dt;$$

$$0 \leq \chi_2(x,0,0,1) \leq v_1''(x) \leq S_2(x,0,0,1),$$

$$\int_{x_0}^{x} S_2(t,0,0,1)\,dt \leq v_1'(x) \leq \int_{x_0}^{x} \chi_2(t,0,0,1)\,dt,$$

$$0 \leq \int_{x}^{x_0} (x-t)\,\chi_2(t,0,0,1)\,dt \leq v_1(x) \leq \int_{x}^{x_0} (x-t)\,S_2(t,0,0,1)\,dt$$

d.h. $y_1(x)$ und $v_1(x)$ sind im Intervall (a, x_0) nichtwachsende Funktionen und also $y_1(x)$ und $v_1(x)$ sind für $x \in (a, x_0)$ positiv.

IV. Mit Rücksicht darauf, dass die Differentialgleichung $L[y]=0$ uas der Klasse $A_0^-(<x_0, b)$ und die Differentialgleichung $M[z]=0$ aus der Klasse $B_0^-(<x_0, b))$ ist, ist der Kern der Integralgleichung (6) (für $l=0$) $A_0(x,t)$ und der Kern der Integralgleichung (7) (für $l=0$) $B_0(x,t)$ eine nichtpositive Funktion für $x_0 \leq t \leq x < b$. Da $P_0(x,0,0,1) \geq 0$, $Q_0(x,0,0,1) \geq 0$ ist und gemäss den Voraussetzungen des Satzes ist auch $\varphi_0(x,0,0,1) \geq 0$, $\psi_0(x,0,0,1) \geq 0$ für $x \in <x_0, b)$, laut den Sätzen 2 und 4 haben wir für die Lösungen $y_1(x)$, $\mu_1(x)$ im Intervall $<x_0, b)$

$$0 \leq \varphi_0(x,0,0,1) \leq y_1(x) \leq P_0(x,0,0,1);$$

$$0 \leq \psi_0(x,0,0,1) \leq \mu_1(x) \leq Q_0(x,0,0,1).$$

Aus diesen Ungleichheiten geht hervor, dass $y_1(x) + \mu_1(x) \geq$ $\geq \varphi_0(x,0,0,1) + \psi(x,0,0,1)$. Da wie vorausgesetz wurde $\varphi_0(x,0,0,1) +$ $+\psi_0(x,0,0,1) > 0$ für $x \in (x_0, b)$ ist, ist auch $y_1(x) +$ $+\mu_1(x) > 0$ für $x \in (x_0, b)$. Wir zeigen, dass $y_1(x) > 0$, $\mu_1(x) > 0$ für $x \in (x_0, b)$ ist. Es sei zum Beispiel

$y_1(\bar{x}_0) = 0$ in irgendeiner Zahl $\bar{x}_0 \in (x_0, b)$. Dann folgt aus der Ungleichung $0 \leq \varphi_0(\bar{x}_0, 0, 0, 1) \leq y_1(\bar{x}_0) = 0$ dass $\varphi_0(\bar{x}_0, 0, 0, 1) = 0$ ist und also $\psi_0(\bar{x}_0, 0, 0, 1) > 0$. Mit Rücksicht darauf, dass $y_1(x) \geq 0$ und $y_1(x) \in C^1(\langle x_0, b))$ ist, ist auch $y_1'(\bar{x}_0) = 0$. Aus dieser Tatsache nach dem Lemma 2 [1] ist dann auch $\mu_1(\bar{x}_0) = 0$, was nichtmöglich ist, weil $\mu_1(\bar{x}_0) \geq \psi_0(\bar{x}_0, 0, 0, 1) > 0$. Ähnlich wird bewiesen, dass $\mu_1(x) > 0$ für $x \in (x_0, b)$ ist.

V. Da die Differentialgleichung $L[y] = 0$ aus der Klasse $A_i^+(\langle x_0, b))$ und die Differentialgleichung $M_1[v] = 0$ aus der Klasse $C_j^+(\langle x_0, b))$ ist und $P_i(x, 0, 0, 1) \geq 0$, $S_j(x, 0, 0, 1) \geq 0$ für $x \in \langle x_0, b)$ ist, gilt für die Lösungen $y_1(x)$ und $v_1(x)$ auf Grund der Sätze 1 und 3 folgendes

$$y_1(x) \geq \frac{(x-x_0)^{i-1}}{(i-1)!} + \int_{x_0}^{x} \frac{(x-t)^{i-1}}{(i-1)!} P_i(t, 0, 0, 1)\,dt \geq 0, \quad v_1(x) \geq \frac{(x-x_0)^{j-1}}{(j-1)!} + \int_{x_0}^{x} \frac{(x-t)^{j-1}}{(j-1)!} S_j(t, 0, 0, 1)\,dt \geq 0$$

wenn $i \geq 1, j \geq 1$,

$$y_1(x) \geq P_0(x, 0, 0, 1) > 0, \; v_1(x) \geq S_0(x, 0, 0, 1) > 0, \text{ für } i = 0, j = 0; \; x \in (x_0, b) \text{ d.h. } y_1(x) > 0,$$
$$v_1(x) > 0 \text{ für } x \in (x_0, b).$$

Damit ist der Beweis des Satzes beendet.

<u>Folgerung 1.</u> Es seien $p_2'(x), p_3(x)$ im Intervall J stetige Funktionen mit der Eigenschaft $p_2(x) \leq 0$, $p_2'(x) \leq p_3(x) \leq 0$ $(p_2(x) \leq 0, 0 \leq p_3(x) \leq p_2'(x))$ für $x \in J$. Dann ist die lineare Differentialgleichung

(15) $$y''' + p_2(x)\, y' + p_3(x)\, y = 0$$

im Intervall J nichtoszillatorisch.

<u>Beweis.</u> Es sei $p_2(x) \leq 0, p_2'(x) \leq p_3(x) \leq 0$ $(p_2(x) \leq 0, 0 \leq p_3(x) \leq p_2'(x))$ im Interval J. Aus den Ungleichungen $p_2(x) \leq 0$, $p_3(x) \leq 0$ $(p_2(x) \leq 0, p_3(x) \geq 0)$ für $x \in J$ ist ersichtlich, dass die Differentialgleichung (15) aus den Klassen $A_i^+(\langle x_0, b))$ $(A_i^+((a, x_0\rangle))$, $i = 1, 2, 3$, ist.

Mit Rücksicht darauf, dass $p_2'(x) \leq p_3(x) \leq 0$ $(0 \leq p_3(x) \leq p_2'(x))$ ist haben wir:

$$\int_t^x [p_2(\xi)+(\xi-t)p_3(\xi)]d\xi \geqq \int_t^x [p_2(\xi)+(\xi-t)p_2'(\xi)]d\xi = \frac{x-t}{1}p_2(x),$$

$$(x-t)p_2(t)+\int_t^x (x-\xi)p_3(\xi)d\xi \geqq (x-t)p_2(t)+\int_t^x (x-\xi)p_2'(\xi)d\xi = \int_t^x p_2(\xi)d\xi$$

für $a < x \leqq t \leqq x_0$

$$\Big(\int_t^x [p_2(\xi)+(\xi-t)p_3(\xi)]d\xi \leqq (x-t)p_2(x),\ (x-t)p_2(t)+\int_t^x (x-\xi)p_3(\xi)d\xi \leqq \int_t^x p_2(\xi)d\xi$$

für $x_0 \leqq t \leqq x < b$).

Aus diesen letzten Ungleichheiten und den Ungleichheiten $p_2(x) \leqq 0$, $p_2'(x) \leqq p_3(x) \leqq 0$ $\big(p_2(x) \leqq 0,\ 0 \leqq p_3(x) \leqq p_2'(x)\big)$ erhalten wir, dass die Differentialgleichung (15) aus den Klassen $A_j^+((a,x_0>)$, $(A_j^+(x_0,b)))$, $j = 0,1,2$, ist. Ähnlich kann bewiesen werden, dass die zu der Differentialgleichung (15) adjungierte Differentialgleichung

$$(15') \qquad u''' + p_2(x)u' + (p_2'(x) - p_3(x))u = 0$$

aus den Klassen $B_i^+(<x_0,b))$, $B_j^+((a,x_0>)$ $\big(B_j^+(<x_0,b))$, $B_i^+((a,x_0>)\big)$, $j = 0,1,2$; $i = 1,2,3$, ist.

Die Voraussetzungen des Satzes 5 sind erfüllt und deshalb ist die Differentialgleichung (15) im Intervall J nichtoszillatorisch. (Diese Folgerung ist ein Spezialfall des Satzes 5.11 [9].

Den ersten Fall der Folgerung bewies auch M. ŠVEC [2].)

<u>Folgerung 2.</u> Es seien $A'(x)$, $b(x)$ stetige Funktionen im Intervall J und es sei $A(x) \leqq 0$, $b(x) \geqq 0$ $(\leqq 0)$ für $x \in J$. Wenn hiebei $A'(x)+b(x) \leqq 0$ $(\geqq 0)$ oder $b(x) - A'(x) \leqq 0$ $(\geqq 0)$ für $x \in J$ ist, dann ist die lineare Differentialgleichung

$$(16) \qquad y''' + 2A(x)y' + (A'(x) + b(x))y = 0$$

im Intervall J nichtoszillatorisch.

(Diese Folgerung ist ein Teil der Ergebnisse, welche von M. GREGUŠ [3] bewiesen wurden).

<u>Beweis.</u> Setzen wir $2A(x) = p_2(x)$, $A'(x) + b(x) = p_3(x)$. Dann ist $A(x) = \frac{1}{2}p_2(x)$, $b(x) = p_3(x) - \frac{1}{2}p_2'(x)$, $b(x) - A'(x) = p_3(x) - p_2'(x)$ und aus den Voraussetzungen der Folgerung erhalten wir, dass

$p_2(x) \leqq 0$, $p_3(x) - \frac{1}{2} p_2'(x) \geqq 0 \,(\leqq 0)$; $p_3(x) \leqq 0 \,(\geqq 0)$ oder $p_3(x) - p_2'(x) \leqq 0 \,(\geqq 0)$ für $x \in \mathcal{J}$. Wir zeigen, dass die Voraussetzungen der Folgerung 1 erfüllt sind. Es sei zuerst $p_2(x) \leqq 0$, $p_3(x) - \frac{1}{2} p_2'(x) \geqq 0 \,(\leqq 0)$, $p_3(x) \leqq 0 \,(\geqq 0)$ für $x \in \mathcal{J}$. Dann folgt aus der Ungleichheit $p_3(x) \leqq 0 \,(\geqq 0)$ und $p_3(x) - \frac{1}{2} p_2'(x) \geqq 0 \,(\leqq 0)$, dass $p_2'(x) \leqq 0 \,(\geqq 0)$ für $x \in \mathcal{J}$ ist und daher $p_3(x) \geqq \frac{1}{2} p_2'(x) \geqq p_2'(x)$ $\left(p_3(x) \leqq \leqq \frac{1}{2} p_2'(x) \leqq p_2'(x)\right)$ d.h $p_3(x) - p_2'(x) \geqq 0 \,(\leqq 0)$ für $x \in \mathcal{J}$.

Wenn $p_2(x) \leqq 0$, $p_3(x) - \frac{1}{2} p_2'(x) \geqq 0 \,(\leqq 0)$, $p_3(x) - p_2'(x) \leqq 0$ $(\geqq 0)$ für $x \in \mathcal{J}$ ist, dann ist $\frac{1}{2} p_2'(x) \leqq p_3(x) \leqq p_2'(x)$ $\left(p_2'(x) \leqq p_3(x) \leqq \frac{1}{2} p_2'(x)\right)$, daraus folgt, dass $p_2'(x) \geqq 0 \,(\leqq 0)$ für $x \in \mathcal{J}$. Dies bedeutet aber, dass $p_3(x) \geqq 0 \,(\leqq 0)$ für $x \in \mathcal{J}$ ist. Damit ist gezeigt, dass die Voraussetzungen der Folgerung 1 erfüllt sind und also die Differentialgleichung (16) im Intervall $\mathcal{J}$ nichtoszillatorisch ist.

<u>Folgerung 3.</u> Es sei c irgendeine Zahl aus dem Intervall (a,b). Es seien $p_2(x)$, $p_3(x)$ stetige Funktionen im Intervall (a,b) und

(17) $$p_2(x) \leqq k + \int_c^x p_3(t)\,dt \leqq 0$$

sei für $x \in (a,b)$, wo k eine nichtpositive Konstante ist. Dann ist die lineare Differentialgleichung

(15) $$y''' + p_2(x)y' + p_3(x)y = 0$$

im Intervall (a,b) nichtoszillatorisch. (Im Falle $k = 0$ erhalten wir ein Kriterium, welches von J. ČERVEŇ [4] abgeleitet wurde.)

<u>Beweis.</u> Wir zeigen, dass die Differentialgleichung (15) aus der Klasse $\mathcal{A}_1^+(<x_0, b))$ und die zu dieser adjungierte Differentialgleichung (15') aus der Klasse $\mathcal{B}_2^+(<x_0, b))$ ist, für jede Zahl $x_0 \in (a,b)$. Gemäss dem zweiten Satz über den Mittelwert der Integralrechnung ist

$$\int_t^x (x-\xi)\, p_3(\xi)\,d\xi = (x-t) \int_t^{\bar{x}} p_3(\xi)\,d\xi\,, \quad t \leqq \bar{x} \leqq x;$$

$$\int_t^x (\xi-t)\, p_3(\xi)\, d\xi = (x-t)\int_{\bar{t}}^{x} p_3(\xi)\, d\xi, \quad t \leq \bar{t} \leq x.$$

Auf Grund dessen und auf Grund der Ungleichkeit (17) haben wir

$$-A_1(x,t) = (x-t)p_2(t) + \int_t^x (x-\xi)p_3(\xi)d\xi = (x-t)\Big[p_2(t) + \int_t^{\bar{x}} p_3(\xi)d\xi\Big] \leq$$

$$\leq (x-t)\Big[k + \int_c^t p_3(\xi)d\xi + \int_t^{\bar{x}} p_3(\xi)d\xi\Big] = (x-t)\Big[k + \int_c^{\bar{x}} p_3(\xi)d\xi\Big] \leq 0$$

und

$$B_2(x,t) = -(x-t)p_2(x) + \int_t^x (\xi-t)p_3(\xi)d\xi = (x-t)\Big[-p_2(x) + \int_{\bar{t}}^x p_3(\xi)d\xi\Big] \geq$$

$$\geq (x-t)\Big[-k - \int_c^x p_3(\xi)d\xi + \int_{\bar{t}}^x p_3(\xi)d\xi\Big] = (x-t)\Big[-k - \int_c^{\bar{t}} p_3(\xi)d\xi\Big] \geq 0$$

für $x \geq t;\ x, t \in (a,b)$. Damit ist gezeigt worden, dass die Differentialgleichung (15) aus der Klasse $\mathcal{A}_1^+(<x_0, b))$ ist und die Differentialgleichung (15´) aus der Klasse $\mathcal{B}_2^+(<x_0, b))$ ist. Daraus folgt, dass die Differentialgleichung (15) im Intervall $<x_0, b)$ nichtoszillatorisch ist (Satz 5).

Da die Differentialgleichung (15) im Intervall $<x_0, b)$ nichtoszillatorisch ist für jede Zahl $x_0 \in (a,b)$ deshalb ist sie nichtoszillatorisch im Intervall (a, b).

Mit ähnlichen Erwägungen kann die nachstehende Folgerung bewiesen werden.

<u>Folgerung 4.</u> Es seien $A'(x), b(x), p(x)$ stetige Funktionen im Intervall $\mathcal{J}$ und es sei $A(x) \leq 0, p(x) \geq 0\ (\leq 0)$, $b(x) \geq 0\ (\leq 0)$ für $x \in \mathcal{J}$. Wenn dabei $A'(x) + b(x) \leq 0\ (\geq 0)$ oder $b(x) - A'(x) - 2A(x)p(x) \leq 0\ (\geq 0)$ für $x \in \mathcal{J}$ ist, dann ist die lineare Differentialgleichung

$$y''' + p(x)y'' + 2A(x)y' + [A'(x) + b(x)]y = 0$$

im Intervall $\mathcal{J}$ nichtoszillatorisch.

(Diese Folgerung bewies auch L. MORAVSKÝ [5] unter der Voraussetzung, dass $p(x)$ in keinem Teilintervall identisch gleich Null ist.)

<u>Bemerkung 3.</u> Wenn in der Folgerungen 1-4 die Voraussetzungen über die betreffenden Koefizienten der gegebenen Differentialgleichungen im Intervall $(-\infty, \infty)$ erfüllt sind, dann

sind die gegebenen Differentialgleichungen im Intervall $(-\infty,\infty)$ nichtoszillatorisch. (Dies folgt aus der Tatsache, dass die Differentialgleichungen im Intervall $\langle x_0,\infty)$ für die beliebige Zahl $x_0 \in (-\infty,\infty)$ nichtoszillatorisch sind.)

Wir haben die lineare Differentialgleichung

(18) $$y''' + k\,x^{s} y = 0\,, \qquad x \geq x_0 > 0,$$

wo k, s gegebene Konstanten sind.

M. RÁB [6] zeigte, dass diese Differentialgleichung unter der Voraussetzung $s < -3$, $|k| \leq \frac{|s+1||s+2||s+3|}{4x_0^{s+3}}$ im Intervall $\langle x_0,\infty)$ nichtoszillatorisch ist.

Wir zeigen, dass diese Differentialgleichung im Intervall $\langle x_0,\infty)$ nichtoszillatorisch ist unter dieser schwächeren Voraussetzung

$$s<-3; \quad |k| \leq \frac{|s+1||s+2||s+3|}{x_0^{s+3}}\,.$$

Die zu der Differentialgleichung (18) adjungierte Differentialgleichung ist

(19) $$z''' - k\,x^{s} z = 0.$$

Es ist bekannt [1], dass wenn die Differentialgleichung (18) im Intervall $\langle x_0,\infty)$ nichtoszillatorisch ist, ist auch die Differentialgleichung (19) im Intervall $\langle x_0,\infty)$ nichtoszillatorisch und umgekehrt. Deshalb ist es möglich, ohne Verlust an der Allgemeinheit vorauszusetzen, dass $k \geq 0$ (weil wenn $k<0$ wäre, würden wir die Aufgaben der Differentialgleichung (18), (19) auswechseln).

Es sei $k \geq 0$. Dann ist die Differentialgleichung (18) aus der Klasse $A_0^-(\langle x_0,\infty))$ und die Differentialgleichung (19) aus der Klasse $B_0^+(\langle x_0,\infty))$. Es sei $y_1(x)$ die Lösung der Differentialgleichung (18) mit der Eigenschaft $y_1(x_0) = y_1'(x_0) = 0$, $y_1''(x_0) = 1$ und es sei $s<-3$, $k \leq \frac{|s+1||s+2||s+3|}{x_0^{s+3}}$.

Mit Rücksicht darauf, dass

$$\int_{x_0}^{\infty} t^{s}(t-x_0)^2\,dt = \frac{2x_0^{s+3}}{|s+1||s+2||s+3|}$$

ist, erhalten wir im Intervall $\langle x_0,\infty)$

$$\varphi_0(x,0,0,1)=\frac{(x-x_0)^2}{2}-k\int_{x_0}^{x}\frac{(x-t)^2}{2}t^6\frac{(t-x_0)^2}{2}dt \geqq \frac{(x-x_0)^2}{2}\left[1-\frac{k}{2}\int_{x_0}^{\infty}t^6(t-x_0)^2dt\right]\geqq 0$$

d.h. $\varphi_0(x,0,0,1)\geqq 0$ für $x \geqq x_0$.

Auf Grund des Satzes 5 folgt aus diesen Tatsachen, dass die Differentialgleichung (18) für oben angeführtes β und k im Intervall $\langle x_0, \infty)$ nichtoszillatorisch ist.

Als weiteres Beispiel nehmen wir die lineare Differentialgleichung

(20) $$y''' + x^2 y'' + x^7 y' + \frac{x+1}{600} y = 0, \quad x \leqq 0.$$

Diese Differentialgleichung ist aus den Klassen $\mathcal{D}_{32}((-\infty,-1\rangle)$, $\mathcal{D}_{32}((-3{\cdot}68,0\rangle)$ und zwar: Die Differentialgleichung (20) ist aus den Klassen $A_3^+((-\infty,-1\rangle)$, $A_3^+((-3{\cdot}68,0\rangle)$ und die zu ihr gehörige Differentialgleichung $M_1[v]=0$ d.h. die Differentialgleichung

(21) $$\left[(v'' + x^2 v' + x^7 v)e^{\frac{x^3}{3}}\right]' - \frac{x+1}{600} e^{\frac{x^3}{3}} v = 0$$

ist aus den Klassen $\mathcal{C}_2^+((-\infty,-1\rangle)$, $\mathcal{C}_2^+((-3{\cdot}68,0\rangle)$.

Deshalb ist die leneare Differentialgleichung (20), auf Grund des Satzes 5, in den Intervallen $(-\infty,-1\rangle$, $(-3{\cdot}68,0\rangle$ nichtoszillatorisch und also ist auch die Differentialgleichung (21) in denselben Intervallen nichtoszillatorisch (siehe [1]). R. M. MATHSEN [7] zeigte die Bedingung, unter welcher die lineare Differentialgleichung der Form $L[y]=0$ im geschlossenen Intervall $\langle c,d\rangle$ nichtoszillatorisch ist. Diese Bedingung ist für die Differentialgleichung (20) im Intervall $\langle -2,0\rangle$ erfüllt, aber im Intervall $\langle -3,0\rangle$ ist diese Bedingung nicht mehr erfüllt.

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NOTE ON $K(Q_n(C))$

by

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Introduction

Let M_n denote the Grothendieck group of Z_2-graded C_n-module. Define A_n to be the Cokernel of the natural restriction

$$M_{n+1} \longrightarrow M_n.$$

We propose in this paper to prove the following theorem:
The diagram

$$\begin{array}{ccc} K(Q_n(C)) & \xrightarrow{\;j'\;} & K(S^n) \\ \uparrow \lambda & & \downarrow \delta \\ A_{n+2} & \xrightarrow{\;a\;} & K^{n+2} \text{ (point)} \end{array}$$

Commutes.

The paper is divided into two parts.

Part I is entirely algebraic and is the study of Clifford algebras. This contains nothing essentially new. Part II is concerned essentially with the proof of the above theorem.

Part I

§ 1: Clifford algebra

Let E be a C-module and let $T(E) = \sum_{i=0}^{\infty} T^i(E)$ $= C \oplus E \oplus E \otimes E \oplus \ldots$ be the tensor algebra over E.

Definition (1,1): A quadratic form is a pair (E,Q), where $Q: E \times E \longrightarrow C$ is a symmetric bilinear form; i.e., the following relations hold

(i) $Q(ax, a'x', y) = a\,Q(x,y) + a'Q(x', y)$ for $a, a' \in C$ and $x, x', y \in E$,

(ii) $Q(x, by+b'y') = b\,Q(x,y) + b'Q(x,y')$ for $b, b' \in C$ and $x, y, y' \in E$,

(iii) $Q(x,y) = Q(y,x)$ for $x, y \in E$.

<u>Definition (1.2):</u> The <u>Clifford algebra</u> of a quadratic form (E,Q) is a pair $(C(Q), i_Q)$, where $C(Q)$ is a C-algebra and $i_Q : E \longrightarrow C(Q)$ is a linear function such that $(i_Q(x))^2 = Q(x,x).1$ for each $x \in E$. We assume also the following universal property: For any linear function $\emptyset$ of E into a C-algebra A with unit such that $\emptyset(x)^2 = Q(x,x).1$ there exists an algebra morphism $\emptyset' : C(Q) \longrightarrow A$ such that $\emptyset' . i_Q = \emptyset$, and $\emptyset'$ is unique.

<u>Proposition (1.1):</u> A Clifford algebra $(C(Q)\ i_Q)$ exists for each quadratic form (E,Q). If $(C(Q), i_Q)$ and $(C(Q'), i_Q')$ are two Clifford algebras, there is an algebra morphism $\emptyset : C(Q) \longrightarrow C(Q')$ such that $i_Q' = \emptyset\, i_Q$. Moreover, $\emptyset$ is an isomorphism and it is unique.

<u>Proof</u>: For the first part, let $C(Q)$ be the tensor algebra $T(E) = \sum_{i=o}^{\infty} T^i(E) = C \oplus E \oplus E \otimes E \oplus \ldots$ modulo the ideal generated by $x \otimes x - Q(x,x).1$. Let i_Q be the composition of the injection $E \longrightarrow T^1(E) \longrightarrow T(E)$ and the projection $T(E) \longrightarrow C(Q)$. If $\emptyset : E \longrightarrow A$ is a linear function with $\emptyset(x)^2 = Q(x,x).1$, then $\emptyset$ factors as $E \longrightarrow T(E) \xrightarrow{\emptyset''} A$ and as $E \xrightarrow{i_Q} C(Q) \xrightarrow{\emptyset'} A$, Moreover, $\emptyset'$ is unique because I, (i_Q) generates $C(Q)$.

Finally, the uniqueness of $(C(Q), i_Q)$ follows from the existence of two algebra morphisms $\emptyset_1 : C(Q) \longrightarrow C(Q')$, and $\emptyset_2 : C(Q') \longrightarrow C(Q)$ with $i_Q' = \emptyset_1\, i_Q$ and $i_Q = \emptyset_2\, i_Q'$. This implies that $i_Q = \emptyset_2\, \emptyset_1\, i_Q$ and $i_Q' = \emptyset_1\, \emptyset_2\, i_Q'$. From the uniqueness property of factorisations, $1 = \emptyset_2\, \emptyset_1$ and $1 = \emptyset_1\, \emptyset_2$. This completes the proof of the proposition.

<u>§ 2: The Z_2 - grading of $C(Q)$:</u> The algebra morphism $\alpha : E \longrightarrow C(Q)$, given by $\alpha(x) = -\,i_Q(x)$, determines an involution $\beta : C(Q) \longrightarrow C(Q)$ with $\beta\alpha(x) = -\alpha(x)$. Usually, we write $\beta(x) = \bar{x}$. Then we denote the subalgebra of $x \in C(Q)$

with $\bar{x} = -x$ by $C(Q)'$. Then $C(Q)^o$ is the image of $\sum_{o \leq i} T^{2i}(E)$ in $C(Q)$, and $C(Q)'$ is the image of $\sum_{o \leq i} T^{2i+1}(E)$ in $C(Q)$. Finally we have $C(Q) = C(Q)^o \oplus C(Q)'$, which is a Z_2-grading.

§ 3: The n-dimensional complex quadric $Q_n(C)$:

An n+1-dimensional complex projective space, $P_{n+1}(C)$, is defined as the space of complex lines through the origin in C^{n+2}.

An n-dimensional complex quadric $Q_n(C)$ in $P_{n+1}(C)$ is given by the equation:

$$\sum_{j=o}^{n+1} t_j^2 = o,$$

where $(t_o, t_1, \ldots, t_{n+1})$ are complex homogeneous corrdinates on $P_{n+1}(C)$. The complex structure on $Q_n(C)$ is induced from the complex structure on $P_{n+1}(C)$. The intersection of $Q_n(C)$ with the plane $t_{n+1} = 1$ is an n-dimensional sphere S^n.

Part II

The Proof of the Theorem.

Theorem: The following diagram

$$\begin{array}{ccc} K(Q_n(C)) & \xrightarrow{\quad j! \quad} & K(S^n) \\ \uparrow \lambda & & \downarrow \delta \\ A_{n+2} & \xrightarrow{\quad \alpha \quad} & K^{n+2}(\text{point}) \end{array}$$

Commutes.

Proof: Let $M = M^o + M'$ be a graded C_{n+2}-module then by the standard contruction of [2] we have an alement $\alpha(M) \in K(B^{n+2}, S^{n+1}) \simeq \tilde{K}(S^{n+2})$. By periodicity theorem [2] this corresponds to an element of $\tilde{K}(S^n)$ and one can check easily thet this element may be obtained by taking the vector bundle M^+ where M_x^+ is the positive-eigenspace of the transformation τ_x of M^o given by

$\tau_x(m) = x\, e_{n+2}(m)$. Here e_{n+2} is a vector of R^{n+2} and we take x a unit vector orthogonal to e_{n+2} (so that $x \in S^n \subset R^{n+1} \subset R^{n+2}$).

Our problem now is to show that we can define a vector bundle over $Q_n(C)$ which restricts to M^+ over S^n. To do this and to avoid algebraic geometry we proceed as follows:

Let H denote the Hopf-bundle of $P_{n+1}(C)$ restricted to $Q_n(C)$. Then we may consider the bundle homomorphism $H^{-1} \otimes M' \xrightarrow{A} M^o$. over $Q_n(C)$. Here M^o, M' denote trivial bundles ($Q_n x M^o$, $Q_n x M'$) and A is Clifford multiplication. I claim that.

(1) A is of constant rank (i.e. independent of $x \in Q_n(C)$). This implies that Ker A and Coker A are vector bundles over $Q_n(C)$ (Atiyah Harvard notes [I])

(2) Coker $A/S^n \cong M^+$.

If we can establish these two points, then we are finished.

In fact, to prove both (1) and (2) it will be sufficient to show that for $x \in S^n$ we have a natural isomorphism Coker $A_x \cong M_x^+$

(3) this will show that dim-Coker A_x = dim $M_x^+ = 1/2$ dim M^o is independent of x for $x \in S^n$. But S^n is sphere, determined once we have picked a vector e_{n+2} of R^{n+2}, so rank A_y is independent of y for all $y \in Q_n$. This prove (1) and then (3) gives (2). But (3) is quite easy. In fact, for $x \in S^n$, we make the standard identification $H \cong 1$ (trivial line bundle), $M^1 \cong M^o$. If we use the decomposition $R^{n+2} \cong R^1 \oplus R^{n+1}$, then A gets replaced by $B: M^o \longrightarrow M^o$.

Moreover we can recall that the map

$$S^n \subset R^{n+2} \longrightarrow Q_n(C) \subset P_{n+1}(C)$$

is give by $\qquad x \longrightarrow e_{n+2} + ix$

I claim that

$$B_x = 1 + i\,\tau_x$$

(recall that it depends linearly on the coordinates and we have arranged to that the value at e_{n+2} is the identity)

Now decomposing M^0 or $M_x^+ \oplus M_x^-$ so that τ_x is $i \oplus -i$ we find that $B_x = 0 \oplus 2$. Hence Coker $B_x \cong M_x^+$ and of course Coker $A_x \cong$ Coker B_x. This completes the proof.

<u>Note:</u> I have avoided algebraic geometry by considering only the vector bundle homomorphism over $Q_n(C)$ given by the matrix A. It is also true that if we work with sheaves over $P_{n+1}(C)$ then we get a sheaf which represents the vector bundle M^+ on $Q_n(C)$. However the proof would be as above, once we know the sheaf was locally free on $Q_n(C)$.

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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS IN NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE HYPERBOLIC TYPE

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The present paper deals with the existence and with uniqueness of solution of a boundary value problems for the equations

$$\frac{\partial^{k_1+\dots+k_m} u}{\partial x_1^{k_1}\dots\partial x_m^{k_m}} = f\left(x_1,\dots,x_m,u,\dots,\frac{\partial^{\gamma_1+\dots+\gamma_m} u}{\partial x_1^{\gamma_1}\dots\partial x_m^{\gamma_m}},\dots\right) \qquad (i)$$

where $\gamma_1+\dots+\gamma_m \leq k_1+\dots+k_m-1$ under the generalized conditions of Krasnosielski-Krein's type from [3] . The criteria of the uniqueness given below for the equations of the type (i) are strictly connected with the results obtained by F. BRAUER [1] , B. PALCZEWSKI and W. PAWELSKI [4] and W. WALTER [5] .

I

In this section we introduce the notations and conceptions used throughout the present article.

a/ We shall denote an arbitrary set of the points $X(x_1,\dots,x_m)$ of a m-dimensional Euclidean space ($m \geq 2$) with the coordinates $0<x_i\leq A_i$ or $0\leq x_i\leq A_i$, $A_i>0$ for $i=1,\dots,m$ by R^0, R respectively. The set of points $X_k(x_1,\dots,x_{k-1},0,x_{k+1},\dots,x_m)$ or $X_{lj}(x_1,\dots,x_{l-1},0,x_{l+1},\dots,x_{j-1},0,x_{j+1},\dots,x_m)$, where $0\leq x_i\leq A_i$ for $i\neq k$ or $i\neq l,j$ will be denoted by R_k, R_{lj} respectively ($1\leq k,l,j\leq m$; $l<j$).

b/ Next, let E^0_{pq} or E_{pq} be the product $R^0\times S_{pq}$, $R\times S_{pq}$ respectively. S_{pq} denotes the Cartesian product of the intervals $(-\infty,+\infty)$ given as follows: $S_{pq} = \prod_{\Delta_{pq}(\gamma)}\{-\infty< v_{\gamma_1\dots\gamma_m}<+\infty\}$, where $\Delta_{pq}(\gamma)=\{(\gamma_1\dots\gamma_m): 0\leq\sum_{j=1}^{m}\gamma_j\leq\sum_{j=1}^{m}k_j-p,\ 0\leq\gamma_j\leq k_j-q,\ \gamma_j\geq 0$ and $k_j\geq 1$ are integers, $j=1,\dots,m\}$ for $p=1$ and $q=0$ or $p=m$ and $q=1$.

c/ The symbol $\Sigma(\nu,\xi)$ denotes a simplex with the $\nu+1$ linearly independent vertices $\Xi_0(0,\ldots,0), \Xi_1(\xi,0,\ldots,0),\ldots, \Xi_{\nu-1}(\xi,\ldots,\xi,0), \Xi_\nu(\xi,\ldots,\xi), \xi>0$ in the ν -dimensional Euclidean space E^ν. Consequently $\Sigma(\nu,\xi)$ is a set of points $P\in E^\nu$ such that $P=\alpha_0\Xi_0+\cdots+\alpha_\nu\Xi_\nu$, where $\alpha_0+\cdots+\alpha_\nu=1, \alpha_i\geq 0$ for $i=0,1,\ldots,\nu$. If $\xi=0$, then $\Sigma(\nu,0)=\Xi_0$.

d/ Let us denote the differential operator $\frac{\partial^{\beta_i}}{\partial x_i^{\beta_i}}$ through $D_{x_i}^{\beta_i}$ for $\beta_i=1,2,\ldots$ and $D_{x_i}^0 v(X)=v(X)$. In our consideration the set of the continuous fonction $v(X)$ with the continuous derivatives $D_{x_1}^{t_1}\ldots D_{x_m}^{t_m}v(X)$ in R for all $t_j, j=1,\ldots,m$ satisfying the relation $(t_1\ldots t_m)\in\Delta_{pq}(\gamma)$ will be denoted by $M_{pq}(R)$ and by $N(R_\lambda)$ the set of the continuous functions $w(X_\lambda)$ with the continuous derivatives $D_{x_1}^{l_1}\ldots D_{x_{\lambda-1}}^{l_{\lambda-1}}D_{x_{\lambda+1}}^{l_{\lambda+1}}\ldots D_{x_m}^{l_m}w(X_\lambda)$ on R for $0\leq l_j\leq k_j, j,\lambda=1,\ldots,m; j\neq\lambda$ will be denoted.

Our aim is to inquire into the existence and uniqueness of solutions of the two following boundary value problems

$$D_{x_1}^{k_1}\ldots D_{x_m}^{k_m}u=f(X,\ldots,D_{x_1}^{\gamma_1}\ldots D_{x_m}^{\gamma_m}u,\ldots), X\in R^0 \tag{1pq}$$

for $\gamma_j, j=1,\ldots,m$ such that $(\gamma_1\ldots\gamma_m)\in\Delta_{pq}(\gamma)$

$$[D_{x_\lambda}^{i_\lambda}u(X)]_{x_\lambda=0}=\sigma_\lambda^{(i_\lambda)}(X_\lambda), X_\lambda\in R_\lambda, i_\lambda=0,1,\ldots,k_\lambda-1;\lambda=1,\ldots,m \tag{2}$$

$$[D_{x_s}^{j_s}\sigma_\lambda^{(i_\lambda)}(X_\lambda)]_{x_s=0}=[D_{x_\lambda}^{i_\lambda}\sigma_s^{(j_s)}(X_s)]_{x_\lambda=0}, X_{\lambda s}\in R_{\lambda s}$$

$$\lambda\neq s, i_\lambda=0,1,\ldots,k_\lambda-1; j_s=0,1,\ldots,k_s-1;\lambda,s=1,\ldots,m$$

where $\sigma_\lambda^{(i_\lambda)}(X_\lambda)\in N(R_\lambda)$ and the function $f(X,\ldots,u_{\gamma_1\ldots\gamma_m},\ldots)$ is continuous in the domain E_{pq}.

Under the solution of the problem $(1pq),(2)$ we understand any function $u(X)\in M_{10}(R)$ satisfying the conditions $(1pq),(2)$.

Hence we immediately obtain that the problem (1_{pq}), (2) and the integro-differential equation

$$u(X) = G(X) + \int_{\Sigma(k_1,x_1)} d\mu_1 \ldots \int_{\Sigma(k_m,x_m)} f(\Xi, \ldots, D_{x_1}^{\gamma_1} \ldots D_{x_m}^{\gamma_m} u(\Xi), \ldots) d\mu_m \tag{3}$$

are mutually equivalent in R, where $\Xi = (\xi_1, \ldots, \xi_m)$ and μ_i for $i = 1, \ldots, m$ denotes the Lebesque measure defined in the Euclidean space E^{k_i}. The function $G(X)$ can be expressed as follows:

$$G(X) = \sum_{j=1}^{m} \sum_{i_1,\ldots,i_j} \sum_{l_1,\ldots,l_j} (-1)^{j-1} \frac{x_{i_1}^{l_1} \cdots x_{i_j}^{l_j}}{l_1! \ldots l_j!} \left[D_{x_{i_1}}^{l_1} \ldots D_{x_{i_j}}^{l_j} u(X) \right]_{\substack{x_{i_1}=0 \\ \vdots \\ x_{i_j}=0}}$$

where $0 \le l_s \le k_{i_s} - 1$, $s = 1, \ldots, j$ and $(i_1, \ldots, i_j)$ is an arbitrary combination of j numbers from the m natural numbers $(1, \ldots, m)$, $i_1 < \ldots < i_j$.

From the equality (3) we have

$$D_{x_1}^{\sigma_1} \ldots D_{x_m}^{\sigma_m} u(X) = D_{x_1}^{\sigma_1} \ldots D_{x_m}^{\sigma_m} G(X) + \int_{\Sigma(k_1-\sigma_1,x_1)} d\mu_1 \ldots \int_{\Sigma(k_m-\sigma_m,x_m)} f(\Xi, \ldots, D_{x_1}^{\gamma_1} \ldots D_{x_m}^{\gamma_m} u(\Xi), \ldots) d\mu_m \tag{3_1}$$

for $X \in R$ and σ_j, $j = 1, \ldots, m$ such that $(\sigma_1 \ldots \sigma_m) \in \Delta_{pq}(\sigma)$,

where

$$\int_{\Sigma(0,x_i)} F(\Xi) d\mu_i = F(\xi_1, \ldots, \xi_{i-1}, x_i, \xi_{i+1}, \ldots, \xi_m)$$

In the conclusion we still formulate the well known Shauder's fixed point theorem (see L. Collatz [2], pp. 353-355):

Theorem 1. Let B be a Banach space, M a closed and convex subset of B and T a continuous operator on M. If moreover, T mapps M into itself ($TM \subseteq M$) and the set of images TM is compact in B then the operator T has at least one fixed point in M.

Theorem 2. If the function $f(X, \ldots, u_{j}, \ldots, j_m, \ldots)$ is defined, continuous and bounded in the domain E_{m_1} then there exists at least one solution of the problem (1_{m1}), (2) in $M_{m_1}(R)$.

Proof. To prove it we shall use the Theorem 1. Then, we have to show that the set $M_{m_1}(R)$ with the norm

$$\| u \| = \max_{X \in R} \sum_{\Delta_{m_1}(\sigma)} | D_{x_1}^{\sigma_1} \ldots D_{x_m}^{\sigma_m} u(X) | \tag{4}$$

forms Banach space. Let the sequence $\{u_n(X)\}_{n=1}^{\infty}$ of the functions $u_n(X) \in M_{m_1}(R)$ be a Cauchy sequence, i.e. $\lim_{n,l\to\infty} \|u_n - u_l\| = 0$. For each $\varepsilon > 0$ there exists a number $N(\varepsilon) > 0$ such that

$$|D_{x_1}^{\alpha_1} \cdots D_{x_m}^{\alpha_m}[u_n(X) - u_l(X)]| < \varepsilon$$

for the all indices $n, l > N(\varepsilon)$ and $(\alpha_1 \dots \alpha_m) \in \Delta_{m_1}(\delta)$ or R.

Hence we conclude that the sequence $\{D_{x_1}^{\alpha_1} \cdots D_{x_m}^{\alpha_m} u_n(X)\}_{n=1}^{\infty}$ converges uniformly in the domain R for each $(\alpha_1 \dots \alpha_m) \in \Delta_{m_1}(\delta)$. If we denote the limit of the sequence $\{u_n(X)\}_{n=1}^{\infty}$ by $u(X)$ then it is obvious that $u(X) \in M_{m_1}(R)$ and $\lim_{n\to\infty} \|u_n - u\| = 0$. The space $M_{m_1}(R)$ is complete and so it is the Banach space. **Because the function f is bounded in E_{m_1}, there exists a closed sphere in the space $M_{m_1}(R)$ such that the operator**

$$Tu = G(X) + \int_{\Sigma(k_1, x_1)} d\mu_1 \cdots \int_{\Sigma(k_m, x_m)} f(\Xi, \dots D_{x_1}^{\alpha_1} \cdots D_{x_m}^{\alpha_m} u(\Xi), \dots) d\mu_m \quad (5)$$

is continuous on M and $TM \subseteq M$.

Now it is sufficient to prave that the set of the images $TM_{m_1}(R)$ is compact. For this purpose let us choose an arbitrary sequence $\{Tu_n(X)\}_{n=1}^{\infty}$ of the functions $Tu_n(X) \in TM_{m_1}(R)$ and put $L = \sup_{E_{m_1}} |f(X, \dots, u_{j_1 \dots j_m}, \dots)|$. From (5) for $X, Y = (y_1, \dots, y_m) \in R$ we obtain the inequality

$$|D_{x_1}^{\alpha_1} \cdots D_{x_m}^{\alpha_m} Tu_n(X) - D_{x_1}^{\alpha_1} \cdots D_{x_m}^{\alpha_m} Tu_n(Y)| \le |D_{x_1}^{\alpha_1} \cdots D_{x_m}^{\alpha_m}[G(X) - G(Y)]| +$$

$$+ \Big|\sum_{i=1}^{m} \int_{\Sigma(k_1-\alpha_1, y_1)} d\mu_1 \cdots \int_{\Sigma(k_{i-1}-\alpha_{i-1}, y_{i-1})} d\mu_{i-1} \Big[\int_{y_i}^{x_i} d\varphi_1 \int_0^{\varphi_1} d\varphi_2 \cdots \int_0^{\varphi_{k_i-\alpha_i-1}} d\xi_i\Big] \times$$

$$\times \int_{\Sigma(k_{i+1}-\alpha_{i+1}, x_{i+1})} d\mu_{i+1} \cdots \int_{\Sigma(k_m-\alpha_m, x_m)} f(\Xi, \dots, D_{x_1}^{\alpha_1} \cdots D_{x_m}^{\alpha_m} u_n, \dots) d\mu_m \Big| \le$$

$$\le |D_{x_1}^{\alpha_1} \cdots D_{x_m}^{\alpha_m}[G(X) - G(Y)]| + L\Big\{\sum_{i=1}^{m} |x_i - y_i| \frac{A_i^{k_i-\alpha_i-1}}{(k_i-\alpha_i-1)!} \prod_{\substack{j=1 \\ j\ne i}}^{m} \frac{A_j^{k_j-\alpha_j}}{(k_j-\alpha_j)!}\Big\}$$

for all $(\delta_1 \dots \delta_m) \in \Delta_{m_1}(\delta)$ and $n = 1, 2, \dots$ The previous estimate guarantees the equicontinuity of the sequence of derivatives $\{ D_{x_1}^{\delta_1} \dots D_{x_m}^{\delta_m} Tu_n(X) \}_{n=1}^{\infty}$ in the domain R. In view of the inequality

$$| D_{x_1}^{\delta_1} \dots D_{x_m}^{\delta_m} Tu_n(X) | \leq \max_{X \in R} | D_{x_1}^{\delta_1} \dots D_{x_m}^{\delta_m} G(X) | + L \cdot \prod_{j=1}^{m} \frac{A_j^{k_j - \delta_j}}{(k_j - \delta_j)!}$$

it is obvious that the sequence $\{ D_{x_1}^{\delta_1} \dots D_{x_m}^{\delta_m} Tu_n(X) \}_{n=1}^{\infty}$ also is uniformly bounded on R .

Let $\{ Tu_{n_k}(X) \}_{k=1}^{\infty}$ be a subsequence of the sequence $\{ Tu_n(X) \}_{n=1}^{\infty}$ which uniformly converges in R to a function $\Psi(X)$, i.e. $\lim_{k \to \infty} Tu_{n_k}(X) = \Psi(X)$, for $X \in R$. There exists a choosen sequence $\{ Tu_{n_k}^{(10 \dots 0)}(X) \}_{k=1}^{\infty}$ (the expression ($10 \dots 0$) contains the m components) of the sequence $\{ Tu_{n_k}(X) \}_{k=1}^{\infty}$ such that the sequence of derivatives $\{ D_{x_1} Tu_{n_k}^{(10 \dots 0)}(X) \}_{k=1}^{\infty}$ uniformly converges on R and

$$\lim_{k \to \infty} Tu_{n_k}^{(10 \dots 0)}(X) = \Psi(X), \lim_{k \to \infty} D_{x_1} Tu_{n_k}^{(10 \dots 0)}(X) = D_{x_1} \Psi(X)$$

Now let us assume that the subsequence $\{ Tu_{n_k}^{(\gamma_1 \dots \gamma_i \dots \gamma_m)}(X) \}_{k=1}^{\infty}$ of the sequence $\{ Tu_{n_k}(X) \}_{k=1}^{\infty}$ along with the sequence of the derivatives $\{ D_{x_1}^{\delta_1} \dots D_{x_i}^{\delta_i} \dots D_{x_m}^{\delta_m} Tu_{n_k}^{(\gamma_1 \dots \gamma_i \dots \gamma_m)}(X) \}_{k=1}^{\infty}$ for all $0 \leq \delta_j \leq \gamma_j$, $j = 1, \dots, m$ uniformly converges on R and that

$$\lim_{k \to \infty} D_{x_1}^{\delta_1} \dots D_{x_i}^{\delta_i} \dots D_{x_m}^{\delta_m} Tu_{n_k}^{(\gamma_1 \dots \gamma_i \dots \gamma_m)}(X) = D_{x_1}^{\delta_1} \dots D_{x_i}^{\delta_i} \dots D_{x_m}^{\delta_m} \Psi(X) \quad (6)$$

γ_j for $j = 1, \dots, m$ are non-negative integers. From the sequence $\{ D_{x_1}^{\delta_1} \dots D_{x_i}^{\delta_i + 1} \dots D_{x_m}^{\delta_m} Tu_{n_k}^{(\gamma_1 \dots \gamma_i \dots \gamma_m)}(X) \}_{k=1}^{\infty}$

we can chooce an uniformly convergent sequence on R . Denote this sequence by

$\{D_{x_1}^{\alpha_1}\ldots D_{x_i}^{\alpha_{i+1}}\ldots D_{x_m}^{\alpha_m} T u_{n_k}^{(\nu_1\ldots\nu_{i+1}\ldots\nu_m)}(X)\}_{k=1}^{\infty}$. Then with respect to the relation $\{T u_{n_k}^{(\nu_1\ldots\nu_{i+1}\ldots\nu_m)}(X)\}_{k=1}^{\infty} \subseteq$

$\subseteq \{T u_{n_k}^{(\nu_1\ldots\nu_i\ldots\nu_m)}(X)\}_{k=1}^{\infty}$ and to (6) we may claim that

$$\lim_{k\to\infty} D_{x_1}^{\alpha_1}\ldots D_{x_i}^{\alpha_{i+1}}\ldots D_{x_m}^{\alpha_m}\{T u_{n_k}^{(\nu_1\ldots\nu_{i+1}\ldots\nu_m)}(X)\}_{k=1}^{\infty} = D_{x_1}^{\alpha_1}\ldots D_{x_i}^{\alpha_{i+1}}\ldots D_{x_m}^{\alpha_m}\psi(X)$$

on R for $0\le \alpha_j \le \nu_j$, $j=1,\ldots,m$.

Continuing in this procedure we see that there exists an uniformly convergent sequence $\{T\tilde{u}_{n_k}(X)\}_{k=1}^{\infty} \subseteq$ $\subseteq \{T u_{n_k}(X)\}_{k=1}^{\infty}$ such that $\lim\limits_{k\to\infty} D_{x_1}^{\alpha_1}\ldots D_{x_m}^{\alpha_m} T\tilde{u}_{n_k}(X) = D_{x_1}^{\alpha_1}\ldots D_{x_m}^{\alpha_m}\psi(X)$ in the domain R for all $(\alpha_1\ldots\alpha_m)\in \Delta_{m_1}(\alpha)$. Hence we easily get that

$$\lim_{k\to\infty} \| T\tilde{u}_{n_k} - \psi \| = 0,\ \psi(X)\in M_{m_1}(R)$$

which is the desired equality. Then the set $TM_{m_1}(R)$ is compact in $M_{m_1}(R)$.

Both the existence of solution of the equation $u = Tu$ and the existence of solution of the problem (1_{m1}), (2) is proved.

In the following theorem we shall prove the uniqueness of the solution of the boundary value problem (1_{10}), (2).

T h e o r e m 3. Let $f(X,\ldots,u_{\gamma_1}\ldots\gamma_m,\ldots)$ be a function defined, continuous and bounded on E_{10} and satisfy the following conditions:

For any $(X,\ldots,u_{\gamma_1}\ldots\gamma_m,\ldots), (X,\ldots,v_{\gamma_1}\ldots\gamma_m,\ldots)\in E_{10}^{0}$

$$|f(X,\ldots,u_{\gamma_1\ldots\gamma_m},\ldots)-f(X,\ldots,v_{\gamma_1\ldots\gamma_m},\ldots)| \le \frac{K}{x_1^{k_1}\ldots x_m^{k_m}}\sum_{\Delta_{10}^{(\nu)}}\{p_{\gamma_1\ldots\gamma_m}\prod_{i=1}^{m}x_i^{\nu_i}[K^{h(k_i-\gamma_i)}]^{-1/m}\times \tag{7}$$

$$\times |u_{\gamma_1\ldots\gamma_m} - v_{\gamma_1\ldots\gamma_m}|\}$$

where $K>0$ and $p_{\gamma_1\ldots\gamma_m}$ are non-negative constants of which at least one is non-vanishing. The function $h(X)$ of the one real variable X is defined as follows:

$$h(X)=\begin{cases}1 & \text{for } X=0\\ 0 & \text{for } X\neq 0\end{cases}$$

Moreover in E_{10}

$$|f(X,\ldots,u_{\gamma_1\ldots\gamma_m},\ldots)-f(X,\ldots,v_{\gamma_1\ldots\gamma_m},\ldots)|\leq\sum_{\Delta_{10}(\gamma)}q_{\gamma_1\ldots\gamma_m}(x_1^{\gamma_1}\ldots x_m^{\gamma_m})^{\alpha}|u_{\gamma_1\ldots\gamma_m}-v_{\gamma_1\ldots\gamma_m}|^{\alpha} \quad (8)$$

for $0<\alpha<1$ and $q_{\gamma_1\ldots\gamma_m}$ are again a non-negative constants of which at least one is non-vanishing. Among the constants K, k_i, α the relation

$$H_i\sqrt[m]{K}(1-\alpha)<k_i(k_i-1)(1-\alpha),\ k_i\geq 1,\ i=1,\ldots,m \quad (9)$$

holds and thereto $H_i>0,\ 1\geq\sum_{\Delta_{10}(\gamma)}p_{\gamma_1\ldots\gamma_m}\prod_{i=1}^{m}A^{-1}(k_i,\gamma_i)$, where

$$A(k_i,\gamma_i)=\begin{cases}H_i\left[\prod_{j=1}^{k_i-\gamma_i-1}(H_i\sqrt[m]{K}+j)\right] & \text{for } \gamma_i\leq k_i-2\\ H_i & \text{for } \gamma_i=k_i-1\\ 1 & \text{for } \gamma_i=k_i\end{cases}$$

Then the problem $(1_{10}),(2)$ has at most one solution $u(X)$.

P r o o f . Let us suppose that the problem (1_{10}), (2) has two different solutions $u_1(X)\neq u_2(X)$ in R and put $M=\sup_{E_{10}}|f(X,\ldots u_{\gamma_1}\ldots\gamma_m,\ldots)|$. Then from the equality (3_1) we obtain

$$x_1^{\delta_1}\ldots x_m^{\delta_m}|D_{x_1}^{\delta_1}\ldots D_{x_m}^{\delta_m}[u_1(X)-u_2(X)]|\leq 2Mx_1^{k_1}\ldots x_m^{k_m}$$

on R for δ_j, $j=1,\ldots,m$ such that $(\delta_1\ldots\delta_m)\in\Delta_{10}(\delta)$. Basing ourselves on the assumption (8) we have the following estimate

$$x_1^{\delta_1}\ldots x_m^{\delta_m}|D_{x_1}^{\delta_1}\ldots D_{x_m}^{\delta_m}[u_1(X)-u_2(X)]|\leq$$

$$\leq x_1^{\delta_1}\ldots x_m^{\delta_m}\int_{\Sigma(k_1-\delta_1,x_1)}d\xi_1\ldots\int_{\Sigma(k_m-\delta_m,x_m)}|f(\Xi,\ldots,D_{x_1}^{\gamma_1}\ldots D_{x_m}^{\gamma_m}u_1(\Xi),\ldots)-f(\Xi,\ldots,D_{x_1}^{\gamma_1}\ldots D_{x_m}^{\gamma_m}u_2(\Xi),\ldots)|\,d\mu_m\leq$$

$$= x_1^{\alpha_1}\dots x_m^{\alpha_m}\int_{\Sigma(k_1-\alpha_1,x_1)} d\mu_1 \dots \int_{\Sigma(k_m-\alpha_m,x_m)} \sum_{\Delta_{10}(\gamma)} q_{\gamma_1\dots\gamma_m} (\xi_1^{\gamma_1}\dots\xi_m^{\gamma_m})^{\alpha} \left| D_{x_1}^{\gamma_1}\dots D_{x_m}^{\gamma_m}[u_1(\Xi)-u_2(\Xi)]\right|^{\alpha} d\mu_m \le$$

$$\le (2M)^{\alpha}\Big(\sum_{\Delta_{10}(\gamma)} q_{\gamma_1\dots\gamma_m}\Big).(x_1^{k_1}\dots x_m^{k_m})^{\alpha+1}$$

Using the same procedures as in the previous estimate we get generally

$$x_1^{\alpha_1}\dots x_m^{\alpha_m}\left|D_{x_1}^{\alpha_1}\dots D_{x_m}^{\alpha_m}[u_1(X)-u_2(X)]\right| \le \Big(\sum_{\Delta_{10}(\gamma)} q_{\gamma_1\dots\gamma_m}\Big)^{1+\alpha+\dots+\alpha^{n}}(2M)^{\alpha^{n+1}}(x_1^{k_1}\dots x_m^{k_m})^{1+\alpha+\dots+\alpha^{n+1}}$$

for $n=1,2,\dots$ and $X\in R$. Hence we may conclude

$$x_1^{\alpha_1}\dots x_m^{\alpha_m}\left|D_{x_1}^{\alpha_1}\dots D_{x_m}^{\alpha_m}[u_1(X)-u_2(X)]\right| \le \Big(\sum_{\Delta_{10}(\gamma)} q_{\gamma_1\dots\gamma_m}\Big)^{1/1-\alpha}(x_1^{k_1}\dots x_m^{k_m})^{1/1-\alpha} \qquad (10)$$

In this case we define the function $Q(X)$ on R by:

$$Q(X)=\begin{cases}\Big[\prod_{i=1}^{m} x_i^{-H_i\sqrt[m]{K}-k_i+1}\Big]\sum_{\Delta_{10}(\gamma)} p_{\gamma_1\dots\gamma_m}\Big\{\prod_{i=1}^{m} x_i^{\gamma_i}\big[K^{h(k_i-\gamma_i)}\big]^{-1/m}\Big\}\left|D_{x_1}^{\gamma_1}\dots D_{x_m}^{\gamma_m}[u_1(X)-u_2(X)]\right| & \text{for } X\in R^\circ\\ 0 & \text{for } X\in R-R^\circ\end{cases}$$

The inequality (10) enables us to find out that

$$0\le Q(X)\le \Big(\sum_{\Delta_{10}(\delta)} q_{\delta_1\dots\delta_m}\Big)^{1/1-\alpha}\Big\{\sum_{\Delta_{10}(\gamma)} p_{\gamma_1\dots\gamma_m}\Big[\prod_{i=1}^{m} K^{h(k_i-\gamma_i)}\Big]^{-1/m}\Big\}\times$$

$$\times\prod_{i=1}^{m} x_i^{[k_i-(k_i-1)(1-\alpha)-H_i\sqrt[m]{K}(1-\alpha)]/1-\alpha}$$

whence we have that $\lim_{R^\circ\ni X\to X_0\in R-R^\circ} Q(X)=0$. Hence it follows that the function $Q(X)$ is continuous on R. Besides we claim that $Q(X)\equiv 0$ on R. Let us assume the contrary, i.e. let there exists a point $Y\in R^\circ$ such that $0<Q(Y)=\max_R Q(X)$. Then by (7) we have

$$\left|D_{x_1}^{\alpha_1}\dots D_{x_m}^{\alpha_m}[u_1(Y)-u_2(Y)]\right|\le \qquad (11)$$

$$\leq \int d\mu_1 \ldots \int\limits_{\Sigma(k_1-\delta_1,\gamma_1)\ \Sigma(k_m-\delta_m,\gamma_m)} \frac{K}{\xi_1^{k_1}\ldots\xi_m^{k_m}} \sum_{\Delta_{10}(\gamma)} p_{\gamma_1\ldots\gamma_m} \prod_{i=1}^{m} \xi_i^{\gamma_i}[K^{h(k_i-\gamma_i)}]^{-1/m} \times$$

$$\times |D_{x_1}^{\gamma_1}\ldots D_{x_m}^{\gamma_m}[u_1(\Xi)-u_2(\Xi)]| d\mu_m < Q(Y)\int d\mu_1 \ldots \int\limits_{\Sigma(k_1-\delta_1,\gamma_1)\ \Sigma(k_m-\delta_m,\gamma_m)} K \times$$

$$\times \prod_{i=1}^{m} \xi_i^{H_i\sqrt[m]{K}-1} d\mu_m \leq K\, Q(Y) \prod_{i=1}^{m} \{ y_i^{H_i\sqrt[m]{K}+k_i-\delta_i-1} B^{-1}(k_i,\delta_i)\}$$

for all δ_j, $j=1,\ldots,m$ fulfilling the condition $(\delta_1\ldots\delta_m)\in\Delta_{10}(\delta)$, where

$$B(k_i,\delta_i) = \begin{cases} \prod\limits_{j=0}^{k_i-\delta_i-1} (H_i\sqrt[m]{K}+j) & \text{for } \delta_i \neq k_i-1 \\ 1 & \text{for } \delta_i = k_i \end{cases}$$

Then from (11) we obtain

$$Q(Y) < Q(Y) \sum_{\Delta_{10}(\delta)} [p_{\delta_1\ldots\delta_m} \prod_{i=1}^{m} A^{-1}(k_i,\delta_i)$$

which is desired contradiction.

The following theorem is a consequence of the previous two theorems.

<u>Theorem 4.</u> Let $f(x,\ldots,u_{\gamma_1\ldots\gamma_m},\ldots)$ be a continuous and bounded function in the domain E_{m1} and satisfy the condition (8). If furthermore the conditions (7) and (9) are satisfied, then the boundary value problem (1_{m1}), (2) has one and only one solution.

<u>Remark 1.</u> With respect to Theorem 3 we may observe on an example proving the failure of the uniqueness in the case if none of the conditions from (9) is satisfied. Consider the Darboux problem

$$u_{xy}(x,y) = f_1(x,y,u,u_x),\ (x,y)\in(0,1\rangle\times(0,1\rangle \qquad \text{(I)}$$

$$u(0,y) = u(x,0) = 0\ ,\ x\in\langle 0,1\rangle,\ y\in\langle 0,1\rangle \qquad \text{(II)}$$

where the function $f_1(x,y,\mu,\mu_x)$ is defined as follows:

$$f_1(x,y,\mu,v)=\begin{cases} 0 & \text{for } -\infty<\mu\le 0,\ -\infty<v\le 0 \\ p_2\frac{\sqrt{K}}{y}v & \text{for } -\infty<\mu\le 0,\ 0<v\le\left(\frac{y}{\sqrt{K}}\right)^{\frac{1}{1-\alpha}}x^{\frac{\alpha}{1-\alpha}} \\ p_2\left(\frac{xy}{\sqrt{K}}\right)^{\frac{\alpha}{1-\alpha}} & \text{for } -\infty<\mu\le 0,\ \left(\frac{y}{\sqrt{K}}\right)^{\frac{1}{1-\alpha}}x^{\frac{\alpha}{1-\alpha}}<v<+\infty \\ p_1\frac{K}{xy}\mu & \text{for } 0<\mu\le\left(\frac{xy}{K}\right)^{\frac{1}{1-\alpha}},\ -\infty<v\le 0 \\ \frac{K}{xy}\left(p_1\mu+p_2\frac{x}{\sqrt{K}}v\right) & \text{for } 0<\mu\le\left(\frac{xy}{K}\right)^{\frac{1}{1-\alpha}},\ 0<v\le\left(\frac{y}{\sqrt{K}}\right)^{\frac{1}{1-\alpha}}x^{\frac{\alpha}{1-\alpha}} \\ p_1\frac{k}{xy}\mu+p_2\left(\frac{xy}{\sqrt{K}}\right)^{\frac{\alpha}{1-\alpha}} & \text{for } 0<\mu\le\left(\frac{xy}{K}\right)^{\frac{1}{1-\alpha}},\ \left(\frac{y}{\sqrt{K}}\right)^{\frac{1}{1-\alpha}}x^{\frac{\alpha}{1-\alpha}}<v<+\infty \\ p_1\left(\frac{xy}{k}\right)^{\frac{\alpha}{1-\alpha}} & \text{for } \left(\frac{xy}{K}\right)^{\frac{1}{1-\alpha}}<\mu<+\infty,\ -\infty<v\le 0 \\ p_1\left(\frac{xy}{k}\right)^{\frac{\alpha}{1-\alpha}}+p_2\frac{\sqrt{K}}{y}v & \text{for } \left(\frac{xy}{K}\right)^{\frac{1}{1-\alpha}}<\mu<+\infty,\ 0<v\le\left(\frac{y}{\sqrt{K}}\right)^{\frac{1}{1-\alpha}}\cdot x^{\frac{\alpha}{1-\alpha}} \\ p_1\left(\frac{xy}{K}\right)^{\frac{\alpha}{1-\alpha}}+p_2\left(\frac{xy}{\sqrt{K}}\right)^{\frac{\alpha}{1-\alpha}} & \text{for } \left(\frac{xy}{K}\right)^{\frac{1}{1-\alpha}}<\mu<+\infty\ \left(\frac{y}{\sqrt{K}}\right)^{\frac{1}{1-\alpha}}x^{\frac{\alpha}{1-\alpha}}<v<+\infty \end{cases}$$

$K>0,\ 0<\alpha<1,\ p_1+p_2=1$

From the easy calculations we find out that the function $f_1(x,y,\mu,v)$ is continuous and bounded in the domain $E_{10}=\langle 0,1\rangle\times\langle 0,1\rangle\times(-\infty,+\infty)\times(-\infty,+\infty)$ and satisfies the conditions:

$$|f_1(x,y,\mu_1,v_1)-f_1(x,y,\mu_2,v_2)|\le\frac{K}{xy}\left(p_1|\mu_1-\mu_2|+p_2\frac{x}{\sqrt{K}}|v_1-v_2|\right) \text{ on } E_{10}^{\circ}$$

$$|f_1(x,y,\mu_1,v_1)-f_1(x,y,\mu_2,v_2)|\le p_1|\mu_1-\mu_2|^{\alpha}+p_2x^{\alpha}|v_1-v_2|^{\alpha} \text{ on } E_{10}$$

If moreover $\sqrt{K}(1-\alpha)<1$ then the problem (I), (II) has one solution $\mu(x,y)\equiv 0$.

If the condition (9) does not hold; i.e. $\sqrt{K}(1-\alpha)\ge 1$ then the problem (I),(II) has in addition to the solution $\mu(x,y)\equiv 0$ a further solution $\mu_0(x,y)=K^{-\frac{1}{2}-\frac{1}{1-\alpha}}(xy)^{\sqrt{K}}$. Really $0<\mu_1(x,y)\le\left(\frac{xy}{K}\right)^{\frac{1}{1-\alpha}}$

and $0 < \mu_{1x}(x,y) \le \left(\frac{y}{\sqrt{k}}\right)^{\frac{1}{1-\alpha}} x^{\frac{\alpha}{1-\alpha}}$.

<u>R e m a r k 2.</u> In the case if both the range of abstract function $D_{x_1}^{\gamma_1}\dots D_{x_m}^{\gamma_m} u(X)$ for $(\gamma_1 \dots \gamma_m) \in \Delta_{10}(\gamma)$ and the range of $f(X, \dots u_{\gamma_1 \dots \gamma_m} \dots)$ are from a Banach space then an analogical result to Theorem 3 holds.

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A COMPLEXITY VALUATION OF THE PARTIAL RECURSIVE FUNCTIONS FOLLOWING THE EXPECTATION OF THE LENGTH OF THEIR COMPUTATIONS ON MINSKY MACHINES

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In this paper we introduce as mentioned above the valuation of the complexity of the partial recursive functions. Further we consider the real - time computable functions and some other sets of the partial recursive functions.

§ 1. Introduction

The method of the complexity valuation of the partial recursive functions which is considered in this paper differs from the methods used by HARTMANIS and STEARNS [3] and by YAMADA [2] mainly from these points of view:

a) The valuation is extended from the unary functions to functions of an arbitrary number of variables. This extension cannot be completely compensated by the transition to the unary function with the help of numbering of the set of all (ordered) n-tuples because then the complexity valuation of an n-ary function would be influenced by the numbering functions which have been applied.

b) The valuation is extended from the recursive functions to the partial recursive functions. The motive for this extension is that some functions which are not recursive (e.g. x - y, x : y comprehended as partial functions from N^2 to N) are in the intuitive sense (also in the sense which will be here introduced) more simple than some recursive functions.

c) Instead of the many - tape TURING machines the MINSKY machines will be used (see e.g. MAĽCEV [1]). The many-tape TURING machine usually realises some (maybe partial or n-ary) alphabetical operator resp. is producing a tape to which only then are given the number functions. The complexity valuation is again influenced by the coordination which has been used. On the contrary to MINSKY machines the number funcions are directly coordinated. This is especially clear when we consider the MINSKY machines as programs for a certain abstract computer. As I was told by Professor Bečvář similar machines were used in the paper [5] .

Besides the MINSKY machines have a more feeble outfit than the many - tape TURING machines (only if we consider also the time necessary for the computing; otherwise they are equivalent). Therefore it can be hoped that the valuation of the functions will be finer than it would be by using the many - tape TURING machines.

d) Real - time computable functions are defined otherwise than in [2] . Because of the above - mentioned motives this notion is also extended on the n-ary functions, (is defined through MINSKY machines) and is liberated from the dependence of coding. By this the condition of increase becomes superfluous. These functions conserve the important properties of the functions from [2] and have also further important properties which we shall be considering later.

e) By the definition of the notion of h-computability (which has an analogical task as the notion of T-computability in [3]) is not required that h be a recursive function. In this way we succeed in the characterisation for example of the so called almost recursive functions as h - computable (for convenient h).

f) One of the most substantial differences between the here used approach and the approach used in [3] is the estimate of the expectation of the length of computing (by a given probability) instead of the estimate of the length of computing separately for every argument (with the values of a given function). This circumstance besides facilitates the comparison of the complexity of the function of different number of variables. Besides the process from [3] valued the complexity of the function according to the values of the argument for which the value of the function has been most difficult to be computed and the remaining of the values of the argument has not been practically expressed, whereas our approach considers all the values of the arguments (with weights which can be chosen arbitrary).
Some of the notions and denotations we are going to use are indicated in § 2. Very important is the calculus with 0 and ∞ and also the modification of probability theory notions which are to be taken narrower than usual, only to be sufficient for the purposes of this paper. In § 3 the MINSKY machines are defined, their instantaneous descriptions and their computations, farther the functions coordinated by the (MINSKY) machines, ϕ^n_Z and $\mathcal{T}^n_Z$. Because we are considering the n-ary functions, an infinite number of functions ϕ^n_Z are coordinated to the machine

and then also an infinite number of functions $\mathcal{T}_{z}^{n}$ which note the lengths of the computings. The notions are introduced also formally but as it is usual in this kind of papers we are going to use mainly more free expression, in case no ambiguity is possible. In § 4 the machines necessary for the futher work are constructed.

In § 5 is introduced the fundamental notion of this paper, the notion of the h - computability and some notions derivated from it of which we indicate the weak H - computability and the quasiordering $\succ$. The weak H - computability facilitates for ur sometimes to avoid the use of the non-recursive functions h as it can be seen from theorem 10.1. For any two functions f, g the relation $f \succ g$ means that f is at least as complex as g. The exact meaning of $\succ$ is introduced in the definition and it is imposible to introduce it here.

§ 6 contains some methods of constructing of machines from the given machines, which are typical for farther proofs. Gradually we cease to define the machines formally and we are content with a description of their work which is a sufficient indication for the construction of a machine with the desired properties. It is only a question of concise proofs. The effectivity of the proofs is not damaged by this.

In § 7 are introduced and characterised the so called almost constant functions. Approximately can be said that these functions can be computed as quickly as the constants. In connection with this we consider also the functions min $(x_1, \ldots, x_n)$ which are (in our sense) the most simple n-ary functions they are more complex than the constants.

In § 8 is considered the closeness of the set of all h - computable functions (for a given h) towards the operation which can be expressed with the help of the superposition with a fixed external constituent. Here belongs for instance the sum of functions because $f(x) + g(x)$ is the superposition of the functions $x + y$, $f(x)$, $g(x)$. We consider directly the set of those external constituents by which we obtain from the h - computable functions again a h - computable function.

In § 9 are introduced the real-time computable functions; examples of such functions and some of their properties are given. An important property is that every h - computable function is a difference of two h - computable and real-time computable functions (the function h must fulfil certain suppositions). This property enables the research of the h - computability without directly constructing new machines, because for real-time computable function the h-computability can be characterised without help of machines. (This method has been chosen in the proofs of some theorems in § 11 but for the present we are not applying it systematically.) For the characterisation of the real-time computable functions it is sufficient to consider the lengths $\mathcal{T}^n_Z$ of the computations, and the functions ϕ^n_Z are not necessary at all.

§ 10 introduces the almost recursive functions. On the whole we can say that they are functions computable as quickly as the recursive functions, i.e. that there exists an algorithm ascertaining if their value (for any arguments) is defined and computing this value if it exists. These functions are characterised with the help of the h - computability as well as with the help of the real - time computability. Farther we consider the closeness of the set $\mathcal{F}_r$ of all the real-time computable functions towards some operations. At the end two theorems are proved on the relations between the set $\mathcal{F}_r$ and the set $\mathcal{F}_y$ of all real-time computable functions in the sense of [2] .

In § 11 the functions are considered which can be intuitively characterised as functions not more complex than the polynomials. § 12 contains different statements which have been used or were necessary but from the thematical point of view they do not belong to this paper. They are chiefly statements from the mathematical analysis. In the symbols register is given the number of a paragraph in which the symbol has been introduced together with a short explaining of meaning but this explanation cannot replace the definition. However, if it is a question of customary denotations we are not going to give again the definition of those notions.

In what follows, the list of the functions effectively admissible for the superposition and of the real-time computable functions will be completed. We shall consider the h - computability of real numbers. We shall consider the connection between $\mathcal{F}_r$ and $\mathcal{F}_y$. Further we shall find the solution of some questions of an algorithmic solvability (resembling to the [3]) and the properties of the sets $\{h\}^F$ and of the relation $\succ$.

I would like to use this opportunity and to express my thanks to RNDr. J. GRUSKA, CSc. for his advice and remarks which have been very helpful.

§ 2. Some Notions and Denotations

We shall use the logical signs $\equiv, \&, \rightarrow, \vee, \neg, \forall, \exists$ in their usual sense i.e. for the equivalence, conjunction, implication, disjunction and negation, the universal and existential quantifiers. Farther we are going to use some denotations from the set theory. Their significations are given in the symbols register. N will denote the set of all natural numbers (including 0), $\mathcal{P}$ the set of all unary total functions from N into N.

Under a function we shall understand a partial function of n variables from N into N (n is an arbitrary natural number) and to comprehend it as a set of ordered (n+1)-tuples. The set of all partial recursive functions will be denoted by $\mathcal{F}$. $\emptyset \in \mathcal{F}$ holds and $\emptyset$ is a unique element of $\mathcal{F}$ for which the number of arguments is not unambiguously determined. Cn(f), resp. Cs(f) denotes the zero supplement of the function f , resp. the zero supplement of the function 1 + f. The functions Cn($\emptyset$), Cs($\emptyset$) are usually considered as unary functions. From [1] we take the denotations of some functions which can be found together with some notes in the symbols register. The denotation $f(x_1,\dots,x_n)$ will denote either the value of the function f by the arguments $x_1,\dots, x_n$ or the function f itself; the significance will be clear from the context. In the second case we take x, y, z,..., x_1, x_2, x_3,... for variables, a, b, c,..., m, n, k,... for constants. E.g. if ax + by means a function then it means a function of two variables x,y. We shall denote the superposition of the functions f, $g_1,\dots, g_n$ by $S^{n+1}(f,g_1,\dots,g_n)$ (under supposition $n \neq 0$). For the minimalisation the signs μ_x, μ_y,... will be used. For the sum and the product the signs Σ, Π will be used. If f(x,y) is a function then Sum(f), Mult(f), Dif(f) are the functions g_1, g_2, g_3 such that

$$g_1(x,y) = \sum_{z=0}^{x} f(z,y) \qquad g_2(x,y) = \prod_{z=0}^{x} f(z,y)$$

$$g_3(x,y) = f(x,y) \dot{-} f(x \dot{-} 1,y).sg(x) .$$

Also in the case of other number of arguments (different from 0) have Sum(f), Mult(f), Dif (f) an analogical signification; we sum or we multiply always according to the first variable. The functions of real variables e.g. x:y, $\sqrt{x}$ are several times considered as partial functions from N into N. However, if they will be used together with the symbol for the whole part, f.i. $[\sqrt{x}]$ we understand them as a whole. (And $[\sqrt{x}]$ is a total function.) If we write some symbol of the number relation between two functions it means that this relation holds for the values of functions by any values of arguments. Instead of $x_1,\dots x_n$ we shall write only X_n, and similarly we shall use the sign B_n (in place of n an other letter can be used, but not instead X, B). We are not excluding by this the case when $n = 0$. We do not exclude it neither in other cases when the forms with 3 dots "..." are used. Instead of $\lim_{x\to\infty}$, $\lim_{(x_1+\dots+x_n)\to\infty}$ we are going to write only lim; analogously for lim inf, lim sup, too. The phrase "for almost all $x_1,\dots,x_n$" means "for all $x_1,\dots x_n$ with the exception of finite number of n-tuples $\langle x_1,\dots,x_n\rangle$".

To the real numbers we add the symbol ∞ and for $0<a<\infty$ $n \in N$ we define

$$\infty \pm 0 = \infty \pm a = \infty + \infty = a \,.\, \infty = \infty : \infty = a : \infty = \infty : a = \infty^n = \infty^{-n} = \infty$$

$$0 : \infty = \infty : 0 = a : 0 = 0 : 0 = 0 \,.\, \infty = 0.$$

We added the element ∞ only to the real numbers and not to N.

We will consider only such probabilities which are defined on the set of all subsets of the set N. There exists a natural bijective coordination between these probabilities and the non-negative real-valued functions p defined on N and such that $\sum_{x=0}^{\infty} p(x) = 1$. These functions will be called probabilities.

We coordinate to every $h \in \mathcal{P}$ a non-negative real number (eventually ∞) and the real function p following the formulas:

$$T(h) = \sum_{x=0}^{\infty} \frac{1}{h(x)} \qquad p(x) = \frac{1}{T(h) \,.\, h(x)}$$

On condition $0 < T(h) < \infty$ p is a probability. We call it the probability coordinated to the function h .

For $0 \leq x < \infty$ $0 \leq y < \infty$ we define

$$x \leq^* y \equiv (y = 0 \vee (x \leq y \,\&\, x \neq 0)).$$

If p_1, p_2 are probabilities coordinated to the functions h_1, $h_2 \in \mathcal{P}$ (and $0 < T(h_1) < \infty$, $0 < T(h_2) < \infty$) then:

$$(\exists\, k \in N)\,(p_1 \leq k.p_2) \equiv (\exists\, k \in N)\,(h_1 \geq^* k.h_2).$$

$\min^{*}(x,y)$ will be the minimum of the numbers x, y in the sense of ordering $\leq^{*}$.

As far as we are going to talk about the expectation of the function of more variables we shall always suppose that the arguments are independent and having the same probabilities.
We shall consider only the non-negative random variables which can acquire also the value ∞ . For their expectation the value ∞ is also acquired (so that for them it will always be defined).

We are going to limit ourselves to the probabilities associated to the functions from $\mathcal{P}$ in our considerations. The results would not change if we took into consideration all the probabilities as can be seen from the theorem 12.1. (We depend chiefly upon the finiteness resp. infiniteness resp. infiniteness of the therms existing in this theorem as can be seen from the definition 5.1.)

§ 3. MINSKY Machines

The MINSKY machines are defined in [1] as non-writing many-tape TURING machines with one-sided infinite (or: prolonged) tapes. On these tapes in the extreme squares the symbols 1 are inserted, the remaining squares are empty . The numbers are coded by the positions of the tapes to the heads (and not by the words inserted on the tapes). Another interpretation of MINSKY machines is also possible.

We understand MINSKY machines (further only "machines") as programs for an abstract computer. This computer will have a potential infinite number of (memory) cells $S_0, S_1, S_2, \ldots$ in every of them one arbitrary natural number can be put (so that every of this cells is itself potentially infinite) and a potential infinite number of cells $q_0, q_1, q_2, \ldots$ for the laying of instructions of the program. (These cells are finite and their contents are not changed during the computing.) We are going to call $q_0, q_1, q_2, \ldots$ states. The operation code of this computer will be

1) to add the number 1 to the content of a cell S_j; the sign is P.

2) to subtact 1 from the content of a cell S_j (only if this content is not zero; if it is zero, then it is not changed); the sign is M.

3) the conditional jump according to zero in a cell S_j; it is without any sign.

The stop occurs if the computer comes into a state which is without instruction. The sequence of instructions is prescribed; the address of the following instruction is always noted on the just working instruction.

We define formally:

<u>D e f i n i t i o n 3.1.</u> a) A machine is a finite set of quadruples of the forms $\langle q_i S_j P q_k \rangle$, $\langle q_i S_j M q_k \rangle$, $\langle q_i S_j q_m q_n \rangle$ which does not contain two different quadruples with the same first element. (The mentioned quadruples will be called plus-, minus-, resp. zero-quadruples.)

b) An instantaneous description is an (ordered) pair

$$\langle q_i \,,\, (u_0, u_1, u_2, \dots) \rangle \tag{3.1}$$

where q_i is a state and $(u_0, u_1, u_2, \dots)$ is a sequence of natural numbers, which has only a finite number of non-zero members.

Instead of (3.1) we shall use a simpler denotation $(q_i;\ u_0, u_1, \dots, u_n)$ in which n is arbitrary but such that $u_{n+1} = u_{n+2} = \dots = 0$.

c) Let Z be a machine, $U_1 = (q_i;\ u_0, u_1, \dots, u_n)$, $U_2 = (q_j;\ v_0, v_1, \dots, v_n)$ are instantaneous descriptions. We shall write $U_1 \rightarrow U_2(Z)$, if there exists $m \in N$ such that for all $k \in N$ $k \neq m$ $u_k = v_k$ and

1) $v_m = u_m + 1$ and $\langle q_i S_m P q_j \rangle \in Z$ or

2) $v_m = u_m \dot{-} 1$ and $\langle q_i S_m M q_j \rangle \in Z$ or

3) $v_m = u_m = 0$ and there exists $p \in N$ such that
$\langle q_i S_m q_p q_j \rangle \in Z$ or

4) $v_m = u_m \neq 0$ and there exists $p \in N$ such that
$\langle q_i S_m q_j q_p \rangle \in Z$

d) By the computation of the machine Z from the instantaneous description U to the instantaneous description V is called a finite sequence $U_0 = U,\ U_1, \dots,\ U_n = V$ such that $U_i \rightarrow U_{i+1}$ (Z). The number n (possibly, 0) is called the length of this computation. We shall write $U \Rightarrow V(Z)$ if there exists a computation of the machine Z from U to V. If this computation is unambigously determined the length of it will be denoted by $L(Z;\ U,V)$. An instantaneous description U is called a final description of the machine Z, if there exists no V such that $U \rightarrow V(Z)$.

e) If there exists a final description V such that $U \Rightarrow V(Z)$ we call the computation of the machine Z from U to V by the computation of the machine Z from U. If for the machine Z and for the instantaneous description U such a V does not exist, then we call by the computation of the machine Z from the instantaneous description U the infinite sequence $(U_0, U_1, U_2, \ldots)$ such that $U_0 = U$ and $U_i \rightarrow U_{i+1}(Z)$ for all $i \in N$. In the first case we put $L(Z;U) = L(Z;\ U,V)$, in the second case $L(Z;U) = \infty$

Remark. The computation of every machine from every instantaneous description exists and is unambiguously determined but it can be finite or infinite and in the first case it can have also a length 0.

Definition 3.2. a) The set of all pairs $\langle nZ \rangle$ where $n \in N$ and Z is a machine we denote by $\mathfrak{Z}$.

b) For $\langle nZ \rangle \in \mathfrak{Z}$ we denote ϕ^n_Z the n-ary function such that $\phi^n_Z(x_1, \ldots, x_n) = y$ if there exists a finite computation of the machine Z from the instantaneous description $(q_1;\ 0, x_1, \ldots, x_n)$ which ends by an instantaneous description $(q_0;\ y, u_1, \ldots, u_n)$ (for arbitrary $m, u_1, \ldots, u_m$).

In the remaining cases the value $\phi^n_Z(x_1, \ldots, x_n)$ is not defined.

c) For $\langle nZ \rangle \in \mathfrak{Z}$ we denote by $\mathcal{T}^n_Z$ the n-ary function with the values from $N \cup \{\infty\}$ defined by the formula

$$(3.2) \qquad \mathcal{T}^n_Z(x_1, \ldots, x_n) = L(Z;\ (q_1;\ 0, x_1, \ldots, x_n)).$$

(In accordance with the above mentioned agreements we will sometimes understand $\mathcal{T}^n_Z$ as a partial function with the values from N.)

It is clear from the definitions that in our interpretation of the machine as a program, the numbers $u_0, u_1, u_2, \ldots$ correspond to the contents of the cells $S_0, S_1, S_2, \ldots$. If we consider $\langle nZ \rangle \in \mathfrak{Z}$, it seems to be sufficiently natural to understand $S_1, \ldots S_n$ as input cells and S_0 as an output cell. Further we shall show that this can always be arranged. We define:

Definition 3.3. We call the cell S_j an input cell (resp. an output cell) if S_j appears only in zero-and minus-quadruples (resp. only in plus-quadruples) of the machine Z . We call it an actual cell of Z , if it appears in quadruples of Z and we call it the working cell if it is actual but neither input cell nor an output one of Z .

For every cell S_j and every machine Z happens to be just one of the four cases: S_j is a working cell of Z , S_j is an actual input cell of Z , S_j is an actual output cell of Z , S_j appears in no quadruple of Z . In [1] has been proved (even if it has not been explicitly formulated) that for the computation of any partial recursive function it is sufficient to have two working cells.

We are giving an example of the machine here. Let be

$$Z = \{\langle q_1 S_1 q_2 q_0\rangle , \langle q_2 S_1 M q_3\rangle , \langle q_3 S_0 P q_1\rangle\} .$$

It holds $\Phi^1_Z = I^1_1$, $\mathcal{T}^1_Z(x) = 3x + 1$. Indeed, the computation of Z from the instantaneous description $(q_1;\ 0,x)$ is

$$((q_1;\ 0,x),\ (q_2;\ 0,x),\ (q_3;\ 0,x-1),\ (q_1;\ 1,x-1),\dots$$
$$\dots,\ (q_2;\ x-1,1),\ (q_3;\ x-1,0),\ (q_0;\ x,0)).$$

Clearly, for every $x \in N$ it holds $\Phi^1_Z(x) = x$, $\mathcal{T}^1_Z(x) = 3x + 1$. In this example S_1 is an input cell and S_0 is an output one of the machine Z . It is clear that for every $n \geq 1$ holds $\Phi^n_Z = I^n_1$ and $\mathcal{T}^n_Z(x_n) = 3x_1 + 1$.

Let us introduce a theorem on the realtion between Φ^n_Z and $\mathcal{T}^n_Z$ at the end.

<u>T h e o r e m 3.1.</u> Let be $\langle nZ\rangle \in \mathfrak{Z}$. Then

a) for every $x_1,\dots,x_n$ such that $\Phi^n_Z(x_n)$ is defined it holds

$$\Phi^n_Z(x_n) \leq \mathcal{T}^n_Z(x_n).$$

b) if for arbitrary $x_1,\dots,x_n$ $\Phi^n_Z(x_n)$ is defined then $\mathcal{T}^n_Z > 0$.

c) if there exist $x_1,\dots,x_n$ such that $\Phi^n_Z(x_n) = \mathcal{T}^n_Z(x_n)$, then $\Phi^n_Z = \mathcal{T}^n_Z$ and both functions are constant and different from 0.

<u>P r o o f .</u> a) It is sufficient to consider that in every step of

the computation the content of S_0 increases at most in one unit. At least $\Phi^n_Z(X_n)$ steps are needed to be done so that the content of S_0 increas from 0 to $\Phi^n_Z(X_n)$.

b) If there exist $x_1,\dots,x_n$ such that $\Phi^n_Z(X_n)$ is defined, then Z contains a quadruple beginning with q_1. Therefore the computation from every instantaneous description with the state q_1 has the length at least 1.

c) In the computation of $\Phi^n_Z(X_n)$ (i.e. in the computation of Z from the $(q_1;\ 0,X_n)$) a plus-quadruple must be used in every step. Therefore the development of the computation does not really depend on the contents of the cells S_1, $S_2,\dots$ (as far as the computation begins from the state q_1).

§ 4. Examples of Machines

In the preceding paragraph the machine Z has been introduced which computed the functions I^n_1 and in the proof of the theorem 3.1. we considered the machines computing constant functions (different from 0). We introduce the machines which will compute the functions $x + y$, $x - y$, $[x : y]$, $D(x,y)$, $[\sqrt{x}]$, $[\log_y x]$, $\min(x_1,\dots,x_n)$, $x \cdot y$, $x \dot{-} yz$ here. The machines will be constructed - as far as possible - so that for the lengths $\mathcal{T}^n_Z$ of their computation (for n determined by the function, an arbitrary $k \in N$ and all the $x_1,\dots,x_n$) will be valid

$$\mathcal{T}^n_Z(x_1,\dots,x_n) \leq k.(1+x_1+\dots+x_n) \tag{4.1}$$

$$\mathcal{T}^n_Z \leq k.(1 + \Phi^n_Z). \tag{4.2}$$

We shall need those estimations especially in §8 and §9. The estimation (4.2) can be fulfilled only if the function Φ^n_Z is total, and so it is impossible to fulfil it for $\Phi^n_Z = \emptyset$. We note that for the functions I^n_m and for the constant functions (with the exception of $\emptyset$) both the estimations (4.1), (4.2) can be fulfilled. Now we are going to construct the machines for the above mentioned functions.

1) T h e f u n c t i o n x + y . Let the machine Z_1 be composed of the quadruples:

$$\langle q_1S_1q_2q_4\rangle \;,\; \langle q_2S_1Mq_3\rangle \;,\; \langle q_3S_0Pq_1\rangle \;,$$
$$\langle q_4S_2q_5q_0\rangle \;,\; \langle q_5S_2Mq_6\rangle \;,\; \langle q_6S_0Pq_4\rangle \;.$$

Clearly $\Phi^2_{Z_1}(x,y) = x + y$, $\mathcal{T}^2_{Z_1}(x,y) = 3x + 3y + 2$ and then the estimations (4.1), (4.2) are fulfilled for $k = 3$.

2) T h e f u n c t i o n x - y . Let us replace in Z_1 the last quadruple by the quadruples $\langle q_6S_0q_7q_8\rangle$, $\langle q_7S_0Mq_4\rangle$

and let us denote the new machine by Z_2. Then it holds $\Phi^2_{Z_2}(x,y) =$ $= x - y$. The estimation (4.1) is fulfilled for $k = 4$, the estimation (4.2) cannot be fulfilled.

3) T h e f u n c t i o n [x : y] . Let the machine Z_3 be composed by the quadruples:

$$\langle q_1S_2q_2q_0\rangle \;,\; \langle q_2S_2q_3q_7\rangle \;,\; \langle q_3S_2Mq_4\rangle \;,\; \langle q_4S_1q_5q_0\rangle \;,$$
$$\langle q_5S_3Pq_6\rangle \;,\; \langle q_6S_1Mq_2\rangle \;,\; \langle q_7S_0Pq_8\rangle \;,\; \langle q_8S_3q_9q_2\rangle \;,$$
$$\langle q_9S_3Mq_{10}\rangle \;,\; \langle q_{10}S_2Pq_8\rangle \;.$$

The machine Z_3 works as follows: At first, it ascertains whether $y \neq 0$. If not then it has finished the computation. If yes then it subtracts y from x and simultaneously forms y in the cell S_3. After every subtraction it transports y back into S_2 and so it continues its work until the whole x is exhausted. The number of accomplished subtractions is formed in S_0. It is equal to $[x : y]$. Therefore it holds $\Phi^2_{Z_3}(x,y) = [x : y]$. For $y = 0$ $\mathcal{T}^2_{Z_3}(x,y)=1$ holds. For $y \neq 0$, $x < y$ it holds $\mathcal{T}^2_{Z_3}(x,y) = 1 + L(Z_3;\ (q_2;\ 0,x,y)) \leqq 5y + 4$. For $x \geqq y > 0$ it holds $L(Z_3;\ 0,x,y),$ $(q_2;\ 1,\ x - y,\ y)) = (5y + 1) + 1 + (3y + 1) = 8y + 3$. Therefore it holds $\mathcal{T}^2_{Z_3}(x,y) \leqq 5y + 4 + [x : y] \cdot (8y + 3) \leqq 5y + 4 + 8x +$ $+ 3x$. So the extimation (4.1) holds for $k = 11$. (The estimation

(4.2) cannot be fulfilled here, nor in the other examples of this paragraph where it will be not expressly mentioned.)

4) T h e f u n c t i o n D(x,y) . (The greatest common divisor.) Let Z_4 be composed of the quadruples:

$\langle q_1 S_1 q_2 q_3 \rangle$, $\langle q_2 S_2 q_4 q_5 \rangle$, $\langle q_3 S_2 q_6 q_0 \rangle$,

$\langle q_5 S_1 q_8 q_0 \rangle$, $\langle q_8 S_1 M q_9 \rangle$, $\langle q_9 S_0 P q_5 \rangle$,

$\langle q_6 S_2 q_{10} q_0 \rangle$, $\langle q_{10} S_2 M q_{11} \rangle$, $\langle q_{11} S_0 P q_6 \rangle$,

$\langle q_4 S_1 q_{12} q_{13} \rangle$, $\langle q_{12} S_2 q_{14} q_{15} \rangle$, $\langle q_{13} S_2 q_{16} q_0 \rangle$,

$\langle q_{14} S_1 M q_{17} \rangle$, $\langle q_{17} S_2 M q_{18} \rangle$, $\langle q_{18} S_0 P q_4 \rangle$,

$\langle q_{15} S_0 q_{19} q_4 \rangle$, $\langle q_{19} S_0 M q_{20} \rangle$, $\langle q_{20} S_2 P q_{15} \rangle$,

$\langle q_{16} S_0 q_2 q_4 \rangle$, $\langle q_{21} S_0 M q_{22} \rangle$, $\langle q_{22} S_1 P q_{16} \rangle$.

At first, we note that $\Phi^2_{Z_4}(x,y) = \Phi^2_{Z_4}(y,x)$, $\mathcal{T}^2_{Z_4}(x,y) = \mathcal{T}^2_{Z_4}(y,x)$. We shall silently use this symmetry in our farther considerations. For $y = 0$ is $\mathcal{T}^2_{Z_4}(x,y) = 3x + 3$, $\Phi^2_{Z_4}(x,y) = x$. Let from now on $x \neq 0$, $y \neq 0$. Then the first nine quadruples express themselves only by the prolongation of the computation by 2 in the computation of D(x,y). For $x > y > 0$ it holds

$$\mathcal{T}^2_{Z_4}(x,y) - \mathcal{T}^2_{Z_4}(x-y,y) = L(Z_4;\ (q_4;\ 0,x,y),\ (q_4;\ 0,x-y,y)) =$$

$= (5y + 2) + (3y + 1) = 8y + 3$. For $x = y > 0$ it holds $\mathcal{T}^2_{Z_4}(x,y) = 2 + (5x + 2) = 5x + 4$, $\Phi^2_{Z_4}(x,y) = x$. By induction following the sum $x + y$ we can prove $\Phi^2_{Z_4}(x,y) = D(x,y)$,

$\mathcal{T}^2_{Z_4}(x,y) \leq 11(1 + x + y)$. Really, let those relations be valid for $x + y \leq K$ and let $u > v > 0$, $u + v = K + 1$. Then $\Phi^2_{Z_4}(u,v) = \Phi^2_{Z_4}(u - v,v) = D(u-v,v) = D(u,v)$, $\mathcal{T}^2_{Z_4}(u,v) = \mathcal{T}^2_{Z_4}(u-v,v) + 8y + 3 \leq 11((u-2) + v + 1) + 8v + 3 \leq 11(u + v + 1)$. If $u = v$

then the formulas mentioned above are also valid and they can be verified without using the inductive assumption. We can easily verify the formulas also for the case $x = 0$ or $y = 0$.

5) The function $[\sqrt{x}]$. The machine Z_5 is composed from the quadruples:

$\langle q_1S_2q_2q_6\rangle$, $\langle q_2S_1q_3q_0\rangle$, $\langle q_3S_1Mq_4\rangle$, $\langle q_4S_2Mq_5\rangle$,

$\langle q_5S_3Pq_1\rangle$, $\langle q_6S_3Pq_7\rangle$, $\langle q_7S_3q_8q_{12}\rangle$, $\langle q_8S_1q_9q_0\rangle$,

$\langle q_9S_1Mq_{10}\rangle$, $\langle q_{10}S_3Mq_{11}\rangle$, $\langle q_{11}S_2Pq_7\rangle$, $\langle q_{12}S_0Pq_1\rangle$.

This machine works so that it subtracts gradually from x the odd numbers 1, 3, 5,... and after every finished subtraction adds 1 into S_0 . The computation is finished by the exhaustion of the content of S_1 . Because $1 + 3 + \ldots + (2y - 1) = y^2$, the number of finished subtractions is equal to $[\sqrt{x}]$. In the computation (from the instantaneous description $(q_1;\ 0,x)$) the machine Z_5 subtracts 1 from the content of S_1 at least in every sixth step and therefore it holds $\mathcal{T}^1_{Z_5}(x) \leqq 6(1 + x)$.

6) The function $[\log_y x]$. Let the machine Z_6 be composed of the quadruples:

$\langle q_1S_2Mq_2\rangle$, $\langle q_2S_2q_4q_3\rangle$, $\langle q_4S_2Pq_5\rangle$, $\langle q_5S_1q_6q_3\rangle$,

$\langle q_6S_1q_7q_{15}\rangle$, $\langle q_7S_2q_{12}q_8\rangle$, $\langle q_8S_3q_9q_{11}\rangle$, $\langle q_9S_3Mq_{10}\rangle$,

$\langle q_{10}S_2Pq_8\rangle$, $\langle q_{11}S_4Pq_7\rangle$, $\langle q_{12}S_1Mq_{13}\rangle$, $\langle q_{13}S_2Mq_{14}\rangle$,

$\langle q_{14}S_3Pq_6\rangle$, $\langle q_{15}S_3q_{16}q_{18}\rangle$, $\langle q_{16}S_3Mq_{17}\rangle$, $\langle q_{17}S_2Pq_{15}\rangle$,

$\langle q_{18}S_4q_{19}q_{21}\rangle$ $\langle q_{19}S_4Mq_{20}\rangle$, $\langle q_{20}S_1Pq_{18}\rangle$, $\langle q_{21}S_1q_{22}q_0\rangle$,

$\langle q_{22}S_0Pq_6\rangle$.

The machine works as follows: At first, it ascertains whether $[\log_y x]$ is defined. If it is not then the computation ends in the state q_3. If it is, then values $[x : y]$, $[[x : y] : y]$, $[[[x : y] : y] : y]$,.... are computed until 0 arises. From the number of the executed divisions $[\log_y x]$ will be determined. It can be shown that the evalution (4.1) is fulfilled for $k = 28$. (This machine is needed only for one statement from §8 which will be proved in §10 without using the machine Z_6. However, this statement gives a better survey of the functions admissible for the superposition.)

7) T h e f u n c t i o n $\min(x_1,\ldots, x_n)$, $n \neq 0$. Let the machine Z_7 be composed of the quadruples:

$$\langle q_1S_1q_2q_0\rangle\ ,\ \langle q_2S_1Mq_3\rangle\ ,\ \langle q_3S_2q_4q_0\rangle\ ,\ \langle q_4S_2Mq_5\rangle\ ,\ \ldots$$

$$\ldots\ ,\ \langle q_{2n-1}S_nq_{2n}q_0\rangle\ ,\ \langle q_{2n}S_nMq_{2n+1}\rangle,\ \langle q_{2n+1}S_0Pq_1\rangle\ .$$

The machine works so that it subtracts a unit from all the numbers $x_1,\ldots, x_n$, then adds the unit to S_0 and repeats the process. The computation is finished when it is impossible to subtract. It holds $\Phi^n_{Z_7} = \min(X_n)$, $\mathcal{T}^n_{Z_7}(X_n) \leqq 2n + (2n + 1)\min(X_n)$. So the estimation (4.1) is fulfilled for $k = 4$, and the estimation (4.2) for $k = 2n + 1$.

8) T h e f u n c t i o n $x.y.$ Let the machine Z_8 contain the quadruples:

$$\langle q_1S_1q_2q_0\rangle\ ,\ \langle q_2S_2q_3q_0\rangle\ ,\ \langle q_3S_2Mq_4\rangle\ ,\ \langle q_4S_1q_5q_8\rangle\ ,$$

$$\langle q_5S_1Mq_6\rangle\ ,\ \langle q_6S_0Pq_7\rangle\ ,\ \langle q_7S_3Pq_4\rangle\ ,\ \langle q_8S_3q_9q_2\rangle\ ,$$

$$\langle q_9S_3Mq_{10}\rangle\ ,\ \langle q_{10}S_1Pq_8\rangle\ .$$

It holds $\Phi^2_{Z_8}(x,y) = x.y$. If $x = 0$, then $\mathcal{T}^2_{Z_8}(x,y) = 1$. If $x \neq 0$ then $\mathcal{T}^2_{Z_8}(x,y) = 1 + 1 + y(1 + (4x + 1) + (3x + 1)) =$ $= 7xy + 3y + 2$. Then the estimation (4.2) holds for $k = 10$. The estimation (4.1) cannot be fulfilled for the function $x . y$, as it can be seen from the teorem 3.1.

9) T h e f u n c t i o n $x \dot{-} y.z.$ Let the machine Z_9 be composed of the quadruples:

$$\langle q_1S_1q_2q_4\rangle\ ,\ \langle q_2S_1Mq_3\rangle\ ,\ \langle q_3S_0Pq_1\rangle\ ,\ \langle q_4S_3q_5q_0\rangle\ ,$$

$$\langle q_5S_3Mq_6\rangle\ ,\ \langle q_6S_2q_7q_{10}\rangle,\ \langle q_8S_0q_9q_0\rangle,\ \langle q_9S_0Mq_{10}\rangle\ ,$$

$$\langle q_{10}S_1Pq_6\rangle\ ,\ \langle q_{11}S_1q_{12}q_4\rangle,\ \langle q_{12}S_1Mq_{13}\rangle,\ \langle q_{13}S_2Pq_{11}\rangle\ ,$$

This machine works so that it puts first x into S_0 and it uses $3x + 1$ steps for them. Then it subtracts a number 1 from z and y from x until z or x becomes equal to 0. One such subtraction needs $8y + 4$ steps. (The renovation of the content of S_2 is included.) The maximal possible number of these subtractions together with the last unfinished one is $[x : y] + 1$ for $y \neq 0$ and $z + 1$ for $y = 0$. In the first case $\mathcal{T}^3_{Z_9}(x,y,z) \leq 3x + 1 + ([x : y] + 1).(8y + 4)$, in the second one $\mathcal{T}^3_{Z_9}(x,y,z) \leq 3x + 1 + (z + 1).(8 . 0 + 1)$. From this we obtain easily the evaluation (4.1) with $k = 12$. It is clear that $\Phi^3_{Z_9}(x,y,z) = x \dot{-} y.z$.

(We could also place the same note there as we did for the function $[\log_y x]$.)

§ 5. h - computability

D e f i n i t i o n 5.1. a) For every $\langle nZ\rangle \in \mathcal{Z}$ and every $h \in \mathcal{P}$ we define

$$T'(h, \langle nZ\rangle) = \sum_{x_1 = 0}^{\infty} \cdots \sum_{x_n = 0}^{\infty} \frac{\mathcal{T}^n_Z(x_1,\dots,x_n)}{h(x_1)\dots h(x_n)} \qquad \text{if } n \neq 0$$

$$T'(h, \langle nZ\rangle) = \sum_{x = 0}^{\infty} \frac{\mathcal{T}^1_Z(0)}{h(x)} \qquad \text{if } n = 0 ,$$

$$T(h) = \sum_{x = 0}^{\infty} \frac{1}{h(x)} .$$

$$T(h, \langle nZ\rangle) = \frac{T'(h, \langle nZ\rangle)}{(T(h))^{(n \dot{-} 1)} + 1} .$$

b) For every function f we define

$$T(h,f) = \inf \left\{ T(h, \langle nZ\rangle) \mid \langle nZ\rangle \in \mathcal{Z} \ \& \ \Phi^n_Z = f \right\} .$$

c) A function f is called h-computable (and we write $f\mathcal{V}h$) if $T(h,f) < \infty$.

In this definition the calculus with 0 and ∞ introduced in §2 is fully used. We shall never use independently the symbol T and therefore no misunderstanding from the denotations $T(h)$, $T(h,f)$, $T(h, \langle nZ \rangle)$ is menacing. In these terms h, f are always functions (and not values of functions). For instance, $T(x) = T(I_1^1) = \infty$, $T(x^2) = \frac{1}{6}\pi^2$. If $n \neq 0$, $g(x_1,\dots,x_{n+1}) = f(x_1,\dots,x_n)$, then $T(h,g) = T(h,f)$ for every $h \in \mathcal{P}$. The definition of $T'(h, \langle nZ \rangle)$ and $T(h, \langle NZ \rangle)$ for $n = 0$ was chosen so that the same statement is fulfilled for $n = 0$, too. For $n \neq 0$, $0 < T(h) < \infty$ it holds

$$T(h, \langle nZ \rangle) = \frac{T'(h, \langle nZ \rangle)}{(T(h))^n} =$$

$$= \sum_{x_1=0}^{\infty} \dots \sum_{x_n=0}^{\infty} \mathcal{T}_Z^n (x_1,\dots,x_n).p(x_1)\dots p(x_n)$$

where $p(x) = 1 : (h(x).T(h))$ is the probability coordinated to the function h . For $n = 0$, $0 < T(h) < \infty$ is $T(h,f) = \mathcal{T}_Z^1(0) = \mathcal{T}_Z^0$. Therefore for $0 < T(h) < \infty$ $T(h, \langle nZ \rangle)$ is the expectation of $\mathcal{T}_Z^n(X_n)$ by the probability which is coordinated to the function h.

In most cases we shall be interested only in the h-computability of a function f and not in the numbering value of $T(h,f)$. This will be facilited by the following theorem.

<u>T h e o r e m 5.1.</u> A function f is h-computable if and only if there exists a pair $\langle nZ \rangle \in \mathcal{Z}$ such that $\Phi_Z^n = f$ and $T'(h, \langle nZ \rangle) < \infty$.

<u>P r o o f .</u> fVh holds if and only if there exists a pair $\langle nZ \rangle$ such that $\Phi_Z^n = f$ and $T(h, \langle nZ \rangle) < \infty$. But

$$T(h, \langle nZ \rangle) < \infty \equiv T'(h, \langle nZ \rangle) < \infty .$$

Indeed, if $T(h, \langle nZ \rangle) = \infty$ then $\mathcal{T}_Z^n \geq 1$ and therefore the numerator of the expression $T(h, \langle nZ \rangle)$ is greater or equal than the denominator of one. Then also $T'(h, \langle nZ \rangle) = \infty$. Conversely, if $T'(h, \langle nZ \rangle) = \infty$ so $T(h) \neq 0$ and then $T(h, \langle nZ \rangle) = \infty$.

Theorem 5.2. Let $h \in \mathcal{P}$, let for almost every $x \in N$ it holds $h(x) = 0$. Then a function f is h-computable if and only if it is partial recursive.

Proof. Let be $M = \{x \in N \mid h(x) \neq 0\}$. If $f \in \mathcal{F}$, then there exists a pair $\langle nZ \rangle \in \mathcal{Z}$ such that $\Phi^n_Z = f$. Because the set M is finite, there exists a machine Z_1 such that $\Phi^n_Z = \Phi^n_Z$ and $\mathcal{T}^n_{Z_1}(X_n) < \infty$ for every $x_1, \ldots, x_n \in M$. Then $T'(h, \langle nZ_1 \rangle) < \infty$ and therefore fVh. Conversely, if fVh for some h, then there exists a pair $\langle nZ \rangle \in \mathcal{Z}$ such that $\Phi^n_Z = f$ and therefore $f \in \mathcal{F}$.

Theorem 5.3. Let $h \in \mathcal{P}$, $T(h) = \infty$. Then a function f is h-computable if and only if $f = \emptyset$.

Proof. If fVh, then there exists $\langle nZ \rangle$ such that $\Phi^n_Z = f$, $T'(h, \langle nZ \rangle) < \infty$. If $f \neq \emptyset$, then $\mathcal{T}^n_Z \geq 1$ and therefore $T'(h, \langle nZ \rangle) \geq (T(h))^{(n \dot{-} 1)+1} = \infty$, which is a contradiction. Conversely, if $f = \emptyset$ then for example $\Phi^1_\emptyset = f$, $T'(h, \langle 1\emptyset \rangle) = 0 < \infty$.

Theorem 5.4. A function $h \in \mathcal{P}$ is h-computable if and only if $h(x) = 0$ for almost all $x \in N$.

Proof. Let $h \in \mathcal{P}$, hVh. There exists a pair $\langle 1Z \rangle \in \mathcal{Z}$ such that $\Phi^1_Z = h$, $T'(h, \langle 1Z \rangle) < \infty$. According to the theorem 3.1. every non-zero member of infinite series $T'(h, \langle 1Z \rangle)$ is greater or equal to 1. Their number must be finite and therefore $h(x) = 0$ for almost all $x \in N$. The converse statement follows from the theorem 5.2.

Definition 5.2. a) For partial recursive functions f, g we define

$f > g$ if for every $h \in \mathcal{P}$ fVh implies gVh,

$f \sim g$ if $f > g$ and $g > f$,

$f \gg g$ if $f > g$ but $g > f$ does not hold.

b) For any function $h \in \mathcal{P}$ we denote the set of all h-computable functions by $\{h\}^F$. For every $H \subseteq \mathcal{P}$ we define

$$H^F = \bigcup_{h \in H} \{h\}^F \qquad H^f = \bigcap_{h \in H} \{h\}^F .$$

A function f is called weakly (resp. strongly) H-computable if $f \in H^F$ (resp. $f \in H^f$).

R e m a r k . We put $\emptyset^f = \mathcal{F}$, $\{h\}^f = \{h\}^F$.

The relation $\succ$ is a quasi-ordering of the set $\mathcal{F}$. (I.e. it is both reflexive and transitive.) As usual we shall write $g \prec f$, $g \prec\prec f$ instead of $f \succ g$, $f \succ\succ g$ respectively. Further some subsets of $\mathcal{F}$ with the help of the relation $\succ$ are defined, without directly using the h-computability.

The following two theorems are corollaries of the theorems 5.2., 5.3.

T h e o r e m 5.5. For every partial recursive function $f \neq \emptyset$ the relations $f \succ 0$, $f \succ\succ \emptyset$ hold.

T h e o r e m 5.6. Let $\mathcal{P}'$ be the set of all $h \in \mathcal{P}$ such that $0 < T(h) < \infty$, let $f, g \in \mathcal{F} - \{\emptyset\}$. Then $f \succ g$ if and only if for every $h \in \mathcal{P}'$ fVh implies gVh.

We can see from the last theorem that the influence of those functions $h \in \mathcal{P}$ to which probabilities are not coordinated is expressed only for the function $\emptyset$.

Let us mention now that $x \geqq^* y$ if and only if $x \geqq y > 0$ or $x = 0$.

L e m m a 1. Let be $h_1, h_2 \in \mathcal{P}$, $h_1 \geqq^* h_2$, $\langle nZ \rangle \in \mathcal{Z}$. Then

$$T'(h_1, \langle nZ \rangle) \leqq T'(h_2, \langle nZ \rangle).$$

P r o o f . If $h_1 \geqq^* h_2$ then for any $x \in N$ $1 : h_1(x) \leqq 1 : h_2(x)$. Now it is sufficient to compare member-wise the infinite series $T'(h_1, \langle nZ \rangle)$, $T'(h_2, \langle nZ \rangle)$.

T h e o r e m 5.7. Let be f a function, $h_1, h_2 \in \mathcal{P}$, $h_1 \geqq^* h_2$. Then h_2-computability of f implies h_1-computability of one. (I.e. $\{h_2\}^F \subseteq \{h_1\}^F$.)

This theorem is an immediate corollary of the lemma 1.

We note that it is impossible to write $\geqq$ instead of $\geqq^*$ in the lemma and in the theorem. For example, $I_1^1 \geqq o$ holds, but $\{o\}^F$ is not a subset of $\{I_1^1\}^F$.

T h e o r e m 5.8. Let be $h_1, h_2 \in \mathcal{P}$ and let there exist (finite) positive real numbers α, β such that $\alpha . h_1 \leqq h_2 \leqq \beta . h_1$. Then every function f is h_2-computable if and only if it is h_1-computable (i.e. $\{h_1\}^F = \{h_2\}^F$).

P r o o f . There exists $m \in N$, $m \neq 0$ such that $m.h_1 \geq^* h_2$. Then $\{h_2\}^F \subseteq \{m.h_1\}^F$. For every pair $\langle nZ \rangle \in \mathcal{Z}$ the equality $T(m.h_1, \langle nZ \rangle) = T(h_1, \langle nZ \rangle)$ holds and so $\{m.h_1\}^F = \{h_1\}^F$. Therefore $\{h_2\}^F \subseteq \{h_1\}^F$. Likewise the second inclusion is proved.

T h e o r e m 5.9. Let be $h \in \mathcal{P}$, $T(h) < \infty$, $h > 0$. Then a function f is h-computable if and only if the function $Cs(f)$ is h-computable.
(We note that $Cs(f)$ is the total function such that $f = Cs(f) - 1$).

P r o o f . If $f = \emptyset$ then $Cs(f) = o$ and clearly oVh. Let now be $f \neq \emptyset$. Let $\langle nZ \rangle \in \mathcal{Z}, \Phi^n_Z = f$, $T'(h, \langle nZ \rangle) < \infty$. Then $1 \leq \mathcal{T}^n_Z < \infty$.

The machine Z can be easily adapted into a machine Z_1 such that $\mathcal{T}^n_Z \leq 1 + 3\mathcal{T}^n_Z$, $\Phi^n_{Z_1} = Cs(f)$. It is sufficient to replace the state q_0 by the state q^1_0 in all quadruples of Z, to add the quadruples $\langle q^2_0 S_0 q^3_0 q_0 \rangle$, $\langle q^3_0 S_0 M q^2_0 \rangle$ and to replace all the states (with the exception of q_0) which are in the third and fourth places of the quadruples of Z but are not in their first places by the state q^2_0. It holds $T'(h, \langle nZ_1 \rangle) < \infty$ and therefore $Cs(f)Vh$. Conversely, from a machine Z_1 such that $\Phi^n_Z = Cs(f)$ can be obtained such a machine Z_2 that $\Phi^n_{Z_2} = f$ and $\mathcal{T}^n_{Z_2} \leq 3\mathcal{T}^n_{Z_1}$. It is sufficient to replace the state q_0 by the state q^1_0 in all quadruples of Z_1 and to add the quadruples $\langle q^1_0 S_0 q^2_0 q^3_0 \rangle$, $\langle q^2_0 S_0 M q_0 \rangle$. If $Cs(f)Vh$, we can assume $T'(h, \langle nZ_1 \rangle) < \infty$. Then $T'(h, \langle nZ_2 \rangle) < \infty$ and so fVh.

R e m a r k . The assumption $T(h) < \infty$ had to be used only for the case $f = \emptyset$.

As a conclusion of this paragraph we shall point out that there is one possible modification of the h-computability. In the definition of $T(h,f)$ only such machines have been considered which have had the property $\Phi^n_Z = f$ and so they computed always the value of the function (i.e., for all $x_1,\dots,x_n$). We could waken this demand in such a way that we could take into consideration all machines which computed the value of the function f with the propability 1 (the probability coordinated to the function h being considered). In a formal way, we could replace the condition $\Phi^n_Z = f$ in the definition of $T(h,f)$ by the condition: for all $x_1,\dots,x_n$ such that $h(x_1) \neq 0,\dots, h(x_n) \neq 0$ it holds $\Phi^n_{Z_1}(x_1,\dots,x_n) = f(x_1,\dots,x_n)$. Then the case $T(h,f)<\infty$ could happen also when f would not be partial recursive. In this paper we are not going to consider this modification.

§ 6. Some Methods of the Constructing of Machines

We shall often construct a new machine from several machines similarly to a new program constructed from subprograms. Therefore it will be necessary to arrange the machines in such a way that they could fulfil some additional conditions. Here we shall describe some methods which will be often used. Because we are interested chiefly in the h-computability of the functions we are not going to be anxious about the linear prolongation of the length of computation. On the other hand we shall try not to prolong the length of computation unnecessarily.

1) The address substitution is the most elementary method of the arrangement of the machines. In the working of the machines all the states q_i as well as all the cells S_j have essentially equal roles. But in the definition of the functions Φ^n_Z, $\mathcal{T}^n_Z$ the states q_1, q_0 and the cell S_0 and the cells $S_1,\dots,S_n$ have a special task. Therefore if we want a machine to work as a subprogram in another machine we must change indices of its states and cells. Such an arrangement, of course, changes substantially the functions Φ^n_Z, $\mathcal{T}^n_Z$ but on the contrary the machine can compute the function Φ^n_Z in the time $\mathcal{T}^n_Z$ if the arguments are put conveniently, if the computing begins with a suitable state and if the result is taken from the suitable cell. In the same time we loosened the states q_0, q_1 and the cells $S_0, S_1,\dots,S_n$.

If we are using more machines in the construction of one machine we suppose always that those machines have no common states, nor cells with the exception of the states and the cells for which it has been expressly introduced.

Often we construct a machine so that its cells and its states are indexed in a more complicated manner than it is allowed by the definition 3.1. (then if we take it rigorously it is not a machine). By an easy address substitution it is possible to replace this indexing by an allowed one.

2) Simultaneous modelling of the work of n machines with separate cells. Let $Z_1,\dots,Z_n$ $(n \geq 1)$ be machines and let every cell S_i arise at most in one Z_j. (The states can be common). Let us construct the machine Z as follows. Its states will be $q^j_{i_1,\dots,i_n}$, where $1 \leq j \leq n$ and q_{i_m} is a state of the machine Z_m for every $m = 1,\dots,n$. In all the quadruples of the machines Z_j (for $j = 1,\dots, n$) we are putting the upper index j to the states in the first places and the index j+1 (if $j < n$) or 1 (if $j=n$) to the states in the third or fourth places. Down we add j-1 indices in front of the original index and n-j indices after the original index by all admissible (i.e. leading to the state of Z) means but we add always the same indices to all states of one quadruple. Let be the machine Z composed by all the quadruples coordinated in this way to the machines $Z_1,\dots,Z_n$.

If this machine Z begins to work from the configuration with the state $q^1_{i_1,\dots,i_n}$, it makes one step of the work of Z_1 from the state q_{i_1} at first, then one step of the work of Z_2 from the state q_{i_2} etc., and also one step of the work of Z_n from the state q_{i_n}. Then follows a farther step of work of Z_1, of the work of Z_2 etc., until Z finds a machine Z_j the work of which was finished already. Then Z stops.

By simple arrangements of Z the following kinds of stops of the machine Z can be obtained:

A) if the computing of one of the machines $Z_1,\dots,Z_n$ is finished. (The original machine Z stops in this way.)

B) if the computings of all the machines $Z_1,\dots,Z_n$ are finished.

C) if the case B happened or if the computing of one of the machines $Z_1,\dots,Z_n$ is finished and it has been ascertained that the value of the competent function is not defined.

By this method the length of the computation can be computed. Let Z_1 be any machine, $Z_2 = \{\langle q_1 S_m P q_1\rangle\}$. Then the machine constructed here (following the point A) after finishing its work shows the length of the computation of Z_1 in the cell S_m.

3) S e p a r a t i o n o f t h e i n p u t , o u t p u t a n d w o r k i n g c e l l s . As has been pointed out in §3 in the consideration of a pair $\langle nZ_0\rangle \in \mathfrak{Z}$ it is natural enough to ask $S_1,\dots,S_n$ to be input cells and S_0 an output cell. We denote by Z_0' the machine arising from Z_0 by address substitution of the cells $S_0, S_1,\dots,S_n$ into $S_0', S_1',\dots,S_n'$. Let for i = 1, ...,n be

$$Z_i = \{\langle q_1 S_i L_2 q_1\rangle , \langle q_2 S_i M q_3\rangle , \langle q_3 S_i' P q_1\rangle\} .$$

Let us now construct the machine Z by application of the method of point 2A (without regard that the machines $Z_1,\dots,Z_n$, Z_0' have common cells S_i, S_i') which will be modelling in every cycle first of all 3 steps of work of Z_1, then 3 steps of work of Z_2 etc. until 3 steps of work of Z_n and finally 1 step of work of Z_0'. Then the next cycle begins. (Every machine $Z_1,\dots, Z_n, Z_0'$ begins its work from the state q_1.) After the end of the work of Z_0' let the machine Z transcribe also the content of S_0' into S_0 and then stops in the same state as Z_0'. (Of course, we must add some other quadruples for this reason.) It holds $\Phi_Z^n = \Phi_{Z_0}^n$, there exist $k_1, k_2 \in N'$ such that $\mathcal{T}_Z^n \leq k_1 \cdot \mathcal{T}_{Z_0}^n + k_2$, $S_1,\dots,S_n$ are input cells of Z and S_0 can be an output cell of Z .

By a small modification we could obtain the machine Z which does not work further with cell $S_j (j = 1,\dots,n)$ if it ascertains that its content is exhausted already.

R e m a r k . It is not convenient to transcribe at first the whole contents of the cells $S_1,\dots,S_n$ into $S_1',\dots,S_n'$ because the evaluation for $\mathcal{T}_Z^n$ needs not to be valid.

4) S i m u l t a n e o u s m o d e l l i n g w i t h c o m m o n i n p u t s . Let the machines $Z_1,\dots,Z_m$ have common

input cells $S_1,\dots,S_n$. The machine Z which modells the work of the machines $Z_1,\dots,Z_n$ can work analogically as the machine Z from the point 3) (without the quadruples which transcribe the result from S_0' into S_0). The new machine Z transcribes the content of any S_j $(j = 1,\dots,n)$ not into a cell S_j' but into m cells S_j^i $(i = 1,\dots,m)$; by this for every machine Z_j special inputs are formed. Instead of the modelling of one step of the machine Z_0' always one step of the machines Z_j' $(j=1,\dots,m)$, which are obtained from Z_j $(j = 1,\dots,m)$ by replacing of S_i $(i=1,\dots,n)$ by S_i^j, will be modelled. For the stop of Z' there are analogical possibilities as in point 2 .

5) <u>C l e a n   m a c h i n e s .</u> The pair $\langle nZ\rangle \in \mathcal{Z}$ is called clean if any finite computation of the machine Z from any instantaneous description $(q_1;\ 0,\ X_n)$ ends by an instaneous description $(q_i;\ y,\ X_n)$ for some q_i and some $y \in N$. If n is known from the context, we call a machine Z clean, if the pair $\langle nZ\rangle$ is clean. Clean machines are proper e.g. when a function f is recursively defined from a function g and so it is necessary to compute the values of g with arguments only slightly changed. We shall find now (for the given $n \in N$) to the machine Z_1 a clean machine Z which will compute the same function whereby the computation time is prolonged only linearly. Let us put $m = 2$, $Z_2 = \{\langle q_1S_1q_1q_1\rangle\}$ in the construction of the point 4 and let us arrange the machine Z described there in such a way that after the end of the modelled work of the machine Z_1, Z shall renovate the contents of the cells $S_1,\dots,S_n$ by adding the contents of the cells $S_1^2,\dots,S_n^2$ to them. (The method from the point 2 does not guarantee that during the modelling of the work of Z_1 the sums of the contents of the cells S_j, S_j^2 will be equal to the original contents of the cells S_j, but this can be easily arranged.) Then Z puts zero in all memory cells with the exception of $S_0, S_1,\dots,S_n$. It holds $\Phi_Z^n = \Phi_{Z_1}^n$ and there exist $k_1,\ k_2 \in N$ such that $\mathcal{T}_Z^n \leq k_1 .\ \mathcal{T}_{Z_1}^n + k_2$.

<u>R e m a r k .</u> Even in this case it would not be proper to transcribe at once the whole inputs data.

We shall now introduce an application of the above mentioned processes.

T h e o r e m 6.1. Let be h_1, h_2 in $\mathcal{P}$, $h = \min^* (h_1,h_2)$. If an unary function f is both h_1-computable and h_2-computable then it is also h-computable.

P r o o f . We can limit ourselves to the case $f \neq \emptyset$. Let be $\Phi^1_{Z_1} = \Phi^1_{Z_2} = f$, $T'(h_1 \langle 1Z_1 \rangle) < \infty$, $T'(h_2, \langle 1Z_2 \rangle) < \infty$

Let a machine Z modell the work of Z_1 and Z_2 until the work of one of machines Z_1, Z_2 is finished. Then, if the value of f is defined, let Z transcribe this value into the cell S_0 and stop in the state q_0. If not, let Z stops in another state. It holds $\Phi^1_Z = f$ and there exist k_1, $k_2 \in N'$ such that $\mathcal{T}^1_Z \leqq k_1.\min(\mathcal{T}^1_{Z_1}, \mathcal{T}^1_{Z_2}) + k_2$. Therefore $k \in N$ exists such that $\mathcal{T}^1_Z \leqq k.\min(\mathcal{T}^1_{Z_1}, \mathcal{T}^1_{Z_2})$. For every $x \in N$ hold

$$\frac{\mathcal{T}^1_Z(x)}{h(x)} \leqq \frac{k.\mathcal{T}^1_{Z_1}(x)}{h_1(x)} + \frac{k.\mathcal{T}^1_{Z_2}(x)}{h_2(x)} .$$

Indeed, the greater member of the right side is greater or equal than the left side. Then $T'(h, \langle 1Z \rangle) \leqq k.T'(h_1, \langle 1Z_1 \rangle) + k . T'(h_2, \langle 1Z_2 \rangle) < \infty + \infty \leqq \infty$.

T h e o r e m 6.2. Let be h_1, h_2 in $\mathcal{P}$, $h = \min(h_1,h_2)$. If an unary function f is both h_1-computable and h_2-computable then it is also h-computable.

P r o o f . Following the theorem 6.1. the function f is $\min^* (h_1,h_2)$ - computable. It holds $h = \min(h_1,h_2) \gtrsim^* \min^* (h_1,h_2)$ and therefore f is h-computable according to the theorem 5.7.

R e m a r k . Theorems 6.1. and 6.2. do not hold for any n-ary function f. Let $f(x,y) = x + y + x^6 . \overline{sg} \mid x + 1 - y \mid$, $h_1(x) = x^3 + x^6 . \overline{sg}\ rest(x,2)$, $h_2(x) = x^3 + x^6 . rest(x,2)$. Then $h(x) = x^3$. The function f is not h-computable because of

$$\sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \frac{f(x,y)}{h(x)h(y)} \geqq \sum_{x=0}^{\infty} \frac{f(x,x+1)}{x^3.(x+1)^3} = \infty .$$

Otherwise, it is h_1-computable and h_2-computable as we shall see latter (f will be real-time computable). Now we note only that

$$\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\frac{f(x,y)}{h_1(x).h_1(y)} \leq \sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\frac{x+y}{x^3.y^3}+\sum_{x=0}^{\infty}\frac{x^6}{h_1(x)h_1(x+1)} \leq$$

$$\leq 4+\sum_{x=0}^{\infty}(x^6 : x^9) < \infty .$$

Analogical consideration can be made for the function h_2 .

§ 7 Almost Constant Functions

D e f i n i t i o n 7.1. An n-ary function f is called almost constant if there exists such an $a \in N$ that for all $x_1,\dots,x_n \in N$ it holds

(7.1) $f(x_1,\dots, x_n) = f(\min(x_1,a),\dots, \min(x_n,a))$.

The set of all almost constant functions will be denoted by $\mathcal{F}_c$.

L e m m a 1. Let us denote for any $n \in N$, $n \neq 0$ and any real number α :

$$A_n(\alpha) = \sum_{x_1=0}^{\infty}\dots\sum_{x_n=0}^{\infty}\frac{\min(x_1,\dots,x_n)}{[1+x_1^{\alpha}]\dots[1+x_n^{\alpha}]} ,$$

$$B_n(\alpha) = \sum_{x_1=0}^{\infty}\dots\sum_{x_n=0}^{\infty}\frac{1}{[1+x_1^{\alpha}]\dots[1+x_n^{\alpha}]} ,$$

$$h_0(x) = (1+x)\left[\log_2(x+2)\right]^2 ,$$

$$A_n = \sum_{x_1=0}^{\infty}\dots\sum_{x_n=0}^{\infty}\frac{\min(x_1,\dots,x_n)}{h_0(x_1)\dots h_0(x_n)} ,$$

$$B_n = \sum_{x_1=0}^{\infty}\dots\sum_{x_n=0}^{\infty}\frac{1}{h_0(x_1)\dots h_0(x_n)} .$$

Then it holds $B_n < \infty$, $A_n = \infty$,

$$B_n(\alpha) < \infty \equiv \alpha > 1 ,$$

$$A_n(\alpha) < \infty \equiv \alpha > 1 + \frac{1}{n} .$$

The proof is given in §12.

<u>Lemma 2.</u> For every $f \in \mathcal{F}_c$ there exist $\langle nZ \rangle \in \mathcal{Z}$ and $k \in N$ that $\Phi_Z^n = f$, $\mathcal{T}_Z^n \leqq k$.

<u>Proof.</u> Let hold (7.1); the machine Z can work in such a way that at first it ascertains (in no more than $n.(2a + 1)$ steps) the values of $\min(x_1,a),\dots, \min(x_n, a)$. According to which of the $(a+1)^n$ cases takes place, the machine Z forms the necessary content of S_0 and finishes the computations conveniently.

<u>Remark.</u> Z need not have working cells.

<u>Definition 7.2.</u> Two computations of a machine Z are called paralell ones if they are of the same length k, if they have the same states in the first places of their i-th members for all natural $i \leqq k$ and if in their every step the contents of the equal cells are changed by the same additive constants.

The paralellism of two computations of Z from the same state can be disturbed only by the using of a zero- or a minus-quadruple if the competent cell is exhausted in one computation but it is not exhausted in the other.

<u>Lemma 3.</u> Let be $\langle nZ \rangle \in \mathcal{Z}$, $k \in N$, let $\mathcal{T}_Z^n \leqq k$. Then $\mathcal{T}_Z^n$, Φ_Z^n are almost constant.

<u>Proof.</u> We put $b_i = \min(x_i, k + 1)$ for $i = 1,\dots,n$. The computations of Z from instantaneous descriptions $(q_1;\ 0,X_n)$, $(q_1;\ 0,B_n)$ are paralell and therefore $\Phi_Z^n(X_n) = \Phi_Z^n(B_n)$, $\mathcal{T}_Z^n(X_n) = \mathcal{T}_Z^n(B_n)$ q.e.d.

<u>Lemma 4.</u> Let $\langle nZ \rangle \in \mathcal{Z}$, let the function $\mathcal{T}_Z^n$ not to be bounded. Then there exists a partition of the index set $\{1,\dots,n\}$ on two disjoint but maybe empty) sets $\{i_1,\dots,i_k\}$, $\{j_1,\dots,j_{n-k}\}$ and numbers $b_1,\dots,b_{n-k}$ such that for all $x_1,\dots,x_n$ such that $x_{j_m} = b_m$ $(m = 1,\dots,n-k)$ holds

$$\mathcal{T}_Z^n(x_1,\dots,x_n) \geqq \min(x_{i_1},\dots,x_{i_k})$$

(for $k = 0$ we put $\min(x_1,\dots,x_k) = \infty$).

P r o o f . If $\mathcal{T}^n_Z$ acquires the value ∞ , the statement is trivial. Let be $\mathcal{T}^n_Z < \infty$. Let us choose a sequence of n-tuples

(7.2.) $$((x_1^{(i)}, \ldots, x_n^{(i)}) \mid i = 1,2,3,\ldots)$$

such that

(7.3.) $$\lim_{i \to \infty} \mathcal{T}^n_Z(x_1^{(i)}, \ldots, x_n^{(i)}) = \infty$$

Let us denote by $i_1, \ldots, i_k$ the indices m for which the sequence $(x_m^{(i)} \mid i = 1,2,\ldots)$ is not bounded and by $j_1, \ldots, j_{n-k}$ the indices m for which that sequence is bounded. (In this case $k \neq 0$, of course.) There exist numbers $b_1, \ldots, b_{n-k}$ such that for every $m = 1, \ldots, n-k$ and for infinite number of i it holds $x_{j_m}^{(i)} = b_m$. We may assume (and we can obtain it by replacing the sequence (7.2) by a convenient subsequence) that for $m = i_1, \ldots, i_k$ the suquence $(x_m^{(i)} \mid i = 1,2,\ldots)$ are increasing and for $m = j_1, \ldots, j_{n-k}$ they are constant with members b_m. In order to make the description more simple let farther be $i_m = m$ for $m = 1, \ldots, k$ and $j_m = k + m$ for $m = 1, \ldots, n-k$.

Let for some $x_1, \ldots, x_k$ hold $\mathcal{T}^n_Z(X_k, B_{n-k}) < \min(X_k)$. For almost all i it holds $x_m^{(i)} \geqq x_m$. For those i the computations of Z from the instantaneous descriptions $(q_1;\ 0, X_k, B_{n-k})$ and $(q_1;\ 0, x_1^{(i)}, \ldots, x_k^{(i)}\ , B_{n-k})$ are paralell and therefore for almost all i

$$\mathcal{T}^n_Z(x_1^{(i)}, \ldots, x_k^{(i)}\ , B_{n-k}) = \mathcal{T}^n_Z(X_k, B_{n-k})$$

which is a contradiction with (7.3).

L e m m a 5. For $n \geqq 1$ holds $\min(X_{n+1}) \prec\prec \min(X_n)$.

P r o o f . Following the section 7 of paragraph 4 and the lemma 1 of this paragraph it holds $\min(X_{n+1})\ V\ [1 + x^{1+\frac{1}{n}}]$ but $\min(X_n)\ V\ [1 + x^{1+\frac{1}{n}}]$ does not hold. Then it is sufficient to prove the relation $\prec$. Let be $\min(X_n) V H$. Then

$$\sum_{x_1=0}^{\infty} \cdots \sum_{x_n=0}^{\infty} \frac{\min(X_n)}{h(x_1)\ldots h(x_n)} < \infty \text{ and } \sum_{x_{n+1}=0}^{\infty} \frac{1}{h(x_{n+1})} < \infty$$

By multiplying and using the comparative criterion we have

$$\sum_{x_1=0}^{\infty} \cdots \sum_{x_n=0}^{\infty} \sum_{x_{n+1}=0}^{\infty} \frac{\min(X_{n+1})}{h(x_1)\ldots h(x_{n+1})} < \infty$$

Then for a machine Z constructed according to the section 7 of §4 such that $\Phi_Z^{n+1} = \min(X_{n+1})$ holds $T'(h, \langle n+1, Z\rangle) < \infty$ and therefore $\min(X_{n+1}) V h$ q.e.d.

<u>Theorem 7.1.</u> Let f be an n-ary function, $n \neq 0$. The following conditions are equivalent:

1) f is almost constant,
2) f is h-computable for $h(x) = (x+1)\left[\log_2 (x+2)\right]^2$,
3) f is h-computable for $h(x) = \left[1 + x^{1+\frac{1}{n}}\right]$
4) $f \prec o$ (remark : $o(x) = 0$ for all $x \in N$) ,
5) $f \prec \min(x_1, \ldots, x_{n+1})$,
6) $f \prec\!\!\prec \min(x_1, \ldots, x_n)$,
7) there exists such a machine Z that $\Phi_Z^n = f$ and $\mathcal{T}_Z^n$ is bounded.

<u>Proof.</u> $1 \to 7$ is lemma 1 ; $7 \to 1$ follows from lemma 3; $7 \to 2$ follows from lemma 1 ; $2 \to 3$ follows from the comparative criterion; $4 \to 5$ follows from this same criterion; $5 \to 6$ follows from the lemma 5. We shall proof $3 \to 4$ and $6 \to 7$ also indirectly.

<u>$\neg 4 \to \neg 3$.</u> Let $\neg f \prec o$, Z is a machine, $\Phi_Z^n = f$. Following the lemma 6 there exist numbers $i_1, \ldots, i_k$, $j_1, \ldots, j_{n-k}$, $b_1, \ldots, b_{n-k}$ (fulfilling the conditions of this lemma) such that for $x_{j_m} = b_m$ $(m = 1, \ldots, n-k)$ it holds $\mathcal{T}_Z^n(X_n) \gtrsim \min(x_{i_1}, \ldots, x_{i_k})$. Then following the lemma 1 $T'(h, \langle nZ\rangle) = \infty$ and therefore f is not h-computable.

<u>$\neg 7 \to \neg 6$.</u> Let $\neg 7$ hold, let be Z a machine such that $\Phi_Z^n = f$. The function $\mathcal{T}_Z^n$ is not bounded and therefore following the lemma 6 it holds $\mathcal{T}_Z^n(X_n) \gtrsim \min(x_{i_1}, \ldots, x_{i_k})$ (the assumptions are the

same as in the lemma 6). The machine Z was arbitrary and therefore $\min(X_n) \prec f$. If $f \ll \min(X_n)$ then $\min(X_k) \ll \min(X_n)$ for some $k \leq n$ which is a contradiction with lemma 5.

E x a m p l e s . The functions $\emptyset$, sg, $\overline{sg}$, rest(k,x), $k \dot{-} x$, min(x,k), k-x are almost constant; the function rest(x,k) is not.

As a conclusion we give here a sufficiently trivial theorem without the proof.

T h e o r e m 7.2. Let f be an n-ary function ($n \neq 0$), $g_1, \ldots, g_n$ m-ary almost constant functions. Then the superposition of the functions f, $g_1, \ldots, g_n$ is an almost constant function.

R e m a r k . The assumption $g_1, \ldots, g_n \in \mathcal{F}_c$ cannot be replaced even for $n = 1$ by the assumption $f \in \mathcal{F}_c$ as can be seen from the example $f = sg$, $g_1 = q$.

§ 8. Functions Admissible for the Superposition

In this paragraph we shall show that the sum and the difference of h-computable functions is again h-computable (on the contrary the product has not this property). We need not to prove such theorems separately for the sum, for the difference and for other functions but we can consider the whole class of all these operations. For this reason we introduce the following definition.

D e f i n i t i o n 8.1. a) An n-ary function f is called admissible for the superposition if $n \neq 0$ and for every $h \in \mathcal{P}$ and every n-tuple of m-ary h-computable functions $g_1, \ldots, g_n$ the superposition of the functions f, $g_1, \ldots, g_n$, is also h-computable.

b) A function f is called effectively admissible for the superposition (sometimes shortly "eas (function) ") if there exists a pair $\langle nZ \rangle \in \mathfrak{Z}$ and $k \in N$ such that $n \neq 0$, $\Phi^n_Z = f$ and for every $x_1, \ldots, x_n \in N$ $\mathcal{T}^n_Z(x_1, \ldots, x_n) \leq k.(1 + x_1 + \ldots + x_n)$.

c) The set of all functions admissible for the superposition resp. effectively admissible for the superposition will be denoted by $\mathcal{F}_s$, resp. $\mathcal{F}_{es}$.

The case $n = 0$ is excluded because for this case we have not defined the superposition; of course, the constant functions (for example $\emptyset$ and o as unary functions) are not excluded. Now it is possible to express the statement "for every $h \in \mathcal{P}$, the sum of h-computable functions is h-computable" by the phrase "the function $x + y$ is admissible for the superposition". As we shall see from the proof of the following theorem, the statement "the function $x + y$ is effectively admissible for the superposition" means that from the machines Z_1, Z_2 such that $T'(h, <nZ_1>) < \infty$, $T'(h, <nZ_2>) < \infty$ a machine Z can be effectively obtained such that

$T'(h, <nZ>) < \infty$ and $\Phi^n_Z = \Phi^n_{Z_1} + \Phi^n_{Z_2}$

<u>T h e o r e m 8.1.</u> Any function effectively admissible for the superpostition is admissible for the superposition.

<u>P r o o f .</u> Let be $<nZ_0> \in \mathcal{Z}$, $n \neq 0$, $k \in N$ and for all X_n

$\mathcal{T}^n_Z(X_n) \leq k.(1 + X_n^+)$. Let $h \in \mathcal{P}$, $<mZ_i> \in \mathcal{Z}$, $T'(h, <mZ_i>) < \infty$

for all $i = 1,\dots,n$. It is necessary to prove that

$S^{n+1}(\Phi^n_{Z_0}, \Phi^m_{Z_1},\dots, \Phi^m_{Z_n}) \vee h$.

Let the machine Z first to modell by the method 4) from §6 the work of the machines $Z_1,\dots,Z_n$ from the common input data in the cells $S_1,\dots,S_m$ and the results are put into the cells $S_0^1,\dots, S_0^n$. If all the machines $Z_1,\dots, Z_n$ finish their work (and all the due values of the functions $\Phi^m_{Z_1},\dots, \Phi^m_{Z_n}$ have been defined) let Z work as a conveniently address substituted machine Z_0. The input data are taken from the cells $S_0^1,\dots,S_0^n$. If some of the values of the functions $\Phi^m_{Z_i}$ are not defined the work of Z ends after the first phase. So we obtain

$$\Phi^m_Z = S^{n+1}(\Phi^n_{Z_0}, \Phi^m_{Z_1},\dots, \Phi^m_{Z_n}).$$

The first phase of the computing of Z has at most $k_1.\max \mathcal{T}^n_{Z_i} + k_2$ steps (for convenient $k_1, k_2 \in N$). The second phase has at most $k.(1 + \Phi^m_{Z_1} + \dots + \Phi^m_{Z_n})$ steps and therefore it holds

$$\mathcal{T}^m_Z \leq (k_1 + k_2 + k).(\mathcal{T}^m_{Z_1} + \dots + \mathcal{T}^m_{Z_n} + 1).$$

If $T(h) < \infty$ then this evaluation implies $T'(h, < nZ >) < \infty$.
If $T(h) = \infty$ then $\Phi^{m}_{Z_1} = \emptyset$ and therefore $\Phi^{m}_{Z} = \emptyset$; clearly Φ^{m}_{Z} V h.

R e m a r k . Let be $g = S^{n+1}(f, g_1, \ldots, g_n)$, $f \in \mathcal{F}_s$. If $H \subseteq \mathcal{P}$ and $g_1, \ldots, g_n \in H^f$ then $g \in H^f$. If $g_i \prec f'$ holds for $f' \in \mathcal{F}$ then $g \prec f'$. However, such statements are not valid for $H^F, \prec\prec$ instead of $H^f, \prec$.

T h e o r e m 8.2. The superposition of functions (effectively) admissible for the superposition is also (effectively) admissible for the superposition.

We are not giving the proof because this can be done from the definition directly.

T h e o r e m 8.3. The following functions are effectively admissible for the superposition:

a) All the almost constant functions with the exception of the 0-ary ones,

b) The functions I^n_m, $x + y$, $x - y$, $x \dot{-} y$, $[x : y]$, $D(x,y)$, $[\sqrt{x}]$, $[\log_y x]$, $\min(x_1, \ldots, x_n)$, $x \dot{-} yz$.

c) The functions s, $|x - y|$, $rest(x,y)$, $[x : k]$, q, l, r, Γ .

P r o o f . a) Following the definition (8.1) and the theorem 7.1.

b) Following the definition 8.1. and with the help of the machines from §4.

c) Following the theorem 8.2. For example $q(x) = x \dot{-} [\sqrt{x}]^2$ so $q = S^4(x \dot{-} yz, x, [\sqrt{x}] , [\sqrt{x}])$ and therefore $q \in \mathcal{F}_{es}$.

R e m a r k . The function $x \cdot y$ is not admissible for the superposition because e.g. $x^4 = S^3(xy, x^2, x^2)$, x^2 V x^4 but $\neg x^4$ V x^4. However, the product of some functions will be expressed with the help of $x \dot{-} yz$. E.g. $[\sqrt{x}] \cdot [\log_2 x] = x \dot{-} (x \dot{-} [\sqrt{x}] \cdot [\log_2 x])$.

Further eas functions can be obtained with the help of some theorems from § 10.

T h e o r e m 8.4. a) If f is admissible for superposition then $f \prec I^1_1$. (Remark : $I^1_1(x) = x$ for all $x \in N$.)

b) For the unary function f the following conditions are equivalent:

1) $f \prec I_1^1$.

2) f is admissible for the superposition.

3) f is effectively admissible for the superposition.

P r o o f . a) Let be f an n-ary function $f \in \mathcal{F}_s$. Then $f = S^{n+1}(f, I_1^n, \ldots, I_n^n)$, $I_k^n \prec I_1^1$ for $k = 1, \ldots, n$ and therefore $f \prec I_1^1$ holds.

b) It is sufficient to prove $1 \rightarrow 3$. Let us take such a function $h \in \mathcal{P}$ that $T(h, I_1^1) < \infty$ and for every unary recursive function f

$$\limsup \frac{f(x)}{x} = \infty \quad \text{implies} \quad \sum_{x=0}^{\infty} \frac{f(x)}{h(x)} = \infty .$$

There exists such a machine Z that $\phi_Z^1 = f$, $T'(h, \langle 1Z \rangle) < \infty$ Then $\limsup (\mathcal{T}_Z^1(x) : x) < \infty$ and therefore f is eas function. (The proof of the existence of the function h with the required properties is given in § 12.)

§ 9. Real - Time Computable Functions

D e f i n i t i o n 9.1. A total function f is called real-time computable (sometimes shortly "rtc (function)") if there exists a pair $\langle nZ \rangle \in \mathcal{Z}$ and $k \in N$ such that $\phi_Z^n = f$ and for all $x_1, \ldots, x_n \in N$ it holds:

$$(9.1.) \qquad \mathcal{T}_Z^n(x_1, \ldots, x_n) \leqq k.(1 + \phi_Z^n(x_1, \ldots x_n)).$$

We denote the set of all real-time computable functions by $\mathcal{F}_r$.

This definition is clearly not equivalent with the definition given in [2] . We shall deal with the comparison later on. In the definition 9.1. it is sufficient to require that the relation (9.1) will hold for almost all $x_1, \ldots, x_n \in N$.

L e m m a 1. For every $\langle nZ \rangle \in \mathcal{Z}$ there exists a machine Z_1 and $m \in N$ such that

1) Z_1 does not use the cell S_0 ,

2) $\mathcal{T}_{Z_1}^n = 2\,\mathcal{T}_Z^n$,

and for every $x_1, \ldots, x_n$ such that $\mathcal{T}_Z^n(x_n) < \infty$

3) the computation of Z_1 from $(q_1;\ 0,X_n)$ ends in the same state as the computation of Z from $(q_1;\ 0,X_n)$,

4) final contents of S_m, S_{m+1} by the computation of Z_1 from $(q_1;\ 0,X_n)$ are equal to $\mathcal{T}^n_Z(X_n)$ and to the final content of S_0 by the computation of Z from $(q_1;\ 0,X_n)$.

P r o o f . If $\mathcal{T}^n_Z = 0$, then it is sufficient to put $Z_1 = \emptyset$. For the remaining cases it is possible to apply the method from the point 2 of §6.

L e m m a 2. For every $k \in N$ there exists a machine Z such that for all $x,\ y \in N$, $1 \leq x < k(y+1)$ it holds $\Phi^2_Z(x,y) = y$, $\mathcal{T}^2_Z(x,y) = (6k+6).(y+1) - 2x$.

P r o o f . We are going to introduce only some instantaneous descriptions through which the computation of Z passes if $1 \leq x < k(y+1)$. From these it will be clear how to construct the machine Z. These instantaneous descriptions are:

$(q_1;\ 0,x,y)$, $(q_2;\ y,x,0,y)$, $(q_3;\ y,x,0,y+1)$, $(q_4;\ y,x,k(y+1)$, $(q_5;\ y,0,ky+k-x)$, $(q_6;\ y,0,ky+k-x-1)$, $(q_7;\ y,ky+k-x-1)$, $(q_8;\ y)$, $(q_9;\ y)$, $(q_0;\ y)$.

The whole length of the computation of the machine Z is

$\mathcal{T}^n_Z(x,y) = (4y+1) + 1 + (ky+2y+k+3) + (3x+1) + 1 + (3ky+3k-3x-2) + (2ky+2k-2x-1) + 1 + 1 = 6ky + 6y - 2x + 6k + 6$, q.e.d.

D e f i n i t i o n 9.2. A pair $\langle nZ\rangle \in \mathcal{Z}$ is called regular output pair with ouptup in every k-th step, if

1) The function Φ^n_Z is total.

2) S_0 is an output cell of Z.

3) In the computing of Z from any $(q_1;\ 0,x_1,\dots,x_n)$ we add unit into S_0 in m-th step just when m is a multiple of k.

4) For all $x_1,\dots,x_n \in N$ it holds

$$\text{rest}\ (\mathcal{T}^n_Z(x_1,\dots,x_n)\ ,\ k) = k - 1\ .$$

If the value of k is not important we shall talk shortly about a regular output pair. If the value of n will be known from the context we shall talk about a regular output machine (with output in every k-th step).

Lemma 3. To any n-ary total function f and to any pair $\langle nZ \rangle \in \mathcal{Z}$ such that $\mathcal{T}^n_Z = k.(1 + f)$ there exists a regular output pair $\langle nZ_1 \rangle$ with output in every $(k+1)$-th step such that $\Phi^n_{Z_1} = f$.

Proof. The machine Z_1 can work in such a way that it always modells at first k steps of the (conveniently readressed) machine Z and in the $(k+1)$-th step adds 1 to S_0 if the computation of Z is not yet finished. After this it is modelling farther k steps of the work of Z, it adds again one unit to S_0 and so on.

Remark. We can replace the $(k+1)$ by any $m \in N$, $m > k$ in the lemma 3.

Theorem 9.1. For any total function f the following conditions are equivalent:

1) f is real-time computable.
2) There exists $k \in N$ and $\langle nZ \rangle \in \mathcal{Z}$ such that $\Phi^n_Z = f$, $\mathcal{T}^n_Z = k.(1 + \Phi^n_Z)$.
3) There exists $k \in N$ and $\langle nZ \rangle \in \mathcal{Z}$ such that $f = [\mathcal{T}^n_Z : k]$.
4) There exists a regular output pair $\langle nZ \rangle$ such that $\Phi^n_Z = f$.

Proof. $1 \to 2$. Let be $\langle nZ_1 \rangle \in \mathcal{Z}$, $\Phi^n_{Z_1} = f$, $\mathcal{T}^n_{Z_1} \leqq k.(1 + \Phi^n_{Z_1})$. Let the machine Z work in the first phase as lemma the machine from lemma 1, in the second phase as a machine from the 2 with the address subtitution of the cells S_0, S_1, S_2, S_3 into S_0, S_m, S_{m+1}, S_{m+2}. The length of the first phase of the computation is $2\mathcal{T}^n_{Z_1}$, of the second one $(6k + 6).(\Phi^n_Z + 1) - 2\mathcal{T}^n_{Z_1}$, together $(6k + 6).(1 + \Phi^n_{Z_1})$. It holds $\Phi^n_{Z_1} = \Phi^n_Z = f$ and therefore $\mathcal{T}^n_Z = (6k + 6).(1 + f)$ q.e.d.

$2 \to 3$ and $4 \to 1$ are trivial.

$3 \to 4$. Let $f = [\mathcal{T}^n_Z : k]$. If $k = 0$ then $f = 0$ and it is rtc.

Therefore we can suppose that $k \neq 0$. The machine Z can be adapted into a machine Z_1 such that $\mathcal{T}^n_{Z_1} = k(1+f)$, $\Phi^n_{Z_1} = f$ in a following way. It holds $\mathcal{T}^n_{Z_1} = \mathcal{T}^n_Z + (k - \text{rest}(\mathcal{T}^n_Z, k))$. The machine Z_1 will work like the machine Z and simultaneously it follows (with the help of a larger number of states) also the function rest (p,k) where p is the number of the steps made already. After the end of the work of Z the machine Z_1 adds the necessary number of steps to fulfil the equation $\mathcal{T}^n_{Z_1} = k(1+f)$. It is sufficient now to apply (to the Z_1) the lemma 3.

<u>Theorem 9.2.</u> A bounded function is real-time computable if and only if it is both total and almost constant.

The proof is easy.

<u>Theorem 9.3.</u> Any superposition of real-time computable functions is real-time computable itself.

<u>Proof.</u> It is sufficient to show that if $\langle nZ_0 \rangle$, $\langle mZ_1 \rangle$, ..., $\langle mZ_n \rangle$ are regular output pairs with the output in every k-th step, there exists a machine Z such that $\Phi^n_Z = S^{n+1}(\Phi^n_{Z_0}, \Phi^m_{Z_1}, \dots, \Phi^m_{Z_n})$ and it holds $\mathcal{T}^n_Z \leqq k_1 \cdot (1 + \Phi^n_Z)$. The machine Z can work so that it always modells k steps of the computations of the machines $Z_1, \dots, Z_n$ from common input data taken from $S_1, \dots, S_m$. The results are put into $S'_1, \dots, S'_n$. Then the machine Z modells one step of the work of Z_0 with using $S'_1, \dots, S'_n$ instead of $S_1, \dots, S_n$. By this procedure the values of the functions $\Phi^m_{Z_1}, \dots, \Phi^m_{Z_n}$ are created quickly enough in order to modell the computation of Z_0 correctly. If some of the machines $Z_1, \dots, Z_n$ ends its work, some useless steps instead of its work can be done. The machine Z' stops if the modelling of the computation of Z_0 is finished.

<u>Theorem 9.4.</u> The following function are real-time computable:

a) I^n_m, $x + y$, $x \dot{-} k$, $[x : k]$, $\min(x,y)$, $[\sqrt{x}]^2$, c, $x.y$, $n(x,y)$,

b) $x!$, x^y.

<u>Proof.</u> In the major part it is sufficient to consider the machines from §4 and the theorems 9.2., 9.3. The function $n(x,y)$

must be computed following the formula $n(x,y) \doteq (x : D(x,y)).y$ (it is not allowed to use the formula $n(x,y) = (x.y) : D(x,y)$. The machines for $x!$, x^y can also be easily constructed but the real-time computability of those functions will follow from the theorems of § 10.

D e f i n i t i o n 9.2. For a function $h \in \mathcal{P}$ and an n-ary function f we define

$$\Sigma(h,f) = \sum_{x_1=0}^{\infty} \cdots \sum_{x_n=0}^{\infty} \frac{f(x_1,\ldots,x_n)}{h(x_1)\ldots h(x_n)} \qquad \text{if } n \neq 0,$$

$$\Sigma(h,f) = \sum_{x=0}^{\infty} \frac{f}{h(x)} \qquad \text{if } n = 0.$$

E.g. it holds $T'(h, \langle nZ\rangle) = \Sigma(h, \mathcal{T}^n_Z)$, $T(h) = \Sigma(h,1)$.

T h e o r e m 9.5. A real-time computable function f is h-computable if and only if $T(h) < \infty$ and $\Sigma(h,f) < \infty$

P r o o f . Let be $\langle nZ\rangle \in \mathcal{Z}$, $\Phi^n_Z = f$, $\mathcal{T}^n_Z \leq k.(1 + f)$. It holds $T'(h, \langle nZ\rangle) \leq k.(\Sigma(h,f) + T(h)) < \infty$ and so $f V h$. Conversely, if $f V h$ then $T(h) < \infty$. Let $T'(h, \langle nZ\rangle) < \infty$. Then $\Sigma(h,f) \leq \Sigma(h, \mathcal{T}^n_Z) = T'(h, \langle nZ\rangle)$.

C o r o l l a r y . If $f_1 < f_2$, f_1 is real-time computable and f_2 is h-computable then f_1 is also h-computable.

T h e o r e m 9.6. Let f be real-time computable, g be both total and almost constant. Then the functions $f \dot{-} g$, $[f : g]$ are real-time computable.

P r o o f . Let be $k \in N$, $g \leq k$. Then $k \dot{-} g \in \mathcal{F}_c$, $k \dot{-} g$ is total and therefore $k \dot{-} g$ is rtc function. Finally, $f \dot{-} g = (f + (k \dot{-} g)) \dot{-} k$ is rtc function as a superposition of rtc functions. For the other functions it holds

$[f : g] \doteq \sum_{m=1}^{k} [f : m] . \overline{sg} \mid g - m \mid$ and so it is a sum of products of rtc functions. Therefore it is a rtc function.

T h e o r e m 9.7. a) Let be $h \in \mathcal{P}$, $T(h) < \infty$ $h > 0$. Then any function f is h-computable if and only if it is a difference of two real-time computable and h-computable functions.

b) Let $H \subseteq \mathcal{P}$, let for every $h \in H$ hold $T(h) < \infty$, $h > 0$. Then any function is weakly H-computable if and only if it is a difference of two real-time computable and weakly H-computable functions.

P r o o f . a) Let be fVh. There exists a pair $\langle nZ \rangle \in \mathcal{Z}$ such that S_0 is an output cell of Z, $\Phi^n_Z = Cs(f)$, $T'(h, \langle nZ \rangle) < \infty$. It holds $\mathcal{T}^n_Z < \infty$. Let the machine Z' be obtained from Z by replacing of any quadruple $\langle q_i S_1 p q_j \rangle$ by the quadruples $\langle q_i S_1 P q_i' \rangle$, $\langle q_i' S_1 P q_j \rangle$. The functions $\mathcal{T}^n_{Z'}$, $\mathcal{T}^n_Z + 1$ are both rtc and h-computable and it holds $\mathcal{T}^n_{Z'} - (\mathcal{T}^n_Z + 1) = (\mathcal{T}^n_Z + Cs(f)) - (\mathcal{T}^n_Z + 1) = Cs(f) - 1 = f$ q.e.d.

b) If f is weakly H-computable there exists $h \in H$ such that fVh. Following the point a) there exist $f_1, f_2 \in \mathcal{F}_r \cap \{h\}^F$ such that $f = f_1 - f_2$. Clearly, $f_1, f_2 \in \mathcal{F}_r \cap H^F$.

Conversely, let $f = f_1 - f_2$, $f_1, f_2 \in H^F \cap \mathcal{F}_r$. Then $\min(1 + f_1, f_2)$ f_2 an rtc function. There exists an $h \in H$ such that $(1 + f_1)$Vh. Then $\min(1 + f_1, f_2)$Vh. It holds $f = f_1 - \min(1 + f_1, f_2)$ and so fVh. Then the sooner is $f \in H^F$.

From the theorem 9.2. and directly from the definition 7.1. the following theorems can be proved (analogically as the theorem 9.7.).

T h e o r e m 9.8. A function is almost constant if and only if it is a difference of two almost constant real-time computable fuctions.

T h e o r e m 9.9. A function is effectively admissible for the superposition if and only if it is a difference of two real-time computable functions effectively admissible for the superposition.

§ 10. Other Properties of the Real-Time Computable Functions

In this section we shall apply the theorems about the real-time computable functions to the so called almost recursive functions and after that we shall consider the rather properties of the set $\mathcal{F}_r$, in particular its closeness against some operations with the functions and the relation to the set $\mathcal{F}_y$ of all yamadan functions (i.e. the functions considered in [2]).

Definition 10.1. A function f will be called almost recursive if it is partial recursive and its domain is a recursive set.

Lemma 1. a) There exists a function $h \in \mathcal{P}$, $h > 0$ such that for any unary recursive function g it holds $\lim(g(x) : h(x)) = 0$.

b) For any function h fulfilling the conditions of the point a) and for every unary recursive function g it holds $\sum_{x=0}^{\infty} (g(x) : h(x)) < \infty$.

Proof. The statement a) is well known. For the statement b) it is sufficient to consider that for almost all $x \in N$ it holds $h(x) \geq x^2 . g(x)$.

Lemma 2. Let $h \in \mathcal{P}$ fulfil the conditions from the lemma 1. Then for every $\langle nZ \rangle \in \mathcal{Z}$ $T'(h, \langle nZ \rangle) < \infty \equiv \mathcal{T}^n_Z < \infty$.

Proof. If $T'(h, \langle nZ \rangle) < \infty$ then because $h > 0$ $\mathcal{T}^n_Z$ cannot reach the value ∞. Conversely, let $\mathcal{T}^n_Z < \infty$. Then the function $\mathcal{T}^n_Z$ is recursive. Let us denote

$$f(x) = \sum_{x_1=0}^{x} \cdots \sum_{x_n=0}^{x} \mathcal{T}^n_Z(x_1, \ldots, x_n) ,$$

$$g(x) = f(x) \dot{-} f(x \dot{-} 1).sg(x).$$

The function g is recursive. It holds $T'(h, \langle nZ \rangle) \leq$

$$\leq \sum_{x_1=0}^{\infty} \cdots \sum_{x_n=0}^{\infty} \frac{\mathcal{T}^n_Z(x_1, \ldots, x_n)}{h(\max(x_1, \ldots, x_n))} \leq \sum_{x=0}^{\infty} \frac{g(x)}{h(x)} < \infty \quad \text{q.e.d.}$$

Theorem 10.1. Let be $h \in \mathcal{P}$, $h > 0$ and let for every unary recursive function g hold $\lim(g(x) : h(x)) = 0$. The following conditions are equivalent:

1) f is almost recursive.
2) $Cs(f)$ is recursive (remark: $Cs(f)$ is the total function such that $Cs(f) - 1 = f$).
3) f is h-computable.
4) f is a difference of two real-time computable functions.

P r o o f . $1 \rightarrow 2$ is in essence a known statement; the same also $4 \rightarrow 1$ if we consider that every real-time computable function is recursive; $3 \rightarrow 4$ is a consequence of the theorem 9.7. There remains to prove $2 \rightarrow 3$. If $Cs(f)$ is recursive then exists a pair $\langle nZ \rangle \in \mathcal{Z}$ such that $\Phi^n_Z = Cs(f)$. It holds $\mathcal{T}^n_Z < \infty$ and therefore following the lemma 2 it holds $T'(h, \langle nZ \rangle) < \infty$.

T h e o r e m 10.2. Let f, g be n-ary real-time computable functions, let
$$\liminf \frac{f(x_1,\dots,x_n)}{g(x_1,\dots,x_n)} > 1 .$$

Then the function $f \dot{-} g$ is real-time computable.

P r o o f . Let be $k_1, k_2 \in N$, $\langle nZ_1 \rangle$, $\langle nZ_2 \rangle \in \mathcal{Z}$, $\Phi^n_{Z_1} = f$, $\Phi^n_{Z_2} = g$, $\mathcal{T}^n_{Z_1} \leqq k_1(1 + f)$, $\mathcal{T}^n_{Z_2} \leqq k_2(1 + g)$. Let be $m \in N$, $m \neq 0$, $\liminf(f:g) > 1 + \frac{1}{m}$. There exists $k_3 \in N$ and $\langle nZ \rangle \in \mathcal{Z}$ such that $\Phi^n_Z = f \dot{-} g$, $\mathcal{T}^n_Z \leqq k_3(\mathcal{T}^n_{Z_1} + \mathcal{T}^n_{Z_2}) + 3f + 3g + 2$ and therefore for convenient $k \in N$ $\mathcal{T}^n_Z \leqq k(1 + f + g)$. It holds $f : g > 1 + (1 : m)$, i.e. $(2m + 1).(f \dot{-} g) > f + g$ for almost all X_n. For almost all X_n $\mathcal{T}^n_Z \leqq k(1 + (2m + 1)(f \dot{-} g)) \leqq k.(2m + 1)(1 + (f \dot{-} g))$ and therefore $f \dot{-} g$ is a rtc function.

C o r o l l a r y . Let be $f(x, X_n)$ real-time computable function, let
$$\liminf \frac{f(x+1, x_1,\dots,x_n)}{f(x, x_1 \dots, x_n)} > 1 .$$

Then $g = Dif(f)$ is real-time computable (remark : $g(x, X_n) = f(x, X_n) \dot{-} f(x \dot{-} 1, X_n).sg(x)$).

T h e o r e m 10.3. Let be $g'(X_n)$, $g(x, y, X_n)$ real-time computable functions, let the function $f(x, X_n)$ arise from g', g by the primitive recursion. Then the function
$$f'(x, X_n) = \sum_{y=0}^{x} (1 + f(y, X_n))$$

is real-time computable. (Remark: $f(0, X_n) = g'(X_n)$, $f(x + 1, X_n) =$ $= g(x, f(x,X_n), X_n)$.)

P r o o f . Let be $k_1 \in N$, let be Z_1, Z_2 machines such that $\Phi^n_{Z_1} = g'$, $\Phi^{n+2}_{Z_2} = g$, $\mathcal{T}^n_{Z_1} \leq k_1(1 + g')$, $\mathcal{T}^n_{Z_2} \leq k_1(1 + g)$.

By applying the methods of §6 we can find $k \in N$ and the machines Z_3, Z_4 for which is S_0 an output cell, which do not use the cell S_1 and the state q_4 and such that for all y, X_n :

1) There exists a computation of Z_3 from $U_1 = (q_1;\ 0, 0, X_n)$ to $U_2 = (q_3;\ q'(X_n) + 1,\ 0,\ X_n,\ 0,\ g'(X_n))$ and it holds $L(Z_3;\ U_1, U_2) \leq k(1 + g'(X_n))$.

2) There exists a computation of Z_4 from $U_3 = (q_2;\ 0, 0, X_n, x, y)$ to $U_4 = (q_3;\ g(x,y,X_n) + 1,\ 0,\ X_n,\ x + 1,\ g(x,y,X_n))$ and it holds $L(Z_4;\ U_3, U_4)\ \ k(1 + g\ (x,y,X_n))$.

3) U_2, U_4 are final instantaneous descriptions of Z_3, Z_4, respectively. Z_3, Z_4 have no common cells and no common states with the exception $S_0, S_2, \ldots, S_{n+1}, S_{n+3}, q_3$. (It is in assent with the above-mentioned agreements.)

Let us put now $Z = Z_3 \cup Z_4 \cup \{ < q_3S_1q_4q_0 >,\ < q_4S_1Mq_2 > \}$ and consider the function Φ^{n+1}_Z, $\mathcal{T}^{n+1}_Z$. Let us consider the computation of Z from $(q_1;\ 0, x, X_n)$. First the value $g'(X_n) = f(0,X_n)$ is computed (by the help of Z_3), then (by the help of Z_4) the values $f(1,X_n), f(2,X_n), \ldots, f(x,X_n)$. These values appear in the cell S_{n+3}. Besides they are also put (together with the added units) into the cell S_0. Therefore it holds $\Phi^{n+1}_Z = f'$. For the length of the computation it holds $\mathcal{T}^{n+1}_Z(x, X_n) \leq k(1 + g'(X_n)) + 2 + k(1 + f(1,X_n)) + 2 + \ldots + k(1 + f(x-1, X_n)) + 2 + k(1 + f(x, X_n)) + 1$.

We have $\mathcal{T}^{n+1}_Z \leq k.f'(x,X_n) + 2x + 1 \leq (k + 2)(1 + f')$ and therefore f' is a rtc function.

By the same method as in the theorem 10.3. the following theorem could be proved.

<u>Theorem 10.4.</u> Let be $f(x,X_n)$ a real-time computable function. Let the function $g(X_n)$ arise from the function $f(x,X_n)$ by the regular minimalisation. Then the functions

$$(10.1) \qquad g'(X_n) = \sum_{y=0}^{g(X_n)} f(y,X_n)$$

is real-time computable. (Remark: $g(X_n)$ is the least natural number u such that $f(u,X_n) = 0$; by the regular minimalisation it must exist.)

We are not giving the proof. We only note that we need not to form the sum of the values $1 + f(y,X_n)$ because all the terms $f(y,X_n)$ in (10.1) with the exception of the last one are non-zero.

<u>Theorem 10.5.</u> If a function $f(x,X_n)$ is real-time computable then the function $g_1 = \mathrm{Sum}(f)$, i.e.

$$g_1(x,X_n) = \sum_{y=0}^{x} f(y,X_n)$$

is also real-time computable.

<u>Proof.</u> Let us put in the theorem 10.3. $g(x,y,X_n) = f(y+1,X_n)$, $g''(X_n) = f(0,X_n)$. Then $f'(x,X_n) = 1 + x + g_1(x,X_n)$ is also rtc. Let us consider the function $\mathrm{sg}\, f(y,X_m)$. It is bounded and rtc and therefore it is almost constant. There exists $a \in N$ such that for $x' = \min(x,a)$, $b_i = \min(x_i,a)$ $(i = 1,\dots, n)$ it holds $\mathrm{sg}\, f(x,X_n) = \mathrm{sg}\, f(x', B_n)$. Let us put $\varphi(x,X_n) = \mathrm{sg}(f(x', B_n).\overline{\mathrm{sg}}(a \dot{-} x))$. Then φ is rtc, $\varphi(x,X_n) = 1$ if $x \geq a$ and $f(x',X_n) \neq 0$. In this case $f(y,X_n) > 0$ for every $y \geq a$. If $\varphi(x,X_n) = 0$ then $x < a$ or $f(x,B_n) = 0$. If $x \geq a$, the second relationship implies $f(y,X_n) = 0$ for every $y \geq a$. Therefore it holds (for every x, X_n) :

$$g_1(x,X_n) = \overline{\mathrm{sg}}\, \varphi(x,X_n). \sum_{k=0}^{a} (f(k,X_n).\mathrm{sg}\,(k \dot{-} x)) +$$

$$+ \varphi(x,X_n) . ((f'(x,X_n) + x.\overline{\mathrm{sg}}\, \varphi(x,X_n) \dot{-} (x+1)).$$

The functions $f'(x,X_n) + x.\overline{\mathrm{sg}}\, \varphi(x,X_n)$, $x + 1$ are rtc functions. Farther for every $x > a$ (and X_n arbitrary) it holds $f'(x,X_n) + x \cdot \overline{sg}\, \varphi(x,X_n) = 1 + x + g_1(x,X_n) + x\, \overline{sg}\, \varphi(x,X_n) \geqq$

$$\geqq x + \sum_{y=a}^{x} (f(y,X_n) + \overline{sg}\, \varphi(y,X_n)) \geqq 2x - a$$ and so

$\liminf ((f'(x,X_n) + x.\overline{sg}\, \varphi(x,X_n)) : (x + 1) \geqq 2 > 1$. Therefore $(f'(x,X_n) + x.\overline{sg}\, \varphi(x,X_n)) \dot{-} (x + 1)$ is a rtc function. All the other functions in the expression of g_1 are clearly rtc and therefore g_1 is also a rtc function.

L e m m a 3. Let $f(x,X_n)$ be a real-time computable function, $f \geqq 2$. Then the function $g_2 = Mult(f)$ is also real-time computable.

P r o o f . Following the theorem 10.3. the function g,

$$g(x,X_n) = 1 + x + \sum_{y=0}^{x} g_2(y,X_n)$$

is rtc. It holds $g_2(x,X_n) = g(x,X_n) \dot{-} (1 + sg\, x.g(x \dot{-} 1, X_n))$.
Now it is sufficient to use the theorem 10.2.

T h e o r e m 10.6. Let be a function $f(x,X_n)$ real-time computable. Then the function $g_2 = Mult(f)$, i.e.

$$g_2(x,X_n) = \prod_{y=0}^{x} f(y,X_n)$$

is also real-time computable.

P r o o f . Let be again $x' = min(x,a)$, $b_i = min(x_i, a)$ for $i = 1,\dots,n$ and let be a chosen such that for every x, X_n $sg(f(x,X_n) \dot{-} 1) = sg(f(x', B_n) \dot{-} 1)$. Let us put
$f'(x,X_n) = 2.sg\, f(x,X_n) + (1 + \overline{sg}(x \dot{-} a)).f(x,X_n)$,

$$\varphi(x,X_n) = \overline{sg}\, \left(\prod_{k=0}^{a} f(k,B_n)\right) + sg(a \dot{-} x) + sg(f(a,B_n) \dot{-} 1)$$

The functions f', φ are rtc. The equality $\varphi(x,X_n) = 0$ holds if and only if $x \leqq a$, $f(a,X_n) \leqq 1$ or when $f(k,X_n) = 0$ for some $k = 0,1,\dots, a$. It holds $f' \geqq 2$ and therefore $Mult(f')$ is a rtc function. We can write the function g_2 also in the following way:

$$g_2(x,X_n) = \overline{sg}\,\varphi(x,X_n).\prod_{k=0}^{a}(f(k,X_n).\overline{sg}(k \dot- x) + sg(k \dot- x)) +$$

$$+ sg\,\varphi(x,X_n).\left[\prod_{y=0}^{x} f'(x,X_n) : 2^{a+1}\right].$$

So g_2 is expressed by the help of superposition of rtc functions and therefore it is also a rtc function.

Corollary. The functions $x!$, x^y are real-time computable.

Proof. It holds $x! = Mult(x+\overline{sg}(x))$, $x^y = \overline{sg}(y) + sg(y).Mult(I_2^2(y \dot- 1, x))$ and $x + y$, $sg(x)$, $\overline{sg}(x)$, $x \dot- 1$, I_2^2 are rtc functions.

Theorem 10.7. Let be $f(x,X_n)$ a real-time computable function, let

$$f'(y,x,X_n) = x \dot- \sum_{z=0}^{y} f(z,X_n).$$

Let the function $g(x,X_n)$ arise from the function $f'(y,x,X_n)$ by the minimalisation. Then the function $g(x,X_n)$ is effectively admissible for the superposition. (Remark:

$$g(x,X_n) = \mu_y(f'(y,x,X_n) = 0) = \mu_y\left(\sum_{z=0}^{y} f(z,X_n) \geq x\right).)$$

Proof. Let x', b_i $(i = 1,\dots,n)$ have the same signification as in the above-mentioned proof. We choose $a \in N$ in such a way that for all x, X_n it holds $sg\,f(x,X_n) = sg\,f(x', B_n)$. The function $\varphi(x,X_n) = sg(f(x',B_n) \,.\, (x \dot- a))$ is almost constant. If $\varphi(x,X_n) \neq 0$ then the value $g(x,X_n)$ is defined and moreover for all $y \geq x$ it holds $f(y,X_n) \geq 1$. (This condition means that the machine Z a description of which we are going to give later on, will not compute uselessly such values of f which would not diminish the content of S_1.) The function

$$\psi(x,X_n) = \overline{sg}\,\varphi(x,X_n).\sum_{z=0}^{y} f(z,X_n) = \overline{sg}\,\varphi(x,X_n).\sum_{k=0}^{a} f(k,B_n)$$

is bounded. Therefore $k_1 \in N$ exists such that for all x, X_n such that $\varphi(x,X_n) = 0$ is $g(x,X_n)$ defined only if $x \leq k_1$. Therefore the function $\overline{sg}\,\varphi(x,X_n) \cdot g(x,X_n)$ is almost constant.

There exists a machine Z_1 and $k \in N$ such that

1) it does not use the cell S_0.
2) the cell S_1 is an output cell of Z_1.
3) for every instantaneous descriptions $U_1 = (q_3;\ 0,0,X_n,z)$, $U_2 = (q_4;\ 0,f(z,X_n),X_n,z+1)$ it holds $L(Z_1;\ U_1,U_2) \leq k(1 + f(z,X_n))$ and U_2 is a final instantaneous description of Z_1.

Let us adapt the machine Z_1 into a machine Z_2 in such a way that every quadruple $\langle q_i S_1 P q_j \rangle$ be replaced by the quadruples $\langle q_i S_1 M q_i' \rangle$ $\langle q_i' S_1 q_j q_0 \rangle$.

Now let be Z a machine such that by its computation from $(q_1;\ 0,x,X_n)$ it computes at first the value $\varphi(x,X_n)$. If $\varphi(x,X_n) \neq 0$ then Z passes to $(q_3;\ 0,x,X_n)$; if $\varphi(x,X_n) = 0$ then Z computes $g(x,X_n)$ directly, as a value of an almost constant function. The machine Z can be constructed in such a way that the above-described phase of its computation has bounded length. From the instantaneous description $(q_3;\ 0,x,X_n)$ let the machine Z compute like the machine $Z_2 \cup \{\langle q_4 S_1 q_5 q_0 \rangle, \langle q_5 S_0 P q_3 \rangle\}$. In this phase Z gradually computes the values $f(0,X_n)$, $f(1,X_n)$, ..., subtract them from S_1 and adds 1 to S_0 by every finished subtraction until the content of S_1 is exhausted. Therefore it holds $\Phi_Z^{n+1} = g$.

We can assume that by the second phase of computation of Z a unit is subtracted from the content of S_1 at least in every k_1-th step. Then there exist $k_2, k_3 \in N$ such that $\mathcal{S}_Z^{n+1} \leq k_2 \cdot x + k_3$ and therefore $g(x,X_n)$ is eas function q.e.d.

<u>T h e o r e m 10.8.</u> Let be $f(x_1,\dots,x_n)$ real-time computable function, $g_1,\dots, g_n$ m-ary functions effectively admissible for the superposition. Let g be the superposition of functions f, $g_1,\dots,g_n$ and let for some $k \in N$ and every $x_1,\dots,x_m$ hold

$$g(x_1,\dots,x_m) \leq k(1 + x_1 + \dots + x_m).$$

Then the function g is effectively admissible for the superposition.

P r o o f . The machine Z can work in such a way that it computes the values $g_1(X_m),\dots, g_n(X_m)$ from common inputs data in $S_1,\dots,S_m$. If all these values are defined then Z computes the value $g(X_m)$ from them. If not then Z stops. For the length of computation of Z the necessary evaluation will be fulfilled if the values $g_i(X_m)$ $(i = 1,\dots,n)$ and $f(g_1(X_m),\dots,g_n(X_m))$ are computed in a competent way. (E.g. by the modelling of computations of convenient machines.)

E x a m p l e s . The functions $(x + 1)^n - x^n$ is rtc because it is a polynomial with natural coefficients. It holds

$$\left[\sqrt[n]{x \dot- 1}\right] = \mu_y\left(\left(\sum_{z=0}^{y} (z+1)^n - z^n\right) \geq x\right)$$

and therefore $\left[\sqrt[n]{x \dot- 1}\right]$ is an eas function. Because the function $x + 1$ is eas the function $\left[\sqrt[n]{x}\right]$ is also eas according to the theorem 8.2. It holds $\left[\sqrt[n]{x}\right]^n \leq x$, x^n is rtc function and therefore $\left[\sqrt[n]{x}\right]^n$ is eas.

T h e o r e m 10.9. Let be $h \in \mathcal{P}$, let m-ary functions $f_1,\dots,f_n$ be h-computable, let f be real-time computable function, g be the superposition of the functions $f, f_1,\dots,f_n$. Then the function g is h-computable if and only if $\sum(h, Cs(g)) < \infty$.

P r o o f . The necessity of the condition $\sum (h,Cs(g)) < \infty$ is trivial. The machine which is needed for the converse implication can be easily constructed. We are not giving the detailed proof.

T h e o r e m 10.10. Let f be an n-ary total function effectively admissible for the superposition, let

$$\liminf \frac{f(x_1,\dots,x_n)}{x_1 + \dots + x_n} > 0.$$

Then the function f is real-time computable.

The proof is trivial.

E x a m p l e . The function $\left[\sqrt[n]{x}\right]^n$ is rtc.

We recall now the definition from [2] .

D e f i n i t i o n 10.2. a) An unary total function f is called Yamadan function if it is increasing, $f(0) > 0$, and if there exists a many-tape TURING machine which prints in every step one symbol of the sequence $(a_1, a_2, a_3, \dots)$ where

$a_m = 1$ if m is a value of the function f,

$a_m = 0$ if m is not a value of the function f

onto its output tape.

b) The set of all Yamadan functions will be denoted by $\mathcal{F}_y$.

We note that following the [3] it is not necessary to ask the printing in every step but it is sufficient to ask the printing (at least) in every k-th step where k is a natural number. Because the Yamadan functions are unary (and increasing) we are going to compare them only with unary rtc functions (mainly with using of operators Sum, Dif).

T h e o r e m 10.11. Let f be unary real-time computable function, $f > 0$. Then the function $g = \mathrm{Sum}(f)$ is Yamadan.

P r o o f . Let be Z a machine with regular output in every k-th step (for $n = 1$), let $\Phi_Z^1 = f - 1$ and let the computation of Z from $(q_1; 0, x)$ end by $(q_0; f(x) - 1, x)$. Let us put $Z' = Z \cup \{\langle q_0 S_1 P q_1 \rangle\}$

We interpret now the machine Z' as a many-tape TURING machine. We add an output tape to Z'. Let Z' print into this tape the symbol 1 by the using of quadruple $\langle q_0 S_1 P q_1 \rangle$ and the symbol 0 always when Z' adds 1 to S_0. Let the computation of Z' begin from $(q_1; 0)$. Then the sequence of symbols

$$0^{f(0)-1} 1 \; 0^{f(1)-1} 1 \; 0^{f(2)-1} 1 \dots$$

arises on the output tape. The machine Z gives a symbol on the output tape in every k-th step. The m-th symbol is 1 if and only if m is a value g for some x. Therefore g is Yamadan function.

T h e o r e m 10.12. Let be $f(x)$ real-time computable function. Let there exist a real number $\varepsilon > 0$ such that for every $x \in N$

$$\frac{f(x+1)}{f(x)} > 1 + \varepsilon .$$

Then the function f is Yamadan.

<u>P r o o f .</u> Following the corollary of theorem 10.2. Dif(f) is rtc function. Dif(f) > 0 and then $f = \text{Sum Dif}(f)$ is Yamadan function according to the theorem 10.10.

§ 11. Polynomial Computability

In this paragraph H_0 means the set of all functions $(1 + x^n)$, $n = 1, 2, 3, \ldots$.

<u>D e f i n i t i o n 11.1.</u> A function f is called polynomially computable if there exists $n \in N$ such that $f \prec x^n$.

<u>T h e o r e m 11.1.</u> Let f be an n-ary function. The following conditions are equivalont:

1) f is polynomially computable.

2) f is weakly H_0 - computable.

3) There exists a machine Z and $k \in N$ such that for almost all $x_1, \ldots, x_n \in N$

$$\mathcal{T}_Z^n(x_1, \ldots, x_n) \leq (x_1 + \ldots + x_n)^k .$$

<u>R e m a r k .</u> Following this theorem the set of all polynomially computable functions is equal to H_0^F .

<u>P r o o f .</u> $1 \to 2$. If $f \prec x^m$ then because $x^m \vee (1 + x^{m+2})$ it holds $f \vee (1 + x^{m+2})$. $2 \to 3$. If $f \vee (1 + x^m)$ then there exists a machine Z such that $\Phi_Z^n = f$, $T'(1 + x^m, \langle nZ \rangle) < \infty$. Then for almost all X_n $\mathcal{T}_Z^n(X_n) \leq (1 + x_1^m) \ldots (1 + x_n^m)$ and therefore for almost all X_n $\mathcal{T}_Z^n(x_n) \leq (x_1 + \ldots + x_n)^{mn+1}$.

$3 \to 1$. Let be $\Phi_Z^n = f$, $\mathcal{T}_Z^n(X_n) \leq (X_n^+)^k$ for almost all X_m . We shall prove $f \prec x^k$. Let be $\sum (h, 1 + x^k) < \infty$. For almost all X_n $\mathcal{T}_Z^n(X_n) \leq n^k.(1 + x_1^k) \ldots (1 + x_n^k)$ and therefore it is sufficient to prove $\sum (h, (1 + x_1^k) \ldots (1 + x_n^k)) < \infty$. It holds $\sum (h, (1 + x_1^k) \ldots (1 + x_n^k)) = (\sum (h, (1 + x^k))^n < \infty$. Therefore $x^k \vee h$ implies $f \vee h$, i.e. $f \prec x^n$ q.e.d.

From the theorem 9.8. follows immediately:

<u>T h e o r e m 11.2.</u> A function f is polynomially computable if

and only if it is a difference of two functions both real-time and polynomially computable.

From the theorem 9.6. follows:

Theorem 11.3. A real-time computable function f is polynomially computable if and only if there exists $k \in N$ such that for almost all $x_1,\ldots,x_n$ it holds $f(x_1,\ldots,x_n) \leq (x_1 + \ldots + x_n)^k$.

From theorems 11.2. and 11.3. follows:

Theorem 11.4. An n-ary total function f is polynomially computable if and only if there exists $k \in N$ such that it holds $f(x_1,\ldots,x_n) \leq (x_1 + \ldots + x_n)^k$ for almost all $x_1,\ldots,x_n$ and the function $f(x_1,\ldots,x_n) + (x_1 + \ldots + x_n)^k$ is real-time computable.

All the almost constant functions and all the functions admissible for the superposition are polynomially computable. We are not going to repeat the known examples of these functions.

Theorem 11.5. The superposition of polynomially computable functions is also polynomially computable.

Proof. Let be $k \in N$, $\langle mZ_0\rangle \in \mathcal{Z}$ $(m \neq 0)$, $\langle nZ_i\rangle \in \mathcal{Z}$ $(i = 1,\ldots,m)$ and let for almost all X_m, X_n $\mathcal{T}^m_{Z_0}(X_m) \leq (X^+_m)^k$, $\mathcal{T}^n_{Z_i}(X_n) \leq (X^+_n)^k$. Let be a machine Z such that

$$\Phi^n_Z = S^{m+1}(\Phi^m_{Z_0}, \Phi^n_{Z_1}, \ldots, \Phi^n_{Z_m})$$ and which works as follows.

In the first phase Z modells the computings of $Z_1,\ldots,Z_n$ from common input data and then from corresponding results the computing of Z_0. If $\Phi^n_Z \neq \emptyset$ then there exists $k_1 \in N$ such that for almost all X_n:

$$\mathcal{T}^n_Z(X_n) \leq k_1.(\mathcal{T}^n_{Z_1}(X_n) + \ldots + \mathcal{T}^n_{Z_m}(X_n)) + \mathcal{T}^m_{Z_0}(\Phi^n_{Z_1}(X_n),$$

$$\ldots, \Phi^n_{Z_m}(X_n)) \leq (X^+_n)^{k^2+1}$$ and therefore $\Phi^n_Z \in H^F_0$.

Theorem 11.6. If the function $f(x,X_n)$ is polynomially computable then the function $g = Sum(f)$, i.e.

$$g(x,X_n) = \sum_{y=0}^{x} f(x,X_n)$$

is also polynomially computable.

P r o o f . Let us consider first the case when f is total. Let be $f = f_1 - f_2$, $f_1, f_2 \in \mathcal{F}_r \cap H_0^F$. Then $g = \mathrm{Sum}(f) = \mathrm{Sum}(f_1) - \mathrm{Sum}(f_2)$. The functions $\mathrm{Sum}(f_1)$ and $\mathrm{Sum}(f_2)$ are rtc and then following the theorem 11.3. they are also polynomially computable. Then g is polynomially computable according to the theorem 11.2. If f is a partial function let be $f_1' = \mathrm{Sum}\ \mathrm{Cs}(f)$, $f_2' = \mathrm{Sum}\ \overline{sg}\ \mathrm{Cs}(f)$. It holds

$$g(x,X_n) = (f_1'(x,X_n) \dot{-} (x+1)) + (0 - f_2'(x,X_n))$$

and therefore $g \in H_0^F$ according to the theorem 11.5.

T h e o r e m 11.7. Let a function $f(x,X_n)$ be polynomially computable. Let the function $g(x,X_n)$ arise from the function $f(x,X_n)$ by the bounded minimalisation, i.e.

$$g(x,X_n) = \mu_y (y \leqq x \ \&\ f(y,X_n) = 0).$$

Then the function g is also polynomially computable.

P r o o f . We express the function g with the help of Sum, Cs, functions from H_0^F, and superpositions. By this $g \in H_0^F$ will be proved. Let be $f_1 = \mathrm{Sum}\ \overline{sg}\ \mathrm{Cs}(f)$, $f_2 = \mathrm{Cs}(f)$, $f_3(x,X_n) = f_1(x,X_n) + sg\ | f_2(x,X_n) - 1 |$. The functions f_3, f give by the bounded minimalisation the same results. Therefore we can assume that f is total. Then it holds

$$g(x,X_n) = \sum_{y=0}^{x} \overline{sg}\ (f(y,X_n) + sg(y) \cdot \sum_{z=0}^{y \dot{-} 1} \overline{sg}\ f(z,X_n)).$$

T h e o r e m 11.8. Let a function $f(x,X_n)$ arise from $g'(X_n)$, $g(x,y,X_n)$ by primitive recursion and let there exists $k \in N$ such that for all x, X_n

$$f(x,X_n) \leqq (2 + x + x_1 + \dots + x_n)^k .$$

If the functions g, g' are polynomially computable then f is also polynomially computable.

The proof will be only indicated. We should find (analogically as in the proof of the theorem 10.3.) a machine Z such that $\Phi_Z^{n+1} = f$ and then prove that for some $h \in H_0$ it holds

$T'(h, \langle n+1, Z\rangle) < \infty$. The substantial part of the work of Z would be taken by the computation of values of g (the value of g' is computed only once). By the computation of $f(x,X_n)$ it is necessary to compute ($y_0 = g'(X_n)$ and) $y_1 = g(0,y_0,X_n)$, $y_2 = g(1,y_1X_n)$, ..., $y_x = g(x,y_{x-1},X_n)$. There exists $m \in N$ such that for the computation of $g(u,v,X_n)$ it is sufficient to make $(2 + u + v + X_n^+)^m$ steps. It holds $y_i \leq (2 + x + X_n^+)^k$ for $i = 1,\dots, x$ and therefore for the computation of $y_1,\dots, y_x$ it sufficient to make $x.(2 + x + (2 + x + X_n^+)^k + X_n^+)^m$ steps. (If $f(x,X_n)$ will be not defined then the computation will be shorter.) The evaluation of the number of remainded steps is easy. We can consider that for convenient $k_1 \in N$ it holds $\mathcal{T}_Z^{n+1}(x,X_n) \leqq (2 + x + X_n^+)^{k_1}$ and then $f \in H_0^F$ according to the theorem 11.1.

R e m a r k . We could not use the well known modus of expression of the primitive recursion with the help of minimalisation because by this expression too big functions are needed. (The function 2^x is not polynomially computable already.)

At the end we shall express H_0^F in the form $\{h\}^F$ and we shall characterise the set H_0^F with the help of quasi-ordering $\prec$ and the superposition.

L e m m a 1. There exists a function $h_1 \in \mathcal{P}$, $h_1 > 0$ such that for any unary recursive function f

$\sum(h_1, f) < \infty$ if and only if there exists $h \in H_0$ such that $\sum(h, f) < \infty$

The proof is given in §12.

T h e o r e m 11.9. Let be h_1 the function from lemma 1. Then a function f is polynomially computable if and only if it is h_1-computable. (I.e. $\{h_1\}^F = H_0^F$.)

P r o o f . Let f be an unary function. Then $f \in H_0^F$ if and only if $Cs(f(c_{n1}(x),\dots,c_{nn}(x)) \in H_0^F$. Therefore it is sufficient to prove the theorem 11.9. for unary total functions. Following the theorem 9.7. it is sufficient to prove 11.9. for any (unary) rtc function f . It holds $fVh_0 \equiv \sum(h_0,f)$,
$f \in H_0^f \equiv (\exists h \in H_0)(\sum(h,f) < \infty)$. But f is recursive and therefore $fVh_0 \equiv f \in H_0^F$ according to lemma 1 .

<u>Theorem 11.10.</u> The set H_0^F of all polynomially computable functions is the smallest set which has the properties:

1) it contains the function I_1^1 (remark: $I_1^1(x) = x$).
2) it contains with any function f all the functions g such that $g \prec f$.
3) it is closed under the superposition.

<u>Proof.</u> It holds $x.y \prec I_1^1$ because $x.y$, I_1^1 are rtc functions and for every $h \in \mathcal{P}$ $\sum (h, x.y) = (\sum (h, I_1^1))^2$. By the superposition all the functions x^2, x^3, x^4, ... can be obtained from $x.y$, I_1^1. If we add to these functions all the functions g, $g \prec x^k$ for some $k \in N$, we obtain the whole set H_0^F. Conversely, the set H_0^F has all there properties from the theorem 11.10. as we can see from the above-mentioned theorems.

§ 12. Supplement

<u>Theorem 12.1.</u> For any probability p there exists a sequence $(h_1, h_2, h_3, \ldots)$ of functions from $\mathcal{P}$ such that $0 < T(h_i) < \infty$ for all $i = 1, 2, 3, \ldots$ and for any n-ary non-negative real-valued function f it holds

$$(12.1.)\quad \lim_{i \to \infty} \frac{\sum (h_i, f)}{(T(h_i))^n} = \sum_{x_1=0}^{\infty} \ldots \sum_{x_n=0}^{\infty} f(X_n).p(x_1)\ldots p(x_n)$$

and the right side of (12.1) is equal to ∞ if and only if $\sum (h_i, f) = \infty$ for almost all i.

<u>Proof.</u> It will be only indicated. Let us put (for $i \in N$, $i \neq 0$ and all $x \in N$) $h_i(x) = [i : p(x)]$. Let p_i be the probability coordinated to h_i for $i = 1, 2, 3, \ldots$. There exist real-valued sequences $(a_1, a_2, a_3, \ldots)$, $(b_1, b_2, b_3, \ldots)$ such that $\lim_{i \to \infty} a_i = \lim_{i \to \infty} b_i = 1$ and for all $i = 1, 2, \ldots$ and all $x \in N$ $a_i.p_i(x) \leq p(x) \leq b_i.p_i(x)$. The left side of (12.1) can be expressed as

$$\lim_{i \to \infty} \sum_{x_1=0}^{\infty} \ldots \sum_{x_n=0}^{\infty} f(X_n).p_i(x_1)\ldots p_i(x_n).$$

Now it is sufficient to use the comparative criterion. (Remark: The function f can reach also the value ∞.)

The following theorem is only another formulation of the lemma 1 from §7.

<u>Theorem 12.2.</u> Let be $h_0 \in \mathcal{P}$, $h_0(x) = (1+x)\left[\log_2(x+2)\right]^2$, $n \in N$, $n \neq 0$, let be α a real number. Then :

1) $\sum (h_0, \min(X_n)) = \infty$,

2) $T(h_0) < \infty$,

3) $\sum ([1 + x^{\alpha}], \min(X_n)) < \infty \equiv \alpha > 1 + \frac{1}{n}$,

4) $T([1 + x^{\alpha}]) < \infty \equiv \alpha > 1$.

<u>Proof.</u> $T(h_0) < \infty$ follows e.g. from the integral criterion. Following the comparative criterion we get the implication $\leftarrow$ in 4); the orther implication is trivial. Analogously we get $3 \rightarrow 1$. We must prove yet the point 3). Let us denote

$$B = \sum_{x_1=0}^{\infty} \sum_{x_2=x_1}^{\infty} \dots \sum_{x_n=x_1}^{\infty} \frac{\min(X_n)}{[1 + x_1^{\alpha}] \dots [1 + x_n^{\alpha}]} .$$

It holds $B \leq \sum ([1 + x^{\alpha}], \min(X_n)) \leq n.B$ and therefore we can consider B instead of $\sum ([1 + x^{\alpha}], \min(X_n))$. It holds

$$B = \sum_{x_1=0}^{\infty} \sum_{x_2=x_1}^{\infty} \dots \sum_{x_n=x_1}^{\infty} \frac{x_1}{[1 + x_1^{\alpha}] \dots [1 + x_n^{\alpha}]} =$$

$$= \sum_{x=1}^{\infty} \left(\frac{x}{[1 + x^{\alpha}]} \cdot \left(\sum_{y=x}^{\infty} \frac{1}{[1 + y^{\alpha}]} \right)^{n-1} \right) .$$

For $\alpha \leq 1$ is trivially $B = \infty$. Let be $\alpha > 1$. Then

$$\lim \left(\frac{x}{[1 + x^{\alpha}]} : \frac{1}{x^{\alpha - 1}} \right) = 1 ,$$

$$\lim \left(\left(\sum_{y=x}^{\infty} \frac{1}{[1 + y^{\alpha}]} \right) : \frac{1}{x^{\alpha - 1}} \right) = \frac{1}{\alpha - 1} ; \quad 0 < \alpha - 1 < \infty$$

and therefore $B < \infty$ if and only if

$$\sum_{x=1}^{\infty} \left(\frac{1}{x^{\alpha-1}} \cdot \left(\frac{1}{x^{\alpha-1}} \right)^{n-1} \right) < \infty$$ and it holds just when

$$\sum_{x=1}^{\infty} x^{-n.(\alpha-1)} < \infty \text{ i.e. } n(\alpha-1) > 1 \text{ , i.e.} \alpha > 1 + \frac{1}{n} \text{ q.e.d.}$$

In the proof of theorem 8.4. the following statement was used.

Theorem 12.3. There exists a function $h \in \mathcal{P}$, $h > 0$, $\sum(h, I_1^1) < \infty$ such that for any unary recursive function f

$$\limsup \frac{f(x)}{x} = \infty \quad \text{implies} \quad \sum_{x=0}^{\infty} \frac{f(x)}{h(x)} = \infty .$$

(We note that $\sum (h, I_1^1) < \infty \equiv T(h, I_1^1) < \infty$ and therefore it is sufficient to prove this theorem.)

Proof. Following [4] for any sequence $(c_1, c_2, c_3, \ldots)$ of positive real numbers such that $\liminf c_x = 0$ there exists a sequence $(a_1, a_2, a_3, \ldots)$ of positive real numbers such that

$$\sum_{x=0}^{\infty} a_x = \infty \qquad \sum_{x=0}^{\infty} c_x \cdot a_x < \infty .$$

Let us order all the unary recursive functions f such that $\limsup(f(x) : x) = \infty$ into a sequence $(f_1, f_2, f_3, \ldots)$ and for any $i = 1, 2, 3, \ldots$ put $c_i(x) = (1 + x) : (1 + f_i(x))$. Then (for any i) $\liminf c_i(x) = 0$ and therefore there exist sequences $(a_i(x) \; x = 0,1,2,\ldots)$ such that

$$\sum_{x=0}^{\infty} a_i(x) = \infty \qquad \sum_{x=0}^{\infty} \frac{1 + x}{1 + f_i(x)} \cdot a_i(x) < \frac{1}{2^i} .$$

Let us put $b_i(x) = a_i(x) : (1 + f_i(x))$. Then

$$\sum_{x=0}^{\infty} (1 + f_i(x)).b_i(x) = \infty , \quad \sum_{x=0}^{\infty} (1 + x).b_i(x) < 2^{-i} .$$

for $b(x) = \sum_{i=1}^{\infty} b_i(x)$ it holds (for all $i = 1,2,3,\ldots$)

$$\sum_{x=0}^{\infty} (1 + f_i(x)).b(x) = \infty , \quad \sum_{x=0}^{\infty} (1 + x).b(x) < 1.$$

Therefore $\sum_{x=0}^{\infty} b(x) < 1$ and then $\sum_{x=0}^{\infty} f_i(x).b(x) = \infty$.

Now it is sufficient to put $h(x) = [2 : b(x)]$ for all $x \in N$.

Let now H_0 have the same meaning as in §11.

<u>Theorem 12.4.</u> There exists a function $h_1 \in \mathcal{P}$, $h_1 > 0$ such that for any unary recursive function f it holds.

$$\sum(h_1,f) < \infty \equiv (\exists h \in H_0)(\sum(h,f) < \infty) .$$

<u>Proof.</u> Let be $(a_0, a_1, a_2, \ldots)$, $a_0 = 0$ an increasing sequence of natural numbers such that for any infinite recursively enumerable set $B = \{ b_0, b_1, b_2, \ldots \}$ (where $b_0 < b_1 < b_2 < \ldots$) it holds $a_i > b_i$ for almost all $i \in N$. Let be $h_1 \in \mathcal{P}$ the function such that

$$h_1(x) = 1 + x^i \quad \text{if} \quad a_{i-1} \leqq x < a_i .$$

If for some $h \in H_0$ it holds $\sum(h,f) < \infty$ then following the comparative criterion it holds also $\sum(h_1,f) < \infty$.

Conversely, let be f an unary recursive function such that $(\forall h \in H_0)(\sum(h,f) = \infty$.(Then also $(\forall h \in H_0)$ (lim sup $\frac{f(x)}{h(x)} = \infty$).

Let us put $b_0 = 0$

b_{i+1} = the least natural number greater than b_i such that $f(x) \geqq x^{i+1}$.

The set $\{ b_0, b_1, b_2, \ldots \}$ is recursively enumerable and therefore for almost all i it holds $b_i < a_i$. If $b_i < a_i$ then

$h_1(b_i) \leq 1 + b_i^i$. But $f(b_i) \geq b_i^i$, so in infinite series $\sum (h_1, f)$ the limit of x-th member is not equal to zero and therefore $\sum (h_1 f) = \infty$.

Symbols Register

The number of the paragraph dealing with a given notion and also a short (maybe not fully exact) explanation of the meaning.

Usual Denotions (§ 2)

$x \in A$	x is an element of A
$\{x \mid \varphi(x)\}$	set of all x with the property φ
$A \times B$	cartesian product
A^n	$A \times A \ldots \times A$ (n-times)
$\langle x_1, \ldots, x_n \rangle$	ordered n-tuple
$A \subseteq B$	set-theoretic inclusion
$(x_1, \ldots, x_n)$ or	
$(x_i \mid i = 1, \ldots, n)$	finite sequence
$(x_1, x_2, x_3, \ldots)$ or	
$(x_i \mid i = 1, 2, \ldots)$	infinite sequence
$\{x_1, \ldots, x_n\}$	finite set
$\rightarrow, \equiv, \vee, \&, \neg, \forall, \exists$	logical signs
$\leq, <$	less or equal, less
∞	infinity
$\sum_{i=1}^{n}, \sum_{i=1}^{n}$	sum, product
$[x]$	whole part
$\lvert x \rvert$	absolute value

Operators (§ 2)

Cn(f)	Null completion
Cs(f)	Cn(1 + f)
Dif(f)	Dif Sum(f) = f for total f

Mult(f)		Multiplication
$S^{n+1}(f,g_1,\dots,g_n)$		Superposition
Sum(f)		Summation
$\mu_y(f(y,X_n) = 0)$		Minimalisation

Number functions (§ 2)

$c(x,y)$		Numbering function; $c(x,y) = \dfrac{(x+y)(x+y+1)}{2} + x$
$c_n(X_n)$		$c_n(X_n) = c(c\dots c(x_1,x_2),\dots,x_n)$
$c_{ni}(x)$		$c_n(c_{n1}(x),\dots,c_{nn}(x)) = x$
$D(x,y)$		Greatest common divisor
$h_0(x)$	§ 7	$h_0(x) = (1 + x)\,[\log_2(x + 2)]^2$.
$I^n_m(X_n)$		$I^n_m(X_n) = x_m$
$l(x)$		Numbering function; $c(l(x),r(x))=x$
$[\log_y x]$		Whole part of logarithm
$\min(X_n)$		Usual minimum
$\min^*(X_n)$		Minimum by the ordering $\leq^*$
$n(x,y)$		Smallest common multiple
$o(x)$		$o(x) = 0$
$o^n(X_n)$		$o^n(X_n) = 0$
$q(x)$		$q(x) = x \dot{-} [\sqrt{x}]^2$
$r(x)$		Numbering function; $c(l(x),r(x)) = x$
$rest(x,y)$		$rest(x,y)=x - [x : y].y$
$s(x)$		$s(x) = x + 1$
$sg(x)$		0 if $x = 0$, 1 otherwise
$\overline{sg}(x)$		$\overline{sg}(x) = 1 - sg(x)$
$\Gamma(x,y)$		$\Gamma(x,y) = rest(l(x),\ 1+(y + 1).r(x))$
$[x : y]$ or $[\frac{x}{y}]$		Whole part of $\frac{x}{y}$; $[x : 0] = 0$
$x!$		Factorial
x^y		Power
$x \dot{-} y$		$x - y$ if $x \geq y$, 0 otherwise
$\lvert x - y \rvert$		$(x \dot{-} y) + (y \dot{-} x)$

Abbreviations

B_n	§2	$b_1, \ldots, b_n$
eas	§8	Effectively admissible for the superposition
rtc	§9	Real-time computable
X_n	§2	$x_1, \ldots, x_n$
X_n^+	§2	$x_1 + \ldots + x_n$

Relations

fVh	§5	f is h-computable
$f < g$	§5	f is simpler than g
$g > f$	§5	g is more complex than f
$\gg, \ll, \sim$	§5	Relations derivated from $\succ$
$U \rightarrow V(Z)$	§3	V arises directly from U by the computations of machine Z
$U \Rightarrow V(Z)$	§3	"$\Rightarrow (Z)$" is reflexive and transitive closure of "$\rightarrow (Z)$"
$x \leq^{*} y$	§2	$0 < x \leq y$ or $y = 0$

Sets

		The set of all ...
$\mathcal{F}$	§2	... partial recursive functions
$\mathcal{F}_c$	§7	... almost constant functions
$\mathcal{F}_{es}$	§8	... functions effectively admissible for the superposition
$\mathcal{F}_r$	§ 9	... real-time computable functions
$\mathcal{F}_s$	§8	... functions admisible for the superposition
$\mathcal{F}_y$	§10	... Yamadan functions
H^F	§5	... weakly H-computable functions
H^f	§ 5	... strongly H-computable functions
$\{h\}^F$	§ 5	... h-computable functions
H_0	§ 11	... functions $1 + x^n$, $n = 1, 2, 3, \ldots$

H_0^F	§11	... polynomially computable functions
N	§2	... natural numbers
$\mathcal{P}$	§2	... unary total functions
$\mathcal{Z}$	§3	... pairs $\langle nZ \rangle$, $n \in N$, Z is a machine
$\emptyset$	§2	Empty set

Other denotations

L(Z; U,V)	§ 3	Length of computation of Z from U to V
L(Z; U)	§3	Length of computation of Z from U
Φ_Z^n	§3	n-ary function being computed by Z
$\mathcal{T}_Z^n$	§3	Computation time of Φ_Z^n (on Z)
$\Sigma\,(h,f)$	§9	$\Sigma\,(h, f(X_n)) = \sum_{x_1=0}^{\infty} \dots \sum_{x_n=0}^{\infty} \frac{f(x_1,\dots,x_n)}{h(x_1)\dots h(x_n)}$
T(h,f)	§5	$\inf \{ T(h, \langle nZ \rangle) \mid \langle nZ \rangle \in \mathcal{Z} \,\&\, \Phi_Z^n = f \}$
T(h)	§5	$\Sigma\,(h,1)$
$T'(h, \langle nZ \rangle)$	§5	$\Sigma\,(h, \mathcal{T}_Z^n)$
$T(h, \langle nZ \rangle)$	§5	Expectation of $\mathcal{T}_Z^n$ by the probability coordinated to h

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ON THE EXISTENCE OF CERTAIN OVERGRAPHS OF GIVEN GRAPHS

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In this paper are studied mixed graphs i.e. graphs which contain directed and undirected edges too without loops and multiple edges. To a give graph G not necessarily connected and finite is constructed its overgraph G_1 with a given diameter and such that after deleting every edge and also after deleting every vertex and all edges incidental with this vertex the diameter will be greater. Moreover the neighbourghoods of every two different vertices of graph G_1 are different. This question was solved also if we suppose that G and G_1 are simultaneously undirected or directed or oriented.

This paper is generalization of one result from the paper [2], where this problem was solved for undirected graphs and diameter 2.

Under a graph $G = (U,H)$ we understand a pair of sets U, H where U is the set of vertices, $H = H_1 \cup H_2$ is the set of edges whereby H_1 is the set of undirected edges (i.e. a subset of the set $\{(x,y) \mid x,y \in U\}$ H_2 is the set of directed edges (i.e. a subset of the set $\{(\overrightarrow{x,y}) \mid x,y \in U\}$.

The following conditions for the graph $G = (U,H)$ will be used:

(1) G does not contain loops nor multiple edges

(2) if $(x,y) \in H$ then $(\overrightarrow{x,y}) \notin H$ and also $(\overrightarrow{y,x}) \notin H$, for every $x,y \in U$

(3) if $(\overrightarrow{x,y}) \in H$ then $(\overrightarrow{y,x}) \notin H$, for every $x,y \in U$.

In the next we suppose that graph G fulfils the conditions (1), (2).

<u>Definition 1.</u> The graph G will be:

a) undirected if it contains undirected edges only
b) directed if it contains directed edges only
c) oriented if it is directed and fulfils the condition (3)
d) tournament if it is oriented and for every two vertices $x,y \in U$ is either $(\overrightarrow{x,y}) \in H$ or $(\overrightarrow{y,x}) \in H$.

The basic notions not defined here were taken over from [1]. The distance from vertex x to vertex y in the graph G we shall denote by $\rho_G(x,y)$. The diameter of the graph G we shall denote by $d(G)$.

<u>D e f i n i t i o n 2.</u> a) A graph is without superfluous edges if after deleting every edge the diameter of this graph will be greater.

b) A graph is without superfluous vertices if after deleting every vertex and edges incidental with this vertex the diameter will be greater.

<u>D e f i n i t i o n 3.</u> The neighbourghood of a vertex u in the graph $G = (U,H)$ will be the ordered trinity of sets (A,B,C), where $A = \{x \mid (u,x) \in H\}$, $B = \{x \mid (\overrightarrow{u,x}) \in H\}$, $C = \{x \mid (\overrightarrow{x,u}) \in H\}$.

<u>L e m m a 1.</u> Let G be a graph which is not necessarily connected and finite. Then there exists a graph R of the diameter at most 2 and such that G is its section graph.

Proof. It is sufficient to construct the graph $R = (V,E)$ where $V = U \cup \{a\}$, $a \notin U$ and $E = H \cup \{(a,x) \mid x \in U\}$.

<u>T h e o r e m 1.</u> Let $k \geq 2$ be a natural number. Let $G = (U,H)$ be a graph which is not necessarily connected and finite. Then there exists a graph G_1 of diameter k and such that:

1) G is its section graph
2) the neighbourghood of every two different vertices differ
3) G_1 is graph without superfluous edges
4) G_1 is a graph without superfluous vertices.

Proof. According to the Lemma 1 we can construct the graph $R = (A_1,E)$ of the diameter 2 and such that G is its section graph. Let $A_1 = \{a^1_j \; j \in J\}$, where J is a set of indices. The graph $G = (U_1, H_1)$ we construct by adding other vertices and edges as follows:

$U_1 = \{w\} \cup \bigcup_{i=1}^{k} A_i$, where $A_i = \{a^i_j \mid j \in J\}$; $H_1 = E \cup E_1 \cup E_2 \cup E_3$ where

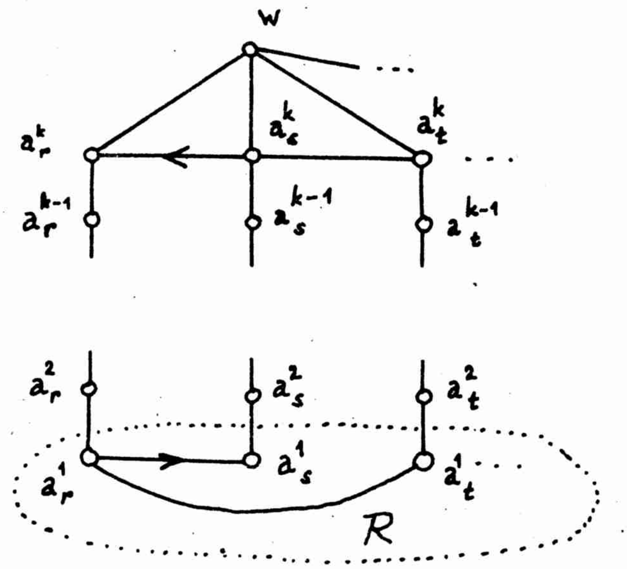
$E_1 = \{(w,x) \mid x \in A_k\} \cup \bigcup_{j \in J} \bigcup_{i=1}^{k-1} \{(x^i_j, x^{i+1}_j) \mid x^i_j \in A_i\}$.

$E_2 = \{(x^k, y^k) \mid (x^1, y^1) \notin E \wedge (\overrightarrow{x^1,y^1}) \notin E \wedge (\overrightarrow{y^1,x^1}) \notin E \; ; \; x^1, y^1 \in A_1\}$

$E_3 = \{(\overrightarrow{x^k, y^k}) \mid (\overrightarrow{y^1, x^1}) \notin E \wedge (x^1, y^1) \notin E \; ; \; x^1, y^1 \in A_1\}$.

The sketch of this construction is in Fig. 1.

First of all we shall prove that $d(G) = k$. Let be $1 \leq i \leq k$, $1 \leq j \leq k$ and $r,s \in J$. If $r=s$ then $\rho(a^i_r, a^j_s) = \rho(a^j_s, a^i_r) < k$.



Let $r \neq s$; let $(\overrightarrow{a_r^1, a_s^1}) \in H$ and $(\overrightarrow{a_s^1, a_r^1}) \notin H$.

If $i+j \leq k-1$ then the path $a_r^i, a_r^{i-1}, \dots, a_r^1, a_s^1, \dots a_s^j)$ has the length at most k. If $i+j \geq k$ then the path $(a_r^i, a_r^{i+1}, \dots, a_r^k, w, a_s^k, \dots a_s^j)$ has the length at most k. Hence $\rho(a_r^i, a_s^j) \leq k$.

If $i+j \leq k-2$ then in the graph R there exists the path (a_s^1, z, a_r^1) of the length 2 and the path $(a_s^j, a_s^{j-1}, \dots, a_s^1, z, a_r^1, \dots a_r^i)$ is of the length at most k. If $i+j \geq k-1$ then the path $(a_s^j, a_s^{j+1}, \dots, a_s^k, a_r^k, \dots, a_r^i)$ has the length at most k. Hence also $\rho(a_s^j, a_r^i) \leq k$.

Other cases may be verified analogously. So $d(G) = k$.

Directly from the construction one can see that the conditions 1) and 2) from the Theorem 1 hold.

The graph G_1 is without superfluous edges because after deleting the edge: (w, a_r^k), for $r \in J$ would be $\rho(w, a_r^1) > k$

(a_r^i, a_r^{i+1}), for $1 \leq i \leq k-1$; $r \in J$ would be $\rho(a_r^i, a_r^{i+1}) > k$

(a_r^j, a_s^j), for $j=1, k$; $r, s \in J$ would be $\rho(a_r^k, a_s^1) > k$

$(\overrightarrow{a_r^j, a_s^j})$, for $j=1, k$; $r, s \in J$ would be $\rho(a_r^k, a_s^1) > k$.

The graph G_1 is without superfluous vertices, because after deleting the vertex (and also edges incidental with this vertex): a_r^i, for $2 \leq i \leq k$; $r \in J$ would be $\rho(w, a_r^{i-1}) > k$

a_r^1, for $r \in J$ would be $\rho(a_r^1, x) > k$, for every $x \in A_1$ such that either

$(a_r^1, x) \in H$ or $(\overrightarrow{a_r^1, x}) \in H_1$.

w, would be $\rho(x,y) > k$, for every $x,y \in A_k$ such that either

$(x,y) \notin H_1$ or $(\overrightarrow{x,y}) \notin H_1$. Hence the theorem holds.

<u>Corollary 1.</u> Let $G = (U,H)$ be a finite graph and $|U| = n$. Then for every two natural numbers $k \geq 2$; $s \geq 1$ there exists a graph G_1 of diameter k, without superfluous edges, without superfluous vertices, with the number of vertices $k(n+s)-1$ and such that G is its section graph and the neighbourghood of every two different vertices differ.

So it is obvious that for a given graph G there exists an infinite number such graphs G_1.

<u>Corollary 2.</u> Let $\mathcal{S}$ be a system of graphs and let $k \geq 2$ be a natural number. Then there exists a graph of diameter k, without superfluous edges and supefluous vertices such, that every graph from the system $\mathcal{S}$ will be its section graph and the neighbourghood of every two different vertices differ.

<u>Remark 1.</u> Let G be a undirected graph. Then graph G_1 constructed by Theorem 1 will be also undirected. If the set of vertices of graph G will be finite, countable, uncountable the set of vertices of the graph G_1 will be also finite, countable, uncountable, respectively.

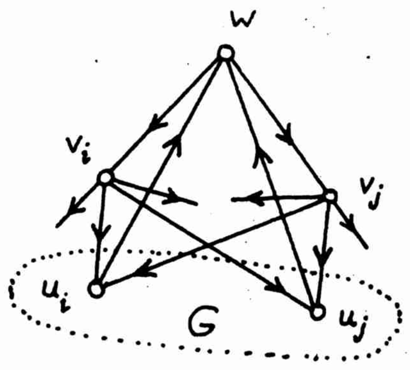
<u>Corollary 3.</u> Let G be a directed graph, not necessarily connected and finite. Then there exists directed graph G_1 of the diameter k, without superfluous edges and superfluous vertices such that G is its section graph and the neighbourghood of every two different vertices differ.

<u>Proof.</u> It is sufficient to replace every undirected edge (u,v) in the graph G_1 from Theorem 1 by a pair of edges $(\overrightarrow{u,v})$, $(\overrightarrow{v,u})$.

<u>Lemma 2.</u> Let G be an oriented graph, not necessarily connected and finite. Then there exists an oriented graph R with the diameter at most 3 and such that G is its section graph.

<u>Proof.</u> Let $G = (U,H)$ and $U = \{u_i \mid i \in J\}$. Then it is sufficient to construct the graph $R = (V,H_1)$, where $V = U \cup \{v_i \mid i \in J\} \cup \{w\}$ and $H_1 = H \cup \{(\overrightarrow{w,v_i}) \mid i \in J\} \cup \{(\overrightarrow{u_i,w}) \mid i \in J\} \cup \{(\overrightarrow{v_i,u_j}) \mid i,j \in J\}$. This construction is illustrated in Fig. 2.

<u>Theorem 2.</u> Let G be an oriented graph, not necessarily connected and finite. Let $d \geq 10$ be a natural number. Then there exists an oriented graph G_1 of diameter d and such that:



1) G is its section graph

2) the neighbbourhood of every two different vertices differ

3) is a graph without superfluous edges

4) is a graph without superfluous vertices.

__Proof.__ According to the Lemma 2 a graph $R = (U,H)$ may be constructe such that G will be its section graph. Let $U = \{u_j \mid j \in J\}$; let k,k be natural number. The vertex set of the graph $G_1 = (U_1 H)$ will be the following set:

$U_1 = U \cup \{w\} \cup U' \cup \bigcup_{i=0}^{\ell-1} X_i \bigcup_{i=0}^{k-1} Y_i \cup A \cup B$, where $U' = \{u'_i \mid i \in J\}$, $X_i = \{x_j^i \mid j \in J\}$,

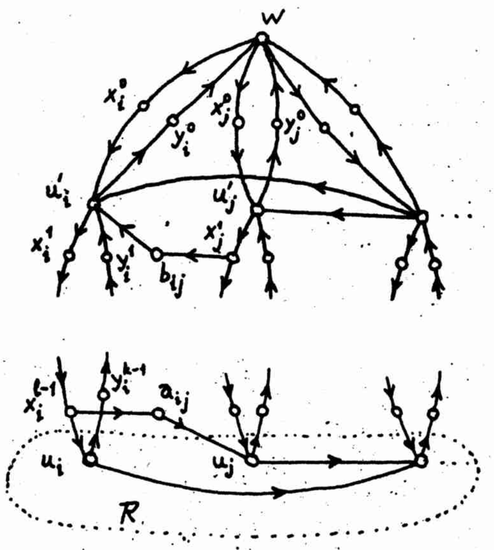
$Y_i = \{y_j^i \mid j \in J\}$,

$A = \{a_{ij} \mid \overrightarrow{(u_i, u_j)} \notin E \wedge \overrightarrow{(u_j, u_i)} \notin E \ ; \ i,j \in J\}$;

$B = \{b_{ij} \mid \overrightarrow{(u_i, u_j)} \notin E \wedge \overrightarrow{(u_j, u_i)} \notin E \ ; \ i,j \in J\}$.

Let $E' = \{\overrightarrow{(u'_j, u'_j)} \ \ \overrightarrow{(u_j, u_i)} \in E \ ; \ i,j \in J\}$. Let H_1 denote the minimal set of edges, that for every i J the following sequences form path: $(w, x_i^0, u'_i, x_i^1, \ldots, x_i^{\ell-1}, u_i)$, $(u_i, y_i^{k-1}, \ldots, y_i^1, u'_i, y_i^0, w)$. Let us denote $H_2 = \{\overrightarrow{(x_i^{\ell-1}, a_{ij})} \ , \ \overrightarrow{(a_{ij}\, u_j)} \ \ a_{ij} \in A \ ; \ i,j \in J\}$,

$H_3 = \{\overrightarrow{(x_j, b_{ij})}, \ (b_{ij}\, u_i) \mid b_{ij} \in B; \ i,j \in J\}$. The set of edges of the graph G_1 will be $H = E \cup E' \cup H_1 \cup H_2 \cup H_3$. The construction of the graph G_1 is illustrated in Fig. 3.



It may be verified that the longest path of the graph G_1 has the length $d = \max(7,\ 2\ell + k - 1,\ 2k + \ell - 1,\ k + \ell + 4)$. Let us put $\ell \geq 5$; $k = 1,2$; then $d = 2\ell + k - 1$. Hence the diameter d of the graph G_1 may be equal to any laue more than 9, for suitable ℓ, k.

From the above mentioned construction it follows that G_1 fulfils the conditions (1) and (2) from this theorem.

Obviously, in the graph G_1 only these edges can be superfluous:

$\overrightarrow{(x_i^{\ell-1}, u)}$, $\overrightarrow{(u'_j, x_j^1)}$ where a_{ij} A; $i,j \in J$,

$\overrightarrow{(u_r, u_s)}$ E, $\overrightarrow{(u'_s, u'_r)} \in E'$, where $r,s \in J$. We will prove that they are not superfluous. Since $d(G) \leq 3$ then there exist $r,s, \in J$ such that

$\overrightarrow{(u_i, u_r)}$ E , $\overrightarrow{(u_j, u_s)} \in E$. After deleting the edge:

$\overrightarrow{(x_i^{\ell-1}, u_i)}$ would be $\rho(x_i^1, x_r^{\ell-1}) > d$

$\overrightarrow{(u_j, x_j^1)}$ would be $\rho(x_s^1, x_j^{\ell-1}) > d$

$\overrightarrow{(u_r, u_s)} \in E$ would be $\rho(x_r^1, x_s^{\ell-1}) > d$

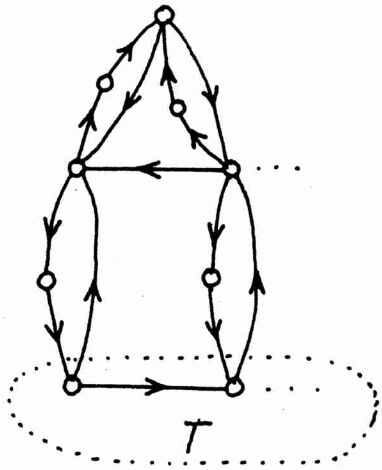
$\overrightarrow{(u_s, u_r)} \in E'$ would be $\rho(x_s^1, x_r^{\ell-1}) > d$, where $d = 2\ell + k - 1$. So the graph G_1 is a graph without superfluous edges. G_1 is also without superluous vertices, because the diameter of the graph formed from the graph G_1 by deloting every vertex and edges incidental with this vertex will not be finite. Hence the theorem is proved.

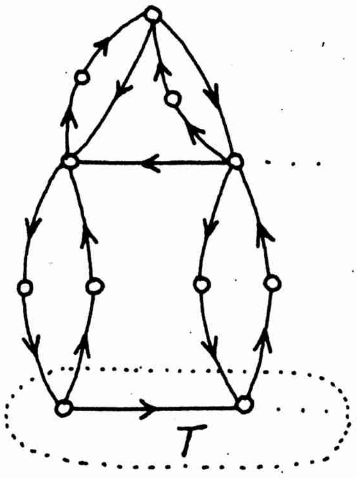
R e m a r k 2. The validity of Theorem 2 for $d < 10$ is not known for us. But the following theorem holds:

T h e o r e m 3. Let T be a tournament, not necessarily finite. Let $d \geq 4$ be a natural number. Then there exists an oriented graph G_1 of the diameter d and such that:

1) T is its section graph
2) the neighbourghood of every two different vertices differ
3) G_1 is a graph without superfluous edges
4) G_1 is a graph without superfluous vertices.

Proof. To a tournament T we may construct by Lemma 2 a graph R of the diameter at most 3 and such that T is its section graph. The graph R may be completed to a tournament T_1 with the diameter at most 3 again. To a graph T_1 we can construct a graph G_1 by constaruction of the Theorem 2. It can be verified that owing to a suitable selection of parameters ℓ , k we shall get a graph with the diameter $d \geq 6$. It is clear that the graph G_1 fulfils the conditions 1), 2), 3) and 4) from this theorem. The sketch of the construction of the graph G_1 for d=4, d=5 is in Fig 4, Fig. 5 , respectively. Hence the theorem holds.





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REAL-TIME COMPUTABLE FUNCTIONS AND ALMOST PRIMITIVE RECURSIVE FUNCTIONS

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Preface

The valutation of the complexity algorithms has been much emphasised recently. One of the methods of this valutation is the introduction of the concept of the real-time computable function.

This paper determines the relations between the real-time computable functions and the primitive recursive functions. The concept of the almost primitive recursive function has been introduced here as well. The almost primitive recursive function is the partial recursive function whose nul completion is the primitive recursive function and whose domain is primitive recursive set. In this paper we have succeeded in determining a simple correlation betwen these and the real-time computable function.

Chapter 1.

CONCEPTS AND DENOTATIONS.

The majority of concepts and denotations employed in this chapter was introduced in [1] , see References.

The concept of the computability of the function as used in this paper refers to the computability by MINSKY machine [1], [2]. MINSKY machine, further only machine, is a device with a potentially infinite number of memory cells S_o, S_1, S_2,..., each of which can be fed one arbitrary natural number. Further it contains potentially infinite number of state q_o, q_1, q_2,..., i.e. finite memory cells wich contain instructions for the machine.

An instantaneous description of the machine Z will be called an expression $< q_i; m_o, m_1,...,m_n >$, where q_i is a state, in which the machine in the given moment is, and numbers m_o, m_1,...,m_n are the contents of the memory cells S_o, S_1,...,S_n with $k > n$ all of the memory cells S_k contain zero.

The operations of the MINSKY machine have one of the following forms:

1. $< q_i, S_j, q_k, q_r >$

2. $< q_i, S_j, P, q_r >$

3. $< q_i, S_j, M, q_r >$

The machine accomplishing the instruction $< q_i, S_j, q_k, q_r >$ goes from the instantaneous description $< q_i; m_o, m_1, \dots, m_n >$ into the instantaneous description $< q_k; m_o, m_1, \dots, m_n >$ if $m_j \neq 0$ or into the instantaneous description $< q_r; m_o, m_1, \dots, m_n >$ if $m_j = 0$. Accomplishing the instruction $< q_i, S_j, P, q_r >$ the machine goes from the instantaneous description $< q_i; m_o, m_1, \dots, m_n >$ into instantaneous description $< q_r; \bar{m}_o, \bar{m}_1, \dots, \bar{m}_n >$, where $\bar{m}_j = m_j + 1$ and $\bar{m}_k = m_k$ if $k \neq j$. Accomplishing the instruction $< q_i, S_j, M, q_r >$ the machine goes from the instantaneous description $< q_i; m_o, m_1, \dots, m_n >$ into the instantaneous description $< q_r; \bar{m}_o, \bar{m}_1, \dots, \bar{m}_n >$ where $\bar{m}_j = m_j \dot{-} 1$ and $\bar{m}_k = m_k$ if $k \neq j$.

A finite set of quadruples of the given form will be also called the machine, where this does not contain two quadruples, having the first symbol equal. The machine stops its work,when it is in the state q_i and there does not exist a quadruple recort of which begins with state q_j.

A step of the work action of the machine will be called the accomplishing of one instruction.

To every machine Z the functions Φ_Z and $\mathcal{T}_Z$ will be mapped. Let Z begins its computation in the instantaneous description $< q_1; 0, x, 0, \dots, 0 >$ and up to stop of its work, it is in the instantaneous description $< q_o; y, m_1, \dots, m_n >$. Then $\Phi_Z(x) = y$ and a value $\mathcal{T}_Z(x)$ is the equal number of steps, which machine Z, has done counting the value of the function $\Phi_Z(x)$.

<u>Definition.</u> A function f(x) is real-time computable if the machine Z exists, that $\Phi_Z = f$ and a number $k \in N$ that $\mathcal{T}_Z \leq k(1 + \Phi_Z)$ holds [1] .

A set of all primitive recursive functions will be designated $\mathcal{R}$ and a set of all primitive recursive unary functions $\mathcal{R}_1$.

A set of all primitive recursive real-time computable functions will be designated $\mathcal{F}_t$.

<u>Definition.</u> An unary total function f is called Yamadan function if it is increasing and there exists a multitape TURING machine, which prints in every step of its computation, one symbol of the sequence $a_0, a_1, a_2, \ldots,$ where $a_m = 1$ if m is a value of the function f and $a_m = 0$ if m is not a value of the function f, onto its output tape.

A set of all primitive recursive Yamadan functions will be designated $\mathcal{F}_y$.

The following theorems, we shall need, are proved in [1] :

The following functions are real-time computable: x, x + y, xy, $[\sqrt{x}]^2$. (Theorem 9.4).

Any superposition of the real-time computable functions is the real-time computable function. (Theorem 9.3).

Let be f(x) real-time computable function, let the g(x) arise from f(x) by the iteration, then the function $\sum_{y=0}^{x} (1+g(y))$ is real-time computable. (Theorem 10.3.).

Let be f(x) real-time computable function, let $f > 0$. Then the function $g(x) = \sum_{y=0}^{x} f(y)$ is Yamadan. (Theorem 10.11).

Chapter 2.

CORRELATION BETWEEN PRIMITIVE RECURSIVE FUNCTIONS AND REAL-TIME COMPUTABLE FUNCTIONS.

A set of all ordered pairs of the unary primitive recursive functions $\langle f_1, f_2 \rangle$, where $f_1 \geqslant f_2$, will be designated $\mathcal{P}$.

The ordered pair of numbers $\langle f_1(x), f_2(x) \rangle$ will be called the value of pair $\langle f_1, f_2 \rangle$ in the point x and designated $\langle f_1, f_2 \rangle (x)$.

If f, g are one-argument partially functions, then the operations +, $\neq$, J are defined in the following way:

$$(f+g)(x) = f(x)+g(x),$$

$$(f \neq g)(x) = f(g(x)) \text{ and}$$

$$g = J(f), \text{ where } g(0) = 0,\ g(x+1) = (f \neq g)(x) = f(g(x)).$$

Let us define the following mappings of the set $\mathcal{P} \times \mathcal{P}$ into the set of all pairs of the primitive recursive functions:

1. The mapping + :

$$\langle f_1, f_2 \rangle + \langle g_1, g_2 \rangle = \langle f_1+g_1, f_2+g_2 \rangle$$

i.e. for all $x \in N$:

$$(\langle f_1, f_2 \rangle + \langle g_1, g_2 \rangle)(x) = \langle f_1(x) + g_1(x), f_2(x) + g_2(x) \rangle .$$

2. The mapping $\neq$:

$$\langle f_1, f_2 \rangle \neq \langle g_1, g_2 \rangle = \langle f_1 \neq (g_1-g_2) + f_1 \neq g_1, f_2(g_1-g_2) + f_1 \neq g_1 \rangle ,$$

i.e. for all $x \in N$:

$$(\langle f_1, f_2 \rangle \neq \langle g_1, g_2 \rangle)(x) = \langle f_1(g_1(x)-g_2(x)) + f_1(g_1(x)), f_2(g_1(x) - g_2(x)) + f_1(g_1(x)) \rangle .$$

3. The mapping J of the set $\mathcal{P}$ into the set of all the ordered pairs primitive recursive functions: Let $\langle f_1, f_2 \rangle \in \mathcal{P}$ then the pair of the functions $\langle \bar{g}_1, \bar{g}_2 \rangle$ is defined:

$$\langle \bar{g}_1, \bar{g}_2 \rangle(0) = \langle 0,0 \rangle , \quad \langle \bar{g}_1, \bar{g}_2 \rangle(x+1) = (\langle f_1, f_2 \rangle \neq \langle g_1, g_2 \rangle)(x),$$

then:

$$(J\langle f_1, f_2 \rangle)(x) = \langle \bar{g}_1, \bar{g}_2 \rangle(x) = \langle \sum_{y=0}^{x} (1+\bar{g}_1(y)) - \sum_{y=0}^{x \dot{-} 1} (1+\bar{g}_2(y)), \sum_{y=0}^{x} (1+\bar{g}_2(y)) - \sum_{y=0}^{x \dot{-} 1} (1+\bar{g}_1(y)) \rangle .$$

<u>Theorem 2.1.</u> The mappings +, $\neq$, J are operations on the set $\mathcal{P}$.

<u>P r o o f.</u> 1. The set $\mathcal{P}$ is closed with respect to the operation +: $(f_1+g_1) - (f_2+g_2) = (f_1-f_2) + (g_1-g_2) \geqslant 0.$

2. The set $\mathcal{P}$ is closed with respect to the operation $\neq$:

$$f_1 \neq (g_1-g_2) + f_1 \neq g_1 - f_2(g_1-g_2) - f_1 \neq g_1 = f_1 \neq (g_1-g_2) - f_2 \neq (g_1-g_2) \geqslant 0.$$

3. The set $\mathcal{P}$ is closed with respect to the operation J:

Let $J(\langle f_1, f_2 \rangle) = \langle g_1, g_2 \rangle$. Then $g_1(0) - g_2(0) = \bar{g}_1(0) + 1 - \bar{g}_2(0) - 1 - \bar{g}_2(0) - 1 + \bar{g}_1(0) + 1 = 0$; and for $x \neq 0$:

$$g_1(x)-g_2(x) = \sum_{y=0}^{x} (\bar{g}_1(y)+1) + \sum_{y=0}^{x-1} (\bar{g}_2(y)+1) - \sum_{y=0}^{x} (\bar{g}_2(y)+1) -$$

$$- \sum_{y=0}^{x-1} (\bar{g}_1(y)+1) = \bar{g}_1(x)+1 - \bar{g}_2(x)-1 = \bar{g}_1(x) - \bar{g}_2(x).$$

By induction will be proved: $\bar{g}_1 - \bar{g}_2 \geqslant 0$.

1. $\bar{g}_1(0) - \bar{g}_2(0) = 0$

2. Let for all $y \leqslant x$, $\bar{g}_1(y)-\bar{g}_2(y) \geqslant 0$, then $\bar{g}_1(x+1) - \bar{g}_2(x+1) =$
$= f_1(\bar{g}_1(x) - \bar{g}_2(x)) + f_1(\bar{g}_1(x)) - f_2(\bar{g}_1(x) - \bar{g}_2(x)) - f_1(\bar{g}_1(x)) \geqslant 0.$

Let us define the mapping Ψ of the set $\mathcal{P}$ into $\mathcal{R}_1$:

$$\Psi(\langle f_1, f_2\rangle) = f_1 - f_2.$$

L e m m a 2.1. The mapping Ψ is the homomorphism of the algebra $(\mathcal{P}, +, \neq J)$ on algebra $(\mathcal{R}_1, + \neq, J)$.

P r o o f . Let $f \in \mathcal{R}_1$, then $\langle f, 0\rangle \in \mathcal{P}$ and $\Psi(\langle f, 0\rangle) = f$ i.e. the mapping Ψ maps the set $\mathcal{P}$ on the set $\mathcal{R}_1$.

1. It will be shown that: $\Psi(\langle f_1, f_2\rangle + \langle g_1, g_2\rangle) =$
$= \Psi(\langle f_1, f_2\rangle) + \Psi(\langle g_1, g_2\rangle).$

$\Psi(\langle f_1, f_2\rangle + \langle g_1, g_2\rangle) = \Psi(\langle f_1+g_1, f_2+g_2\rangle) =$
$= f_1+g_1-f_2-g_2 = (f_1-f_2) + (g_1-g_+) = \Psi(\langle f_1, f_2\rangle) + \Psi(\langle g_1, g_2\rangle).$

2. It will be shown that: $\Psi(\langle f_1, f_2\rangle \neq \langle g_1, g_2\rangle) =$
$= \Psi(\langle f_1, f_2\rangle) \neq \Psi(\langle g_1, g_2\rangle).$

$\Psi(\langle f_1, f_2\rangle \neq \langle g_1, g_2\rangle) = \Psi(\langle f_1 \neq (g_1-g_2) + f_1 \neq g_1,$
$f_2 \neq (g_1-g_2) + f_1 \neq g_1\rangle) = f_1 \neq \Psi(\langle g_1, g_2\rangle) - f_2 \neq \Psi(\langle g_1, g_2\rangle) =$
$= \Psi(\langle f_1, f_2\rangle) \neq \Psi(\langle g_1, g_2\rangle).$

3. It will be shown that: $\Psi(J\langle f_1, f_2\rangle) = J(\Psi(\langle f_1, f_2\rangle)).$
Let $J(\langle f_1, f_2\rangle) = \langle g_1, g_2\rangle$, $\Psi(\langle g_1, g_2\rangle) = g$,

$\Psi(\langle f_1, f_2\rangle) = f$ and $J(f) = g'$.

Then $\Psi(\langle g_1, g_2\rangle(x)) = g_1(x) - g_2(x) = g(x).$
$g(0) = g_1(0)-g_2(0) = \bar{g}_1(0)+1 - \bar{g}_2(0)-1-\bar{g}_1(0)-1+\bar{g}_2(0)+1 = 0,$

$$g(x+1) = g_1(x+1) - g_2(x+1) = \sum_{y=0}^{x+1} (\bar{g}_1(y)+1) - \sum_{y=0}^{x} (\bar{g}_2(y)+1) +$$

$$+ \sum_{y=0}^{x+1} (\bar{g}_2(y)+1) - \sum_{y=0}^{x} (\bar{g}_1(y)+1) = \bar{g}_1(x+1) - \bar{g}_2(x+1) =$$

$$= f_1(\bar{g}_1(x)-\bar{g}_2(x)) + f_1(\bar{g}_1(x)) - f_2(\bar{g}_1(x)-\bar{g}_2(x)) - f_1(\bar{g}_1(x)) =$$

$$= (\Psi(<f_1,f_2>) \neq \Psi(<g_1,g_2>))(x) = (f \neq g)(x).$$

$$J(\Psi(<f_1,f_2>)) = J(f) = g' \text{ where } g'(0) = 0, \quad g'(x+1) = (f \neq g)(x).$$

Thus we showed $g = g'$, that had to be proved.

The homomorphism Ψ determines on the algebra $(\mathcal{P}, +, \neq, J)$ the congruence that will be designated $\vdots$:

$<f_1, f_2> \vdots <g_1 g_2>$, if $f_1 - f_2 = g_1 - g_2$.

The factor algebra $(\mathcal{P}, +, \neq, J)$ with respect to the congruence $\vdots$ will be designated $(\mathcal{D}, +, \neq J)$. The elements of the set $\mathcal{D}$ will be called complexes. The monomorphism of the algebra $(\mathcal{D}, +, \neq, J)$ into the algebra $(\mathcal{R}_1, +, \neq, J)$ induced by the mapping Ψ will be designated φ. The monomorphism φ is obviously an isomorphism.

Let us designate:

$$\bar{\mathcal{K}} = \{<f_1,f_2> \mid <f_1,f_2> \ \vdots\ <x+1, 0>\},$$

$$\bar{\bar{\mathcal{K}}} = \{<f_1,f_2> \mid <f_1,f_2> \ \vdots\ <x, [\sqrt{x}]^2>\}.$$

Then $\varphi(\bar{\mathcal{K}}) = s$ and $\varphi(\bar{\bar{\mathcal{K}}}) = q$. It follows from the isomorphism of the algebras $(\mathcal{R}_1, +, \neq, J)$ and $(\mathcal{D}, +, \neq, J)$, where $\mathcal{R}_1$ is generated by the functions $s(x) = x+1$, $q(x) = x \div [\sqrt{x}]^2$, from the Robinson's theorem, that the algebra $(\mathcal{D}, +, \neq, J)$ will be generated by the complexes $\bar{\mathcal{K}}, \bar{\bar{\mathcal{K}}}$.

<u>L e m m a 2.2.</u> Let f, g are total functions, for which holds:

1. f is the real-time computable function i.e. there exists the machine Z_1 and number $k_1 \in N$ so that $\Phi_{Z_1} = f$ and $\mathcal{T}_{Z_1} \leq k_1(1+f)$,

2. there exists the machine Z_2 and number $k_2 \in N$ so that $\Phi_{Z_2} = g$ and $\mathcal{T}_{Z_2} \leq k_2(1+f)$;

then $f + g$ is real-time computable function.

<u>P r o o f .</u> It will be constructed the machine $Z_0 = \{<q_0, S_1, q_2, q_1>, <q_2, S_1, M, q_3>, <q_3, S_1', P, q_4>, <q_4, S_1'', P, q_1>\}$

and the machines Z_1', Z_2' in the following way: if Z_1 worked with memory cells S_0, $S_1, \dots, S_{r_1}$, we shall change S_i to S_i' in all quadruples of the machine Z_1, when $i \in \{0,1,\dots,r_1\}$. If the machine Z_2 worked with memory cells S_0, $S_1, \dots, S_{r_2}$, we shall change S_i to S_i'' in all quadruples, when $i \in \{0,1,\dots,r_2\}$. For the machines Z_1' and Z_2',

$$\tau_{Z_1'} \leq k_1(1+f) \quad \text{and} \quad \tau_{Z_2'} \leq k_2(1+f)$$

will hold. It will be constructed the machine Z from the machines Z_0, Z_1', Z_2' so that in each cycle of the work of the machine Z, we let the four steps of the machine Z_0 work and after one step Z_1' and Z_2' (what can be accomplished by the appropriate renomination of the steps of the machines Z_0, Z_1', Z_2'). The machine Z will curry on working as long as the machines Z_1' and Z_2' go on. The work of the machine Z consists in the simultaneous work Z_1' and Z_2', whilo the input data will copied from S_1 as quickly as needed. At the places S_0' and S_0'' will be formed the values of the functions f and g. The quadruples, which at the end of the work of Z add contents of the memory cells S_0' and S_0'' into S_0, will be added to Z. Thus a machine Z will be formed for which:

1. $\Phi_Z = f+g$,

2. $\tau_Z \leq 6(\max(k_1(1+f), k_2(1+f)) + 4 + 3(1+f+g) \leq k(1+f+g)$, where $k = 6k_1+6k_2+7$, that means the function $f+g$ is real-time computable;

hold.

T h e o r e m 2.2. Each complex $\mathcal{K}$ contains pair $\langle f_1, f_2 \rangle$, where f_1, f_2 are increasing and $\langle f_1, f_2 \rangle \in \mathcal{F}_t \times \mathcal{F}_t$ holds.

P r o o f . Let $\langle f_1, f_2 \rangle \in \mathcal{K}$. Let us define functions:

$$\bar{f}_1(x) = \sum_{y=0}^{x} (f_1(y)+1) + \left(\sum_{y=0}^{x \dot- 1} (f_2(y)+1)\right) \cdot sg(x) \text{ and}$$

$$\bar{f}_2(x) = \sum_{y=0}^{x} (f_2(y)+1) + \left(\sum_{y=0}^{x \dot- 1} (f_1(y)+1)\right) \cdot sg(x)$$

Therefore $\bar{f}_1(x) - \bar{f}_2(x) = f_1(x) - f_2(x)$; $\bar{f}_1 \geq \bar{f}_2$ and $\langle f_1, f_2 \rangle \equiv \langle \bar{f}_1, \bar{f}_2 \rangle$ holds. It is necessary to prove, that $\bar{f}_1, \bar{f}_2$ are increasing.

$\bar{f}_1(1) - \bar{f}_1(0) = f_1(1) + 1 + f_2(0) + 1 - f_1(0) - 1 - f_2(0) - 1 =$

$= f_1(1) + 1 > 0.$

For $x > 0$.

$$\bar{f}_1(x+1) - \bar{f}_1(x) = f_1(x+1) + 1 + \sum_{y=0}^{x} (f_1(y)+1) + \sum_{y=0}^{x} (f_2(y)+1) -$$

$$- \sum_{y=0}^{x} (f_1(y)+1) - \sum_{y=0}^{x-1} (f_2(y)+1) = f_1(x+1) + 1 + f_2(x) + 1 > 0.$$

In this way we may prove, that the $\bar{f}_2$ is increasing.

The complexes $\bar{\mathcal{K}}$ and $\bar{\bar{\mathcal{K}}}$ contain pairs of the real-time computable functions, because the functions $x+1,\ 0,\ x, [\sqrt{x}]^2 \in \mathcal{F}_t$.

1. Let $\mathcal{K}_1 + \mathcal{K}_2 = \mathcal{K}_3$ and $\langle f_1, f_2 \rangle \in \mathcal{K}_1 \cap (\mathcal{F}_t \times \mathcal{F}_t)$, $\langle g_1, g_2 \rangle \in \mathcal{K}_2 \cap (\mathcal{F}_t \times \mathcal{F}_t)$, then $\langle f_1+g_1, f_2+g_2 \rangle \in \mathcal{F}_t \times \mathcal{F}_t$.

2. Let $\mathcal{K}_1 \neq \mathcal{K}_2 = \mathcal{K}_3$.

Let $\langle f_1, f_2 \rangle \in \mathcal{K}_1 \cap (\mathcal{F}_t \times \mathcal{F}_t)$, $\langle g_1, g_2 \rangle \in \mathcal{K}_2 \cap (\mathcal{F}_t \times \mathcal{F}_t)$

and f_1, f_2, g_1, g_2 be increasing. It will be shown that

$\langle f_1 * (g_1 - g_2) + f_1 * g_1,\ f_2 * (g_1 - g_2) + f_1 * g_1 \rangle \in \mathcal{F}_t \times \mathcal{F}_t$.

The functons $f_1 * (g_1 - g)$, $f_1 * g_1$ satisfy the conditions of the Lemma 2.2. Then the function $f_1 \neq g_1 \in \mathcal{F}_t$. It must be proved that the machine Z exists and $k \in N$, that

$\Phi_Z = f_1 \neq (g_1 - g_2)$ and $\mathcal{T}_Z \leq k(1 + f_1 \neq g_2)$ hold.

The construction of the machine Z : It will be constructed the machine Z^*, composed of two blocks A, B. Block A is constructed of the machines Z_1 and Z_2 from which $\Phi_{Z_1} = g_1$, $\mathcal{T}_{Z_1} \leq k_1(1+g_1)$ and $\Phi_{Z_2} = g_2$, $\mathcal{T}_{Z_2} \leq k_2(1+g_2)$ hold, in this way as in the proof of the Lemma 2.2. After the completion of the block A will be computed the values of the functions $g_1(x)$, $g_2(x)$ in memory cells S_0', S_0'' and the inequality $\mathcal{T}_A \leq k'(1+g_1)$ will hold.

The block B substracts the contents of memory cells S_0', S_0'' and the result is written instead of S_0. Then $\mathcal{T}_Z = \mathcal{T}_A + \mathcal{T}_B =$

$= k'(1+g_1) + k''(1+g_1+g_2) \leq (k' + 2k'')(1+g_1) = \bar{k}(1+g_1)$.

We presume that the machine Z, writes after each k steps one digit to S_0. Apart from this no number is added or subatracted from S_0 [1].

Let Z_0 be a machine and $k_0 \in N$ that $\Phi_{Z_0} = f_1$ and $\mathcal{T}_{Z_0} \leq \leq k_0(1+f_1)$ hold. The machine Z, which will simulate k steps of the work Z' and then one step of the work Z_0, will be constructed. It may be presumed that f_1 is increasing i.e. $x \leq f_1(x)$ for all $x \in N$, value of the function can be formed at a satisfary speed and will be exist k that $\mathcal{T}_Z \leq k(1+f_1 \neq g_1)$ holds. The functions $f_1 \neq (g_1-g_2)$, $f_1 \neq g_1$ satisfy the conditions of the Lemma 2.2 so and that $f_1 \neq (g_1-g_2) + f_1 \neq g_1 \in \mathcal{F}_t$. It can be proved in the same way $f_2 \neq (g_1-g_2) + f_1 \neq g_1 \in \mathcal{F}_t$.

It is proved, that $\mathcal{K}_3 = \mathcal{K}_1 \neq \mathcal{K}_2$ contains a pair of the real-time computable functions.

3. Let $J(\mathcal{K}_1) = \mathcal{K}_2$.

Let $\langle f_1, f_2 \rangle \in \mathcal{K}_1 \cap (\mathcal{F}_t \times \mathcal{F}_t)$, then the functions

$$\bar{g}_1(0) = 0,\ \bar{g}_1(x+1) = f_1(\bar{g}_1(x) - \bar{g}_2(x)) + f_1(\bar{g}_1(x)),$$

$$\bar{g}_2(0) = 0,\ \bar{g}_2(x+1) = f_2(\bar{g}_1(x) - \bar{g}_2(x)) + f_1(\bar{g}_1(x)).$$

are generated by primitive recursion from the real-time computable functions an according the Theorem 10.3 [1] it holds that the functions $\sum_{y=0}^{x} (1+\bar{g}_1(y))$, $\sum_{y=0}^{x} (1+\bar{g}_2(y))$ are real-time computable. Because the sum of the real-time computable functions is the real-time computable function, it holds:

$$\sum_{y=0}^{x} (1+\bar{g}_1(y)) + \sum_{y=0}^{x\dot{-}1} (1+\bar{g}_2(y)),\ \sum_{y=0}^{x} (1+\bar{g}_2(y)) + \sum_{y=0}^{x\dot{-}1} (1+\bar{g}_1(y)) \rangle \in \mathcal{F}_t \times \mathcal{F}_t .$$

Due to the fact that algebra $(\mathcal{D}, +, \neq, J)$ is generated by the complexes $\mathcal{K}$ and $\mathcal{K}$, each complex contain a pair $\langle f_1, f_2 \rangle$, where $f_1, f_2 \in \mathcal{F}_t$.

<u>Theorem 2.3.</u> Each complex $\mathcal{K}$ contains a pair $\langle f_1, f_2 \rangle$ where f_1, f_2 are Yamadan functions.

P r o o f . Let $< f_1, f_2 > \in \mathcal{X} \cap (\mathcal{F}_t \times \mathcal{F}_t)$. It will be denoted $f_1, f_2 > 0$. Let us define:

$$\bar{f}_1(x) = \sum_{y=0}^{x+1} f_1(y) + \sum_{y=0}^{x} f_2(y),$$

$$\bar{f}_2(x) = \sum_{y=0}^{x+1} f_2(y) + \sum_{y=0}^{x} f_1(y).$$

According to theorem 10.10 [1] the functions $\bar{f}_1, \bar{f}_2$ are Yamadan. It is clear, that if $f(x) \in \mathcal{F}_y$, then $g(x)$: $g(0) < f(0)$ and $g(x+1) = f(x)$ is Yamadan. Let us construct the functions g_1, g_2 in the following way:

$$g_1(0) = f_1(0), \quad g_1(x+1) = \bar{f}_1(x) \quad \text{and}$$

$$g_2(0) = f_2(0), \quad g_2(x+1) = \bar{f}_2(x).$$

Because $f_1(0) < \bar{f}_1(0)$ and $f_2(0) < \bar{f}_2(0)$ are the functions $g_1, g_2 \in \mathcal{F}_y$.
It holds: $g_1(0) - g_2(0) = f_1(0) - f_2(0)$,

$$g_1(x+1) - g_2(x+1) = \bar{f}_1(x) - \bar{f}_2(x) = \sum_{y=0}^{x+1} f_1(y) + \sum_{y=0}^{x} f_2(y) - \sum_{y=0}^{x+1} f_2(y) - \sum_{y=0}^{x} f_1(y) = f_1(x+1) - f_2(x+1)$$

so and that $< g_1, g_2 > \in \mathcal{X} \cap (\mathcal{F}_y \times \mathcal{F}_y)$.

T h e o r e m 2.4. Each primitive recursive function may be expressed as the difference of the two real-time computable primitive recursive functions.

P r o o f . On the basis of the isomorphism of the algebras $(\mathcal{D}, +, \neq, J)$ and $(\mathcal{R}_1, +, \neq J)$ it will be corresponded complex $\mathcal{X} = \varphi^{-1}(f)$ to each function $f \in \mathcal{R}_1$. According to Theorem 2.2 exists $< \bar{f}_1 \bar{f}_2 > \in \mathcal{X}$ that $\bar{f}_1, \bar{f}_2 \in \mathcal{F}_t$ and $f = \bar{f}_1 - \bar{f}_2$.

T h e o r e m 2.5. Each primitive recursive function may be expressed as the difference of the two primitive recursive Yamadan functions.

P r o o f . On the basis of the isomorphism of the algebras

$(\mathcal{D}, +, \neq J)$ and $(\mathcal{R}_1, +, \neq J)$ it will be corresponded complex $\mathcal{X} = \varphi^{-1}(f)$ to each function $f \in \mathcal{R}_1$. According to Theorem 2.3 exists $< f_1, f_2 >$ that $f_1, f_2 \in \mathcal{F}_y$ and $f = f_1 - f_2$.

Chapter 3.

ALMOST PRIMITIVE RECURSIVE FUNCTIONS

It will be designated X_m a n-tuple of the natural numbers $(x_1, x_2, \ldots, x_n)$.

D e f i n i t i o n . A function $f(X_n)$ is called an almost primitive recursive function if it can be obtained by a finite number of applications of substitution and primitive recursion from the functions $\underline{0}$, s, $\emptyset$, I^n_m, where $\underline{0}$ is unary function identically equal 0, and $\emptyset$ is 0-ary newhere defined function.

Similary, will be designated $\underline{1}$ the 0-ary function identically equal 1.

The class of the all almost primitive recursive functions will be designated $\mathcal{P}$. The algebra $(\mathcal{P}, S, R)$, where S is operation of the substitution and R is the operation of primitive recursion, by the functions $\underline{0}$, $\emptyset$, s, I^n_m.

An operator Cs will be defined in following way: Let $f(X_n)$ be a partial recursive function, then Cs f = g, where

$$g(X_n) = \begin{cases} f(X_n) + 1 & \text{for all } X_n, \text{ where } f(X_n) \text{ is defined,} \\ 0 & \text{elsewhere.} \end{cases}$$

A value of the function $f(X_n)$ in the point X_n will be designated $(Cs\ f)(X_n)$.

L e m m a 3.1. If f is an almost primitive recursive function, the Cs f is the primitive recursive function.

P r o o f . 1. It will be shown, that Cs $\underline{0}$, Cs s, Cs $\emptyset$ and Cs I^n_m are primitive recursive functions. In fact: Cs $\underline{0}$ = $\underline{1}$, Cs s = s+$\underline{1}$, Cs $\emptyset$ = $\underline{0}$ and Cs $I^n_m = I^n_m + \underline{1}$.

2. It will be shown, if $f = S(g, g_1, \ldots, g_m)$ and Cs g, Cs g_1, Sc $g_2, \ldots,$ Cs $g_m \in \mathcal{R}$, then Cs $f \in \mathcal{R}$. Let

$$l = \prod_{i=1}^{m} sg(Cs\ g_i) . S(Cs\ g,\ (Cs\ g_1) \dot{-} 1, \ldots, (Cs\ g_m) \dot{-} \underline{1})$$

It follows from the suppositions of the theorem and the form of the record of the function, that the function l is primitive recursive.

The value of the function $l(X_n) = 0$ in two following cases:

1. if the value of the function $\prod_{i=0}^{m} sg(Cs\ g_i) = 0$ in the point X_n, i.e. at least one of the values $g_1(X_n), g_2(X_n), \ldots, g_m(X_n)$ is not defined, i.e. if $f(X_n)$ is not defined, or if

2. the value of the function $S(Cs\ g,\ (Cs\ g_1) \dot{-} \underline{1}, \ldots, (Cs\ g_m) \dot{-} \underline{1}) = 0$ in the point X_n, i.e. if $g(g_1(X_n), \ldots, g_m(X_n))$ is not defined, i.e. $f(X_n)$ is not defined.

If the function $f(X_n)$ is defined, then $((Cs\ g_1) \dot{-} \underline{1})(X_n) =$ $= g_1(X_n), \ldots, ((Cs\ g_m) \dot{-} \underline{1})(X_n) = g_m(X_n)$ and $\prod_{i=1}^{m} sg(Cs\ g_i) = 1$.

In this case:

$$l(X_n) = (Cs\ g)\ (g_1(X_n), \ldots, g_m(X_n)) = g(g_1(X_n), \ldots, g_m(X_n)) + 1.$$

Summarized as follows:

$$l(X_n) = \begin{cases} 0 & \text{if } f(X_n) \text{ is not defined} \\ g(g_1(X_n), \ldots, g_m(X_n)) + 1 & \text{if } f(X_n) \text{ is defined,} \end{cases}$$

i.e. $l = Cs\ f$.

3. It will be shown if $f = R(g_1, g_2)$ and $Cs\ g_1$, $Cs\ g_2$, then $Cs\ f$.

Let $l = R(r_1, r_2)$, where

$$l(X_n, 0) = r_1(X_n) = (Cs\ g_1)(X_n),$$

$$l(X_n, y+1) = r_2(X_n, y, l(X_n, y)) = sg(l(X_n, y)).(Cs\ g_2)(X_n, y, l(X_n, y)).$$

It follows from the suppositions of the theorem and the form of the record of the function, that the function l is primitive recursive. By means of induction we can prove, that $l = Cs\ f$:

1. The value $l(X_n) = (Cs\ g_1)(X_n) = 0$ if $g_1(X_n)$ is not defined, i.e. the value $f(X_n, 0)$ is not defined. If $f(X_n, 0)$ is defined, then $g_1(X_n)$ is defined and $l(X_n, 0) = g_1(X_n) + 1$.

Summarized as follows:

$$l(X_n,0) = \begin{cases} 0 & \text{if } f(X_n,0) \text{ is not defined} \\ g_1(X_n) + 1 & \text{if } f(X_n,0) \text{ is defined,} \end{cases}$$

so and that $(Cs\ f)\ (X_n,0) = l(X_n,0)$.

2. Let us suppose for all $t \leq y$, $(Cs\ f)\ (X_n,t) = l(X_n,t)$ holds. The value $l(X_n,y+1) = 0$ in two following cases:

i. if $sg(l(X_n,y)) = 0$, i.e. $f(X_n,y)$ is not defined, that means, the value $f(X_n,y+1)$ is not defined, or if

ii. $(Cs\ g_2)(X_n,y,l(X_n,y)) = 0$, i.e. $g_2(X_n,y,f(X_n,y))$ is not defined, so and that $f(X_n,y+1)$ is not defined.

If $f(X_n,y+1)$ is defined, then $sg(l(X_n,y)) = 1$ and $l(X_n,y+1) = g_2(X_n,y,f(X_n,y)) + 1$.

Summarized as follows:

$$l(X_n,y+1) = \begin{cases} 0 & \text{if } f(X_n,y+1) \text{ is not defined} \\ g_2(X_n,y,f(X_n,y)) + 1 & \text{elsewhere,} \end{cases}$$

i.e. $(Cs\ f)(X_n,y+1) = l(X_n,y+1)$.

<u>T h e o r e m 3.1.</u> The function $f(X_n)$ is almost primitive recursive if and only if the primitive recursive function $g(X_n)$ exists, that for all X_n, $f(X_n) = g(X_n) - 1$ holds.

<u>P r o o f .</u> The functions $f = Cs\ f - \underline{1}$, $Cs\ f \in \mathcal{R}$.
It will be shown, that $r(x) = x-1$ is almost primitive recursive:
Let

$k = R(0,\emptyset)$
$$k(x) = \begin{cases} 0 & \text{if } x = 0 \\ \text{not defined} & \text{if } x \neq 0, \end{cases}$$

$d = S(k,q)$, where $q = 1 \dot{-} x$,
$$d(x) = \begin{cases} \text{not defined} & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

$r = d.l$, where $l = x \dot{-} 1$,
$$r(x) = \begin{cases} \text{not defined} & \text{if } x = 0 \\ x \dot{-} 1 & \text{if } x \neq 0 \end{cases}$$

i.e. $r(x) = x-1$.

Let $g(X_n) \in \mathcal{R}$, then $f(X_n) = g(X_n) - 1 \in \mathcal{P}$, because $f = S(r,g)$.

<u>Theorem 3.2.</u> The difference between two primitive recursive functions is the almost primitive recursive function.

<u>Proof.</u> Let $f = g_1 - g_2$, where $g_1, g_2 \in \mathcal{R}$ and $f = (g_1 + \underline{1} \dot{-} g_2) - \underline{1}$. It follows from the preceding theorem, that $f \in \mathcal{P}$, because $g_1 + \underline{1} \dot{-} \dot{-} g_2 \in \mathcal{R}$.

<u>Theorem 3.3.</u> The following conditions are equivalent for the each unary partially function f:

1. f is an almost primitive recursive function.

2. f is the differonce between two primitive recursive realtime computable functions.

3. f is the difference between two primitive recursive Yamadan functions.

<u>Proof.</u> a. The equivalence of the conditions 1 and 2: Let $f \in \mathcal{P}$, then $f = g - \underline{1}$, where $g \in \mathcal{R}$. According to Theorem 2.4 $g_1, g_2 \in \mathcal{F}_t$ exists, that $g = g_1 - g_2$ holds; then $f = g_1 - (g_2 + \underline{1})$. The converse is the corollary of the Theorem 3.2.

b. The equivalence of the conditions 1 and 3; Let $f \in \mathcal{P}$, then $f = g - \underline{1}$, where $g \in \mathcal{R}$. According to Theorem 2.5 $g_1, g_2 \in \mathcal{F}_y$ exists, from its follows $g_2 + \underline{1} \in \mathcal{F}_y$, that $f = g_1 - (g_2 + \underline{1})$ is a difference between two Yamadan functions.

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