(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXII, 1969)

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE

MATHEMATICA XXII

1969

SLOVENSKÉ PEDAGOGICKÉ NAKLADATEĽSTVO
BRATISLAVA

(ACTA F. R. N. UNIV. COMEN. – MATHEMATICA XXII, 1969)

UNIVERSITAS COMENIANA

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BRATISLAVA

Dňa 10. mája toho roku sa dožíva 70. narodenín akademik O. B o r ů v k a, profesor Prírodovedeckej fakulty Univerzity J.E. Purkyně v Brne. S jeho menom sú spojené mnohé veľké úspechy československej matematiky na vedeckom poli a bohatá pedagogická činnosť. Prof. O. Borůvka sa zapísal do myslí pracovníkov Univerzity J.A. Komenského svojou nezištnou prácou, venovanou rozvoju matematického bádania a pedagogickej práce na našich katedrách. Slávnostné číslo Acta fac.rer.nat. Univ. Comen. venované nášmu vzácnemu jubilantovi je skromným prejavom našej úcty a vďaka za jeho celoživotné dielo.

Redakčná rada

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K SEDEMDESIATINÁM AKADEMIKA OTAKARA BORŮVKU

M. KOLIBIAR

Dňa 10. mája t.r. sa dožíva sedemdesiatin významný československý matematik, akademik OTAKAR BORŮVKA, profesor Prírodovedeckej fakulty univerzity v Brne. Matematický život v Brne a v Bratislave je veľmi úzko zviazaný s týmto menom a mnohé úspechy brnenskej a bratislavskej matematickej školy sa dosiahli jeho pričinením.

Prof. O. BORŮVKA študoval na českej vysokej škole technickej v Brne, potom na Prírodovedeckej fakulte Masarykovej univerzity v Brne. Dva roky študoval u prof. E. CARTANA na Sorbone a potom ešte jeden rok u prof. BLASCHKEHO v Hamburgu.

Od svojho vynikajúceho učiteľa M. LERCHA a od prof. E. ČECHA, ktorý v tom čase pôsobil v Brne, získal lásku ku klasickej analýze a modernej diferenciálnej geometrii. Jeho práce z diferenciálnej geometrie sú zamerané jednak na projektívnu diferenciálnu geometriu, jednak na plochy konštantnej krivosti vo viacrozmerných priestoroch. O. BORŮVKA bol prvým v Československu, ktorý pracoval Cartanovými metódami a tým prispel k ich rozšíreniu u nás. Jeho práce z diferenciálnej geometrie sú často citované a niektoré výsledky sú uvádzané v učebniciach diferenciálnej geometrie. Na jeho práce o analytických korešpondenciách nadväzuje celý rad prác, v ktorých sa buduje sústavná teória korešpondencií. Napr. geometrická škola v Bologni vyšla z týchto Borůvkových prác.

Druhým významným okruhom práce prof. Borůvku je algebra. Ku koncu tridsiatych rokov publikoval práce o grupoidoch, ktorými sa začína budovať rozsiahla časť modernej algebry. Je dobre si uvedomiť, že teória algebraických systémov s jednou binárnou operáciou do značnej miery odzrkadľuje aj všeobecnejšiu situáciu, keď je operácií viac a nemusia pritom byť binárne. Takto Borůvkove práce do istej miery predstihli a myšlienkove pripravili teóriu abstraktných algebier, ktorá sa v posledných rokoch rozvíja. Veľkou zásluhou prof. Borůvku je jeho rozpracovanie teórie rozkladov v množine. Táto teória má mimoriadny význam nielen pre matematiku, ale pre vedu vôbec. V súvislosti s teóriou grupoidov spočíva význam Borůvkových výsledkov aj v tom, že ukázal, že abstraktné jadro viacerých algebraických viet a konštrukcií tvoria vlastne množinovo-teoretické výsledky, týkajúce sa rozkladov množín. Niet divu, že jeho knižka o tejto problematike bola v matematickom svete veľmi priaznivo prijatá a v dôsledku toho vyšla aj v Nemecku.

Na začiatku svojej vedeckej dráhy publikoval prof. O. BORŮVKA niekoľko prác z klasickej analýzy. Neskoršie prešiel na štúdium diferenciálnych rovníc, kde vypracoval osobitné metódy. Dôležitým aparátom jeho teórie sú špeciálne funkcie, tzv. disperzie, ktoré popisujú rozptyl nulových bodov integrálov danej lineárnej diferenciálnej rovnice. Vypracoval tiež teóriu transformácií lineárnych diferenciálnych rovníc, umožňujúcu napr. transformovať rovnicu do jednoduchšieho tvaru. Teóriu diferenciálnych rovníc rozvíja prof. BORŮVKA v seminári, ktorý pracuje už celý rad rokov a ktorého účastníci široko rozvinuli myšlienky svojho učiteľa a ukázali ich nosnosť.

Za jeho prácu sa mu dostalo viacero pôct a vyznamenaní. Tak napr. v r. 1934 dostal cenu Českej akadémie, bol vyznamenaný štátnou cenou Klementa Gottwalda a stal sa riadnym členom Československej akadémie vied.

Charakteristickým rysom práce prof. O. Borůvku je, že zapája do práce na svojej problematike čo najširší okruh pracovníkov. Vďaka svojim osobným vlastnostiam podarilo sa mu vždy získať kvalitných spolupracovníkov a počet tých matematikov, ktorí mu vďačia za svoj vedecký rast, je nemalý. Aj matematici v Brarislave a na Slovensku sú mu mimoriadne zaviazaní. V období, keď matematické katedry v Bratislave boli len v zárodku, prof. O. BORŮVKA po dobu 11 rokov pravidelne prednášal na Prírodovedeckej fakulte Univerzity Komenského v Bratislave. Avšak prednášky netvorili jedinú náplň jeho činnosti v Bratislave. S veľkou obetavosťou a mimoriadnym taktom získaval pre vedeckú prácu mládež, sústavne ju viedol. Na výsledky tejto svojej práce môže pozerať s radosťou, rovnako ako jeho žiaci s vďakou a láskou myslia na svojho "staršieho priateľa", lebo patrí ku kúzlu jeho osobnosti, že pôsobí viac priateľským taktom, než učiteľskou autoritou.

A keď na každoročných výletoch, ktoré prof. BORŮVKA organizuje pre matematikov z celej republiky, zaznie družná pieseň, je to prejav neustálej mladosti nášho jubilanta, mladosti, ktorá nachádza trvalé pramene v spojení s mladými.

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A REMARK TO THE ASYMMETRY OF FUNCTIONS

LADISLAV MIŠÍK

To Professor Otakar Borůvka on the occasion of his 70th birthday

In the paper [4] T. ŚWIATKOWSKI studied the properties of the asymmetry of real functions which are defined on a topological space. The fundamental theorem of his paper is the theorem 1. The first part of this theorem is used to obtain the theorem the special cases of which are W.H. YOUNG's theorem [5] about the asymmetry points of arbitrary real functions of a real variable and M. KULBACKA's theorem [2] about the approximative asymmetry points of such functions. The purpose of this paper is to show that the first part of the theorem 1 by T. ŚWIATKOWSKI is a simple consequence of one theorem on ordered sets. Further on we shall show how easy we can get W.H. YOUNG's and M. Kulbacka's theorems from this theorem on ordered sets.

Definition 1. Let A and B be two ordered sets and let B be a complete lattice, i.e. for every C, $\emptyset \neq C \subset B$, there exists sup C and inf C. Let φ be a function from A into B. Let $\mathcal{E}$ be a system of down-ward directed subsets of A. Then we put:

$$\limsup \varphi(\mathcal{E}) = \inf \{ \inf \{ \sup \{ \varphi(a) : a \in E,\ a \leqq b \} : b \in E \} : E \in \mathcal{E} \}$$

and

$$\liminf \varphi(\mathcal{E}) = \sup \{ \sup \{ \inf \{ \varphi(a) : a \in E,\ a \leqq b \} : b \in E \} : E \in \mathcal{E} \}.$$

Definition 2. Let B an ordered set and $\mathfrak{m}$ a transfinite cardinal number. A subset A of B is called a $\mathfrak{m}$-ideal iff the following conditions are satisfied:

1. For a, $b \in B$, $a \leqq b$, $b \in A$ is a an element of A.

2. For every subset C of A the cardinal number of which is not greater than $\mathfrak{m}$ sup C belongs to A if it exists.

If $A \subset B$, then the minimal $\mathfrak{m}$-ideal over A exists always and we shall denote it by $[A]_{\mathfrak{m}}$.

<u>Theorem 1.</u> Let A be an ordered set and let B be a complete lattice. Let $\mathfrak{E}$ be a system of down-ward directed subsets of A which is $\mathfrak{m}$ -separable, i.e. there exists a subset H of A of the cardinal number $\mathfrak{m}$, $H \subset \bigcup \{E : E \in \mathfrak{E}\}$, for which the following condition is satisfied: for every $E \in \mathfrak{E}$ and every $a \in E$ there exists an element $h \in H \cap E$ so that $h \leqq a$. Let ψ be a function from A into B. Then $\liminf \psi(\mathfrak{E}) \in [\psi(\bigcup \mathfrak{E})]_{\mathfrak{m}}$ whereby $\psi(\bigcup \mathfrak{E}) = \{\psi /a/ : a \in E$ for at least one $E \in \mathfrak{E}\}$.

<u>Proof.</u> Evidently $\psi /h/ \in \psi(\bigcup \mathfrak{E})$ for every $h \in H$. There exists $c = \sup \{\psi /h/ : h \in H\}$ because $\{\psi /h/ : h \in H\} \subset B$ and B is a complete lattice. The cardinal number of the set $\{\psi /h/ : h \in H\}$ is not greater than $\mathfrak{m}$ and therefore $c \in [\psi(\bigcup \mathfrak{E})]_{\mathfrak{m}}$.

It is evident: $\{\inf \{\psi /a/ : a \in E, a \leqq b\} : b \in H \cap E\} \subset \{\inf\{\psi /a/ : a \in E, a \leqq b\} : b \in E\}$ for every set $E \in \mathfrak{E}$. Therefore $\sup \{\inf \{\psi /a/ : a \in E, a \leqq b\} : b \in H \cap E\} \leqq \sup\{\inf \{\psi /a/ : a \in E, a \leqq b \{ : b \in E\}$ for every set $E \in \mathfrak{E}$. Now let $b \in E$. Then there exists an element $h \in H \cap E$ for which $h \leqq b$. Hence we have $\inf \{\psi /a/ : a \in E, a \leqq b\} \leqq \inf\{\psi /a/ : a \in E, a \leqq h\}$. Therefore it follows that: $\sup \{\inf \{\psi /a/ : a \in E, a \leqq b\} : b \in E\} = \sup \{\inf\{\psi /a/ : a \in E, a \leqq b\} : b \in H \cap E\}$ for every set $E \in \mathfrak{E}$. Further we have: $\sup \{\inf \{\psi /a/ : a \in E, a \leqq b\} : b \in E\} = \sup\{\inf\{\psi /a/ : a \in E, a \leqq b\} : b \in H \cap E\} \leqq \sup \{\psi /b/ : b \in H \cap E\} \leqq \sup \{\psi /b/ : b \in H\} = c$ for every set $E \in \mathfrak{E}$. Hence $\liminf \psi(E)\, c$ and further $\liminf \psi(\mathfrak{E}) \in [\psi(\bigcup \mathfrak{E})]_{\mathfrak{m}}$.

Let X be a topological space and Y a Hausdorff topological space which satisfies the second countability axion of Hausdorff /i.e. axiom /10/ of [1], p. 229/ /1/, i.e. that there exists a countable basis of open sets. Let $\mathfrak{M}$ be a σ -structure of subsets of X. Let f be a $\mathfrak{M}$ -measurable function from X into Y, i.e. $f^{-1}/G/ \in \mathfrak{M}$ for every open subset G of Y. Let φ be a function from $\mathfrak{M}$ into 2^X /2^X is the set of all subsets of X/. The function f is said-to be φ -regular at the point $x \in X$ iff for every $y \in Y$ and every open neighbourhood U of y there exists an open neighbourhood U' of this point y such that $U' \subset U$ and $x \in \varphi /f^{-1}/U'//$. The set of all points at which f is φ -regular we shall denote by $\varphi /f/$. Let $\varphi^* /f/ = X - \varphi /f/$. Then it holds:

/1/ In the theorem 1 T. ŚWIATKOWSKI ought to suppose that Y satisfies the second countability axiom of Hausdorff instead of the supposition that Y is a separable space.

Corollary /The first part of T. Świątkowski's theorem/. Let $\mathfrak{N}$ be a σ-ideal [2] of subsets of X. For every $\mathfrak{M}$-measurable function f we have $\varphi^*/f/ \in \mathfrak{N}$ if $X - \varphi/M/ \in \mathfrak{N}$ for every $M \in \mathfrak{M}$.

Proof. Now we must put: A = the system of all open sets in Y, $B = 2^X$ with ordering by set-inclusion, $\psi/a/ = X - \varphi/f^{-1}/a//$ for every $a \in A$, $\xi = \{\{U : U$ is an open subset of Y which contains $y\} : y \in Y\}$. Now we shall prove that $\varphi^*/f/ = \liminf \psi(\xi)$.

An element x is of $\varphi^*/f/$ iff f is not φ-regular at x. But f is not φ-regular at x iff there exists an element $y \in Y$ and an open set U containing y with the property: $x \notin \varphi/f^{-1}/U'// = X - \psi/U'/$, i.e. $x \in \psi/U'/$ for all open subsets U' of U containing y. The last assertion is equaivalent to the following one: there exists an element $y \in Y$ and an open set U containing y with the property: $x \in \inf\{\psi/U'/ : U'$ is an open subset of U containing $y\}$. From that we obtain: $x \in \varphi^*/f/$ iff there exists an element $y \in Y$ for which $x \in \sup\{\inf\{\psi/U'/ : U'$ is an open subset of U containig $y\} : U$ is an open subset of Y containing $y\}$. Now the following assertion is evident: $x \in \varphi^*/f/$ iff $x \in \sup\{\sup\{\inf\{\psi/U'/ : U'$ is an open subset of U containing $y\} : U$ is an open subset of Y containing $y\} : y \in Y\}$. This gives $\varphi^*/f/ = \liminf \psi(\xi)$.

$\psi/U/$ is of $\mathfrak{N}$ for every nonempty open set U since f is $\mathfrak{M}$-measurable and $X - \varphi/f^{-1}/U// \in \mathfrak{N}$. Therefore $\psi(\bigcup \xi) \subset \mathfrak{N}$ and further $[\psi(\bigcup\xi)]_{\aleph} \subset \mathfrak{N}$. But $\varphi^*/f/ \in [\psi(\bigcup\xi)]_{\aleph}$ according to theorem 1. This gives $\varphi^*/f/ \in \mathfrak{N}$.

By using theorem 1 for the case: $A = 2^{(-\infty,\infty)}$ and $B = 2^{(-\infty,\infty)}$ with ordering by set-inclusion, $\xi = \{\{/y - \varepsilon, y + \varepsilon/ : \varepsilon > 0 : y \in /-\infty,\infty/\}$ and $\psi/X/$ is the set of all one-sided limit points of the set $f^{-1}/X/$, whereby X is a subset of $/-\infty,\infty/$ and f is a real function of a real variable, we get W.H. Young's theorem for the function f since $\psi/X/$ is a countable set for every $X \subset /-\infty,\infty/$.

2/ σ-ideal is $\aleph_0$-ideal.

Let A be a subset of $/-\infty, \infty/$. The point x is said to be the right /left/ approximative limit point of A iff the set $A \cap /x,\infty/$ $//-\infty,x/ \cap A/$ has at the point x the outer upper density / [3] , p. 128/ positive, i.e.

$$\overline{\lim_{\varepsilon \to 0+}} \frac{|A \cap /x,x+\varepsilon/|}{\varepsilon} > 0 \quad \overline{\lim_{\varepsilon \to 0+}} \frac{|A \cap /x - \quad ,x/|}{\varepsilon} > 0/ \text{ /3/}.$$

P r o p o s i t i o n . Let A be a subset of $/-\infty, \infty/$. The set of all its one-sided approximative limit points is of the first Baire's category and of Lebesgue's measure 0.

P r o o f . Let A^+ $/A^-/$ be the set of all right /left/ approximative limit points of A which are not left /right/ approximative limit points of A. Let $A_n^+ = \{ x : x \in /-\infty, \infty/,$

$$\overline{\lim_{\varepsilon \to 0+}} \frac{|A \cap /x,x+\varepsilon/|}{\varepsilon} > \frac{1}{n} \text{ and } \frac{|A \cap /x-\eta,x/|}{\eta} < \frac{1}{2n} \text{ for all } \eta$$

which satisfy the inequality $0 < \eta \leq \frac{1}{n} \}$. We shall prove that A_n^+ are nowhere dense.

If the set A_n^+ was not nowhere dense, there should exist an open interval J of the length nor greater than $\frac{1}{n}$ which is a subset of the closure of A_n^+ . Let $x \in J \cap A_n^+$. There exists an element $y \in J$ which satisfies the inequalities $y > x$ and $\frac{|A \cap /x,y/|}{y - x} > \frac{1}{n}$. The function $\frac{|A \cap /x,y/|}{y - x}$ as a function of y is continuous and therefore there exists an element $z \in J \cap$ $\cap A_n^+$ with the following properties: $z > y$ and $\frac{|A \cap /x,z/|}{z - x} > \frac{1}{n}$. The inequality $\frac{|A \cap /x,z/|}{z - x} > \frac{1}{n}$ is in contradiction with the definition of A_n^+ because $0 < z - x \leq \frac{1}{n}$ and $z \in A_n^+$, i.e. that there ought to be $\frac{|A \cap /x,z/|}{z - x} <$ $< \frac{1}{2n}$.

Since the equality $A^+ = \bigcup \{ A_n^+ : n = 1, 2, 3, \ldots \}$ is evident the set A^+ is of the first Baire's category in $/-\infty, \infty/$. Analogously we can prove

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/3/ $|B|$ denotes the Lebesgue's outer measure of the set B.

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that the set A^- is of the first Baire's category in $/-\infty, \infty/$. This proves the first assertion of our proposition.

It is evident that no point of A^+ can be a point of outer density / [3] p. 128/ neither of the set A nor of the set $/-\infty, \infty/ - A$. From the equality $A^+ = /A^+ \cap A/ \cup /A^+ \cap //-\infty, \infty/ - A/$ and the well-known density theorem / [3] , p. 129/ it follows that $|A^+| = 0$. Analogously $|A^-| = 0$. Also the Lebesgue's measure of the set of all one-sided approximative limit points of A is equal to zero.

If we apply the theorem 1 to the case: $A = 2^{/-\infty, \infty/}$, $B = 2^{/-\infty, \infty/}$ which we order by set-inclusion, $\xi = \{\{ /y - \varepsilon , y + \varepsilon / : \varepsilon > 0\} : y \in /-\infty, \infty/ \}$ and $\psi /X/$ is the set of all one-sided approximative limit points of the set $f^{-1}/X/$, whereby X is a subset of $/-\infty, \infty/$ and f is a real function of a real variable, we get from our proposition M. Kulbacka's theorem, i.e. the theorem: The set of all approximative asymmetry points of a real function of a real variable is of the first Baire's category and of the Lebesgue's measure zero /4/.

/4/ The second part of this assertion was given and proved for the first time by A. Császár, see footnote of [2] on p. 91.

R E F E R E N C E S

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REMARK ON THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF THE DIFFERENTIAL EQUATIONS

MARKO ŠVEC, Bratislava

To Professor Otakar Borůvka on the occasion of his 70^th^ birthday

In this remark we deal with the asymptotic behaviour of the solutions of the equation

$$(E) \qquad y^{(n)} = B(t, y, y', \dots, y^{(n-1)})$$

If the function $B(t, u)$, $u = (u_0, u_1, \dots, u_{n-1})$ is small, in certain sense then the asymptotic behaviour of all solutions of (E) can be as the behaviour of the solutions of the equation $z^{(n)} = 0$, i.e.

$$y(t) = t^p[A + \varphi(t)], \ \lim_{t\to\infty} \varphi(t) = 0 \quad \text{where}$$
$p = 0, 1, \dots, n-1$.

This case was studied by authors [1] - [3]. Our aim is to study the problem of the existence of such numbers p and A_i, $i = 0, 1, \dots, n-1$ and of such solution of (E) that the formulae

$$(P) \qquad \lim_{t\to\infty} \frac{y^{(i)}(t)}{t^{p-i}} = A_i$$

hold. The number of such solutions is discused also. For linear nonhomogeneous differential equation such problem was discused by T.G. Hallam [5] - [7]. In nonlinear case [4], [8], [9], [11].

We will prove the following theorem:

T h e o r e m 1. Let $B(t, u)$ be a continuous function in the range

$$\Omega : t > a \qquad |u_i| < \infty, \ i = 0, 1, \dots, n-1$$

Let N be a set of all functions $\varphi(t)$ such that $\varphi(t)$ is continuous for $t>a$ and $\lim\limits_{t\to\infty}\varphi(t)=0$ Let ρ, A_i, $i=0,1,\dots,n-1$, be the real numbers such that the following equations are fulfilled:

$$/1/\quad \lim \frac{B(t, t^{\rho}[A_0+\varphi_0(t)], t^{\rho-1}[A_1+\varphi_1(t)], \dots, t^{\rho-n+1}[A_{n-1}+\varphi_{n-1}(t)]}{t^{\rho-n}} =$$

$$= (\rho-n+1)A_{n-1}$$

$$/2/\quad A_{i+1} = (\rho-i)\,A_1,\ i=0,1,\dots,n-1.$$

where $\varphi_i(t)\in N$,

Let $F(t,u)$ be a function continuous on Ω , non decreasing in each of its variables u_i, $i=0,1,\dots,n-1$ and such that

$$/3/\quad |B(t,u)| \leq F(t,u) \qquad \text{for } (t,u)\in\Omega,$$

$$/4/\quad V(K)=\int^{\infty}\frac{F(t, Kt^{\rho}, Kt^{\rho-1}, \dots, Kt^{\rho-n+1})}{t^{\rho-n+1}}\,dt<\infty \text{ for } K>0$$

$$/5/\quad \lim_{K\to\infty}\frac{V(K)}{K}=0$$

Then the equation (E) has the solution $y(t)$ having the property (P) If $\rho>n-1$ then there exist at least n solutions of (E) having the property (P). If $0\leq K-1<\rho\leq K=n-1$, there exist at least such solutions of (E).

P r o o f . Let H be the set of all functions $f(t)$ having the continuous derivative of order $n-1$ on $J=[t_0,\infty)$, $t_0>1$ and such that

$$/6/\quad \lim_{t\to\infty}\frac{f^{(i)}(t)}{t^{\rho-i}} = a_i,\ i=0,1,\dots,n-1$$

where $a_i\in R^1$. Let for $f(t)\in H$

$$/7/\quad \|f\| = \max_i\left\{\sup_J \frac{|f^{(i)}(t)|}{t^{\rho-i}}\right\}$$

be the norm. Then H is a Panach space. The strong convergence in H implies the quasi-convergence (q -convergence) [Yee [10]]

Let now
$$M = \left\{ f \in H \,\middle|\, \lim_{t\to\infty} \frac{f^{(j)}(t)}{t^{\rho-j}} = A_j,\ j = 0,1,\dots,n-1 \right\}$$

and

/8/
$$M_K = \left\{ f \in M \,\middle|\, \|f\| \leq K \right\}$$

It is evident that $M_K \subset H$ and M_K is convexe and closed and for every $f \in M_K$ there exist the functions $\varphi_i(t) \in N,\ i = 0,1,\dots,n-1$ such that $f^{(i)}(t) = t^{\rho-1}[A_i + \varphi_i(t)]$.

A. Let $\rho > n-1$. Let $C_0, C_1, \dots, C_{n-1}$ be arbitrary real numbers . We define operator T on M as follows: If $f \in M$, then

/9/
$$Tf = V(t) = \sum_{j=0}^{n-1} C_j \frac{(t-t_0)^j}{j!} + \int_{t_0}^{t} \frac{(t-\tau)^{n-1}}{(n-1)!} B(\tau, f(\tau))\, d\tau$$

Then

/10/
$$V^{(i)}(t) = \sum_{j=i}^{n-1} C_j \frac{(t-t_0)^{j-i}}{(j-i)!} + \int_{t_0}^{t} \frac{(t-\tau)^{n-i-1}}{(n-i-1)!} B(\tau, f(\tau))\, d\tau$$

We will prove some lemmas:

a/
$$TM \subset M$$

By use of l'Hospital's rule and, /10/, /1/, /2/ we have

$$\lim_{t\to\infty} \frac{V^{(i)}(t)}{t^{\rho-i}} = \lim_{t\to\infty} \frac{B(t, f(t))}{(\rho-i)(\rho-i-1)\dots(\rho-n+1)}$$

$$= \frac{1}{(\rho-i)(\rho-i-1)\cdots(\rho-n+2} A(n-1) = A_i,\ i = 0,1,\dots,n-1 .$$

b/ There exists such $K_0 > 0$ that $TM_{K_0} \subset M_{K_0}$

It follows from /10/ that

$$\left| \frac{v^{(i)}(t)}{t^{\rho-1}} \right| \leq \sum_{j=0}^{n-1} |C_j|\, t^{j-\rho} + \frac{t^{n-i-1}}{t^{\rho-1}} \int_{t_0}^{t} |B(\tau, f(\tau))]\, d\tau$$

$$/11/ \quad \leq \sum_{j=0}^{n-1} |c_j| + \int_{t_0}^{\infty} \frac{|B(\tau, f(\tau))|}{\tau^{p-n+1}} d\tau \leq$$

$$\leq \sum_{j=0}^{n-1} |c_j| + \int_{t_0}^{\infty} \frac{F(\tau, K\tau^p, K\tau^{p-1}, \ldots, K\tau^{p-n+1})}{\tau^{p-n+1}} d\tau$$

$i = 0, 1, \ldots, n-1$ Now from this and /5/ follows the existence of such that $\|Tf\| = \|V\| \leq K_0$.

c/ T is quasi-continuous on M_{K_0}.

Let $f_s, f \in M_{K_0}, f_s$ q -converge to f. Then using the same reasoning as above we get
$$\frac{\|V_s^{(i)}(t) - V^{(i)}(t)\|}{t^{p-i}} = \int_{t}^{\infty} \frac{|B(\tau, f_s(\tau)) - B(\tau, f(\tau))|}{\tau^{p-n+1}} d\tau$$

Because $|B(\tau, f_s(\tau)) - B(\tau, f(\tau))| \leq 2F(\tau, K\tau^p, K\tau^{p-1}, \ldots, K\tau^{p-n+1})$

by use of Lebegues theorem we have from the above inequality

$\|V_s(t) - V(t)\| \to 0$ as $s \to \infty$.

d/ The convexe and closed envelope S of TM_{K_0} is q -compoct.

Let $g(t) \in TM_{K_0}$ Then $\frac{|g^{(i)}(t)|}{t^{p-i}} \leq K_0$, $i = 0, 1, \ldots, n-1$

That means that the functions $g^{(i)}(t) \in (TM_{K_0})^{(i)}$ are uniformly bounded on every closed finite subinterval of $\mathcal{J}$. Therefore for $i = 0, 1, \ldots, n-2$ they are also equicontinuous on every closed subinterval of $\mathcal{J}$. We prove that the functions $g^{(n-1)}(t) \in (TMK_0)^{(n-1)}$ are equicontinuous on every closed subinterval of $\mathcal{J}$.

Let $g(t) = Tf$, $f \in M_{K_0}$. Then

$$g^{(n-1)}(t) = C_{n-1} + \int_t B(\tau, f(\tau)) d\tau$$

Using /3/ and monotonocity of $F(t, u)$ we have that for

$t_1, t_2 \in [\alpha, \beta] \subset \mathcal{J}$ holds $|g^{(n-1)}(t_2) - g^{(n-1)}(t_1)| = |$

$$\left|\int_{t_1}^{t_2} B(\tau, f(\tau)) d\tau\right| \leq \left|\int_{t_1}^{t_2} F(\tau, K_0\tau^p, K\tau^{p-1}, \ldots, K_0\tau^{p-n+1}) d\tau\right| =$$

$$= \max_{[\alpha,\beta]} F(\tau, K_0\tau^p, K\tau^{p-1}, \ldots, K_0\tau^{p-n+1}) |t_2 - t_1|.$$

This completes the statement that the functions of TM_{K_0} and their derivatives of order i, $i = 0, 1, \dots, n-1$ are uniformly bounded and equaicontinuous on every closed subinterval of J.

Make now the convex and closed envelope S of TM_{K_0}. Then the functions of S are also uniformly bounded and equaicontinuous on every closed subinterval af J from what follows that S is q- -compact / See [10] .

Thus S is convex, closed and q -compact set and $TS \subset TM_{K_0} \subset S$ because $TM_{K_0} \subset S \subset M_{K_0}$. Following the theorem 2 from [10] T has at least one fixed point in M_{K_0} which is the solution of (E) having the property (P).

It is easy to see that in this cese the set of solutions of (E) having the property (P) depend on n parameters.

B. Let $0 \leq K-1 < p < K \leq n-1,\ K = 1, 2, \dots, n-1.$

Let M be the same set of functions as in part A. We define operator T on M as follows: If f M, then

$$/12/\quad Tf = v(t) = \sum_{j=0}^{k-1} c_j \frac{(t-t_0)^j}{j!} - \int_{t_0}^{t} \frac{(t-\tau)^{K-1}}{(K-1)!} \int_{\tau}^{\infty} \frac{(\tau-s)^{n-k-1}}{(n-k-1)!} B(s,f(s))\,ds\,d\tau$$

For the derivatives of V we have the formulae

$$v^{(i)}(t) = \sum_{j=i}^{K-1} c_j \frac{(t-t_0)^{j-i}}{(j-i)!} - \int_{t_0}^{t} \frac{(t-\tau)^{k-i-1}}{(k-i-1)!} \int_{\tau}^{\infty} \frac{(\tau-s)^{n-K-1}}{(n-K-1)!} B(s,f(s))\,ds\,d\tau$$

$$\underline{0 \leq i \leq K-1}$$

$$/13/\quad v^{(i)}{}_t = -\int_{\tau}^{\infty} \frac{(t-s)^{n-i-1}}{(n-i-1)!} B(s,f(s))\,ds, \quad K \leq i \leq n-1.$$

It can be proved in the same way as in part A that $TM \subset M$.
We omit the proof.

We will prove that there exists such $K_0 > 0$ that $TM_{K_0} \subset M_{K_0}$
From /13/ we get

$$\left| \frac{v^{(i)}(t)}{t^{p-i}} \right| \leq \sum_{j=0}^{K-1} |c_j| + \frac{t^{K-i-1}}{t^{p-i}} \int_{t_0}^{t} \int_{\tau}^{\infty} (s-\tau)^{n-K-1} B(s,f(s))\,ds\,d\tau$$

$$\le \sum_{j=0}^{k-1} |c_j| + \frac{1}{K-p}\int_{t_0}^{\infty} s^{n-p-1} F(s, Ks^{p}, \dots, Ks^{p-n+1})\, ds$$

for $0 \le i \le K-1$ and

$$\left| \frac{v^{(i)}(t)}{t^{p-i}} \right| \le \sum_{j=0}^{K-1} |c_j| + \int_{t_0}^{\infty} s^{n-p-1} F(s, Ks^{p}, \dots, Ks^{p-n+1})\, ds$$

for $K \le i \le n-1$

Thus

$$\|v\| \le \sum_{j=0}^{K-1} |c_j| + \frac{1}{K-p}\int_{t_0}^{\infty} s^{n-p-1} F(s, Ks^{p}, \dots, Ks^{p-n+1})\, ds$$

Respecting /5/ we get the existence of such $K_0 > 0$ that $TM_{K_0} \subset M_{K_0}$.

By the same reasoning as in part A we get that T is $\mathcal{Q}$ -continuous on M_{K_0} the convexe and closed envelope S of TM_{K_0} is $\mathcal{Q}$ -compact and that $TS \subset S$. So using the theorem 2 from [10] we have that T has at least one fixed point in M_{K_0}

The set of solutions of (E) having the property (P) depend in this case at least on K parameter $c_j, j = 0, 1, \dots, K-1$.

C. Let $p = K, K = 0, 1, \dots, n-1$.

Let $c_j, j = 0, 1, \dots, K-1,$ be arbitrary real numbers. Let M denote the same as above. Then we define operator T on M as follows:

If $f \in M$ then

$$Tf = v(t) = \sum_{j=0}^{K-1} c_j \frac{(t-t_0)^j}{j!} + A_K \frac{(t-t_0)^K}{K!} - \int_{t_0}^{t} \frac{(t-\tau)^{K-1}}{(K-1)!} \int_{\tau}^{\infty} \frac{(\tau-s)^{n-k-1}}{(n-k-1)!} B(s, f(s))\, ds\, d\tau$$

Then

$$v^{(i)}(t) = \sum_{j=0}^{K-1} c_j \frac{(t-t_0)^{j-i}}{(j-i)!} + A_K \frac{(t-t_0)^{K-i}}{(K-i)!} - \int_{t_0}^{t} \frac{(t-\tau)^{K-i-1}}{(K-i-1)!} \int_{\tau}^{\infty} \frac{(\tau-s)^{n-K-1}}{(n-K-1)!} B(s, f(s))\, ds\, d\tau$$

$0 \le i \le K-1,$

$$v^{(k)}(t) = A_k - \int_t^{\infty} \frac{(t-s)^{n-k-1}}{(n-k-1)!} B(s, f(s))\, ds,$$

$$v^{(i)}(t) = -\int_t^{\infty} \frac{(t-s)^{n-i-1}}{(n-i-1)!} B(s, f(s))\, ds,\ k < i \leq n-1.$$

The statement and its proof is the same as in part B. D. Let $\rho < 0$

In this case we define the operator T on M as follows:
If $f \in M$ then

$$Tf = v(t) = -\int_t^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} B(s, f(s))\, ds,$$

and

$$v^{(i)}(t) = -\int_t^{\infty} \frac{(t-s)^{n-i-1}}{(n-i-1)!} B(s f(s))\, ds,$$

$$i = 1, 2, \ldots, n-1.$$

The proof the existence of a fixed point of T in M can be made in the same way as above. In this case we get at least one solution of (E) having the property (P).

<u>Remark</u>. Let exist a continuous fonction $F(t)$ for $t > a$ and let

/14/ $|B(t, u)| \leq F(t)$ for all $(t, u) \in \Omega$

and

/14/ $\int_t^{\infty} n - \rho - 1 \quad F(t)\, dt < \infty$

Then, if we substitute /3/ by /14/, /4/ by /15/ /the condition /5/ is now fulfilled/ the validity of the theorem 1 remains.

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11. ---------- Sur un probleme aux limites /to appear/.

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TWO REMARKS ON THE PROPERTIES OF SOLUTIONS OF A NONLINEAR DIFFERENTIAL EQUATION

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To Professor Otakar Borůvka on the occasion of his 70th birthday

1. Let us consider the differential equation

$$y''(x) + f(x)\,|y(x)|^{\alpha}\,\mathrm{sgn}\,y(x) = 0, \qquad (r)$$

where $0 < \alpha < 1$ and the function $f(x)$ is continuous on the interval $\langle 0, \infty)$. In the case $f(x) > 0$ it was proved in [1] that all the solutions of (r) are oscillatory if and only if $\int^{\infty} x^{\alpha} f(x)dx = \infty$. In the following theorem we shall omit the assumption $f(x) > 0$ and it will be proved that all solutions of (r) are oscillatory also in the case if $f(x)$ "alternates". For $\alpha > 1$ this problem was solved in the papers [6] and [4]. Let us note that from the known theorems on the extension of solutions it follows that all solutions of the equation (r) can be extended over the whole interval $\langle 0, \infty)$ [see 5].

<u>Theorem 1.</u> Let there exist a number β satisfying $0 \le \beta \le \alpha$ such that

$$\int^{\infty} x^{\beta} f(x)dx = \infty, \qquad (1)$$

then all solutions of (r) are oscillatory.

<u>Proof.</u> Suppose that the equation (r) has a nonoscillatory solution $y(x)$. Without loss of generality, it can be supposed that $y(x) > 0$ for $x \ge x_0 > 0$. The substitution $t = 1/x$, $y = xu$ transforms (r) into the form

$$u''(t) + f(x)x^{3+\alpha}|u(t)|^{\alpha}\,\mathrm{sgn}\,u(t) = 0 \qquad (2)$$

$$(u' = \frac{du}{dt},\ x = 1/t).$$

It is evident that to the solution $y(x)$ of the equation (r) corresponds the solution $u(t)>0$ for $0<t\leq t_0 = 1/x_0$ of the equation (2). Let us denote $1+\alpha-\beta = \gamma$, then from (2) after an arrangement we have

$$\left(\frac{\dot{u}(t)}{u^{\alpha}(t)}\right)^{\cdot} t^{\gamma} + \alpha\frac{\dot{u}^2(t)}{u^{\alpha+1}(t)}\, t^{\gamma} = -f(1/t)t^{-\beta-2} \quad \text{for } 0<t\leq t_0 .$$

Hence integrating over the interval $<t,\ t_0>(t_0>0)$ and using the second mean value theorem for integral $\int_{t_0}^{t}\frac{\dot{u}(z)}{u^{\alpha}(z)}\, z^{\gamma-1}dz$ we get

$$-\frac{\dot{u}(t)}{u^{\alpha}(t)}\, t^{\gamma} + \frac{\gamma}{1-\alpha}\, t_0^{\gamma-1}u^{1-\alpha}(\nu) + \alpha\int_{t}^{t_0}\frac{\dot{u}^2(z)}{u^{\alpha+1}(z)}\, z^{\gamma}dz = c_1 - \int_{t}^{t_0} f(1/z)z^{-\beta-2}dz, \quad (3)$$

where $c_1 = \dfrac{\dot{u}(t_0)}{u^{\alpha}(t_0)}\, t_0^{\gamma} + \dfrac{\gamma}{1-\alpha}\, t_0^{\gamma-1}\, u^{1-\alpha}(t_0),\ t\leq\nu\leq t_0.$

According to (1) $\int_{t}^{t_0} f(1/z)z^{-\beta-2}dz\to\infty$ for $t\to 0^+$, hence by (3) we have

$$-\frac{\dot{u}(t)}{u^{\alpha}(t)}\, t^{\gamma}\to -\infty \text{ for } t\to 0^+. \quad (4)$$

Hence $u(t)>0$ in some interval $(0,T>(T<t_0)$, therefore $u(t)$ is a bounded function over this interval. If we use the given transformation we get

$$\frac{y'(x)}{y(x)}\, x^{\beta} = \frac{u(t)-x^{-1}\dot{u}(t)}{x^{\alpha}u^{\alpha}(t)}\, x^{\beta} = u^{1-\alpha}(t)t^{\alpha-\beta} - \frac{\dot{u}(t)}{u^{\alpha}(t)}\, t^{\gamma}.$$

The last equality, preceding considerations and (4) give for all $x\geq x_1> x_0$ the following inequality $\frac{y'(x)}{y^{\alpha}(x)}\, x^{\beta} < -1$. Integrating this inequality over

over the interval $<x_1,\ x>$ we get

$$y^{1-\alpha}(x) < -\frac{1-\alpha}{1-\beta}\, x^{1-\beta} + c_2, \text{ where } c_2 = \frac{1-\alpha}{1-\beta}\, x_1^{1-\beta} + y^{1-\alpha}(x_1).$$

From this it is evident that for sufficiently large x, $y(x)<0$. But this is a contradiction and thus all solutions of (r) are oscillatory.

2. The solution $y(x)$ of (r) is said to be left (right) singular at the point x_0 if $y(x_0) = y'(x_0) = 0$ and in every neighbourhood of x_0 there exist two numbers $x_1, x_2 < x_0$ $(x_1, x_2 > x_0)$ such that $y(x_1) \neq 0$, $y(x_2) = 0$. The solution $y(x)$ of (r) is said to be singular, if there exists a point $x_0 \in \langle 0, \infty)$ such that this solution is left (right) singular at x_0.

In what follows it will be given an example of the function $f(x)$, under which equation (r) has a singular solution. The function $f(x)$ will be constructed in the same way as in [3] , where for $\alpha = 3$ was proved the existence of a solution which cannot be extended into the whole interval $\langle 0, \infty)$. For the sake of the simplicity of the calculation we put in (r) $\alpha = 1/5$. Then the following assertion holds:

There exists a constant $c > 0$ such that the equation

$$U''(t) + cU^{1/5}(t) = 0 \tag{5}$$

has a solution $U(t)$, which satisfies the following boundary conditions

$$U(0) = U(1) = 1, U'(0) = U'(1) = 0 \tag{6}$$

and has exactly two zeros in the interval $(0,1)$.

This assetion follows, besides another, from the fact that all solutions of (5) are periodic with the period

$T = A^{-2/5}\left(\frac{48}{5c}\right)^{1/2}\int_0^1 (1 - z^{6/5})^{-1/2}dz$, where A is amplitude of a solution.

If we put $A = 1$, $T = 1$ then the existence of a number $c > 0$ follows such that (6) holds.

In the following a lemma, proved in [3] , will be introduced and adapted to our purposes.

<u>L e m m a 1.</u> For every positive integer n there exists a function $q_n(t)$, continuous in the interval $\langle 0,1 \rangle$ such that the equation

$$U''(t) + (c + q_n(t))\, U^{1/5}(t) = 0 \tag{7}$$

has a solution $U_n(t)$ having two zeros in the interval $(0,1)$ and satisfying the boundary conditions

$$U_n'(0) = U_n'(1) = 0,\ U_n(0) = 1,\ U_n(1) = \left(\frac{n}{n+1}\right)^5 \tag{8}$$

Moreover the function $q_n(t)$ are such that $q_n(0) = q_n(1) = 0$ and $\lim_{n\to\infty} q_n(t) = 0$ uniformly on the interval $\langle 0,1\rangle$.

P r o o f . Let $U(t)$ be a solution of (5) satisfying (6) and $0 < t_0 < 1$ such that $U(t) > 0$ in the interval $\langle t_0, 1\rangle$. Define $U_n(t)$ as follows

$$U_n(t) = \begin{cases} U(t) \text{ for } 0 \leq t \leq t_0 \\ \left(\frac{n}{n+1}\right)^5 - 1 + U(t) - \int_t^1 (t - s) f_n(s)ds \text{ for } t_0 \leq t \leq 1. \end{cases} \tag{9}$$

The functions $f_n(t)$ will be chosen to be continuous in the interval $\langle t_0, 1\rangle$, bounded for all n by some constant and satisfying the boundary conditions

$$\int_{t_0}^1 (t_0 - s)f_n(s)ds = \left(\frac{n}{n+1}\right)^5 - 1, \quad \int_{t_0}^1 f_n(s)ds = 0 \tag{10}$$

$$f_n(t_0) = 0, \quad f_n(1) = \frac{c}{n+1} .$$

The first three conditions from (10) ensure the continuity of functions $U_n(t)$, $U_n'(t)$ and $U_n''(t)$ for every n in the interval $\langle 0,1\rangle$. It is also evident that t_0 can be so chosen in order that $U_n(t) > 0$ in the interval $\langle t_0, 1\rangle$. Then $U_n(t)$ has two zeros in the interval$(0,1)$ and satisfies (8). If we define $q_n(t) = -(c + U_n^{-1/5}(t)\, U_n''(t))$, then $q_n(t)$ is continuous function in the interval $\langle 0,1\rangle$ for every n and $q_n(0) = q_n(1) = 0$, which follows from (6) and the fourth condition of (10). Thus $U_n(t)$ is a solution of (7) satisfying (8). It remains only to prove that the functions with the requested properies can be found.

It turns out that it can be e.g. the functions of the form $f_n(t) = a_n(t - t_0)^3 + b_n(t - t_0)^2 + c_n(t - t_0)$. The third condition of (10) is satisfied and the rest ones give us the following linear system

$$\begin{aligned} &a_n/5\ (1 - t_0)^5 + b_n/4\ (1 - t_0)^4 + c_n/3\ (1 - t_0)^3 = 1 - \left(\frac{n}{n+1}\right)^5 \\ &a_n/4\ (1 - t_0)^4 + b_n/3\ (1 - t_0)^3 + c_n/2\ (1 - t_0)^2 = 0 \\ &a_n(1 - t_0)^3 + b_n(1 - t_0)^2 + c_n(1 - t_0) = \frac{c}{n+1} \end{aligned} \tag{11}$$

Since the determinant of the system (11) $D \neq 0.$, we can calculate the coefficients a_n, b_n and c_n. In additon to a constant K exists such that $\max(|a_n|, |b_n|, |c_n|) \leq K/n$ for $n = 1, 2, 3, \dots$, which follows from the right-hand side of the system (11). If we choose therefore a_n, b_n, c_n in such a way, all requiremnts on $f_n(t)$ are satisfied. From this, by the definiton of $q_n(t)$ and by (7) it follows that $\lim_{n\to\infty} q_n(t) = 0$ uniformly in the interval $\langle 0,1 \rangle$ (see [3]). Thus the lemma is proved.

By using the preceding considerations we are going to construct a function $f(x)$ in order to be continuous in the interval $\langle 0,\infty)$ and equation (r) to have for $\alpha = 1/5$ a singular solution.

Let the sequence $\{\nu_n\}$ be such that $\nu_1 = 0$, $\nu_n = \sum_{k=1}^{n-1} \frac{1}{k^2}$, then $\lim_{n\to\infty} \nu_n = \pi^2/6$. For $x \in \langle 0, \pi^2/6)$ let us define $f(x)$ and $y(x)$ as follows

$$
\begin{aligned}
f(x) &= c + q_n(n^2(x - \nu_n)) \quad \text{for} \quad \nu_n \leq x \leq \nu_{n+1} \\
y(x) &= 1/n^5 U_n(n^2(x - \nu_n)) \quad \text{for} \quad \nu_n \leq x \leq \nu_{n+1}
\end{aligned}
\tag{12}
$$

The function $f(x)$ defined in this manner is continuous in the interval $\langle 0,\pi^2/6)$. Because $\lim_{n\to\infty} q_n(t) = 0$ uniformly on $\langle 0,1 \rangle$, it follows that $\lim_{x\to\pi^2/6} f(x) = c$. Therefore the function $f(x)$ can be continuously extended on the whole interval $\langle 0,\infty)$. By (8) also the functions $y(x)$, $y'(x)$ are continuous in the interval $\langle 0,\pi^2/6)$ and by (12) in every of the intervals $\langle \nu_n, \nu_{n+1} \rangle$ $y(x)$ is a solution of equation (r) for $\alpha = 1/5$. It is evident that $y(\nu_n) = 1/n^5$ and by (9) in any of the intervales $\langle \nu_n, \nu_{n+1} \rangle$ $y(x)$ and $y'(x)$ has at least one zero. From this and from the possibility of the extension of the solution $y(x)$ it follows that $y(\pi^2/6) = y'(\pi^2/6) = 0$. Thus the solution $y(x)$ of (r) is left singular at the point $\pi^2/6$.

From the preceding example it is evident that the continuity of a positive function $f(x)$ does not eliminate a singular solution of (r). However if we require some additional assumptions, then (r) has no such a solution.

T h e o r e m 2. Let there exist a number $x_1 > x_0$ ($x_1 < x_0$) such that for $x \in (x_0, x_1)$ the function $f(x)$ is nonnegative, continuously differentiable and the following inequality holds

$$f(x) \geqslant \frac{x_0 - x}{3 + \alpha} f'(x). \tag{12}$$

Under these assumptions equation (r) has no right (left) singular solution at the point x_0.

P r o o f. Let $y(x)$ be such a solution (the proof for $x_1 < x_0$ is similar). Transform the equation (r) by $t = 1/(x - x_0)$, $y = (x - x_0)u$. Then to the solution $y(x)$ corresponds an oscillatory solution $u(t)$ of the equation

$$\ddot{u}(t) + q(t)\,|u(t)|^{\alpha}\,\mathrm{sgn}\,u(t) = 0, \tag{13}$$
$$\left(\dot{u} = \frac{du}{dt}\right)$$

where $q(t) = t^{-3-\alpha} f(x_0 + 1/t)$. From the proof of Theorem 1 of [2] it follows for the equation (13) the following assetion: If the function $q(t) \geqslant 0$ is continuously differentiable, where $\dot{q}(t) \leqslant 0$ for $t \in (c, \infty)$ /c is a suitable number) and $\int^{\infty} tq(t)dt < \infty$, then all the solutions of (13) are nonoscillatory (besides a trivial one). Using this fact our theorem will be proved easily, since the assumption of the continuous differentiability of $q(t)$ and the convergence of $\int_{t_0}^{\infty} tq(t)dt$ are satisfied and by (13) we get

$$\dot{q}(t) = -\,t^{-4-\alpha}\left\{(3 + \alpha)f(x) + (x - x_0)f'(x)\right\} \leqslant 0.$$

Therefore equation (13) has no oscillatory solution and this contradiction proves the assertion.

From the preceding considerations it follows that if $f'(x)$ is continuous in the interval $\langle 0, \infty)$, then equation (r) has no singular solution.

The existence of a singular solution of (r) is closely related with the uniqueness of solutions of (r). It is evident that uniqueness of (r) can fail only on the x-axis. In the points where a singular solution of (r) exists there fails the uniqueness. In the next we prove that uniqueness can fail only in these points.

Let x_0 be a point on the x-axis in which does not exist a singular solution. Suppose that equation (r) has two nonnegative solutions $v(x)$, $z(x)$ such that $v(x_0) = z(x_0) = 0$, $v'(x_0) = z'(x_0)$ and in a neighbourhood of x_0 for $x > x_0$, $u(x) > v(x)$ holds (similarly for $x < x_0$). Then from (r) we get.

$$u(x) - v(x) = \int_{x_0}^{x}\int_{x_0}^{t} f(\gamma)\left\{ |v(\gamma)|^{\alpha} \operatorname{sgn} v(\gamma) - |u(\gamma)|^{\alpha} \operatorname{sgn} u(\gamma)\right\} d\gamma dt \leq 0,$$

and hence $u(x) \leq v(x)$, which contradicts the assumption. Therefore if such solutions exist they have to intersect infinitely many times in every neighbourhood of x_0. However, this is impossible what follows e.g. from the following lemma.

L e m m a 2. Let the function $f(x) \geq 0$. Let $u(x)$, $v(x)$ and $z(x)$ be solutions of (r) such that $0 \leq u(x) < v(x)$, $u(x) < z(x)$ for $x \in (a,b)$ $(a \geq 0)$. Then solutions $z(x)$ and $v(x)$ can intersect in this interval only once.

P r o o f . Let x_1, x_2 be such points that $a < x_1 < x_2 < b$, $v(x_1) = z(x_1)$, $v(x_2) = z(x_2)$ and for $x \in (x_1,x_2)$ hold $u(x) < v(x) < z(x)$. Consider the function

$$F(x(= (z(x) - v(x))\,(v'(x) - u'(x)) - (v(x) - u(x))\,(z'(x) - v'(x)).$$

Since

$$F'(x) = (z(x) - v(x))\,(f(x)u^{\alpha}(x) - f(x)v^{\alpha}(x)) -$$
$$- (v(x) - u(x))\,(f(x)v^{\alpha}(x) - f(x)z^{\alpha}(x)),$$

then integrating over the interval $\langle x_1,x_2\rangle$, for the sake of concavity, from the preceding equality we get

$$F(x_2) - F(x_1) =$$
$$= \int_{x_1}^{x_2} f(x)\left\{(v(x) - u(x))(z^{\alpha}(x) - v^{\alpha}(x)) - (z(x) - v(x))(v^{\alpha}(x) - u^{\alpha}(x)\right\} dx \leq 0.$$

On the orther hand we have

$$F(x_1) = - (v(x_1) - u(x_1))(z'(x_1) - v'(x_1)) < 0,$$
$$F(x_2) = - (v(x_2) - u(x_2))(z'(x_2) - v'(x_2)) > 0,$$

but this contradicts (14). Thus the solutions $z(x)$ and $v(x)$ can intersect at most once in the interval (a,b).

If we put $u(x) \equiv 0$, then by Lemma 2 it is evident that solutions $v(x)$ and $z(x)$ cannot intersect more than once, unless $v(x)$ intersects the x-axis and this proves our assertion.

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THE DECOMPOSITION OF A DIRECTED GRAPH INTO QUADRATIC FACTORS CONSISTING OF CYCLES

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To Professor Otakar Borůvka on the occasion of his 70^{th} birthday

We understand under a graph in this paper a finite graph without loops.

Let G be a directed graph. In the following we denote by V_G the set of vertices of G; by E_G the set of edges of G; by $E_G(\rightarrow v)$ $[E_G(v \rightarrow)]$ the set of such edges of G that are oriented to [from] the vertex v, respectively.

Under an indegree [outdegree] of a vertex v we understand the cardinal number of the set $E_G(\rightarrow v)$ $[E_G(v\rightarrow)]$ and we shall denote it by $\pi_G(v)$ $[\omega_G(v)]$ respectively. The sum $\sigma_G(v) = \pi_G(v) + \omega_G(v)$ we call as usual the degree of v in G. We say that a vertex v is a spring in G if $\pi_G(v) = 0$; if $\omega_G(v) = 0$, v is called a receiver in G.

G is said to be a ρ-graph if for every $v \in V_G$ we have $\pi_G(v) = \omega_G(v)$. It is clear that a degree of each vertex of a ρ-graph is an even number. Under a ρ-factor of a graph G we understand a factor of G, which is a ρ-graph.

Let G be a directed graph and $V_G = \{v_1, v_2, \ldots, v_p\}$. Let F be a directed graph constructed as follows:

(1) $V_F = \{u_1, u_2, \ldots, u_p, w_1, w_2, \ldots, w_p\}$;

(2) $E_F = E_G$ and moreover for $i = 1, 2, \ldots p$ holds:

$$[e \in E_G(\rightarrow v_i)] \Rightarrow [e \in E_F(\rightarrow u_i)]\ ;\ [f \in E_G(v_i \rightarrow)] \Rightarrow [f \in E_F(w_i \rightarrow)].$$

Then we say that the graph F arose from the graph G by the η-transformation and we write $F = \eta(G)$. It is obvious that every $e \in E_F$ is oriented from a vertex $w_x \in W = \{w_1, w_2, \dots w_p\}$ to a vertex $u_y \in U = \{u_1, u_2, \dots, u_p\}$. Hence F is a bipartite graph. The set W is a set of springs of F and U is a set of receivers of F. It is clear, that $\sigma_F(u_i) = \mathcal{R}_F(u_i) = \mathcal{R}_G(v_i)$; $\sigma_F(w_i) = \omega_G(v_i)$ holds. The following two assertions are obvious (see e.g. BERGE [1] corollary to Theorem 2, chapter XI):

L e m m a 1. Let G be a regular ρ -graph of degree 2n with p vertices, then $F = \eta(G)$ is a bipartite regular graph of degree n with 2p vertices in which every edge is oriented from some spring to some receiver (i.e. F does not contain any cycle).

L e m m a 2. A directed graph G is decomposable into quadratic ρ -factors if and only if G is a regular ρ -graph.

R e m a r k 1. Directly from the definition of the ρ -factor it is clear that every component of a quadratic ρ -factor is a cycle. Therefore a decomposition into quadratic ρ -factors means nothing else than a decomposition into quadratic factors composed of cycles only. It is well-known (see e.g. KÖNIG [2], p.29 that every ρ -graph which need not be regular and only a ρ -graph is decomposable into cycles. The Lemma 2 is dealing with the case when we can distribute these cycles into n quadratic factors.

L e m m a 3. Every ρ -tournament can be decomposed into quadratic ρ -factors.

P r o o f. Lemma 3 follows from Lemma 2.

Let G be a regular ρ -graph of the degree 2n ($n > 1$) with p vertices, $V_G = \{v_1, v_2, \dots v_p\}$ and let the graph F arose from G by an η -transformation; $V_F = \{u_1, u_2, \dots, u_p, w_1, w_2, \dots w_p\}$. Let H be a graph which arose from F by adding another of the edges $h_1, h_2, \dots, h_p$ such that the edge h_i is oriented from u_i to w_i. It is clear that G is decomposable into n Hamiltonian cycles if and only if there exists such a decomposition $L^* = \{L(1), L(2), \dots, L(n)\}$ of F into linear factors that for every $i = 1, 2, \dots n$ we have: The quadratic factor $L(0) \cup L(i)$ where $L(0)$ is the linear factor of H containing all edges and only the edges from $\{h_1, h_2, \dots, h_p\}$ is a Hamiltonian cycle of H.

L e m m a 4. If the decomposition $L^* = \{L(1), L(2), \dots, L(n)\}$ has the required property then for the number of circuits $q_{i,j}$ in the quadratic factor $Q_{i,j} = L(i) \cup L(j)$ for every $i \neq j$; $i,j \in \{1,2,\dots,n\}$ we have $q_{i,j} \equiv p \pmod 2$.

P r o o f. Let $L^* = \{L(1), L(2), \dots, L(n)\}$ be such a decomposition of F into linear factors that $L(0) \cup L(i)$ is a Hamiltonian cycle of H for every $i = 1,2,\dots,n$ (where $L(0)$ contains the edges of the set $\{h_1, h_2, \dots, h_p\}$. Let $i \neq j$ be two arbitrary numbers from $\{1,2,\dots,n\}$. We denote by $C_{i,j}$ the cubic graph $L(0) \cup L(i) \cup L(j)$ and by $q_{x,y}$ the number of the circuits in its quadratic factor $L(x) \cup L(y)$. In [3] it is proved: $q_{0,i} + q_{0,j} + q_{i,j} \equiv p \pmod 2$. In our case we have $q_{0,i} = q_{0,j} = 1$, from which the assertion of our lemma follows immediately.

T h e o r e m. Let G be a φ-tournament with $2n+1$ vertices. Its decomposition into Hamiltonian cycles does not exist if every decomposition $L^* = \{L(1), L(2), \dots, L(n)\}$ of $F = \eta(G)$ into linear factors has the following property: at least one its quadratic factor $Q_{i,j} = L(i) \cup L(j)$ contains an even number of circuits.

P r o o f of this theorem follows from Lemma 4.

R e m a r k 2. We hope that simplifications following from our consideration and from our Theorem (the problem of finding a decomposition of a φ-graph into Hamiltonian cycles is equivalent to the problem of finding such a decomposition of some bipartite regular graph into linear factors $L_0, L_1, \dots, L_n$ [whereby L_0 is a given its linear factor] that every quadratic factor $L_0 \cup L_i$ is a Hamiltonian circuit - the orientation of edges is not essential - and we may exclude from our considerations all the decompositions in which at least one quadratic factor $Q_{i,j}$ has an even number of circuits) will be useful at solving following unsolved problem: Is it possible to decompose every φ-tournament with $2n+1$ vertices into n Hamiltonian cycles?

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A REMARK ON NON DIRECT PRODUCT OF MEASURES

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To Professor Otakar Borůvka on the occasion of his 70^{th} birthday

The paper [1] contains a result asserting the following:

(A) Let λ be a non negative finitely additive set function defined on an algebra $\mathcal{A} \otimes \mathcal{B}$ of subsets of $X \times Y$ such that (i) $\lambda(X \times Y) = 1$, (ii) λ is a non direct product of μ and ν where μ and ν are non negative finitely additive set functions defined on the algebras $\mathcal{A}$ and $\mathcal{B}$ respectively Then under assumption that one of the functions μ and ν is compact and the other is σ -additive, the function λ is σ -additive, the function λ is σ -additive.

The notion of the finiteness is substantial in the definition of non direct product of measures introduced in [1] . This note contains a result analogical to (A) , and based on (A) ,for the case of infinite measures. Further the applications of the obtained result, to the problem of σ -additivity of a function $\lambda(E \times F)$, which is supposed to be σ -additivity as a function of F when E is fixed and as a function of E when F is fixed, are given.

If nothing else is said, the notions are used such as in [4] , $\mathcal{A}$ and $\mathcal{B}$ will denote rings of subsets of X and Y respectively. If τ is a non negative set function defined and additive on $\mathcal{A}$ (The values of τ are supposed to be real numbers or ∞) , we shall write shortly that τ is f.a. function. Moreover if $\mathcal{A}$ is an algebra and if $\tau(X) = 1$, τ is said to be normed finitely additive, shortly written τ is n.f.a. function.

A collection K of subsets of X is said to be compact if for any sequence of $C_n \in K$, $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$ whenever $C_1 \cap C_2 \cap \dots C_n \neq \emptyset$ for n = 1,2,... . If τ is a f.a. function defined on $\mathcal{A}$ then τ is termed

/..

compact if there exists a compact collection $\mathcal{K}$ of subsets of X such that for any $E \in \mathcal{A}$ for which $\tau(E) < \infty$ and for any $\varepsilon > 0$ there exist $K \in \mathcal{K}$ and $F \in \mathcal{A}$ such that $E \supset K \supset F$ and $\tau(E-F) < \varepsilon$.

If τ is finite f.a. function, then the above definition coincides with that one in [5] (cf.also [6] p. 225)

Note that if X is a Hausdorff topological space and C the class of its compact subsets, then any regular Borel measure defined on the σ-ring generated by C is compact. Recall that the notion of the regularity and that one of Borel measure is used in the same sense as in [4] with the only difference that X need not be locally compact space.

Further $\mathcal{A} \times \mathcal{B} = \{ E \times F : E \in \mathcal{A}, F \in \mathcal{B} \}$. The symbol $\mathcal{A} \otimes \mathcal{B}$ denotes the ring and $\mathcal{A} \otimes_\sigma \mathcal{B}$ the σ-ring generated by $\mathcal{A} \times \mathcal{B}$.

In accordance with [1] n.f.a. function defined on $\mathcal{A} \otimes \mathcal{B}$ is said to be a non direct product of n.f.a. functions μ and ν defined on $\mathcal{A}$ and $\mathcal{B}$ respectively if

$$\lambda(E \times Y) = \mu(E) \quad \text{for any } E \in \mathcal{A},$$
$$\lambda(X \times F) = \nu(F) \quad \text{for any } F \in \mathcal{B}.$$

Further we put $\mathcal{A}^* = \{ E : E \cap A \in \mathcal{A}, \text{ for any } A \in \mathcal{A} \}$:
The following properties follow immediately, therefore the proofs are omitted.

(a) $\mathcal{A}^*$ is an algebra and $\mathcal{A}^* \supset \mathcal{A}$

(b) If $E \in \mathcal{A}^* \times \mathcal{B}^*$, then $(P \times Q) \cap E \in \mathcal{A} \otimes \mathcal{B}$ for any $P \in \mathcal{A}$, $Q \in \mathcal{B}$.

If λ is f.a. function on $\mathcal{A} \otimes \mathcal{B}$ and $\mathcal{F}$ denotes the collection of all $P \times Q \in \mathcal{A} \times \mathcal{B}$ for which $0 < \lambda(P \times Q) < \infty$, then we define $\lambda^{P \times Q}(E) = \lambda((P \times Q) \cap E)$ for any $E \in \mathcal{A}^* \otimes \mathcal{B}^*$. This definition is correct as it can be seen from (b), and moreover

(c) $\lambda^{P \times Q}$ is additive on $\mathcal{A}^* \otimes \mathcal{B}^*$.

$\bar{\lambda}^{P \times Q}$ denotes the n.f.a. function which can be obtained from λ. Thus

$$\bar{\lambda}^{P \times Q}(E) = \frac{\lambda^{P \times Q}(E)}{\lambda(P \times Q)} \quad \text{for any } E \in \mathcal{A}^* \otimes \mathcal{B}^*.$$

We have

$$\bar{\lambda}^{P_x Q}(E) = \frac{\lambda^{P_x Q}(E)}{\lambda(P_x Q)} = \bar{\lambda}((P_x Q)_\cap E).$$

Hence

$$(d) \qquad \bar{\lambda}^{P_x Q}(E) = \bar{\lambda}((P_x Q)_\cap E).$$

If λ is defined on $\mathcal{A} \times \mathcal{B}$ the symbols ${}_E\lambda$ and λ_F have the same meaning as in [2] or [3] , i.e. ${}_E\lambda(F) = \lambda(E \times F)$ for every $F \in \mathcal{B}$ while $E \in \mathcal{A}$ is supposed to be fixed. λ_F is defined in a similar way.

The notion of a semifinite set function, which is a bit more general than that of σ -finite, will be used in the following sence.

A non negative monotone set function λ defined on a system $\mathcal{E}$ of sets is called semifinite on $\mathcal{E}$ if $\lambda(E) = LUB\{\lambda(F): F \subset E, \lambda(F) < \infty\}$ for any $E \in \mathcal{E}$. For the measures the notion of semifiniteness appears in [7] .

<u>L e m m a 1.</u> Let λ be f.a. function defined on $\mathcal{A} \otimes \mathcal{B}$ and let the following assumptions be satisfied:

(1) For any $P_x Q \in \mathcal{F}$ the function $\bar{\lambda}^{P_x Q}$ is a non direct product of a measure $\mu^{P,Q}$ and a measure $\nu^{P,Q}$ defined on $\mathcal{A}$ and $\mathcal{B}$ respectively.

(2) At least one of the measures $\mu^{P,Q}, \nu^{P,Q}$ is compact for any $P_x Q \in \mathcal{F}$.

(3) λ is a semifinite on $\mathcal{A} \otimes \mathcal{B}$.

Then λ is a measure on $\mathcal{A} \otimes \mathcal{B}$.

<u>P r o o f .</u> $\bar{\lambda}^{P_x Q}$ is n.f.a. function on $\mathcal{A}^* \otimes \mathcal{B}^*$ for any $P_x Q \in \mathcal{F}$. According to (1) and (2) and the result (A) $\bar{\lambda}^{P_x Q}$ is a measure on $\mathcal{A}^* \otimes \mathcal{B}^*$. Thus $\lambda^{P_x Q}$ is a measure too. The system $\{\lambda^{P_x Q}\}$ where $P_x Q \in \mathcal{F}$ is increasingly directed system of measures on $\mathcal{A}^* \otimes \mathcal{B}^*$. Hence the function $\mathcal{T}(E) = \underset{P_x Q \in \mathcal{F}}{LUB}\, \lambda^{P_x Q}(E)$ is a measure on $\mathcal{A}^* \otimes \mathcal{B}^*$ (see [7] p. 87, or [8]) . We shall prove that $\mathcal{T}$ coincides with λ on $\mathcal{A} \otimes \mathcal{B}$.

It is sufficient to prove that τ coincides with λ on $\mathcal{A} \times \mathcal{B}$. If $0 < \lambda(A \times B) < \infty$, then $A \times B \in \mathcal{F}$ and we have

$$(e) \quad \tau(A \times B) \geq \lambda^{A \times B}(A \times B) = \lambda(A \times B), \text{ for any } A \times B \in \mathcal{F}.$$

But $\lambda^{P \times Q}(A \times B) \leq \lambda(A \times B)$ for any $P \times Q \in \mathcal{F}$. Hence

$$\tau(A \times B) \leq \lambda(A \times B).$$

The last and (e) give

$$(f) \quad \tau(A \times B) = \lambda(A \times B) \quad \text{for any } A \times B \in \mathcal{F}.$$

Now let $\lambda(A \times B) = \infty$. The semifiniteness of τ, the property (f) and the fact that τ is monotone give

$$\infty = \lambda(A \times B) = LUB\{\lambda(E \times F) : E \times F \subset A \times B,\ \lambda(E \times F) < \infty\} \leq LUB\{\tau(E \times F) : E \times F \subset A \times B\} \leq \tau(A \times B)$$

Hence $\tau(A \times B) = \infty = \lambda(A \times B)$. Since the case $\lambda(A \times B) = 0$ is evident, the proof is finished.

L e m m a 2. Let λ be non negative, monotone set function defined on $\mathcal{A} \times \mathcal{B}$. Let ${}_E\lambda$ be semifinite on $\mathcal{B}$ for every $E \in \mathcal{A}$. Then λ is semifinite on $\mathcal{A} \times \mathcal{B}$. Moreover if ${}_E\lambda$ is σ-finite on $\mathcal{B}$ (i.e for any $F \in \mathcal{B}$ there exists a sequence $\{F_n\}_{n=1}^{\infty}$, $F_n \in \mathcal{B}$ such that ${}_E\lambda(F_n) < \infty$ and $F \subset \bigcup_{n=1}^{\infty} F_n$), then λ is σ-finite on $\mathcal{A} \times \mathcal{B}$.

P r o o f. Let ${}_E\lambda$ be semifinite. Let $E \times F \in \mathcal{A} \times \mathcal{B}$

$$\lambda(E \times F) = {}_E\lambda(F) = \text{L U B}\ \{{}_E\lambda(G) : F \supset G,\ G \in \mathcal{B},\ \lambda(G) < \infty\} =$$

$$\text{L U B}\ \{\lambda(E \times G) : F \supset G,\ \lambda(E \times G) < \infty\} \leq \text{L U B}\ \{\lambda(H \times G) : H \times G \subset E \times F,$$
$$H \in \mathcal{A},\ G \in \mathcal{B},\ \lambda(H \times G) < \infty\} \leq \lambda(E \times F).$$

Thus $\lambda(E \times F) = \text{L U B}\ \lambda(Z)$ where Z runs over all those sets belonging to $\mathcal{A} \times \mathcal{B}$ for which $\lambda(Z) < \infty$. This proves the semifiniteness of λ on $\mathcal{A} \times \mathcal{B}$.

Now, let ${}_E\lambda$ be σ-finite. Let $E \times F \in \mathcal{A} \times \mathcal{B}$. There exists a sequence $\{F_n\}_{n=1}^{\infty}$, $F_n \in \mathcal{B}$ for which ${}_E\lambda(F_n) < \infty$, $F \subset \bigcup_{n=1}^{\infty} F_n$. Thus $E \times F_n \in \mathcal{A} \times \mathcal{B}$ for n = 1,2,...,

$$\bigcup_{n=1}^{\infty} E \times F_n \supset E \times F,\ \lambda(E \times F_n) = {}_E\lambda(F_n) < \infty .$$

Thus λ is σ-finite on $\mathcal{A} \times \mathcal{B}$.

Theorem. Let $(X, \mathcal{S})$, $(Y, \mathcal{T})$ be measurable spaces. Let λ be non negative set function defined on $\mathcal{S} \times \mathcal{T}$ satisfying the following conditions:

(1') For any $E \times F \in \mathcal{T}$ the function ${}_E\lambda$ is σ-additive on $\mathcal{T}$ and λ_F is σ-additive on $\mathcal{S}$.

(2') For any $E \times F \in \mathcal{T}$ at least one of ${}_E\lambda$, λ_F is compact.

(3') For any $E \times F \in \mathcal{S} \times \mathcal{T}$ at least one of ${}_E\lambda$, λ_F is semifinite.

Then λ is σ-additive on $\mathcal{S} \times \mathcal{T}$. Moreover, if the semifiniteness in (3') is substituted by σ-finiteness then on $\mathcal{S} \otimes_{\sigma} \mathcal{T}$ there exists just one measure μ which is an extension of the function λ.

Proof. From the condition (1'), the finite additivity of λ on $\mathcal{S} \times \mathcal{T}$ follows. The last implies the existence of the unique extension of λ which is finitely additive on $\mathcal{S} \times \mathcal{T}$. We shall use the same symbol λ for this extension. Now let $P \times Q \in \mathcal{T} \subset \mathcal{S} \times \mathcal{T}$. Let us form $\bar{\lambda}^{P \times Q}$ and define

$$\mu^{P,Q}(E) = \lambda((E \cap P) \times Q) \quad \text{for any } E \in \mathcal{S}^*,$$
$$\nu^{P,Q}(E) = \bar{\lambda}(P \times (F \cap Q)) \quad \text{for any } F \in \mathcal{T}^*.$$

If $P \times Q \in \mathcal{T}$ then $\bar{\lambda}^{P \times Q}$ is non direct product of $\mu^{P,Q}$ and $\nu^{P,Q}$, because if $E \in \mathcal{S}^*$ then (in view of (d))

$$\bar{\lambda}^{P \times Q}(E \times Y) = \bar{\lambda}((E \cap P) \times Q) = \mu^{P,Q}(E)$$

and if $F \in \mathcal{T}^*$

$$\bar{\lambda}^{P \times Q}(X \times F) = \bar{\lambda}(P \times (F \cap Q)) = \nu^{P,Q}(F),$$

$\mu^{P,Q}$ and $\nu^{P,Q}$ are measures. The last follows from (1´). For any $P \times Q \in \mathcal{F}$ at least one of $\mu^{P,Q}$, $\nu^{P,Q}$ is compact. In fast, let e.g. λ_Q be compact. Let $E \in \mathcal{S}^*$ be any set. Then $E \cap F \in \mathcal{S}$. Since $\lambda((E \cap P) \times Q) < \infty$, there exists to any $\varepsilon > 0$ a set $K \in \mathcal{K}_Q$ ($\mathcal{K}_Q$ denotes the compact collection corresponding to the measure λ_Q) and a set $F \in \mathcal{S}^*$ such that $E \cap P \supset K \supset F \cap P$ and $\lambda((E \cap P - F \cap P) \times Q) < \varepsilon \, \lambda(P \times Q)$. Hence $\lambda((E \cap P - F \cap P) \times Q) < \varepsilon$ which means $\mu^{P,Q}(E - F) < \varepsilon$. Thus the compactness of $\mu^{P,Q}$ on $\mathcal{S}^*$ is proved. The semifiniteness of λ on $\mathcal{S} \otimes \mathcal{T}$ follows from Lemma 2. The assumptions of Lemma 1 are fulfilled. Using Lemma 1, we have that λ is σ-additive.

If the σ-finiteness instead of semifiniteness is assumed in (3´), then λ is σ-finite on $\mathcal{S} \times \mathcal{T}$. This implies σ-finiteness of λ on $\mathcal{S} \otimes \mathcal{T}$ and the uniqueness of the extension on $\mathcal{S} \otimes_\sigma \mathcal{T}$ follo s from the well-known measure extension theorem. The proof is finished.

C o r o l l a r y . Let X be a Hausdorff topological space and $\mathcal{S}$ the system of all Borel sets in X. Let $(Y, \mathcal{T})$ be a measurabl. space and λ a non negative set function defined on $\mathcal{S} \times \mathcal{T}$ and such that

$(1´´)_E$ λ is σ-additive for any $E \in \mathcal{S}$

(2´´) λ_F is inner regular Borel measure for any $F \in \mathcal{T}$

Then there exists just one measure μ on $\mathcal{S} \otimes_\sigma \mathcal{T}$ such that μ is the extension of λ.

P r o o f . The condition (1´´) and (2´´) immediately imply that (1´) of Theorem is satisfied. Similarly (2´) follows from (2´´). Moreover (2´´) implies the semifiniteness of λ_F for any $F \in \mathcal{T}$, hence (3´) is satisfied. Thus the assumptions of Theorem are fulfilled and Corollary follows.

Another proof of the corollary for the case of the locally compact space, and related results are given in [3] . The case when both spaces are locally compact was solved in [2] .

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A NOTE ON MEASURES IN CARTESIAN PRODUCTS

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To Professor Otakar Borůvka on the ocassion of his 70th birthday

In the paper [1] the following problem is dealt with. Let measurable spaces $(S, \mathcal{S})$ and $(T, \mathcal{T})$ and a function λ on the system of the sets of the form $E \times F$, $E \in \mathcal{S}$, $F \in \mathcal{T}$, be given. Suppose that λ $(E \times F)$ for every fixed $F \in \mathcal{T}$ is sigma additive on $\mathcal{S}$ as a function of E and for every fixed $E \in \mathcal{S}$ is sigma additive on $\mathcal{T}$ as a function of F. The question arises whether λ is sigma additive as a function of $E \times F$.

In the paper [1] the affirmative answer to the question is given in case of the families $\mathcal{S}, \mathcal{T}$ of Baire sets in locally compact spaces S, T and in the futher conditions of regularity in case of Borel sets.

In this note the affirmative answer is given in case that $(S, \mathcal{S})$ is an arbitrary measurable space and $\mathcal{T}$ is a family of Baire sets (in conditions of regularity - a family of Borel sets) in a locally compact space T. From this we obtain the result of the paper [1].

Our terminology in measure theory is drawn from [2] and [4].

In the sequel T denotes a locally compact (Hausdorff) topological space. Further $\mathcal{T}_0, \mathcal{T}$ denotes the family of all Baire, Borel sets, respectively, in T.

Let us denote

$$\mathcal{S} \times \mathcal{T}_0 = \{E \times F : E \in \mathcal{S}, F \in \mathcal{T}_0\}, \quad \mathcal{S} \times \mathcal{T} = \{E \times F : E \in \mathcal{S}, F \in \mathcal{T}\}.$$

If λ is a function on $\mathcal{S} \times \mathcal{T}_0$ the symbol ${}_E\lambda$ for an arbitrary $E \in \mathcal{S}$ will denote the function on $\mathcal{T}_0$ defined by the equality ${}_E\lambda(F) = \lambda(E \times F)$.

Similarly for $F \in \mathcal{T}_0$ we shall denote by the symbol λ_F the function on $\mathcal{S}$, for which $\lambda_F(E) = \lambda(E \times F)$. Analogous definitions we take for the function λ on $\mathcal{S} \times \mathcal{T}$.

Further, $\mathcal{S}\otimes\mathcal{T}_0$, $\mathcal{S}\otimes\mathcal{T}$ will denote the ring of sets generated by $\mathcal{S}\times\mathcal{T}_0$, $\mathcal{S}\times\mathcal{T}$, respectively. It is known [4, Theorem 33 E] that the ring $\mathcal{S}\otimes\mathcal{T}_0$ ($\mathcal{S}\otimes\mathcal{T}$) consists of all sets of the form

$$(1)\qquad G = \bigcup_{i=1}^{k} E_i \times F_i ,$$

where $\{E_i \times F_i\}$ is a finite family of the sets from $\mathcal{S}\times\mathcal{T}_0$ ($\mathcal{S}\times\mathcal{T}$), E_i being mutually disjoint. The sigma ring generated by $\mathcal{S}\otimes\mathcal{T}_0$ or $\mathcal{S}\times\mathcal{T}_0$ ($\mathcal{S}\otimes\mathcal{T}$ or $\mathcal{S}\times\mathcal{T}$) we shall denote $\mathcal{S}\otimes_\sigma\mathcal{T}_0$ ($\mathcal{S}\otimes_\sigma\mathcal{T}$).

If λ is an additive function on $\mathcal{S}\times\mathcal{T}_0$ ($\mathcal{S}\times\mathcal{T}$), then on $\mathcal{S}\otimes\mathcal{T}_0$ ($\mathcal{S}\otimes\mathcal{T}$) there exists one and only one additive frunction ν such that ν and λ coincide on $\mathcal{S}\times\mathcal{T}_0$ ($\mathcal{S}\times\mathcal{T}$). The function ν is defined by the equality

$$(2)\qquad \nu(G) = \sum_{i=1}^{k} \lambda(E_i \times F_i),$$

for every set G expressed in the form (1) [4, Exercise 8.5].

<u>Theorem 1.</u> Let λ be a function defined on $\mathcal{S}\times\mathcal{T}_0$ with the following properties:

(i) The values of λ are nonnegatve real numbers or ∞ and $_E\lambda(C) < \infty$ for every compact set $C \in \mathcal{T}_0$.

(ii) For every set $E \in \mathcal{S}$ the function $_E\lambda$ is sigma additive on $\mathcal{T}_0$.

(iii) For every set $F \in \mathcal{T}_0$ the function λ_F is sigma additive on $\mathcal{S}$.

Then the function λ is sigma additive on $\mathcal{S}\times\mathcal{T}_0$.

<u>Proof.</u> It is easy to show that the function λ is additive on $\mathcal{S}\times\mathcal{T}_0$ [4, Theorem 33 D, Exercise 7.5] and hence the function ν defined by the equality (2) is additive on the ring $\mathcal{S}\otimes\mathcal{T}_0$.

We shall prove that the function ν is sigma additive on the ring $\mathcal{S}\otimes\mathcal{T}_0$ and our result then will follow.

It is known [4, Theorem 52 G or 2, Theorem 60.1] that every Baire measure on $\mathcal{T}_0$ is regular.

Let $G = E \times F \in \mathcal{S}\times\mathcal{T}_0$, $\nu(G) < \infty$. From the regularity of $_E\lambda$ it follows that for every $\varepsilon > 0$ there exists a compact set $C \in \mathcal{T}_0$ such that $C \subset F$ and ${}_E\lambda(F - C) = \lambda(E \times (F - C)) < \varepsilon$.

If $G \in \mathcal{S} \otimes \mathcal{T}_0$ is an arbitrary set of the form (1), $\nu(G) < \infty$, for each $i = 1,2,\dots,k$ we can find a compact set $C_i \in \mathcal{T}_0$ such that we have $C_i \subset F_i$ and

$$_{E_i}\lambda(F_i - C_i) = \lambda(E_i \times (F_i - C_i)) < \frac{\varepsilon}{k}, \quad i = 1,2,\dots k.$$

Then we have

$$E_i \times C_i \subset E_i \times F_i, \quad \bigcup_{i=1}^{k} E_i \times C_i \subset G = \bigcup_{i=1}^{k} E_i \times F_i$$

and

$$\nu\Big[\bigcup_{i=1}^{k} E_i \times F_i\Big] - \Big[\bigcup_{i=1}^{k} E_i \times C_i\Big]) = \nu\Big(\bigcup_{i=1}^{k}\big[E_i \times (F_i - C_i)\big]\Big) =$$

$$= \sum_{i=1}^{k} \nu(E_i \times (F_i - C_i)) < \varepsilon.$$

To prove the sigma additivity of ν, let $\{G_n\}_{n=1}^{\infty}$, $G_n \in \mathcal{S} \otimes \mathcal{T}_0$, $n = 1,2,\dots$, be a decreasing sequence of the sets of the form

$$G_n = \bigcup_{i=1}^{k_n} E_i^n \times F_i^n, \quad E_i^n \times F_i^n \neq \emptyset, \; i = 1,2,\dots,k_n, \; n = 1,2,\dots.$$

with

$$\nu(G_n) > \varepsilon > 0, \qquad n = 1,2,\dots$$

Denote $F = \bigcup_{i=1}^{k_1} F_i^1$. Then we have

$$0 < \varepsilon < \nu(G_n) = \sum_{i=1}^{k_n} \nu(E_i^n \times F_i^n) = \sum_{i=1}^{k_n} \lambda(E_i^n \times \bigcup_{i=1}^{k_n} F_i^n) \leq$$

$$\leq \sum_{i=1}^{k_n} \lambda_F(E_i^n) = \lambda_F\Big(\bigcup_{i=1}^{k_n} E_i^n\Big).$$

The sequence $\Big\{\bigcup_{i=1}^{k_n} E_i^n\Big\}_{n=1}^{\infty}$ forms a decreasing sequence of the sets with the intersection $\bigcap_{n=1}^{\infty}\Big(\bigcup_{i=1}^{k_n} E_i^n\Big)$. Since

$$\lambda_F\Big(\bigcup_{i=1}^{k_n} E_i^n\Big) > \varepsilon > 0, \; n = 1,2,\dots,$$

so from the sigma additivity of λ_F it follows

$$\lambda_F\left(\bigcap_{n=1}^{\infty}\left(\bigcup_{i=1}^{k_n} E_i^n\right)\right) \geq \varepsilon > 0,$$

i.e. the intersection $\bigcap_{n=1}^{\infty}\left(\bigcup_{i=1}^{k_n} E_i^n\right)$ is nonempty.

For every n there exist compact sets $C_i^n \subset F_i^n$, $i = 1,2,\dots, k_n$, such that

$$\nu_1\left(G_n - \bigcup_{i=1}^{k_n} E_i^n \times C_i^n\right) = \nu(G_n - Y_n) < \frac{\varepsilon}{2^n} \qquad n = 1,2,\dots,$$

where $Y_n = \bigcup_{i=1}^{k_n} E_i^n \times C_i^n$. Denote $X_n = Y_1 \cap \dots \cap Y_n$, then $m \leq n$ implies $X_n \subset X_m$ and

$$\nu(G_n - X_n) = \nu\left(\bigcup_{i=1}^{n}(G_n - Y_i)\right) \leq \sum_{i=1}^{n} \nu(G_n - Y_i) \leq$$

$$\leq \sum_{i=1}^{n} \nu(G_i - Y_i) \leq \varepsilon \sum_{i=1}^{n} 2^{-i} < \varepsilon .$$

Hence it follows

$$0 < \varepsilon < \nu(G_n) = \nu(X_n) + \nu(G_n - X_n) < \nu(X_n) + \varepsilon,$$

and $\nu(X_n) > 0$, i.e. the sets X_n are nonempty, and $X_{n+1} \subset X_n$.

Define the projection P_S of $G \subset S \times T$ into S as the mapping $P_S G = \{s \in S :$ for which there exists $t \in T$ such that $(s,t) \in G\}$.

Clearly $P_S G_n = \bigcup_{i=1}^{k_n} E_i^n$, $P_S Y_n = \bigcup_{i=1}^{k_n} E_i^n$, and also $P_S X_n = P_S Y_n = P_S G_n = \bigcup_{i=1}^{k_n} E_i^n$ (we suppose that $F_i^n \neq \emptyset$, $i = 1,2,\dots,k_n$, $n = 1, 2,\dots$).

Now $P_S \bigcap_{n=1}^{\infty} X_n \subset \bigcap_{n=1}^{\infty} P_S X_n$, and if $s \in \bigcap_{n=1}^{\infty} P_S X_n$, then $s \in P_S X_n$, $n = 1,2,\dots$, the sets $(X_n)_s$ form a decreasing sequence of nonempty compact sets, hence the intersection $\bigcap_{n=1}^{\infty} (X_n)_s = \left(\bigcap_{n=1}^{\infty} X_n\right)_s$ is nonempty and $s \in P_S \bigcap_{n=1}^{\infty} X_n$. Thus we have ,

$$P_S \bigcap_{n=1}^{\infty} X_n = \bigcap_{n=1}^{\infty} P_S X_n = \bigcap_{n=1}^{\infty} P_S G_n = \bigcap_{n=1}^{\infty} \Big\{ \bigcup_{i=1}^{k_n} E_i^n \Big\} .$$

Since we have shown that $\bigcap_{n=1}^{\infty} \{ \bigcup E_i^n \}$ is a nonempty set, the set $P_S \bigcap_{n=1}^{\infty} X_n = \bigcap_{n=1}^{\infty} \Big\{ \bigcup_{i=1}^{k_n} E_i^n \Big\}$ contains a point s_0, therefore there exists a point t_0 in T such that $(s_0, t_0) \in \bigcap_{n=1}^{\infty} X_n$.

Since

$$G_n \supset X_n, \quad n = 1,2,\ldots,$$

it follows $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$. Thus if $\bigcap_{n=1}^{\infty} G_n = \emptyset$, we must have $\lim_{n\to\infty} \nu(G_n) = 0$.

If $\nu(G) = \infty$ and $G = \bigcup_{n=1}^{\infty} G_n$, $G_n \in \mathcal{S} \otimes \mathcal{T}_0$ $n = 1,2,\ldots,$ we must show that $\nu(G) = \sum_{n=1}^{\infty} \nu(G_n) = \infty$. Let $G = E \times F = \bigcup_{n=1}^{\infty} E_n \times F_n$, $E_i \times F_i$ and $E_j \times F_j$ mutually disjoint. Since ${}_E\lambda(F) = \lambda(E \times F) = \nu \cdot (E \times F)$ is sigma finite, we have $F = \bigcup_{i=1}^{\infty} F \cap H_i$, ${}_E\lambda(E \cap H_i) < \infty$, $i = 1,2,\ldots$, i.e. $\lambda(E \times F \cap H_i) < \infty$ and $E \times F = E \times F \cap \bigcup_{i=1}^{\infty} H_i =$

$$= \bigcup_{i=1}^{\infty} (E \times F \cap H_i), \quad \nu(E \times F) = \sum_{i=1}^{\infty} \nu(E \times F \cap H_i). \text{ Now}$$

$$E \times (F \cap H_i) = \bigcup_{n=1}^{\infty} E_n \times (F_n \cap H_i), \quad \nu(E \times (F \cap H_i)) < \infty \; i = 1,2,\ldots,$$

$$\nu(E \times (F \cap H_i)) = \sum_{n=1}^{\infty} \nu(E_n \times (F_n \cap H_i)),$$

$$\nu(E \times F) = \sum_{i=1}^{\infty} \nu(E \times (F \cap H_i)) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \nu(E_n \times (F_n \cap H_i)) =$$

$$= \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \nu(E_n \times (F_n \cap H_i)) = \sum_{n=1}^{\infty} \nu(E_n \times F_n).$$

From it we have at once that $\nu(G) = \sum_{n=1}^{\infty} \nu(G_n)$ for G of the form (1) and $\nu(G) = \infty$

Clearly ν is sigma finite on the ring $\mathcal{S} \otimes \mathcal{T}_0$.

<u>Corollary 1.</u> If λ satisfies the conditions of Theorem 1, then on $\mathcal{S} \otimes_\sigma \mathcal{T}_0$ there exists one and only one measure μ_0 and on $\mathcal{S} \otimes_\sigma \mathcal{T}$ one

and only one measure μ such that for

$E \times F \in \mathcal{P} \times \mathcal{T}_0$ $\mu(E \times F) = \mu_0(E \times F) = \lambda(E \times F)$.

Corollary 2. Let λ be a function on $\mathcal{P} \times \mathcal{T}$ with the properties:

(i) The values of λ are nonnegative real numbers or ∞, and $\lambda(E \times C) < \infty$ for every campact $C \in \mathcal{T}$.

(ii) For every $E \in \mathcal{P}$ $_E\lambda$ is additive and regular on $\mathcal{T}$.

(iii) For every $F \in \mathcal{T}$ λ_F is sigma additive on $\mathcal{P}$.

Then λ is sigma additive on $\mathcal{P} \times \mathcal{T}$ and on $\mathcal{P} \otimes_\sigma \mathcal{T}$ there exists one and only one measure μ coinciding with λ on $\mathcal{P} \times \mathcal{T}$.

Theorem 2. Let λ be a fucntion on $\mathcal{P} \times \mathcal{T}_0$ with the following properties:

(i) The values of λ are complex numbers and the function ν defined by the relation (2) is bounded on $\mathcal{P} \otimes \mathcal{T}_0$,

(ii) For every $E \in \mathcal{P}$ $_E\lambda$ is sigma additive on $\mathcal{T}_0$.

(iii) For every $F \in \mathcal{T}_0$ λ_F is sigma additive on $\mathcal{P}$.

Then λ is sigma additive on $\mathcal{P} \times \mathcal{T}_0$ and on $\mathcal{P} \otimes_\sigma \mathcal{T}_0$ there exists one and only one sigma additive function μ_0, which coincides with λ on $\mathcal{P} \times \mathcal{T}_0$ and on $\mathcal{P} \otimes_\sigma \mathcal{T}$ one and only one sigma additive function μ coinciding with λ on $\mathcal{P} \times \mathcal{T}_0$.

Corollary 3. Let λ be a complex function on $\mathcal{P} \times \mathcal{T}$ with the following properties:

(i) the function ν is bounded on $\mathcal{P} \otimes \mathcal{T}$.

(ii) For every $E \in$ $_E\lambda$ is additive and regular on $\mathcal{T}$.

(iii) For every $F \in \mathcal{T}$ λ_F is sigma additive on $\mathcal{P}$.

Then λ is sigma additive on $\mathcal{P} \otimes \mathcal{T}$ and on $\mathcal{P} \otimes_\sigma \mathcal{T}$ there exists one and only one sigma additive function μ coinciding with λ on $\mathcal{P} \times \mathcal{T}$.

The proofs of Corollaries 1,2, 3 and of Theorem 2 are similar to those in [1] and hence omitted.

Remark 1. It is easy to give an example that the condition of boundedness (i) in Theorem 2 cannot be omitted because in general the function ν may fail to be bounded.

R e m a r k 2. In general, the condition of regularity (ii) in Corollaries 2 and 3 cannot be omitted. An example can be found in [7, p. 102] of a function λ for which ${}_S\lambda$ is a probability measure and λ_T is a probability measure but λ is not sigma additive on $\mathcal{S} \times \mathcal{T}$ (cf. also [4] p. 214 and [5] p. 167).

R e m a r k 3. T. NEUBRUNN has pointed out to me that in [5] similar problems for probabilities are treated with. For this we refer to [6] .

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ON SOME PROBLEMS CONCERNING DISJOINTNESS IN LATTICE ORDERED GROUPS

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To Professor Otakar Borůvka on the occasion of his 70^{th} birthday

The concept of disjointness was used by several authors for studying the structure of lattice ordered linear spaces and lattice ordered groups (F. RIESZ [9] , KANTOROVIČ - VULICH - PINSKER [7] , NAKANO [8] , CONRAD [2] ŠIK [10]). In this note there are investigated some problems on the existence of lattice ordered groups with certain properties concerning the disjointness proposed by G. Birkhoff ([1] , Problem 117) and Ch. Holland [4] .

§ 1. Basic concepts and notations

Let G be a lattice ordered group; the group operation and lattice operations are denoted by + and $\wedge$, $\vee$, respectively. Elements $x, y \in G$ are said to be disjoint, if $x \wedge y = 0$. A subset $X \subset G$ is disjoint, if $x > 0$ for every $x \in X$ and any two distinct elements of X are disjoint. For any $Y \subset G$ we denote

$$\perp Y = \{z \in G : |z| \wedge |y| = 0 \text{ for each } y \in Y\} .$$

Instead of $\perp Y$ we write also Y^* (cf. [1] , Chap. XIII; in [7] and [10] the set $\perp Y$ is called "the component generated by Y") .

Let $\Theta(G)$ be the system of all l-ideals of G; the set $\Theta(G)$ is partially ordered by inclusion. It is well-known that $\Theta(G)$ is a distributive lattice with lattice operations $A \wedge B = A \cap B$, $A \vee B = A + B$ for any $A, B \in \Theta(G)$. G is a simple lattice ordered group, if $\Theta(G) = \{\{0\}, G\}$. An l-ideal J is closed if from $\{x_\alpha\} \subset J$, $\vee x_\alpha = x \in G$ it follows $x \in J$; J is said to be "closed", if $\perp(\perp J) = J$ (cf. [1] , p. 307) .

If A, B, A_λ $(\lambda \in \Lambda)$ are lattice ordered groups, then we denote by

$A \times B$, $\prod_{\lambda \in \Lambda} A_\lambda$ and $\Gamma_{\lambda \in \Lambda} A_\lambda$ the direct product, complete direct product and lexicographic product (in the last case we assume that Λ is linearly ordered), respectively; the lexicographic product of two factors is denoted by $A \circ B$ (cf. [1] and [3]).

§ 2. Complements of "closed" l-ideals

The following theorem is well-known ([1], Chap. XIII, Thm. 27):

(RIESZ-BIRKHOFF) In a complete l-group G, the following are equivalent conditions on an l-ideal J;

(i) J is complemented, (ii) $J = \perp(\perp J)$, (iii) J is closed.

G. Birkhoff ([1], p. 318) raised the following problem;

"Are there incomplete l-groups in which: (a) every "closed" l-ideal is complemented, or (b) the correspondence $K \longrightarrow (K^{*})^{*}$ of Theorem 19 is a lattice endomorphism? What are they? What does this mean for Θ (G)?" The correspondence of Thm. 19 concerns the mapping $K \longrightarrow (K^{*})^{*}$ where K runs over the set $\Theta(G)$.)

E x a m p l e 1. Let R_0 be the additive group of all rational numbers with the natural ordering. R_0 is an incomplete l-group. Since R_0 is a simple l-group, each l-ideal of G fulfils both conditions (i) and (ii) and thus (i)$\Leftrightarrow$(ii) for any $J \in \Theta(G)$. More generally, we have:

P r o p o s i t i o n 1. Let G be a linearly ordered group. For any $J \in \Theta(G)$ the following conditions are equivalent: (i) ; (ii) ; (iv) $J \in \{\{0\}, G\}$.

P r o o f. Obviously, (iv)$\Rightarrow$(i), (ii). Let $J \neq \{0\}$. If (ii) is valid, then $J = \perp(\perp J) = \perp\{0\} = G$, hence (iv) holds; since (i)$\Rightarrow$(ii) in any lattice ordered group ([1], p. 315), the proof is complete.

C o r o l l a r y 1. Let α be an infinite cardinal. There exists an incomplete lattice ordered group G such that card $G = \alpha$ and (i)$\Rightarrow$(ii) for any $J \in \Theta(G)$.

P r o o f. Let I be a linearly ordered set, card $I = \alpha$ and for any $i \in I$ let $G_i = R_0$. Denote $H = \Gamma_{i \in I} G_i$ and let G be the system of all elements $g \in H$ such that the set $I(g) = \{i \in I : g(i) \neq 0\}$ is finite. Then card $G = \alpha$, G is an incomplete l-group and G is linearly ordered; hence by Prop. 1 (i)$\Leftrightarrow$(ii) for any $J \in \Theta(G)$.

If (i) $\Longleftrightarrow$ (ii) for any $J \in \Theta(G)$, the implication (iii) $\Longrightarrow$ (i) need not hold:

E x a m p l e 2. Let $G = R_0 \circ R_0$, $J_0 = \{(0,x) : x \in R_0\}$. Since G is linearly ordered, (i) $\Longleftrightarrow$ (ii) for any $J \in \Theta(G)$ and J_0 does not satisfy (i) ; obviously J_0 fulfils (iii) .

If the conditions (i) , (ii) and (iv) are equivalent for any $J \in \Theta(G)$ the lattice ordered group G need not be linearly ordered:

E x a m p l e 3. There exist simple lattice ordered groups that are not linearly ordered (Holland [4] ; cf. also § 4) ; let G have the mentioned properties. Clearly (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iv) for any l-ideal $J \in \Theta(G)$. (An l-group with these properties cannot be complete, since any complete l-group that is not linearly ordered is a nontrivial direct product, thus it is not simple.)

Let us remark that if an l-ideal $J \in \Theta(G)$ has a complement, then this complement equals $\perp J$.

We can generalize Proposition 1 as follows:

T h e o r e m 1. Let I be a non-empty set and for any $i \in I$ let G_i be a linearly ordered group. Let $G = \prod_{i \in I} G_i$, $J \in \Theta(G)$. Then the following conditions are equivalent : (i) ; (ii) ; (v) there exists a subset $I_1 \subset I$ such that $J = \{g \in G : g(i) = 0 \text{ for } i \in I_1\}$.

P r o o f . Assume that (v) holds. Put $I_2 = I \setminus I_1$,

(1) $J_1 = \{g \in G : g(i) = 0 \text{ for each } i \in I_2\}$.

Then J_1 is a complement of J and hence (v) $\Rightarrow$ (i) , (ii) .

For any subset $X \subset G$ let I (X) be the set of all $i \in I$ such that there is $x \in X$ satisfying $x(i) \neq 0$. Suppose that $J \in \Theta(G)$ fulfils (ii) , put $I_2 = I(J)$, $I_1 = I \setminus I_2$ and let J_1 have the same meaning as in (1) . Then $J_1 = \perp J$,

$$J = \perp(\perp J) = \{g \in G : g(i) = 0 \text{ for each } i \in I_1\} ,$$

thus (ii) $\Rightarrow$ (v) . Since (i) $\Rightarrow$ (ii) , the proof is complete.

The conclusion of Thm. 1 cannot be generalized for subdirect products of linearly ordered groups:

Example 4. Let G be the set of all continuous real functions defined on the closed interval $[-1,1] = I$. Then G is a subdirect product of linearly ordered groups $G_i = R$ $(i \in I)$, where R is the additive group of all real numbers with the natural order. Let $I_1 = [-1,0]$, $I_2 = [0,1]$, $J_k = \{f \in G : f(i) = 0 \text{ for each } i \in I_k\}$ $(k = 1,2)$. Obviously $\perp J_1 = J_2$, $\perp J_2 = J_1$, hence (ii) holds for J_1. If $J \in \Theta(G)$, $J \wedge J_1 = \{0\}$, then $J \subset J_2$, thus $J \vee J_1 = J + J_1 \subset J_2 + J_1$. For any $g \in J_2 + J_1$ we have $G(0) = 0$, whence $J \vee J_1 \neq G$ and therefore J_1 has no complement.

Lemma 1. $\perp J \in \Theta(G)$ for any $J \in \Theta(G)$.

Proof. It is easy to verify that the set $\perp X$ is a convex l-subgroup of G for any subset $X \subset G$. Since each element $g \in \perp J$ can be written as a difference $g = x - y$, $x \geq 0$, $y \geq 0$, $x,y \in \perp J$, it suffices to prove that from $x \in \perp J$, $x > 0$, $t \in G$ it follows $-t + x + t \in \perp J$. Let $z \in J$. Then $x \wedge |z| = 0$, hence

$$(-t + x + t) \wedge |z| = -t + [x \wedge (t + |z| - t)] + t = 0.$$

This implies $-t + x + t \in \perp J$.

Let us consider the following condition on a lattice ordered group G:

(a) (ii) $\Longrightarrow$ (i) for any $J \in \Theta(G)$.

Theorem 2. A complete direct product $G = \prod_{i \in I} G_i$ of lattice ordered groups G_i satisfies the condition (a) if and only if each factor G_i fulfils this condition.

Proof. For any subset $J \subset G$ denote

$$J(G_i) = \{g(i) : g \in J\},$$

$$\perp(J(G_i)) = \{x \in G_i : |x| \wedge |y| = 0 \text{ for any } y \in J(G_i)\}.$$

At first we shall prove that

$$\perp J = \prod \perp (J(G_i)). \tag{2}$$

If $h \in \perp J$, $g \in J$, $i \in I$, then $|h| \wedge |g| = 0$, hence

$$|h(i)| \wedge |g(i)| = |h|(i) \wedge |g|(i) = (|h| \wedge |g|)(i) = 0,$$

thus $h(i) \in \perp(J(G_i))$ and $h \in \prod \perp(J(G_i))$. Conversely, let $h \in \prod \perp(J(G_i))$, $g \in J$. Then we have $|h(i)| \wedge |g(i)| = 0$ for each $i \in I$ and therefore $|h| \wedge |g| = 0$, $h \in \perp J$.

If we put $K = \prod \perp(J(G_i))$, we get from (2) (since $K(G_i) = \perp(J(G_i))$)

(3) $\perp(\perp J) = \prod \perp(\perp(J(G_i)))$.

Let us now assume that each factor G_i satisfies (a) and let $J \in \Theta(G)$ $\perp(\perp J) = J$. Then it follows from 3

(4) $$J(G_i) = \perp(\perp(J(G_i)))$$

and obviously $J(G_i)$ is an l-ideal of the lattice ordered group G_i. According to (4) and since (a) holds for G_i the l-ideal $J(G_i)$ is complemented in the lattice $\Theta(G_i)$ and its complement equals $\perp(J(G_i))$. Hence any element $x^i \in G_i$ can be written in the form

(5) $$x^i = y^i + z^i, \quad y^i \in J(G_i), \; z^i \in \perp(J(G_i)) .$$

Let $g \in G$ and denote $g(i) = x^i$. There exist elements $y, z \in G$ such that $y(i) = y^i$, $z(i) = z^i$ for each $i \in I$, where y^i, z^i are as in (5) . From (2) , (3) and (4) it follows $y \in J$, $z \in \perp J$, hence $g \in J + \perp J$. Thus $G = J + \perp J$. According to lemma 1 $\perp J \in \Theta(G)$ and therefore $J \vee \perp J = G$. Obviously $J \wedge \perp J = \{0\}$, thus $\perp J$ is a complement of J.

Conversely, suppose that G fulfils (a) . Let $i_0 \in I$, $J_{i_0} \in \Theta(G_{i_0})$, $\perp(\perp J_{i_0}) = J_{i_0}$. For any $i \in I$, $i \neq i_0$ put $J_i = \{0\}$ and let $J = \prod J_i \; (i \in I)$. Then J is an l-ideal of G and according to (3) we have $\perp(\perp J) = J$. Hence $\perp J$ is a complement of J, thus

(6) $$G = J + \perp J.$$

Let $g^{i_0} \in G_{i_0}$. There is an element $g \in G$ such that $g(i_0) = g^{i_0}$ (other components of g can be arbitrary). By (6) there exist elements $h \in J$, $k \in \perp J$ satisfying $g = h + k$. Then $g^{i_0} = h(i_0) + k(i_0)$ and according to (3) $h(i_0) \in \perp(\perp(J(G_i))) = J_{i_0}$, $k_{i_0} \in \perp J_i$. This implies $G_{i_0} = J_{i_0} + \perp J_{i_0}$ and hence according to lemma 1 $\perp J_{i_0}$ is a complement of J_{i_0} in the lattice $\Theta(G_i)$.

From Proposition 1 and Thm. 2 we get as a corollary a part of Thm. 1:

<u>C o r o l l a r y 2.</u> If G is a complete direct product of linearly ordered groups, then G fulfils (a) .

Moreover, since any simple lattice ordered group satisfies (a) , it follows:

Corollary 3. Any complete direct product of simple l-groups satisfies the condition (a) .

Proposition 2. A lattice ordered group G fulfils (a) if and only if $\Theta(G)$ is a Stone lattice.

Proof. A Stone lattice is a distributive pseudocomplemented lattice S with 0 and 1 satisfying $s^* \vee s^{**} = 1$ for any $s \in S$, where s^* is the pseudocomplement of s, i.e., s^* is the greatest element of the set $\{x \in S : x \wedge s = 0\}$ ([1] , p. 130) . The lattice $\Theta(G)$ is complete, distributive and pseudocomplemented (the pseudocomplement of $J \in \Theta(G)$ being the l-ideal $\perp J$). Assume that $\Theta(G)$ satisfies (a) , $J \in \Theta(G)$ and denote $K = \perp(\perp J)$. Then $\perp(\perp K) = K$ and thus by (a) $K \vee \perp K = G$. Since $\perp K = \perp J$, we get $\perp(\perp J) \vee \perp J = G$ for any $J \in \Theta(G)$ and hence $\Theta(G)$ is a Stone lattice. Conversely, if $\Theta(G)$ is a Stone lattice, then by the definition $\perp J \vee \perp(\perp J) = G$ and clearly $\perp J \wedge \perp(\perp J) = \{0\}$; hence (a) is valid.

Let us remark that if each l-ideal $H \in \Theta(G)$, $H \neq G$ satisfies (a) then G need not satisfy (a) :

Example 5. Let $G = X \circ (Y \times Z)$, where any of lattice ordered groups X, Y, Z equals R. Then G contains only there non-trivial l-ideals, namely $J_1 = \{(0,y,0) : y \in Y\}$, $J_2 = \{(0,0,z) : z \in Z\}$ and $J_3 = \{(0,y,z) : y \in Y,\ z \in Z\}$.

J_1 and J_2 are linearly ordered, hence they fulfil (a) ; J_3 is a direct product of linearly ordered groups and according to Thm. 1 J_3 satisfies (a) as well. But G does not fulfil (a) , since $\perp J_1 = J_2$, $\perp J_1 = J_1$, $J_1 \vee J_2 = J_3 \neq G$.

§ 3. The mapping $J \longrightarrow (J^*)^*$

In this section we are dealing with the correspondence

$$J \longrightarrow (J^*)^*, \tag{7}$$

where J runs over the set $\Theta(G)$. Since we shall be interested mainly in the properties of the lattice $\Theta(G)$ (rather than with those of the l-group G itself) the notation J^* instead of $\perp J$ will be more convenient for our purposes; this notation coincides with that used for pseudocomplemented lattices (cf. Proposition 2) . We shall consider the condition:

(b) The correspondence (7) is an endomorphism of the lattice $\Theta(G)$.

<u>Theorem 3.</u> The conditions (a) and (b) are equivalent.

<u>Proof.</u> Since $J^{*} \wedge J^{**} = \{0\}$ for any $J \in \Theta(G)$, the condition (a) is equivalent with

(a_1) $\quad J^{*} \vee J^{**} = G$ for any $J \in \Theta(G)$.

The condition (b) means that for any $A, B \in \Theta(G)$ the following two conditions hold:

(b_1) $$(A \vee B)^{**} = A^{**} \vee B^{**},$$

(b_2) $$(A \wedge B)^{**} = A^{**} \wedge B^{**}.$$

Let $J \in \Theta(G)$. If $J_1 \in \Theta(G)$, $J_1 \subset (J^{*} \vee J^{**})^{*}$, then $J_1 \perp J^{*}$ thus $J_1 \subset J^{**}$, and at the same time $J_1 \perp J^{**}$; this implies $J_1 = \{0\}$. Hence $(J^{*} \vee J^{**})^{*} = \{0\}$ and therefore

$$(J^{*} \vee J^{**})^{**} = G.$$

Assume that (b_1) is fulfilled. Then we have

$$J^{*} \vee J^{**} = J^{***} \vee J^{****} = (J^{*} \vee J^{**})^{**} = G;$$

this proves that $(b_1) \Longrightarrow (a_1)$.

Conversely, suppose that (a_1) holds. Then (since $\Theta(G)$ is distributive) J^{*} belongs to the center of the lattice $\Theta(G)$ for any $J \in \Theta(G)$ and therefore (the center of a lattice S being a sublattice of S, cf. [1], p. 66, Thm. 10) $A^{*} \wedge B^{*}$ and $A^{*} \vee B^{*}$ are elements of the center of $\Theta(G)$ for any $A, B \in \Theta(G)$. Thus these elements are complemented and the system $\Theta_0(G) = \{J^{*} : J \in \Theta(G)\}$ is a Boolean algebra; J^{**} is the complement of J^{*}. It is easy to verify that

$$(A \vee B)^{*} = A^{*} \wedge B^{*}$$

(cf. also [1], p. 129) and thus

$$(A \vee B)^{**} = [(A \vee B)^{*}]^{*} = [A^{*} \wedge B^{*}]^{*} = A^{**} \vee B^{**},$$

hence (b_1) holds.

The complete lattice $\Theta(G)$ satisfies the infinite distributive law

$$A \wedge (\bigvee B_{\alpha}) = \bigvee (A \wedge B_{\alpha})$$

for any $A, B_\alpha \in \theta(G)$ (cf. [3] , p. 117) , hence $\theta(G)$ is a Brouwerian lattice ([1] , p. 304) . It is known that for a Brouwerian lattice the condition (a_1) implies that

$$(A \wedge B)^* = A^* \vee B^*$$

for any A,B ([1] , p. 130) . From this we get

$$(A \wedge B)^{**} = [(A \wedge B)^*]^* = [A^* \vee B^*]^* = A^{**} \wedge B^{**}$$

and therefore G fulfils (b_2) .

From the results of § 2 and from Thm. 3 it follows:

<u>Corollary 4.</u> There exist incomplete l-groups G for which the correspondence $J \longrightarrow (J^*)^*$ is a lattice endomorphism on the lattice $\theta(G)$.

<u>§ 4. Simple lattice ordered groups</u>

In this section the group operation on G is written multiplicatively; e is the neutral element of G. An element $g \in G$, $g > e$ is said to be insular, if there exists a conjugate g_1 od g such that $g \wedge (h^{-1} g_1 h) = e$ for all $h \in G$, $h > e$. By using this concept Holland characterized the class of all lattice ordered groups that can be represented as groups of automorphisms of an ordered set with bounded support; as a corollary he obtained the following assertion ([4] , Corollary 2) :

If G is a simple lattice ordered group with an insular element, then for every $g \in G$, $g > e$ there is an infinite collection of pairwise disjoint conjugates of g.

In [4] it is remarked also that any simple nontotally ordered l-group contains an infinite collection of pairwise disjoint elements (this follows from results in [2]) and there is formulated the question, whether the conclusion of Corollary 2 follows from the weaker hypothesis that G is simple and not totally ordered.

The answer to this question is negative. There exists a simple nontotally ordered l-group containing an element g_0 such that no pair of conjugates of g_0 is disjoint. For proving this we use one of the series of examples of simple l-groups given in [4] (some topological properties of this l-group were studied in [5]) .

<u>Example 6.</u> Let G_1 be the set of all automorphisms of the linearly ordered set R of all reals. The group operation is the composition

of endomorphisms, the operation $\vee$ is defined by $x(f \vee g) = (xf) \vee (xg)$ and the operation $\wedge$ is defined dually. Then G_1 is a lattice ordered group. Let G be the set of all $f \in G_1$ satisfying

$$(8) \qquad (x + 1)f = xf + 1$$

for any $x \in R$. G is an l-subgroup of G_1; it is proved in [4] that G is simple. Clearly G is not totally ordered.

An element h_0 of a lattice ordered group H is called a weak unit, if $h \wedge h_0 > e$ whenever $h \in H$, $h > e$. It is easy to verify that any conjugate h_1 of a weak unit h_0 is a weak unit, too.

Let $c \in R$, $c > 0$. The automorphism f_0 of R defined by $xf_0 = x + c$ for any $x \in R$ satisfies the condition (8), hence $f_0 \in G$. If $g \in G$, $g > e$, then $xg \geq xe = x$ for any $x \in R$ and there exists $x_1 \in R$ such that $x_1 g > x_1$. Therefore

$$x(f_0 \wedge g) = (x + c) \wedge (xg) \geq x, \quad x_1(f_0 \wedge g) > x_1$$

and thus $f_0 \wedge g > e$. The element f_0 is a weak unit of G and all conjugates f_0 are weak units. Hence no pair of conjugates of f_0 can be disjoint.

It is now natural to ask whether each simple nontotally ordered l-group contains an element g such that there exists an infinite disjoint collection of conjugates of g.

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BANDS OF SOLUTIONS OF SOME SPECIAL DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

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To Professor Otakar Borůvka on the occasion of his 70^{th} birthday

In this paper we consider the differential equation of the third order of the form

$$\text{(a)} \quad (py')'' + (py')' + qy' + ry = 0$$

and we supose that $p = p(x) > 0$, $q = q(x) \leq 0$, $q'(x)$, $r = r(x) \geq 0$ are continuous fonctions of $x \in (-\infty, \infty)$ and that $r(x) - q'(x) \geq 0$ and that $r - q' \equiv 0$ does not hold in any interval.

In this paper we derive some properties of the solutions of the differential equation (a) we introduce the so called "bands of solutions" of (a) and derive some properties of the solutions which follow from the properties of the bands. Furthemore we prove the existence of the solutions without the zero-points in $(-\infty, \infty)$ of (a) and we derive some consequences of this existence. At the end we derive one sufficient condition for the solution of (a) to be unoscillatory on some interval (a, ∞).

1. Fundamental properties of the solutions.

The solutions y_1, y_2, y_3 of (a) are linearly independent and form the fundamental system of solutions, if the determinant (Wronskian)

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1, & y_2, & y_3 \\ y_1', & y_2', & y_3' \\ (py_1')', & (py_2')', & (py_3')' \end{vmatrix}$$

is at least at one point different from zero.

<u>Remark 1.</u> The Wronski determinant of the fundamental system of (a) is $W(x) = p(x_0)\cdot W(x_0)/p(x)e^{x-x_0}, -\infty < x_0 < \infty$, since $(pW)' = -pW$.

<u>Remark 2.</u> The solutions of the differential equation (a) with the property

$$y(a) = y'(a) = 0, \ (py')'(a) \neq 0,$$

or

$$y(a) = (py')'(a) = 0, \ y'(a) \neq 0.$$

or

$$y'(a) = (py')'(a) = 0, \ y(a) \neq 0, \ -\infty < a < \infty,$$

are dependent.

This follows from the existence and uniqueness theorem.

The following integral identity

$$(1) \quad (py')' + py' + qy + \int_a^x [r - q']y\,dt \qquad = \text{const.}$$

holds for the solutions of differential equation (a). We get the integral identity 1 by integrating equation (a) term by term from a to x, $-\infty < a < \infty, \quad -\infty < x < \infty$.

<u>Theorem 1.</u> The solution y of the differential equation (a) with the property $y(a) = y'(a) = 0, (py')'(a) \neq 0$ has no zero point for $x < a, -\infty < a < \infty$.

<u>Proof.</u> Suppose that $y(a) = y'(a) = 0, (py')(a) > 0$. The integral identity (1) for $y(x)$ is

$$(1') \quad (py')' + py' + qy + \int_a^x [r(t) - q'(t)]y(t)\,dt = (py')'(a) > 0.$$

Suppose that $(py')'(x_1) = 0$, where $x_1 < a$ is the first zero point of $(py')'(x)$ on the left of a. Referring to $(1')$ we obtain

$$p(x_1)\, y'(x_1) + q(x_1)\, y(x_1) - \int_{x_1}^{a} [\lambda(t) - q'(t)]\, y(t)\, dt = (py')'(a) > 0$$

This leads to a contradiction, since the expression on the left of the last equation is negative.

<u>Remark 3.</u> In theorem 1 we have proved also that $y'(x)$ and $(py')'(x)$ have no zero points for $x < a$.

<u>Corollary 1.</u> Let y_1, y_2 be two solutions of (a) with the properties $y_1(a) = y_1'(a) = 0, (py_1')'(a) \neq 0$ and $y_2(a) = (py_2')'(a) = 0, y_2'(a) \neq 0.$ Then the function $W(x) = y_1 y_2' - y_1' y_2 \neq 0$ for $x > a$.

<u>Proof.</u> Suppose $w(x_1) = 0,\ x_1 > a$. Then the equations

$$c_1 y_1(x_1) + c_2 y_2(x_1) = 0$$
$$c_1 y_1'(x_1) + c_2 y_2'(x_1) = 0$$

have the nontrivial solution C_1^*, C_2^* i.e. the solution $y^*(x) = C_1^* y_1(x) + C_2^* y_2(x)$ has the double zero point at x_1 and $y^*(a) = 0$, which is a contradiction with the statement of theorem 1. Thus $w(x) \neq 0$ for $x > a$.

<u>Remark 4.</u> Let y_1, y_2 be two arbitrary solutions of (a) then $w(x) = y_1 y_2' - y_1' y_2$ is the solution of the differential equation $[p(pw)'' + p(pw)' + pqw]' + p(pw)'' + p(pw)' + (q - \lambda).pw = 0.$

2. The bands of solutions and its properties

<u>Definition.</u> Let y_1, y_2 be two independent solution of (a) with the properties

$$y_1(a) = y_1'(a) = 0,\ (py_1')'(a) = 1$$

and

$$y_2(a) = (py_2')'(a) = 0,\ y_2'(a) = 1;\ -\infty < a < \infty.$$

The set of solutions $y = C_1 y_1(x) + C_2 y_2(x)$ with the property

$y(a)=0$ is said to be the band of solutions of the differential equation (a) at the point a or shortly the band at the point a.

Corollary 2. The band of solutions fulfils the following differential equation of the second order

$$\begin{vmatrix} y_1 , & y_2 , & y \\ y_1' , & y_2' , & y' \\ (py_1')', & (py_2')', & (py')' \end{vmatrix} = 0$$

i.e.

$$(b) \qquad w(py')' - (pw)'y' + [(pw)'' + (pw)' + qw]y = 0,$$

where $w = w(x) = y_1 y_2' - y_1' y_2 \neq 0$ for $x > a$ which follows from corollary 1.

Remark 5. Since the differential equation (b) is of the second order, the zero points of all the solutions of (b) are simple in (a,α) and separate (if there exist) each other.

Theorem 2. Let $x_1 > a$ be the first zero point of the solution y_1 of (a) (with double zero point at a). Then every solution y of the band at the point a has between a and x_1 just one zero point.

Proof. Suppose the contrary, i.e. $y(x) \neq 0$ in $(a, x_1 >$. Then

$$\left(\frac{y_1}{y}\right)' = \frac{y_1'y - y_1y'}{y^2} = \frac{y_1'(c_1y_1 + c_2y_2) - y_1(c_1y_1' + c_2y_2')}{y^2} = -c_2\frac{w(x)}{y^2}$$

If we integrate this equation between a and x_1 we get

$$\frac{y_1(x_1)}{y(x_1)} - \lim_{x\to a} \frac{y_1(x)}{y(x)} = \lim_{x \to a+} \int_x^{x_1} -\frac{c_2 w}{y^2}\, dt.$$

The expression on the left is zero. The integral on the right exists, since the function under the integral sign is continuous and different from zero in (a, x_1) and possesses a limit at a. Then we obtain

$$0 = -c_2 \int_a^{x_1} \frac{w(t)}{y^2(t)}\, dt \neq 0.$$

This is a contradiction. Then y must have in (a, x_1) at least one zero point. Since the zero point in (a, x_1) of each solution $y(x)$ of the band seperate each other, then there is only one.

T h e o r e m 3. If every solution of the differential equation (a) with the double zero point has at least one zero point on the right of the double zero point, then every solution with one zero point is oscillatory (has infinite number of zero points on the right of this zero point).

P r o o f. Let $y(x)$ be the solution of a with the property $y(x_1) = 0, -\infty < x_1 \infty$. Let y_1 be the solution of (a) with double zero point at x_1 i.e. $y_1(x_1) = y_1'(x_1) = 0, (py_1')'(x_1) \neq 0$ and let $y_1(x_2) = 0, \quad x_2 > x_1$. The solutions $y(x)$ and $y_1(x)$ betong to the same band at x_1 and according to theorem 2, the solution $y(x)$ must have at least one zero point on the interval (x_1, x_2). Let $y(\bar{x}_2) = 0, \quad x_1 < \bar{x}_2 < x_2$. Let $\bar{y}_1(x)$ be the solution of (a) with the double zero point at $\bar{x}_2$ and let $\bar{y}_1(x_3) = 0, \quad x_3 > \bar{x}_2$. The solutions $y(x), \bar{y}_1(x)$ belong to the same band at $\bar{x}_2$ and according to theorem 2, $y(x)$ must have one zero point on the interval $< \bar{x}_2, x_3)$ and so on. From this process it follows, that $y(x)$ has infinite number of zeros on $< x_1, \infty)$.

Similarly it can be proved the following.

T h e o r e m 4. If one solution of the differential equation (a) has in some interval $(\alpha, \infty), -\infty < \alpha < \infty$ infinite number of zero points, then every solution of a with one zero point, is oscillatory on (α, ∞).

3. E x i s t e n c e o f s o l u t i o n s w i t h o u t z e r o p o i n t s

L e m m a 1. Let y_1 be the solution of (a) with the double zero point at a and let $y_1(x) \neq 0$ for $x > a$. Then every solution of (a) with double zero point at $\alpha > a$ has not the zero point for $x > \alpha$.

The proof follows from the properties of bands of solutions. From theorem 1, it follows that $y_1(x) \neq 0$ for $x < a$ also. Let $\alpha > a$ and let $y(\alpha) = y'(\alpha) = 0$, $(py')'(\alpha) \neq 0$. If $y(x_1) = 0, x_1 > \alpha$, then $y_1(x)$ must have at least one zero point on the right of a which is a contradiction.

Theorem 5. Let the differential equation (a) have at least one solution with the double zero point, which has no zeros different from this double zero point. Then there exists at least one solution of (a) without zeros on $(-\infty, \infty)$.

Proof. Let $y_1(x)$, $y_2(x)$ be two solutions of (a) with the properties $y_1(x_1) = y_1'(x_1) = 0, (py_1')'(x_1) \neq 0$ and $y_2(x_2) = y_2'(x_2) = 0, (py_2')'(x_2) \neq 0$. Let $-\infty < x_1 < x_2 < \infty$. Let $y_1(x) \neq 0$ for $x \neq x_1$. According to lemma 1, $y_2(x) \neq 0$ for $x \neq x_2$. Therefore there exist constants C_1, C_2 such that $y(x) = C_1 y_1(x) + C_2(x) \neq 0$ for $x \in (-\infty, \infty)$.

Theorem 6. The differential equation (a) has at least one solution without zero points in $(-\infty, \infty)$.

Proof. Let y_1, y_2, y_3 be the fundamental system of solutions of the differential equation (a) with the properties: $y_1(a) = y_1'(a) = 0, (py_1')'(a) = 1; y_2(a) = (py_2')'(a) = 0, y_2'(a) = 1; y_3'(a) = (py_3')'(a) = 0, y_3(a) = 1$. From theorem 1, it follows that $y_1(x) > 0$ for $x < a$. Let $a < x_1 < x_2 < \ldots < x_n < \ldots$ be the sequence of numbers which diverges to $+\infty$. From the sequence of solutions of (a) $\{\mu_n(x)\}_{n=1}^{\infty}$, $\mu_n(x) = C_1^n y_1 + C_2^n y_2 + C_3^n y_3$ with the properties: $\mu_n(x_n) = \mu_n'(x_n) = 0, (p\mu_n')'(x_n) > 0$ and $\mu_n^2(a) + \mu_n'^2(a) + (p\mu_n')'^2(a) = 1$. It is evidently possible. From theorem 1, it follows that $\mu_n(x) > 0$ for $x < x_n$. Form the sequence of numbers:

$$(2) \quad \{\mu_n(a)\}_{n=1}^{\infty}, \{\mu_n'(a)\}_{n=1}^{\infty}, \{(p\mu_n')'(a)\}_{n=1}^{\infty}$$

Each sequence from (2) is evidently bounded, therefore we can choose from (2) such sequences

3 $\{u_{n_k}(a)\}_{n_k=1}^{\infty}$, $\{u'_{n_k}(a)\}_{n_k=1}^{\infty}$, $\{(pu'_{n_k})'(a)\}_{n_k=1}^{\infty}$

which converge.

Let u_0, u'_0, u''_0 be the limits of these sequences. Let $u(x)$ be the solution of (a) with the properties:

$$u(a) = u_0, \quad u'(a) = u'_0, \quad (pu')'(a) = u''_0.$$

This solution is not trivial, since $u_0^2 + u'^2 + u''_0 = 1$.

The solutions $u_{n_k}(x)$ and $u(x)$ can be written in the form

$$u_{n_k}(x) = (pu'_{n_k})'(a)\,\mathcal{J}_1(x) + u'_{n_k}(a)\,\mathcal{J}_2(x) + u_{n_k}(a)\,\mathcal{J}_3(x)$$

and

$$u(x) = u''_0\,\mathcal{J}_1(x) + u'_0\,\mathcal{J}_2(x) + u_0\,\mathcal{J}_3(x)$$

From these relations it follows that

$$\lim_{n_k\to\infty} u_{n_k}(x) = u(x) \qquad \text{for every } x \in (-\infty,\infty).$$

To prove that $u(x)$ cannot have the simple zero points, suppose the contrary. Let $u(\xi) = 0$, $u'(\xi) \neq 0$ $-\infty < \xi < \infty$. Then there exists in the neighborhood of ξ such a point ξ_1 for which $u(\xi_1) < 0$. It follows from the preceding, that $\lim_{n_k\to\infty} u_{n_k}(\xi_1) = u(\xi_1) < 0$. This is not possible, since $u_{n_k}(\xi) > 0$ for n_k sufficiently great. Therefore $u(x)$ cannot have the simple zero points. Let $u(x)$ have the double zero point at τ. It is $u(x) > 0$ for $x \neq \tau$, since every solution of (a) can have at most one double zero point. Then there exists in the neighborhood of τ such a point τ_1 for which $u'(\tau_1) > 0$. It follows from the preceding, that

$$\lim_{n_k\to\infty} u'_{n_k}(\tau_1) = u'(\tau_1) > 0.$$

But his is not possible, since from theorem 1 and remark 3 it follows that $u'_{n_k}(\tau_1) < 0$ for n_k sufficiently great. We can prove also that $(pu')'(x)$ cannot be negative in any point. The proof is complete.

Corollary 2. From the proof of the preceding theorem it follows that the differential equation (a) has at least one solution $u(x)$ of the properties: $u(x) \neq 0$, $\operatorname{sgn} u(x) = \operatorname{sgn} (pu')'(x) \neq \operatorname{sgn} u'(x)$ for all $x \in (-\infty, \infty)$.

Corollary 3. The solution $u(x)$ of (a) with the properties as in corollary 2 has the following property:

$$u'(x) \to 0, \ (pu')'(x) \to 0 \qquad \text{for } x \to \infty .$$

Proof. Let $u(x) > 0, u'(x) < 0, (pu')'(x) > 0$ for $x \in (-\infty, \infty)$.

From the differential equation (a), it follows that $(pu')'' = -(pu')' - qu' - pu \leq 0$ and therefore $(pu')' > 0$ is a decreasing function. Suppose now $(pu')'(x) > k^2 > 0$. Integrating this inequality from a to x we get

$$p(x)u'(x) > p(a)u'(a) + k^2(x-a).$$

From this inequality it follows that there is $u'(x) > 0$ for sufficiently great x. But this is a contradiction, therefore $(pu')' \to 0$ for $x \to \infty$. Similarly it can be proved that $u'(x) \to 0$ for $x \to \infty$.

Lemma 2. Let $u(x)$ be the solution of (a) with the properties as in corollary 2 and 3 and let $p(x)$ be bounded on $(-\infty, \infty)$ and $\liminf_{x \to \infty} q(x) = 0$. Then for this solution the following integral indentity holds:

$$(4) \qquad (pu')' + pu' + qu = \int_x^\infty [r(t) - q'(t)] u(t)\, dt.$$

Proof. Let $\{x_n\}_{n=1}^\infty$ be a sequence of real numbers which diverge to $+\infty$. For every x_n and for the solution $u(x)$ the integral identity (1) holds

$$(pu')' + pu' + qu + \int_{x_n}^x [r(t) - q'(t)] u(t)\, dt = (pu')'(x_n) + p(x_n)u'(x_n) + q(x_n)u(x_n),$$

i.e.

$$(p\mu')' + p\mu' + q\mu = (p\mu')'(x_n) + p(x_n)\mu'(x_n) + q(x_n)\mu(x_n) + \int_x^{x_n} [r - q']\mu \, dt.$$

If we take the limit for $x_n \to \infty$, we get the identity (4), since

$$\lim_{x_n \to \infty} (p\mu')'(x_n) = 0, \ \lim_{x_n \to \infty} p(x_n)\mu'(x_n) = 0, \ \lim_{x_n \to \infty} q(x_n)\mu(x_n) = 0.$$

Theorem 7. Let $\int_{x_0}^{\infty} [r(t) - q'(t)] \, dt$ diverge and let the supposition of lemma 2 be fulfilled. Then the solution $\mu(x)$ of (a) with the properties as in corollary 2 and 3 fulfils the condition $\mu(x) \to 0$ for $x \to \infty$.

Proof. Suppose that $\mu(x) > k^2 > 0$ for $x \in (-\infty, \infty)$. Then

$$(5) \quad \int_{x_0}^{\infty} [r(t) - q'(t)] \mu(t) \, dt > \int_{x_0}^{\infty} [r(t) - q'(t)] k^2 dt \to +\infty .$$

This is a contradiction because the integral on the left in (5) exists, which follows from identity 4.

4. Unoscillatory Solutions

Now we derive one sufficient condition for the solutions of (a) to be unoscillatory on (a, ∞), i.e. for every solution of (a) to have at most two simple zeros or at most one double zero point.

Theorem 8. Let $p'(x)$ be continuous on (a, ∞) and let

$$q(x) - p'(x) - 1 + (x - a)[r(x) - q'(x)] < 0$$

for $x > a$. Then every solution of (a) is unoscillatory on (a, ∞).

Proof. It is sufficient to prove that $y(x)$ of the property $y(a) = y'(a) = (py')'(a) = 1$ has no zeros on the right of (a). Then from remark 5 and theorem 2, it follows the statement of theorem 8. The integral identity (1) for $y(x)$ is of the form

$$(py')' + py' + qy + \int_a^x [r - q'] y \, dt = (py')'(a) = 1 .$$

Integrating it from a to x we get

$$(6) \qquad py' + py + \int_a^x [q(t) - p'(t) - 1 + (x-t)(r(t) - q'(t))]\, y(t)\, dt = 0$$

Suppose that $x_1 > a$ is the first zero point of y on the right. Then $y(x_1) = 0,\ y'(x_1) \leqq 0$. From (6) we get

$$0 > p(x_1) \cdot y'(x_1) + \int_a^{x_1} [q(t) - p'(t) - 1 + (x-t)(r(t) - q'(t))]\, y(t)\, dt = 0,$$

which is a contradiction.

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ZUR THEORIE DER ZASSENHAUSSCHEN VERFEINERUNGEN ZWEIER REIHEN VON ZERLEGUNGEN I

Gleichbasig halbverkettete Verfeinerungen

VÁCLAV HAVEL, Brno

Herrn Professor Otakar Borůvka zu seinem 70. Geburtstag gewidmet

Im weiteren verwenden wir bis auf wenige Abweichungen die Terminologie aus [1] Wir wollen eine Vertiefung des Verfeinerungssatzes von O. BORUVKA, ([1] S. 65) gewinnen.

Die Menge $G \neq \emptyset$ sei fest gegeben. Wir beginnen mit zwei Hilfssätzen

<u>Hilfssatz 1</u> Es seien $\bar{\bar{X}} \bar{\bar{Y}} \bar{\bar{Z}}$ Zerlegungen auf G Es seien weiter $\bar{x} \in \bar{\bar{X}}, \bar{y} \in \bar{\bar{Y}}, \bar{x} \,\chi\, \bar{y}$ gegeben. Dann sind $(\bar{x} \cap \bar{y}) \sqsubset (\bar{\bar{X}} \smile \bar{\bar{Z}}), (\bar{x} \cap \bar{y}) \sqsubset (\bar{\bar{Y}} \frown \bar{\bar{Z}})$ verknüpfte Hüllen.

<u>Beweis.</u> Für $\bar{a} \in (\bar{x} \cap \bar{y}) \sqsubset (\bar{\bar{X}} \frown \bar{\bar{Z}})$ gilt offensichtlich $\bar{a} \subset \bar{x}$, $\bar{a} \,\chi\, \bar{y}$. Es gilt auch $\bar{a} \subset \bar{z}$ für ein geeignetes $\bar{z} \in \bar{\bar{Z}}$. Daraus folgt aber $\bar{a} \cap \bar{y} \in \bar{\bar{X}} \smile (\bar{\bar{Y}} \smile \bar{\bar{Z}})$, so dass $\bar{a} \cap \bar{y} = \bar{x} \cap \bar{t} = \bar{a} \cap \bar{t}$ für ein geeignetes (und zwar eindeutig bestimmtes)[3] $\bar{t} \in \bar{\bar{Y}} \in \bar{\bar{Z}}$. Die Rollen von $\bar{\bar{X}} \frown \bar{\bar{Z}}, \bar{\bar{Y}} \frown \bar{\bar{Z}}$ sind vertauschbar, so dass der Beweis beendet ist.

1/ Für Inzidenz (nichtleeres Durchschneiden) gebrauchen wir gelegentlich die Bezeichnung χ

2/ Die Symbole $\cap, \cup, \frown, \smile$ bezeichnen mengentheoretische bezw. verbandstheoretische Durchschnitte und Vereinigungen

3/ Die Eindeutigkeit ist leicht erweisbar

<u>H i l f s s a t z   2.</u> Es seien Zerlegungen $\bar X \geq \bar A$, $\bar Y \geq \bar B$ auf G gegeben; wir setzen $A^\circ = \bar A \smile (\bar X \frown \bar B)$, $B^\circ = \bar B \smile (\bar Y \frown \bar A)$. Die beiden Hüllen $\tilde A = (\bar x \frown \bar y) [A^\circ$, $\tilde B = (\bar x \cap \bar y) [B^\circ$ sind für jede Wahl von $\bar x \in \bar X, \bar y \in \bar Y, \bar x \mathcal{I} \bar y$ dann und nur dann verknüpft, wenn

$$(1) \qquad A^\circ \frown \bar Y = B^\circ \frown \bar Y$$

<u>B e w e i s.</u> Es gelte (1) und seien $\bar x \in \bar X, \bar y \in \bar Y, \bar x \mathcal{I} \bar y$ gegeben. Wählen wir einen beliebigen $\hat A$ -Block[4/] $\tilde a$, so folgt $\tilde a \mathcal{I} \bar x, \tilde a \mathcal{I} \bar y$ und wegen $\bar X \geq A^\circ$ sogar $\tilde a \subset \bar x$. Aus $\tilde a \mathcal{I} \bar y$ ergibt sich nun $\tilde a \cap \bar y \in A^\circ \frown \bar Y$, also wegen (1) existiert ein geeignetter (und zwar eindeutig bestimmter) $\tilde B$ -Block $\tilde b$, so dass $\tilde a \cap \bar y = \tilde b \cap \bar x = \tilde a \cap \tilde b$. Es existiert also genau ein $\tilde B$ -Block $\tilde b \mathcal{I} \tilde a$. Die Rollen beider Hüllen sind vertauschbar, es existiert also auch zu jedem $\tilde B$ -Block genau ein mit ihm inzidenter $\tilde A$ -Block.

Sind die entsprechenden Hüllen $\tilde A, \tilde B$ für jede Wahl von $\bar x \in \bar X$, $\bar y \in \bar Y$, $\bar x \mathcal{I} \bar y$, verknüpft, so existiert zu jedem $\tilde A$ -Block $\tilde a$ genau ein $\tilde B$ -Block $\tilde b \mathcal{I} \tilde a$. Weiter gilt wieder $\tilde a \subset \bar x, \tilde b \subset \bar y$, $\tilde a \cap \bar y = \tilde b \cap \bar x = \tilde a \cap \tilde b$ so dass jeder $(\tilde A \frown \bar Y)$ -Block mit einem $(\tilde B \frown \bar Y)$ Block zusammenfällt. Daraus folgt schon der Rest des Beweises.

Im folgenden untersuchen wir zwei Reihen $(\bar A_\mu)$, $(\bar B_\nu)$ von Zerlegungen auf G, wo

$$(2) \qquad \bar G_{max} = \bar A_0 \geq \bar A_1 \geq \ldots \geq \bar A_\alpha \geq \bar A_{\alpha+1} = \bar A_\alpha \frown \bar B_\beta = \bar V,$$
$$\bar G_{max} = \bar B_0 \geq \bar B_1 \geq \ldots \geq \bar B_\beta \geq \bar B_{\beta+1} = \bar A_\alpha \frown \bar B_\beta = \bar V.$$

Unter den oberen bzw. unteren Zassenhausschen Verfeinerungen von (2) versteht man nun die Reihen $(\overset{\smile}{A}_{\mu,\nu})$ $(\overset{\smile}{B}_{\nu,\mu})$ bzw. $(\overset{\frown}{A}_{\mu,\nu})$ $(\overset{\frown}{B}_{\nu,\mu})$ wo

$$(2a) \qquad \overset{\smile}{A}_{\mu,\nu} = \bar A_{\mu-1} \frown (\bar A_\mu \smile \bar B_\nu),$$
$$\overset{\smile}{B}_{\nu,\mu} = \bar B_{\nu-1} \frown (\bar B_\nu \smile \bar A_\mu),$$

4/ Elemente einer Zerlegung nennen wir ihre Blöcke

(2b) $\hat{A}_{\mu,\nu} = \bar{A}_{\mu} \smile (\bar{A}_{\mu-1} \frown \bar{B}_{\nu}),$

$\hat{B}_{\nu,\mu} = \bar{B}_{\nu} \smile (\bar{B}_{\nu-1} \frown \bar{A}_{\mu})$

für $\mu = 1, \ldots \alpha + 1;\ \nu = 1, \ldots \beta + 1.$

Es ist gut bekannt, dass es sich tatsächlich um Verfeinerungen von (2) handelt ([4]), S. 8-9 und S. 14)

S a t z 1. Die oberen Zassenhausschen Verfeinerungen von (2) sind jedenfalls gleichbasig halbverkettet.

S a t z 2. Im Falle

(3) $\hat{A}_{\mu\nu} \frown \bar{B}_{\nu-1} = \hat{B}_{\nu\mu} \frown \bar{A}_{\mu-1}$ für jedes $\mu = 1, \ldots \alpha+1;\ \nu = 1, \ldots \beta+1$

sind die unteren Zassenhausschen Verfeinerungen von (2) gleichbasig halbverkettet.

B e w e i s. Es seien $(\breve{K}_{\mu,\nu}), (\breve{L}_{\nu\mu})$ bzw. $(\hat{K}_{\mu,\nu}), (\hat{L}_{\nu,\mu})$ lokale Ketten von $(\breve{A}_{\mu,\nu}), (\breve{B}_{\nu,\mu})$ bzw. $(\hat{A}_{\mu,\nu}), (\hat{B}_{\nu,\mu})$ bezüglich einer gewählten Basis $\bar{a} \in \bar{V}$. Wir haben also

$$\bar{a} \subset \bar{a}_{\mu-1} \in \bar{A}_{\mu-1}, \quad \bar{a} \subset \bar{b}_{\nu-1} \in \bar{B}_{\nu-1},$$

$$\bar{a} \subset \breve{a}_{\mu,\nu} \in \breve{A}_{\mu,\nu}, \quad \bar{a} \subset \breve{b}_{\nu,\mu} \in \breve{B}_{\nu,\mu},$$

$$\bar{a} \subset \hat{a}_{\mu,\nu} \in \hat{A}_{\mu,\nu} \quad \bar{a} \subset \hat{b}_{\nu,\mu} \in \hat{B}_{\nu,\mu},$$

$$\bar{a}_0 = \bar{b}_0 = G$$

$$\breve{K}_{\mu,\nu} = \breve{a}_{\mu,\nu-1} \sqcap \breve{A}_{\mu\nu}, \quad \breve{L}_{\nu,\mu} = \breve{b}_{\nu,\mu-1} \sqcap \breve{B}_{\nu,\mu},$$

$$\hat{K}_{\mu,\nu} = \hat{a}_{\mu,\nu-1} \sqcap \hat{A}_{\mu,\nu}, \quad \hat{L}_{\nu,\mu} = \hat{b}_{\nu,\mu-1} \sqcap \hat{B}_{\nu,\mu}$$

$$\breve{a}_{\mu 0} = \hat{a}_{\mu 0} = \bar{a}_{\mu-1}, \quad \breve{b}_{\nu 0} = \hat{b}_{\nu 0} = \bar{b}_{\nu-1},$$

$$\breve{K}_{\alpha+1,\beta+1} = \breve{L}_{\beta+1,\alpha+1} = \hat{K}_{\alpha+1,\beta+1} = \hat{L}_{\beta+1,\alpha+1} =$$

$$= \bar{a} \sqcap \bar{V}$$

$$(\mu = 1, \ldots, \alpha+1;\ \nu = 1, \ldots, \beta+1)$$

a/ Aus $\breve{\bar a}_{\gamma,\nu-1} \in \breve{\bar A}_{\gamma,\nu-1} = \bar A_{\gamma-1} \cap (\bar A_\gamma \cup \bar B_{\gamma-1})$ folgt $\bar a_{\gamma-1} \cap \bar b_{\nu-1} \subset \breve{\bar a}_{\gamma,\nu-1}$; ähnlich bekommen wir $\bar b_{\nu-1} \cap \bar a_{\gamma-1} \subset \bar b_{\nu,\gamma-1}$, so dass $\bar a_{\gamma-1} \cap \bar b_{\nu-1} \subset \breve{\bar a}_{\gamma,\nu-1} \cap \breve{\bar b}_{\nu,\gamma-1}$ Andererseits gilt aber $\breve{\bar a}_{\gamma,\nu-1} \subset \bar a_{\gamma-1}$, $\bar b_{\nu,\gamma-1} \subset \bar b_{\nu-1}$ woraus $\bar a_{\gamma-1} \cap \bar b_{\nu-1} = \breve{\bar a}_{\gamma,\nu-1} \cap \bar b_{\nu,\gamma-1} = s\breve K_{\gamma,\nu} \cap s\breve L_{\nu,\gamma}$ folgt. Für $\breve x \in \breve K_{\gamma,\nu}$ bzw. $\breve y \in \breve L_{\nu,\gamma}$ besteht die

Inzidenz mit einem $\breve L_{\nu,\gamma}$ -Block bzw. $\breve K_{\gamma,\nu}$ -Block genau dann, wenn $\breve x \in (\bar a_{\gamma-1} \cap \bar b_{\nu-1})\, [\, \breve K_{\gamma,\nu}$ bzw. $\breve y \in (\bar a_{\gamma-1} \cap \bar b_{\nu-1})\, [\, \breve L_{\nu,\gamma}$. Die Halbverknüpfung von $\breve K_{\gamma,\nu}$ und $\breve L_{\nu,\gamma}$ folgt also aus der Verknüpfung der Hüllen $(\bar a_{\gamma-1} \cap \bar b_{\nu-1})\, [\, \breve{\bar A}_{\gamma,\nu} = (\bar a_{\gamma-1} \cap \bar b_{\nu-1})\, [\, \breve K_{\gamma,\nu}$,

$$(\bar a_{\gamma-1} \cap \bar b_{\nu-1})\, [\, \breve{\bar B}_{\nu,\gamma} = (\bar a_{\gamma-1} \cap \bar b_{\nu-1})\, [\, \breve L_{\nu,\gamma};$$

eine solche Verknüpfung beider Hüllen ergibt sich aus dem

<u>H i l f s s a t z 1.</u> Die Halbverknüpfung von $\breve K_{\alpha+1,\beta+1}, \breve L_{\beta+1,\alpha+1}$ ist trivial.

b/ Aus $\overset{\frown}{\bar a}_{\gamma,\nu-1} \in \overset{\frown}{\bar A}_{\gamma,\nu-1} = \bar A_\gamma \cup (\bar A_{\gamma-1} \cap \bar B_{\nu-1})$ folgt $\bar a_\gamma \cup (\bar a_{\gamma-1} \cap \bar b_{\nu-1}) \subset \overset{\frown}{\bar a}_{\gamma,\nu-1}$ und also auch $\bar a_{\gamma-1} \cap \bar b_{\nu-1} \subset \overset{\frown}{\bar a}_{\gamma,\nu-1}$. Ähnlich erweist sich $\bar b_{\nu-1} \cap \bar a_{\gamma-1} \subset \overset{\frown}{\bar b}_{\nu,\gamma-1}$. Hieraus und aus $\overset{\frown}{\bar A}_{\gamma,\nu-1} \leq \bar A_{\gamma-1}$, $\overset{\frown}{\bar a}_{\gamma,\nu-1} \subset \bar a_{\gamma-1}$ ergibt sich endlich $\bar a_{\gamma-1} \cap \bar b_{\nu-1} = \overset{\frown}{\bar a}_{\gamma,\nu-1} \cap \bar b_{\nu,\gamma-1} = s\overset{\frown}{K}_{\gamma,\nu} \cap s\overset{\frown}{L}_{\nu,\gamma}$. Nun besteht für $\overset{\frown}{x} \in \overset{\frown}{K}_{\gamma,\nu}$ bzw. $\overset{\frown}{y} \in \overset{\frown}{L}_{\nu,\gamma}$ die Inzidens mit einem $\overset{\frown}{L}_{\nu,\gamma}$ -Block bzw. $\overset{\frown}{K}_{\gamma,\nu}$ -Block gerade dann, wenn $\overset{\frown}{x} \in (\bar a_{\gamma-1} \cap \bar b_{\nu-1})\, [\, \overset{\frown}{K}_{\gamma,\nu}$ bzw. $\overset{\frown}{y} \in (\bar a_{\gamma-1} \cap \bar b_{\nu-1})\, [\, \overset{\frown}{L}_{\nu,\gamma}$. Die Halbverknüpfung zwischen $\overset{\frown}{K}_{\gamma,\nu}, \overset{\frown}{L}_{\nu,\gamma}$ folgt also aus der Verknüpfung zwischen $(\bar a_{\gamma-1} \cap \bar b_{\nu-1})\, [\, \overset{\frown}{\bar A}_{\gamma,\nu} = (\bar a_{\gamma-1} \cap \bar b_{\nu-1})\, [\, \overset{\frown}{K}_{\gamma,\nu}$,

$$(\bar a_{\gamma-1} \cap \bar b_{\nu-1})\, [\, \overset{\frown}{\bar B}_{\nu,\gamma} = (\bar a_{\gamma-1} \cap \bar b_{\nu-1})\, [\, \overset{\frown}{L}_{\nu,\gamma}$$

die sich nach (3) aus dem Hilfsatz (2) ergibt. Die Halbverknüpfung zwischen $\overset{\frown}{K}_{\alpha+1,\beta+1}, \overset{\frown}{L}_{\beta+1,\alpha+1}$ ist wieder trivial.

<u>Z u s a t z .</u> Sind umgekehrt bei jeder Wahl der Basis $\bar{a} \in \bar{V}$ sämtliche Paare von Hüllen $(\bar{a}_{\gamma-1} \cap \bar{b}_{\nu-1}) \sqsubset \bar{K}_{\gamma,\nu}$, $(\bar{a}_{\gamma-1} \cap \bar{b}_{\nu-1}) \sqsubset \bar{L}_{\nu,\gamma}$ verknüpft, so folgt nach Hilfssatz 2 die Bedingung (3)

<u>H i l f s s a t z 3.</u> Es seien $\bar{X} \geq \bar{A}, \bar{Y} \geq \bar{B}$ Zerlegungen auf G. Die Zerlegungen $\bar{x} \sqcap \bar{A}, \bar{y} \sqcap \bar{B}$ sind für jede Wahl von $\bar{x} \in \bar{X}$, $\bar{y} \in \bar{Y}$, $\bar{x} \mathbin{\mathrm{I}} \bar{y}$ halbverknüpft, wenn und nur wenn

$$(4) \quad \bar{A} \frown \bar{Y} = \bar{B} \frown \bar{X}$$

<u>B e w e i s .</u> Aus (4) folgt sofort $\bar{A} \frown \bar{Y} = \bar{B} \frown \bar{X} = \bar{A} \frown \bar{B}$ und die $(\bar{A} \frown \bar{B})$ -Klassen in $\bar{x} \cap \bar{y}$ sind also genau die Durchschnitte der $(\bar{A} \frown \bar{Y})$ -Blöcke, bzw. der $(\bar{B} \frown \bar{X})$ -Blöcke mit $\bar{x} \cap \bar{y}$. Diese Betrachtung ist auch umkehrbar, so dass der Hilfssatz bewiesen ist.

<u>Z u s a t z .</u> Gerade im Falle $\bar{X}\bar{B} = \bar{A}\bar{Y}$ oder $\bar{B}\bar{X} = \bar{Y}\bar{A}$ [5)] wird die Halbverknüpfung aus dem vorigen Hilfssatz die Verknüpfung. Der Beweis ist in [2] § 32, gegeben.

Man kann also in Satz 2 die Bedingung (3) durch

$$5 \quad \bar{A}_{\gamma,\nu} \frown \bar{B}_{\nu,\gamma-1} = \bar{A}_{\gamma,\nu-1} \frown \bar{B}_{\nu,\gamma}$$

für jedes $\gamma = 1, \ldots, \alpha + 1$; $\nu = 1, \ldots, \beta + 1$ ersetzen und die neue Form des Satzes ist auch gülltig.

5/ Diese Relationsprodukte sind da im Sinne von 3 S. 14 gemeint.

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ПРИМЕНЕНИЕ ПРОСТРАНСТВЕННЫХ МАТРИЦ В ТЕОРИИ АЛГЕБРАИЧЕСКИХ ПОВЕРХНОСТЕЙ 4-ГО ПОРЯДКА

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Профессору Отакару Борувке по случаю 70-ой годовщины со дня его рождения

Статья является вкладом в теорию и геометрические приложения пространственных матриц. В ней продолжаются исследования автора из работы [1] в которой с использованием пространственных матриц и их детерминантов найдены основные эквиангармонические комитанты одной, двух и трех троичных алгебраических форм 4-ой степени и дано геометрическое истолкование нулевого значения этих комитантов для соответствующей системы плоских алгебраических кривых 4-той степени.

В предлагаемой работе результаты из [1] расширены для системы одной, двух, трех и четырех алгебраических поверхностей четвертой степени. Внимание автора в этой области намерено на алгебраические образы высших степеней и он уверен в возможности разработания теории алгебраических образов k-той степени, для четного k, при помощи k- мерных матриц аналогично тому, как это наглядно сделано для квадратических образов при помощи двумерных матриц. Ясно, что чем k больше, теория будет более богатой и сложной и тем настойчивее выступает необходимость найти более наглядные методы исследования.

Эквиангармонический совместный инвариант трех троичных алгебраических форм четвертой степени

$$\overset{j}{f} = (\overset{j}{a}x) = (\overset{j}{b}x) = (\overset{j}{c}x),\ j = 1,2,3 \qquad (1)$$

где $\overset{j}{a}, \overset{j}{b}, \overset{j}{c}$ – идеальные параллельные ковариантные векторы, можно выразить в форме (см. [1])

$$|A| = \left|\overset{1}{A}\right|\overset{2}{A}\left|\overset{3}{A}\right|_{\substack{\pm \\ (i_1)(i_3)}}^{\substack{\pm(i_2) \\ \pm(i_4)}} \quad j i_1 i_2 i_3 \ldots = 1,2,3 \qquad (2)$$

где

$$\overset{j}{A} = \left\| \begin{array}{c|c|c} \overset{j}{A}_{11 i_3 i_4} & \overset{j}{A}_{12 i_3 i_4} & \overset{j}{A}_{13 i_3 i_4} \\ \hline \overset{j}{A}_{21 i_3 i_4} & \overset{j}{A}_{22 i_3 i_4} & \overset{j}{A}_{23 i_3 i_4} \\ \hline \overset{j}{A}_{31 i_3 i_4} & \overset{j}{A}_{32 i_3 i_4} & \overset{j}{A}_{33 i_3 i_4} \end{array} \right\|_{(i_3)}^{\to (i_4)} \quad j; i_3, i_4 = 1,2,3 \qquad (3)$$

4 – мерные матрицы порядка три, составленные из коэффициентов $\overset{j}{a}_{i_1 i_2 i_3 i_4}$ форм $\overset{j}{f}$. Инвариант (2) является гипердетерминантом порядка три, сигнатуры ($j\, i_1 i_2 i_3 i_4$) 5 – мерной матрицы A, для которой матрицы $\overset{j}{A}$ являются двумерными сечениями ориентации ($j\ i_1\ i_2$). Геометрическое истолкование нулевого значения этого гипердетерминанта дано в теореме $\overline{V}$ работы [1].

Может случиться, что некоторые из форм $\overset{j}{f}$ равны. Тогда инвариант (2) получит специальную форму. Если какие-нибудь две из этих форм равны, в следствие того, что индекс j в детерминанте (2) неалтернативен и, значит, можно изменить порядок 4 – мерных матриц $\overset{1}{A}, \overset{2}{A}, \overset{3}{A}$ в этом детерминанте без изменения его значения, инвариант (2) получит форму

$$\left|\overset{1}{A}\right|\overset{2}{A}\left|\overset{3}{A}\right|_{\substack{\pm \\ (i_1)(i_3)}}^{\substack{\pm j \\ (\pm i_2) \\ \pm i_4}} \quad j, i_1, i_2, i_3, i_4 = 1,2,3 \qquad (4)$$

где индекс $j = 1$ принадлежит двум одинаковым формам.

Если все три формы $\overset{1}{f}, \overset{2}{f}, \overset{3}{f}$ одинаковы, то инвариант (2) получает форму

$$|\overset{1}{A}|\overset{1}{A}|\overset{1}{A}|^{\overset{+}{j}\;\overset{\pm}{(i_2)}\;\overset{\pm}{(i_4)}}_{\overset{\pm}{(i_1)}\;\overset{\pm}{(i_3)}} \qquad j, i_1, i_2, i_3, i_4 = 1,2,3 \qquad (5)$$

Если некоторая из форм $\overset{1}{f}, \overset{2}{f}, \overset{3}{f}$ равна другой умноженной на численный множитель, этот фактор выступает также как сомножитель в определителе (4) или (5) и поэтому этот случай не требует специального исследования.

Рассмотрим теперь 4 четверичные алгебраические формы четвертой степени

$$\overset{j}{\varphi} = (\overset{j}{a}x) = (\overset{j}{b}x) = (\overset{j}{c}x) = (\overset{j}{d}x); \; j = 1,2,3,4 \qquad (6)$$

где $\overset{j}{a}, \overset{j}{b}, \overset{j}{c}, \overset{j}{d}$ идеальные параллельные ковариантные векторы. Из коэффициентов $\overset{j}{a}_{i_1 i_2 i_3 i_4}$ форм $\overset{j}{\varphi}$ составим 5-мерную матрицу четвертой степени

$$A = \|\overset{1}{A}|\overset{2}{A}|\overset{3}{A}|\overset{4}{A}\|^{\overset{}{j}\;(i_2)\;(i_4)}_{(i_1)\;(i_3)} \qquad j, i_1, i_2, i_3, i_4 = 1,2,3,4 \qquad (7)$$

и ей гипердетерминант

$$|A| = |\overset{1}{A}|\overset{2}{A}|\overset{3}{A}|\overset{4}{A}|^{\overset{\pm}{(i_2)}\;\overset{\pm}{(i_4)}}_{\overset{\pm}{(i_1)}\;\overset{\pm}{(i_3)}} \qquad j, i_1, i_2, i_3, i_4 = 1,2,3,4 \qquad (8)$$

где $\overset{j}{A}$ 4 - мерные матрицы четвертой степени форм $\overset{j}{\varphi}$. Эти матрицы являются двумерными сечениями ориентации $(j i_1 i_2)$ 5 - мерной матрицы A.

Для гипердетерминанта (8) верна следующая

<u>Т е о р е м а I.</u> Гипердетерминант (8) есть совместный инвариант веса 4 форм (6).

Д о к а з а т е л ь с т в о. Теорему докажем так, что гипердетерминант (8) выразим в форме, в которой известен инвариант и его вес.

5- мерный гипердетерминант (8) обозначим через Q и разложим его в алгебраическую сумму 4! 4-мерных гипердетерминантов следующим образом

$$Q = |A| = \sum (-1)^{\mathcal{T}_{i_1}} \left|\begin{array}{c|c|c} \overset{1}{A}_{i_1^{(1)} 1 i_3 i_4} & ----- & \overset{4}{A}_{i_1^{(4)} 1 i_3 i_4} \\ \hline & & \\ ----- & ----- & ----- \\ & & \\ \hline \overset{1}{A}_{i_1^{(1)} 4 i_3 i_4} & ---- & \overset{4}{A}_{i_1^{(4)} 4 i_3 i_4} \end{array}\right| \begin{array}{l} \to \overset{\pm}{(i_2)} \\ \to \overset{\pm}{(i_4)} \\ \\ \downarrow \overset{\pm}{(i_4^{(k)})} \; \downarrow \overset{\pm}{(i_3)} \end{array} \qquad (9)$$

где $\overset{j}{A}_{i_1^{(j)} i_2 i_3 i_4}$ соответствующие двумерные сечения ориентации $(j i_1 i_2)$ матрицы (7), последовательность

$$\{ i_1^{(j)} / j = 1, 2, 3, 4 \} \qquad (10)$$

является некоторой перестановкой чисел 1, ..., 4, число $\mathcal{T}_{i_1}$ равно числу инверсий в перестановке (10) и суммирование распространено на все эти перестановки.

Элементы $\overset{j}{a}_{i_1 i_2 i_3 i_4}$ выразим в форме символических произведений

$$\overset{j}{a}_{i_1 i_2 i_3 i_4} = \overset{j}{a}_{i_1} \overset{j}{a}_{i_2 i_3 i_4}, \; j = 1, \dots, 4 \qquad (11)$$

Теперь мы можем из каждого трехмерного сечения ориентации $(j i_1^{(j)})$ матриц гипердетерминантов в сумме (9) вынести за знаки матриц множитель $\overset{j}{a}_{i_1}$. Нетрудно теперь видеть, что

$$Q = R . Q_1 \qquad (12)$$

где

$$R = | \overset{j}{a}_{i_1} | \qquad (13)$$

и

$$Q = \left|\overset{j}{a}_{i_2 i_3 i_4}\right| \qquad (14)$$

есть гипердетерминант 4-мерной матрицы четвертой степени из символических элементов $\overset{j}{a}_{i_2 i_3 i_4} / j, i_2, i_3, i_4 = 1, \dots, 4.$

Разложивши гипердетерминант Q_1 в алгебраическую сумму кубических детерминантов получим

$$Q_1 = \sum (-1)^{J_{i_2}} \left|\overset{\overset{+}{\frown}}{\overset{j}{a}}_{i_2 \overset{\pm}{i_3} \overset{\pm}{i_4}}\right| =$$

$$= \sum (-1)^{J_{i_2}} \left|\overset{1}{A}_{i_2^{(1)} i_3 i_4}\right| \cdots \left|\overset{4}{A}_{i_2^{(4)} i_3 i_4}\right| \quad \longrightarrow \overset{\pm}{(j_{i_2})} \quad \longrightarrow \overset{\pm}{(i_4)} \quad \downarrow \overset{\pm}{(i_3)} \qquad (15)$$

где $\overset{j}{A}_{i_2^{(j)} i_3 i_4}$ двумерные матрицы с элементами $\overset{j}{a}_{i_2^{(j)} i_3 i_4}$, последовательность $i_2^{(1)}, \dots, i_2^{(4)}$ есть перестановка чисел 1, ..., 4 и J_{i_2} равно числу инверсий соответствующей перестановки.

Элементы $\overset{j}{a}_{i_2 i_3 i_4}$ снова разложим в символические произведения.

$$\overset{j}{a}_{i_2 i_3 i_4} = \overset{j}{a}_{i_2} \overset{j}{a}_{i_3 i_4};\ j, i_2, i_3, i_4 = 1, \dots, 4.$$

Теперь из каждого двумерного сечения ориентации $(j i_2^{(j)})$ матриц кубических определителей в сумме (15) можно вынять фактор $\overset{j}{a}_{i_2}$. После небольшого оформления получаем

$$Q_1 = Q_2 \cdot R^2 \qquad (16)$$

где

$$Q = \left|\overset{1}{A}_{i_3 i_4}\right| \overset{2}{A}_{i_3 i_4}\left| \overset{3}{A}_{i_3 i_4}\right| \overset{4}{A}_{i_3 i_4}\right| \quad \longrightarrow \overset{\pm}{(j)} \quad \longrightarrow \overset{\pm}{(i_4)} \quad \downarrow \overset{\pm}{(i_3)}$$

После подстановки (16) в (12) имеем

$$Q = Q_2 \cdot R^2 \qquad (17)$$

Разложим наконец кубический детерминант Q_2 в сумму обыкновенных определителей. Имеет место соотношение

$$Q_2 = \left| \overset{j}{a}_{\overset{\pm}{i_3}\,\overset{\pm}{i_4}} \right| = \sum (-1)^{\mathcal{T}_{i_3}} \left| \overset{j}{a}_{\overset{+}{i_3}\,\overset{\pm}{i_4}} \right| ,$$

значит,

$$Q_2 = \sum (-1)^{\mathcal{T}_{i_3}} \begin{vmatrix} \overset{1}{a}_{i_3^{(1)}1} & \cdots & \overset{1}{a}_{i_3^{(1)}4} \\ \cdots & \cdots & \cdots \\ \overset{4}{a}_{i_3^{(4)}1} & \cdots & \overset{4}{a}_{i_3^{(4)}4} \end{vmatrix} \qquad (18)$$

где

$$\left\{ i_3^{(j)} / j = 1, \ldots, 4 \right\} \qquad (19)$$

и $\mathcal{T}_{i_3}$ имеют аналогическое значение как в (10).

Элементы $\overset{j}{a}_{i_3 i_4}$ разложим также в символические произведения

$$\overset{j}{a}_{i_3 i_4} = \overset{j}{a}_{i_3} \cdot \overset{j}{a}_{i_4} ; \quad j, i_3, i_4 = 1, \ldots, 4.$$

После этого окончательного разложения элементов можем из j -тых строк обыкновенных определителей в сумме (18) вынять фактор $\overset{j}{a}_{i_3}$. Соотношение (18) тогда получит форму

$$Q_2 = Q_3 \sum (-1)^{\mathcal{T}_{i_3}} \prod_{j=1}^{4} \overset{j}{a}_{i_3(j)}$$

где Q_3 есть детерминант (13), значит

$$Q_3 = R$$

и суммирование распространено на все перестановки (19).

Далее

$$\sum (-1)^{\mathcal{T}_{i_3}} \prod_{j=1}^{4} \overset{j}{a}_{i_3(j)} = R$$

из чего следует

$$Q_2 = R^2$$

Подставляя в (17) вместо Q_2 окончательно получаем

$$Q = R^4 \tag{20}$$

где R есть определитель (13). И так

$$Q = |A| = |\overset{j}{a}_{i_1}|^4; \quad j, i_1 = 1,\dots,4.$$

Определитель (13) составленный из координат векторов $\overset{j}{a}$, которые могут быть также идеальные, параллельные или непараллельные, однако не что иное как фактор Аронгольда второго рода

$$[\overset{1}{a}\overset{2}{a}\overset{3}{a}\overset{4}{a}]$$

И так имеет место равенство

$$|A| = [\overset{1}{a}\overset{2}{a}\overset{3}{a}\overset{4}{a}]^4. \tag{21}$$

Следовательно, в силу первой основной теоремы символического метода Аронгольда, выражение (21) является совместным инвариантом веса 4 форм (6). Тем доказательство теоремы окончено.

И так имеем

$$|\overset{1}{A}|\overset{2}{A}|\overset{3}{A}|\overset{4}{A}|_{\overset{\pm}{(i_1)},\ \overset{\pm}{(i_3)}}^{\overset{+}{(j)},\ \overset{\pm}{(i_2)},\ \overset{\pm}{(i_4)}} = [\overset{1}{a}\overset{2}{a}\overset{3}{a}\overset{4}{a}]^4 \tag{22}$$

где $j, i_1, i_2, i_3, i_4 = 1,\dots, 4$.

Если в дальнейшем мы будем соблюдать ориентацию индексов в 5-мерных матрицах и сигнатуры их детерминантов как в (22), стрелки будем опускать.

Так например матрицу (7) запишем в форме

$$A = \| \overset{1}{A} | \overset{2}{A} | \overset{3}{A} | \overset{4}{A} \|$$

и ей гипердетерминант (8) просто

$$|A| = | \overset{1}{A} | \overset{2}{A} | \overset{3}{A} | \overset{4}{A} |$$

и соотношение (22) получит форму

$$| \overset{1}{A} | \overset{2}{A} | \overset{3}{A} | \overset{4}{A} | = [\overset{1}{a} \overset{2}{a} \overset{3}{a} \overset{4}{a}]^4 \tag{23}$$

Аналогичное соотношение верно также для n -арных форм четвертой степени

$$\overset{j}{\psi} = (\overset{j}{a}x) = (\overset{j}{b}x) = (\overset{j}{c}x) = (\overset{j}{d}x) = \ldots; \; j = 1, \ldots, \tag{24}$$

Построим пятимерную матрицу

$$A = \| \overset{1}{A} | \overset{2}{A} | \cdots | \overset{n}{A} \| \tag{25}$$

порядка n из коэффициентов форм $\overset{j}{\psi}$ аналогично как мы построили матрицу (7) из коэффициентов форм (6) и ей гипердетерминант

$$|A| = | \overset{1}{A} | \overset{2}{A} | \cdots | \overset{n}{A} | \tag{26}$$

Для гипердетерминанта (26) верна

Т е о р е м а II. Гипердетерминант (26) равен четвертой степени фактора Аронгольда второго рода форм (24):

$$| \overset{1}{A} | \overset{2}{A} | \ldots | \overset{n}{A} | = [\overset{1}{a} \overset{2}{a} \ldots \overset{n}{a}]^4 \tag{27}$$

Доказательство этой теоремы аналогично доказательству которое мы привели для $j = 4$ и по этому мы его опускаем.

Из свойств фактора Аронгольда второго рода и из (27) вытекает следующая

Т е о р е м а III. Гипердетерминант (26) есть совместный инвариант веса 4 форм (24).

Соотношение (27) может иметь значительное место в теории алгебраических образов четвертой степени по следующим причинам:

1) Выражение инварианта в форме степени фактора Аронгольда второго рода очень просто и для некоторых целей выгодно, но для расчетов совсем непригодно, так-как в нем появляются координаты идеальных векторов которые не имеют геометрического значения - значение имеют только их произведения. В выражении этих инвариантов через гипердетерминант пространственной матрицы появляются, напротив, непосредственно коэффициенты соответствующих алгебраических форм.

2) Выражая инвариант в форме гипердетерминанта пространственной матрицы может использовать свойства пространственных матриц и их определителей.

3) Выражение инварианта в форме гипердетерминанта пространственной матрицы одинаково просто как и его выражение в форме фактора Аронгольда второго рода.

В дальнейшем мы будем заниматься опять четырьмя кватерными алгебраическими формами (6). Найдем геометрическое значение гипердетерминанта (8) в случае если все формы (6) разные и также в случае, если две или даже все равны. Для этого воспользуемся следующим принципом КЛЕБША:

пусть равенство нулю инварианта

$$[ab]^{\alpha}[ac]^{\beta}\ldots$$

бинарной формы

$$\varphi = (ax)^{k} = (bx)^{k} = (cx)^{k} = \ldots$$

означает, что k точек определенных уравнением $\varphi = 0$ обладают некоторым проективным свойством (E). Тогда контравариант

$$[abu]^{\alpha} \cdot [acu]^{\beta}\ldots$$

терматной формы

$$\Psi = (ax)^2 = (bx)^2 = (cx)^2 = \ldots$$

обращается в нуль для тех и только тех прямых μ, которые пересекают кривую $\Psi = 0$ в точках, обладающих свойством (E).

Этот принцип КЛЕБША не трудно обобщить и для случая проективного пространства P_n, при произвольном n.

Т е о р е м а IV. Эквиангармонический контравариант поверхности

$$(ax)^4 = (bx)^4 = (cx)^4 = (dx)^4 = 0 \qquad (28)$$

пространства P_3 имеет в форме гипердетерминанта 5-мерной матрицы порядка 4 уравнение

$$|A^4|A^4|A^4|A^4|U| = 0 , \qquad (29)$$

где A^4 есть 4-мерная матрица формы $(ax)^4$ и U 4-мерная матрица элементы которой равны

$$U_{i_1 i_2 i_3 i_4} = U_{i_1} U_{i_2} U_{i_3} U_{i_4}$$

причем $U_{i_1}, \ldots, U_{i_4}$ координаты плоскости μ.

Д о к а з а т е л ь с т в о. 4 точки прямой, которые определены уравнением

$$(ax)^4 = (bx)^4 = 0$$

образуют эквиангармоническую четверку точек если

$$[ab]^4 = 0.$$

Из этого на основании принципа КЛЕБША следует, что необходимым и достаточным условием для того, чтобы прямая пересекала кривую 4-той степени

$$(ax)^4 = (bx)^4 = (cx)^4 = 0 \qquad (30)$$

пространства P_2 в 4 эквиангармонических точках, является следующее условие

$$[abu]^4 = 0 \qquad (31)$$

Уравнение (29) есть линейчатое уравнение эквиангармонического контраварианта кривой (28). Условием аполярности кривых (28) и (29) тогда, очевидно, будет

$$[abc]^4 = 0. \tag{32}$$

Если снова используемся принципом КЛЕБША то для поверхности (28) получим уравнение ей эквиангармонического контраварианта в форме

$$[abcu]^4 = 0 \tag{33}$$

Уравнение (31) можно, на основании (27) выразить в форме

$$|\overset{1}{A}|\overset{2}{A}|\overset{3}{A}|u| \quad = 0 \tag{34}$$

Так как идеальные векторы a, b, c параллельны, имеет место

$$\overset{1}{A} = \overset{2}{A} = \overset{3}{A}$$

из чего следует, что уравнение (32) можно записать в форме

$$|\overset{1}{A}|\overset{1}{A}|\overset{1}{A}|u| \quad = 0,$$

чем доказательство завершено.

Т е о р е м а V. Для того, чтобы поверхность (28) была аполярной своему эквиангармоническому контраварианту необходимо и достаточно исполнение условия

$$|\overset{1}{A}|\overset{1}{A}|\overset{1}{A}|\overset{1}{A}| \quad = 0 \tag{35}$$

Д о к а з а т е л ь с т в о. Пусть уравнение эквиангармонического контраварианта поверхности (28) написано в форме (34). Для того, чтобы поверхность (28) была аполярной огибающей (29) необходимо и достаточно, чтобы сумма произведений соответствующих коэффициентов уравнения поверхности (28) и уравнения (29) была нулевой. Эту сумму можем сформировать так, что в уравнение (29) положим $u_{i_1}u_{i_2}u_{i_3}u_{i_4} = a_{i_1 i_2 i_3 i_4}$, то есть

$$u = \overset{1}{A}.$$

Таким образом получим условие (35), чем теорема доказана.

Т е о р е м а VI. Необходимое и достаточное условие для того, чтобы поверхность (28) была аполярной своему эквиангармоническому контраварианту можно выразить также в форме

$$\left|\overset{1}{A}\right| = 0 \qquad (36)$$

где $\left|\overset{1}{A}\right|$ есть гипердетерминант 4-мерной матрицы поверхности (28).

Д о к а з а т е л ь с т в о. Так как все 4-мерные сечения матрицы 5-мерного гипердетерминанта порядка 4 в уравнении (35) равны и индекс неалтернативен, из свойств определителей пространственных матриц следует

$$\left|\overset{1}{A}\right|\overset{1}{A}\left|\overset{1}{A}\right|\overset{1}{A}\right| = 4! \left|\overset{1}{A}\right|$$

где $\left|\overset{1}{A}\right|$ имеет значение из теоремы. И так условие (35) можно выразить более просто в форме (36). Тем теорема доказана.

Остаются случаи, в которых некоторые или даже все из идеальных векторов a, b, c, d взаимно непараллельны. В этих случаях из упомянутых векторов

1) три взаимно параллельны и один с ними непараллелен,

2) два взаимно параллельны но с первыми двумья непараллельны,

3) два взаимно параллельны и два как взаимно так и с первыми двумья параллельны,

4) любые две пары взаимно непараллельны.

Кроме предшествующих ковариантов и инвариантов можно образовать еще следующие типы:

а) ковариант $\left|\overset{1}{A}\right|\overset{1}{A}\left|\overset{2}{A}\right|u\right|$ (37)

$\left|\overset{1}{A}\right|\overset{2}{A}\left|\overset{3}{A}\right|u\right|$ (38)

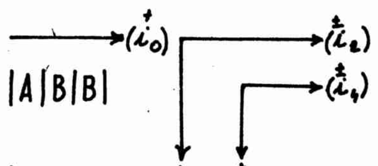
в) инварианты $\left|\overset{1}{A}\right|\overset{1}{A}\left|\overset{1}{A}\right|\overset{2}{A}\right|$ (39)

$\left|\overset{1}{A}\right|\overset{1}{A}\left|\overset{2}{A}\right|\overset{2}{A}\right|$ (40)

$\left|\overset{1}{A}\right|\overset{1}{A}\left|\overset{2}{A}\right|\overset{3}{A}\right|$ (41)

$\left|\overset{1}{A}\right|\overset{2}{A}\left|\overset{3}{A}\right|\overset{4}{A}\right|$ (42)

Геометрическое значение ковариантов (37) и (38) получим использованием принципа КЛЕВША из известного геометрического значения инвариантов (2) и (4), которые содержатся в теоремах $\overline{VI}$ и $\overline{V}$ работы [1]. Так например пусть (E_1) обозначает свойство из теоремы $\overline{VI}$ статьи [1]. Если в инварианте



положим

$$B = \overset{1}{A}\,, \quad A = \overset{2}{A}\;(\overset{\pm}{i_1})\;(\overset{\pm}{i_3}) \qquad (43)$$

и согласно сказанному выше опустим стрелки, получим

$$|\overset{2}{A}|\overset{1}{A}|\overset{1}{A}| \qquad (44)$$

Так как индекс i_0 неалтёрнативен, инвариант (44) можно выразить в форме

$$|\overset{1}{A}|\overset{1}{A}|\overset{2}{A}| \qquad (45)$$

Соотношение (27) позволяет нам использовать принцип КЛЕВША для гипердетерминанта (45). Из него следует, что уравнение

$$|\overset{1}{A}|\overset{1}{A}|\overset{2}{A}|u| = 0 \qquad (46)$$

выполнено теми и только теми плоскостями u, которые пересекают алгебраические поверхности степени 4 с матрицами $\overset{1}{A}$ и $\overset{2}{A}$ /обозначение из (43) / в кривых имеющих свойство (E_1). И так геометрическое значение уравнения (46) можно сформулировать в форме следующей теоремы.

<u>Т е о р е м а VII.</u> Уравнение (46) есть уравнение огибающей плоскостей пересекающих алгебраические поверхности четвертого порядка с матрицами $\overset{1}{A}$ и $\overset{2}{A}$ в кривых имеющих взаимное свойство (E_1) из теоремы $\overline{VI}$ статьи [1] /с использованием обозначения (43) /.

Если в инварианте (46) заменим матрицу u постепенно матрицами $\overset{1}{A}, \overset{2}{A}, \overset{3}{A}$ получим инварианты (39), (40) и (41). Геометрическое значение последних выражено следующей теоремой.

Т е о р е м а VIII. Нулевое значение инвариантов (39), вернее (40), вернее (41) является необходимым и достаточным условием для того, чтобы алгебраическая поверхность 4-ой степени с матрицей соответственно $\overset{1}{A}$ вернее $\overset{2}{A}$ вернее $\overset{3}{A}$ была аполярной поверхности (46).

Доказательство этой теоремы аналогично доказательству теоремы $\overline{V}$.

Свойство из теоремы $\overline{V}$ из [1] обозначим через (E_2). Используя обозначения

$$A = \overset{1}{A}, \quad B = \overset{2}{A}, \quad C = \overset{3}{A}$$

для нулевых значений (38) и (42) верны следующие теоремы.

Т е о р е м а IX. Уравнение

$$|\overset{1}{A}|\overset{2}{A}|\overset{3}{A}|u| = 0$$

есть уравнение огибающей плоскостей пересекающих поверхности четвертого порядка с матрицами $\overset{1}{A}, \overset{2}{A}, \overset{3}{A}$ в кривых имеющих свойство (E_2).

Т е о р е м а X. Равенство нулю инварианта (42) есть необходимое и достаточное условие для того, чтобы каждая из поверхностей четвертого порядка с матрицами $\overset{1}{A}, \overset{2}{A}, \overset{3}{A}, \overset{4}{A}$ была аполярной поверхности

$$|\overset{j_1}{A}|\overset{j_2}{A}|\overset{j_3}{A}|u| = 0,$$

где j_1, j_2, j_3 произвольные три из чисел 1, ..., 4, которые могут и совпадать.

Доказательства этих теорем аналогичны доказательствам предшествующих предложений.

Вообще можно сказать, что для алгебраических поверхностей четвертой степени число которых не больше 4, основным совместным ковариантом является ковариант (38) и совместным инвариантом - инвариант (42). Из последних и их геометрического значения (постепенным отождествлением некоторых из этих поверхностей) вытекают коварианты и соответственно инварианты для трех, двух и одной алгебраических поверхностей четвертой степени и также их геометрические значения.

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