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Regularly connected trivalent graphs without non-trivial cuts of cardinality 3

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Troughout the present paper we mean by a graph a non-empty finite graph.

Convention: the set of vertices (or edges) G will be denoted by V(G) (or E(G)). We shall deal especially with trivalent graphs, i.e. regular graphs of the third degree. Such a graph will be called a C-graph if and only if it is connected and remains connected even after the deleting of fewer than three of any of its edges (according to the terminology used used in [4], a C-graph is a regularly connected and regular graph of the third degree). Examples of the simplest C-graphs are given in Fig. 1 (representing all non-isomorphi C-graphs with fewer than 8 vertices).

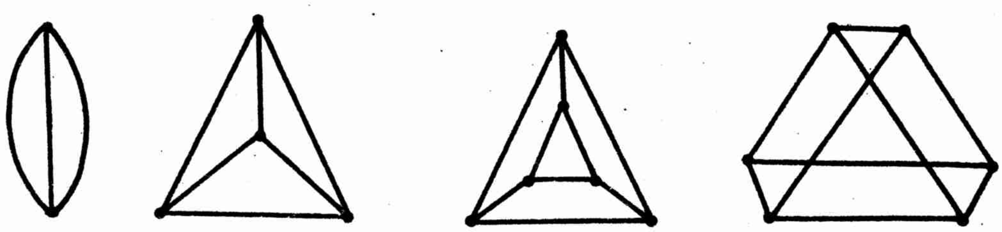


Fig. 1
C-graphs with fewer than 8 vertices

Let G be any graph and let R be such a subset of the set E(g) that has the following properties: after deleting all edges of the set R we get from G a graph which has one component more than G and if we delete from G all edges of R with the exception of arbitrary one edge, we get a graph with the same number of components as G. The set of edges with the above properties will be called an edge-cut of G. When speaking of cardinality of the edge-cut R (symbol $|R|$) we mean the number of its edges. As usually a single edge of the edge-cut of cardinality 1 is called a bridge (sea [3]).

It follows directly from the definition of the C-graph that it is connected and does not contain either a loop, or a bridge, or an edge-cut of cardinality 2. Hence it follows that if a C-graph has more than two vertices it does not contain a two-gon (i.e., it does not contain multiple edges). Since, if two edges, e, f, were incident at the same two vertices $u \neq v$ of a connected 3-valent graph G with more than two vertices, the third edge incident at u together with the third edge incident at v would form an edge-cut of cardinality 2.

It is known (see [4], [5]) that after deleting all edges of any edge-cut of a graph, one of its components is decomposed into exactly two components. These are called the banks of the cut. It is also known that each edge of the edge-cut joins two vertices belonging to different banks of the cut. Further, the following holds:

<u>Theorem 1.</u> Let R be an edge-cut of the graph G and let K be a circuit of G. If p denotes the number of edges from $R \cap K$ then we have $p \equiv 0 \pmod 2$.

<u>Proof</u>. If $p = 0$, it is not necessary to prove anything. Let $p > 0$. If we delete from the circle K all p edges belonging to R, the circuit K will split into p paths (an isolated vertex is considered here as the path of a zero lenght). Each of these paths is joined to another path by at most two edges from R. Running along the circuit K, the path belonging to one bank of the cut R must alternate with the path belonging to the other bank (as the edge of R joins two vertices of different banks and each of the considered paths belongs entirely to one of the banks of the cut R). Hence it follows that the number of such paths is even and $p \equiv 0 \pmod 2$, q.e.d.

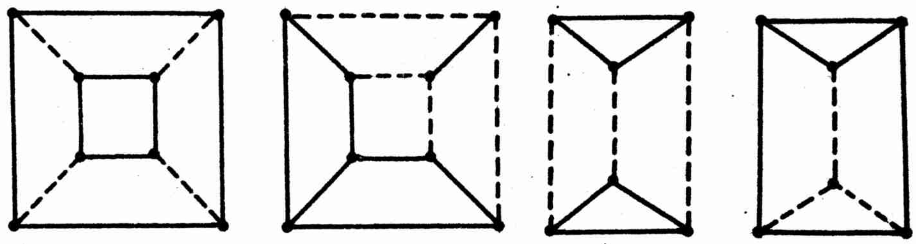
The edge-cut of a graph is said to be non-trivial if each of both its banks contains a circuit as a subgraph. In the reserve case it is called a trivial edge-cut. A trivial (or non-trivial) edge-cut will be called henceforward a t-cut (or u-cut).

T h e o r e m 2. In a 3-valent graph no t-cuts of cardinality 1 and 2 can exist. An edge-cut of cardinality $m \geqq 3$ is a t-cut if and only if the number of vertices of at least one of the banks is exactly $m-2$.

P r o o f. The validity of the first assertion of the theorem is evident. Let us prove the validity of the second assertion. Let R be such an edge-cut of cardinality $m \geqq 3$ of the 3-valent graph G that one of its banks contains exactly $m-2$ vertices. With respect to the number p of the edges of the bank the following holds: $2p = 3(m-2)-m$. Hence $p=m-3$ and thus the number of edges of the bank is one less than the number of its vertices. As the bank of the cut is a connected graph, it follows that the considered bank is a tree. Hence it is a t-cut.

Let Q be any t-cut of cardinality $m \geqq 3$ in the 3-valent graph G and let B be that of its banks which does not contain a circuit. In such a case B is a tree. Let b be the number of vertices of the bank B. The number of its edges is $b-1$ and $2(b-1) = 3b-m$ holds. Hence, if Q is a t-cut of cardinality m, the number of vertices of one of its banks is $m-2$. This proves the theorem.

Examples of t-cuts and u-cuts of cardinality 3 and 4 in a C-graph are given in Fig. 2 (the edges of the cut are in a dashed line).



cardinality 3 — t-cut, u-cut

cardinality 4 — t-cut, u-cut

Fig. 2
Examples of edge-cuts in C-graphs

Let G be a C-graph and let $e \neq f$ be edges of $E(G)$; let u_1, u_2 be the endpoints of the edge e; v_1, v_2 the endpoints of the edge f. Let G be a graph arising from G as follows: (1) the paths of length one $\{u_1, e, u_2\}$

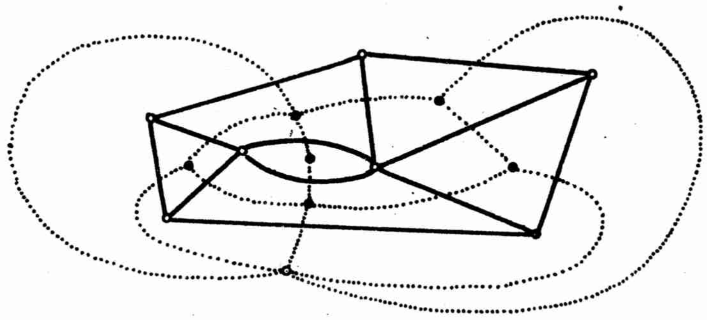
and $\{v_1, f, v_2\}$ are substituted by the paths of lenght two $\{u_1, e_1, u_0, e_2, u_2\}$ and $\{v_1, f_1, v_0, f_2, v_2\}$ (where u_0, v_0 are new vertices and e_1, e_2, f_1, f_2 are new edges) and (2) two new vertices u_0, v_0 are joined by another new edge g. We shall say that the graph G', arising as described from G, is the H-extension of G on the edges e, f. The reverse process, i.e the chance of G' into G is called the H-reduction of the edge g of G'.

<u>Theorem 3.</u> Let G be any C-graph and let G' the graph which arises by an H-extension of G on any two of its edges; then G' is a C-graph and every C-graph can be constructed from the C-graph containing exactly two vertices by repeated H-extensions.

<u>Proof</u>. The first assertion of the theorem is evidently valid. The proof of the validity of the second assertion may be found in paper [4].

We say that a graph G is a planar graph if it can be realized in a plane in such a way that its vertices are points and its edges simple arcs joining the corresponding pairs of points and no two arcs have any common point apart from the end-point (which is the vertex of the graph). A graph G constructed in this way decomposes the plane into connected regions called faces. The set of faces arising in a given plane from G will be denoted by $S(G)$. Instead "the face $s \in S(G)$ is bounded by elements of G" we shall say (as in the special case of polyhedra) that the face is incident at vertices and edges from G. The following is evident: If G is a connected planar fraph without cut points and loops containing at least one circuit), then each face is incident at all elements of a circuit and only at elements of this circuit. In such a case each edge of $E(G)$ is incident at exactly two elements of $V(G)$ an also at exactly two elements of $S(G)$. For any $s \in S(G)$, $e \in E(G)$, $v \in V(G)$ we have: [s is incident at e; e is incident at v] $\Rightarrow$ [s is incident at v] . If a face is incident at the vertex v, then there exist exactly two edges incident both at the face and the vertex v. The triple of sets $\{V(G), E(G), S(G)\}$ with thus defined incidence will be called the P-complex of the graph G and will be detoned by $P(G)$.

Let G be a connected planar graph and $P(G)$ its P-complex. Let us assign to it a topologically dual complex $P(G^*)$ so that within each face $s \in S(G)$ we choose a vertex $\overset{*}{v}$; to each edge $e \in E(g)$ we assign an edge $\overset{*}{e} \in E(G^*)$ cutting the edge e, which joins the vertices $\overset{*}{x}, \overset{*}{y}$ chosen of the faces lying on both sides of the edge e (see Fig. 3).



vertices $\in V(G)$ vertices $\in V(G^*)$

Fig. 3

If $P(G^*)$ is the topologically dual P-complex of the P-complex $P(G)$, we say that the graph G^* is conjugate with the graph G. It is known that if G^* is conjugate with G, then G is conjugate with G^* and there exists a one-to-one mapping c of the set $V(G) \cup E(G) \cup S(G)$ onto the set $V(G^*)$ $E(G^*) \cup S(G^*)$ with the following properties:

(1) $c(e) \in E(G^*)$ $\forall$ $e \in E(G)$ for any

(2) $c(v) \in S(G^*)$ $\forall$ $v \in V(G)$ for any

(3) $c(s) \in V(G^*)$ $\forall$ $s \in S(G)$ for any

(4) the edge $c(e) \in E(G^*)$ is in G^* incident at the vertices $c(s_i)$, $c(s_j)$ and at the faces $c(v_p)$, $c(v_q)$, if and only if the edge e in the graph G is incident at the faces s_i, s_j and the vertices v_p, v_q.

Theorem 4. Let G, G^* be conjugate graphs and $P(G), P(G^*)$ their P-complexes. Let c be a mapping with the properties (1),(2),(3),(4). Then the c-images of the edges of any circuit of G form an edge-cut of the graph G^* and the c-images of the edges of any edge-cut of G form the set of edges of a circuit of G^*.

Proof. It follows directly from the definition of the P-complex that if two faces $s \neq t$ of $S(G)$ were incident at two edges e_p, e_g then these edges would form in G an edge-cut of cardinality 2 (see Fig.4). This, however, is not possible, since G is a C-graph. The theorem follows.

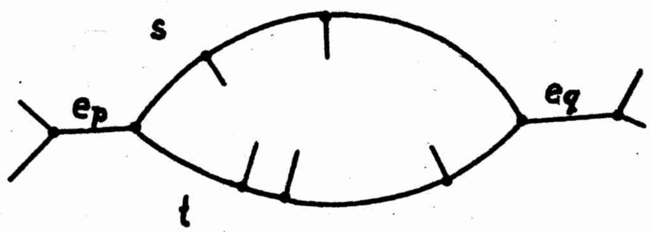


Fig. 4

T h e o r e m 6. Let R be any edge-cut of cardinality n in the planar C-graph G and let P(G) be its P-complex. Then S(G) contains exactly n faces incident at edges from R. These faces can be detoned by $s_1, s_2, \dots, s_n$ and the edges of R can be denoted by $e_1, e_2, \dots, e_n$ so that for all $i \in \{1, 2, \dots, n\}$ the following holds: the edge e_i is incident at the faces s_i, s_{i+1} (we put $s_{n+1} = s_1$).

P r o o f. The theorem is a simple corollary of Theorem 4.

T h e o r e m 7. Let G be any planar C-graph and P(G) its P-complex. Let G' be a graph arising by an H-extension on its edges $e \neq f$, then G' is planar if and only if e, f are incident at the same face of S(G); by an H-reduction of G on each of its edges there always arises a planar graph. The proof is evident.

N o t e 1. It follows directly from Theorem 7 that if, in case of extensions of planar C-graphs, we bear in mind that both edges with the H-extension be incident at the same face (this face is in the case of H-extension decomposed into two new faces) then starting from a C-graph with two vertices (see Fig. 1) we can construct in this way successively all planar C-graphs with a given number of vertices without the necessity to construct non-planar C-graphs.

N o t e 2. If by a 3-valent polyhedron we mean such an Euler polyhedron in the sense of Steinitz (see [7] ,p.113) where each vertex is incident at exactly three edges, we evidently have: Any planar C-graph with more than two vertices is a graph of a 3-valent polyhedron and the graph of such a polyhedron is always a planar C-graph. Hence Theorem 7 enables us to construct all 3-valent polyhedra.

When studying the decomposition of planar 3-valent graphs into three linear factors (which play an important role in the well-known four-colour problem) we can restrict - as it is known - to such graphs which contain more than three vertices and do not contain a u-cut of cardinality 3. Through the method mentioned in note 1 enables us to construct successively all planar C-graphs with the given number of vertices, this way of construction does not seem to be convenient, as we obtain a (for us useless) byprodukct in the form of planar C-graphs with u-cuts of cardinality 3. We should like to avoid these. If, however, in constructing planar C-graphs with the method of repeated H-extensions, we avoid at every step a planar C-graph with a u-cut of cardinality 3 (in other words: we avoid a planar C-graph with a triangle), it may happen that we do not obtain some graphs with the required property by our method. Of course, there is still the possibility that there exist more graphs than one planar C-graph without a u-cut of cardinality 3, wherein the H-reduction cannot be accomplished without a u-cut of cardinality 3 arising (such graphs would inevitably be avoided in the mentioned procedure). In the following we shall deduce theorems which will enable us to avoid such difficulties.

<u>C o n v e n t i o n :</u> Fot the sake of simplification we shall call in the following a planar C-graph with more than 4 vertices, not containing any u-cut of cardinality 3, a Y-graph.

<u>T h e o r e m 8.</u> Let G be any Y-graph and R any its u-cut of cardinality 4, then no two edges of R are adjacent.

<u>P r o o f</u>. If two edges e,f of a u-cut R (where $|R| = 4$) were incident at the same vertex v, then the bank with the vertex v would contain, apart from a circuit, also an edge which is one edge of the bank incident at the vertex v (let us denote this edge by g - as two other edges incident at v belong to R). If in the cut R both edges e,f were substituted by a single edge g, we should evidently obtain a u-cut of cardinality 3. This is a contradiction to the assumption that G is a Y-graph. This proved the theorem.

<u>T h e o r e m 9.</u> Let G be such a Y-graph wherein each edge belongs to a u-cut of cardinality 4, then G is isomorhic with the graph of a (3-dimensional) cube.

<u>P r o o f</u>. G contains a u-cut of cardinality 4. Hence it contains four edges, no two of which are adjacent (see Theorem 8). This mean that G has at least 8 vertices.

<u>A s s e r t i o n:</u> G does not contain any triangle. Proof of the assertion: Three edges incident at the vertice of a triangle and not belonging to this triangle would form such an edge-cut of cardinality 3 that one of its banks would be the considered triangle and the other bank would contain at least 5 vertices. According to the Theorem 2 it would 2 it would be a u-cut of cardinality 3, which is not possible in a Y-graph. This proves the validity of the assertion.

A s s e r t i o n: G contains at least one quadrangle. Proof: Let us assume, on the contrary, that G does not contain a quadrangle. It is known (see [6] , p.25) that the P-complex of a 3-valent graph without bridge, not containing either a triangle or a quadrangle, must have at least 12 such faces that are bounded by a pentagon. Hence G contains a pentagon.

Let K_0 be any pentagon Of G, let e_1 be any of its edges and R_1 any u-cut of cardinality 4 containing e_1. According to the Theorem 1 R_1 contains an even number of edges from K_0. Therefore there is in K_0 another edge (denoted by e_3) belonging to R_1. According to the Theorem 8 e_3 is not adjancet at e_1 and the refore there is in K_0 exactly one edge (denoted by e_2) adjacent both at e_1 and e_3 (since K_0 is a pentagon). The remaining two edges of K_0 will be denoted by e_4, e_5 so that we have: e_4 is adjacent at e_3 and e_5 is adjacent at e_1. It is evident (see Theorem 8) that none of the edges e_2, e_4, e_5 belongs to R_1.

Denote the faces of $S(G) \in P(G)$ as follows: s_0 is the face bounded by K_0; s_i $(i \in \{1,2,\dots,5\})$ is the face adjacent at s_0 on the edge e_i. Denote the vertices and edges of G as follows: denote by $v_{i,j}$ the vertex of K_0 incident at the edges e_i, e_j and by $f_{i,j}$ (see Theorem 6) the edge incident at $v_{i,j}$ and not belonging to K_0.

Fig 5

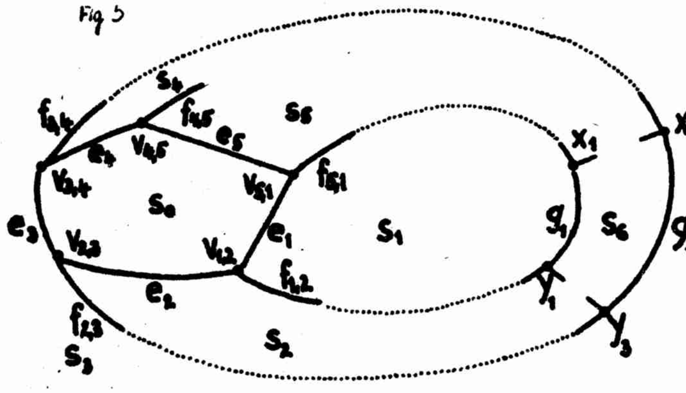


Fig. 5

Apart from the edges e_1, e_3 R_1 contains other two edges, both incident in P(G) at the same face (denoted by s_6). Denote these edges by g_1, g_3 so that the following holds: g_1 is incident at s_1, s_6 and g_3 is incident at s_3, s_6. The endpoints of the edge g_i $(i \in \{1,3\})$ will be denoted by x_i, y_i (Fig.5).

Denote by R_2 any u-cut of cardinality 4 containing e_2. R_2 contains from K_0 another edge: either e_4 or e_5. Without loss of generality we may assume that R_2 contains e_4.

From Theorem 8 it follows that $f_{2,3} \neq g_3 \neq f_{3,4}$; $f_{5,1} \neq g_1 \neq f_{1,2}$, hence $s_4 \neq s_6 \neq s_2$; $s_6 \neq s_5$. The edge cut R_2 contains another edge from the circuit, which bounds s_2 (denote it by g_2) and contains an edge (denote it by g_4) belonging to the circuit bounding s_4. We evidently have $g_2 \neq g_4$, hence $R_2 = \{ e_2, e_4, g_2, g_4 \}$. As $f_{1,2} f_{2,3}$ are adjacent at e_2 and e_2, g_2 belong to R_2 it necessarily follows (see Theorem 8) that $f_{1,2} \neq g_2 \neq f_{2,3}$. In a similar way we get $f_{3,4} \neq g_4 \neq f_{4,5}$. Hence the following holds: no edge of R_2 is incident at s_2, s_6 and g_4 is incident at s_4, s_6.

<u>A s s e r t i o n:</u> The distance of vertices $v_{1,2}, y_1$ similarly as the distance of vertices $v_{2,3}, y_3$ (or of the vertices $v_{3,4}, x_3$) is 1. Proof: If there were on the circuit bounding s_1 between the vertices $v_{1,2}, y_1$ a vertex z_1, it would mean that $\{ f_{1,2}, g_1, g_2 \}$ is an edge-cut, each bank of which contains more than one vertex. According to the Theorem 2 it is a u-cut which is a contradiction to the assumption that G is a Y-graph. Hence $f_{1,2}$ joins the vertices $v_{1,2}, y_1$. In the same way it can be proved that $f_{2,3}$ joins $v_{2,3}$ and $f_{3,4}$ joins $v_{3,4}, x_3$. Then, of course, each of the faces s_2, s_4 is incident at exactly four edges. This is a contradiction to the assumption that G does not contain a quadrangle. This proves the validity of the assertion that G contains a quadrangle.

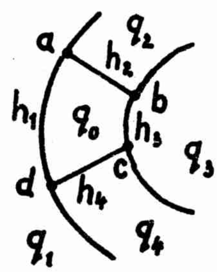


Fig. 6

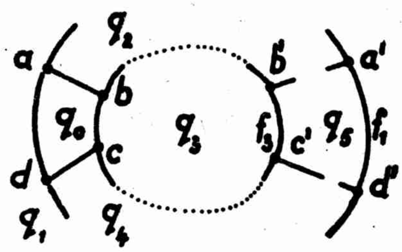


Fig. 7

Let therefore K be a quadrangle bounding the face q_0 of P(G). Denote the vertices of K by a,b,c,d and the edges of K by h_1,h_2,h_3,h_4 and the faces adjacent at q_0 by q_1,q_2,q_3,q_4 in such a way as shown in Fig. 6.

Let R_3 be a u-cut of cardinality 4 containing h_3. This cut contains also h_1 and apart from it two other edges f_1,f_3, both incident in P(G) at the same face (denoted by q_5); f_1 is incident at q_1,q_5 and f_3 is incident at q_3,q_5 and we have: $q_2 \neq q_5 \neq q_4$. The endpoints of the edges f_1,f_3 will be denoted in the same way as in Fig. 7.

The edge h_2 belongs together with h_4 to the same u-cut of cardinality 4 (denoted by R_4). For reasons mentioned in the proof of the foregoing assertion, R_4 contains apart from the edges h_2,h_4 also an edge (denoted by f_2), which in P(G) is incident at q_2,q_5. R_4 contains besides only one more edge (denoted by f_4), which in P(G) is incident at q_4,q_5. Similarly as in the proof of the foregoing assertion we find that the distance between the vertices x,x' (where $x \in \{a,b,c,d\}$) is 1. Whence it follows that q_1 as well as q_3 are incident at exactly four edges. If f_2 were not incident at a', it would mean that f_1,f_2 with the edge joining the vertices a,a' form a u-cut of cardinality 3. This is not possible, as G is a Y-graph. Therefore f_2 is incident at a'. For the same reasons f_2 is incident at b' and f_4 joins the vertices c',d'. In other words: G is isomorphic with the graph of a three-dimensional cube, q.e.d.

<u>Theorem 10.</u> Let G be a Y-graph with the following property: By an 4-reduction of any of its edges we obtain a graph which is not a Y-graph; then G is isomorphic with the graph of a three-dimensional cube.

<u>Proof.</u> With respect to graph G with the asumed property, evidently the following holds: any edge of it belongs to the u-cut of cardinality 4. Such a graph according to the Theorem 9 is isomorphic with the graph of a three-dimensional cube. This proves the theorem.

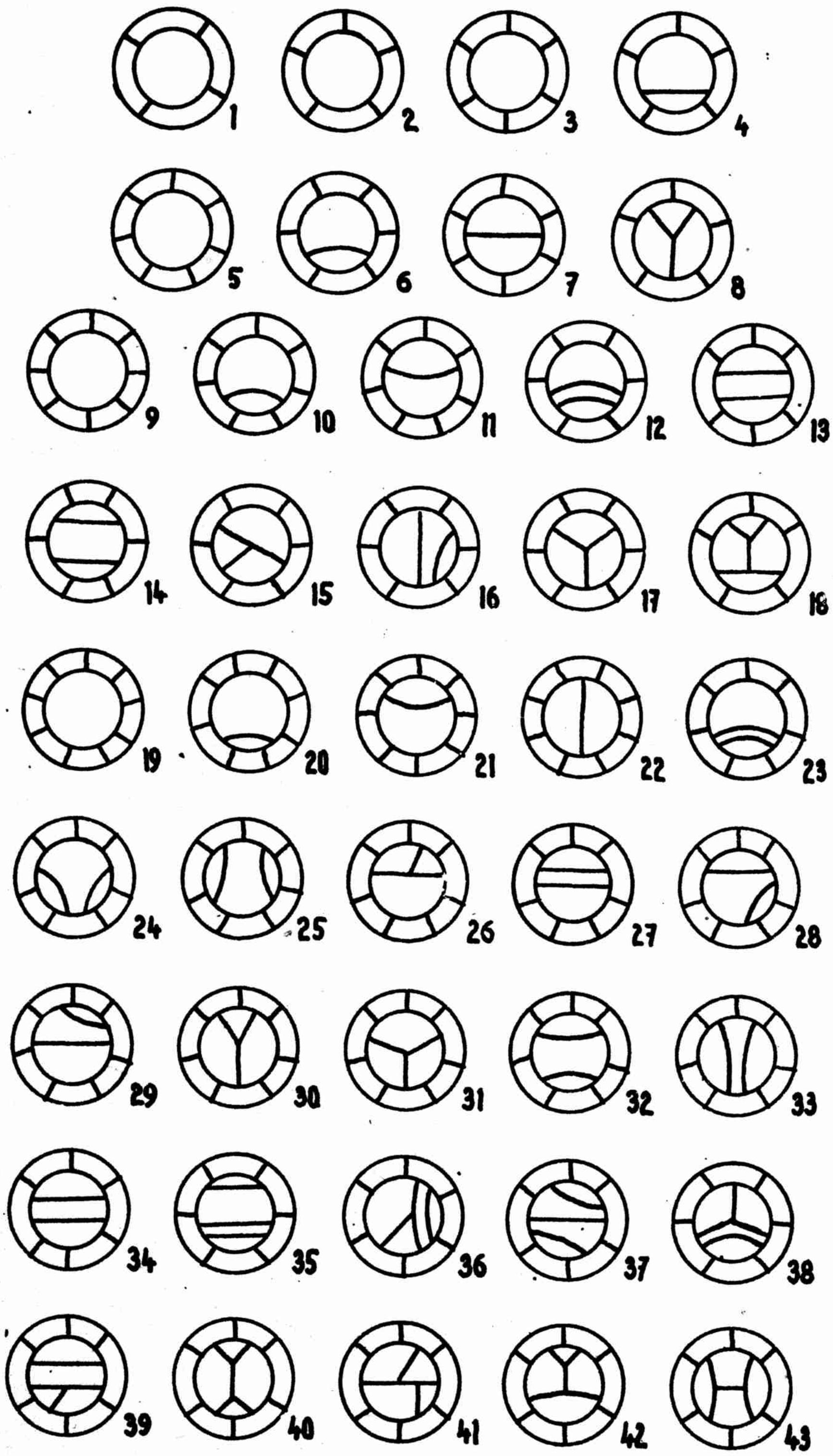
<u>Theorem 11.</u> Any Y-graph with the minimum number of vertices is isomorphic with the graph of a three-dimensional cube. Any Y-graph with more than 8 vertices can be constructed from such a graph by repeated H-extensions, which ar always done on a pair of non-adjacent edges incident at the same face of the P-complex of the graph that is being extended.

<u>Proof.</u> The validity of the first assertion of the theorem is evident. The validity of the second assertion follows from the fact that every Y-graph with more than 8 vertices can be always, with the help of an H-reduction, reduced to a Y-graph (see Theorem 10).

Fig. 8 shows the all possible non-isomorphic Y-graphs with the number of vertices less than 20. The following table shows how many faces nounded by an n-gon are contained in a P-complex of this or that Y-graph in Fig.8. Grace's list of 3-valent polyhedra without triangles with fewer than 20 vertices is a little more extensive (there are 55 - Grace does not exclude polyhedra with a u-cut of cardinality 3 - see [2]).

N o t e 3. The table of Y-graphs easily proves the well-known fact that the number of individual n-gons in polyhedron does not determine the polyhedron - hence neither the Y-graph - uniquely. Thus, e.g., Y-graphs 37,38,39 and 40 in Fig. 8 agree as regards to these data, but no two of them are isomorphic.

N o t e 4. A list of convex 3-valent polyhedrons containing no triangle (up to 18 vertices) is given in [1]; cyclically 4-connected polyhedrons (i.e. those containing a non-trivial edge-cut u-cut of cardinality 3) are in this list marked by stars. As Grace [2] and Lederberg [1] have noted, their list may still be incomplete. From our results it follows that it is not such a case (but in [1] in Tab. 2 at polyhedron No. 505 a star has been omitted, therefore the total number of cyclically 4-connected convex trivalent polyhedra up to 18 vertices is - in accordance with our results - only 43 and not 44 as stated in [1]).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Y-graps with fewer than 20 vertices

Y-graph	Number of faces bounded by an n-gon; n = 4	5	6	7	8	9	Y-graph	Number of faces bounded by an n-gon; n = 4	5	6	7	8	9
1	6	-	-	-	-	-	23	7	-	2	2	-	-
2	5	2	-	-	-	-	24	6	2	1	2	-	-
3	6	-	2	-	-	-	25	5	4	-	2	-	-
4	4	4	-	-	-	-	26	5	4	-	2	-	-
5	7	-	-	2	-	-	27	6	1	3	1	-	-
6	5	2	2	-	-	-	28	5	3	2	1	-	-
7	4	4	1	-	-	-	29	5	3	2	1	-	-
8	3	6	-	-	-	-	30	5	3	2	1	-	-
9	8	-	-	-	2	-	31	4	5	1	1	-	-
10	6	2	-	2	-	-	32	4	5	1	1	-	-
11	5	3	1	1	-	-	33	4	5	1	1	-	-
12	6	-	4	-	-	-	34	6	-	5	-	-	-
13	5	2	3	-	-	-	35	5	2	4	-	-	-
14	4	4	2	-	-	-	36	5	2	4	-	-	-
15	4	4	2	-	-	-	37	4	4	3	-	-	-
16	4	4	2	-	-	-	38	4	4	3	-	-	-
17	3	6	1	-	-	-	39	4	4	3	-	-	-
18	2	8	-	-	-	-	40	4	4	3	-	-	-
19	9	-	-	-	-	2	41	3	6	2	-	-	-
20	7	2	-	-	2	-	42	3	6	2	-	-	-
21	6	3	-	1	1	-	43	2	8	1	-	-	-
22	6	2	2	-	1	-							

R E F E R E N C E S

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ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
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Berechnung einer bestimmten Konstante aus der Theorie der Lürothschen Reihen

A. DÁVID

Jede reelle Zahl $x \in (0,1)$ kann man in der Form de Lürothschen Reihe

$$x = \frac{1}{d_1+1} \quad \frac{1}{s_1} \cdot \frac{1}{d_2+1} + \dots + \frac{1}{s_1 \dots s_{n-1}} \cdot \frac{1}{d_n+1} + \dots$$

eindeutig darstellen,

wo $s_i = d_i(d_i+1)$, $i = 1, 2, \dots$ $d_i = d_i(x)$

natürliche Zahlen, die sogenanten Lürothschen Ziffern der Zahl sind. Siehe [1] S. 116-122.

Die Lürothschen Entwicklungen stehen (mit Rücksicht auf ihre Eigenschaften) an der Grenze zwischen dekadischen (g-adischen) Entwicklungen und Kettenbrüchen.

A. Chinčin hat das folgende Ergebnis bewiesen:

Wenn $$x = \cfrac{1}{c_1 + \cfrac{1}{c_2 + \dots}}$$

der Kettenbruch von $x \in (0,1)$ ist, dann gilt für fast alle $x \in (0,1)$

$$\lim_{n\to\infty} \sqrt[n]{c_1(x) \dots c_n(x)} = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k(k+2)}\right)^{\frac{\ln k}{\ln 2}} = k_0 = 2{,}685452\dots \quad (1)$$

Über eine detailierte Berechnung von Chinčins Ergebnis ist in der Arbeit [5] ein analogisches Ergebnis für Lürothsche Reihen bewiesen: Für fast alle $x \in (0,1)$ gilt die Gleichheit

$$\lim_{n\to\infty} \sqrt[n]{d_1(x) \dots d_n(x)} = \prod_{k=1}^{\infty} k^{\frac{1}{k(1+k)}} = c_0$$

---

ln u bezeichnet den natürlichen Logaritmus von u.

Berechnung der Konstante C_0

Untersuchen wir das endliche Produkt

$$S_N = \prod_{k=1}^{N} k^{\frac{1}{k(1+k)}} = 1^{\frac{1}{1.2}} \cdot 2^{\frac{1}{2.3}} \cdots N^{\frac{1}{N(1+N)}} .$$

Dann ist

$$(1) \qquad \ln S_N = \sum_{k=1}^{N} \frac{\ln k}{k(1+k)} .$$

Da die Reihe

$$\sum_{k=1}^{\infty} \frac{\ln k}{k(1+k)}$$

schwach konvergent ist, war es notwendig folgendes abzuschätzen

1. Die Abweichung von

$$\sum_{k=1}^{N} \frac{\ln k}{k(1+k)}$$

2. Maschinenabweichungen (das Normalisieren der Zahl, Ziffernverschiebungen u.s.w.).
3. Den Rest

$$\sum_{k=N+1}^{\infty} \frac{\ln k}{k(1+k)}$$

(durch ein bestimmtes Integral) ausendrücken.

1. Abweichungsabschätzung von (1)

Bezeichnen wir mit $E(x)$ die absolute Abweichung von x,
mit $RE(x)$ die relative Abweichung von x.

Die Berechnung wurde an dem Rechenautomat ODRA 1013 durchgefürhrt
Für diesen Automat gilt

$$E\,(k(1+k)) \leqq 10^{-9}\,, \quad RE\,(\ln k) \leqq 10^{-9}$$

woraus
$$RE\,\big(k(1+k)\big) \leqq \frac{10^{-9}}{k(1+k)}\,, \quad E(\ln k) \leqq 10^{-9} \cdot \ln k.$$

Dann ist

$$RE\left(\frac{\ln k}{k(1+k)}\right) \leqq RE(\ln k) + RE\big(k(1+k)\big) \leqq 10^{-9} + 10^{-9} \cdot \frac{1}{k(1+k)}\,,$$

$$E\left(\frac{\ln k}{k(1+k)}\right) \leqq 10^{-9}\left(1+\frac{1}{k(1+k)}\right) \cdot \frac{\ln k}{k(1+k)} < 10^{-9} \cdot \frac{7}{6} \cdot \frac{\ln k}{k(1+k)}\,,$$

denn für $k = 1$ ist $\ln k = 0$ und $E\,(\ln S_1) = 0,$

für $k = 2$ ist

$$1 + \frac{1}{k(1+k)} = \frac{7}{6}\,,$$

für $k > 2$ ist

$$1 + \frac{1}{k(1+k)} < \frac{7}{6}\,.$$

Aus (1) bekommen wir

$$(2) \qquad E(\ln S_N) < \sum_{k=2}^{N} \frac{7}{6} \cdot 10^{-9}\, \frac{\ln k}{k(1+k)} = \frac{7}{6} \cdot 10^{-9} \cdot \ln S_N$$

2. Maschinenabweichungen

Die Abweichung $a_k = 10^{-9}$ ist bei jedem Glied der Reihe (1) (durch das Normalisieren und die Verschiebungen Akkumulator ↔ Speicher) entstanden.

Durch die Summierung aller Glieder von (1) bekomt man

$$\sum_{k=1}^{N} a_k = N \cdot 10^{-9}$$

Die Totalabweichung der Reihe (1) ist also nach (2)

$$E_N = N \cdot 10^{-9} + \frac{7}{6} \cdot 10^{-9} \cdot \ln S_N\,.$$

3. Abschätzung von $\sum_{k=N+1}^{\infty} \frac{\ln k}{k(1+k)}$

Untersuchen wir die Funktion

$$f(x) = \frac{\ln x}{x(1+x)}$$

im Intervall $[N, \infty)$

Es ist $f(x)>0$ für $x>1$; $\lim_{x\to\infty} f(x) = 0$, ferner

$$f'(x) = \frac{1+x-(2x+1)\ln x}{x^2(1+x)^2} < 0$$

für $x>e$.

Die Funktion f ist also im Intervall (e, ∞) positiv, fallend und deshalb gilt für $N \geq 3$ die Ungleichheit

$$\sum_{k=N+1}^{\infty} \frac{\ln k}{k(1+k)} < \int_N^{\infty} \frac{\ln x}{x(1+x)}\, dx.$$

Die Funktion

$$\int \frac{\ln x}{x(1+x)}\, dx$$

kann man nicht durch elementare Funktionen darstellen. Darum schätzen wir das Integral in (3) nach oben folgendermassen ab:

Es gilt für $x>e$

$$\frac{\ln x}{x(1+x)} - \frac{\ln x}{x^2+x+\frac{1}{4}} = \frac{1}{4}\,\frac{\ln x}{x(1+x)(x+\frac{1}{2})^2} < \frac{1}{4}\,\frac{\ln x}{x^3},$$

also

$$\frac{\ln x}{x(1+x)} < \frac{\ln x}{(x+\frac{1}{2})^2} + \frac{1}{4}\,\frac{\ln x}{x^3}.$$

Daraus folgt

$$(4)\qquad \int_N^{\infty} \frac{\ln x}{x(1+x)}\, dx < \int_N^{\infty} \frac{\ln x}{(x+\frac{1}{2})^2}\cdot dx + \frac{1}{4}\int_N^{\infty} \frac{\ln x}{x^3}\, dx =$$

$$= \frac{\ln N}{N+\frac{1}{2}} - 2\ln\frac{N}{N+\frac{1}{2}} + \frac{1}{8}\,\frac{\ln N}{N^2} + \frac{1}{16N^2}.$$

Für die gesamte Abschätzung der Zahl $\ln C_0$ bekommen wir

$$\ln S_N - E_N < \ln C_0 < \ln S_N + E_N + \int_N^\infty \frac{\ln x}{x(1+x)}\,dx$$

und nach (2), (3), (4) haben wir

$$-\frac{7}{6}\,10^{-9}\ln S_N - N.\,10^{-9} < \ln C_0 < \ln S_N + N.\,10^{-9} + \frac{\ln N}{N+\frac{1}{2}} -$$

$$- 2\ln\frac{N}{N+\frac{1}{2}} + \frac{1}{8}\,\frac{\ln N}{N^2} + \frac{1}{16N^2} + \frac{7}{6}\,10^{-9}\ln S_N .$$

Die untere und obere Grenze für die Zahl C_0 ist aus der Tabelle I ersichtlich:

$$2{,}19965460 < C_0 < 2{,}20016886.$$

Daraus $C_0 = 2{,}19991175 \pm 0{,}0002571$.

Die Zahl C_0 ist also kleiner als die Chinčinsche Konstante $k_0 = 2{,}685452 \ldots.$

Tabelle I

N	ln S_N	Total-abweichung	Untere Grenze	Obere Grenze
1044	0,78092145	0,00761525	2,18348103	2,20017224
2937	0,78547226	0,00306479	2,19343611	2,20016886
10195	0,78752729	0,00102374	2,19793240	2,20018363
25694	0,78809653	0,00048549	2,19914981	2,20021771
50000	0,78829046	0,00033643	2,19952286	2,20026298
74048	0,78836126	0,00031302	2,19962569	2,20031437
102595	0,78840294	0,00032740	2,19965460	2,20037488
104099	0,78840434	0,00032877	2,19965436	2,20037767
106744	0,78840680	0,00033131	2,19965394	2,20038286
119000	0,78842198	0,00034461	2,19966039	2,20041885
120000	0,78842291	0,00034580	2,19966023	2,20042102

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On sums of the prime powers

by

TIBOR ŠALÁT AND ŠTEFAN ZNÁM, Bratislava

In the book [1] (see [1], p. 30) is proved that for each $a > -1$ there exist such constants $c_{11}, c_{12} > 0$ that for any $x \geq 2$ it holds

$$c_{11} \frac{x^{1+a}}{\log x} < \sum_{p \leq x} p^a < c_{12} \frac{x^{1+a}}{\log x} .$$

In our article we give a sharpening of this result for the case $a > 0$.

Let us denote $S_a(x) = \sum_{p \leq x} p^a$ (for arbitrary $x \geq 2$).

<u>Theorem.</u> For arbitrary $a > 0$ we have

$$\lim_{x \to \infty} \frac{S_a(x) \log x}{x^{1+a}} = \frac{1}{1 + a} .$$

<u>Proof.</u> For $a = 0$ our assertion follows immediately from prime number theorem. Now, let us suppose that $a > 0$.

Denote by the symbol $\{p_k\}_1^\infty$ the increasing sequence of all primes. Let $0 < \varepsilon < 1$. Choose $x_0 \geq 2$ so that for $k > \pi(x_0)$ it holds

$$(1) \quad (1 - \varepsilon) k \log k < p_k < (1 + \varepsilon) k \log k.$$

(The existence of such x_0 ensures the equality

$$\lim_{k \to \infty} \frac{p_k}{k \log k} = 1 \; ; \quad \text{see [2], p. 153).}$$

Then we can write

(2) $$S_a(x) = \sum_{p\le x_0} p^a + \sum_{x_0<p\le x} p^a = C_0 + \sum_{x_0\le p\le x} p^a = C_0 + S_a^{*}(x).$$

From (1) it follows

(3) $$(1-\varepsilon)^a \sum_{\pi(x_0)<h\le\pi(x)} h^a\log^a h \le S_a^{*}(x) \le$$

$$(1+\varepsilon)^a \sum_{\pi(x_0)<h\le\pi} h^a\log^a h .$$

The funktion $f(t) = t^a\log^a t$ fulfils the assumptions of theorem 4 of [3] (p.8), from which it follows

$$\sum_{\pi(x_0)<h\le\pi(x)} h^a\log^a h = \int_{\pi(x_0)}^{\pi(x)} t^a\log^a t\,dt + O(\pi^{a}(x)\log^a\pi(x)) =$$

$$= I_a + O(x^a), \text{ where } I_a = \int_{\pi(x_0)}^{\pi(x)} t^a\log^a t\,dt.$$

With help of the integration per partes we get

$$I_a = \left[\frac{t^{1+a}}{1+a}\log^a t\right]_{\pi(x_0)}^{\pi(x)} - \frac{a}{1+a}\int_{\pi(x_0)}^{\pi(x)} t^a\log^{a-1}t\,dt =$$

$$= \frac{1}{1+a}\pi^{a+1}(x)\log^a\pi(x) + C_1 + I_a^{*},$$ where I_a^{*} is the second integral.

Obviously

$$I_a^{*} = O(\pi(x)\pi^{a}(x)\log^{a-1}\pi(x)) = O\left(\frac{x^{1+a}}{\log^2 x}\right) = o\left(\frac{x^{1+a}}{\log x}\right).$$

If we substitute in the inequality (3), we get

$$(4)\qquad S_a^*(x) \leq (1+\varepsilon)^a \left[\frac{1}{1+a} \pi^{a+1}(x) \log^a \pi(x) + C_1 + o\left(\frac{x^{1+a}}{\log x}\right)\right];$$

$$(4^*)\qquad S_a^*(x) \geq (1-\varepsilon)^a \left[\frac{1}{1+a} \pi^{a+1}(x) \log^a \pi(x) + C_1 + o\left(\frac{x^{1+a}}{\log x}\right)\right].$$

Divide the inegualities (4*) and (4*) by $\frac{x^{1+a}}{\log x}$. If $x \longrightarrow \infty$ then we get

$$\limsup_{x\to\infty} \frac{S_a^*(x)}{\left(\frac{x^{1+a}}{\log x}\right)} \leq (1+\varepsilon)^a \left[\frac{1}{1+a} \lim_{x\to\infty} \frac{(x)\log^a (x)}{\left(\frac{x^{1+a}}{\log x}\right)}\right].$$

$$\liminf_{x\to\infty} \frac{S_a^*(x)}{\left(\frac{x^{1+a}}{\log x}\right)} \geq (1-\varepsilon)^a \left[\frac{1}{1+a} \lim_{x\to\infty} \frac{\pi^{a+1}(x(\log^a \pi(x)}{\left(\frac{x^{1+a}}{\log x}\right)}\right].$$

From the prime number theorem it follows that

$$\lim_{x\to\infty} \frac{\pi^{a+1}(x)\log^a \pi(x)\log x}{x^{1+a}} = 1, \text{ hence}$$

$$(1-\varepsilon)^a \frac{1}{1+a} \leqq \liminf_{x\to\infty} \frac{S_a^*(x)}{\left(\frac{x^{1+a}}{\log x}\right)} \leqq \limsup_{x\to\infty} \frac{S_a^*(x)}{\frac{x^{1+a}}{\log x}} \leqq$$

$$\leqq (1+\varepsilon)^a \frac{1}{1+a} .$$

Since our estmations aré valid for each $\varepsilon > 0$, we have

$$(5) \quad \lim_{x\to\infty} \frac{S_a^*(x) \log x}{x^{1+a}} = \frac{1}{1+a} .$$

If $x \to \infty$, then $\frac{x^{1+a}}{\log x} \to \infty$, too. Hence from (2) and (5) it follows that

$$\lim_{x\to\infty} \frac{S_a(x) \log x}{x^{1+a}} = \frac{1}{1+a} \; ; \quad \text{q.e.d.}$$

R E F E R E N C E S

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The basic notions of this paper were taken from [1] and [4] . The directed edge from the vertex u to v is denoted as $(\overrightarrow{u,v})$. $\pi(i)$ means the number of edges incoming to the vertex i. In this paper it is proved that the maximal number of 3-cycles is in such a tournament in which for each vertex the number of incoming and outcoming edges is equal-so called ς-tournament (see also [2] and [3]). As a special case of ς-tournaments apper so called homogeneous tournaments - i.e. such tounaments in which each of edges is situated in an equal number of 3-cycles. It is proved here that in a homogeneous tournament every edge is situated in an equal number of 4-cycles and also in an equal number of 5-cycles. There were investigated the turnable tournaments as well (ξ-tournaments, see [4]). This paper is documented with the table of all homogeneous ξ-tournaments with less than 100 vertices. There is shown an example of a homogeneous tournament which is not a ξ-tournament too.

The author would like to show his gratitude to Prof. KOTZIG for his valuable advices.

T h e o r e m 1. In the tournament with n vertices the number of 3-cycles is equal to the number

$$\binom{n}{3} - \sum_{i=1}^{n} \binom{\pi(i)}{2}$$

P r o o f. All the triangles with number $\binom{n}{3}$ can be divided into two parts:

1) Those ones which are presented by 3-cycles, these triangles let be p.

2) Those ones in which exists such a vertex that both edges are incoming to it, these triangles are $\sum_{i=1}^{n} \binom{\pi(i)}{2}$

From this follows the statement of the theorem already.

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On homogeneous tournaments

JÁN PLESNÍK

Corollary 1. Let G be a tournament with n vertices and let p be the number of 3-cycles in G. Then it holds:

a) for n odd: $0 \leq p \leq \frac{n(n-1)(n+1)}{24}$

b) for n even: $0 \leq p \leq \frac{n(n-2)(n+2)}{24}$

where for every n there exists such a tournament in which the number of 3-cycles is zero (so called acyclic tournament) and also there exists such a tournament in which the number of 3-cycles is equal to the mentioned upper bound.

Proof. Because $p = \binom{n}{3} - \sum_{i=1}^{n} \binom{\pi(i)}{2}$ it is sufficient to investigate the extrems of this function by the assumption that $\sum_{i=1}^{n} \binom{\pi(i)}{2} = \binom{n}{2}$.

Recurrently it is possible to construct an acyclic tournament for arbitrary n, by which $\pi(i) = i - 1$, $= 1, 2, \ldots, n$. And also recurrently it is possible to construct a tournament with $\pi(i) = \frac{n}{2} - 1$, $i = 1, 2, \ldots, \frac{n}{2}$, $\pi(i) = \frac{n}{2}$, $i = \frac{n}{2} + 1, \ldots, n$, if n is even; $\pi(i) = \frac{n-1}{2}$, $i = 1, 2, \ldots, n$ if n is odd (ϱ-tournament). Now it is sufficient to use the formula from the Theorem 1 and we get the mentioned bounds.

Theorem 2. The tournament in which every edge is situated in an equal number of m-cycles ($m \geq 3$) is ϱ-tournament.

Proof. Let every edge be situated in k m-cycles of the tournament with n vertices ($k > 0$). Let x be an arbitrary vertex. Let the number of edges incoming to x be p and let the number of outcoming ones be q. Because each of edges is situated in k m-cycles the vertex x is situated in $kp = kq$ m-cycles. Then $p = q$ and $n = 2p + 1$.

Remark 1. According to the [4] a homogeneous tournament in which each of edges is situated in k 3-cycles ($k > 0$) has $4k - 1$ vertices.

Theorem 3. Let $A = \| a_{ij} \|$ be a vertex incidence matrix of ϱ-tournament with n vertices ($n = 2r + 1$), where $a_{ij} = 1$, if it contains the edge $(\overrightarrow{i,j})$ in the opposite case $a_{ij} = 0$. Then it holds : $A^2 + A = K$, where $K = \|k_{ij}\|$ and k_{ij} is the number of 3-cycles in which is situated the edge $(\overrightarrow{i,j})$ or $(\overrightarrow{j,i})$ then we put $k_{ij} = k_{ji}$ and $k_{ii} = 0$.

<u>Proof.</u> Let $E(G)$ denote the set of edges of the tournament G and let $(\overrightarrow{i,j}) \in E(G)$. Let us define the following sets:

$$C_1 = \{ x \mid (\overrightarrow{j,x}) \in E(G) \wedge (\overrightarrow{x,i}) \in E(G)\}$$
$$C_2 = \{ x \mid (\overrightarrow{i,x}) \in E(G) \wedge (\overrightarrow{x,j}) \in E(G)\}$$
$$C_3 = \{ x \mid (\overrightarrow{i,x}) \in E(G) \wedge (\overrightarrow{j,x}) \in E(G)\}$$
$$C_4 = \{ x \mid (\overrightarrow{x,i}) \in E(G) \wedge (\overrightarrow{x,j}) \in E(G)\}$$

Because G is ρ-tournament it holds: $|C_1| = k$, $|C_1| + |C_4| = r$, $|C_2| + |C_3| + 1 = r$, $|C_1| + |C_3| = r$, $|C_2| + |C_4| + 1 = r$. Hence $|C_1| = k_{ij}$, $|C_2| = k_{ij}$, $|C_3| = r - k_{ij}$, $|C_4| = r - k_{ij}$. If we denote $A^m = \|{}^m a_{ij}\|$ then $|C_2| = {}^2a_{ij}$, $|C_1| = {}^2a_{ji}$ (see also [1]) and then ${}^2a_{ij} = k_{ij} - 1$, ${}^2a_{ji} = k_{ij}$, then obviously ${}^2a_{ii} = 0$. From that follows the statement of the theorem already.

<u>Remark 2.</u> From the proof of the Theorem 2 follows that in ρ-tournament every edge is situated in one 3-cycle at least, otherwise $|C_2| = -1$, what is not possible. From the abovesaid follows as well that every edge is not situated in more than in r 3-cycles.

<u>Theorem 4.</u> In the homogeneous tournament G where every edge is situated in k 3-cycles in $2k(k - j)$ 4-cycles and in $k(4k - 3)(k - 1)$ 5-cycles.

<u>Proof.</u> According to the Theorem 3 it holds: $A^2 + A = K - kE$, where A is the vertex incidence matrix of the tournament G, $K = \|k\|$, and E is the identity matrix. Then for $m \geq 3$ is $A^m = KA^{m-2} - kA^{m-1} - A^{m-1}$, i.e.

${}^m a_{ij} = k\sum_{x=1}^{4k-1} {}^{m-2}a_{xj} - k\, {}^{m-2}a_{ij} - {}^{m-1}a_{ij}$. For $m = 3$ is ${}^3a_{ij} = k\sum_{x=1}^{4k-1} a_{xj} - ka_{ij} - {}^2a_{ij}$. Because $\sum_{x=1}^{4k-1} a_{xj} = 2k - 1$ then ${}^3a_{ij} = k(2k - 1) - ka_{ij} - (k - a_{ij}) = 2k(k - 1) - (k - 1)a_{ij}$ for $i \neq j$ and ${}^3a_{ii} = k(2k - 1)$. Analogically we will find out that for $i \neq j$ is ${}^4a_{ij} = k(4k - 3)(k - 1) + (2k - 1)a_{ij}$ and ${}^4a_{ii} = 2k(k - 1)(2k - 1)$. Also for $i \neq j$ is ${}^5a_{ij} = 8k^4 - 18k^3 + 15k^2 - 4k + (k^2 - 3k + 1)a_{ij}$ and ${}^5a_{ii} = 8k^4 - 18k^3 + 13k^2 - 3k$. Because for $i \neq j$ is $a_{ij} \neq a_{ji}$ then every edge belongs to $2k(k - 1)$ 4-cycles and to $2k(k - 1)(2k - 1)$ 5-cycles.

<u>Theorem 4</u>. If every edge of the homogeneous tournament G in k 3-cycles is situated then also in $2k(k - 1)$ 4-cycles and in $k(4k - 3)(k - 1)$ 5-cycles is situated.

<u>R e m a r k 3.</u> In this homogeneous tournament the number of 3-cycles is equal to $\frac{1}{3}\sum_{i=1}^{4k-1} {}^{3}a_{ii} = \frac{1}{3}\,k(2k - 1)\,(4k - 1)$, the number of 4-cycles is equal to

$$\frac{1}{4}\sum_{i=1}^{4k-1} {}^{4}a_{ii} = \frac{1}{4}\,2k(k - 1)\,(2k - 1)$$ and the number pf 5-cycles is equal to

$$\frac{1}{5}\sum_{i=1}^{4k-1} {}^{5}a_{ii} = \frac{1}{5}\,k(k - 1)\,(4k - 3)\,(2k - 1)\,(4k - 1).$$

<u>R e m a r k 4.</u> ρ-tournament with 5 vertices can serve as an example of such a tournament in which every edge belongs to an equal number of 5-cycles but it is not a homogeneous tournament.

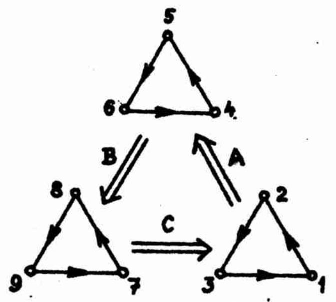

Fig.9

<u>R e m a r k 5.</u> The graph (see Fig. 9) as an example of ρ-tournament with 9 vertices in which every edge is situated in 9 4-cycles but it is not a homogeneous tournament. Whereby following denotation is used:

$$A = \{ (\overrightarrow{x,y}) \mid x \in \{1,2,3\} \wedge y \in \{4,5,6\}\}$$
$$B = \{ (\overrightarrow{x,y}) \mid x \in \{4,5,6\} \wedge y \in \{7,8,9\}\}$$
$$C = \{ (\overrightarrow{x,y}) \mid x \in \{7,8,9\} \wedge y \in \{1,2,3\}\}$$

Because the graph is symetric it is sufficient to verify the statement e.g. for the edges $(\overrightarrow{1,2})$ and $(\overrightarrow{1,4})$. It is easy to find that the edge $(\overrightarrow{1,2})$ is situated in one 3-cycle an the edge $\overrightarrow{(1,4)}$ in 3 3-cycles and that each of them is situated in 9 3-cycles.

Now let us consider a special tournament so called ξ-tournament (turnable tournament) i.e. a tournament in which it is possible to denote the vertices by symbols $v_1, v_2, \ldots, v_n$ so that it includes the edge $\overrightarrow{(v_i, v_j)}$ only if it includes at the same time the edge $\overrightarrow{(v_{i_1}, v_{j_1})}$ for $i, j, i_1, j_1 \in \{1, 2, \ldots, n\}$ and $i_1 - j_1 \equiv i - j \pmod n$. The notion of ξ-tournament was introduced into the paper [4] where the mentioned tournaments were investigated.

Let $A = \| a_{ij} \|$ be a vertex incidence matrix of ξ-tournament with n vertices. According to the paper [4] every ξ-tournament is also ϱ-tournament and then $n = 2r + 1$. It is easy to find out that A is a cyclic matrix of the following form:

$$\left\| \begin{array}{cccccccc} 0 & a_1 & a_2 & \ldots & a_r & \bar{a}_r & \ldots & \bar{a}_2 & \bar{a}_1 \\ \bar{a}_1 & 0 & a_1 & \ldots & a_{r-1} & a_r & \ldots & \bar{a}_3 & \bar{a}_2 \\ \bar{a}_2 & \bar{a}_1 & 0 & a_1 & \ldots & \ldots & \ldots & \ldots & \bar{a}_3 \\ & & & & & & & & \\ a_2 & a_3 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & a_1 \\ a_1 & a_2 & \ldots & \ldots & \ldots & \ldots & \ldots & \bar{a}_1 & 0 \end{array} \right\|$$

where $\bar{a}_i = 1 - a_i$

We define that the edge $\overrightarrow{(v_i, v_j)}$ is of the lenght d if $j - i \equiv d \pmod n$. According to the Theorem 3 is $A^2 + A + \bar{A}^2 + A = 2K$, where $\bar{A}$ is the matrix transponed to A. Then $k_{ij} = \frac{1}{2}({}^2a_{ij} + {}^2a_{ji} + 1)$, $i \neq j$. For ξ-tournament obviously is ${}^2a_{ij} = {}^2a_{i_1 j_1}$ if $i - j \equiv i_1 - j_1 \pmod n$ and the edge of the length d $(1 \leq d \leq r)$ is situated in k_{nd} (briefly k_d) 3-cycles. Then obviously hold the following formulas:

For $i = 1, 2, \ldots, r$:

for i odd:

$${}^2a_{ni} = 2(\bar{a}_1 a_{i+1} + \ldots + \bar{a}_{r-i} a_r) + 2(\bar{a}_{r-i+1}\bar{a}_r + \ldots + \bar{a}_{r-\frac{i-1}{2}}\bar{a}_{r-\frac{i+3}{2}}) +$$

$$2(a_1 a_{i-1} + \ldots + a_{\frac{i-1}{2}} a_{\frac{i-1}{2}+1}) + \bar{a}_{r-\frac{i+1}{2}}$$

for i even:

$$^2a_{nj} = 2(\bar a_1 a_{i+1} + \dots + \bar a_{r-i} a_r) + 2(a_{r-i+1} a_r + \dots + a_{r-\frac{i}{2}} a_{r-\frac{i}{2}+1}) + 2(a_1 a_{i-1} + \dots + a_{\frac{i}{2}-1} a_{\frac{i}{2}+1}) + a_{\frac{i}{2}}$$

For $r < i < 2r + 1 = n$ we put $j = n - i$ and $1 \le j \le r$, then holds:

for j odd:

$$^2a_{nj} = 2(a_1 \bar a_{j+1} + \dots + a_{r-j} \bar a_r) + 2(a_{r-j+1} a_r + \dots + a_{r-\frac{j-1}{2}} a_{r-\frac{j+3}{2}}) + 2(\bar a_1 \bar a_{j-1} + \dots + \bar a_{\frac{j-1}{2}} \bar a_{\frac{j-1}{2}}) + a_{r-\frac{j+1}{2}}$$

for j even

$$^2a_{nj} = 2(a_1 \bar a_{j+1} + \dots + a_{r-j} \bar a_r) + 2(a_{r-j+1} a_r + \dots + a_{r-\frac{j}{2}} a_{r-\frac{j}{2}+1}) + 2(\bar a_1 \bar a_{j-1} + \dots + \bar a_{\frac{j}{2}+1} a_{\frac{j}{2}-1}) + \bar a_{\frac{j}{2}}$$

$$^2a_{nn} = 0$$

Because $^2a_{in} = {}^2a_{n,n-i}$ holds for $i = 1,2,\dots,r$ then from the above-mentioned formulas it follows:

for i odd:

$$k_{ni} = [(a_1 \bar a_{i+1} + \bar a_1 a_{i+1}) + \dots + (a_{r-i} \bar a_r + \bar a_{r-i} a_r)] + (a_{r-i+1} a_r + \bar a_{r-i+1} \bar a_r) + \dots + (a_{r-\frac{i-1}{2}} a_{r-\frac{i+3}{2}} + \bar a_{r-\frac{i-1}{2}} \bar a_{r-\frac{i+3}{2}}) + (\bar a_1 \bar a_{i-1}) + \dots + (a_{\frac{i-1}{2}} a_{\frac{i-1}{2}+1} + \bar a_{\frac{i-1}{2}} \bar a_{\frac{i-1}{2}+1}) + 1$$

for i even:

$$k_{ni} = [(a_1 \bar a_{i+1} + \bar a_1 a_{i+1}) + \dots + (a_{r-i} \bar a_r + \bar a_{r-i} a_r)] + (a_{r-i+1} a_r + \bar a_{r-i+1} \bar a_r) + \dots + (a_{r-\frac{i}{2}} a_{r-\frac{i}{2}+1} + \bar a_{r-\frac{i}{2}} a_{r-\frac{i}{2}+1}) + (a_1 a_{i-1} + \bar a_1 \bar a_{i-1}) + \dots + (a_{\frac{i}{2}-1} a_{\frac{i}{2}+1} + \bar a_{\frac{i}{2}+1} \bar a_{\frac{i}{2}-1}) + 1$$

$$k_{nn} = 0$$

In the formulas for k_{ni} in square brackets there exist pairs of the form: $(a_j\bar{a}_{i+j} + \bar{a}_j a_{i+j})$. The value of each such a pair is obviously 1 just then if $a_j \neq a_{i+j}$. It is advantageous by counting k for little i ; so e.g. for $i = 1$: $k_{n1} = \zeta_1 + 1$, where $\zeta_1 = (a_1\bar{a}_2 + \bar{a}_1 a_2) + \dots + (a_{r-1}\bar{a}_r + \bar{a}_{r-1}a_r)$ is in fact the number of the alternations in the sequence $(a_1, a_2, \dots, a_r)$ which is called the character of the ξ-tournament. This result is possible to find in the paper [4] , where this result was investigated more generally. Because $k_{ni} = {}^2a_{ni} + a_i$ then from the formulars for ${}^2a_{ni}$ follows the following theorem.

<u>Theorem 5.</u> In ξ-tournament with $2r + 1$ vertices and the character $(a_1, a_2, \dots, a_r)$ let mean k the number of 3-cycles in which the edge of the lenght d (and then each of the lenght d) is situated. Then it holds:

a) for d odd: $a_{r-\frac{d+1}{2}} + a_d = k_d + 1 \pmod 2$

b) for d even:

$$a_{\frac{d}{2}} + a_d = k_d \pmod 2$$

<u>Corollary 1.</u> In the homogeneous ξ-tournament with the number of vertices $4k - 1$ and the character $(a_1, a_2, \dots, a_{2k-1})$ let be $j, 2j, 2k - j \in \{1, 2, \dots, 2k - 1\}$. Then it holds:

a) for k even: $a_j = a_{2j}$, $a_{2j-1} \neq a_{2k-j}$

b) for k odd: $a_j \neq a_{2j}$, $a_{2j-1} = a_{2k-j}$

<u>Theorem 6.</u> Let p, q, t, s be integer numbers. If for n one of the following conditions holds then there does not exist a homogeneous ξ-tournament with n vertices:

a) $n = 8p(4t - 1) - 12t + 3$ $(p, t \geq 1)$

b) $n = 8t - 1$, where t is the solution of equation $4t = 2^p(2s - 1) + s$ and $p \geq 0$, $s \geq 1$

c) $n = 8t - 5$, where t is the solution of equation $4t = 2^{2p+1}(2s - 1) + s + 2$ and $p \geq 0$, $s \geq 1$

d) $n = ts$, $4^p + 1 = qs$ $(s > 1, p \geq 1)$

P r o o f. a) According to [4] (Theorem 4) if $n = s(8p - 3)$ where p,s are natural numbers there does not exist a homogeneous ξ-tournament with n vertices. From this the statement follows already.

b) According to the corollary 1a) if for even k exist such i,j,p that $j = 2i - 1$ and $2^p j = 2k - i$, then the mentioned system of the recurrent formulas is contraversial.

c) This part is proved analogically like the preceding part by using the corollary 1b).

d) This statement is taken directly from the paper [4] (Theorem 6).

R e m a r k 6. According to the corollary 1 characters of homogeneous ξ-tournaments can be constructed. Then it is still necessary to verify these characters by formulas for k_{ni}. The corollary 1 is not a sufficent condition; this may be seen from the example of ξ-tournament with 31 vertices and with character (11111 10111 01000) which fulfils the condition a) of the corollary 1 ($k = 8$), but $k_1 = 6 \neq 8$.

R e m a r k 7. According to the Theorem 8 of the paper [4] there exists a homogeneous ξ-tournamnet with $4k - 1$ vertices for every prime number $4k - 1$. List of all the characters of such tournaments with the number of the vertices less than 100 one can find in the following theorem.

T h e o r e m 7. Every homogeneous ξ-tournament with the number of vertices $n < 100$ is isomorfic to one of the following tournaments:

k = 1 ,	n = 3 :	(1)
2 ,	7	(110)
3 ,	11	(10111)
5 ,	19	(10011 1101)
6 ,	23	(11110 10110 0)
8 ,	31	(11011 01111 00010)
11,	43	(10010 10011 10111 11000 1)
12,	47	(11110 11110 01010 11100 100)
15,	59	(10111 01010 01001 11011 11001 1111)
17,	67	(10010 10011 00011 11010 11111 10010 001)
18,	71	(11111 10111 01001 10111 00011 01011 01000)
20,	79	(11011 00111 10100 10111 11101 10000 11000 1010)
21,	83	(10110 01011 11000 11000 10101 11111 10100 11101 1)

P r o o f. The abovementioned characters are constructed with the accordance to the remark 6. Fot those k,n which do not appear here there do not exist homogeneous ξ -tournaments. It is possible to eliminate some of them immediately according to the Theorem 6 and the other ones according to the formulas for k_{ni}.

R e m a r k 8. It is possible to put the matrix of every homogeneous tournament with $n = 4k - 1$ vertices to the following form (see also [5]):

$$
\left\|\begin{array}{ccc}
 & 0 & \\
 & 0 & \\
T_1 & \vdots & A \\
 & 0 & \\
1\,1\,.\,.\,1 & 0 & 0\,.\,.\,0\,0 \\
 & 1 & \\
B & \vdots & T_2 \\
 & 1 & \\
 & 1 &
\end{array}\right\|
$$

Where T_1, T_2 are ϱ -tournaments and A, B are some matrices. If we choose an arbitrary vertex w of the homogeneous tournament G, then the set $V_1 = \{x \mid (\overrightarrow{w,x}) \in E(G)\}$ is the vertex set of the tournament T_1 and $V_2 = \{x \mid (\overrightarrow{x,w}) \in E(G)\}$ is the set of vertices of the tournament T_2. If we consider the fact that every edge $(\overrightarrow{w,x})$ where x is the vertex of the tournament T_1 is situated in k 3-cycles, then from that it follows immediately that T_1 is ϱ -tournament. So if we wish to find a tournament G it is sufficient to eliminate the search on the matrix A where for T_1, T_2 we choose gradually all the tournaments from the system of nonisomorfic ϱ -tournaments with $2k - 1$ vertices.

R e m a r k 9. It is not known whether there exists homogeneous tournament of n vertices for each $n = 4k - 1$ if n is not a prime number. But there is known a homogeneous tournament with 15 vertices which was invented by A.KOTZIG. The following matrix is a vertex incidence matrix of such tournament.

$$
\begin{Vmatrix}
0&1&1&0&0&0&1&0&1&1&1&0&1&0&0\\
0&0&0&1&1&1&0&0&1&0&1&1&1&0&0\\
0&1&0&0&1&1&0&0&1&1&0&0&0&1&1\\
1&0&1&0&0&1&0&0&0&1&0&1&1&0&1\\
1&0&0&1&0&0&1&0&1&1&0&1&0&1&0\\
1&0&0&0&1&0&1&0&0&0&1&0&1&1&1\\
0&1&1&1&0&0&0&0&0&0&1&1&0&1&1\\
1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&1&1&1&0&1&1&0&0&0&1\\
0&1&0&0&0&1&1&1&0&0&0&1&1&1&0\\
0&0&1&1&1&0&0&1&0&1&0&0&1&1&0\\
1&0&1&0&0&1&0&1&1&0&1&0&0&1&0\\
0&0&1&0&1&0&1&1&1&0&0&1&0&0&1\\
1&1&0&1&0&0&0&1&1&0&0&0&1&0&1\\
1&1&0&0&1&0&0&1&0&1&1&1&0&0&0
\end{Vmatrix}
$$

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O jednom špeciálnom prípade p-sústav

ŠTEFAN PORUBSKÝ

Sústavu zvyškových tried

$$a_i \pmod{n_i}, \quad 0 \leq a_i < n_i, \quad i = 1,2,\dots,k$$

nazývame presne pokrývajúca sústava (alebo aj p-sústava), ak každé celé číslo patrí do práve jednej z týchto tried.

Moduly n_i $(i=1,\dots,k)$ nemusia byť rovnaké čísla. Je dokázané [1], že v sústave (1) musia vystupovať aspoň dve triedy podľa najväčšieho z modulov n_i $(i=1,\dots,k)$. Ďalej je známe, že nutná podmienka, aby vzťahy (1) udávali p-sústavu je $(n_i, n_j) > 1$ pre všetky $i,j=1,\dots,k$.

Nech v ďalšom $m_1 < m_2 < \dots < m_h$ sú všetky navzájom rôzne čísla, ktoré vystupujú ako moduly tried p-sústavy (1).

V tejto poznámke si budeme všímať presne pokrývajúce sústavy, v ktorých $(m_i, m_j) = d$, kde $1 < d \leq m_1$ pre každé $i \neq j$, $i,j = 1,\dots,h$. Pre každé prirodzené číslo h zrejme existuje aspoň jedna skupina čísiel $m_1,\dots,m_k$ s uvedenou vlastnosťou, napr. $p_1 d,\dots p_h d$, kde $p_1 < p_2 < \dots p_h$ sú prvočísla.

Najmenšie nezáporné číslo patriace do niektorej z tried (1) nazveme najmenší nezáporný zvyšok tej triedy.

Dokážme najprv tieto dve pomocné vety:

L e m m a 1. Nech v p-sústave platí $(m_i, m_j) = d$ pre všetky $i \neq j$. Potom medzi zvyškovými triedami modulo m_j, pre každé $j = 1,\dots,h$ sa vyskytuje aspoň jedna trieda s najmenším nezáporným zvyškom $< d$.

D ô k a z. Nech existuje aspoň jeden modul m_r $(1 \leq r \leq h)$ v p-sústave (1) tak, že vo všetkých zvyškových triedach podľa tohto modulu je najmenší nezáporný zvyšok $\geq d$. Vyberme jednu z týchto tried, napr.

$$b \pmod{m_r}, \quad b \geq d, \quad 1 \leq r \leq h.$$

Nech $b \equiv c \pmod d$, $0 \le c < d$. Číslo c nemôže patriť do žiadnej z tried podľa modulu m_r, pretože $c < d$. Nakoľko ale (1) je p-sústava, musí byť číslo c obsiahnuté v nejakej triede podľa modulu m_j, kde $m_j \ne m_r$. Potom ale triedy

$$c \pmod{m_j}$$
$$b \pmod{m_r}$$

majú spoločné riešenie, čo je v spore s tým, že (1) je p-sústava. To plynie z toho, že $(m_j, m_r) = d \mid (b - c)$ a z toho, že d možno pomocou m_j a m_r lineárne vyjadriť.

<u>L e m m a 2.</u> Ak v p-sústave s h $(h \ge 1)$ navzájom rôznymi modulmi $m_1, \dots, m_h$ je $(m_i, m_j) = d$ pre každé $i \ne j$, $i,j = 1,2,\dots,h$, potom $h \le d$.

<u>D ô k a z.</u> Z definície p-sústavy vyplýva, že žiadne dve triedy nemôžu mať rovnaký najmenší nezáporný zvyšok. Ak by bol počet rôznych modulov v p-sústave $> d$, potom by existoval taký modul m_r, že všetky triedy podľa tohto modulu by mali najmenší nezáporný zvyšok $\ge d$, čo je v spore s lemmou 2.

Pristúpime teraz k dôkazu týchto viet:

<u>V e t a 1.</u> Nech d,h, kde $1 < h \le d$ a $m_1 < m_2 < \dots m_h$ sú prirodzené čísla, pričom $(m_i, m_j) = d$ pre $i \ne j$, $i,j = 1,\dots,h$. Potom existuje aspoň jedna p-sústava, v ktorej čísla m_i $(i = 1,\dots,h)$ vystupujú ako moduly.

<u>D ô k a z.</u> Priraďme všetky čísla $0,1,\dots, d-1$ k daným modulom $m_1, m_2, \dots, m_h$ tak, aby ku každému modulu bolo priradené aspoň jedno z týchto čísiel a aby čísla priradené modulu m_i tvorili dizjunktnú množinu s množinou čísiel priradených modulu m_j pri $i \ne j$ (inak by sa mohlo stať, že by sme v (2) mali $a_f \pmod{m_j}$ a tiež $a_f \pmod{m_i}$, $i \ne j$, potom pomocou (2) získaná sústava (3) by asi nebola p-sústava). Nech tomuto priradeniu prislúchajúca sústava zvyškových tried má tvar

$$\begin{array}{ll} a_1 \pmod{m_1} & \\ \vdots & \\ a_f \pmod{m_j} & d > f \ge j, a_1, \dots, a_d < d \qquad (2) \\ \vdots & \\ a_d \pmod{m_h} & \end{array}$$

Vytvorme teraz pomocou sústavy (2) nasledujúcu sústavu zvyškových tried

$$\begin{array}{ll} a_1, a_1+d, \ldots, a_1+\left(\frac{m_1}{d}-1\right)d & (\mathrm{mod.}\ m_1) \\ \vdots & \\ a_f, a_f+d, \ldots, a_f+\left(\frac{m_j}{d}-1\right)d & (\mathrm{mod.}\ m_j \\ \vdots & \\ a_d, a_d+d \ldots, a_d+\left(\frac{m_n}{d}-1\right)d & (\mathrm{mod.}\ m_n) \end{array} \qquad (3)$$

Ľahko nahliadneme, že zvyškové triedy modulu m_j (pre každé $j = 1,\ldots,h$) v (3) obsahujú práve všetky čísla zo zvyškovej triedy a_f (mod d) pre $f = 1,\ldots,d$. Obdobne ľahko sa dá presvedčiť, že ľubovoľné dve triedy z (3) sú navzájom dizjunktné a nakoľko čísla a_f $(f = 1,\ldots,d)$ tvoria úplnú sústavu zvyškov (modulo d), je sústava (3) presne pokrývajúca.

P o z n á m k a. Ak predpoklad $1 < h \leq d$ nie je splnený, potom na základe lemmy 2 presne pokrývajúca sústava s uvedenou vlastnosťou neexistuje.

V e t a 2. Nech (1) je p-sústava, v ktorej $(n_i, n_j) = d$ pre každé $n_i \neq n_j$ a nech $m_1, m_2,\ldots,m_h$ sú všetky navzájom rôzne moduly zvyškových tried v tejto p-sústave. Potom v uvedenej p-sústave vystupuje práve $r_i\frac{m_i}{d}$ zvyškových tried s modulom m_i $(i = 1,\ldots,h)$, pričom číslo r udáva počet zvyškových tried (modulo m_i), ktorých najmenší nezáporný zvyšok je $< d$.

D ô k a z. Na základe lemmy 1 v každej p-sústave (s vlastnosťou $(m_i, m_j) = d$) pre $i \neq j$ musí vystupovať aspoň jeden taký systém zvyškových tried, že každé z čísiel $0,1,\ldots,d-1$ patrí do práve do jednej z nich, pričom sú v ňom zastúpené všetky rôzne moduly. Podľa dôkazu vety 1 možno ku každému takémuto systému priradiť aspoň jednu p-sústavu. Ukážme, že práve jednu.

Aby zo systému zvyškových tried (2) vznikla p-sústava, musia zvyškové triedy modulo m_j $(j = 1,\ldots,h)$ obsahovať všetky čísla triedy a_j (mod d). Keby totiž niektoré z týchto čísiel, napr. c, patrilo do zvyškovej triedy modulo m_i, kde $m_i \neq m_j$, potom zvyškové triedy

$$c \pmod{m_i}$$

$$a_f \pmod{m_j}$$

by mali spoločný prvok, čo je v spore s definíciou p-sústavy. Spomedzi prvkov triedy a_f (mod d) sú nekongruetné (modulo m_j) práve čísla $a_f, a_f+d,\ldots,a_f+\left(\frac{m_j}{d}-1\right)d$. Teda p-sústava obsahujúca systém /2/ musí obsahovať aj systém /3/. Nakoľko /3/ už tvorí p-sústavu, nie je možné, aby p-sústava obsahujúca systém /2/ obsahovala zvyškové triedy iného typu než triedy uvedené v /3/. Teda /2/ skutočne určuje práve jednu p-sústavu.

Vidíme, že každá zvyšková trieda a_f (mod m_j) $(f=1,\dots,d; j=1,\dots,h)$ s najmenším nezáporným zvyškom $< d$ v (2) viaže celkove práve $\frac{m_j}{d}$ zvyškových tried v (3) s tým istým modulom m_j. Ak označíme r počet zvyškových tried modulo m_j, ktorých najmenší nezáporný zvyšok je $< d$, dostávame tvrdenie vety.

Uveďme ešte vetu, ktorá súvisí zo všeobecnejším prípadom p-sústav, než aké sme v tomto príspevku sledovali.

<u>V e t a 3.</u> Nech v p-sústave (1) sú $m_1 < \dots < m_h$ všetky navzájom rôzne moduly zvyškových tried a nech $(m_i, m_h) = d$ pre každé $i \neq h$. Potom v tejto p-sústave sa vyskytuje práve $r_h \frac{m_h}{d}$ zvyškových tried (modulo m_h), kde číslo r_h udáva počet zvyškových tried (modulo m_h) s najmenším nezáporným zvyškom $< d$.

Dôkaz tejto vety vyplýva z dôkazov viet 1 a 2, ak úvahy redukujeme len na modul m_h.

Š. Znám v [2] vyslovil túto hypotézu:

V každej presne pokrývajúcej sústave sa vyskytuje aspoň p zvyškových tried s rovnakým modulom, kde p je najmenší prvočíselný deliteľ najväčšieho modulu m_h tejto p-sústavy.

Z vety 2 príp. 3 vyplýva platnosť tejto hypotézy pre p-sústavy, v ktorých $(m_i, m_j) = d$ pre $m_i \neq m_j$ resp. $(m_i, m_h) = d$ pre $m_i \neq m_h$. Skutočne

$$r_h \cdot \frac{m_h}{d} \geq \frac{m_h}{d} \geq p$$

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On certain classes of graphs of diameter two without superfluous edges

FERDINAND GLIVJAK

Under a graph we mean here an undirected graph without loops and multiplied edges. This paper presents the continuation of the study of the graphs of diameter two without superfluous edges. In the papers [3], [4] and [5] we have investigated only the graphs without triangles and by this way the whole problem became more simple.

We investigate here 8 classes of the graphs according to the kind of k-angles they include (for $k = 3,4,5$). For the three simplest classes of these we put necessary and sufficient conditions for G to belong to someone of them. In the next we solve a problem of the existence of a graph from some other classes with given numbers of vertices and the minimum degree, given numbers of vertices an edges respectively.

At the end it is proved that for an undirected graph G without loops and multiplied edges there exists a graph R without the superfluous edges of diameter two and such that the neighbourghoods of each of two vertices of the graph R differ and the graph G is the section graph of the graph R.

I. Definitions and notations

A graph shal be denoted by $G = (U,H)$, where U is the set of vertices and H is the set of edges of the graph G. If $U = \{z_0, z_1, \dots, z_k\}$, $H = \{(z_0, z_i)\ \ i = 1,2,\dots,k\}$, then we call G a star. The symbol $\Omega(v)$ will denote the neighbourghood of the vertex v. The graph $G = (U,H)$ will be called η -irreducible if for every $x,y \in U$, $x \neq y$ is $\Omega(x) \neq \Omega(y)$.

<u>Definition 1.</u> The edge (u,v) of the graph $G = (U,H)$ is superfluous if for the subgraph $G_1 = (U,H_1)$ where $H_1 = H - \{(u,v)\}$ it holds $d(G) = d(G_1)$.

D e f i n i t i o n 2. We say that the graph G belongs to the class B if $d(G) = 2$ and moreover G does not contain any superfluous edge.

D e f i n i t i o n 3. We say that the graph G contains an exact s-angle if there exist s vertices $u_1, u_2, \ldots, u_s \in U$ which formed a circle K of the lenght s where for every two vertices x,y of K it holds $\rho_G(x,y) = \rho_K(x,y)$.

R e m a r k 1. If the graph G belongs to the class B, then it cannot contain any exact s-angle, for $s > 5$.

D e f i n i t i o n 4. Let be $G \in B$. Then we say that G belongs to the class

1) B_0, if G does not contain an exact s-angle, for $s = 3,4,5$.

2) B_1, if G contains at least one triangle and does not contain any exact s-angle, for $s = 4,5$.

3) B_2, if G contains at least one exact 4-angle and does not contain any exact s-angle, for $s = 3,5$.

4) B_3, if G contains at least one exact 5-angle and does not contain any exact s-angle, for $s = 3,4$.

5) B_4, if G contains at least one exact s-angle, for $s = 3,4$ and does not contain any exact 5-angle.

6) B_5, if G contains at least one exact s-angle, for $s = 3,5$ and does not contain any exact 4-angle.

7) B_6, if G contains at least one exact s-angle, for $s = 4,5$ and does not contain any exact 3-angle.

8) B_7, if G contains at least one exact s-angle, for $s = 3,4,5$.

R e m a r k 2. Let $G = (U,H)$ be a graph of the diameter 2 ; let the edge $h = (u,v)$ be a superfluous one. Then h is situated in a triangle (otherwise after removing the edge h would hold $\rho(u,v) > 2$).

II. Results

We can easy verify two following assertions.

A s s e r t i o n 1. Let $G = (U,H)$ be a graph of diameter two. Then the edge $h = (u,v)$ is superfluous if and only if

1) for every $x \in \Omega(u)$, $x \neq v$ there exists in the graph G either the way (x,z,v), $z \neq u$ of the lenght 2, or the edge (x,v).

2) for every $y \in \Omega(v)$, $y \neq u$ there exists in the graph G either the way (y,z,u), $z \neq v$ of the lenght 2, or the edge (y,u).

3) in the graph G there exists the way (v,z,u), of the lenght 2.

Assertion 2. Let be $G = (U,H) \in B$. Let have G a bridge or an articulation. Then G is a star.

Theorem 1. Graph $G \in B_0$ if and only if G is a star.

Proof. If G is a star, then obviously $G \in B_0$. Let be $G \in B_0$, $h \in H$. Then according to the remark 1 and to the definition of the set B_0 the edge h is not situated in s-angle, where $s \geq 3$. E.i. h is a bridge. According to the Assertion 2 is the graph G a star.

Theorem 2. Set $B_1 = \emptyset$.

Proof (by contradiction). Let be $G = (U,H) \in B_1$. Let be $h=(a,b) \in H$. Let us denote $M_0 = \{x \mid x \in \Omega(a \wedge x \in \Omega(b)\}$

$$M_1 = \{x \mid x \in \Omega(a) \wedge x \notin \Omega(b) \mid (x \neq b)\}$$

$$M_2 = \{x \mid x \notin \Omega(a) \wedge x \in \Omega(b) \mid (x \neq a)\}$$

$$M_3 = \{x \mid x \notin \Omega(a) \wedge x \notin \Omega(b)\}$$

From the Remark 1, Assertion 2 and from the assumption $G \in B_1$ follows that h is situated in a triangle, e.i. $M_0 \neq \emptyset$.

1) Let be $M_1 = M_2 = \emptyset$, then the edge h is obviously superfuous.

2) Let $M_1 \neq \emptyset$, $M_2 \neq \emptyset$. Let be $x \in M_1$, $y \in M_2$.

a) Let be $(x,y) \in H$, then G includes exact 4-angle (a,b,y,x) what cannot happen.

b) $\rho(x,y) = 2$, then there exist the edges (x,z), (z,y) where either $z \in M_3$ (and then G includes exact 5-angle (x,z,y,b,a) what cannot happen), or $z \in M_0$. Therefore, for every $x \in M_1$, $y \in M_2$ there exists such $z \in M_0$, that (x,z), $(z,y) \in H$. But then the edge h is superfluous, what is a contradiction with the assumption.

3) Let be $M_1 = \emptyset$, $M_2 \neq \emptyset$.

a) Let be $M_3 = \emptyset$. Then for each $z \in M_o$ the edge (a,z) is a superfluous one, what is a contradiction with the assumption.

b_1) Let $M_3 \neq \emptyset$ and let for every $x \in M$ exist such $y \in M_o$, that $(x,y) \in H$. Then the edge h is superfluous, what is impossible.

b_2) Let be $M_3 \neq \emptyset$ and let exist such $u \in M_2$, that for every $y \in M_o$ is $\rho(u,y) = 2$. Then for every $z \in M_3$ is $\rho(u,z) = 2$. It is clear, that for every $z \in M_3$ there exists such $z_1 \in M_3$ that $\rho(z,z_1)=1$ (otherwise would hold $\rho(a,z) > 2$). If it holds $\rho(u,z) = 1$ then would also hold $\rho(u,z_1) = 1$ (otherwise G include exact 4-angle (u,z,z_1,b) what is impossible. Let $w \in M_o$ for which exists $v \in M_3$ such that $\rho(w,v)=1$. This vertex w exists, else would hold $\rho(a,z) > 2$ for all $z \in M_3$. The decomposition of the vertices into the sets M_i, $i = 1,2,3$, it is possible to arrange with the accordance to an arbitrary edge. Let make such a decomposition of the vertices according to the edge (w,b). Then will be the vertex $u \in M_2$, $v \in M_1$. According to the part 2 of this proof it is impossible. E.i. the theorem holds.

<u>Theorem 3.</u> Graph G belongs to the class B_2 if and only if G is bipartite.

Proof. If G is bipartite, then it obviously belongs to the set B_2. Let be $G = (U,H) \in B_2$, $h = (a,b) \in H$. Let the symbols M_o, M_1, M_2, M_3 have the same meaning as in theorem 1. It is obvious that $M_o = \emptyset$ otherwise G would include an exact triangle. Also $M_3 = \emptyset$, because if there exists a vertex $x \in M_3$, then there would have to exist such vertices $u \in M_1$, $v \in M_2$ that $(x,u) \in H$, $(x,v) \in H$. Then the graph G would include either exact 5-angle (a,b,v,x,u) or exact 3-angle (x,u,v) what cannot happen. It is clear that $M_1 \neq \emptyset$, $M_2 \neq \emptyset$ holds. Let be $x \in M_1$. Then it must hold $\rho(x,y) = 1$ for all $y \in M_2$. Let be $y \in M_2$. Then obviously for all $x \in M_1$ holds $\rho(x,y) = 1$. Hence G is a bipartite graph.

<u>Remark 3.</u> The graphs of the class B_3 were investigated in the paper [6] . It was proved that there exist at most 4 such graphs. There are, as follows: 5-angular, Peterson's graph with 50 vertices constructed in paper [6] The problem of the fourth graph remains unsolved.

The graphs of the class B_6 were investigated in papers [1] , [2], [3] , [4] , [5] . There were already done the estimations for the number of edges, the maximal degree and there were described the extension and reduction of these graphs is borders of the same class. Further there were solved some problems dealing with the existence of the graphs from the class B_6, which are irreducible, by means of the described operation.

In the following part we shall describe the construction of certain graphs of the class B_7 and the class B_4. The set of vertices of these graphs will be $U = P \cup Q \cup R \cup S \cup T \cup Z \cup A \cup$, where $P = \{x_1, x_2, \dots, x_p\}$, $Q = \{y_1, y_2, \dots, y_q\}$ $R = R_o \cup R_1 \cup R_2$, where $R_k = \{w_1^k, w_2^k, \dots, w_r^k\}$ for $k = 0,1,2$. $S = S_1\ S_2$, where $S_k = \{u_1^k, u_2^k, \dots, u_s^k\}$ for $k=1,2$; $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ $T = \{v_1, v_2, \dots, v_t\}$, $Z = \{b_1, b_2, \dots, b_z\}$. Let assume that $t > 0$, $q > 0$.

The set of the edges H of this graph (by given parameters) we describe with the help of the neighbourghood of separate vertices. Let denote $|U| = n$, $|H| = m$. The basical decomposition of the sets of vertices is illustrated by picture No. 10.

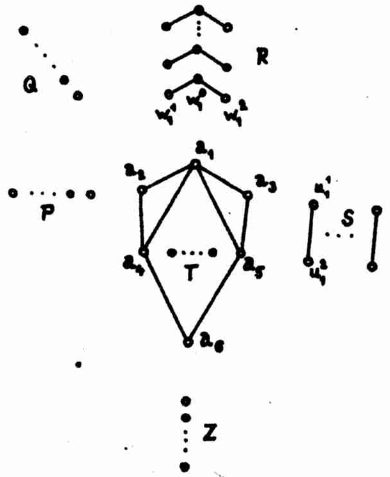

Fig. 10

$\Omega(a_1) = \{a_2, a_3, a_4, a_5\} \cup T \cup P \cup U \cup S_1 \cup S_2$

$\Omega(a_2) = \{a_1, a_4\} \cup Q$

$\Omega(a_3) = \{a_1, a_5\} \cup Q$

$\Omega(a_4) = \{a_1, a_2, a_6\} \cup P \cup R_o \cup R_1 \cup Z$

$\Omega(a_5) = \{a_1\ a_3, a_6\} \cup R_o \cup R_2 \cup Z$, $\Omega(a_6 (= \{a_4, a_5\} \cup T \cup S_2$

$\Omega(w_i^0) = \{a_4, a_5\} \cup T \cup S_2 \cup \{w_i^1, w_i^2\}$ for $i = 1,2,\dots,r$

$\Omega(w_i^1) = \{w_i^0, a_4\} \cup Q$ for $i = 1,2,\dots,r$

$\Omega(w_i^2) = \{w_i^0, a_5\} \cup Q$ where $i = 1,2,\dots,r$

$\Omega(x_i) = \{a_1, a_4\} \cup Q$ where $i = 1,2,\dots,p$

$\Omega(y_i) = \{a_2, a_3\} \cup T \cup P \cup S_1 \cup R_1 \cup R_2$, for $i = 1,2,\dots,q$

$\Omega(u_i^1) = \{u_i^2, a_1\} \cup Q$; $\Omega(u_i^2) = \{u_i^1, a_1, a_6\} \cup Z \cup R_0$, where $i = 1,2,\dots,s$

$\Omega(v_i) = \{a_1, a_6\} \cup Z \cup Q \cup R_0$, $i = 1,2,\dots,t$

$\Omega(b_i) = \{a_4, a_5\} \cup T \cup S_2$, $i = 1,2,\dots,z$.

Directly from the construction of this graph (by known parameters) follows that it is a graph of the diameter 2. We shall show, that it has no superfluous edges. According to the remark 2 it is sufficient to investigate just the edges situated in a triangle. The following edges are not superfluous, because after removing the edge

(a_2, a_1)	it would be	$\rho(a_2, a_5) > 2$
(x_i, a_1)	it would be	$\rho(x_i, a_5) > 2$ for $i = 1,2,\dots,p$
(w_j^1, w_j^0)	it would be	$\rho(w_j^1, a_5) > 2$ for $j = 1,2,\dots,r$
(z, a_4)	where $z = a_2, x_i, w_j^1$	it would be $\rho(z, a_6) > 2$
(a_4, a_1)	it would be	$\rho(a_4, a_3) > 2$
(a_4, w_j^1)	it would be	$\rho(a_4, w_j^2) > 2$

The superfluousness of other edges can be veryfied analogically therefore we shall not write about them. The following table gives us a summary of the degrees of separate vertices.

vertices	number	degree
a_1	1	2s+p+t+4
$a_2, a_3, R_1, R_2, P, S_1$	2r+p+s+2	q+2
a_4	1	2r+p+z+3
a_5	1	2r+z+3
a_6, Z	z+1	t+s+2
R_o	r	t+s+4
Q	q	2r+t+p+s+2
S_2	s	r+z+3
T	t	q+z+2

<u>Remark 4.</u> From the above table it is easy to derive a formula for the number of vertices and number of edges of this graph.

(1) $n = 3r+2s+p+q+z+t+6$.

(2) $n = 2rq+pq+qs+zt+zs+rt+rs+qt+4s+2p+2t+6r+2q+2z+8$.

Let d be the minimum degree of the graph. Then it is easy to verify, that if

(1) $s \neq 0$, then $d = \min(q+2, t+s+2, r+z+3)$

(2) $s = 0$, then $d = \min(q+2, t+2, 2r+z+3)$.

<u>Remark 5.</u> If $t = 0$ then also $q = 0$, otherwise there either would arise a superfluous edge or $d(G) > 2$. If $q = 0$ then also $r = 0$ because else (w, a) 2. If $q = 0$ (e.i. also $r = 0$) then the construction in this way gives us the graphs of the class B_4. If $q \neq 0$ then we graphs of the class B_7.

<u>Theorem 4.</u> a) For every $n \geq 6$ there exists a graph from the class B_4 of n vertices.

b) For every $m \geq 12$ there exists a graph from the class B_4 of m edges.

<u>Proof.</u> a) If we put $q=r=p=s=z=0$ in the equation (1) of the Remark 4, then $n = t + 6$, where $t \geq 0$.

b) Put in the equation (2) of the Remark 4 $q=r=s=0$. Then $m=zt+2p+2t+2z+8$.

If $z = 0$, $t = 2$ then $m = 12+2p$; if $z = 1$, $t = 1$, then $m = 13 + 2p$, where $p \geq 0$. The proof of the theorem is finished.

Theorem 5. For every $d \geq 3$, $n \geq 3d - 1$ there exists a graph of the class B_7 of n vertices and the minimum dehree d.

Proof. Choose separate parameters in such a way that the graph constructed by the described construction will be of the minimum degree $d = q + 2$, i.e. $q = d - 2$, $t = d - 2 + t_1$, $t_1 \geq 0$; $z = d - 3 + z_1$, $z_1 \geq 0$; $s \geq 0$; $r \geq 0$; $p \geq 0$; Equation (1) of the Remark 4 changes into the shape $n = 3d - 1 + p + z_1 + t_1 + 2s + 3r$. So n runs over all the values starting from $3d - 1$.

Theorem 6. For every $d \geq 3$, $m \geq 4d^2 + 3$ there exists a graph from the class B_7 with m edges and with the minimum degree d.

Proof. Let us choose the parameters in the described construction in such a way, that the minimum degree $d = q + 2$; i.e. $Q = D - 2$, $t=d-2+t_1$, $t_1 \geq 0$; $z = d - 3 + z_1$, $z_1 \geq 0$; $p \geq 0$; $s \geq 0$; $r \geq 0$; Then the equation (2) of the Remark 4 changes into the shape :

$$m = 2d^2 - 3d + 4 + d(3r + p + 2s + z_1 + 2t_1)+s(z_1+r-1) + t_1(r+z_1-3).$$

Futher we put $r = 1$, $t_1 = 0$, $z_1 = 1$. Then it will hold

$m = d(2d + 1 + p + 2s) + s + 4$. let be $p = 0$ and let $s = 0,1,2,\ldots, d - 1$.

Then $m \equiv i \pmod d$, where $i = 4,5,\ldots,d - 1,0,1,2,3$. If be $s = d - 1$, $p = 0$ then be $m = 4d^2 + 3$. It is easy to verify that m obtains all numbers than $4d^2 + 3$.

Theorem 7. a) For every $n \geq 10$ there exists a graph from the class B_7 with n vertices and minimum degree 2.

b) For every $m \geq 26$ there exists a graph from the class B_7 of m edges and the minimum degrees 2.

Proof. Let be $r = 0$, $q \neq 0$, $s \neq 0$, $t \neq 0$ in the described construction. Eliminate the edge (y_i, a_3) and the edge (y_i, a_j^2), where $i=1,\ldots,q$; $j = 1,\ldots,s$. Analogously as in the basic construction we can verify that the arised graph will be from the class B_7. About the number of vertices the formula (1) of Remark 4 holds. Hence part a) of theorem holds. The number of edges will be the same as in basic construction for $r = 0$. If we put $q = 1$, $s = 1$, $t = 1$ then $m = 3p + 4z +18$. If $z = 0$ then

$4z + 18 \equiv 0 (\mathrm{mod}\ 3)$; If $z = 1$ then $4z + 18 \quad 1 \ (\mathrm{mod}\ 3)$; If $z = 2$ then $4z+18 \quad 2 \ (\mathrm{mod}\ 3)$. So the theorem holds for $m \geq 26$.

Finaly we shall prove, that an arbitrary graph without the loops and multiplied edges can became a section graph from the class B (even of its subclass B_7).

<u>L e m m a 1.</u> Let be $G = (U,H)$ a graph, which is not necessarily connetcted and finite. Then there exists a graph R of diameter at most 2 and such that G is its section graph.

Proof. It is enough to put $R = (V,E)$ whereby $V = U \cup \{a\}$, where $a \notin U$ and $E = H \cup \{(a,x) \mid x \in U\}$.

<u>T h e o r e m 8.</u> Let be $G = (U,H)$ a graph, which is not necessarily connected and finite. Then there exists an η-irredicible graph G_1 without superfluous edges of the diameter two and such that G is its section graph.

<u>P r o o f.</u> Let be $R = (V,E)$, $V = \{a_1, a_2, \ldots, a_n\}$ a graph constructed by Lemma 1. The graph G_1 we construct from the graph R by adding the set of vertices $B = \{w, b_1, \ldots, b_n\}$ and the set of edges (w,b_i), (b_i,a_i), $i = 1,2,\ldots,n$; and if the edge (a_i,a_j) was not in the graph R, then we add the edge (b_i,b_j) for $i \neq j$, $i,j = 1,2\ldots,n$. It is possible to verify that graph so constructed is of the diameter 2 and moreover for $x,y \in V \cup B$, $x \neq y$ is $\Omega_{G_1}(x) \neq \Omega_{G_1}(y)$. All edges are not superfluous, because after removing the edge (w,b_i) it would be $\rho_{G_1}(w,a_i) > 2$, for $i = 1,2,\ldots,n$.

(b_i,b_j) it would be $\rho_{G_1}(a_j,b_i) > 2$, for $i \neq j$, $i,j = 1,2,\ldots,n$.

(a_i,a_j) it would be $\rho_{G_1}(a_j,b_i) > 2$, for $i \neq j$, $i,j = 1,2,\ldots,n$.

(a_i,b_i) it would be $\rho_{G_1}(w,a_i)$. 2, for $i = 1,2,\ldots,n$.

So the theorem holds.

<u>C o r o l l a r y 1.</u> Let $G = (U,H)$ be a graph (not necessarily connected and finite). Then there exists an η-irreducible graph from the class B_7 and such that G is its section graph.

<u>P r o o f.</u> If G does not contain exact k-angle, for $K = 3,4,5$, then we extend it by new vertices and edges in such a way that it will contain exact k-angle for $k = 3$ and 4 and 5 as well. According to the Theorem 8 from the arised graph we construct the required graph from the class B_7.

<u>C o r o l l a r y 2.</u> There exists an infinite number of η-irreducible graphs from the class B_7.

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On the solutions of the differential equation

by

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In a recent paper (1), the successive approximate solutions $y_n(n=1,2,3,..)$ of the diferential equation

$$\frac{d^2y}{dx^2} - (x+ax^m)y = 0$$

have been discussed only in the two cases $m = 2$ and 3.
Expressions connecting the solutions y_n and y_{n+1} were given in these two cases.

It is the object of the present paper to deal with the same problem when m takes any positive integral value ($m \geq 2$).

For this purpose we collect together some results mainly for subsequent use.

1. Preliminary Results

L e m m a 1. The complete solution of the differential equation

$$\frac{d^2y}{dx^2} - xy = 0$$

is $y = c_1 \,\mathrm{Ai}\, x + c_2 \,\mathrm{Bi}\, x = u(x)$

where $Ai\,x$, $Bi\,x$ are the two Airy's integrals and c_1, c_2 are arbitrary constants.

<u>Lemma 2.</u> Let the notation be as in Lemma 1, and let $u^{(n)}$ denote the nth derivate of u with respect to x. Let further H^{-1} be the inverse operator of the operator

$$\frac{d^2}{dx^2} - x$$

then

$$H^{-1} \; u^{(n)} = \frac{u^{(n+1)}}{n+1} .$$

In (1), the following relations have been required:

$$x\,u^{(n)} = u^{(n+2)} - n\,u^{(n-1)} ,$$

$$x^2\,u^{(n)} = u^{(n+4)} - 2(n+1)\,u^{(n+1)} + (n-1)_2\,u^{(n-2)} .$$

$$x^3\,u^{(n)} = u^{(n+6)} - 3(n+2)\,u^{(n+3)} + \omega(n)\,u^{(n)} - (n-2)_3\,u^{(n-3)} ,$$

where $\omega(n) = 3n^2 + 3n + 2$.

For the present situation, we require a similar expression for $x^m u^{(n)}$ in general.

<u>Lemma 3.</u>

$$x^m\,u^{(n)} = a^{(n)}_{m,1}\,u^{(2m+n)} + a^{(n)}_{m,2}\,u^{(2m+n-3)} + \ldots$$

$$+ a^{(n)}_{m,i}\,u^{(2m+n-3i+3)} + \ldots$$

where $a^{(n)}_{m,1} = 1$

$$a^{(n)}_{m,2} = -\,m(m+n-1),$$

$$a^{(n)}_{m,i} = a^{(n)}_{m-1,i} - \left\{2m+n-3i+4\right\} a^{(n)}_{m-1,i-1} \cdot$$

This result is true for $m = 1,2,3$ and may be established in general by mathematical induction on m [2].

For the convenience of notation and typing, we introduce the coefficients $\alpha(m,n,i)$ where

$$\alpha(m,n,i) = a^{(n)}_{m,i+1}$$

In this notation.

$$\alpha(1,n,o) = 1 , \quad \alpha(1,n,1) = -n$$

$$\alpha(2,n,o) = 1 , \quad \alpha(2,n,1) = -2(n+1) = -2(n+1) , \quad \alpha(2,n,2) = (n-1)_2$$

$$\alpha(3,n,o) = 1 , \quad \alpha(3,n,1) = -3(n+2) ,$$

$$\alpha(3,n,2) = \omega(n) , \quad \alpha(3,n,3) = -(n-2)_3 .$$

Moreover, by means of this notation Lemma 3 will be written

L e m m a 4.

$$x^m u^{(n)} = \alpha(m,n,o)\, u^{(2m+n)} + \alpha(m,n,1)\, u^{(2m+n-3)} + \dots$$

$$= \sum_{i=o}^{k(m,n)} \alpha(m,n,i)\, u^{(2m+n-3i)}$$

where $\alpha(m,n,o) = 1,$

$\alpha(m,n,1) = -m(m+n-1),$

$\alpha(m,n,i) = \alpha(m-1,n,i) - (2m+n-3i+1)\ \alpha(m-1,n,i-1).$

and $k(m,n) = [(2m + n)|3]$ *

L e m m a 5. For $i \geq 2$,

$$\alpha(m+1,n,i) = -\sum_{r=1}^{m} (2r+n-3i+3)\ \alpha(r,n,i-1).$$

---

*Whereever the symbol [r] appears, it denotes the integral part of the number r.

This follows directly from the last relation of Lemma 4. In fact, we have

$$\alpha(m,n,i) = \alpha(m-1,n,i) - (2m+n-3i+1) \quad \alpha(m-1,n,i-1).$$

If we replace m by $r+1$, and rearrange the terms, we obtain

$$\alpha(r+1,n,i) - \alpha(r,n,i) = (2r+n - 3i+3) \quad \alpha(r,n,i-1).$$

Writing this relation for $r = 1,2,\ldots,m$, then adding all tohether, we get

$$\alpha(m+1,n,i) - \alpha(1,n,i) = -\sum_{r=1}^{m} (2r+n-3i+3) \quad \alpha(r,n,i-1)$$

and the lemma follows at once if we observe that

$$\alpha(1,n,i) = 0 \qquad \text{for} \quad i \geq 2.$$

<u>Lemma 6.</u> With the above notation

$$- \alpha(m+1,n,1) = (m+1)(m+n) ,$$

$$6\alpha(m+1,n,2) = m(m+1) \left\{ 3 m^2 + (6n - 7)m + 3 n^2 - 9n + 4\right\}$$

$$- 6\alpha(m+1,n,3) = (m-3)_6 + (3n+2) (m-2)_5 + n(3n-1) (m-1)_4 +$$
$$+ n(n-1) (n-2) (m)_3$$

The first of these relations in obvious. The last two relations follow directly if in Lemma 5 we take $i=1,2$ in succession.

2. Successive approximations of the differential equation

$$\frac{d^2y}{dx^2} - (x+ax^m)y = 0$$

In this section we apply the method of successive approximations (already used in the previous paper) for the differential equation

$$\frac{d^2y}{dx^2} - (x + ax^m)y = 0 \tag{2.1}$$

For this purpose we write (2.1) into the form

$$\frac{d^2y}{dx^2} - xy = ax^m y \tag{2.2}$$

If a is small enough, then for a first approximation, the above equation may be written

$$\frac{d^2y}{dx^2} - xy = 0 .$$

The solution of this equation is, by Lemma 1, $y = u(x) = u$.
This may be regardes as a first approximation y_1.

Thus $y_1 = u$.

For a second approximation, we put $y = y_1 = u$ in the R.H.S. of equation (2.2), thus having

$$\frac{d^2y}{dx^2} - xy = a\, x^m\, u \tag{2.3}$$

Then by using Lemma 4 (with n = o) for the R.H.S., equation (2.3) will be written

$$\frac{d^2y}{dx^2} - xy = a \sum_{i=o}^{k(m,o)} \alpha(m,o,i)\; u^{(2m-3i)} \tag{2.4}$$

The complementary function of this equation is evidently $y = u$, while the particular integral is by Lemma 2,

$$a \; Ⓗ^{-1} \left\{ \sum_{i=o}^{k(m,o)} \alpha(m,o,i)\; u^{(2m-3i)} \right\}$$

$$= a \sum_{i=o}^{k(m,o)} \alpha(m,o,i)\; \frac{u^{(2m-3i+1)}}{2m-3i+1}$$

Thus the complete primitive of equation (2.4) gives directly the second approximate solution y_2, namely

$$y_2 = u + a \sum_{i=o}^{k(m,o)} \frac{\alpha(m,o)}{2m-3i+1} u^{(2m-3i+1)} \cdot$$

i.e. $$y_2 = y_1 + a \sum_{i=o}^{k(m,o)} \frac{\alpha(m,o,i}{2m-3i+1} u^{(2m-3i+1)} .$$

For a third approximation, we put $y = y_2$ in the R.H.S. Of (2.2), thus we have

$$\frac{d^2y}{dx^2} - xy = ax^m u + a^2 \sum_{i=o}^{k(m,o)} \frac{\alpha(m,o,i)}{2m-3i+1} x^m u^{(2m-3i+1)} \qquad (2.5)$$

Then using Lemma 4 twice for the R.H.S. of (2.5), we have

$$\frac{d^2y}{dx^2} - xy = a \sum_{i=o}^{k(m,o,i)} \alpha(m,o,i) u^{(2m-3i)}$$

$$+ a^2 \sum_{i=o}^{k(m,o)} \frac{\alpha(m,o,i)}{2m-3i+1} \sum_{j=o}^{k(m,2m-3i+1)} \alpha(m,2m-3i+1,j) u^{(4m-3i-3j+1)} .$$

The complete solution of this equation gives the third approximation y_3 in the form

$$y_3 = u + a \sum_{i=o}^{k(m,o)} \frac{\alpha(m,o,i)}{2m-3i+1} u^{(2m-3i+1)}$$

$$+ a^2 \sum_{i=o}^{k(m,o)} \sum_{j=o}^{k(m,2m-3i+1)} \frac{(m,o,i)\,\alpha(m,o,i) \quad (m,2m-3i+1,j}{(2m-3i+1)(4m-3i-3j+2)} u^{(4m-3i-3j+2)}$$

$$\text{i.e.} \quad y_3 = y_2 + a^2 \sum_{i=o}^{k(m,o)} \; \sum_{j=o}^{k(m,2m-3i+1)} \frac{\alpha(m,o,i) \;\; (m,2m-3i+1,j)}{(2m-3i+1)(4m-3i-3j+2)} \, u^{(4m-3i-3j+2)}$$

Thus we have shown the following lemma.

<u>L e m m a 7.</u> The first three approximate solutions of the differential equation

$$\frac{d^2y}{dx^2} - (x+ax^m)y = o \qquad \text{(a being small enough)}$$

are

$y_1 = u,$

$$y_2 = y_1 + a \sum_{i=o}^{k(m,o)} \frac{\alpha(m,o,i)}{2m-3i+1} \, u^{(2m-3i+1)} \; ,$$

$$y_3 = y_2 + a^2 \sum_{i=o}^{k(m,o)} \; \sum_{j=o}^{k(m,2m-3i+1)} \frac{\alpha(m,o,i) \; \alpha(m,2m-3i+1,j)}{(2m-3i+1)(4m-3i-3j+2)} \, u^{(4m-3i-3j+2)}$$

In a similar way one can easily obtain the fourth approximation y_4. This will be given in the following Lemma.

<u>L e m m a 8.</u>

$$y_4 = y_3 + a^3 \sum_{i_1=o}^{k_1} \sum_{i_2=o}^{k_2} \sum_{i_3=o}^{k_3} \frac{\alpha\; m,o,i_1)\alpha(m,2m-3i_1+1,i_2) \; \alpha(m,4m-3i_1-3i_2+2,i_3}{(2m-3i_1+1)(4m-3i_1-3i_2+2)(6m-3i_1-3i_2-3i_3+3)} \times$$

$$. \; u^{(6m-3i_1-3i_2-3i_3+3)} .$$

where $k_1 = k(m,o)$,

$k_2 = k(m,2m-3i_1+1),$

and $k_3 = k(m,4m-3i_1-3i_2+2)$.

3. Simplification of the forms of the approximate solutions

The approximate solutions already obtained in the previous article may be simplified if we introduce the following symbols.

$$\beta_1 = \frac{\alpha(m,o,i_1)}{2m-3i_1+1},$$

$$\beta_2 = \frac{\alpha(m,2m-3i_1+1,i_2)}{4m-3i_1-3i_2+2}$$

$$\beta_3 = \frac{\alpha(m,4m-3i_1-3i_2+2,i_3)}{6m-3i_1-3i_2-3i_3+3}.$$

In this notation Lemmas 7, 8 can be simplified.

<u>Lemma 8*.</u> The first four approximate solutions of the differential equation

$$\frac{d^2y}{dx^2} - (x + ax^m)y = o \qquad \text{(a small enough)}$$

may be written

$$y_1 = u,$$

$$y_2 = y_1 + a \sum_{i_1=o}^{k_1} \beta_1 \, u^{(2m-3i_1+1)},$$

$$y_3 = y_2 + a^2 \sum_{i_1=o}^{k_1} \sum_{i_2=o}^{k_2} \beta_1 \, \beta_2 \, u^{(4m-3i_1-3i_2+2)}$$

$$y_4 = y_3 + a^3 \sum_{i_1=o}^{k_1} \sum_{i_2=o}^{k_2} \sum_{i_3=o}^{k_3} \beta_1 \, \beta_2 \, \beta_3 \, u^{(6m-3i_1-3i_2-3i_3+3)}$$

4. The approximate solutions y_n, y_{n+1}

In this section, we express y_{n+1} in terms of y_n. For this purpose, we introduce the following

$$k_1 = k(m,o) ,$$

$$k_{s+1} = k(m, 2sm - 3(i_1+i_2+\ldots+i_s) + s) ; \qquad s=1,2,\ldots,n-1.$$

and

$$\beta_1 = \frac{\alpha(m,o,i_1)}{2m-3i_1+1},$$

$$\beta_{s+1} = \frac{\alpha(m, 2sm-3(i_1+i_2+\ldots+i_s)+s, i_s)}{2sm-3(i_1+i_2+\ldots+i_s)+s} \qquad \text{for } s = 1,2,\ldots,n-1$$

By means of this notation, we state the following

<u>Theorem 1.</u> Let y_n and y_{n+1} be the n^{th} and $(n+1)$ th approximate solutions of the differential equation

$$\frac{d^2y}{dx^2} - (x+ax^m)y = o \qquad (a \text{ small enough})$$

Then $y_1 = u$,

$$y_{n+1} = y_n + a^n F_{m,n}(u) \qquad n \geq 1,$$

Where

$$F_{m,n}(u) = \sum_{i_1=o}^{k_1} \sum_{i_2=o}^{k_2} \ldots \sum_{i_n=o}^{k_n} \left\{ \prod_{s=1}^{n} \beta_s \right\} u^{(2sm-3 \sum_{s=1}^{n} i_s+s)} \qquad (*)$$

This theorem is true for $n=1,2,3$ and may be established in general by mathematical induction on n. The proof is omitted.

Although $F_{m,n}(u)$ is written in a compact form, yet such a form depends indeed on n processes of summation. However, $F_{m,n}(u)$ consists of a finite number of terms involving derivatives of u. Moreover, we observe that the highest derivative appearing in $F_{m,n}(u)$ is of order

$$2nm + n = n(2m + 1)$$

(such a term corresponds to $i_1=i_2=\ldots=i_n=o$).
Furthermore, the derivatives of u diminishes by 3 and its multiples. This suggests an expression of $F_{m,n}(u)$ which consists of terms of the form

$$u^{(n(2m+1)-3i)}$$

Thus, Theorem 1 may be restated as follows:

Theorem 2. Let y_n and y_{n+1} be the n^{th} and $(n+1)^{th}$ approximate solutions of the differential equation

$$\frac{d^2y}{dx^2} - (x+ax^m)y = o$$

Then $y_1 = u$,

$$y_{n+1} = y_n + a^n F_{m,n}(u) \qquad n \geq 1,$$

where $$F_{m,n}(u) = \sum_{i=o}^{p} (-)^i c_{m,n,i} \frac{u^{(n(2m+1)-3i)}}{n(2m+1)-3i} \qquad (**)$$

with suitable expressions for $c_{m,n,i}$ and with $p = p(m,n)$ depending on both m and n.

In fact,

(i) If $n = 3N$, $p = (2m+1)N - 1$

(ii) If $n = 3N + 1$, $p = (2m+1)N + u$

$$\text{where} \quad \mu = \begin{cases} \left[\dfrac{2m+1}{3}\right] & \text{if} \quad 2m+1 \not\equiv o \pmod 3 \\[2ex] \left[\dfrac{2m+1}{3}\right] - 1 & \text{if} \quad 2m+1 \equiv o \pmod 3 \end{cases}$$

(iii) If $n = 3N+2$, $p = (2m+1(N +$

$$\text{where} \quad \mu = \begin{cases} 2\left[\dfrac{2m+1}{3}\right] & \text{if} \quad 2m+1 \not\equiv o \pmod 3 \\[2ex] 2\left[\dfrac{2m+1}{3}\right] - 1 & \text{if} \quad 2m+1 \equiv o \pmod 3 \end{cases}$$

5. On the coefficients $c_{m,n,i}$

We prove the following

<u>Lemma 9.</u>

$$c_{m,n,o} = \frac{1}{(2m+1)^n \; n!} \, ,$$

$$c_{m,n+1,i} = \sum_{r=o}^{k(m,\overline{2m+1}\,n-3i)} \frac{\alpha(m, \overline{2m+1}\,n-3i+3r, r)}{(2m+1)n-3i+3r} \, c_{m,n,i-r}$$

<u>Proof.</u> Byusing a process similar to that one used in the previous paper (see §§4,5; Theorem 1,2), we can easily show that

$$y_{n+2} = y_{n+1} + a^{n+1} \, Ⓗ^{-1}\left\{x^m \, F_{m,n}(u)\right\} . \tag{5.1}$$

On the other hand

$$y_{n+2} = y_{n+1} + a^{n+1} \, F_{m,n+1}(u) \tag{5.2}$$

Comparing (5.1) and (5.2), we get

$$Ⓗ^{-1}\left\{x^m \, F_{m,n}(u)\right\} + F_{m,n+1}(u) \tag{5.3}$$

Also $x^m F_{m,n}(u) = \sum_{i=o}^{p} (-)^i \frac{c_{m,n,i}}{n(2m+1)-3i} . x^m u^{(n(2m+1)-3i)}$

$$= \sum_{i=o}^{p} (-)^i \frac{c_{m,n,i}}{n(2m+1)-3i} \sum_{j=o}^{k(m,\ n(2m+1)-3i} \alpha(m,n(2m+1)-3i,j) \times$$

$$. u^{(2m+n(2m+1)-3i-3j)}$$

Operating by $(H)^{-1}$, we get

$$(H)^{-1}\left\{x^m F_{m,n}(u)\right\} = \sum_{i=o}^{p} \sum_{j,o}^{k(m,n(2m+1)-3i)} (-)^i \alpha(m,n(2m+1)-3i,j) \times$$

$$. c_{m,n,i} \frac{u^{((2m+1)(n+1)-3i-3j)}}{(2m+1)(n+1)-3i-3j} \qquad (5.4)$$

Moreover

$$F_{m,n+1}(u) = \sum_{i=o}^{k(m,n+1)} (-)^i c_{m,n+1,i} \frac{u^{((n+1)(2m+1)-3i)}}{(n+1)(2m+1)-3i} \qquad (5.5)$$

Then if we combine together (5.3), (5.4), (5,5) and compare the coefficients of $u^{((n+1)(2m+1)-3i)}$, we get

$$c_{m,n+1,i} = \sum_{r=o}^{k(m,\overline{2m+1}\ n-3i)} (-)^r \frac{\alpha(m,\overline{2m+1}\ n-3i+3r,r)}{(2m+1)n-3i + 3r} c_{m,n,i-r} .$$

This proves Lemma 9 (ii).

For $i=o$, the above expresion reduces to

$$c_{m,n+1,o} = \frac{\alpha(m,\overline{2m+1}\ n,o)}{(2m+1)n} c_{m,n,o}$$

but $\alpha(m,n,o) = 1$ for all m,n, and therefore

$$c_{m,n+1,o} = \frac{1}{(2m+1)n} c_{m,n,o}$$

Then by a repeated application of this recurrence formula and remarkong that $c_{m,1,o} = 1$, we get directly

$$c_{m,n+1,o} = \frac{1}{(2m+1)^n \, n!}$$

This completes the proof of the lemma.

6. Special Cases

In this case, we deal with the two special cases $m = 2,3$ which have been discussed already [(i), §§ 4,5].

The case $m = 2$

In this case,

$\alpha(2,n,o) = 1, - \alpha(2,n,1) = 2(n+1), \quad \alpha(2,n,2) = n(n-1),$

$\alpha(2,n,i) = o \qquad \text{for } i \geq 2.$

Also Lemma 9, will be

$$c_{2,n+1,o} = \frac{1}{5^n \, n!},$$

$$c_{2,n+1,i} = \frac{\alpha(2,5n-3i,o}{5n-3i} c_{2,n,i} - \frac{\alpha(2,5n-3i+3,1)}{5n-3i+3} c_{2,n,i-1} + \frac{\alpha(2,5n-3i+6,2)}{5n-3i+6} c_{2,n,i-2}$$

$$= \frac{1}{5n-3i} c_{2,n,i} + 2\,\frac{5n-3i+4}{5n-3i+3} c_{2,n,i-1} + (5n-3i+5)\, c_{2,n,i-2} .$$

Now if we write

$$c_{2,n+1,i} = a_{n+1,i} \qquad i = o,1,2,\ldots$$

$$F_{2,n}(u) = F_n \; u(x)$$

The above results will be written

$$a_{n+1,o} = \frac{1}{5^n \quad n!},$$

$$a_{n+1,i} = \frac{1}{5n-3i} a_{n,i} + 2 \frac{5n-3i+4}{5n-3i+3} a_{n,i-1} + (5n-3i+5) a_{n,i-2}$$

$$F_{2,n}(u) = \sum_{i=o}^{p} (-)^i c_{2,n,i} \frac{u^{(5n-3i)}}{5n-3i}$$

$$= \sum_{i=o}^{p} (-)^i a_{n,i} \frac{u^{(5n-3i)}}{5n-3i}$$

$$= F_n \; u(x)$$

[see. (1), § 4 Theorem 1. and Cor. 1,2].

The case m = 3

In this case

$$\alpha(3,n,o) = 1 \quad , \quad \alpha(3,n,1) = -3(n+2),$$
$$\alpha(3,n,2) = \omega(n) \quad , \quad \alpha(3,n,3) = -n(n-1)(n-2),$$
$$\alpha(3,n,i) = o \qquad \text{for } i \geq 3.$$

Then Lemma 9, will be

$$c_{3,n+1,o} = \frac{1}{7^n \quad n!},$$

$$c_{3,n+1,i} = \frac{\alpha(3,7n-3i,o)}{7n-3i} c_{3,n,i} - \frac{\alpha(3,7n-3i+3,1)}{7n-3i+3} c_{3,n,i-1}$$

$$+ \frac{\alpha(3,7n-3i+6,2)}{7n-3i+6} c_{3,n,i-2} - \frac{\alpha(3,7n-3i+9,3)}{7n-3i+3} c_{3,n,i-3}$$

but $\alpha(3,7n-3i,o) = 1$,

$$\alpha(3,7n-3i+3,1) = -3(7n-3i+5),$$

$$\alpha(3,7n-3i+6,2) = \omega(7n-3i+6),$$

$$\alpha(3,7n-3i+9,3) = -(7n-3i+9)(7n-3i+7)_2 ,$$

and therefore if we write

$$c_{3,n+1,i} = b_{n+1,i} \qquad \text{for } i = o,1,2,\ldots$$

and $F_{3,n}(u) = G_n(u)$,

we have
$$b_{n+1,o} = \frac{1}{7^n\, n!},$$

$$b_{n+1,i} = \frac{1}{7n-3i}\, b_{n,i} + 3\,\frac{7n-3i+5}{7n-3i+3}\, b_{n,i-1} +$$

$$+ \frac{\omega(7n-3i+6)}{7n-3i+6}\, b_{n,i-2} + (7n-3i+7)_2\, b_{n,i-3} .$$

$$F_{3,n}(u) = \sum_{i=o}^{q} (-)^i\, c_{3,n,i}\, \frac{u^{(7n-3i)}}{7n-3i}$$

$$= \sum_{i=o}^{q} (-)^i\, b_{n,i}\, \frac{u^{(7n-3i)}}{7n-3i} = G_n(u)$$

[see [1], § 5, Theorem 2 and Cor. 1,2].

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On the absolute continuity of product measures

T. NEUBRUNN

1. A condition concerning the absolute continuity of the product measure $\nu_1 \times \nu_2$ with respect to $\mu_1 \times \mu_2$ for a sufficiently general case will be discussed here in the part 4. At first in the part 3. a simple completion of a well-known result related to the connection of the absolute continuity of the types $\nu <<_\varepsilon \mu$ and $\nu << \mu$ is given.

2. Unless otherwise stated notations are as in [3] . $(X, \mathcal{S})$ denotes a measurable space.

Definition 1. If ν and μ are measures defined on $\mathcal{S}$ then ν is said to be absolutely continuos with respect to μ in case, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that the relations $E \in \mathcal{S}$ and $\mu(E) < \delta$ imply $\nu(E) < \varepsilon$ Briefly denoted $\nu <<_\varepsilon \mu$.

Definition 2. The measure ν is said to be dominated by μ (briefly $\nu << \mu$) in case every μ -null set is ν -null, that is $\mu(E) = 0$ implies $\nu(E) = 0$ for every $E \in \mathcal{S}$.

One can immediately verify that $\nu <<_\varepsilon \mu$ implies $\nu << \mu$ In general the converse is not true (See [3] , p. 128 ex 12).

In case of a finite measure ν the two above defined notions coincide (See [1] , p. 149).

3. We shall give a necessary and a sufficient condition for the connection for of the relations $\nu <<_\varepsilon \mu$ and $\nu << \mu$ in case of a nonatomic measure ν and an arbitrary measure μ .

Lemma 1. If ν is a nonatomic and μ an arbitrary measure on $\mathcal{S}$, then a necessary condition for $\nu <<_\varepsilon \mu$ is:

(1) $\nu(E) = \infty$ implies $\mu(E) = \infty$ for every $E \in \mathcal{S}$

P r o o f. Assume that there exists $E \in S$ such that $\nu(E) = \infty$ and $\mu(E) < \infty$. Since ν is nonatomic, we may form a sequence of pairwise disjoint sets $E_n \in S$, such that $E_n \subset E$, $\nu(E_n) = 1$, $(n=1,2,3\ldots)$ (See 3, p.174). Putting $F_k = \bigcup_{k=n}^{\infty} E_m$, we have

$$\nu(F_k) = \sum_{n=k}^{\infty} \nu(E_n) = \infty, \quad \mu(F_k) < \mu(E) < \infty$$

for $k = 1,2,3,\ldots$

Since
$$\sum_{n=1}^{\infty} \mu(E_n) = \mu(F_1) < \mu(E) < \infty,$$

we have $\lim_{k\to\infty} \mu(F_k) = \lim_{k\to\infty} \sum_{n=k}^{\infty} \mu(E_n) = 0$

Thus $\lim_{k\to\infty} \mu(F_k) = 0$ and $\nu(F_k) = \infty$, for $k = 1,2,3,\ldots$

So $\nu \ll_\varepsilon \mu$ does not hold. This completes the proof.

L e m m a 2. Let ν be a nonatomic measure. Then, under the condition (1), the implication $\nu \ll \mu \Longrightarrow \nu \ll_\varepsilon \mu$ holds.

P r o o f. The proof uses the same reasoning as the proof of the statement $\nu \ll \mu$ implies $\nu \ll_\varepsilon \mu$ in the case of a finite measure ν (See [3] p. 125).

In fact, let $\nu \ll \mu$ hold and let $\nu \ll_\varepsilon \mu$ be not true. Then there exists $\varepsilon > 0$ and a sequence $\{E_n\}_{n=1}^{\infty}$ of E_n such that $E_n \in S$.

Thus
$$\mu\left(\bigcup_{k=n}^{\infty} E_n\right) \leq \sum_{n=k}^{\infty} \mu(E_n) \leq \sum_{n=k}^{\infty} \frac{1}{2^n} < \infty$$

The last and (1) imply $\nu\left(\bigcup_{n=k}^{\infty} E_n\right) < \infty$ for $k = 1,2,3,\ldots$

Putting $B = \limsup_{n\to\infty} E_n = \lim_{k\to\infty} \left(\bigcup_{n=k}^{\infty} E_n\right)$

and taking in account that $\nu\left(\bigcup_{k=1}^{\infty} E_k\right) < \infty$ and $\bigcup_{n=k}^{\infty} E_n > \bigcup_{n=k+1}^{\infty} E_n$

(as follows from (1) and from the choice of E_n), we get

$$\nu(B) = \lim_{k\to\infty} \nu\left(\bigcup_{n=k}^{\infty} E_n\right) \geq \limsup_{n\to\infty} \nu(E_n) \geq \varepsilon$$

On the other hand

$$\mu(B)=\lim_{k\to\infty}\mu\left(\bigcup_{n=k}^{\infty}E_n\right)=\lim_{k\to\infty}\sum_{n=k}^{\infty}\mu(E_n)=0$$

4. In what follows some questions related to product spaces will be discussed. Two measurable spaces $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ will be considered. Let μ_1 and μ_2 be measures defined on $\mathcal{S}$ and $\mathcal{T}$ respectively Further let ν_1 be a measure absolutely continuous with respect to μ_1 and ν_2 a measure absolutely continuous with respect to μ_2. A question arises: Is then $\nu_1 \times \nu_2$ absolutely continuous with respect to $\mu_1 \times \mu_2$? The answer is positive if μ_i $(i = 1,2)$ are σ-finite and ν_i finite. A question how does the situation look like for more general cases is posed in [1]([1] : p.169 ex # 1).

In view of lemma 1 and 2 it is natural to expect that discussing this question, the relation between the sets $E \in \mathcal{S}\times\mathcal{T}$ for which $\nu_1\times\nu_2(E)=\infty$ and those for which $\mu_1\times\mu_2(E)=\infty$, will play more importatnt role than the finitness or σ-finitness of the measures themselves. The following notion will be useful.

D e f i n i t i o n 3. If $(X, \mathcal{S})$ is a measurable space, μ and ν measures defined on $\mathcal{S}$, then ν is said to be μ-attainable on a set $E \in \mathcal{S}$ of finite ν-measure, if there exists a sequence $\{E_n\}_{n=1}^{\infty}$ of sets such that $E_n \subset E_{n+1} \subset E$, $\mu(E_n)<\infty$, (n=1,2,3,...) and $\lim_{n\to\infty}\nu(E_n)=\nu(E)$

If ν is μ-attainable on each set E of finite ν-measure, then ν is said to be μ-attainable.

Obviously if ν is arbitrary and μ finite, then ν is μ-attainable. It is easy to give an example of two measures ν and μ such that ν is μ-attainable and μ is not σ-finite. It suffices to put $\nu=\mu$ where is semifinite but not σ-finite. (For the notion of semifiniteness see [1] p.87).

Now we explain briefly how the product measure $\mu\times\nu$. (μ, ν are arbitrary measures on $\mathcal{S}$ and $\mathcal{T}$ respectively) will be defined. When μ, ν are finite, the product is defined in the usual way ([1]:p.143, [3] p.145). In case of arbitrary measures we shall use the method developed in [2] (see also (1), p.127), where the product $\mu\times\nu$ is defined as $\mu\times\nu = l.u.b.\,\tau^{P\times Q}$ where $P\times Q$ runs over the measurable rectangles such that $\mu(P)<\infty, \mu(Q)<\infty$ and $\tau^{P\times Q}$ is the product, taken in the usual sense, of finite measures μ^P, ν^Q which are defined as $\mu^P(E)=\mu(P\cap E)$, $\nu^Q(E)=\nu(Q\cap E)$ on $\mathcal{S}$ and $\mathcal{T}$ respectively.

The proof of the following known lemma will be omitted. (See e.g. [4] pp. 178,179). The fact that the measures appearing in [4] are probabilities is not substantial for the proof.

Lemma 3. Let $(X,\mathcal{S})$, $(Y,\mathcal{T})$ be measurable spaces, $\{\nu_i\}$ and $\{\mu_i\}$ $(i=1,2)$ finite measures defined on $\mathcal{S}$ and $\mathcal{T}$ respectively. Let $\nu_i \ll \mu_i$. Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$

Teorem 1. Let $(X,\mathcal{S})$, $(Y,\mathcal{T})$ be measurable spaces. Let ν_i and μ_i be measures (not necessary finite) defined on $\mathcal{S}$ and $\mathcal{T}$ respectively and such that ν_i is μ_i -attainable and $\nu_i \ll \mu_i$ $(i=1,2)$. Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$.

Proof. Let M denote the set of all measurable rectangles $P \times Q$ such that $\mu_1(P) < \infty$, $\mu_2(Q) < \infty$. Put N for the set of all those $P \times Q$ for which $\nu_1(P) < \infty$, $\nu_2(Q) < \infty$. Since ν_i $(i=1,2)$ are μ_i -attainable, we can construct for each $P_o \times Q_o \in N$ a sequence of rectangles $\{P_n \times Q_n\}_{n=1}^{\infty}$ such that $P_n \times Q_n \in M \cap N$, $P_n \times Q_n \subset P_{n+1} \times Q_{n+1} \subset P_o \times Q_o$, $(n=1,2,3,\ldots)$ and

$$(2) \qquad \lim_{n\to\infty} \nu_1 \times \nu_2 (P_n \times Q_n) = \nu_1 \times \nu_2 (P_o \times Q_o)$$

holds. (P_n and Q_n are chosen such that $\nu_1\left(P_o - \bigcup_{n=1}^{\infty} P_n\right) =$

$$= \nu_2\left(Q_o - \bigcup_{n=1}^{\infty} Q_n\right) = 0, \quad P_n \subset P_{n+1} \subset P_o, \; Q_n \subset Q_{n+1} \subset Q_o$$

Owing to the fact that $\{P_n \times Q_n\}_{n=1}^{\infty}$ is an increasing sequence and also to the fact that is a restriction of $\nu_1 \times \nu_2$ to the rectangle $P_n \times Q_n$ (see [2]), we get for each $E \in \mathcal{S} \times \mathcal{T}$

$$\lim \nu_1^{P_n} \times \nu_2^{Q_n} (E) = \lim_{n\to\infty} \nu_1 \times \nu_2 ((P_n \times Q_n) \cap E)$$

$$(3) \qquad = \nu_1 \times \nu_2 ((\lim_{n\to\infty} (P_n \times Q_n)) \cap E) = \nu_1 \times \nu_2 \left(\left(\bigcup_{n=1}^{\infty} P_n \times \bigcup_{n=1}^{\infty} Q_n\right) \cap E\right)$$

Putting $A = \bigcup_{n=1}^{\infty} P_n$, $B = \bigcup_{n=1}^{\infty} Q_n$ and taking into account that $A \times B \subset P_o \times Q_o$, we have

$$(4) \qquad P_o \times Q_o = (A \times B) \cup (P_o - A) \times Q_o \cup A \times (Q_o - B)$$

Since $\nu_1(P_o - A) = \nu_2(Q_o - B) = 0$, we get from (4)

$$(5) \qquad \nu_1 \times \nu_2 (P_o \times Q_o) = \nu_1 \times \nu_2 (A \times B) = \nu_1 \times \nu_2 \left(\bigcup_{n=1}^{\infty} P_n \times \bigcup_{n=1}^{\infty} Q_n\right)$$

(5)

From the fact $A \times B \subset P_0 \times Q_0$ and from (5) one see that $P_0 \times Q_0$ and

$\bigcup_{n=1}^{\infty} P_n \times \bigcup_{n=1}^{\infty} Q_n$ are equal in measure. Hence for every $E \in S \times T$

(6) $$\nu_1 \times \nu_2 \left(E \cap (P_0 \times Q_0) \right) = \nu_1 \times \nu_2 \left(E \cap \left(\bigcup_{n=1}^{\infty} P_n \times \bigcup_{n=1}^{\infty} Q_n \right) \right)$$

Thus (3) and (6) give

(7) $$\lim_{n \to \infty} \nu_1^{P_n} \times \nu_2^{Q_n} (E) = \nu_1 \times \nu_2 \left(\left(\bigcup_{n=1}^{\infty} P_n \times \bigcup_{n=1}^{\infty} Q_n \right) \cap E \right) = \nu_1^{P_0} \times \nu_2^{Q_0} (E)$$

Since $\nu_1^{P_n} << \mu_1^{P_n}$ and $\nu_2^{Q_n} << \mu_2^{Q_n}$ for every n and since $P_n \times Q_n$ $\in M \cap N$ we get, using Lemma 3

(8) $$\nu_1^{P_n} \times \nu_2^{Q_n} << \mu^{P_n} \times \mu_2^{Q_n}$$

Now, if $Z \in S \times T$ and $\mu_1 \times \mu_2 (Z) = 0$ we have $\mu_1^P \times \mu_2^Q (Z) = 0$ for each $P \times Q \in M$. If $P \times Q \in M \cap N$ then according to (8) also $\nu_1^P \times \nu_2^Q (Z) = 0$. Hence according to (8) and (7) $\nu_1^{P_0} \times \nu_2^{Q_0} (Z) = 0$ for $P_0 \times Q \in N$. Thus $\nu_1 \times \nu_2 (Z) = \sup_{P \times Q \in N} \nu_1^P \times \nu_2^Q (Z) = ($

The proof is complete.

The preceding theorem generalizes the known fact that if ν_1 and ν_2 are finite and μ_1, μ_2 σ-finite then under the assumption $\nu_1 << \mu_1$, $\nu_2 << \mu_2$ the affirmation $\nu_1 \times \nu_2 << \mu_1 \times \mu_2$ holds.

<u>Theorem 2.</u> Let (X, S), (Y, T) be measurable spaces $\{\nu_i\}$ and $\{\mu_i\}$ $(i = 1, 2)$ measures defined on S and T respectively. Let the following conditions hold:

(I) $\quad \nu_i \ll_\varepsilon \mu_i$

(II) $\quad \nu_i$ is μ_i -attainable

(III) At least one of the measures ν_i is nonatomic. Then a necessary and sufficient condition for $\nu_1 \times \nu_2 \ll_\varepsilon \mu_1 \times \mu_2$ is: $\nu_1 \times \nu_2(E) = \infty$ imply $\mu_1 \times \mu_2(E) = \infty$ for every $E \in \mathcal{S} \times \mathcal{T}$

The proof immediately follows from lemma 1, theorem 1 and the following

<u>Lemma 4</u>. If $(X, \mathcal{S}, \mu)$, $(Y, \mathcal{T}, \nu)$ are measure spaces and at least one of the measures μ, ν is nonatomic, then $\mu \times \nu$ is a nonatomic measure on $\mathcal{S} \times \mathcal{T}$.

<u>Proof.</u> Suppose μ to be nonatomic. Since the product $\lambda = \mu \times \nu$ is defined as $\lambda = \underset{P \times Q \in \mathcal{L}}{\text{l.u.b.}}\, \mu^P \times \nu^Q$ where $\mathcal{L}$ is the set of all the measurable rectangles with finite sides and since the measure μ^P is evidently nonatomic, it suffices to prove these two statements:

a) If a measure λ is l.u.b. of a system of finite nonatomic measures then λ is nonatomic.

b) Lemma 4 is valid for the case of finite measures.

As to the a), let λ_i $(i \in \mathcal{Y})$ be a system of finite nonatomic measures and λ = l.u.b. λ_i a measure. If $E \in \mathcal{S}$ and $\lambda(E) > 0$ then there exists $i_0 \in \mathcal{Y}$ such that $\lambda_{i_0}(E) > 0$. Since λ_{i_0} is nonatomic, the existence of a set $F \in \mathcal{S}$ follows such that $0 < \lambda_{i_0}(F) < \lambda_{i_0}(E)$ Hence also $\lambda_{i_0}(F - E) > 0$ Consequently $\lambda(F) > 0$ and $\lambda(F-E) > 0$. Since $\lambda(F) + \lambda(E-F) = \lambda(E)$ we have $0 < \lambda(F) < \lambda(E)$ and a) is proved.

b) Let $E \in \mathcal{S} \times \mathcal{T}$, $\lambda(E) > 0$, $(\lambda = \mu \times \nu, \mu, \nu$ are finite). We have to prove the existence of t $(0 < t < \lambda(E))$ and $F \in \mathcal{S} \times \mathcal{T}$ $F \subset E$ such that $\lambda(F) = t$. If E is a measurable rectangle $E = A \times B$ then such F exists for each t $(0 < t < \lambda(E))$ It. suffices to choose $F = A_1 \times B$ where $A_1 \subset A$, $A_1 \in \mathcal{S}$ is such that $\mu(A_1) = \frac{t}{\nu(B)}$

Then $0 < \mu(A_1) < \mu(A)$ and $t = \mu(A_1)\,\nu(B) < \mu(A)\,\nu(B)$

Now, let $E \in \mathcal{S} \times \mathcal{T}$ be an arbitrary set. Since, as it is well known (cf.e.g. [3] p. 145)

$$\lambda(E) = \inf\left\{ \sum_{n=1}^{\infty} \mu(A_n)\,\nu(B_n) : E \subset \bigcup_{n=1}^{\infty} A_n \times B_n,\ A_n \in \mathcal{S},\ B_n \in \mathcal{T} \right\}$$

there exists n_0 such that $\lambda(E \cap (A_{n_0} \times B_{n_0})) > 0$. If $\lambda(E \cap (A_{n_0} \times B_{n_0})) < \lambda(E)$ then it suffices to put $F = E \cap (A_{n_0} \times B_{n_0})$ and the proof is finished. If $\lambda(E \cap (A_{n_0} \times B_{n_0})) = \lambda(E)$, we may suppose without loss of generality that $E \subset A_{n_0} \times B_{n_0}$. Then E is equal in measure to $A_{n_0} \times B_{n_0}$. Choose $L \subset A_{n_0} \times B_{n_0}$, such that $L \in \mathcal{S} \times \mathcal{T}$ $0 < \lambda(L) < \lambda(A_{n_0} \times B_{n_0})$ and put $F = E \cap L$. Evidently $\lambda(F) > 0$, $\lambda(F) < \lambda(E)$.
This completes the proof.

Note that is general under the assumptions (I) (II) (III) the statement: $\nu_1 \times \nu_2(E) = \infty$ impies $\mu_1 \times \mu_2(E) = \infty$ for every $E \in \mathcal{S} \times \mathcal{T}$, may be false. Hence $\nu_1 \ll_\varepsilon \mu_1$, $\nu_2 \ll_\varepsilon \mu_2$ may be true and $\nu_1 \times \nu_2 \ll_\varepsilon \mu_1 \times \mu_2$ may be false.

<u>Example.</u> Put $X = \{1, 2, 3, \ldots\}$, $\mathcal{S} = 2^X$, $Y = \langle 0, 1 \rangle$, $\mathcal{T}$ = the system of all Lebesgue measurable sets contained in $\langle 0, 1 \rangle$. Define ν_1 on $\mathcal{S}$ such that $\nu_1(\{n\}) = 2^n$ for arbitrary one point set $\{n\}$, (n is any positive integer. Further, let μ_1 on $\mathcal{S}$ be defined such that for every one point set $\{n\}$, $\mu_1(\{n\}) = 1$. Evidently $\nu_1 \ll_\varepsilon \mu_1$ Let ν_2 and μ_2 be defined on $\mathcal{T}$ as $\nu_2 = \mu_2$ – the Lebesgue measure on $\mathcal{T}$ We have $\nu_2 \ll_\varepsilon \mu_2$. Take $E = \bigcup_{n=1}^{\infty} E_n \times F_n$ where $E_n = \{n\}$, $F_n \in \mathcal{T}$ such that $\nu_2(F_n) = \frac{1}{2^n}$. Then $\nu_1 \times \nu_2(E) = \infty$, $\mu_1 \times \mu_2(E) = 1$

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