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PROBLEMS OF OPTIMIZATION OF NUMERICAL MATHEMATICS.

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1. Modern computational techniques are putting forward new problems in numerical analysis. At present numerical mathematics can be considered as a set od constructive mathematical methods transforming given information into desired ones (see e.g. BABUŠKA [1966], HENRICI [1964], BABUŠKA, SOBOLEV [1965], BABUŠKA, PRÁGER, VITÁSEK [1966]). The classic concepts as for example that of method are beginning to have new meaning. The first place is being occupied by algorithms and the methods are rather comprehend as a class of algorithms of certain kind. Concerning algorithms the following requirements arise:

a) sufficient generality of algorithms

this requires the algorithm to be applicable to a sufficiently wide class of problems. For example the algorithm of integration by Cotes' formulae of highest order is not sufficiently general as it is applicable only to the narrow class of analytic functions.

b) Sufficient universal efficiency;

this means that the algorithm should treat the given informations „approximately" as well as the optimal algorithm (see below).

c) sufficiently good realizability

By realizability we mean, that the fact, that the computer does not work in the field of real numbers (the rounding off) should not have a great effect on the result. Especially this is the problem of numerical stability (see Babuška, Práger, Vitásek [1966]).

In this paper we will study some aspects concerning the universal efficiency. In order to illustrate this problem we will restrict us here only to very special cases.

2. Let a Banach space B be given and let $\varphi \in B^*$. Our task will be to calculate the value $\varphi(f)$ for a given $f \in B$. The principal idea of (linear) numerical methods of calculation of the value of the functional φ is the following. A matrix of functionals $\Phi \equiv \{\varphi_j^{(n)}\}$, $j = 1, \ldots, n$, $n = 1, 2, \ldots$, $\varphi_j^{(n)} \in B^*$, is given (these functionals will be called *calculable functionals*). Now it is necessary to construct the functionals $\varphi_n = \sum_{j=1}^{n} C_j^{(n)} \varphi_j^{(n)}$ in such a way that $\varphi_n(f) \to \varphi(f)$ for $n \to \infty$. In practical cases we take $\varphi_n(f) \approx \varphi(f)$ for sufficiently great n. There is a number of problems connected with this task.

1) Problem of the estimate for the upper bound of error.

Here the upper bound of the quantity $\varepsilon_n(\varphi, \varphi_n, B) = ||\varphi - \varphi_n||_{B^*}$ is to be estimated.

This problem bears in fact a classic character and is intensively investigated at present (especially it concerns not only the estimate of order, but also of the corresponding constants); in the case of integration of periodic functions see e.g. Sobolev [1965], [1967], Jagerman [1966], Agahanov [1965], Ehlich [1966], Babuška [1965], Čarušnikov [1966] and others.

2) Problem of the estimate for the lower bound of error.

Here the lower bound of the quantity

$$\eta_n(\varphi, \Phi, B) = \inf_{\alpha_k^{(n)},\ k = 1, \ldots, n} ||\varphi - \sum_{k=1}^{n} \alpha_k^{(n)} \varphi_k^{(n)}||_{B^*}$$

is to be estimated. Also this question is intensively studied at present. See e.g. Sobolev [1965], [1967], Babuška, Sobolev [1965], Bachvalov [1963] and many others. The quantity gives the maximal accuracy at obtainable on the ground of given information.

3) Problem of the optimal formula.

The task is to construct the functionals φ_n in such a way that

$$\varepsilon_n(\varphi, \varphi_n, B) = \eta_n(\varphi, \Phi, B)$$

See e.g. Babuška, Sobolev [1965], Sobolev [1965], [1967], Golomb, Weinberger [1959] etc. The concrete construction of optimal formulae is very difficult and is known only in special cases. In connection with these difficulties formulae are studied, which are asymptotically optimal or optimal by order. See e.g. Babuška, Sobolev [1965], Sobolev [1965]. From the point of view of numerical practice the problem of optimal formulae encounters some difficulties. Beyond the difficulties connected with the construction of optimal formulae there is also the problem of how to choose the space B in a concrete case. We will now illustrate the practical importance of this problem by a simple example.

Let
$$\varphi(f) = \int_0^1 f(x)\,\mathrm{d}x$$

Let Φ be a matrix of the functional,such that $\varphi_{n+1}(f) = \frac{1}{n} \sum_{s=0}^{n} a_s^{(n)} f\left(\frac{s}{n}\right)$ holds. If $\|f\|_B^2 = f^2(0) + \int_0^1 (f')^2\,\mathrm{d}x$, then the optimal formula will be the trapezoid-rule. At the same time it is known, that the trapezoid-rule is scarcely used in practice.

The question of how to lower the risk of choosing the space B in a concrete case is the question of universality of the formula.

4) Problem of universal optimality by order.

Let $\mathfrak{A}$ be a given system of Banach spaces B embedded in a Banach space B_0. Let us have a matrix of calculable functionals $\varphi_j^{(n)} \in B_0^*$ and a matrix of coefficients $\Psi = \{C_j^{(n)}\}$, $j = 1, \ldots, n$; $n = 1, 2, \ldots$. We will use the following notation:

$$\mathfrak{A}_{\Psi}^{\Phi,\varphi} = E\left[B \in \mathfrak{A}, \quad \frac{\|\varphi - \sum_{j=1}^{n} C_j^{(n)} \varphi_j^{(n)}\|_{B^*}}{\eta_n(\varphi, \Phi, B)} \leq C(B)\right]$$

[where $C(B)$ depends on B, φ, Φ, Ψ but not on n]. We will say that the formula $\varphi_n = \sum_{j=1}^{n} C_j^{(n)} \varphi_j^{(n)}$ is universally optimal by order with respect to $\mathfrak{A}_{\Psi}^{\Phi,\varphi}$. Further let us have two formulae given by the matrices $\Psi_i = \{{}^iC_j^{(n)}\}$, $i = 1, 2. \ldots$ [i.e. $\varphi_{n,i} = \sum_{j=1}^{n} {}^iC_j^{(n)}\varphi_j^{(n)}$]. We will say that the formula given by with the matrix Ψ_1 is comparable or better or not worse respect to $\mathfrak{A}$ than the formula given by the matrix Ψ_2, if $\mathfrak{A}_{\Psi_1}^{\Phi,\varphi} \gtreqless \mathfrak{A}_{\Psi_2}^{\Phi,\varphi}$ or $\mathfrak{A}_{\Psi_1}^{\Phi,\varphi} \supset \mathfrak{A}_{\Psi_2}^{\Phi,\varphi}$ or $\mathfrak{A}_{\Psi_1}^{\Phi,\varphi} \supseteq \mathfrak{A}_{\Psi_2}^{\Phi,\varphi}$, respectively. The problem of universal optimality lies in

a) characterization of $\mathfrak{A}_{\Psi}^{\Phi,\varphi}$ for a given formula,

b) characterization of $\mathfrak{A}$ in such a that the best formula exist,

c) construction of an algorithm leading to this best formula and an estimate of the quantities η_n and $C(B)$ as functions of B.

3. In this part we will give some illustrative assertions concerning the universal optimality. Let us have the task to calculate a functional over the Hilbert space of periodic functions and let us ask, what is (in the intuitive sense) understood the concept of this space. Its intuitive meaning can be perhaps expressed in the following manner.

Definition 1. *We will say that a Hilbert space H of 2π-periodic complex functions has the property P, if the following properties are fullfiled.*

P_1: *H is dense in $C_{2\pi}$.*

P_2: if $f \in H$ *then also* $g(x) = f(x + c) \in H$ *for every real* c *and* $\|f\| = \|g\|$.

P_3: *H is imbedded in $C_{2\pi}$.*

Now the following theorem holds.

Theorem 1.[1]

Let H have the property P. Then

1) $e^{ikx} \in H$, $k = \ldots, -1, 0, 1, \ldots;$

2) $(e^{ikx}, e^{ilx}) = \lambda_k^2$ for $k = l$

$\qquad\qquad\quad = 0$ *for* $k \neq l$;

3) $\sum_{m=-\infty}^{+\infty} \lambda_k^{-2} < \infty$.

It is easy to prove also the inverse theorem.

Theorem 2.

Let K be the set of all sequences Λ, $\Lambda \equiv \{\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots\}$ for which $\lambda_k > 0$, $k = \ldots, -1, 0, 1, \ldots$ and $\Sigma\, \lambda_k^{-2} < \infty$.

Let M be a linear space of all trigonometric polynomials with the scalar product $(e^{ikx}, e^{ilx}) = \lambda_k^2$ *for* $k = l$

$\qquad\qquad\quad = 0$ *for* $k \neq l$.

Let H_Λ denote the complete envelope of M in the given norm; then H_Λ has the property P.

[1] It was M. PRÁGER who has drawn my attention to this theorem.

Now we will introduce various systems of spaces with the property P. Let $\mathfrak{A}$ be the system of all Hilbert spaces with the property P.
Let $\mathfrak{A}_1$ be of all H_Λ, $\Lambda \in K_1 \subset K$, such that if $\Lambda \in K_1$, then
1) $\lambda_k = \lambda_{-k}$, $\quad k = 0, 1, 2, \dots$
2) $\lambda_{k+1} \geq \lambda_k \quad k \geq 0$
3) $$\sum_{t=0}^{\infty} \frac{\lambda^2[\alpha n]}{\lambda^2[\alpha n] + t(2n+1)} \leq D, \quad 0 \leq \alpha \leq 4.$$
D does not depend on n (but depends on Λ.)

Let $\mathfrak{A}_2$ be the system of all H_Λ, $\Lambda \in K_2 \subset K_1$ satisfying.

$$\lambda_k \leq C + |k|^\beta \ , \quad \beta > 0 \ .$$

Now let $\quad \Phi \equiv \{\varphi_j^{(n)}\}, \quad j = 1, 2, \dots, 2n+1, \quad n = 1, 2, \dots$

$$\varphi_j^{(n)}(f) = f\left(\frac{2\pi}{2n+1} j\right)$$

be a matrix of calculable functionals and let us turn to the problem of computation of the functional

$$\varphi(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, \zeta(x)\, \mathrm{d}x, \quad \zeta(x) \in L_2 \ .$$

Then the formula becomes[2)]

$$\varphi_n(f) = \sum_{j=1}^{2n+1} C_j^{(n)}(\zeta)\, \varphi_j^{(n)}$$

Now the question is how to choose the coefficients $C_j^{(n)}(\zeta)$. The following theorem holds.

Theorem 3.

A necessary and sufficient condition that there should exist such $C_j^{(n)}(\zeta)$ that the formula

$$\varphi_n(f) = \sum_{j=1}^{2n+1} C_j^{(n)}(\zeta)\, \varphi_j^{(n)}$$

should be universally optimal by order with respect to $\mathfrak{A}$, is that $\zeta(x)$ should be a trigonometric polynomial. The coefficients are uniquelly determined except for a finite number of indices n and are given by

$$^*C_j^{(n)} = \frac{1}{2n+1}\, \zeta\left(\frac{2\pi}{2n+1} j\right)$$

If $\zeta(x)$ is a more general function, then it follows from theorem **3** that

[2)] To simplify formally the following assertions we have restricted us to an odd number of points used.

a formula, which would be universally optimal by order with respect to $\mathfrak{A}$ does not exist. In connection with what has been said above the question arises whether it is possible to restrict the system of spaces $\mathfrak{A}$ in such a way that universally optimal — by — order formula should exist. This is solved by the following theorem.

Theorem 4.

If $\zeta(x) \in L_2$, then $*C_j^{(n)}(\zeta)$ exist so that the formula

$$\varphi_n(f) = \sum_{j=1}^{2n+1} *C_j^{(n)}(\zeta)\, \varphi_j^{(n)}$$

is universally optimal by order with respect to $\mathfrak{A}_1$. Except for a finite number of indices n, the coefficients are uniquelly determined and we have

$$*C_j^{(n)} = \frac{1}{2n+1} S_n \left(\frac{2\pi}{2n+1} j\right)$$

where $$S_n = \sum_{k=-n}^{+n} d_k e^{ikx} \quad \textit{and} \quad \zeta(x) = \sum_{k=-\infty}^{+\infty} d_k e^{ikx} .$$

By theorem 3 and 4 the universally optimal — by — order formula is uniquelly determined. It is clear that should we further restrict the system of spaces $\mathfrak{A}$, then the formula can be determined non uniquelly. In this connection the following theorem holds.

Theorem 5.

Let $\zeta(x) \in L_2$. Then the formula given by theorem 4 is not the only formula universally optimal by order with respect to $\mathfrak{A}_2$.

Returning once more to the formula given by theorem 4 we see that it is not optimal in any $H \in \mathfrak{A}_1$ but is universally optimal by order. It is also easy to see that in fact this formula is obtainable by means of the classic (interpolation) method using trigonometric polynomials. From this point of view the connection between the classic (interpolation) theory of quadrature formulae and the theory based on optimization of formulae is well visible. But we will not go further in the study of this problem.

Using the simplest examples, I have given some typical theorems concerning the form of the universal optimality by order. This problem can of course be substantionally extended to include the problem of calculation of functionals as well as operators.

4. In the conclusion let us give some numerical results. Let us compute

$$I = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\alpha \sin x} \cos x \, dx$$

for different values of α.

As the integrand is obviously a 2π-periodic function, I can be written in the form

$$I = \int_{-\pi}^{+\pi} e^{\alpha \sin x} \zeta(x) \, dx$$

where $\zeta(x) = \cos x \qquad \text{for } |x| < \frac{\pi}{2}$

$$\zeta(x) = 0 \qquad \text{for } \frac{\pi}{2} \leq x \leq \pi, \quad -\pi \leq x \leq -\frac{\pi}{2}$$

$\Bigl($here we make use of the symetry of $f = e^{\alpha \sin x}$ with respect to the point $x = \pm \frac{\pi}{2}\Bigr)$. Now the integrand has the form studied in theorem 4. In the following Table together with various formulae the quadrature error is given in dependence on the number of vaules of the function $e^{\alpha \sin x}$ (for $\alpha = 1, 5, 7$) used in the calculation. Besides the trapezoid-rule and the Simpson formula also the Romberg formula (see BAUER, RUTISHAUSER, STIEFL [1963]) according to BAUMAN algorithm [1961] is given under the notation Romberg. Two other modified methods are given as Romberg 1 and Romberg 2. The formula Romberg 1 is that of Bulirsch—Romberg (see BULIRSCH [1964]) and the formula Romberg 2 is that of Bulirsch—Stoer (see BULIRSCH, STOER [1965]). The last one is given for comparison although it is not a linearone.

The computation has been carried out on the *ICT* **1900** with a double precission of word.

BIBLIOGRAPHY

[1] С. А. АГАХАНОВ *(1965): Оточности некоторых квадратурных и кубатурных формул. Сиб. мат. ж. VI, 1, 1—15.*

[2] I. BABUŠKA (1965): *Über optimale Formeln zur numerischen Berechnung linearer Funktionale. Apl. Mat. 441—445.*

[3] I. BABUŠKA (1966): *О некоторых вопросах оптимизации численных методов матем. анализа. Сб. Некоторые вопросы вычислительной и прикладной математики. Новосибирск 23—39.*

[4] I. BABUŠKA, M. PRÁGER, E. VITÁSEK (1966): *Numerical Processes in Differential Equations. Wiley 1966.*

[5] I. BABUŠKA, S. L. SOBOLEV (1965): *Оптимизация численных методов. Apl. Mat. 96—130.*

[6] Н. С. БАХВАЛОВ, *(1936): Об оценке количества вычислительной работы необходимой при приближенном решении задач. Доп. к книге Годунов, Рабенький, Введение в теорию разностных схем. Москва.*

[7] F. L. BAUER, H. RUTISHAUSER, E. STIEFEL (1963): *New Aspect in Numerical Quadrature. Proc. Symp. in Appl. Math. 15, Am. Math. Soc.*

[8] R. Baumann (1961): *Algol-Manual der Alcor Gruppe, Sonderdruck aus Elektronische Rechenanalogen H/ 5/6 (1961) und H 2 (1962). R. Odenbourg, München.*

[9] R. Bulirsch (1964): *Bemerkung zur Romberg-Integration. Num. Math. No 1.*

[10] R. Bulirsch, I. Stoer (1965): *Fehlerabschätzung und Extrapolation mit rationalen Funktionen bei Verfahren von Richardson Typus. Berecht 6403. München, Math. F.*

[11] В. Д. ЧАРУШНИКОВ *(1966): Об одной минимаксной задаче, теории кубатурьных формул. ДАН СССР т. 160, № 1, 36—39.*

[12] H. Ehlich (1966): *Untersuchungen zur numerischen Fourieranalyse. Math. Zeitsch. 9, 380—420.*

[13] M. Coolomb, H. F. Weinberger (1959): *Optimal Approximation and Errors Baunas Proceeding of a Symposium Conducted by the Math. Res. Anter US Army at the Univ. Wisconsin, Madison Apr. 21—23, 1958. The University of Wisconsin Press.*

[14] P. Henrici (1964): *Elements of numerical Analysis. J. Wiley.*

[15] D. Jagermann (1966): *Investigation of Modified. Mid. Point Quadrature Formula, Math. of Comp. Vol. 20, 79—84.*

[16] С. Л. СОБОЛЕВ: *(1965): Лекции по теории кубатурных формул. Новосибирск.*

[17] I. BABUŠKA (1968): *Problems of optimization and numerical stablity in computations. Apl. Mat. 388—404.*

[18] I. BABUŠKA (1968a): *Über universal optimale Quadraturformeln, 304—338, 388—404.*

[19] K. SEGETH (1968): *On universally optimal huadrature formulae involving values oj derivatives of integrand. Preprint, Math. Inst. df Czechoslovak akademy of Sc.*

Number of points	1	2	3	4	5	6
4	0.17 0	−0.40 −1	0.76 −3	0.19 −2	−0.12 0	−0.11 −1
6	0.72 −1	−0.62 −2	0.26 −5		0.12 −1	−0.11 −1
8	0.40 −1	−0.16 −2	0.63 −8	−0.85 −5	0.57 −3	0.93 −3
10	0.26 −1	−0.61 −3	0.11 −10			
12	0.18 −1	−0.28 −3	0.15 −13		−0.15 −3	−0.53 −4
14	0.13 −1	−0.15 −3	0.15 −16			
16	0.99 −2	−0.87 −4	−0.54 −19	−0.38 −7	0.71 −5	−0.44 −6
18	0.79 −2	−0.54 −4	0.60 −18			
20	0.64 −2	−0.35 −4	0.16 −18			
22	0.53 −2	−0.24 −4	−0.16 −18			
24	0.44 −2	−0.17 −4	−0.54 −19		−0.12 −6	0.27 −7
26	0.38 −2	−0.12 −4	−0.49 −18			
28	0.32 −2	−0.90 −5	−0.27 −18			
30	0.28 −2	−0.69 −5	−0.27 −18			
32	0.25 −2	−0.53 −5	−0.54 −19	0.12 −9	−0.70 −10	−0.19 −9
34	0.22 −2	−0.41 −5	−0.16 −18			
36	0.20 −2	−0.33 −5	0.54 −19			
38	0.18 −2	−0.27 −5	0.54 −19			
40	0.16 −2	−0.22 −5	0.16 −18			
42	0.14 −2	−0.18 −5	0.16 −18			
44	0.13 −2	−0.15 −5	0.38 −18			
46	0.12 −2	−0.12 −5	0.38 −18			
48	0.11 −2	−0.10 −5	0.16 −18		0.17 −10	−0.76 −12
50	0.10 −2	−0.88 −6	0.38 −18			
52	0.94 −3	−0.75 −6	0.38 −18			
54	0.87 −3	−0.65 −6	0.38 −18			
56	0.81 −3	−0.56 −6	0.38 −18			
58	0.76 −3	−0.49 −6	0.38 −18			
60	0.71 −3	−0.42 −6	0.38 −18			
62	0.66 −3	−0.37 −6	0.38 −18			
64	0.62 −3	−0.33 −6	0.60 −18	−0.62 −13	−0.13 −12	0.12 −14

Table *1.* *The calculation of I for $\alpha = 1$ according to various formulae.* 1) *trapezoid-rule,* 2) *Simpson formula,* 3) *universal formula,* 4) *Romberg formula,* 5) *Romberg formula 1,* 6) *Romberg formula 2.*

Number of points	1	2	3	4	5	6
4	0.98 1	0.37 1	0.20 1	−0.95 0	0.88 1	−0.21 2
6	0.37 1	−0.12 1	0.11 0		−0.65 1	−0.21 2
8	0.20 1	−0.61 0	0.51 −2	0.35 −1	−0.39 0	0.15 1
10	0.13 1	−0.21 0	0.19 −3			
12	0.86 0	−0.84 −1	0.57 −5		0.39 0	0.24 0
14	0.63 0	−0.41 −1	0.13 −6			
16	0.48 0	−0.23 −1	0.25 −8	−0.26 −3	−0.35 −1	−0.48 −1
18	0.38 0	−0.14 −1	0.39 −10			
20	0.31 0	−0.88 −2	0.51 −12			
22	0.25 0	−0.59 −2	0.55 −14			
24	0.21 0	−0.41 −2	0.49 −16		0.52 −3	−0.56 −3
26	0.18 0	−0.30 −2	−0.26 −17			
28	0.16 0	−0.22 −2	−0.30 −17			
30	0.14 0	−0.17 −2	−0.26 −17			
32	0.12 0	−0.13 −2	0.00 0	0.22 −6	0.50 −4	0.44 −4
34	0.11 0	−0.99 −3	−0.13 −17			
36	0.94 −1	−0.79 −3	−0.43 −18			
38	0.85 −1	−0.63 −3	0.87 −18			
40	0.76 −1	−0.51 −3	0.87 −18			
42	0.69 −1	−0.42 −3	0.87 −18			
44	0.63 −1	−0.35 −3	0.22 −17			
46	0.58 −1	−0.29 −3	0.87 −18			
48	0.53 −1	−0.25 −3	0.17 −17		−0.20 −5	0.65 −8
50	0.49 −1	−0.21 −3	0.22 −17			
52	0.45 −1	−0.18 −3	0.13 −17			
54	0.42 −1	−0.15 −3	0.22 −17			
56	0.39 −1	−0.13 −3	0.26 −17			
58	0.36 −1	−0.12 −3	0.26 −17			
60	0.34 −1	−0.10 −3	0.30 −17			
62	0.32 −1	−0.88 −4	0.87 −18			
64	0.30 −1	−0.77 −4	0.35 −17	0.45 −9	0.23 −7	−0.64 −9

Table 2. *The calculation of I for $\alpha = 5$ according to various formulae.* 1) *trapezoid-rule,* 2) *Simpson formula,* 3) *universal formula,* 4) *Romberg formula,* 5) *Romberg formula* 1, 6) *Romberg formula* 2.

Number of points	1	2	3	4	5	6
4	0.77 2	0.52 2	0.21 2	−0.11 2	0.98 2	0.57 3
6	0.29 2	−0.39 1	0.21 1		−0.23 2	0.57 3
8	0.15 2	−0.58 1	0.15 0	0.32 0	−0.19 2	0.22 2
10	0.94 1	−0.25 1	0.97 −2			
12	0.64 1	−0.99 0	0.52 −3		0.49 1	0.36 1
14	0.47 1	−0.46 0	0.22 −4			
16	0.36 1	−0.25 0	0.79 −6	−0.80 −3	−0.79 −1	−0.52 0
18	0.28 1	−0.15 0	0.23 −7			
20	0.23 1	−0.92 −1	0.56 −9			
22	0.19 1	−0.62 −1	0.12 −10			
24	0.16 1	−0.43 −1	0.21 −12		−0.40 −1	−0.25 −1
26	0.13 1	−0.31 −1	0.32 −14			
28	0.12 1	−0.23 −1	0.24 −16			
30	0.10 1	−0.17 −1	−0.12 −16			
32	0.88 0	−0.13 −1	0.52 −17	−0.12 −4	0.26 −2	0.15 −1
34	0.78 0	−0.10 −1	0.87 −18			
36	0.70 0	−0.81 −2	0.35 −17			
38	0.63 0	−0.65 −2	0.11 −16			
40	0.57 0	−0.53 −2	0.95 −17			
42	0.51 0	−0.43 −2	0.61 −17			
44	0.47 0	−0.36 −2	0.19 −16			
46	0.43 0	−0.30 −2	0.69 −17			
48	0.39 0	−0.25 −2	0.15 −16		−0.60 −4	0.41 −5
50	0.36 0	−0.21 −2	0.21 −16			
52	0.33 0	−0.18 −2	0.16 −16			
54	0.31 0	−0.16 −2	0.17 −16			
56	0.29 0	−0.14 −2	0.23 −16			
58	0.27 0	−0.12 −2	0.21 −16			
60	0.25 0	−0.10 −2	0.26 −16			
62	0.24 0	−0.89 −3	0.87 −17			
64	0.22 0	−0.79 −3	0.30 −16	0.29 −7	0.36 −6	−0.37 −7

Table 3. *The calculation of I for $\alpha = 7$ according to various formulae.* 1) *trapezoid-rule,* 2) *Simpson formula,* 3) *universal formula,* 4) *Romberg formula,* 5) *Romberg formula* 1, 6) *Romberg formula 2.*

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ALGEBRAIC ELEMENTS IN THE TRANSFORMATION THEORY OF 2nd ORDER LINEAR OSCILLATORY DIFFERENTIAL EQUATIONS

O. BORŮVKA, Brno

1. In the last fifteen years, I have developed a transformation theory of ordinary 2nd order linear homogeneous differential equations in the real domain. It is a qualitative theory of global character. This theory deals with the effect of processes connected with the transformations of the variables on the integrals of the mentioned differential equations.

The origin of the transformation theory of 2nd order linear differential equations is due to E. E. KUMMER, who was the first to find the 3rd order non-linear differential equation which forms the basis of the transformation theory (1834). This equation is:

$$(Qq) \qquad -\{X, t\} + Q(X)\, X'^2 = q(t);$$

Q and q are given functions of a variable, X the unknown function and the symbol $\{X, t\}$ denotes the Schwarz derivative of X at the point t:

$$\{X, t\} = \frac{1}{2}\frac{X'''(t)}{X'(t)} - \frac{3}{4}\frac{X''^2(t)}{X'^2(t)}.$$

Kummer's ideas have prepared the way for more extensive investigations into the transformations of linear differential equations of the nth order in connection with the equivalence problem. The most important results in this field are due to E. LAGUERRE, F. BRIOSCHI, G. H. HALPHEN, A. R. FORSYTH, S. LIE and P. APPELL, in whose works we occasionally also find information about transformations of 2nd order differential equations in the complex domain.

The transformation theory in the real domain which I have developed may perhaps at first sight appear only as a special case of the linear differential equations of the nth order ($n \geq 2$). One is, nevertheless, necessarily led to a systematic treatment of this case $n = 2$. This is due to the fact that the linear differential equations of the 2nd order not only occupy a special position among those of the n (≥ 2)th order, since only in case of $n = 2$ two differential equations are always equivalent, but the results concerning transformations of 2nd order differential equations are most useful even for a general n. A systematic investigation of this special case leads, moreover, to a considerable enrichment of the classical theory of the 2nd order differential equations, both as to the formation of new notions and as to the development of the method.

2. The kernel of the mentioned transformation theory of 2nd order differential equations consists in investigating the connections between the solutions of the 2nd order linear differential equations

$$\text{(q)} \qquad y'' = q(t)\, y, \qquad \ddot{Y} = Q(T)\, Y \qquad \text{(Q)}$$

and Kummer's non-linear 3rd order differential equations (Qq), (qQ). The functions q, Q, which I shall occasionally call *carriers* of the differential equations (q), (Q), are generally only supposed to be continuous in their (open) intervals of definition $j = (a, b)$, $J = (A, B)$. A fundamental piece of information about the mentioned connections, which was already known to Kummer, is that the solutions $X(t)$, $x(T)$ of the differential equations (Qq), (qQ) transform all the integrals Y, y of the linear differential equations (Q), (q), in the sens of the following formulas:

$$\text{(1)} \qquad y(t) = \frac{Y[X(t)]}{\sqrt{|X'(t)|}}, \qquad Y(T) = \frac{y[x(T)]}{\sqrt{|\dot{x}(T)|}}.$$

3. Let us now first introduce some basic notions essetial to any further research into the transformation theory in question.

Consider a differential equation (q) in an (open) interval $j = (a, b)$. The carrier q is only assumed to be continuous. The *integral space* r of the differential equation (q) is understood to be the set of all the integrals of (q). The ***basis*** (u, v) ***of the differential equation*** (q) stands for a sequence of two linearly independent integrals u, v of (q). The basis of the ***integral space*** r is a basis of the differential equation (q).

One of the most important notions of the transformation theory is the notion of a phase, about which I shall now say a few words.

We discern phases of a ***basis*** (u, v) of the differential equation (q) and phases of the ***differential equation*** (q).

By a ***phase of the basis*** (u, v) of the differential equation (q) we mean any function α continuous in the interval j and satisfying in the latter, except for the zeros of the integral v, the equation $\operatorname{tg} \alpha(t) = u(t) : v(t)$.

It is easily understood that the phases of the basis (u, v) form a countable system, the so called *phase-system* of the basis (u, v) and that the singular phases of the system differ by integer multiples of the number π.

A phase of the *differential equation* (q) is understood to be a phase of any basis of the differential equation (q).

Every phase α of the differential equation (q) has, in the interval j, the following properties:

$$1. \quad \alpha \in C^3, \qquad 2. \quad \alpha' \neq 0.$$

By means of a phase α of the differential equation (q), the carrier q of the latter is uniquely defined, in the sense of the formula

$$q(t) = -\{\alpha, t\} - \alpha'^2(t). \tag{2}$$

The notion of a phase is closely connected with that of a phase function:

A *phase function* in the interval j is understood to be a function with the above properties 1., 2. A phase function α is a phase of the differential equation (q) with the carrier q defined in the sense of the formula (2).

A phase function α is called *elementary* if its values at any two points t, $t + \pi \in j$ are connected in the following way: $\alpha(t + \pi) = \alpha(t) + \pi \,.\, \operatorname{sgn} \alpha'$.

The phases I have spoken about are the so called first phases of the basis (u, v) or the differential equation (q). Besides these, one analogously defines the second phases, namely by means of the equation $\operatorname{tg} \beta(t) = u'(t) : v'(t)$. Since we shall, in what follows, not deal with the latter, we shall simply always refer to phases instead of first phases.

4. Let us now restrict our consideration to oscillatory differential equations (q). The term "oscillatory" means that the integrals of the differential equation (q) vanish, infinitely many times, in both directions towards the endpoints a, b of the interval $j = (a, b)$.

We shall start our considerations with the theorem that the differential equation (q) is oscillatory if, and only if, its phases are unbounded on both sides, from above and from below.

The phases α of an oscillatory differential equation (q) have, therefore, besides the properties 1. and 2., even the following one:

$$3. \quad \lim_{t \to a+} \alpha(t) = -\infty \,.\, \operatorname{sgn} \alpha', \qquad \lim_{t \to b-} \alpha(t) = \infty \,.\, \operatorname{sgn} \alpha'.$$

We see that a phase function unbounded on both sides is a phase of an oscillatory differential equation (q), i.e. the one with the carrier q defined in the sens of formula (2).

Oscillatory differential equations (q) have, furthermore, the characteristic property that they allow, in their intervals of definition, certain privileged functions, i.e. the so called central dispersions $\ldots, \varphi_{-2}(t), \varphi_{-1}(t), \varphi_0(t), \varphi_1(t), \varphi_2(t), \ldots$. The *central dispersion with the index* $\nu = 0, \pm 1, \pm 2, \ldots$ of the differential equation (q) is understood to be the function $\varphi_\nu(t)$ defined in the interval j as follows:

The value $\varphi_n(t)$ or $\varphi_{-n}(t)$ of the central dispersion φ_n or φ_{-n} $(n = 1, 2, \ldots)$ is, at every point $t \in j$, the n^{th} number conjugated with t on the right or on the left with regard to the differential equation (q). In other words: If one considers an integral y of the differential equation (q), vanishing at the point t, then $\varphi_n(t)$ or $\varphi_{-n}(t)$ is the n^{th} zero of y on the right or on the left of t. $\varphi_0(t)$ stands for the function t. The function φ_1 is also called the *fundamental dispersion* of the differential equation (q) and is briefly denoted by φ.

Every central dispersion φ_ν has, in the interval j, the following properties:

1. $\varphi_\nu(t) > \varphi_{\nu-1}(t)$, **2**, $\varphi_\nu \in C^3$, **3.** $\varphi_\nu'(t) > 0$, **4.** $\lim\limits_{t\to a+} \varphi_\nu(t) = -\infty$, $\lim\limits_{t\to b-} \varphi_\nu(t) = \infty$

We see that every central dispersion φ_ν is an increasing phase function, unbounded on both sides.

Moreover, we can show that:

Every central dispersion φ_ν and every phase α of the differential equation (q) are connected, at every point $t \in j$ by the so called *Abelian relation*

$$\alpha\varphi_\nu(t) = \alpha(t) + \nu\pi \,.\, \mathrm{sgn}\, \alpha'. \tag{3}$$

Instead of $\alpha[\varphi_\nu(t)]$ we simply write $\alpha\varphi_\nu(t)$.

Forming, in (3), on both sides the Schwarz derivative, one receives, with regard to (2), the relation

$$-\{\varphi_\nu, t\} + q(\varphi_\nu)\, \varphi_\nu'^2 = q(t).$$

We see that every central dispersion φ_ν satisfies Kummer's differential equation (qq) and, consequently, transforms every integral Y of the differential equation (q) into an integral y of the same differential equation (q) in the sense of formula (1).

The central dispersion φ_ν are the so called central dispersions of the first kind of the differential equation (q). Besides these, one also definies central dispersions of the 2^{nd}, 3^{rd} and 4^{th} kind of the differential equation (q). In what follows we shall, however, not meet with the latter and will therefore simply refer only to central dispersions instead of to central dispersions of the 1^{st} kind.

5. Let us now make a closer study of the transformation theory of oscillatory differential equations and, for this purpose, first briefly describe a constructive integration theory of Kummer's differential equation (Qq): One first defines, constructively, certain functions continuously dependent on three parameters, i.e. the so called *general dispersions* of the differential equations (Q), (q), and then shows that the latter are exactly the integrals of Kummer's differential equation (Qq).

Let, then, (q), (Q) be arbitrary oscillatory differential equations in the intervals $j = (a, b)$, $J = (A, B)$. Their integral spaces will be denoted by r or R.

Let $t_0 \in j$, $T_0 \in J$ be arbitrary numbers. Choose in the integral space r a basis (u, v) and in the integral space R a basis (U, V) such that

$$u(t_0)\, V(T_0) - v(t_0)\, U(T_0) = 0. \tag{4}$$

It is easily understood that the choice of the latter depends on two arbitrary parameters. Let us now define, by means of the bases (u, v), (U, V), a linear representation p of the integral space r on the integral space R by making correspond, to every integral $y \in r$ of (q), $y = \lambda u + \mu v$, the integral $py = Y = \lambda U + \mu V$ of (Q), formed with the same constants λ, μ. The quotient $\chi p = w : W$ of the wronskians w or W of the basis (u, v) or (U, V) is allcalled the *characteristic* of the linear representation p. The latter has, with regard to the relation (4), the following property: Every integral $y \in r$ of (q), vanishing at t_0, is in the linear representation p represented on an integral $Y \in R$ of (Q), vanishing at T_0. In other words: $y(t_0) = 0$ always yields $py(T_0) = 0$ With regard to this property, we call the linear representation p *normalized with respect to the numbers* t_0, T_0.

Let us, moreover, consider the numbers conjugated, both on the left and on the right, with t_0, with respect to the differential equation (q): $\ldots, t_{-2} = \varphi_{-2}(t_0), t_{-1} = \varphi_{-1}(t_0), t_0 = \varphi_0(t_0), t_1 = \varphi_1(t_0), t_2 = \varphi_2(t_0), \ldots$, and, similarly, the analogous numbers with respect to the differential equation (Q): $\ldots$, $T_{-2} = \Phi_{-2}(T_0)$, $T_{-1} = \Phi_{-1}(T_0)$, $T_0 = \Phi_0(T_0)$, $T_1 = \Phi_1(T_0)$, $T_2 = \Phi_2(T_0)$, $\ldots$. Every interval $j_\nu = [t_\nu, t_{\nu+1})$ or $j'_\nu = (t_{\nu-1}, t_\nu]$ for $\nu = 0, \pm 1, \pm 2, \ldots$ is called the ν^{th} *right* or *left-hand side basic interval* of the differential equation (q) with respect to the number t_0; the intervals $J_\nu = [T_\nu, T_{\nu+1})$ or $J'_\nu = (T_{\nu-1}, T_\nu]$ are called analogously. We see: every number $t \in j$ lies in a determined basic interval j_ν or j'_ν and, vice versa, every basic interval j_ν or j'_ν contains exactly one zero of every integral of (q). An analogous statement holds, of course, for every number $T \in J$ and for every integral Y of (Q).

Now we shall define, in the interval j, a function X as follows:

Let $t \in j$ be an arbitrary number and y an integral of the differential equation (q) vanishing at the point t. The number t lies in a determined right-hand side ν^{th} basic interval j_ν.

The value $X(t)$ of the function X at the point t is, according to whether $\chi p > 0$ or $\chi p < 0$, given as follows:

In case $\chi p > 0$, $X(t)$ is a zero of the integral py of (Q), namely the one lying in the right-hand side ν^{th} basic interval J_ν.

In case of $\chi p < 0$, $X(t)$ is a zero of the integral py of (Q), namely the one lying in the left-hand side $-\nu^{\text{th}}$ basic interval $J'_{-\nu}$.

The function X is called *general dispersion of the differential equations* (q), (Q) *with respect to the numbers* t_0, T_0 *and the linear representation* p. At the point t_0 it obviously takes on the value $T_0 : X(t_0) = T_0$.

It is obvious that the general dispersions we have just defined continuously depend on three arbitrary parameters: one is the arbitrarily chosen initial value T_0 and the two others are the parameters of the corresponding normalized linear representation p.

From the properties of the general dispersions, which can be deduced from the above construction, we shall only mention the following:

Let X be a general dispersion of the differential equations (q), (Q) and p the corresponding linear representation of the integral space r on the integral space R.

1. The set of values of the function X is the interval $J : X(j) = J$.

2. The function X is a phase function.

3. There holds $\operatorname{sgn} X' = \operatorname{sgn} \chi p$. Consequently, the function X increases or decreases according to whether $\chi p > 0$ or $\chi p < 0$.

4. The function X may be expressed by means of two phases $\alpha(t)$, $A(T)$ of the differential equations (q) or (Q) in the following way:

$$X(t) = A^{-1}\alpha(t).$$

Vice versa, the function $A^{-1}\alpha(t)$ formed by means of arbitrary phases α, A of the differential equations (q) or (Q) is a general dispersion of the differential equations (q), (Q).

Moreover, there holds the following theorem:

5. The general dispersions of the differential equations (q), (Q) are exactly the integrals of Kummer's differential equation (Qq).

6. The above considerations, and especially the constructive integration theory we have just outlined, hold for differential equations (q), (Q) in arbitrary (open) intervals j, J. Let us now restrict our considerations to the case $j = J = (-\infty, \infty)$ and, consequently, deal only with oscillatory differential equations (q), (Q) in the interval $j = (-\infty, \infty)$. That is exactly the case when algebraic elements enter the transformation theory and algebraic theorems, particularly those from the group theory, allow us to learn new facts about the integrals of Kummer's differential equation (Qq).

The prototype of the differential equations to be considered is the differential equation (-1), i.e. $y'' = -y$ in the interval $j = (-\infty, \infty)$. The integrals of this differential equation obviously have, in both directions, an infinite number of π-equidistantly displaced zeros, i.e. arranged so that the difference between any two neighbouring zeros of every integral is always the same, namely π. Hence it follows that the fundamental dispersion φ of the differential equation

(-1) is linear, $\varphi(t) = t + \pi$, and more generally, that the following formula holds for the central dispersion φ_ν:

$$\varphi_\nu(t) = t + \nu\pi \qquad (\nu = 0, \pm 1, \pm 2, \ldots).$$

If, furthermore, α is a phase of the differential equation (-1), then the Abelian relation (3) yields

$$\alpha(t + \pi) = \alpha(t) + \pi \,.\, \operatorname{sgn} \alpha'.$$

We see that all the phases of the differential equation (-1) are elementary.

7. Let us now consider the set $\mathfrak{G}$ formed of all the phase functions-unbounded on both sides, i.e. both from above and from below- in the interval $j = (-\infty, \infty)$. We see, first, that the function $\alpha\beta(t)$ composed of two arbitrary elements α, $\beta \in \mathfrak{G}$, is again an element of $\mathfrak{G}$. With regard to this, we shal now introduce, in the set $\mathfrak{G}$, a multiplication consisting in composing functions. For any two phase functions α, $\beta \in \mathfrak{G}$, the product $\alpha\beta$ is therefore understood to be the composed function $\alpha[\beta(t)]$. The set $\mathfrak{G}$ is obviously, with regard to this multiplication, a semi-group. The latter evidently contains the unit element 1, i.e. the phase function $\varepsilon(t) = t$; furthermore, there exists, to every element $\alpha(t) \in \mathfrak{G}$, the inverse element $\alpha^{-1}(t)$, namely the function $\alpha^{-1}(t)$ inverse to the function $\alpha(t)$. Thus we have shown that the set $\mathfrak{G}$, together with the considered multiplication, forms a group. Let us call it the *phase group* $\mathfrak{G}$.

The phase group $\mathfrak{G}$ consists, according to its definition, exactly of the phases of all the oscillatory differential equations (q) in the interval $j = (-\infty, \infty)$. To discern the phases of the singular differential equations (q), we shall now introduce, in the phase group $\mathfrak{G}$, a relation $\mathscr{R}$ in the following way: the relation $\alpha \mathscr{R} \beta$ expresses that the phase function β is a phase of the same differential equation (q) as the phase function α. It is easily verified that this relation $\mathscr{R}$ is reflexive, symmetrical and transitive and therefore forms an equivalence relation. Consequently, there exists, on the phase group $\mathfrak{G}$, a decomposition $\bar{R}$ such that any two elements α, $\beta \in \mathfrak{G}$ are phases of the same differential equation (q) if, and only if, they lie in the same element $\bar{a} \in \bar{R}$.

Let now $\mathfrak{E}$ be that element of $\bar{R}$ in which the unit element $\varepsilon(t) = t$ of $\mathfrak{G}$ is contained. The formula (2) shows that the phase function $\varepsilon(t)$ is a phase of the above differential equation (-1). Consequently, the element $\mathfrak{E} \in \bar{R}$ consists of all the phases of the differential equation (-1) and can be shown to be an undergroup of $\mathfrak{G} : \mathfrak{E} \subset \mathfrak{G}$. This undergroup will be called the *fundamental undergroup* of $\mathfrak{G}$. Furthermore, there holds the following theorem: ***The decomposition $\bar{R}$ coincides with the right-hand side class decomposition of the phase group $\mathfrak{G}$ with regard to $\mathfrak{E}$:***

$$\bar{R} = \mathfrak{G}/_r\mathfrak{E}.$$

The set of all the oscillatory differential equations (q) in the interval $j = (-\infty, \infty)$ therefore admits an one-one representation on the right-hand side class decomposition $\mathfrak{G}/_r\mathfrak{E}$, namely the one that makes correspond, to every differential equation (q), the element $\bar{q} \in \mathfrak{G}/_r\mathfrak{E}$ consisting of the phases of (q).

We shall now consider the undergroup of $\mathfrak{G}$ consisting of all the *elementary*, phase functions; let us denote it by $\mathfrak{H}$. Since, as we know, all the phases of the differential equation (-1) are elementary and form the fundamental undergroup $\mathfrak{E}$, we see, first, that $\mathfrak{H}$ is an overset of $\mathfrak{E}$. A further investigation which I cannot describe here in detail, shows that the elementary phase functions generally depend on arbitrary periodic functions with period π whereas the elements of $\mathfrak{E}$ depend only on three parameters. It follows that $\mathfrak{H}$ is a *proper* overset of $\mathfrak{E}$. It can, moreover, be shown that $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$. Hence there hold, between the groups $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{E}$, the relations:

$$\mathfrak{G} \supset \mathfrak{H} \supset \mathfrak{E}, \tag{5}$$

the overgroups as well as the subgroups in question being proper.

Let us now consider the righ-hand side class decomposition $\bar{H}$ of the phase group $\mathfrak{G}$ with respect to the subgroup $\mathfrak{H}$: $\bar{H} = \mathfrak{G}/_r\mathfrak{H}$.

First, the relations (5) yield the formula:

$$(\bar{H} =)\ \mathfrak{G}/_r\mathfrak{H} \geq \mathfrak{G}/_r\mathfrak{E}\ (= \bar{R}),$$

by which the decomposition $\bar{H}$ is a covering of $\bar{R}$, in other words, each element of $\bar{H}$ is the set-sum of some elements of $\bar{R}$ ([1]). Furthermore, the following theorem applies:

The elements of $\bar{R}$, contained in an arbitrary element $\mathfrak{H}\alpha \in \bar{H}$ $(\alpha \in \mathfrak{G})$, consist of phases of all the differential equations (q) whose fundamental dispersion φ is the same.

Finally, let us note that cardinal number of the set of the elements of $\bar{R}$ contained in an arbitrary element $\mathfrak{H}\alpha \in \bar{H}$ is always the same and equal to that of the continuum. Consequently: the cardinal number of the set of all the differential equations (q) whose fundamental dispersion φ is the same does not depend on the latter and is always equal to the cardinal number $\aleph$ of the continuum ([2]).

8. We shall now return to the general dispersions of two differential equations (q), (Q), namely to the integrals of Kummer's differential equation (Qq). As we have said above, every general dispersion X of the differential equations (q), (Q) transforms all the integrals Y of the differential equation (Q) to integrals y of the differential equations (q), the transformation being expressed by the first formula (1).

It can, first, be easily seen that the general dispersions of the differential equations (q), (Q) form elements of the phase group $\mathfrak{G}$. Indeed, every general dispersion X of (q), (Q) is, as we know, a phase function, whose set of values coincides with the interval of definition J of (Q). But since $J = (-\infty, \infty)$, the general dispersion X is a phase function unbounded on both sides and hence an element of the phase group $\mathfrak{G}$. It is, besides, easy to show that the general dispersion X is a phase of the differential equation (q_X), the relation between the functions q_X, q, Q being as follows:

$$q_X(t) = q(t) - [1 + Q(X)]\, X'^2(t).$$

We shall now determine the general dispersions of the differential equations (q), (Q) in the phase group $\mathfrak{G}$ by means of the following theorem:

Let α be a phase of the differential equation (q) *and A be one of* (Q). *The integral space $(X)_{(Qq)}$ of Kummer's differential equation* (Qq), *i.e. the set of all general dispersions of the differential equations* (q), (Q) *is given by the following formula:*

$$(X)_{(Qq)} = A^{-1}\mathfrak{E}\alpha; \tag{6}$$

$\mathfrak{E}$ naturally stands for the fundamental subgroup of $\mathfrak{G}$.

This theorem yields a number of results of which I shall only mention a few, so as not to spoil the general outline by too many details.

It may first be shown [by means of (6)], that the integral spaces $X_{(Qq)}$, $(X)_{(Q_1q_1)}$ of two arbitrary Kummer's differential equations (Qq), $(Q_{1}q_{1})$ have the same cardinal numbers and can be one-one represented on each other in the sense of formula:

$$X_1 = Z^{-1}Xz.$$

In this formula: $X \in (X)_{(Qq)}$, $X_1 \in (X)_{(Q_1q_1)}$, Z standing for a fixed integral of (QQ_1) and z for one of (qq_1).

Let us next consider the case of two *coinciding* differential equations (Q), (q) and Kummer's corresponding differential equation (qq). Every integral of this differential equation transforms, in the sense of formula (1), every integral Y of the differential equation (q) into an integral y of the same differential equation (q). From the above theorem it follows that:

The integral space $(X)_{(qq)}$ of Kummer's differential equation (qq) is the subgroup of $\mathfrak{G}$ conjugated with $\mathfrak{E}$:

$$(X)_{(qq)} = \alpha^{-1}\mathfrak{E}\alpha; \tag{7}$$

α naturally denotes an arbitrary phase of the differential equation (q).

Consequently:

The integral spaces $(X)_{(qq)}$, $(X)_{(q_1q_1)}$ of two arbitrary differential equations (qq), (q_1q_1) are isomorphous, the isomorphism being given by the following formula:

$$X_1 = z^{-1}Xz; \tag{8}$$

z denotes a fixed integral of the differential equation (qq_1).

A further consideration now permits to investigate, more closely, the algebraic structure of the integral space $(X)_{(qq)}$ of every differential equation (qq). One proceeds by first finding out the structure of the integral space of the differential equation $(-1, -1)$, i.e. of the group $\alpha^{-1}\mathfrak{E}\alpha$ $(\alpha \in \mathfrak{E})$ and then, by means by formula (8), passing to the differential equation (qq). One finds, particularly, that the increasing integrals contained in the integral space $(X)_{(qq)}$ of (qq) form a normal subgroup $\mathfrak{A}$ of index 2, the center of this subgroup coinciding with the infinite cyclic group formed by all the central dispersions φ_ν of the differential equation (q).

Herewith I have arrived at the conclusion of my lecture. Let me only add the remark, addressed particularly to those who take a special interest in the above considerations, that the latter form part of my book "Lineare Differentialtransformationen 2. Ordnung". This book will be published by the Deutscher Verlag der Wissenschaften, Berlin (DDR), in 1967.

BIBLIOGRAPHY

[1] Borůvka O.: *Grundlagen der Gruppoid- und Gruppentheorie. VEB Deutscher Verlag der Wiss., Berlin (DDR), 1960.*

[2] Borůvka O.: *Sur l'ensemble des équations différentielles linéaires ordinaires du deuxième ordre qui ont la même dispersion fondamentale. Bul. Instit. Polit. din Iaşi, IX (XIII), 1963, 11–20.*

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
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ON THE CONVERGENCE OF DIFFERENCE SCHEMES FOR CLASSICAL AND WEAK SOLUTIONS OF THE DIRICHLET PROBLEM[1)]

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I. Introduction

In the past much work has been done on convergence of sequences of solutions of difference analogs of the Dirichlet problem for second order uniformly elliptic equations and in particular Laplace's equation and Poisson's equation (c.f. FORSYTHE and WASOW [4], HUBBARD [5] and literature cited therein). Usually some rather restrictive conditions concerning smoothness of the solution of the continuous problem have been imposed in order to obtain the results. There have been, however, several studies of convergence properties under less stringent assumptions. Interesting results along these lines have been obtained for rectangular domains by WASOW [10], WALSH and YOUNG [9], and NITSCHE and NITSCHE [7] and for piecewise analytic boundaries with corners by LAASONEN [6]. Other important work has been done by CEA [3] who studied self-adjoint equations with bounded and measurable coefficients and obtained theorems on convergence of difference approximations to weak solutions in L_2.

In this paper some recent results of the author, the author and HUBBARD, and the author, HUBBARD and ZLÁMAL will be presented. Only indications of the proofs will be given since all of this work will be published elsewhere in complete detail. All the results share the common property that the smoothness conditions are much weaker than those classically assumed.

Although many of the results have been extended to equations with variable

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coefficients and various difference approximations, in order to minimize detail, I will consider only the Laplace operator and one of its simplest difference analogs.

II. Continuous and discrete problems.

Let R be a bounded region in N-dimensional Euclidean space with boundary δR. We shall, in the usual manner, consider the space as having been covered by hypercubes of side h and call the corners mesh points. Those mesh points in R shall be called R_h and the intersection of δR with the edges of the cubes will be called δR_h.

We shall denote the Laplace operator by Δ and the difference analog by Δ_h. The operator Δ_h will be defined for functions on $\bar{R}_h = R_h \cup \delta R_h$ as follows. When a point $x \in R_h$ has its $2N$ nearest neighbors also in $\bar{R}_h$ then Δ_h is the usual $2N + 1$ point approximation to Δ. We consider, at the remaining points of R_h, Δ_h to be defined as a locally $O(1)$ operator (bounded independent of h for smooth functions) and such that the matrix arrising from Δ_h operating on functions vanishing on δR_h is symmetric and of positive type. This is just one of the standard formulations which is globally second order for problems with smooth solutions.

We shall concern ourselves with approximating solutions to two problems. First, the solution u of the classical Dirichlet problem, which satisfies

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } R \\ u &= f \qquad \text{on } \delta R \end{aligned} \tag{2.1}$$

where f is a given continuous function on δR. That is, u is to be continuous on the closure, satisfy $\Delta u = 0$ in R and its restriction to δR should be equal to f. Conditions on δR in order for this problem to be solvable are, of course, well known.

The other problem to be discussed is a weak formulation of the problem for the inhomogeneous equation with homogeneous boundary values. More precisely we define the class

$$T = \{\varphi | \varphi \in C^{2+\alpha}(R) \cap C^0(\bar{R});\ \varphi(x) = 0,\ x \in \delta R;\ \Delta\varphi \in C_0^\alpha(R);\ \text{for some } \alpha\}.$$

In words, each member must have Hölder continuous second partial derivatives in R, be continuous on $\bar{R}$ (the closure of R), vanish on δR and its Laplacian must have compact support in R. Note that T contains the standard "test functions". We then want to consider the solution u, belonging to the real Banach space L_p, $1 \leq p < \frac{N}{N-2}$, of the equation

(2.2) $$\int_R u\,\Delta\varphi\,\mathrm{d}x = \int_R \varphi F\,\mathrm{d}x, \qquad \varphi \in T,$$

for a given $F \in L_1$. (If F and δR are sufficiently regular then the "weak solution" u will be the classical one, having zero boundary values.)

We shall consider the following approximating problems as analogs of (2.1) and (2.2) respectively:

(2.1h) $$\begin{aligned} \Delta_h u_h(x) &= 0, & x &\in R_h \\ u_h(x) &= f(x), & x &\in \delta R_h \end{aligned}$$

and

(2.2h) $$\begin{aligned} \Delta_h u_h(x) &= F_h(x), & x &\in R_h \\ u_h(x) &= 0, & x &\in \delta R_h. \end{aligned}$$

In (2.2h) F_h is defined as

$$F_h(x) = \frac{1}{h^N} \int_{S_h(x)} F(y)\,\mathrm{d}y$$

where $S_h(x)$ is the (normally oriented) hypercube of side h and center x, and F is extended to be zero outside R.

We shall in the sequel use the notation $\overset{\frown}{V(x)}$ to mean the extension of a function $V(x)$ defined on R_h, as constant over $S_h(x) \cap R$ and zero outside $R \cap [\bigcup_{x \in R_h} S_h(x)]$.

III. Some results on convergence.

We call a domain R which has no "unstable" boundary points a regular domain (c.f. Brelot [1]). [This condition admits quite general domains and in particular problem (2.1) is always solvable for such regions.]

Theorem 1. *Let R be a regular domain and u the solution of* (2.1). *Then if u_h is the solution of* (2.1h), *$\tilde{u}_h \to u$ uniformly on R as $h \to 0$.*

Although there are several theorems on convergence of difference approximations in the literature, it is not clear what the most general known result is for the classical Dirichlet problem. In any case this theorem gives a quite general result. The proof is quite simple and relies on an approximation theorem of the type studied by Walsh [8]. The appropriate theorem is given in Brelot [1]. To extend this theorem to more general second order operators an approximation theorem of Browder [2] is used. More restrictions must be placed on the domain in this case (he calls the resulting domains "firmly regular") but the result is still quite general.

The following existence and uniqueness theorem is easily proved.

Theorem 2. *Let R be a regular domain and let $F \in L_1$. Then there exists a unique $u \in L_p$, $1 \leq p < \frac{N}{N-2}$, such that (2.2) holds.*

Such a theorem can also be proved for operators with variable coefficients for "firmly regular" domains provided the coefficients and those of the formal adjoint satisfy some smoothness conditions.

From our point of view here, an interesting method of proof makes use of the difference approximations. We obtain the following convergence theorem as a byproduct.

Theorem 3. *Let R be a regular domain and $u \in L_p$, $1 \leq p < \frac{N}{N-2}$, be the solution of (2.2). If u_h is the solution of (2.2h) then $\tilde{u}_h \to u$, strongly in L_p, $1 \leq p < \frac{N}{N-2}$, as $h \to 0$.*

The proof involves showing that the functions $\tilde{u}_h$ are uniformly bounded in L_p for all $1 \leq p < \frac{N}{N-2}$. By the weak compactness of bounded sets in L_p, $1 < p < \infty$, we obtain a weak limit point which is then shown to satisfy (2.2). The uniqueness tells us that $\tilde{u}_h \to u$, weakly in L_p as $h \to 0$. An additional argument can then be employed to show the strong convergence. The extension of this theorem to operators with variable coefficients, though true, is not a triviality.

It is interesting to note here that even in two dimensions there are problems of the form (2.2) whose solutions are not continuous. This is true only for $F \in L_1$, and $F \notin L_p$, $p > 1$. If $F \in L_p$, $p > 1$ and $N = 2$ then u will be continuous.

IV. Some results on rates of convergence.

In this section we shall consider regions R whose boundaries are no worse than of class C^2 (or piecewise C^2). We have the following:

Theorem 4. *Let $\delta R \in C^2$ and suppose that the solution u of (2.1) is of class $C^{m+\lambda}(\bar{R})$, $m = 0, 1, \ldots$, $0 \leq \lambda \leq 1$. Then if u_h is the solution of (2.1h) it follows that*

$$(4.1) \qquad \max_{x \in R_h} |u_h(x) - u(x)| \leq K(\varepsilon) \begin{cases} h^{m+\lambda-\varepsilon} + h^{2-\varepsilon}; & m = 0, 1, 2 \\ h^2; & m \geq 3 \end{cases}$$

where ε is an arbitrary positive number and $K(\varepsilon)$ depends on ε and u but not on h.

The proof of this theorem is based on some delicate estimates of the behavior of the discrete Green's function.

It should be pointed out here that an order h^2 estimate is essentially achieved when $u \in C^{2+0}(\bar{R})$. Previous results required that $u \in C^{4+0}(\bar{R})$ in order to obtain a second order error estimate. The present theorem yields a great deal more information than other theorems on this subject. The author has subsequently become aware of a paper of Bahualov (Vestnik Moskov. Univ. Meh. Astronom. Fiz. Chem. (1959) pp. 171—195) which essentially contains this result.

In the important case $N = 2$ the results are better, in that piecewise C^2 boundaries are treated. We have

Theorem 5. *Let $N = 2$ and $\delta R \in C^2$ piecewise with no reentrant cusps, i.e, R is composed of a finite number of C^2 arcs meeting at (interior) angles π/α_i, $i = 1, \ldots, k$, $\alpha_i > 1/2$. Then (4.1) holds.*

We now consider the case of problem (2.2). It is possible to obtain rate of convergence estimates even assuming no more than that $F \in L_1$. In this case we obtain only interior L_p estimates.

Theorem 6. *Let $\delta R \in C^2$ and u and u_h be solutions of* (2.2) *and* (2.2h) *for a given $F \in L_1$. Then if $\Psi \in C_0^\infty(R)$ the following estimate holds for $N = 2$.*

$$(4.2) \qquad ||(\tilde{u}_h - u)\,\Psi||_{L_p} \leqq K(p, \Psi)\,||F||_L \begin{cases} h; & 1 \leq p < \dfrac{N}{N-1} \\ h|\ln h|; & p = \dfrac{N}{N-1} \\ h^{\frac{2}{p}};\ 2 < p < \infty \end{cases}$$

where $K(p, \Psi)$ is a constant depending on p and Ψ but not on h. The notation $||\cdot||_{L_p}$ is just the usual L_p-norm, $1 \leq p < \infty$.

This result is obtained from a careful estimation of the difference between the discrete and continuous Green's functions. Theorem 4 is used in the derivation of this estimate. Since the analysis is based on the knowledge of the discrete and continuous fundamental solutions, the result only has been obtained for the Laplace operator. A similar result should be true in the more general case.

Other results of this type have been obtained. For example, when F is Hölder continuous with exponent α the estimates go up to $h^{1+\alpha}$ on compact subsets. Also if F is smooth on an open subset Ω of R, local maximum norm estimates can be obtained on compact subsets of Ω. This type of result shows that the local properties of elliptic operators are carried over to local convergence properties of corresponding difference approximations.

Finally, we consider the case where more precise knowledge of F is given. In particular we suppose that F is smooth, except at the origin O, (an arbitrary point of $\bar{R}$) and for simplicity that δR is smooth. For convenience we suppose that O lies at the center of a mesh hypercube for every h. We also prefer here to state the hypotheses on the solution u itself, rather than as conditions on F.

Theorem 7. *Let u be the solution of* (2.2) *and F be such that*

$$u \in C^{4+0}(\bar{R} - 0)$$

$$(4.3) \qquad |D^k u(x)| \leq K \begin{cases} 1; & k \leq m \\ |x|^{m+\lambda-k}; & m+1 \leq k \leq 4, \end{cases}$$

$k = 0, 1, \ldots, 4$, *where $|x|$ is the distance from x to 0 and D^k stands for an arbitrary partial derivative of order k. In* (4.3) *m is an integer (not necessarily positive) less than or equal to 3, $0 < \lambda \leq 1$ and $m + \lambda > 2 - N$. Then if u_h is the solution of* (2.2h) *we have the estimates, for $x \in R_h$,*

$$(4.4) \quad |u_h(x) - u(x)| \leq K(\varepsilon) \begin{cases} h^{m+\lambda+N-2-\varepsilon}|x|^{\varepsilon+2-N} & 2 - N < m + \lambda \leq 4 - N \\ h^2|x|^{m+\lambda-2}, & 4 - N < m + \lambda < 2 \\ h^2, & 2 < m + \lambda \end{cases}$$

where ε is an arbitrary positive number and $K(\varepsilon)$ depends on ε but not on h. If $N > 3$ then the last inequality is valid for $2 \leq m + \lambda$.

The proof of this result is again based on the Green's function method. It involves the construction of certain majorants and the development of some new discrete inequalities suggested by known continuous ones.

Again it should be pointed out that this type of result displays the local effect of singularities on the convergence rate of difference analogs of elliptic problems. Note that we still get convergence away from the origin for any function whose singularity is not as bad as that of the fundamental solution and quadratic convergence even allowing bad behavior at the origin.

BIBLIOGRAPHY

[1] Brelot, M.: *Sur l'approximation et la convergence dans la theorie des fonctions harmoniques ou holomorphes. Bull. Soc. Math. France 73, 55–70 (1945).*

[2] Browder, F. E.: *Approximation by solutions of partial differential equations. Am. J. Math. 84, 134–160 (1962).*

[3] Céa, J.: *Sur l'approximation des problemes aux limites elliptiques. Compte rendus 254, 1729–1731 (1962).*

[4] FORSYTHE, G. AND WASOW, W.: *Finite difference methods for partial differential equations. Wiley, N. Y. (1960).*

[5] HUBBARD, B. E.: *Remarks on the order of convergence in the discrete Dirichlet problem. Numerical Solution of Partial Differential Equations (edited by J. H. Bramble) Academic Press, New York and London, 21—23 (1966).*

[6] LAASONEN, P.: *On the degree of convergence of discrete approximations for the solutions of the Dirichlet problem. Ann. Acad. Sci. Fenn. Ser. A. 246, 1—19 (1957).*

[7] NITSCHE, J. AND NITSCHE, J. C. C.: *Error estimates for the numerical solution of elliptic differential equations. Arch. Rar. Mech. Anal. 5, 293—306 (1960).*

[8] WALSH, J. L.: *The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions. Bull. Am. Math. Soc. 35, 499—544 (1929).*

[9] WALSH, J. L. AND YOUNG, D.: *On the degree of convergence of solutions of difference equations to the solution of the Dirichlet problem. J. Math. Phys. 33, 80—93 (1954).*

[10] WASOW, W.: *On the truncation error in the solution of Laplace's equation by finite differences. J. Res. Nat. Bur. Standards, 48, 345—348 (1952).*

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
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NONLINEAR FUNCTIONAL ANALYSIS AND NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS.[1]

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Introduction: The two basic approaches to fundamentally nonlinear problems in partial differential equations are on the one hand, variational methods (the direct method of the calculus of variations, the MORSE theory, and the LUSTERNIK—SCHNIRELMAN theory) and on the other hand, the theory of nonlinear operators in Banach spaces (the SCHAUDER fixed point theorem, the LERAY—SCHAUDER theory of the degree for compact displacements). In the past few years, we have seen a merging of these two lines of ideas in their applications to partial differential equations through the theory of monotone operators from a Banach space X to its conjugate space X^*, i.e. operators T such that for all u and v in the domain of T, we have

$$(Tu - Tv,\ u - v) \approx 0,$$

(where (w, v) denotes the pairing between the functional w and the element v). On the one hand, every operator T which is the derivative (or subderivative) of a convex functional on X is monotone, and on the other hand, the consideration of monotone (or quasi-monotone, or semi-monotone) operator equations falls within the framework of nonlinear functional analysis, i.e. the study of nonlinear operators and nonlinear operator equations in Banach spaces.

It is our object in the present paper to give a survey of some recent work by the writer on this type of functional analysis and its applications to various types of abstract differential equations in Hilbert and Banach spaces. We refer the reader to an earlier survey ([6]) for a development of the basic ideas in the application of monotone operators to such topics as:

(1) The existence of solutions for variational boundary problems for nonlinear elliptic differential operators of the form

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$$A(u) = \sum_{|\alpha| \leq m} D^{\alpha}(A_{\alpha}(x, u, \ldots, D^m u)).$$

(2) The corresponding existence theorems for parabolic operators of the form:

$$\frac{\partial u}{\partial t} + A_t(u) = 0.$$

(3) Nonlinear equations of evolution in Hilbert and Banach spaces arising from initial-boundary value problems of various types.

Section **1** below presents the results of [14] on nonlinear equations of evolution in Hilbert space and the generalized method of steepest descent for monotone operators in Hilbert space. Section 2 develops the results of the extension of this theory as carried through in [16] to Banach spaces, both for monotone operators T from a Banach space X to its dual space X^* and for J-monotone operators T from a Banach space X to X. Section **3** discusses the general method developed in [15] for proving the existence of periodic solutions for classes of nonlinear equations of evolution in infinite dimensional spaces comparable to the classes of differential equations treated in Sections **1** and **2**.

We remark that the method of steepest descent and its generalizations have close links with the ideas of the calculus of variations, and the results presented below are connected with extensions of the results given in BROWDER [7] on the application of the Lusternik—Schnirelman principle to the proof of the existence of infinitely many eigenfunctions for nonlinear elliptic eigenvalue problems.

Section 1: Let H be a real Hilbert space, T an operator (generally nonlinear) with domain and range in H. We consider three inter-related problems concerning such operators T:

(I) The existence for a given w in H of solutions u of the equation $Tu = w$.

(II) The existence for a given u_0 of solutions of the nonlinear equation of evolution

$$\frac{du}{dt} = -T(u), \qquad t \geq 0,$$

with $u(0) = u_0$.

(III) For a suitably chosen perturbation term $R(t, u)$ which converges to zero as $t \to +\infty$, the convergence as $t \to +\infty$ of solutions of the equation

$$\frac{du}{dt} = -T(u) + R(t, u)$$

to solutions v_0 of the stationary equation $Tv_0 = 0$.

We denote this last problem as that of the generalized method of steepest descent for the operator T.

We recall that an operator T is said to be hemicontinuous if it is continuous from each line segment in $D(T)$ to the weak topology of H.

Theorem 1.1: *Let T be a monotone operator in the Hilbert space H such that either: (i) $D(T) = H$, and T maps H hemicontinuously into H; or (ii) $T = L + T_0$ where L is a maximal accretive closed linear operator in H and T_0 is a hemicontinuous monotone mapping of H into H which maps bounded subsets into bounded subsets.*

Suppose that there exists $R > 0$ such that for u in $D(T)$ with $||u|| = R$, $(Tu, u) \geq 0$.

Then the set of solutions u of the equation $Tu = 0$ is a nonempty closed convex subset K of H.

Theorem 1.2: *Let T be a hemicontinuous locally bounded operator from H to H such that for a fixed constant c in R^1 and all u and v of H,*

$$(Tu - Tv, u - v) \leq c||u - v||^2.$$

Then there exists one and only one strongly continuous, veakly once-differentiable function u from $R^+ = \{t | \, t \in R^1, t \geq 0\}$ to H such that u is a solution of the differential equation

$$\frac{du}{dt} = Tu, \qquad t \geq 0,$$

with the initial condition $u(0) = u_0$, for a given u_0 in H.

In addition, if T is continuous, then u is strongly C^1.

Theorem 1.3: *Let H be a Hilbert space, f a mapping of $R^+ \times H$ into H such that the following three conditions are satisfied:*

(1) *f is locally bounded (i.e. bounded on some neighborhood of each point of $R^+ \times H$). For each fixed t in R^+, $f(t, \cdot)$ is a hemicontinuous mapping of H into H. For each fixed u in H, $f(\cdot, u)$ is continuous from R^+ to the weak topology of H.*

(2) *There exists a continuous function c from R^+ to R^1 such that for all t in R^+ and all u and v in H:*

$$(f(t, u) - f(t, v), u - v) \leq c(t) \, ||u - v||^2.$$

(3) *For each u in H, $f(t, u)$ is weakly once differentiable from R^+ to H, and there exists a continuous function q from $R^+ \times R^+$ to R^+ such that for all u and t:*

$$\left\| \left(\frac{\partial}{\partial t}\right) f(t, u) \right\| \leq q(t, ||u||).$$

Then for any u_0 in H, there exists one and only one function u from R^+ to H

which is weakly continuously once-differentiable and which satisfies the differential equation

$$\frac{du}{dt} = f(t, u), \qquad t \geq 0,$$

and the initial condition $u(0) = u_0$.

Theorems (**1.2**) and (**1.3**) are sharpenings (under more restrictive hypotheses on the dependence of f on t) of an existence theorem given in BROWDER [3] with the additional assumption that $f(t, u)$ is bounded for t and u ranging through a bounded set of $R^+ \times H$. The interest of this strenghthening lies primarily in the fact that it is obtained through a new a priori estimate for solutions of these equations of evolution from which one obtains much stronger control over the solutions of these equations. This is brought out more clearly in the following theorems on nonlinear evolution equations containing an unbounded linear operator L.

Definition: *Let H be a Hilbert space, $\{L(t) | t \in R^+\}$ a family of closed, densely defined linear operators in H, T_0 a mapping of $R^+ \times H$ into H. If we set $T_t(u) = L(t)\, u + T_0(t, u)$, then by a sharp solution u on R^+ of the equation of evolution*

$$\frac{du}{dt} = T_t(u), \qquad t \geq 0,$$

we mean a strongly continuous function u from R^+ to H with u weakly once continuously differentiable from R^+ to H, $u(t)$ in the domain of $L(t)$ for each t in R^+ and with $L(t)\, u(t)$ weakly continuous from R^+ to H, and such that for all t in R^+,

$$\frac{du}{dt}(t) = L(t)\, u(t) + T_0(t, u(t)).$$

Theorem 1.4: *Let H be a Hilbert space, L a maximal dissipative linear operator in H, T_0 a mapping of $R^+ \times H$ into H which maps bounded sets into bounded sets. Suppose that T_0 satisfies the following three conditions:*

(1) *For each fixed t in R^+, $f(t, \cdot)$ is a hemicontinuous mapping of H into H. For each fixed u in H, $f(\cdot, u)$ is continuous from R^+ to the weak topology of H.*

(2) *There exists a continuous function c from R^+ to R^1 such that for all t in R^+ and all u and v in H:*

$$(T_0(t, u) - T_0(t, v), u - v) \leq c(t)\, ||u - v||^2.$$

(3) *For each fixed u in H, $f(t, u)$ is weakly once-differentiable on R^+ in t, and there exists a continuous function q from $R^+ \times R^+$ to R^+ such that for all t in R^+ and all u in H,*

$$\left\| \left(\frac{\partial}{\partial t} T_0 \right)(t, u) \right\| \leq q(t, ||u||).$$

Then for each u_0 in $D(L)$, there exists one and only sharp solution u on R^+ of the equation of evolution

$$\frac{du}{dt} = Lu + T_0(t, u), \qquad t \geq 0,$$

with $u(0) = u_0$.

As an illustration of the basic a priori bounds from which these results are derived, we have the following:

Theorem 1.5: *Let L and T_0 satisfy the conditions of Theorem 1.4 and let u be a sharp solution of the differential equation*

$$\frac{du}{dt} = Lu + T_0(t, u).$$

Let $C(t) = \int_0^t c(s)\, ds$. *Then:*

(I) *For all t in R^+,*

$$||u(t)|| \leq \exp(C(t))\, ||u(0)|| + \int_0^t \exp(C(t) - C(s))\, ||T_0(s, 0)||\, ds.$$

(II) *If $q(t, r)$ is nondecreasing in r (as we may always assume) and if* $||u(s)|| \leq$ $\leq M(s)$ *for all s in R^+, then*

$$\left|\left| \frac{du}{dt}(t) \right|\right| \leq \exp(C(t))\, ||T_0(0, u(0)) + Lu(0)|| + \int_0^t \exp(C(t) - C(s))\, q(s, M(s))\, ds.$$

Combining these apriori estimates with the corresponding existence theorems, we obtain the following general result on the generalized method of steepest descent for monotone operators in Hilbert spaces:

Theorem 1.6: *Let H be a Hilbert space, T a monotone operator with domain in H and values in H which lies in one of the two following classes:*

(*a*) *T is a locally bounded hemicontinuous mapping of H into H.*

(*b*) *$T = L + T_0$, where L is a maximal accretive linear operator in H, and T_0 is a hemicontinuous monotone mapping of H into H which carries bounded subsets into bounded subsets.*

Suppose that there exists $R > 0$ such that $(Tu, u) \geq 0$ for all u in $D(T)$ with $||u|| = R$.

Let c be a C^1 function from R^+ to R^+ which is non-increasing and such that $c(t) \to 0$ *as* $t \to +\infty$, $\int_0^\infty c(s)\, ds = +\infty$.

Let v_0 be any element of H with $||v_0|| < R$, u_0 any element of $D(L)$ with $||u_0|| \leq R$.

Then:

(1) *The equation of evolution*

$$\frac{du}{dt} = -T(u) - c(t)\,\{u - v_0\}, \qquad t \geq 0,$$

has one and only one sharp solution u on R^+ with $u(0) = u_0$.

(2) *As $t \to +\infty$, this solution converges strongly in H to a solution w_0 of the equation $Tw = 0$. This limit is characterized as that solution of $Tw = 0$ in the ball $B_R = \{u|\ ||u|| \leq R\}$ closest to the given element v_0.*

Section 2: We now turn to the generalizations and extensions of the results of Section 1 to more general Banach spaces than Hilbert space, as given by the writer in BROWDER [16]. These extensions are of two kinds:

(1) The consideration of monotone operators T from X to X^*.

(2) The consideration of J-monotone operators T from X to X, for a duality mapping J of X into X^*.

We shall consider case (1) first.

Definition: *Let X be a Banach space, with $X \subset H \subset X^*$ for a Hilbert space H, in the sense that we are given continuous linear injections of each space on a dense subset of its successor and the pairing between two elements w and u of H with w in X and u in X^* coincides with the H inner product.*

Let f be a mapping of $R^+ \times X$ into X^*.

Then a function u from R^+ to X is said to be a sharp solution on R^+ of the equation of evolution

$$\frac{du}{dt} = f(t, u), \qquad t \geq 0$$

if u satisfies the following three conditions:

(1) *u is continuous from R^+ to the weak topology of X.*

(2) *As a function from R^+ to H, u is continuous to the strong topology of H and satisfies a Lipschitz condition in H on each finite interval. u is strongly once-differentiable in H a.e. on R^+ and $\left\|\frac{du}{dt}(t)\right\|_H$ is essentially bounded on each finite interval.*

(3) *The differential equation*

$$\frac{du}{dt}(t) = f(t, u(t))$$

holds a.e. on R^+.

To abbreviate these hypotheses, we use the following notation: If Y is a Banach space, $C_s^0(R^+, Y)$ and $C_w^0(R^+, Y)$ denote the functions from R^+ to Y continuous to the strong and weak topologies of Y, respectively; $C_s^1(R^+, Y)$

and $C^1_w(R^+, Y)$ denote the continuously once-differentiable functions from R^+ to the strong and weak topologies of Y, respectively; $L^\infty_{loc}(R^+, Y)$ is the family of strongly measurable functions from R^+ to Y whose norm is bounded on each finite interval; $\frac{dv}{dt}$ denotes the distribution derivative. Then the assumptions of the above definition may be rewritten:

(1) $u \in C^0_w(R^+, X)$; (2) $u \in C^0_s(R^+, H)$, and $\frac{du}{dt} \in L^\infty_{loc}(R^+, H)$.

(3) $\frac{du}{dt} = f(t, u(t))$, on R^+.

Theorem 2.1: *Let X be a reflexive separable Banach space with $X \subset H \subset X^*$ for a given Hilbert space H. Let T be a hemicontinuous monotone mapping of X into X^* which carries bounded sets of X into bounded sets in X^*. Suppose that $(Tu, u) \to +\infty$ as $||u||_X \to \infty$.*

Then for each u_0 in X such that $T(u_0)$ lies in H, there exists one and only sharp solution of the differential equation

$$\frac{du}{dt} = f(t, u), \qquad t \geq 0,$$

on R^+ such that $u(0) = u_0$.

We omit the detailed statement of the corresponding time-dependent result, and pass directly to the generalized method of steepest descent:

Theorem 2.2: *Let X be a reflexive separable Banach space with $X \subset H \subset X^*$ for a given Hilbert space H. Let T be a hemicontinuous monotone mapping of X into X^* such that T maps bounded subsets of X into bounded subsets of X^* while $(Tu, u) \to +\infty$ as $||u||_X \to +\infty$.*

Let c be a C^1 non-increasing function from R^+ to R^+ such that $c(t) \to 0$ as $t \to +\infty$, $\int_0^\infty c(s)\, ds = +\infty$. Let v_0 be an arbitrary element of H, u_0 any element of X such that $T(u_0)$ lies in H.

Then:

(a) The differential equation

$$\frac{du}{dt} = -T(u) - c(t)\{u - v_0\}, \qquad t \geq 0,$$

has one and only one sharp solution u on R^+ with $u(0) = u_0$.

(b) As $t \to +\infty$, $u(t)$ converges weakly in X to a solution w_0 of the equation $Tw = 0$. Moreover, $u(t)$ converges strongly in H to w_0. The limit element w_0 is uniquely characterized as the solution of the equation $Tw = 0$ closest to v_0 in H.

The existence theorems, Theorem 2.1 and its time-dependent generalization which we have not stated, apply directly to the treatment of initial boundary value problems of parabolic type ([3]) and (especially for the time-independent case) give a significant strengthening of the parabolic existence theorems under hypotheses on T which are essentially weaker than those considered in the treatment of *variational* rather than *sharp* solutions. The previous hypotheses (though they can be put in a much more general-looking and untransparent form) have the same force essentially as the following simple assumption:

There exists an exponent p with $1 < p < +\infty$ such that for suitable positive constants c and c_0,

$$||Tu||_{X^*} \leq c\{||u||_X^{p-1} + 1\};$$
$$(Tu, u) \geq c_0||u||_X^p - c.$$

In Theorem (2.1), however, we need only assume that T maps bounded sets of X into bounded sets in X^* and that $(Tu, u) \to +\infty$ as $||u||_X \to +\infty$.

Similar considerations apply to the existence theorem which we derive for the abstract wave equation of the form

$$u_{tt} = -Au - T(u_t) - S(u)$$

where T and S are mappings of X into X^*, and u_t, u_{tt} denote the first and second derivatives of u with respect to t. We introduce a class of *sharp* solutions as follows:

Definition: *Let X be a Banach space, H a Hilbert space with $X \subset H \subset X^*$. Let A be a non-negative closed self-adjoint operator in H, $A^{\frac{1}{2}}$ its non-negative square root. Let T and S be mappings of X into X^*.*

Then a function u from R^+ to X is said to be a sharp solution on R^+ of the differential equation

$$u_{tt} = -Au - T(u_t) - S(u)$$

if u satisfies the following five conditions:

(1) *u lies in $C_w^1(R^+, X)$ and in $C_s^1(R^+, H)$.*

(2) *u_{tt} lies in $L_{\mathrm{loc}}^\infty(R^+, H)$.*

(3) *For each t in R^+, $u(t)$ and $u_t(t)$ lie in the domain of $A^{\frac{1}{2}}$, and $A^{\frac{1}{2}}u$ lies in $C_w^1(R^+, H)$, $A^{\frac{1}{2}}u_t$ lies in $C_w^0(R^+, H)$.*

(4) *For each t in R^+, $Au(t)$ lies in X^*, i.e. there exists $y(t)$ in X^* which we denote by $Au(t)$ such that for all w in $D(A^{\frac{1}{2}}) \cap X$, we have*

$$(A^{\frac{1}{2}}u(t), A^{\frac{1}{2}}w) = (y(t), w).$$

Furthermore Au lies in $C_w^0(R^+, X^*)$.

(5) *For almost all t in R^+,*

$$u_{tt}(t) = -Au(t) - T(u_t(t)) - S(u(t)).$$

Theorem 2.3: *Let X be a reflexive separable Banach space, H a Hilbert space with $X \subset H \subset X^*$. Let A be a non-negative closed self-adjoint linear operator in H such that $D(A^{\frac{1}{2}}) \cap X$ is dense both in X and $D(A^{\frac{1}{2}})$, where the latter is given the graph norm. Let T be a hemicontinuous mapping of X into X^* which maps bounded sets into bounded sets, S a Lipschitz mapping of H into H, (where both T and S may be nonlinear). Suppose that $(Tu, u) \to +\infty$ as $||u||_X \to +\infty$ and that T is monotone.*

Then for each u_0 in $D(A^{\frac{1}{2}}) \cap X$ and for each u_1 in $D(A^{\frac{1}{2}}) \cap X$ such that $T(u_1)$ lies in H, there exists one and only one sharp solution u on R^+ of the differential equation

$$u_{tt} = -Au - T(u_t) - S(u)$$

which satisfies the initial conditions

$$u(0) = u_0, \qquad u_t(0) = u_1.$$

Abstract wave equations of the above form with S linear but with time-dependent terms were studied by LIONS and STRAUSS [24] who obtained *variational* solutions for similar initial value problems but under growth conditions for T like those disccused above in connection with the first order case.

We now turn from operators T mapping X into X^* to the consideration of J-monotone operators T from X to X. These are defined as follows:

Definition: *Let X be a Banach space, q a continuous strictly increasing function from R^+ to R^+ such that $q(0) = 0$ and $q(r) \to +\infty$ as $r \to +\infty$. Then a mapping J of X into X^* is said to be a duality mapping with gauge function q if the following conditions hold for all u in X:*

$$(Ju, u) = ||u|| \,.\, ||Ju||; \qquad ||Ju|| = q(||u||).$$

Definition: *Let X be a Banach space, J a duality mapping of X into X^*. If T is a mapping with domain $D(T)$ in X and with range in X, then T is said to be J-monotone if for all u and v in $D(T)$,*

$$(T(u) - T(v), J(u - v)) \geq 0.$$

The definition of J-monotone mapping was first given and applied in BROWDER [10] and results on J-monotone mappings have been established in BROWDER [15] and BROWDER—FIGUEIREDO [19]. The concept of J-monotonicity is intimately linked to that of non-expansiveness of a mapping from X to X, where U is said to be *non-expansive* if for all u and v of X,

$$||U(u) - U(v)|| \leq ||u - v||.$$

For every non-expansive mapping U, $T = I - U$ is J-monotone for any duality mapping J of X into X^*. On the other hand, if the differential equation

$$\frac{du}{dt} = -T(u), \qquad t \geq 0,$$

has a solution u on R^+ with $u(0) = u_0$ for every u_0 in $D(T)$ and if we set $U(t)\, u_0 = u(t)$, then the non-expansiveness of all the operators $U(t)$ is equivalent to the J-monotonicity of T. In particular, if L is a closed densely defined linear operator in X, then L is the generator of a C_0 semigroup of nonexpansive linear operators $U(t)$, (i.e. $||U(t)|| \leq 1$, $t > 0$) if and only if L satisfies both of the following conditions:

(1) $(-L)$ is J-monotone for any duality mapping J of X into X^*.

(2) $(-L + I)$ has all of X as its range.

We present results on J-monotone operators T of two types. First, with mild regularity assumptions on T and very weak assumptions on the space X. Second, with weak assumptions on the operator T (comparable to those in the Hilbert space case) but with fairly drastic restrictions on the Banach space X.

Definition: *A mapping T of X into X is said to be weakly once-differentiable at u_0 in X if there exists a bounded linear operator B such that for all x in X and all y in X^*,*

$$(T(u_0 + hx), y) = (T(u_0), y) + h(Bx, y) + R_{x,y}(h)$$

where for each fixed y in X^*,

$$h^{-1}R_{x,y}(h) \to 0, \qquad \text{as} \qquad h \to 0,$$

uniformly in x on the unit ball of X.

Theorem 2.4: *Let X be a Banach space with a continuous duality mapping J of X into X^*. Let T be a nonlinear mapping of X into X which is weakly once differentiable and locally Lipschitzian at each point of X. Suppose that there exists a constant c in R^1 such that for all u and v in X:*

$$(T(u) - T(v), J(u - v)) \leq c||u - v|| \,.\, ||J(u - v)||.$$

Then for each u_0 in X, there exists one and only one strongly C^1 function u from R^+ to X which satisfies the differential equation

$$\frac{du}{dt} = T(u), \qquad t \geq 0,$$

with $u(0) = u_0$.

Theorem 2.5: *Let X be a Banach space with a continuous duality mapping J of X into X^*. Let L be a closed densely defined linear operator in L which is the infinitesimal generator of a C_0 semigroup of nonexpansive linear operators in X. Let T_0 be a nonlinear mapping of X into X which is weakly once-differentiable and locally Lipschitzian in a neighborhood of each point of $D(L)$, and such* *that T_0 maps bounded subsets of X into bounded subsets of X. Suppose also that there exists a constant c in R^1 such that for all u and v in X,*

$$(T_0(u) - T_0(v), J(u - v)) \leq c\|u - v\| \,.\, \|J(u - v)\|.$$

Then for each u_0 in $D(L)$, there exists one and only one strongly C^1 function u from R^+ to X with $u(t)$ in $D(L)$ for all tvin R^+ such that u satisfies the differential equation

$$\frac{du}{dt} = Lu + T_0(u), \qquad t \geq 0,$$

and the initial condition $u(0) = u_0$.

Theorem 2.6: *Let X be a Banach space with a continuous duality mapping J of X into X^*. Let L be a closed linear operator in X which is the infinitesimal generator of a C_0 semigroup of nonexpansive operators in X. Let T_0 be a mapping of $R^+ \times X$ into X which maps bounded subsets of $R^+ \times X$ into bounded subsets of X. Suppose that for each fixed t in R^+, $T_0(t, u)$ is weakly once-differentiable and locally Lipschitzian on a neighborhood of each point of $D(L)$. Suppose further that both of the following conditions are satisfied:*

(a) There exists a continuous function c from R^+ to R^1 such that for all u and v in $D(L)$ and all t in R^+,

$$(T_0(t, u) - T_0(t, v), J(u - v)) \leq c(t)\, \|u - v\| \,.\, \|J(u - v)\|.$$

(b) For each fixed u in $D(L)$, $T_0(t, u)$ is weakly once differentiable in t on R^+. There exist two continuous functions k: $R^+ \to R^+$ and q: $R^+ \times R^+ \to R^+$ such that for all u in $D(L)$ and all t in R^+,

$$\left\| \left(\frac{\partial}{\partial t} T_0\right)(t, u) \right\| \leq k(t)\, \|Lu\| + q(t, \|u\|).$$

Then for each u_0 in $D(L)$, there exists one and only one strongly C^1 function u from R^+ to X with $u(t)$ lying in $D(L)$ for all t in R^+ such that u is a solution of the differential equation

$$\frac{du}{dt} = Lu + T_0(t, u), \qquad t \geq 0,$$

and the initial condition $u(0) = u_0$ holds.

For this case, we obtain the following variant of the method of steepest descent:

Theorem 2.7: *Let X be a Banach space with a continuous duality mapping J of X into X^*. Let T be a mapping with domain and range in X which lies in one of the following two classes:*

(1) *T is a J-monotone mapping of X into X which is weakly once-differentiable and locally Lipschitzian at each point of X.*

(2) $T = -L + T_0$, *where* L *is a closed linear operator in* X *which is the infinitesimal generator of a* C_0 *semigroup of nonexpansive operators in* X, T_0 *is a nonlinear J-monotone mapping of X into* X *which carries bounded sets into bounded sets and such that* T_0 *is weakly once-differentiable and locally Lipschitzian on a neighborhood of each point of* $D(L)$.

Suppose that there exists $R > 0$ *such that for* u *in* $D(T)$ *with* $||u|| = R$, $(Tu, Ju) \geq 0$.

Let c *be a nonincreasing* C^1 *function from* R^+ *to* R^+ *with* $c(t) \to 0$ *as* $t \to +\infty$, $\int_0^\infty c(s)\, ds = +\infty$. *Let* u_0 *be any element of* $D(T)$ *with* $||u_0|| \leq R$, *and let* v_0 *be any element of* X *with* $||v_0|| < R$.

Then:

(*a*) *The differential equation*

$$\frac{du}{dt} = -T(u) - c(t)\,\{u - v_0\}$$

has one and only one solution u *on* R^+ *with* $u(0) = u_0$.

(*b*) *For each such solution* u *on* R^+, *we have*

$$||T(u(t))|| \to 0$$

as $t \to +\infty$.

(*c*) *Suppose that in addition to the preceding hypotheses,* T *satisfies the following condition*:

(*C*) *For each* $M > 0$, *there exists a compact mapping* C *of* X *into* X *and a continuous strictly increasing function* p *from* R^+ *to* R^+ *with* $p(0) = 0$ *such that for all* u *and* v *of* $D(T)$ *with* $||u|| \leq M$, $||v|| \leq M$,

$$||T(u) - T(v)|| \geq p(||u - v||) - ||C(u) - C(v)||.$$

Then $u(t)$ *converges strongly in* X *as* $t \to +\infty$ *to a solution* v_0 *of the equation* $Tv_0 = 0$.

As a consequence of Theorem 2.7, we have the following existence theorem for solution of the equation $Tv = w$.

Theorem 2.8: *Let* X *be a Banach space with a continuous duality mapping* J *of* X *into* X^*, *and let* T *be a J-monotone mapping which is in one the two classes* (1) *or* (2) *of Theorem* (2.7). *Then*:

(1) *Let* B_R *be the closed ball of radius* $R > 0$ *about the origin in* X, S_R *its boundary. If for some* $R > 0$, $(Tu, Ju) \geq 0$ *for all* u *in* $D(T) \cap S_R$, *then* 0 *lies in the strong closure of* $T(B_R \cap D(T))$. *In particular, if* $T(B_R \cap D(T))$ *is closed in* X, *then the equation* $Tv = 0$ *has a solution* v_0 *with* $||v_0|| \leq R$.

(2) *Suppose that T is J-coercive, i.e.*

$$(Tu, Ju)/||Ju|| \to +\infty, \qquad (||u|| \to +\infty).$$

Then the range of T is dense in X.

(3) *If T is J-coercive and satisfies condition (C) of part (c) of Theorem* **2.7**, *then the range of T is the whole of X.*

(4) *If X is reflexive and T is J-coercive as well as demiclosed (i.e. for any weakly convergent sequence $u_j \to u$ with Tu_j converging strongly to w, u lies in $D(T)$ and $Tu = w$), then the range of T is all of X.*

(5) *If X is strictly convex and T is J-coercive, the set*

$$K_w = \{v | v \in D(T), Tv = w\}$$

, for a fixed w in X, is a closed convex subset of X.

We now restrict the class of Banach spaces X, and thereby can eliminate the regularity conditions imposed upon T in the preceding results. Our basic hypothesis upon X is the following:

Definition: *X is said to satisfy the conditions (P) if the following two conditions hold:*

(1) *There exists a duality mapping J of X into X^* which is continuous and also weakly continuous (i.e. continuous in the weak topology of X and X^*).*

(2) *There exists an increasing sequence $\{F_j\}$ of finite dimensional subspaces of X whose union is dense in X, and a corresponding sequence $\{P_j\}$ of projections of X such that the range of each P_j is the corresponding F_j and for each j, $||P_j|| = 1$.*

The properties (P) were applied in Browder—Figueiredo [19] to obtain an existence theorem for nonlinear functional equations involving J-monotone operators. Aside from Hilbert spaces, the most important class of concretely defined Banach spaces which satisfy the conditions (P) are the sequence spaces l^p for $1 < p < +\infty$, as was shown in [10]. The restrictive condition in the pair of conditions (P) is the first which does not hold for any L^p space with $p \neq 2$ on the line. Property (2) seems to hold for all examples of separable Banach spaces familiar to the writer.

Theorem 2.9: *Let X be a reflexive Banach space which is strictly convex and satisfies the conditions (P). Let J be any duality mapping of X into X^*. Let T be a mapping of X into X which is hemicontinuous and locally bounded, and for which there exists a constant c in R^1 such that for all u and v of X*

$$(T(u) - T(v), J(u - v)) \leq c||u - v|| \,.\, ||J(u - v)||.$$

Then for each u_0 in X, there exists one and only weakly C^1 function u from R^+ to X which satisfies the differential equation

$$\frac{du}{dt} = T(u), \qquad t \geq 0,$$

and the initial condition $u(0) = u_0$.

Theorem 2.10: *Let X be a reflexive strictly convex Banach space which satisfies the conditions (P), J a duality mapping of X into X^*. Let L be a closed linear operator in X which is the infinitesimal generator of a C_0 semigroup of nonexpansive operators in X. Let T_0 be a mapping of X into X which is hemicontinuous, maps bounded subsets of X into bounded subsets of X, and for which there exists a constant c in R^1 such that for all u and v of X,*

$$(T_0(u) - T_0(v), J(u - v)) \leq c\|u - v\| \cdot \|J(u - v)\|.$$

Then for each u_0 in $D(L)$, there exists one and only one sharp solution u on R^+ of the differential equation

$$\frac{du}{dt} = Lu + T_0(u), \qquad t \geq 0,$$

with $u(0) = u_0$.

By a *sharp solution,* we mean a function u from R^+ to X which lies in $C_w^1(R^+, X)$ with $u(t)$ in $D(L)$ for all t in R^+ and with Lu in $C_w^0(R^+, X)$.

A time dependent generalization of Theorem 2.10 is the following:

Theorem 2.11: *Let X be a reflexive strictly convex Banach space which satisfies the conditions (P) and let J be a duality mapping of X into X^*. Let L be a closed linear operator in X which is the infinitesimal generator of a C_0 semigroup of nonexpansive linear operators in X. Let T_0 be a mapping of $R^+ \times X$ into X which carries bounded sets into bounded sets and satisfies the following three conditions:*

(1) *For each fixed t in R^+, $T_0(t, \cdot)$ is a hemicontinuous mapping of X into X. For each fixed u in X, $T_0(\cdot, u)$ is a continuous mapping from R^+ to the weak topology of X.*

(2) *There exists a continuous function c from R^+ to R^1 such that for all u and v in X and all t in R^+:*

$$(T_0(t, u) - T_0(t, v), J(u - v)) \leq c(t)\, \|u - v\| \cdot \|J(u - v)\|.$$

(3) *For each fixed u in $D(L)$, $T_0(t, u)$ is weakly once differentiable in t from R^+ to X, and its derivative satisfies the inequality*

$$\left\| \left(\frac{\partial}{\partial t} T_0\right)(t, u) \right\| \leq q(t, \|u\|)$$

for all u in $D(L)$ and a continuous function q from $R^+ \times R^+$ to R^+.

Then for each u_0 *in* $D(L)$, *there exists one and only one sharp solution* u *on* R^+ *of the differential equation*

$$\frac{du}{dt} = Lu + T_0(t, u), \qquad t \geq 0,$$

with $u(0) = u_0$.

The variant of the generalized method of steepest descent which holds for this case is the following:

Theorem 2.12: *Let* X *be a reflexive strictly convex Banach space which satisfies the conditions* (P), *and let* J *be any duality mapping of* X *into* X^*. *Let* L *be a closed linear operator in* X *which is the infinitesimal generator of a* C_0 *semigroup of nonexpansive operators in* X. *Let* T_0 *be a mapping of* X *into* X *which is hemicontinuous and locally bounded. If* L *is unbounded, we suppose in addition that* T_0 *maps bounded sets of* X *into bounded sets of* X.

Suppose that T_0 *is* J*-monotone, and that there exists* $R > 0$, *such that* $(Tu, Ju) \geq 0$ *for all* u *in* $D(T)$ *with* $||u|| = R$.

Let c *be a continuous nonincreasing* C^1 *function from* R^+ *to* R^+ *such that* $c(t) \to 0$ *as* $t \to +\infty$, $\int_0^\infty c(s)\,ds = +\infty$. *Let* u_0 *be any element of* $D(L)$ *with* $||u_0|| \leq R$, *and let* v_0 *be any element of* X *with* $||v_0|| < R$.

Then:

(*a*) *There exists exactly one sharp solution* u *on* R^+ *of the differential equation*

$$\frac{du}{dt} = Lu - T_0(u) - c(t)\,\{u - v_0\}, \qquad t \geq 0,$$

with $u(0) = u_0$.

(*b*) *For each such solution,*

$$||-Lu(t) + T_0(u(t))|| \to 0$$

as $t \to +\infty$.

(*c*) *For each such solution,* $u(t)$ *converges strongly in* X *to a solution* v_0 *of the equation* $Tv_0 = 0$, *as* $t \to +\infty$.

A consequence of Theorem (2.12) is the following existence theorem for solutions of nonlinear functional equations involving J-monotone operators.

Theorem 2.13: *Let* X *be a reflexive strictly convex Banach space which satisfies the conditions* (P), J *a duality mapping of* X *into* X^*. *Let* T *be a mapping with domain and range in* X *which lies in one of the following two classes*:

(*a*) T *is a hemicontinuous locally bounded* J*-monotone mapping of* X *into* X.

(*b*) $T = -L + T_0$, *where* L *is a closed linear operator in* X *which is the infinitesimal generator of a* C_0 *semigroup of nonexpansive operators in* X, *and*

T_0 *is a hemicontinuous J-monotone mapping of X into X which carries bounded sets into bounded sets.*

Then:

(1) *If for a given* $R > 0$ *and for all u in* $D(T)$ *with* $\|u\| = R$, $(Tu, Ju) \geq 0$, *then the set*

$$K = \{v | v \in D(L), Tv = 0, \|v\| \leq R\}$$

is a nonempty closed convex subset of X.

(2) *If T is J-coercive, then the range of T is all of X.*

The existence of a solution u_0 of the equation $Tu_0 = 0$ in case (a) was previously established in Browder—Figueiredo [19].

Let us turn finally to nonlinear equations of evolution involving J-monotone operators without a differentiability assumption on the dependence of f on t.

First, we have the following theorem which extends the similar result in Hilbert space proved in Browder [4]:

Theorem 2.14: *Let X be a reflexive strictly convex Banach space which satisfies the conditions (P), and let J be a duality mapping of X into* X^*. *Suppose that f is a mapping of* $R^+ \times X$ *into X which carries bounded subsets of* $R^+ \times X$ *into bounded sets in X. Suppose that f satisfies the following two conditions*:

(1) *For each fixed t in* R^+, $f(t, \cdot)$ *is a hemicontinuous mapping of X into X. For each fixed u in X,* $f(\cdot, u)$ *is a continuous mapping of* R^+ *into the weak topology of X.*

(2) *There exists a continuous function c from* R^+ *to* R^1 *such that for all t in* R^+ *and all u and v of X,*

$$(f(t, u) - f(t, v), J(u - v)) \leq c(t) \|u - v\| \,.\, \|J(u - v)\|.$$

Then for each u_0 *in X, there exists one and only one solution u in* $C^1_w(R^+, X)$ *of the differential equation*

$$\frac{du}{dt} = f(t, u), \qquad t \geq 0,$$

which satisfies the initial condition $u(0) = u_0$.

A corresponding extension of the existence theorems for mild solutions of nonlinear equations of evolution in Hilbert space involving unbounded linear operators, as proved in Browder [4] and Kato [21], is based upon the following natural extension of the definition of *mild solution*:

Definition: *Let X be a Banach space,* $\{L(t) \mid t \in R^+\}$ *a family of closed linear operators in X, f a mapping of* $R^+ \times X$ *into X. Suppose that the time-dependent linear problem*

$$\frac{du}{dt}(t) = L(t)\, u(t), \qquad t \geq s,$$

$u(s) = u_0$

has one and only one strongly continuous solution $u(t) = U(t, s)\, u_0$ for each $s \geq 0$ and each u_0 in $D(L(s))$, where $U(t, s)$ is a bounded linear operator in X for each s and t in R^+ with $s \leq t$.

Then a function u from R^+ to X is said to be a mild solution of the nonlinear differential equation

$$\frac{du}{dt} = L(t)\, u + f(t, u), \qquad t \geq 0,$$

if u is a strongly continuous function from R^+ to X which is a solution of the nonlinear integral equation:

$$u(t) = U(t, 0)\, u_0 + \int_0^t U(t, s)\, f(s, u(s))\, ds, \qquad t \geq 0.$$

Theorem 2.15: *Let X be a reflexive strictly convex Banach space which satisfies the conditions (P), J a duality mapping of X into X^*. Let $\{L(t) \mid t \in R^+\}$ be a family of closed linear operators in X, with each $L(t)$ the infinitesimal generator of a C_0 semigroup of nonexpansive operators in X. Suppose also that for each $s \geq 0$ and each u_0 in $D(L(s))$, the time-dependent linear problem*

$$\frac{du}{dt}(t) = L(t)\, u(t), \qquad t \geq s,$$

$u(s) = u_0$

has one and only one solution u in $C^1_w((s, \infty); X)$.

Let f be a mapping of $R^+ \times X$ into X which maps bounded subsets of $R^+ \times X$ into bounded subsets of X and satisfies the two conditions (1) *and* (2) *of Theorem* (2.14).

Then there exists for each u_0 in X, one and only one mild solution u on R^+ of the nonlinear equation of evolution

$$\frac{du}{dt} = L(t)\, u + f(t, u), \qquad t \geq 0,$$

with $u(0) = u_0$.

Section 3: We now turn to the problem of the existence of periodic solutions of equations of the form

$$\frac{du}{dt} = f(t, u) \tag{3.1}$$

where $f(t, u)$ is periodic in t of period p, i.e. $f(t + p, u) = f(t, u)$ for all t in R^+. We seek to find periodic solutions of period p. We shall present here some of the simpler results given in Browder [15].

Definition: *A function V from the Banach space X to R^+ is said to be a Lyapounov function for the equation* (**3.1**), *where f is a mapping of $R^+ \times X$ into X, if the following conditions are valid*:

(**1**) *V is a convex function on X, with $V(0) = 0$, $V(u) > 0$ for $u \neq 0$, and the level sets of V are bounded and uniformly convex, i.e. given $R > 0$, $d > 0$ there exists $R_1 < R$ such that if $V(u_0) \leq R$, $V(u_1) \leq R$, with $||u_0 - u_1|| \geq d$, then*:

$$V((u_0 + u_1)/2) \leq R_1.$$

(**2**) *There exists a continuous mapping S of X into X^* which is a subderivative of V, i.e. for all u and v in X*

$$V(u) - V(v) \geq (S(v), u - v).$$

(**3**) *For each pair u and v in X and all t in R^+,*

$$(f(t, u) - f(t, v) \mathrm{S} (u - v)) \leq 0.$$

(**4**) *There exists $R_0 > 0$ such that for all t in R^+ and all u in X with $||u|| \geq R_0$,*

$$(f(t, u), S(u)) \leq 0.$$

Theorem 3.1: *Let X be a reflexive Banach space, f a mapping of $R^+ \times X$ into X such that for all u_0 in X, the differential equation*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(t, u), \qquad t \geq 0,$$

has exactly one solution with $u(0) = u_0$.

Suppose that $f(t, u)$ is periodic in t of period $p > 0$, and suppose that there exists a Lyapounov function for this equation in the sense of the above definition.

Then the equation (**3.1**) *has a periodic solution of period p.*

As an application of this result, we have the following:

Theorem 3.2: *Let X be a uniformly convex Banach space, J a duality mapping of X into X^*. Let f be a mapping of $R^+ \times X$ into X such that the equation*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(t, u)$$

has one and only one solution on R^+ with $u(0) = u_0$, for any given u_0 in X. Suppose further that for each t in R^+,

$$(f(t, u) - f(t, v), J(u - v)) \leq 0,$$

and that there exists $R > 0$ such that for all t in R^+ and all u in X with $||u|| \geq R$,

$$(f(t, u), Ju) \leq 0.$$

Then if $f(t, u)$ is periodic in t of period $p > 0$, there exists a solution of the differential equation which is periodic of period p.

Extensions are given in [ ] to more general nonlinear equations of evolution of the types considered in Sections **1** and **2** above. The proofs are all based upon the following simple fixed point theorem:

Theorem 3.3: *Let X be a reflexive Banach space, V a convex continuous function from X to R^+ such that $V(0) = 0$, $V(u) > 0$ for $u \neq 0$. Suppose that the level sets of V are bounded and uniformly convex. Let U be a mapping of a closed convex subset C of X into C such that for all u and v of C,*

$$V(U(u) - U(v)) \leq V(u - v).$$

Then U has a fixed point in C.

The proof of Theorem **3.3** uses an argument of Brodski and Milman [1], which was applied in the case in which V is a function of the norm in an uniformly convex space by Browder [9] and Kirk [22]. Similar fixed point theorems with weakened hypotheses can be established in Hilbert spaces and Banach spaces having weakly continuous duality mappings J by using the fact that for every nonexpansive mapping U, $T = I - U$ is J-monotone, (cf [10], [17]).

Theorem **3.2** is an extension of a result in Hilbert space given by the writer in [8].

We remark in conclusion that the most general form of aplication of Theorem (**3.3**) to initial value problems can be put in the following abstract form:

Theorem 3.4: *Let X be a reflexive Banach space, C a closed convex bounded subset of X. Let $\{U(t, s) \mid t \geq s\}$ be a family of transition operators on C, i.e. for $r \leq s \leq t$, $U(t, r) = U(t, s)\, U(s, r)$, where each $U(t, s)$ is a (possibly) non-linear nonexpansive mapping of C into itself. Suppose further that there exists a convex function V from X to R^+ such that $V(0) = 0$, $V(u) > 0$ for $u \neq 0$, and with the level surfaces of V uniformly convex, such that for all t and s, $(s \leq t)$ and all u and v in C,*

$$V(U(t, s)\, u - U(t, s)\, v) \leq V(u - v).$$

Suppose that the transition operators $U(t, s)$ are periodic of period $p > 0$, in the sense that for every $s < t$ in R^+, $U(t + p, s + p) = U(t, s)$.

Then there exists u_0 in C such that for every t in R^+, $U(t, 0)\, u_0$ is periodic in t of period p.

BIBLIOGRAPHY

[1] M. S. Brodski and D. P. Milman, *On the center of a convex set, Dokladi Akad. Nauk SSSR, (N. S.), 59 (1948), 837–840.*

[2] F. E. Browder, *Nonlinear elliptic boundary value problems, Bull. Amer. Math. Soc., 69 (1963), 862–874.*

[3] *Strongly nonlinear parabolic boundary value problems*, Amer. Jour. Math., 86 (1964), 339–357.

[4] *Nonlinear equations of evolution*, Annals of Math., 80 (1964), 485–523.

[5] *Nonlinear initial value problems*, Annals of Math., 82 (1965), 51-87.

[6] *Existence and uniqueness theorems for solutions of nonlinear boundary value problems*, Proceed. Amer. Math. Soc. Symposia on App. Math. Vol. 17 (1965), 24–49.

[7] *Infinite dimensional manifolds and nonlinear elliptic eigenvalue problems*, Annals of Math., 82 (1965), 459–477.

[8] *Existence of periodic solutions for nonlinear equations of evolution*, Proc. Nat. Acad. Sci. U. S. A., 53 (1965), 1100–1103.

[9] *Nonexpansive nonlinear operators in Banach spaces*, Proc. Nat. Acad. Sci. U. S. A., 54 (1965), 1041–1044.

[10] *Fixed point theorems for nonlinear semicontractive mappings in Banach spaces*, Arch. Rat. Mech. and Anal., 21 (1966), 259–266.

[11] *Problemes nonlineaires*, Lecture Notes, University of Montreal NATO Seminar, Summer 1965.

[12] *On the unification of the calculus of variations and the theory of monotone nonlinear operators in Banach spaces*, (to appear in Proc. Nat. Acad. Sci. U. S. A., August 1966).

[13] *Existence and approximation theorems for solutions of nonlinear variational inequalities*, (to appear in Proc. Nat. Acad. Sci. U. S. A.).

[14] *Monotone operators and nonlinear equations of evolution*, (to appear).

[15] *Periodic solutions of nonlinear equations of evolution in infinite dimensional spaces*, (to appear).

[16] *Nonlinear equations of evolution and the method of steepest descent in Banach spaces*, (to appear).

[17] *Convergence of approximants to fixed points of nonexpansive mappings in Banach spaces*, (to appear).

[18] *Periodic solutions of nonlinear partial differential equations*, (to appear).

[19] F. E. Browder and D. G. de Figueiredo, *J-monotone nonlinear mappings in Banach spaces*, Inding. Math. (to appear).

[20] P. Hartman, *Generalized Lyapunov functions and functional equations*, Annali di mat. pura ed. appl., (IV), 69 (1965), 305–320.

[21] T. Kato, *Nonlinear evolution equations in Banach spaces*, Proceed. Amer. Math. Soc. Symposia in App. Math., Vol. 17 (1965), 50–67.

[22] W. A. Kirk, *A fixed point theorem for mappings which do not increase distance*, Amer. Math. Monthly, 72 (1965), 1004–1006.

[23] J. L. Lions, *Sur certaines equations paraboliques nonlineaires*, Bull. Soc. Mat. France, 93 (1965), 155–175.

[24] J. L. Lions and W. A. Strauss, *Some nonlinear evolution equations*, Bull. Soc. Mat. France, 93 (1965), 43–96.

[25] G. J. Minty, *On a "monotonicity" method for the solution of nonlinear equations in Banach spaces*, Proc. Nat. Acad. Sci. U. S. A., 50 (1963), 1038–1041.

[26] M. I. Visik, *Solvability of boundary problems for quasilinear equations of higher order*, Mat. Sbornik, 59 (1962), 289–335.

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XVII — 1967

FORMALLY NORMAL ORDINARY DIFFERENTIAL OPERATORS[1)]

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1. Introduction. We shall present some results concerning the spectral theory of ordinary differential operators which commute in a formal way with their formal adjoints. Let L denote the ordinary linear differential expression given by

$$L = \sum_{k=0}^{n} p_k D^k,$$

where D represents the operation id/dx, the p_k are complexvalued functions of class C^∞ on an open interval $a < x < b$ of the real axis, and $p_n(x) \neq 0$ for $a < x < b$. The formal adjoint L^+ of L is given by

$$L^+ = \sum_{k=0}^{n} D^k \bar{p}_k.$$

We say that L commutes formally with L^+, and write $LL^+ = L^+L$, if $LL^+u = L^+Lu$ for all $u \in C^\infty(a, b)$; such an L is said to be formally normal. If L is formally normal we can ask whether it determines, in some natural way, a normal operator in the Hilbert space $L_2(a, b)$, or perhaps in a Hilbert space containing $L_2(a, b)$ as a subspace. We shall indicate below that, in general, this occurs only in rather special cases, and for these cases the spectral theory is easy. We exhibit a large class of formally normal L which determine no normal operators in $L_2(a, b)$, or in any larger Hilbert space. Some details concerning the spectra of these operators are presented. First, we present some abstract results on formally normal operators, which form the basis for the work on ordinary differential operators.

The work reported on here is due to R. E. Balsam [1], G. Biriuk and E. A. Coddington [2], and E. A. Coddington [3], [4], [5].

2. Formally normal operators in a Hilbert space. A *formally*

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normal operator in a Hilbert space $\mathfrak{H}$ is a linear, closed operator with domain $\mathfrak{D}(N)$ dense in $\mathfrak{H}$ such that $\mathfrak{D}(N) \subset \mathfrak{D}(N^*)$ and

$$\|Nf\| = \|N^*f\|, \qquad f \in \mathfrak{D}(N).$$

A *normal operator* in $\mathfrak{H}$ is a formally normal operator N such that $\mathfrak{D}(N) = \mathfrak{D}(N^*)$. For N formally normal define $\overline{N}$ to be the restriction of N^* to $\mathfrak{D}(N)$; thus $\overline{N} = N^*|\mathfrak{D}(N)$. Then $\overline{N} \subset N^*$, in the sense that the graph $\mathfrak{G}(\overline{N})$ of $\overline{N}$ is contained in the graph $\mathfrak{G}(N^*)$ of N^*, and similarly $N \subset \overline{N}^*$. (We note that a symmetric operator in $\mathfrak{H}$ is a formally normal N having the property that $N = \overline{N}$, and a self-adjoint operator is a normal operator such that $N = N^*$.)

If N is formally normal in $\mathfrak{H}$ it can be shown that

$$\mathfrak{D}(\overline{N}^*) = \mathfrak{D}(N) + \mathfrak{M}, \qquad \mathfrak{M} = \nu(I + N^*\overline{N}^*),$$
$$\mathfrak{D}(N^*) = \mathfrak{D}(N) + \overline{\mathfrak{M}}, \qquad \overline{\mathfrak{M}} = \nu(I + \overline{N}^*N^*),$$

where $\nu(A)$ represents the null space of A, and both sums above are direct ones. The following result tells precisely when N has a normal extension in $\mathfrak{H}$. (See [2] and [3]).

Theorem 1. *A formally normal operator N in a Hilbert space $\mathfrak{H}$ has a normal extension N_1 in $\mathfrak{H}$ if and only if*

(1) $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$, *a direct sum,*
(2) $\mathfrak{G}(\overline{N}^*|\mathfrak{M}_1) \perp \mathfrak{G}(\overline{N}^*|\mathfrak{M}_2)$,
(3) $\overline{N}^*\mathfrak{M}_2 = \mathfrak{M}_1$,
(4) $\|\overline{N}^*\varphi\| = \|N^*\varphi\|, \ \varphi \in \mathfrak{M}_1$,

and

$$\mathfrak{D}(N_1) = \mathfrak{D}(N) + \mathfrak{M}_1, \qquad N \subset N_1 \subset \overline{N}^*.$$

The first two conditions imply that N_1 is a closed operator such that $N \subset N_1 \subset \overline{N}^*$, and the last two guarantee that N_1 is normal. It follows from (3), and the fact that $\overline{N}^*\mathfrak{M} = \overline{\mathfrak{M}}$, that $\mathfrak{M}_1 \subset \mathfrak{M} \cap \overline{\mathfrak{M}}$, and moreover $\dim \mathfrak{M}_1 = \dim \mathfrak{M}_2$. Thus a necessary condition for N to have a normal extension is that $\dim \mathfrak{M}$ be even.

It is not true that every formally normal N in $\mathfrak{H}$ has a normal extension in $\mathfrak{H}$; in fact, this is not even true if N is symmetric. However, every symmetric N has a self-adjoint (and hence normal) extension in the larger Hilbert space $\mathfrak{H} \oplus \mathfrak{H}$. We ask whether a similar result is valid for formally normal N.

We shall now alter our notation slightly in order to deal with the several Hilbert spaces in what follows. Let us assume N_1 is now a maximal formally normal operator in a Hilbert space $\mathfrak{H}_1$ (N_1 has no proper formally normal extensions in $\mathfrak{H}_1$), and suppose that

$$\mathfrak{D}(\overline{N}_1^*) = \mathfrak{D}(N_1) + \mathfrak{M}^1,$$

where dim $\mathfrak{M}^1$ is finite. It is this case which occurs for ordinary differential operators. In the following we shall refer to two such formally normal operators, N_1 in $\mathfrak{H}_1$, and N_2 in $\mathfrak{H}_2$, with

$$\mathfrak{D}(\overline{N}_2{}^*) = \mathfrak{D}(N_2) + \mathfrak{M}^2,$$

and it will be true that $\mathfrak{M}^1 = \overline{\mathfrak{M}^2}$; and $\mathfrak{M}^2 = \overline{\mathfrak{M}^2}$. We let

$$M_1 = \overline{N}_1{}^*|\mathfrak{M}^1, \qquad M_2 = \overline{N}_2{}^*|\mathfrak{M}^2;$$

thus M_i maps $\mathfrak{M}^i$ into $\mathfrak{M}^i$, $i = 1, 2$. Also, we use the abbreviations

$$\alpha(M_i) = M_i + M_i{}^{*-1},$$
$$\beta(M_i) = M_i{}^*M_i - M_i{}^{*-1}M_i{}^{-1}.$$

In these notations, the following result characterizes when a maximal formally normal operator has a normal extension in a larger Hilbert space (See [2]).

Theorem 2. *A maximal formally normal operator N_1 in a Hilbert space $\mathfrak{H}_1$ has a normal extension $\mathcal{N}_1$ in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ if and only if*

(1) $\mathfrak{M}^1 = \overline{\mathfrak{M}^1}$,

(2) *there exists a formally normal operator N_2 in the Hilbert space $\mathfrak{H}_2$ such that*

$$\mathfrak{M}^2 = \overline{\mathfrak{M}^2}, \qquad \dim \mathfrak{M}^2 = \dim \mathfrak{M}^1,$$

and a one-to-one map C of $\mathfrak{M}^1$ onto $\mathfrak{M}^2$ satisfying

(3) $\alpha(M_1) + C^*\alpha(M_2)\, C = 0$,

(4) $\beta(M_1) + C^*\beta(M_2)\, C = 0$.

If $\mathcal{N} = N_1 \oplus N_2$, then

$$\mathfrak{D}(\mathcal{N}_1) = \mathfrak{D}(\mathcal{N}) + (I + C)\,\mathfrak{M}^1, \qquad \mathcal{N} \subset \mathcal{N}_1 \subset \overline{\mathcal{N}}^*.$$

A necessary condition for N_1 to have a normal extension in $\mathfrak{H}$ is given by condition (1), but it is not known whether this is sufficient. An N_2 and C satisfying (2), (3), (4) exist if $M_1M_1{}^* = M_1{}^*M_1$, if $M_1{}^2 = uI$, $|u| = 1$, and in almost all cases if dim $\mathfrak{M}^1 = 2$. Such an N_2 can be defined in terms of a conjugation operator J, which is an operator on $\mathfrak{H}_1$ satisfying $J^2 = I$, and $(Jf, Jg) = (g, f)$ for all $f, g \in \mathfrak{H}_1$. Then $N_2 = J\overline{N}_1J$ on $\mathfrak{H}_2 = \mathfrak{H}_1$ will work in the above cases (See [2]).

3. Formally normal ordinary differential operators. Let us now return to the differential expression

$$L = \sum_{k=0}^{n} p_kD^k, \qquad D = id/dx,$$

which we considered in the Introduction. For our Hilbert space we take $\mathfrak{H}_1 = L_2(a, b)$. We suppose that

$$||Lu|| = ||L^+u||, \qquad u \in C_0^\infty(a, b),$$

where $C_0^\infty(a, b)$ denotes the set of all complex-valued functions of class C^∞ on $a < x < b$ which vanish outside compact subsets of this interval. This restriction on L is equivalent to the condition $LL^+ = L^+L$. Let us define N_1 to be the operator in $\mathfrak{H}_1$ which is the closure of L defined on $C_0^\infty(a, b)$, in the sense that $\mathfrak{G}(N_1)$ is the closure of $\mathfrak{G}(L|C_0^\infty(a, b))$ in $\mathfrak{H}_1 \oplus \mathfrak{H}_1$. This operator N_1 is formally normal in $\mathfrak{H}_1$ and is called the *minimal operator* in $\mathfrak{H}_1$ associated with L. The operator $\overline{N}_1{}^*$ is just L on $\mathfrak{D}(\overline{N}_1{}^*)$, which is the set of all $f \in \mathfrak{H}_1$ such that $f \in C^{n-1}(a, b)$, $f^{(n-1)}$ is absolutely continuous, and $Lf \in \mathfrak{H}_1$. The operator $N_1{}^*$ is described in the same way with L replaced by L^+. Hence $\overline{N}_1$ is L^+ on $\mathfrak{D}(N_1)$. We have

$$\mathfrak{D}(\overline{N}_1{}^*) = \mathfrak{D}(N_1) + \mathfrak{M}^1, \qquad \mathfrak{D}(N_1{}^*) = \mathfrak{D}(N_1) + \overline{\mathfrak{M}}^1,$$

where

$$\mathfrak{M}^1 = \{\varphi \in \mathfrak{H}_1 \,|(LL^+ + I)\,\varphi = 0,\, L\varphi \in \mathfrak{H}_1\},$$
$$\overline{\mathfrak{M}}^1 = \{\psi \in \mathfrak{H}_1 \,|(LL^+ + I)\,\psi = 0,\, L^+\psi \in \mathfrak{H}_1\}.$$

Thus we see that $\mathfrak{M}^1$ and $\overline{\mathfrak{M}}^1$ consist of solutions to a homogeneous differential equation of order $2n$, and from this it follows that $0 \leqq \dim \mathfrak{M}^1 = \dim \overline{\mathfrak{M}}^1 \leqq 2n$.

The simplest example occurs when all the coefficients p_k are constants. Here there are three cases:

(i) a, b both finite,
(ii) only one of a, b finite,
(iii) $a = -\infty$, $b = +\infty$.

In all cases $\mathfrak{M}^1 = \overline{\mathfrak{M}}^1$, and $\dim \mathfrak{M}^1$ is $2n$, n, and 0 in cases (i), (ii), (iii) respectively. The N_1 for case (iii) as a normal extension. If n is odd, the N_1 of case (ii) is thus normal, and the N for the other cases have the N of case (iii) as a normal eptension. If n is odd, the N, of case (ii) has no normal eptension in $\mathfrak{H}_1$; see the remarks just after the statement of Theorem 1.

We now interpret Theorem 1 for our differential operator N_1. The conditions of that theorem can be given a more conventional form by means of two bracketed expressions:

$$\langle uv\rangle = (Lu, v) - (u, L^+v),\quad u \in \mathfrak{D}(\overline{N}_1{}^*),\ v \in \mathfrak{D}(N_1{}^*),$$
$$[uv] = (Lu, Lv) - (L^+u, L^+v),\quad u, v \in \mathfrak{D}(\overline{N}_1{}^*) \cap \mathfrak{D}(N_1{}^*).$$

It can be shown that these expressions depend only on $u, u', \ldots, u^{(n-1)}$ and $v, v', \ldots, v^{(n-1)}$ in the vicinity of a and b, and they are limiting values of certain semi-bilinear forms in these quantities. Theorem 1 can be rephrased in these terms so as to display the domain of a normal extension of N_1 described by certain boundary conditions.

Theorem 3. *The minimal operator N_1 associated with L has a normal extension $\widehat{N}_1$ in $\mathfrak{H}_1 = L_2(a, b)$ if and only if* $\dim \mathfrak{M}^1 = 2p$, *and there exist linearly independent $\alpha_1, \ldots, \alpha_p \in \mathfrak{M}^1 \cap \overline{\mathfrak{M}^1}$ satisfying*

$$\langle \alpha_j \alpha_k \rangle = [\alpha_j \alpha_k] = 0, \qquad (j, k = 1, \ldots, p).$$

Then $N_1 \subset \widehat{N}_1 \subset \overline{N}_1{}^*$ *and*

$$\mathfrak{D}(\widehat{N}_1) = \{u \in \mathfrak{D}(\overline{N}_1{}^*) \mid \langle u \alpha_j \rangle = 0, j = 1, \ldots, p\}.$$

There is an obvious choice of L whose corresponding N_1 has a good chance of having a normal extension in $\mathfrak{H}_1$, namely those L which can be represented as polynomials in some formally symmetric differential expression Λ:

$$L = \sum_{k=0}^{n} c_k \Lambda^k, \qquad \Lambda = \Lambda^+,$$

where the c_k are complex numbers, some of which may be zero. If the minimal operator for Λ has a self-adjoint extension S in $\mathfrak{H}_1$, then clearly

$$\widehat{N}_1 = \sum_{k=0}^{n} c_k S^k$$

will be a normal extension of N_1 in $\mathfrak{H}_1$. In any case, if L is a polynomial in $\Lambda = \Lambda^+$, it will be true that N_1 has a normal extension $\mathscr{N}_1$ in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_1$. This is due to the fact that the minimal operator S_1 for Λ is symmetric and always has a self-adjoint extension S in $\mathfrak{H}$, and then

$$\mathscr{N}_1 = \sum_{k=0}^{n} c \mathscr{S}_k{}^k$$

is a normal extension of N_1 in $\mathfrak{H}$.

We now might ask: can every L be represented as a polynomial in a formally symmetric Λ, and does every formally normal differential operator N_1 have a normal extension in a larger space? For L of order one or two the answer is yes, but for L of order $n \geqq 3$ the answer is no. The simplest example which shows this is the L defined by

$$Lu = u''' + u'' - 3x^{-2}u' + (3x^{-3} - 2x^{-2})\, u,$$

with $\mathfrak{H}_1 = L_2(0, \infty)$. This determines a formally normal N_1 with $\dim \mathfrak{M}^1 = 1$, but with $\dim (\mathfrak{M}^1 \cap \overline{\mathfrak{M}^1}) = 0$. It is maximal formally normal, but not normal, and has no normal extensions in any Hilbert space $\mathfrak{H} \supset \mathfrak{H}_1$. Recall the necessary condition (1) of Theorem 2.

In spite of the fact that not every L, such that $LL^+ = L^+L$, is a polynomial in a formally symmetric Λ, the following interesting result is valid (see [1]).

Theorem 4. *For L of orders 1, 2 or 3 the formally normal operator N_1 has a normal extension $\hat{N}_1$ in $\mathfrak{H}_1$ if and only if*

$$L = c_3\Lambda^3 + c_2\Lambda^2 + c_1\Lambda + c_0, \qquad \Lambda = \Lambda^+,$$

where the c_k are constants (some of which may be 0), and

$$\hat{N}_1 = c_3S^3 + c_2S^2 + c_1S + c_0I,$$

for some self-adjoint extension S of the minimal operator for Λ.

We remark that the example mentioned above is typical for a third order L which is not a polynomial in a formally symmetric Λ. Any N_1, for such an L on a maximal interval of definition, is such that it has no normal extension in any Hilbert space $\mathfrak{H} \supset \mathfrak{H}_1$. Examples of higher order operators, to be given in the next section, further support the conjecture that N_1 has a normal extension in some $\mathfrak{H} \supset \mathfrak{H}_1$ if and only if L is a polynomial in some $\Lambda = \Lambda^+$. Also, one might conjecture that an analogue of Theorem 4 is valid for L of arbitrary order.

The spectral theory of normal operators $\mathscr{N}_1$ in $\mathfrak{H} \supset \mathfrak{H}_1$, which have the form

$$\mathscr{N}_1 = \sum_{k=0}^{n} c_k \mathscr{S}^k,$$

where $\mathscr{S}$ is self-adjoint in $\mathfrak{H}$, is completely determined by the spectral theory for $\mathscr{S}$. If $\mathscr{S}$ has the spectral resolution

$$\mathscr{S} = \int_{-\infty}^{\infty} \lambda \, dE(\lambda),$$

then
$$\mathscr{N}_1 = \int_{-\infty}^{\infty} p(\lambda)\, dE(\lambda), \qquad p(\lambda) = \sum_{k=0}^{n} c_k \lambda^k.$$

Because of Theorem 4, and the remarks following it, we see that results concerning the spectra of self-adjoint extensions of symmetric ordinary differential operators assume added importance.

4. Some special formally normal differential operators. We have investigated in detail a large class of L, for which $LL^+ = L^+L$, but which can not be expressed as polynomials in a formally symmetric Λ. Let m, n be relatively prime positive integers with $m > n \geqq 2$, and let $g = (m-1)(n-1)/2$. We define the differential expressions L_m and L_n by

$$L_m = x^{-m} \prod_{k=0}^{m-1} (\delta - kn + g),$$

$$L_n = x^{-n} \prod_{k=0}^{n-1} (\delta - km + g),$$

where $\delta = xd$, and d stands for d/dx. These operators have the form

$$L_m = d^m + a_1 x^{-1} d^{m-1} + \ldots + a_m x^{-m},$$
$$L_n = d^n + b_1 x^{-1} d^{n-1} + \ldots + b_n x^{-n},$$

where the a_k and b_k are real constants. Let $L = L_m + L_n$ if one of m, n is even; otherwise let $L = iL_m + L_n$. Then it is true that $LL^+ = L^+L$. If N_1 is the minimal operator in $\mathfrak{H}_1 = L_2(0, \infty)$ for L, then N_1 is formally normal, but not normal; moreover it has no normal extension in $\mathfrak{H}_1$ or in any Hilbert space $\mathfrak{H} \supset \mathfrak{H}_1$ (see [5]). The example in Section **4** is the case $m = 3$, $n = 2$.

The specific nature of the spectrum of N_1 has been determined for each pair of integers m, n. Recall that the resolvent set of N_1 is the set $\varrho(N_1)$ of all $\lambda \in \mathbf{C}$ (the complex numbers) such that $(N_1 - \lambda I)^{-1}$ exists as a bounded operator on all of $\mathfrak{H}_1$, and the spectrum $\sigma(N_1) = \mathbf{C} - \varrho(N_1)$. The point spectrum $\sigma_p(N_1)$ is the set of all $\lambda \in \mathbf{C}$ such that $\dim \nu(N_1 - \lambda I) > 0$; the continuous spectrum $\sigma_c(N_1)$ is the set of all $\lambda \in \sigma(N_1)$ such that $N_1 - \lambda I$ is one-to-one, the range of $N_1 - \lambda I$ is dense in $\mathfrak{H}_1$, but is not all of $\mathfrak{H}_1$; the residual spectrum $\sigma_r(N_1)$ is the set of all $\lambda \in \sigma(N_1)$ such that $N_1 - \lambda I$ is one-to-one, and the range of $N_1 - \lambda I$ is not dense. The essential spectrum $\sigma_e(N_1)$ is the set of all $\lambda \in \mathbf{C}$ such that the range of $N_1 - \lambda I$ is not closed.

There are three cases according as

(a) m odd, n even,

(b) m odd, n odd,

(c) m even, n odd.

For cases (a) and (c) let $p(r) = r^m + r^n$, and for case (b) let $p(r) = ir^m + r^n$. The curve C in $\mathbf{C}$, defined by

$$C = \{\lambda \in \mathbf{C} \mid \lambda = p(it),\ -\infty < t < \infty\},$$

plays an essential role; in fact $\sigma_e(N_1) = C$. In all cases the point spectrum is empty. If $m > 2n$ in cases (a) and (b), and $m > 3n$ in case (c), we have $\sigma(N_1) = \sigma_r(N_1) = \mathbf{C}$. In the remaining situations the spectrum is more interesting, and depends on certain arithmetic relationships between m and n. The set $\mathbf{C} - C$ consists of two components, which we may call I and II, letting I denote that component which contains the positive real axis. In case (a), for example, if $n < m < 2n + 1$, $\sigma(N_1) = C \cup I$ if $m = 2k + 1$ with k even, whereas $\sigma(N_1) = C \cup \mathrm{II}$ if k is odd. The distribution of $\sigma(N_1)$ between $\sigma_c(N_1)$ and $\sigma_r(N_1)$ is further complicated. As an example, if $m = 3$, $n = 2$, we have $\sigma(N_1) = C \cup \mathrm{II}$, $\sigma_e(N_1) = \sigma_c(N_1) = C$, $\sigma_r(N_1) = \mathrm{II}$ (see [5]).

REFERENCES

[1] R. E. BALSAM, *Normal ordinary differential operators, Dissertation, University of California, Los Angeles (1965).*

[2] G. BIRIUK and E. A. CODDINGTON, *Normal extensions of unbounded formally normal operators, J. Math. Mech., 13 (1964), 617–638.*

[3] E. A. CODDINGTON, *Normal extensions of formally normal operators, Pacific J. Math., 10 (1960), 1203–1209.*

[4] E. A. CODDINGTON, *Formally normal operators having no normal extensions, Can. J. Math., 17 (1965), 1030–1040.*

[5] E. A. CODDINGTON, *The spectra of some formally normal ordinary differential operators, to appear.*

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
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PROBLEMS IN LINEAR CONTROL THEORY

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1.

Given a Banach space X and a real $T > 0$ let $A: t \to A(t)$ be a function of $t \in [0, T]$ with values in the space of linear (possibly unbounded) operators in X.

We shall assume the existence of the Green's function (evolution operator) associated with A. By this we mean a function $G : t, s \to G(t, s)$ defined for $0 \leq s \leq t \leq T$, with values in the space $\mathscr{L}(X, X)$ of linear bounded operators in X, strongly continuous in the two variables jointly and satisfying the conditions:

$$G(t, s)\, G(s, r) = G(t, r), \qquad 0 \leq r \leq s \leq t \leq T,$$
$$G(s, s) = 1 \qquad \text{(the identity in } X)$$

$$\frac{\partial G(t, s)\, x}{\partial t} = A(t)\, G(t, s)\, x, \qquad x \in D(A(s))$$

$$\frac{\partial G(t, s)\, x}{\partial s} = -G(t, s)\, A(s)\, x, \qquad x \in D(A(s))$$

where $\partial/\partial t$, $\partial/\partial s$ denote strong derivatives and $D(A(s)) \subset X$ is the domain of $A(s)$.

There are various known sufficient conditions for the existence of Green's function (T. KATO [9], J. KISYNSKI [10], E. T. POULSEN [14]).

Let $1 \leq p \leq \infty$. Given a Banach space E we denote by $L^p(0, T; E)$ the Banach space of all E-valued, strongly measurable functions f defined in $[0, T]$, such that

$$|f|_p = \left(\int_0^T |f(t)|_E^p \mathrm{d}t\right)^{1/p} < \infty \qquad \text{if } p < \infty$$

$$|f|_\infty = \text{ess sup}\, \{|f(t)|_E : 0 \leq t \leq T\} < \infty, \qquad \text{if } p = \infty.$$

If $c : t \to c(t)$ belongs to $L^1(0, T; X)$ then

$$\int_0^t G(t, s)\, c(s)\, ds \in X, \qquad 0 \leq t \leq T$$

the integral understood in the sense of Bochner.

Beside X we shall, consider another Banach space U and the space $\mathscr{L}(U,X)$, of linear bounded operators from U into X.

Let $B : t \to B(t)$ belong to $L^{p'}(0, T; \mathscr{L}(U, X))$ with $p' = p(p-1)^{-1}$ for $1 < p < \infty$, $p' = 1$ for $p = \infty$, $p' = \infty$ for $p = 1$.

If $u : t \to u(t)$ belongs to $L^p(0, T; U)$ then $t \to B(t)\, u(t)$ will belong to $L^1(0, T; X)$ and

$$\int_0^t G(t, s)\, B(s)\, u(s)\, ds \in X, \qquad 0 \leq t \leq T.$$

Summing up, if G exists and if $v \in X$, $u \in L^p(0, T; U)$, $B \in L^{p'}(0, T; \mathscr{L}(U,X))$, $c \in L^1(0, T; X)$, we may define

$$(1.1)\ x(t, u, v) = G(t, 0)\, v + \int_0^t G(t, s)\, B(s)\, u(s)\, ds + \int_0^t G(t, s)\, c(s)\, ds, \quad 0 \leq t \leq T.$$

We shall denote by V, W, and $\mathscr{U}$ three convex, bounded, closed subsets of X, X and $L^p(0, T; U)$ respectively and consider the following:

Problem P. *Given* X, U, p, T, A *(or rather* G*)*, B, c, V, W, $\mathscr{U}$, *determine whether there are* $v \in V$, $u \in \mathscr{U}$ *such that* $x(T, u, v) \in W$.

A few comments before we go further.

Equation (1.1) can be considered as the Bochner integral version of the linear differential equation

$$(1.2) \qquad dx/dt - A(t)\, x = B(t)\, u(t) + c(t)$$

with initial condition

$$(1.3) \qquad x(0, u, v) = v.$$

Sufficient conditions in order that (1.1) yield (1.2) are known (T. Kato [9], J. Kisynski [10], E. T. Poulsen [14]).

The problem we are dealing with is a typical one in linear control theory where x represents the state of some physical system, u, v are controls, permanent and initial, respectively, and it is required to determine such controls from given sets $\mathscr{U}$, V which transfer x from V into W in a given time interval $[0, T]$ along a trajectory of (1.2).

If dim $X < \infty$ then (1.1) is in fact equivalent to the ordinary differential equation (1.2) and $G(t, s) = \Phi(t)\, \Phi^{-1}(s)$ where $\Phi(t)$ is any fundamental matrix associated with A. However control problems involving partial differential equations (distributed parameter controls) require that also infinite dimensional spaces X be considered (A. G. Butkovskii [3], P. K. C. Wang [16]).

2.

The linear operator

$$\Gamma_T : x \to G(T, 0)\, x$$

from X into itself is bounded, therefore the image $\Gamma_T V$ of V is a bounded convex subset of X.

Also the linear operator

$$\Lambda_T : u \to \int_0^T G(T, s)\, B(s)\, u(s)\, \mathrm{d}s$$

from $L^p(0, T; U)$ into X is bounded and the image $\Lambda_T \mathcal{U}$ of $\mathcal{U}$ is a bounded convex subset of X.

Therefore $W - \Gamma_T V - \Lambda_T \mathcal{U}$ is a bounded convex subset of X.

By virtue of (1.1) Problem P reduces then to establish whether

$$(2.1) \qquad -\int_0^T G(T, s)\, c(s)\, \mathrm{d}s \in -W + \Gamma_T V + \Lambda_T \mathcal{U}.$$

Let us first consider the weaker relation

$$(2.2) \qquad -\int_0^T G(T, s)\, c(s)\, \mathrm{d}s \in \overline{-W + \Gamma_T V + \Lambda_T \mathcal{U}},$$

the closure of $-W + \Gamma_T V + \Lambda_T U$.

Recall that for any bounded subset $C \subset X$ a supporting function $h_C(x')$ is defined in the dual space X' by

$$h_C(x') = \sup_{x \in C} \langle x, x' \rangle$$

We need the following lemmas.

Lemma 1.

$$(2.3) \qquad h_{\overline{C}}(x') = h_C(x'), \qquad x' \in X'$$

Proof. Since $C \subset \overline{C}$ it follows $h_C(x') \leq h_{\overline{C}}(x')$ by definition. Conversely, for a fixed $x' \in X'$ let $x_k \in \overline{C}$ be such that $\lim_k \langle x_k, x' \rangle = \sup_{x \in \overline{C}} \langle x, x' \rangle = h_{\overline{C}}(x')$. Now choose $\chi_k \in C$, $|\chi_k - x_k|_X < k^{-1}$.

Then $\langle x_k, x' \rangle = \langle \chi_k, x' \rangle + \langle x_k - \chi_k, x' \rangle \leq h_C(x') + k^{-1}|x'|_{X'}$, and letting $k \to \infty$ we have $h_{\overline{C}}(x') \leq h_C(x')$.

Lemma 2. *If C is a bounded convex set $\subset X$, then*

$$(2.4) \qquad \langle \chi, x' \rangle \leq h_C(x'), \qquad x' \in X' \Leftrightarrow \chi \in \overline{C}.$$

Proof. $\chi \in \overline{C}$ means $\langle \chi, x' \rangle \leq \sup_{x \in \overline{C}} \langle x, x' \rangle = h_{\overline{C}}(x') = h_C(x')$ by lemma 1. Let $\chi \notin \overline{C}$, i.e. let $\{\chi\} \cap \overline{C}$ be void. Since $\{\chi\}$, $\overline{C}$ are convex, closed sets and $\{\chi\}$ is compact the "strict separation" theorem holds, i.e. there are two real numbers $\varepsilon > 0$, c and some $\chi' \in X'$ such that $\langle x, \chi' \rangle \leq c - \varepsilon < c \leq \langle \chi, \chi' \rangle$ $x \in \overline{C}$, hence $h_{\overline{C}}(\chi') \leq \langle \chi, \chi' \rangle$ and $h_C(\chi') < \langle \chi, \chi' \rangle$ by lemma **1**.

By applying (2.4) to (2.2) we have

Theorem 1. *The inequality*

$$(2.5)\ \langle -\int_0^T G(T, s)\, c(s)\, ds, x' \rangle \leq h_{-W+\Gamma_T V+\Lambda_T U}(x'), \qquad x' \in X'$$

is equivalent to (2.2), *therefore it is equivalent to* (2.1) *iff the set* $-W + \Gamma_T V + + \Lambda_T \mathcal{U}$ *is closed.*

3.

We are now going to indicate some criteria for the validity of

$$(3.1)\ -W + \Gamma_T V + \Lambda_T \mathcal{U} = \overline{-W + \Gamma_T V + \Lambda_T \mathcal{U}}.$$

This can be insured by

$$(3.2)\ W = \overline{W}, \qquad \Gamma_T V = \overline{\Gamma_T V}, \qquad \Lambda_T \mathcal{U} = \overline{\Lambda_T \mathcal{U}},$$

plus an additional assumption namely that

(3.3) *X is a reflexive Banach space.*

We recall in fact that in a Banach space X: *i*) all bounded weakly closed subset are weakly compact iff X is reflexive; *ii*) convex sets are weakly closed iff they are closed; *iii*) any finite sum of weakly compact sets is weakly closed. The implication (3.2) + (3.3) $\Rightarrow$ (3.1) then follows from the fact that all sets involved are convex and bounded.

Now $W = \overline{W}$ by assumption. Also $\Gamma_T V = \overline{\Gamma_T V}$ since Γ_T, as a linear operator continuous in the norm topology of X is also weakly continuous and V is, by assumption, weakly compact. On the contrary the validity of $\Lambda_T \mathcal{U} = \overline{\Lambda_T \mathcal{U}}$ requires some further assumption on $\mathcal{U}$. In particular the case $p = 1$ has to be put aside since there are examples of $\Lambda_T \mathcal{U} \neq \overline{\Lambda_T \mathcal{U}}$ in $L^1(0, T; U)$ even for $U = R$, the real number system.

Therefore we shall consider, from now on, only the case $1 < p \leq \infty$ and make a further assumption, namely

$$U = \varrho \mathcal{U}_1$$

with given $\varrho > 0$ and $\mathcal{U}_1 = \{u : |u|_p \leq 1\}$, the unit ball of $L^p(0, T; U)$. What we have to show is then that $\Lambda_T \mathcal{U}_1$ is (weakly) closed, or, equivalently, weakly compact.

Since Λ_T is continuous (in the norm hence) in the weak topologies of $L^p(0, T; U)$, X, we have weak compactness of $\Lambda_T \mathcal{U}_1$ when also $\mathcal{U}_1$ is weakly compact, which is equivalent to the assumption that

(3.4) *$L^p(0, T; U)$ is a reflexive Banach space*[1].

We thus have

Theorem 2. *Let X be a reflexive Banach space and let V, W be convex, bounded, closed subsets of X.*

Then Problem P has solutions if, $\mathscr{U} = \varrho\mathscr{U}_1$, $\varrho > 0$, $\mathscr{U}_1$ the unit ball of $L^p(0, T; U)$, $1 < p < \infty$ and U is such that $L^p(0, T; U)$ be reflexive.

Let us now turn to the case $p = \infty$.

We have (P. L. Falb [6]).

Lemma 3. *If U is such that $L^p(0, T; U)$ is reflexive, $1 < p < \infty$, then the unit ball $\mathscr{U}_1$ of $L^\infty(0, T; U)$ is a weakly compact subset of $L^p(0, T; U)$.*

Proof. Clearly $\mathscr{U}_1$ is a bounded subset of $L^p(0, T; U)$. Further if a sequence $u_k \in \mathscr{U}_1$ converges in $L^p(0, T; U)$ towards some $v \in L^p(0, T; U)$ then $v \in \mathscr{U}_1$, i.e. $\mathscr{U}_1$ is a closed subset of $L^p(0, T; U)$. In fact $u_k \to v$ in measure, hence $u_{k_n} \to v$ a.e. in $[0, T]$ for some subsequence u_{k_n}. Since $|u|_U \leq 1$ is closed, $|v(t)|_U \leq 1$ a.e. in $[0, T]$, i.e. $v \in \mathscr{U}_1$. Since $\mathscr{U}_1$ is also convex it is also weakly closed in $L^p(0, T; U)$, hence is weakly compact in $L^p(0, T; U)$ as $L^p(0, T; U)$ is reflexive.

From this follows

Theorem 2′. *Let X, V, W be as in Theorem 2.*

Then Problem P has solutions if $\mathscr{U} = \varrho\mathscr{U}_1$, $\varrho > 0$, $\mathscr{U}_1$ the unit ball of $L^\infty(0, T; U)$, provided that $L^p(0, T; U)$, $1 < p < \infty$ be reflexive, and

(3.5) *$B \in L^{1+\alpha}(0, T; \mathscr{L}(U, X))$, for some $\alpha > 0$.*

Proof. In fact (3.5) allows to consider Λ_T as a mapping of $L^{1+1/\alpha}(0, T; U)$ into X, continuous (in the norm, hence) in the weak topologies and by lemma **3** ($p = 1 + 1/\alpha$) it follows, again, that $\Lambda_T\mathscr{U}_1$ is a weakly compact subset of X.

Assumption (3.5) is actually stronger than $B \in L^1(0, T; \mathscr{L}(U, X))$ which would be the natural one in the case $u \in L^\infty(0, T; U)$. It can be avoided, however, at the expense of heavier assumptions on U, X, by using a particular case of the well-known Alaoglu's theorem, namely

Lemma 4. *If $L^\infty(0, T; U) = (L^1(0, T; U'))'$, then the unit ball $\mathscr{U}_1$ of $L^\infty(0, T; U)$ is weakly * compact.*

Let u_k be any sequence in $\mathscr{U}_1$. We may assume that u_k converges weakly * towards some $u \in \mathscr{U}_1$, i.e.

$$\int_0^T \langle v, u_k\rangle \, dt \to \int_0^T \langle v, u\rangle \, dt \qquad \text{for all } v \in L^1(0, T; U'). \tag{3.6}$$

This will imply $\Lambda_T u_k \to \Lambda_T u$ strongly in X in some cases, for instance when

[1] Recall that the reflexivity of $L^p(0, T; U)$ depends on U, but not on p, $1 < p < \infty$.

U, X are both finite dimensional: $\dim U = m$, $\dim X = n$. In fact $\Lambda_T u_k$, $\Lambda_T u$ are n-vectors with components, respectively

$$\int_0^T \langle v_j, u_k \rangle \, dt, \quad \int_0^T \langle v_j, u \rangle \, dt, \qquad j = 1, 2, \ldots, n$$

where v_j denotes the j. th row of the n by m matrix $G(T, t)\, B(t)$.

We thus have (H. A. ANTOSIEWICZ [1]).

Theorem 2''. *Let V, W, be convex, bounded, closed subsets of X, $\dim X = n$. Then Problem P has solutions if $\mathcal{U} = \varrho \mathcal{U}_1$, $\varrho > 0$, $\mathcal{U}_1$ the unit ball of $L^\infty(0, T; U)$,* $\dim U = m$.

4.

We shall now write the right hand side of (2.5) under the assumption $\mathcal{U} = \varrho\mathcal{U}_1$ in a more explicit form. We have

$$h_{-W+\Gamma_T V+\varrho\Lambda_T \mathcal{U}_1}(x') = h_{-W}(x') + h_{\Gamma_T V}(x') + \varrho h_{\Lambda_T U_1}(x')$$

with

$$h_{\Gamma_T V}(x') = \sup_{v \in V} \langle v, x' G(T, 0)\rangle$$

and

$$h_{\Lambda_T U_1}(x') = \Big(\int_0^T |x' G(T, s)\, B(s)|_{U'}^{p'} ds\Big)^{1/p'}.$$

Therefore (2.5) becomes

$$(4.1) \quad \langle -\int_0^T G(T, s)\, c(s)\, ds, x'\rangle \leq \sup_{w \in -W} \langle w, x'\rangle + \sup_{v \in V} \langle v, x' G(T, 0)\rangle +$$

$$+ \varrho \Big(\int_0^T |x' G(T, s)\, B(s)|_{U'}^{p'} ds\Big)^{1/p'}, \quad x' \in X'.$$

This inequality already appeared in the literature in many particular instances, both finite (H. A. ANTOSIEWICZ [1], R. CONTI [4], R. GABASOV—F. M. KIRILLOVA [8], W. T. REID [15]) and infinite dimensional (W. MIRANKER [11], G. MOCHI [12]).

5.

Some existence theorems for certain typical optimum control problems can be drawn from (4.1) along the lines followed by H. A. ANTOSIEWICZ [1] in the finite dimensional case.

a) Let ϱ_0 be the infimum of ϱ's such that (4.1) holds and let $\varrho_k \downarrow \varrho_0$ be a sequence of such ϱ's. Then (4.1) must hold also with $\varrho = \varrho_0$ and we have

Theorem 3. *Under the assumptions of Theorems 2, 2', 2'' if Problem P has a solution, then it also has a solution v, u with minimum $|u|_p$.*

Sometimes $|u|_P$ is called the "effort" associated with the control system and Theorem **3** states that under the assumptions of Theorems 2, 2′, 2″ there is a solution of the minimum effort control problem (W. A. PORTER—J. P. WILLIAMS [13]) as soon as the corresponding control problem has solutions.

b) Another typical problem in optimum control theory is the so-called "final value" problem (A. V. BALAKRISHNAN [2]). For instance it is required to minimize $|x(T, u, v) - w^0|_X$ for a given $w^0 \in X$. To this purpose we may assume the set W to be a closed ball of radius $\varepsilon > 0$ with center at w^0, i.e. $W = \{w^0\} + \varepsilon X_1$, X_1 the unit ball of X. Then $-W = \{-w^0\} + \varepsilon X_1$, and $h_{-W}(x') = -\langle w^0, x'\rangle + \varepsilon |x'|_{X'}$. Substituting into (4.1), the same argument we used for ϱ, applied to the infimum of ε's for which (4.1) holds leads to

Theorem 4. *Under the assumptions of Theorems 2.2', 2″ if Problem P with $W = \{w^0\} + \varepsilon X_1$ has a solution, then it also has a solution v, u such that $|x(T, u, v) - w^0|_X$ is minimum.*

c) In a similar way we could consider an "initial value" problem by taking $V = \{v^0\} + \sigma X_1$, $\sigma > 0$. Then $h_{\Gamma_T V}(x') = \langle v^0, x'G(T, 0)\rangle + \sigma |x'G(T, 0)|_{X'}$, etc.

d) The best known problem in optimum control theory is perhaps the "minimum time" problem: to find solutions yielding the minimum time T of transfer from V to W.

Since both sides of (4.1) are continuous functions of T, denoting by T_0 the infimum of T's for which (4.1) holds and by $T_k \downarrow T$ a sequence of such T's we obtain

Theorem 5. *Under the assumptions of Theorems 2,2', 2″ if Problem P has a solution, then it also has a solution such that T is minimum.*

For an infinite dimensional X particular cases of this Theorem were obtained by Y. V. EGOROV [5], H. O. FATTORINI [7], A. V. BALAKRISHNAN [2].

REFERENCES

[1] H. A. ANTOSIEWICZ, *Archive Rat. Mech. Anal., 12 (1963), 313.*

[2] A. V. BALAKRISHNAN, *Jour. SIAM Control, 3 (1965), 152.*

[3] A. G. BUTKOVSKII, *Teoriya optimal'nogo upravleniya sistemami s raspredelennymi parametrami, Izdat. Nauka, Moskva, 1965.*

[4] R. CONTI, *Jour. Diff. Equations, 1 (1965), 427.*

[5] Y. V. EGOROV, *Matem. Sbornik, 64 (106) (1964), 79.*

[6] P. L. FALB, *Jour. Math. Anal. Appl., 9 (1964), 12.*

[7] H. O. FATTORINI, *Jour. SIAM Control, 2 (1964), 54.*

[8] R. GABASOV—F. M. KIRILLOVA, *Avtom. i Telem., 25 (1964), 1058.*

[9] T. KATO, *A. M. S. Symposia in Appl. Math., vol. 17 (1965), 50.*

[10] J. Kisynski, *Studia Math.*, *23 (1964)*, *285*.
[11] W. Miranker, *Jour. Math. Anal. Appl.*, *10 (1965)*, *378*.
[12] G. Mochi, *Boll. Un. Mat. Ital.*, *(3) 21 (1966)*, *35*.
[13] W. A. Porter—J. P. Williams, *Jour. Math. Anal. Appl.*, *13 (1966)*, *251*.
[14] E. T. Poulsen, *Math. Zeitsch.*, *90 (1965)*, *286*.
[15] W. T. Reid, *Duke Math. Jour.*, *29 (1962)*, *591*.
[16] P. K. C. Wang, *Advances in Control Theory, edited by C. T. Leondes, vol. 1, Academic Press, 1964, 75*.

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ON LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ODD ORDER

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In my paper I will consider a differential equation of the n-*th* order, where n is odd of the following form:

$$y^{(n)} + 2A(x)\, y' + [A'(x) + b(x)]\, y = 0. \tag{a}$$

Suppose that $A'(x)$ and $b(x)$ are continuous functions of $x \in (-\infty, \infty)$.
The adjoint differential equation to the equation (a) is of the form

$$z^{(n)} + 2A(x)\, z' + [A'(x) - b(x)]\, z = 0. \tag{b}$$

Between the solutions of the differential equations (a) and (b) hold some relations, for instance:

If $y_1, y_2, \ldots, y_{n-1}$ are linearly independent solutions of the equation (a) then

$$z(x) = \begin{vmatrix} y_1 & y_2 & \ldots & y_{n-1} \\ y_1' & y_2' & \ldots & y_{n-1}' \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-2)} & y_2^{(n-2)} & \ldots & y_{n-1}^{(n-2)} \end{vmatrix}$$

is the solution of the equation (b).

If $y(x)$ is the solution of the differential equation (a) with the property

$$y(a) = y'(a) = \ldots = y^{(n-2)}(a) = 0, \qquad y^{(n-1)}(a) \neq 0,$$
$$a \in (-\infty, \infty) \qquad \text{and} \qquad y(\bar{x}) = 0, \qquad \bar{x} \neq a,$$

then the solution $z(x)$ of the differential equation (b) with the property

$$z(\bar{x}) = z'(\bar{x}) = \ldots = z^{(n-2)}(\bar{x}) = 0, \qquad z^{(n-1)}(\bar{x}) \neq 0$$

has also the property $z(a) = 0$.

We can deduce more of such relations.

The solutions of the differential equation (a), respectively (b) fulfil the following integral identities:

$$(1) \qquad y^{(n-1)} + 2Ay + \int_a^x (b - A')\, y\, dt = \text{const.}$$

$$(2) \qquad yy^{(n-1)} + Ay^2 + \int_a^x [by^2 - y'y^{(n-1)}]\, dt = \text{const.}$$

respectively

$$(1') \qquad z^{(n-1)} + 2Az - \int_a^x (A' + b)\, z\, dt = \text{const.}$$

$$(2') \qquad zz^{(n-1)} + Az^2 - \int_a^x [bz^2 + z'z^{(n-1)}]\, dt = \text{const.}$$

In the following I will introduce the results concerning the solutions without zeros of the differential equations (a) and (b) and the criterion of the divergence of the solutions without zeros of the differential equation (b). In the special case for $n = 3$ I will quote further results concerning this problems.

At the end I will deal with the existence of solutions of certain boundary problems chiefly of the third order.

I. First I will introduce two lemmas.

Lemma 1. *Let $A(x) \leqq 0$, $b(x) \geqq 0$ for $x \in (-\infty, \infty)$. Let $y(x)$ be the solution of the differential equation (a) with properties $y^{(i)}(a) = 0$, $i = 0, 1, \ldots, k-1, k+1, \ldots, n-1$, $y^{(k)}(a) \neq 0$, $1 \leq k \leq n-1$. Then neither $y(x)$ nor its derivatives $y^{(i)}(x)$, $i = 1, 2, \ldots, n-1$ have no zeros to the left side of a.*

Lemma 2. *Let the assumptions of Lemma 1 be satisfied and let $z(x)$ be the solution of the differential equation (b) with the properties $\overset{(i)}{z}(a) = 0$, $i = 0, 1, \ldots, k-1, k+1, \ldots, n-1$, $z^{(k)}(a) \neq 0$. Then $z^{(i)}(x)$, $i = 0, 1, \ldots, n-1$ have no zeros to the right of a and at the same time $z^{(i)}(x) \to \pm\infty$ for $x \to \infty$, $i = 0, 1, \ldots, n-3$. Here $z^{(i)}(x) \to +\infty$ if $z^{(k)}(a) > 0$ and $z^{(i)}(x) \to -\infty$ if $\overset{(k)}{z}(a) < 0$.*

The proof of Lemma 1 follows from the identity (2) and that of Lemma 2 from the identity (2′).

Remark 1. Similarly as Lemma 2 it can be proved that every non-trivial solution $z(x)$ of the differential equation (b) with properties

$$z(a) = 0, \qquad z^{(i)}(a) \geqq 0, \qquad i = 1, 2, \ldots, n-1, \; -\infty < a < +\infty,$$

has no zero point to the right, and no point of zero of the derivatives up to the order $n - 1$.

Theorem 1. *Let $A(x) \leqq 0$, $b(x) \geqq 0$ for $x \in (-\infty, \infty)$. Then the differential equation (a) $[(b)]$ has at least one solution $u(x)$ $[v(x)]$ without zero in the interval $(-\infty, \infty)$ for which holds*

$$\operatorname{sgn} u(x) = \operatorname{sgn} u''(x) = \ldots = \operatorname{sgn} u^{(n-1)}(x) \neq \operatorname{sgn} u'(x) = \operatorname{sgn} u'''(x) = \\ = \ldots = \operatorname{sgn} u^{(n-2)}(x)$$

$$[\operatorname{sgn} v(x) = \operatorname{sgn} v'(x) = \operatorname{sgn} v''(x) = \ldots = \operatorname{sgn} v^{(n-1)}(x)]$$

for all $x \in (-\infty, \infty)$ *at the same time* $u'(x) \to 0$, $u''(x) \to 0$, $\ldots$, $u^{(n-2)}(x) \to 0$ *for* $x \to \infty$ $[v(x) \to \pm\infty, \ldots, v^{(n-3)}(x) \to \pm\infty$ *for* $x \to \infty]$.

The solution $u(x)$ without zero-points of the properties mentioned in Theor. 1 can be obtained as the limit of the sequence of the solutions $\{u_k(x)\}_{k=1}^{\infty}$ with $u_k^{(i)}(x_k) = 0$, $u_k^{(n-1)}(x_k) > 0$, $i = 0, 1, \ldots, n-2$, where $\{x_k\}_{k=1}^{\infty}$ is a suitable sequence of numbers which diverges to infinity. The integral identity (2) for the solution u_k is of the form

$$u_k u_k^{(n-1)} + A u_k^2 = \int_x^{x_k} [b u_k^2 - u_k' u_k^{(n-1)}] \, \mathrm{d}t.$$

It can be shown that the solution $u(x)$ fulfils the analogical identity:

$$uu^{n-1} + Au^2 = \int_x^{\infty} [bu^2 - u'u^{(n-1)}] \, \mathrm{d}t. \tag{3}$$

Theorem 2. *Let* $A(x) \leqq 0$, $b(x) \geqq 0$ *for* $x \in (-\infty, \infty)$ *and let* $\int_a^{\infty} b \, \mathrm{d}t$ *diverge,* $-\infty < a < \infty$. *Let* $z(x)$ *be the solution of the differential equation* (*b*) *with the properties* $z^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-2$, $z^{(n-1)}(a) > 0$. *Then the following holds*: $z^{(n-1)}(x) + 2A(x)\, z(x) \to \infty$ *for* $x \to \infty$.

The statement follows from Lemma 2 and from the identity (1′).

Lemma 3. *Let the assumtions of Lemma 2 hold and in additicn let* $A' + b \geqq 0$ *and* $\int_a^{\infty} t^{n-1}[A'(t) + b(t)] \, \mathrm{d}t$ *diverge,* $-\infty < a < \infty$.

Let $z(x)$ *be the solution of the differential equation* (*b*) *with the properties* $z^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-2$, $z^{(n-1)}(a) > 0$. *Then it holds*: $z^{(n-1)}(x) + 2A(x)\, z(x) \to \infty$ *for* $x \to \infty$.

The statement follows from Lemma 2 and from the identity (1′).

Theorem 3. *Let the assumptions of Lemma 3 be satisfied. Then there exists at least one solution of the differential equation* (*a*) $y(x) \neq 0$ *for* $x \in (-\infty, \infty)$ *which has the following properties*: $y, y', \ldots, y^{(n-1)}$ *are monotonous function of* $x \in (-\infty, \infty)$, $\operatorname{sgn} y = \operatorname{sgn} y'' = \ldots = \operatorname{sgn} y^{(n-1)} \neq \operatorname{sgn} y' = \operatorname{sgn} y''' = \ldots = \operatorname{sgn} y^{(n-2)}$ *for* $x \in (-\infty, \infty)$ *and* $y \to 0$, $y' \to 0$, $\ldots$, $y^{(n-1)} \to 0$ *for* $x \to \infty$.

Let $n = 3$. Then the statement of Theorem 2 and Theorem 3 can be sharpened in the following way:

Theorem 4. *Let the assumptions of Theorem 2 and Remark 2 respectively Theorem 3 for* $n = 3$ *be fulfilled and let* $b(x) \equiv 0$ *do not hold in any interval.*

Then there exists just one solution of the differential equation (a) with the following properties: y, y', y'' are monotonous functions of $x \in (-\infty, \infty)$, $\operatorname{sgn} y = \operatorname{sgn} y'' \neq \neq \operatorname{sgn} y'$ *for $x \in (-\infty, \infty)$ and $y \to 0$, $y' \to 0$, $y'' \to 0$ for $x \to \infty$.*

Theorem 5. *Let $n = 3$. Let $A(x) \leqq 0$, $b(x) \geqq 0$, $A'(x) + b(x) \geqq 0$ for $x \in (-\infty, \infty)$ and let $b(x) \equiv 0$ do not hold in any interval.*

If the differential equation (a) has one oscillatory solution in the interval (a, ∞), $-\infty < a < +\infty$ (i.e. the solution has an infinite number of zero-points there) then all solutions of the differential equation (a) are oscillatory in the interval (a, ∞) with one exception of the solution y (up to the linear dependence) which has the following properties: $y(x) \neq 0$, $\operatorname{sgn} y(x) = \operatorname{sgn} y''(x) \neq \operatorname{sgn} y'(x)$ *for $x \in (-\infty, \infty)$, y, y', y'' are monotonous functions of $x \in (-\infty, \infty)$* and *$y' \to 0$, $y'' \to 0$ for $x \to \infty$.* [1].

The question is about the solution without zeros in the case $A(x) \geqq 0$. For $n = 3$ hold the following results [2]:

Theorem 6. *Let $n = 3$. Let $b(x) \geqq 0$ for $x \in (-\infty, \infty)$ and $b(x) \equiv 0$ do not hold in any interval. Then the solution of the differential equation (a) has at least one solution without zero points in $(-\infty, \infty)$.*

Theorem 7. *Let the assumptions of Theorem 6 be fulfilled and let $\int\limits_{x_0}^{\infty} b \, \mathrm{d}t$ diverge. Then the differential equation (a) has at least one solution without zero-points for which holds $\liminf\limits_{\langle x_0, \infty)} y(x) = 0$.*

If $b(x) \geqq m > 0$ for $x \in (x_0, \infty)$ then $y(x)$ belongs to the class L^2.

M. Zlámal [3] proved the following theorem:

Let $n = 3$. Let $A(x) \geqq m > 0$, $A'(x) + b(x) \geq m$ and $b(x) - A'(x) \geqq 0$ for $x \in (x_0, \infty)$.

Then every solution of the differential equation (a) is either oscillatory in (x_0, ∞) or non oscillatory and then $\lim\limits_{x \to \infty} y = \lim\limits_{x \to \infty} y' = 0$ and $y(x)$ is of the class L^2.

In the paper [2] is shown that under given assumptions all solutions of the differential equation (a) are oscillatory in (x_0, ∞) with the exception of one (up to the linear dependence) which has the mentioned properties.

Now we shall devote our attention to the differential equation (b). According to Theorem 1, the equation (b) has at least one solution without zero points and every solution of the differential equation (b) of the properties given in Lemma 2 and in the Remark 1 has not on the right side of a zero and there are no zeros of the derivatives up to the order $n - 1$. In the following we give the criteria for the rate of divergence of these solutions to the infinity.

Theorem 8. *Let $A(x) \leqq 0$, $b(x) \geqq 0$ for $x \in (-\infty, \infty)$ and let $b(x) \equiv 0$ do not* *hold in any interval. Let $f(x)$ be a non-negative function with the continuous n-th derivative of properties* $f^{(n)} + 2Af' + [A' - b]f \leqq 0$ *for* $x \in (x_0, \infty)$, $-\infty < < x_0 < \infty$.

Then there exists for every non-trivial solution $z(x)$ of the differential equation (b) of the properties $z(a) = 0$, $z^{(i)}(a) \geqq 0$, $i = 1, \ldots, n - 1$, $a \geqq x_0$, such a $\alpha \geqq a$ and such a constant $k > 0$ that for $x > \alpha$ holds $z(x) - kf(x) > 0$.

Corollary 1. Let $f(x) = e^x$. Then the non-trivial solution of the differential equation (b) of the properties $z(a) = 0$, $z^{(i)}(a) \geqq 0$ diverges to $+\infty$ faster then e^x if $A(x) \leqq 0$, $b(x) \geqq 0$, $1 + 2A + A' - b \leqq 0$ for $x \in (x_0, \infty)$, $x_0 \leqq a$ and at the same time $b(x) \equiv 0$ does not hold in any interval.

For $n = 3$ and the case $A(x) \geqq 0$ hold the following criteria:

M. Zlámal [3] proved:

Let $A(x) \geqq 0$, $A'(x)$ and $b(x) \geqq 0$ be continuous functions of x for $0 < x_0 \leqq x$. Futher let on (x_0, ∞) $M = \limsup \dfrac{A(x)}{\sqrt{x}} < \infty$, $m = = \limsup \sqrt{x}\,[A'(x) - b(x)] < 0$. Then every non-trivial solution $y(x)$ fo the differential equation (b) is either oscillatory or diverges into $\pm\infty$ faster then a certain positive power x_0. The solution $y(x)$ is oscillatory then, and only then when for every $x \in (x_0, \infty)$ holds $yy'' - \dfrac{1}{2}y'^2 + Ay^2 < 0$.

If $b(x) \geqq d > 0$, then every scillatory solution of the differential equation (b) belongs to the class L^2.

Theorem 9. *Let $n = 3$ and let $A(x) \geqq 0$, $b(x) \geqq 0$, $A'(x) - b(x) \leqq 0$ at the same time $b(x) \equiv 0$ do not hold, in any interval and let $\int_{x_0}^{\infty} b\,\mathrm{d}t$ diverge. Further let $f(x)$ be a non-negative function with continuous third derivative on (x_0, ∞) for which holds on (x_0, ∞) the inequality $f''' + 2Af' + (A' - b)f \leqq 0$. Then for every solution $z(x)$ of the differential equation (b) without zeropoints on the interval (α, ∞), $\alpha \geqq x_0$ holds: $|z| - kf > 0$ for $x \in (\alpha, \infty)$, k is a suitable constant.*

The proof is given in the paper [2].

II. We devide this section in two parts. In the first section we shall deal with certain non—homogenous boundary value problems chiefly of the 3[d] order, and in the second section we shall show some results of the so called Sturm boundary problems of the 3[d] order.

Let the boundary problem

(4) $L(y) = 0$.

(5) $Ui(y) = 0$, $i = 1, 2, \ldots, n$, be given where $L(y)$ is a linear differential operator of the n^{th} order, $n \geqq 2$, with continuous coefficients $p_0 \neq 0$ (coefficient of the highest derivative) $p_1, \ldots, p_n$ on the interval $\langle a_1, a_m\rangle$, Ui, $i = 1, 2, \ldots, n$ are linearly independent forms of $y(a_1), \ldots, y^{(n-1)}(a_1), \ldots,$

$y(a_m), \ldots, y^{(n-1)}(a_m)$, $a_1 < a_2 < \ldots < a_m$, $m \geqq 2$. Suppose that the problem (4), (5) is unsolvable, i.e. its only solution is trivial. Then the following theorem [4] holds.

Theorem 10. *For an arbitrary point* $\xi \in (a_k, a_{k+1})$ *the function* $y = G_k(x, \xi)$ *may be constructed (the particular Green's function) which has the following properties:*

1. $G_k(x, \xi)$, $\dfrac{\partial}{\partial x} G_k(x, \xi), \ldots, \dfrac{\partial^{n-2}}{\partial x^{n-2}} G_k(x, \xi)$ *are continuous functions of* $x \in \in \langle a_1, a_m \rangle$.

2. *The function* $\dfrac{\partial^{n-1}}{\partial x^{n-1}} G_k(x, \xi)$ *is on* $\langle a_1, a_m \rangle$ *everywhere continuous with the exception of the point* ξ, *where it has a discontinuity of the first order with a jump of the discontinuity* $\dfrac{1}{p_0(\xi)}$, *i.e.*

$$\frac{\partial^{n-1}}{\partial x^{n-1}} G_k(\xi + 0, \xi) - \frac{\partial^{n-1}}{\partial x^{n-1}} G_k(\xi - 0, \xi) = \frac{1}{p_0(\xi)}.$$

3. *The function* $G_k(x, \xi)$ *is the solution of the equation* (4) *on the intervals* $\langle a_1, \xi)$, $(\xi, a_m\rangle$ *and satisfies the boundary conditions* (5).

4. *The function* $G_k(x, \xi)$ *is by the properties* 1., 2., 3., *uniquely defined.*

Theorem 11. *Let* $G_k(x, \xi)$, $k = 1, 2, \ldots, m-1$, *be the particular Green's functions belonging to the problem* (4), (5). *Then the solution of the problem*

(4′) $L(y) = r(x)$.

(5) $U_i(y) = 0, \quad i = 1, 2, \ldots, n$

where $r(x)$ *is the continuous function on* $\langle a_1, a_m \rangle$ *is given by the formula*

$$y(x) = \sum_{k=1}^{m-1} \int_{a_k}^{a_{k+1}} G_k(x, \xi)\, r(\xi)\, \mathrm{d}\xi. \tag{6}$$

Lemma 4. *Let* $n = 3$. *Let* $A(x) \leqq 0$, $A'(x)$, $b(x) \geqq 0$ *be such continuous functions of* $x \in (-\infty, \infty)$ *that* $b(x) - A'(x) \leqq 0$ *and* $b(x) \equiv 0$ *does not hold in any interval. Then every solution of the differential equation* (a) *has at most two points or one double zero-point* [2].

Lemma 5. *Let* $n = 3$. *Let* $A(x) \leqq 0$, $A'(x)$, $b(x) \geqq 0$ *be such continuous functions of* $x \in (-\infty, \infty)$ *that* $A'(x) + b(x) \leqq 0$ *and* $b(x) \equiv 0$ *does not hold in any interval. Then every solution of the differential equation* (a) *has only two zero points or one double zero point* [2].

Theorem 12. *Let* $n = 3$. *Let the coefficients of the differential equation* (a) *fulfil the assumptions of Lemma* 4, *resp.* 5. *Then the boundary problem*

$$y''' + 2A(x)\,y' + [A'(x) + b(x)]\,y = r(x),$$
$$y(a_1) = y(a_2) = y(a_3) = 0, \qquad a_1 \leq a_2 < a_3 \in (-\infty, \infty)$$

has only one solution given by the formula for $m = 3$.

The proof follows from Lemma 4 and 5 and from Theorem 11.

Lemma 6. *Let $n = 3$. Let the assumption of Lemma 5 be satisfied. Then every solution of the differential equation (a) has at most three zero points of the first derivative. If the solution $y(x)$ has exactly three zero points of the first derivative, then $y(x)$ has exactly two zeros.*

Theorem 13. *Let $n = 3$. Let the coefficients of the differential equation (a) fulfill the assumptions of Lemma 5.*

Then the boundary problem

$$y''' + 2A(x)\,y' + [A'(x) + b(x)]\,y = r(x),$$
$$y^{(i)}(a) = y^{(i)}(b), \qquad i = 0, 1, 2; \qquad a < b$$

with periodical boundary conditions has just one solution of the form (6) for $m = 2$.

Remark 3. M. Gera of Bratislava in his dissertation devotes his attention to the problems of the higher orders of periodic boundary conditions.

Now let us consider the differential equation (a) for $n = 3$ in case that the coefficients are continuous functions of the parameter $\lambda \in (\Lambda_1, \Lambda_2)$ i.e. in the form

$$(\bar{a}) \qquad y''' + 2A(x, \lambda)\,y' + [A'(x, \lambda) + b(x, \lambda)]\,y = 0.$$

The following oscillatory theorem [2] holds:

Theorem 14. *Let the coefficients of the equation $(\bar{a})$ $A = A(x, \lambda)$, $A' = \dfrac{\partial}{\partial x} A(x, \lambda)$, $b = b(x, \lambda) > 0$ be continuous functions of $x \in (-\infty, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$.*

Further let $|A(x, \lambda)| \leqq k$, $|A'(x, \lambda)| \leqq k$, $k > 0$, for all $x \in (-\infty, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$ and let $\lim\limits_{\lambda \to \Lambda_2} b(x, \lambda) = +\infty$ uniformly for all $x \in (-\infty, \infty)$.

Let $a < b \in (-\infty, \infty)$ be given numbers and let $y(x, \lambda)$ be a solution of the differential equation $(\bar{a})$ with the property $y(a, \lambda) = 0$. Then with the increasing $\lambda \to \Lambda_2$ increases also the number of zero points of the solution y in (a, b) to the infinity and at the same time the distance of every two neighbouring zero points converges to zero.

Remark 4. G. Samsone [5] proved also the oscillatory theorem which can be formulated for the equation $(\bar{a})$ as follows:

Let $A = A(x, \lambda)$, $A' = \dfrac{\partial}{\partial x} A(x, \lambda)$, $b = b(x, \lambda) \geqq 0$ be continuous functions of $x \in (-\infty, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$ and $\lim\limits_{\lambda \to \Lambda_2} A(x, \lambda) = +\infty$ hold uniformly for all $x \in (-\infty, \infty)$.

Let $b(x, \lambda) \equiv 0$ do not hold in any interval. Then the statement of the previous theorem holds.

With the help of the oscillatory theorems the existence of eigenvalues and of eigenfunctions of the following boundary problem can be proved:

Theorem 15. *Let the coefficients of the differential equation* ($\bar{a}$) *fulfil the assumptions of oscillatory theorems. Let* $a \leqq b < c \in (-\infty, \infty)$ *be given numbers. Further let* $\alpha(\lambda)$, $\alpha_1(\lambda)$, $\beta(\lambda)$, $\beta_1(\lambda)$ *be continuous functions of the parameter* $\lambda \in (\Lambda_1, \Lambda_2)$ *for which holds* $|\alpha| + |\alpha_1| \neq 0$, $|\beta| + |\beta_1| \neq 0$, *at the same time either* $\beta(\lambda) \equiv 0$, or $\beta(\lambda) \neq 0$ *for all the* $\lambda \in (\Lambda_1, \Lambda_2)$.

Then there exists such a natural number ν *and such a sequence of the parameter* λ *(eigenvalues)*:

$$\lambda_\nu,\ \lambda_{\nu+1},\ \ldots,\ \lambda_{\nu+p},\ \ldots$$

to which belongs the functional sequence (eigenfunctions)

$$y_\nu,\ y_{\nu+1},\ \ldots,\ y_{\nu+p},\ \ldots$$

of such property that $y_{\nu+p} = y(x, \lambda_{\nu+p})$ *is the solution of the differential equation which fulfils the following boundary conditions*

$$\begin{gathered} y(a, \lambda_{\nu+p}) = 0, \\ \alpha_1(\lambda_{\nu+p})\, y(b, \lambda_{\nu+p}) - \alpha(\lambda_{\nu+p})\, y'(b, \lambda_{\nu+p}) = 0, \\ \beta_1(\lambda_{\nu+p})\, y(c, \lambda_{\nu+p}) - \beta(\lambda_{\nu+p})\, y'(c, \lambda_{\nu+p}) = 0 \end{gathered}$$

and $y(x, \lambda_{\nu+p})$ *has in* (a, c) *exactly* $\nu + p$ *points of zero.*

BIBLIOGRAPHY

[1] M. Greguš: *Über die asymptotischen Eigenschaften der Lösungen der linearen Differentialgleichung dritter Ordnung, Annali di Matematica pura ed applicata IV. LXIII, 1963, 1–10.*

[2] M. Greguš: *Über die lineare homogene Differentialgleichung dritter Ordnung, Wiss. Zeitschr. Univ. Halle, Math., Nat., XII/3, 1963, 265–286.*

[3] M. Zlámal: *Asymptotic properties of the solutions of the third Order linear differential equation, Spisy Přír. fak. MU, Brno, 1951, 159–167.*

[4] M. Greguš: *Über das Randwertproblem der n-ten Ordnung in m-Punkten, Acta F. R. N. Univ. Comen. IX., 2., 1964, 49–55.*

[5] G. Sansone: *Studi sulle equazioni differenziali lineari omogenee di terzo ordine nel campo reale, Revista Matem. y Fisica Teorica, Serie A, 1948, 195–253.*

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XVII — 1967

INVARIANT MANIFOLDS FOR FLOWS

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The purpose of this paper is to present a geometric approach to the theory of invariant manifolds of differential systems. Let the concept of an invariant manifold of a differential system be illustrated by the following simple and rather typical example (which is frequently met in applications to electrical systems).

(1) $$\dot{x} = Ax, \qquad \dot{\varphi} = 0,$$

(2) $$\dot{x} = Ax + f_1(x, \varphi, t), \qquad \dot{\varphi} = f_2(x, \varphi, t).$$

Here x, f_1 are n-vectors, A is an $n \times n$-matrix, φ is a coordinate vector on an m-dimensional torus Φ. Assume that the real parts of the characteristic numbers of A are different from zero. The subset of $E_n \times \Phi \times E_1$, which consists of all points $(0, \varphi, \check{t})$, $\check{\varphi} \in \Phi$, $\check{t} \in E_1$, is obviously invariant with respect to (1), i.e. if $x(\check{t}) = 0$, then $x(t) \equiv 0$ for a solution (x, φ) of (1). If f_1, f_2 are sufficiently small, then a similar situation holds for (2), more precisely there exists a map p from $\Phi \times E_1$ to E_n such that if $\tilde{x} = p(\tilde{\varphi}, \check{t})$, then there exists a solution (x, φ) of (2) defined on E_1, $x(\check{t}) = \tilde{x}$, $\varphi(\check{t}) = \tilde{\varphi}$ and $x(t) = p(\varphi(t), t)$ for $t \in E_1$. The map p is unique and the set P of all $(\tilde{x}, \tilde{\varphi}, \check{t})$, $\tilde{x} = p(\tilde{\varphi}, \check{t})$ is the invariant manifold of (2). The behaviour of the solutions of (2) near P is similar to the behaviour of solutions of (1) near the plane $\tilde{x} = 0$.

Usually it is assumed that the perturbation f_i, $i = 1, 2$ is small in that sense that it fulfils one of the following conditions

(I) $f_i(0, \varphi, t) = 0, \qquad \|f_i(x_1, \varphi, t) - f_i(x_2, \varphi, t)\| \leqq L \, \|x_1 - x_2\|, \qquad i = 1, 2$

L being small (which is usually fulfilled in the way that f_i contains higher powers in x only),

(II) $$f_i(x, \varphi, t, \varepsilon) = \varepsilon g_i(x, \varphi, t), \qquad i = 1, 2,$$

g_i fulfilling some boundedness conditions, ε being a parameter, which is at our disposal and which may be chosen sufficiently small,

(III) $$f_i(x, \varphi, t, \varepsilon) = h_i(x, \varphi, t/\varepsilon), \qquad i = 1, 2,$$

$h_i(x, \varphi, \tau)$ being periodic or almost periodic in τ, the average of h_i with respect to τ being zero and ε being again the small parameter.

Or it may be assumed that f_i is a sum of three terms, each of which fulfils one of conditions (I), (II), (III). Theorems of the above type were proved for a large number of various situations, for example the matrix A need not be constant, there may appear a small parameter ε on the right hand side of some rows of (2) or on the left hand sides of some rows (i.e. at derivatives of some components of x or φ) and recently similar theorems were proved for equations with time-lags or for functional differential equations.

The unifying theory may be obtained by a geometric approach. The basic concept is the one of a flow, which is more general then the concept of a dynamical system. It differs from the concept of a dynamical system in the following way: the solution $y(t, \check{y}, \check{t})$ which passes through $\check{y}$ in the moment $\check{t}$ need not exist on the whole real axis but on some interval $\langle \check{t}, t_1 \rangle$, $t_1 > \check{t}$ and uniqueness of solutions is required with t increasing only. The values of the solutions y are from a metric space or from a Banach space. By a flow Y we shall mean a set of functions fulfilling some axioms and in special cases Y may be the set of all solutions of a differential equation or of a functional differential equation. The elements y of a flow Y will be called solutions.

The conditions which guarrantee the existence and uniqueness of an invariant manifold for a flow, cannot be stated in detail here. They may be described roughly as follows: the space Y, where the solutions y of the flow Y take their values from, may be represented as a product of two spaces X and Φ and the x- and φ-components of the solutions y satisfy some inequalities.

General Theorem: *If the above conditions are satisfied, then there exists a unique map p from $\Phi \times E_1$ to X such that if $\check{x} = p(\check{\varphi}, \check{t})$, then there exists a solution $y = (x, \varphi)$ from the flow, which is defined on the whole real axis $x(\check{t}) = \check{x}$, $\varphi(\check{t}) = \check{\varphi}$ and $x(t) = p(\varphi(t), t)$ for $t \in E_1$. Again the set P of all $(\check{x}, \check{\varphi}, \check{t})$, $\check{x} = p(\check{\varphi}, \check{t})$ is an invariant subset of the flow Y and it is possible to describe the behaviour of solutions from Y near P.*

Let several features of the General Theorem be emphasized.

(i) If the flow Y fulfils the conditions from the General Theorem, then every flow Z, which is sufficiently close to Y, fulfils conditions of the same type and therefore there exists an invariant subset of Z. The fact that flows Y and Z are close is described by two numbers $\zeta > 0, T > 0$, ζ being small and T being large and it is required that the following inequalities hold

$$(3) \qquad \|y(t, \check{u}, \check{t}) - z(t, \check{u}, \check{t})\| \leq \zeta \quad \text{for} \quad \check{t} \leq t \leq \check{t} + T,$$

$$(4) \qquad \|y(t, \check{u}, \check{t}) - y(t, \check{v}, \check{t}) - z(t, \check{u}, \check{t}) + z(t, \check{v}, \check{t})\| \leq \zeta \, \|\check{u} - \check{v}\| \quad \text{for} \quad \check{t} \leq t \leq \check{t} + T.$$

Assume that flows Y and Z are the sets of all solutions of

$$\dot{y} = f(y, t), \tag{5}$$

$$\dot{z} = g(z, t), \tag{6}$$

the space Y where the solutions y and z take their values from being a Banach space.

Theorem CDP (Continuous Dependence on a Parameter): *If f and g fulfil some boundedness conditions, then flows Y and Z are close in the above sense if $\| \int_t^{t+\Delta} [f(y, \sigma) - g(y, \sigma)] \, d\sigma$ is sufficiently small for all y, t and $0 < \Delta \leqq 1$.*

Theorem CDP may be applied in the special case that

$$\dot{y} = h(y, t/\varepsilon), \tag{7}$$

$$\dot{z} = h_0(z), \tag{8}$$

$h_0(z) = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} h(z, \sigma) \, d\sigma$ the limit being uniform with respect to z and t. This way the averaging principle is included into the above theory without any transformation of coordinates.

(ii) The theory of invariant manifolds (or subsets) may be developed for metric spaces. It is clear that the norm of the difference of y and z in (3) is to be replaced by the distance; (4) in the case of metric spaces is formulated in a more complicated way. Usually the invariant subset is a product of a torus with an Euclidian space, but in the above theory the invariant subset may be a general complete metric space Φ. The theory simplifies considerably, if Φ has the following property:

(A) if Ψ is a continuous map from Φ to Φ, if Ψ^{-1} exists and fulfils a Lipschitz condition, then $\Psi(\Phi) = \Phi$. It is very easy to prove that every finite-dimensional manifold has the property (A) and there exist spaces having property (A), which are not manifolds.

(iii) There are no periodicity or almostperiodicity conditions in General Theorem. If it happens that the flow Y is periodic [i.e. if f is periodic in t in the case that Y is the set of solutions of (5)], it is verified easily that the invariant subset remains an invariant subset, if it is shifted in the time by the period of the flow; as the invariant manifold is unique, it is necessarily periodic. In a similar way almostperiodicity may be treated.

(iv) General Theorem may be applied if the behaviour of solutions near the invariant subset is like the behaviour of solutions of a differential system near a saddle point; the case that the invariant subset is exponentially stable is the most simple one.

(v) Systems with discrete time — i.e. transformations — are included in General Theorem.

(vi) General Theorem may be applied in case of singular perturbations.

As one of the applications of the above theory the following result may be mentioned: It is well known that solutions of differential equations with time lags or of functional differential equations cannot be prolonged with t decreasing in general. It may be deduced from the above theory that the solutions of a functionally perturbed ordinary differential equation, which are defined on E_1, fill up an $(n + 1)$-dimensional manifold, if the unperturbed equation is a (nonlinear) ordinary differential equation in E_n or in an n-dimensional manifold the right hand side of which fulfils some boundedness conditions. The reason is in the very simple structure of the flow which corresponds to the unperturbed equation considered as a functional equation: the x-component of any solution from this flow tends to zero extremely rapilly. Of course the necessary boundedness conditions are not fulfilled by equation $\dot{x}(t) = Ax(t) + \varepsilon Bx(t - 1)$ — it is well known that there exist solutions $x_j = e^{\lambda_j t}$, $j = 1, 2, 3, \ldots$ — but the above result always applies, if a functional perturbation term is added to the right hand side of $\dot{\varphi} = g(\varphi)$, φ being a coordinate vector on a compact-manifold (and some smoothness conditions being fulfilled).

Finally let some results on the Van der Pol Perturbation of a Vibrating String be described. Consider the problem

(9) $\quad u_{tt} - u_{xx} = \varepsilon h(u)\, u_t, \qquad 0 \leqq x \leqq 1, \qquad u(t, 0) = u(t, 1) = 0,$

h having similar properties as $1 - u^2$. This problem may be transformed to an ordinary differential equation in a function space of the type (7). There are no time-independent solutions of the averaged equation (8), which are continuous (in the space variable), but there exists an infinity of discontinuous ones. Some of them are exponentially stable, other ones are unstable so that the picture rendered by the averaged equation is rather complicated. For the unperturbed equation it may be proved that there exist smooth solutions, tending with $t \to \infty$ to periodic ones, which are discontinuous. Thus it is shown that there exist discontinuous periodic solutions of (9) and that discontinuous solutions appear in a natural way, if (9) is examined.

REFERENCES

[1] J. Kurzweil, *Exponentially Stable Invariant Manifolds, Continuous Dependence on Parameters and Averaging Principle, Czech. Math. J. 16 (1966), 3, 4.*

[2] J. Kurzweil, *Invariant Manifolds for Flows, Proc. Int. Symp. on Dynamical Systems held in Mayaguez, Puerto Rico, December 1965.*

[3] J. Kurzweil, *Van der Pol Perturbation of the Equation of a Vibrating String, Czech. Math. J. 17 (1967).*

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VECTORS OF GEVREY CLASSES AND APPLICATIONS

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Introduction.

In several problems in partial differential equations one is led to study the space of functions u defined in a domain Ω of R^n with smooth boundary Γ and which satisfy conditions of the following type (we take here the simplest possible case):

(1) $\left(\int_\Omega |\Delta^k u|^2 \, dx\right)^{1/2} \leq c L^k M_k \qquad \forall\, k,$

(2) $\Delta^k u = 0 \quad$ on $\quad \Gamma \qquad \forall\, k,$

where c and L are suitable constants (which depend on u) and M_k is a given sequence — For example, if

(3) $M_k = (2k)!$

then (1) (2) imply that u is analytic in $\bar{\Omega} = \Omega \cup \Gamma$ (assuming Γ to be real-analytic). A much more general result of this type will be reported in Section 4 below.

Once one is led to study classes of functions satisfying conditions of type (1) (2), it is natural to put this question in a more general framework and to replace in (1) (2) Δ by an unbounded operator A in a Banach space E, condition (2) being then replaced by

($\tilde{2}$) $u \in$ domain of A, $Au \in$ domain of A, and so on, and condition (1) being replaced by

($\tilde{1}$) $||A^k u|| \leq c L^k M_k \qquad \forall\, k,$

(where $||\quad||$ denotes the norm in $\quad$).

In Sections 1,2 we give some (simple) remarks on the spaces defined by

(1) Expositery lecture. All details and other results are contained in the book [4] by E. Magenes and the A.

$\tilde{(1)}$ $\tilde{(2)}$ (the so — called "vectors of Gevrey class" when $\{M_k\}$ is a Gevrey sequence) when $(-A)$ is the infinitesimal generator of a semi-group. [This contains (1) (2) by taking $E = L^2(\Omega)$, $A = -\Delta$, the domain of A consisting of those functions u which are zero on Γ].

The plan is as follows:

1. Domains $D(A^\infty; M_k)$.
2. A criterion of non triviality.
3. The semi group on $D(A^\infty; M_k)$.
4. The case when A is an elliptic operator.
5. Transposition.
6. Cauchy problem.
7. Some examples.

Bibliography

1. Domains $D(A^\infty; M_k)$.

Let E be a reflexive Banach space, norm $\| \quad \|$; let A be an unbounded operator given in E; we assume (for semi-group theory we refer to [2], [10]):
(1.1) $(-A)$ is the infinitesimal generator of a continuous semi-group $G(t)$ in E. Let $D(A)$ be the domain of A. We set

$$D(A^\infty) = \{u \mid A^k u \in D(A) \qquad \forall\, k\};$$

it is well known [2], [10] that $D(A^\infty)$ is dense in E.

Let now $\{M_k\}$ be a given sequence of positive numbers.

We define

$$(1.2)\quad \begin{cases} D(A^\infty; M_k) = \{u \mid u \in D(A^\infty); \text{ there exist constants } c \text{ and } L \text{ (de-} \\ \text{pending on } u) \text{ such that } \|A^k u\| \leq c L^k M_k \ \forall\, k\}. \end{cases}$$

Example 1.1.

If $M_k = (k!)^\alpha$, $\alpha > 1$, the coresponding $D(A^\infty; M_k)$ space is called: the space of vectors of Gevrey class α.

Example 1.2.

If $M_k = k!$, the coresponding $D(A^\infty; M_k)$ is the space of analytic vectors. (See [8])

Remark 1.1

Definition 1.2 is purely algebraic. There is a "natural" locally convex topology on $D(A^\infty; M_k)$: firstly, fix L in (1.2) (but not C) and call $D^L(A^\infty; M_k)$

the coresponding space; provided with the norm $\sup_{k\geq 0} \frac{1}{L^k M_k} ||A^k u||$, it is a Banach space; then $D(A^\infty; M_k) =$ inductive limit of $D^{L_n}(A^\infty; M_k)$, $L_n \to +\infty$. For details see [4].

Remark 1.2.

Hypothesis (1.1) is perfectly useless in Definition (1.2). But it will be useful in the proofs below.

The "natural questions" are now:

(i) when is $D(A^\infty; M_k) \neq \{O\}$?

(ii) what is the "abstract" interest of $D(A^\infty; M_k)$?

(iii) how can one characterice, in "concrete" situations, the spaces $D(A^\infty; M_k)$ in "concrete" terms?

Partial answers to these questions are respectively given in Sections 2, 3, 4 below — some applications being given in Sections 5, 6, 7.

2. A criterion of non triviality.

Theorem 2.1. *Let $\{M_k\}$ be a non quasi-analytic sequence*[1] [1] [7]. *Then $D(A^\infty; M_k)$ is dense in E.*

Proof. 1) If $\{M_k\}$ is non quasi-analytic, one can find a sequence ϱ_n of functions with the following properties [7] [9]

$$(2.1) \begin{cases} \varrho_n \in D_{M_k},\ \varrho_n(t) = 0 \text{ if } t \leq 0 \text{ or if } t \geq \varepsilon_n,\ \varepsilon_n \to 0 \text{ if } n \to \infty, \\ \varrho_n \geq 0,\ \int_0^\infty \varrho_n(t)\, dt = 1. \end{cases}$$

2) Define next $G(\varrho_n) \in L(E;E)$ by

$$(2.2)\ G(\varrho_n)\, e = \int_0^\infty G(t)\, e \,.\, \varrho_n(t)\, dt,\ e \in E$$

One checks easily that $G(\varrho_n)\, e \in D(A^\infty)$ and that

$$(2.3)\ A^k G(\varrho_n)\, e = G(\varrho_n^{(k)})\, e\ \forall\ k.$$

Thanks to the fact that $\varrho_n \in D_{M_k}$ it follows that $G(\varrho_n)\, e \in D(A^\infty; M_k)$.

3) Let now e be arbitrarely given in E; by (2.1) $G(\varrho_n)\, e \to e$ in E, and by 2), $G(\varrho_n)\, e \in D(A^\infty; M_k)$, hence the result follows.

Remark 2.1. It can happen that $D(A^\infty; M_k)$ is dense in E even with $M_k = 1\ \forall\ k$ example: assume that A has a complete set in E of eigenvectors ω_n then $A\omega_n = \lambda_n \omega_n$ hence $||A^k \omega_n|| \leq ||\omega_n||\, \lambda_n^k$, i.e. belongs to $D(A^\infty; 1)$.

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(2) This means: let D_{M_k} be the space of C^∞ scalar functions φ on R with compact support and satisfying $|\ldots| \leq |\varphi^{(k)}(t)| \leq cL^k M_k\ \forall k$ then $D_{M_k} \neq \{O\}$.

But in can happen that $D(A^\infty; M_k) = \{O\}$ if M_k is quasi — analytic; example: $E = L^p(0, \infty)$, $A = \frac{\mathrm{d}}{\mathrm{d}x}$, $D(A) = \left\{ f \mid f, \frac{\mathrm{d}f}{\mathrm{d}x} \in L^p(0, \infty),\ f(0) = 0 \right\}$.

3. The semi-group on $D(A^\infty; M_k)$.

Theorem 3.1. *The necessary and sufficient condition for $u \in E$ to be in $D(A^\infty; M_k)$ is that the function*

(3.1) $G(.)\, u =$ "$t \to G(t)\, u$"

is of class M_k with values in E, i.e.

$$(3.2)\quad \begin{cases} \text{for every finite } T \text{ there exist constants } C_1 \text{ and } L_1 \text{ (depending on } T \text{ and} \\ u\text{) such that} \\ \left\| \dfrac{\mathrm{d}^k}{\mathrm{d}t^k} G(t)\, u \right\| \leq C_1 L_1^k M_k \quad \forall\ k,\ t \in [0, T]. \end{cases}$$

Remark 3.1. This property justifies the terminology introduced in Examples 1.1 and 1.2.

Proof of Theorem 3.1.

1) (3.2) implies (1.2) (with $C = C_1$, $L = L_1$). Obvious, take $t = 0$ in (3.2) and use $\frac{\mathrm{d}^k G(t)}{\mathrm{d}t^k} \cdot u\,|_0 = (-1)^k A^k u$.

2) (1.2) implies (3.2). Obvious too. Indeed $\frac{\mathrm{d}^k}{\mathrm{d}t^k} G(t)\, u = (-1)^k G(t)\, A^k u$ hence, for $t \in [0, T]$

$$\left\| \frac{\mathrm{d}^k G(t)}{\mathrm{d}t^k} u \right\| \leq \sup_{t \in [0, T]} \|G(t)\|_{L(E;E)} \|A^k u\|,$$

hence 3.2 follows.

It follows easily from Theorem 3.1 that (see [4] for details).

Theorem 3.2. *For every t, $G(t)$ is a continuous linear mapping from $D(A^\infty; M_k)$ into itself; the semi group $G(t)$ in $D(A^\infty; M_k)$ is C^∞ (and of infinitesinal generator $-A$).*

One can also show [4] that if for a suitable constant d

(3.3) $M_{k+j} \leq d^{k+j} M_k M_j \quad \forall\ k, j$

then for every $u \in D(A^\infty; M_k)$ the function $t \to G(t)\, u$ is of class M_k in $t \geq 0$ with values in $D(A^\infty; M_k)$ (i.e., for every finite T, there exists a bounded set B in $D(A^\infty; M_k)$ and a constant L such that $\frac{1}{L^k M_k} \frac{\mathrm{d}^k}{\mathrm{d}t^k} G(t)\, u \in B \ \forall\ k$, $t \in [0, T]$).

4. The case when A is an elliptic operator.

Let us recall first a classical definition: a complex-valued function φ defined on a compact set of R^n is said of Gevrey order $\beta > 1$ (resp. real analytic) if for suitable constants c and L one has

$$|D^p\varphi(x)| \leq cL^{p_1+\cdots+p_n}(p_1!\,p_2!\,\ldots\,p_n!)^\beta \quad (\text{resp. } \beta = 1)$$

$\forall\, p = \{p_1, \ldots, p_n\}$, $\forall\, x \in$ compact set of definition of φ.

Let Ω be a bounded open set of R^n, of boundary Γ; we assume

(4.1) $\begin{cases} \Gamma \text{ is a } (n-1) \text{ dimensional variety, of Gevrey order } \beta \text{ (resp. real} \\ \text{analytic)} \end{cases}$

Let A be a differential operator in Ω; we assume that

(4.2) A is an elliptic operator of order $2m$ (and properly elliptic if $n = 2$) and that

(4.3) the coefficients of A are of Gevrey order β (resp. real analytic) in $\bar{\Omega}$.

We are going to characterize $D(A^\infty; M_k)$, taking

(4.4) $E = L^2(\Omega)$.

(4.5) $D(A) = \{u \mid u \in H^{2m}(\Omega) \cap H_0^m(\Omega)\}$ (that is: $D^pu \in L^2(\Omega)\ \forall\, p$, $|p| \leq$ $\leq 2m$, $D^pu = 0$ on $\Gamma\ \forall$, $|p| \leq m - 1$),

and when we choose

(4.6) $M_k = [(2km)!]^\beta$.

One can prove (see [5], [6], [4]):

Theorem 4.1. *We assume the hypotheses* (4.1), (4.2), (4.3) *to hold choosing* $D(A)$ *and* M_k *by* (4.5) (4.6) *one has*

(4.7) $\begin{cases} D(A^\infty; M_k) \equiv \textit{ functions of Gevrey order } \beta \textit{ in } \bar{\Omega} \textit{ (resp. real analytic)} \\ \textit{which satisfy the boundary conditions “}A^ku \in H_0^m(\Omega)\ \forall\, k\textit{”.} \end{cases}$

Remark 4.1. Under the hypothesis (4.2), $-A$ is the infinitesimal generator of a semi-group in E and even of an analytical semi-group. [2], [10].

One can replace $E = L^2(\Omega)$ by $L^p(\Omega)$, $1 < p < \infty$, $p \neq 2$, without changing $D(A^\infty; M_k)$.

Remark 4.2. The same result holds true for other boundary conditions than the Dirichlet boundary conditions considered above. — See [4].

Remark 4.3. If u satisfies $||A^ku|| \leq cL^k((2km)!)\ \forall\, k$ and no boundary conditions, then one can conclude that u is real analytic on every compact subset of Ω; see [3]; this result in contained in Theorem 4.1.

Remark 4.4. A more general result is proved in [4] when we also consider “non-zero boundary conditions”.

5. Transposition

Since E is assumed to be a reflexive Banach Space (actually "reflexive" is used here for the first time — and in a non essential manner!) all what we said in Sections 1, 2, 3 is valid after replacing

E by E' = dual of E

$G(t)$ by $G^*(t)$ = adjoint of $G(t)$

A by A^*, A^* being the adjoint of A in the sense of unbounded operators in E or the (opposite to the) infinitesimal generator of the adjoint semi-group $G^*(t)$.

Consequently:

(5.1) $G^*(t)$ is a semi-group in $D(A^{*\infty}; M_k)'$.

If we make the hypothesis (see Theorem 1.1):

(5.2) $D(A^{*\infty}; M_k)$ is dense in E'

then we can identify E to a sub-space of the dual $D(A^{*\infty}; M_k)'$ of $D(A^{*\infty}; M_k)$; summing up, we have

(5.3) $D(A^\infty; M_k) \subset E \subset D(A^{*\infty}; M_k)!$

Taking the adjoint of (5.1) we obtain:

(5.4) $[G^*(t)]^*$ is a semi-group in $D(A^{*\infty}; M_k)'$.

But one easily checks that $(G^*(t))^*$ is an extension of $G(t)$, that we can still denote by $G(t)$. Therefore:

$$(5.5)\quad \begin{cases} G(t) \text{ is a semi-group in } D(A^{*\infty}; M_k)', \text{ which is } C^\infty \text{ and whose infinite-} \\ \text{simal generators is } -A. \end{cases}$$

For more details, see [4].

Remark 5.1. In the applications, $D(A^{*\infty}; M_k)'$ is not a space of distributions but a space of functionals (analytic functionals of Gervey's functionals). Structure theorems for the elements of $D(A^{*\infty}; M_k)'$ are given in [4].

6. Cauchy problem.

If $-A$ is the infinitesimal generator of a semi-group $G(t)$, then the unique solution of the Cauchy problem

$$(6.1)\quad Au + u' = 0 \qquad \left(u' = \frac{du}{dt}\right),$$

$$(6.2)\quad \begin{cases} u(t) \in D(A), \\ u(0) = u_0 \end{cases}$$

is given by

(6.3) $u(t) = G(t)\, u_0$.

See [2], [10].

Thanks to Theorem 3.2 and its "transposed" version (5.5) we obtain:

Theorem 6.1. *We assume that* (5.2) *holds true — For u_0 given in $D(A^\infty; M_k)$ (resp. in $D(A^{*\infty}; M_k)'$) the Cauchy problem* (6.1), (6.2) *admits a unique solution, given by* (6.3), *which is C^∞ from $t \geq 0 \to D(A^\infty; M_k)$ (resp. $D(A^{*\infty}; M_k)'$). Moreover, in case* (3.3) *holds true, the solution $u(t)$ is of class M_k.*

Remark 6.1. In case $G(t)$ is analytic (see Remark 4.1) then, even starting with $u_0 \in D(A^{*\infty}; M_k)'$ (i.e. with an extremely general Cauchy data), one has

$$u(t) \in D(A^\infty; M_k) \ \forall\, t > 0.$$

See [4].

7. Some examples.

We take the two as simple as possible cases.

7.1. *Heat equation.*

Combining results of Sections 4 and 6 we obtain the following result: let u_0 be given in Ω, satisfying

$$(7.1)\quad \begin{cases} u_0 \text{ is of Gevrey order } \beta \text{ (resp. real analytic) in } \overline{\Omega}, \text{ and } \Delta^k u_0 = 0 \text{ on} \\ \Gamma \ \forall\, k. \end{cases}$$

Then the solution of

$$(7.2)\quad -\Delta u + \frac{\partial u}{\partial t} = 0 \text{ in } \Omega \times]0, \infty[,$$

$$(7.3)\quad u(x, t) = 0 \text{ if } x \in \Gamma,\ t > 0,$$

$$(7.4)\quad u(x, 0) = u_0(x),\ x \in \Omega$$

is of Gevrey order β in x (resp. real analytic if $\beta = 1$) and of Gevrey order 2β in t.

We have just to take: $M_k = [(2k)!]^\beta$ in the general theory.

Moreover in this case Remark 6.1 applies —

7.2. *Wave equation.*

We consider now

$$(7.5)\quad -\Delta u + \frac{\partial^2 u}{\partial t^2} = 0 \text{ in } \Omega \times]0, \infty[,$$

$$(7.6)\quad u(x, t) = 0 \text{ if } x \in \Gamma,\ t > 0,$$

$$(7.7)\quad \begin{cases} u(x, 0) = u_{00}(x),\ x \in \Omega, \\ \dfrac{\partial u}{\partial t}(x, 0) = u_{01}(x),\ x \in \Omega. \end{cases}$$

Writing (7.5) as a first order system in t one can apply semi-group theory. One obtains:

$$(7.8) \begin{cases} \text{if } u_{01} \text{ and } u_0, \text{ satisfy conditions analogous to (7.1) for } u_0, \text{ then } u(x, t) \\ \text{is of Gevrey order } \beta \text{ in } x \text{ and in } t. \end{cases}$$

See [4] Vol 3 for technical details.

BIBLIOGRAPHY

[1] T. Carleman: *Les fonctions quasi analytiques-Gauthies-Villars, Paris, 1926.*

[2] E. Hille—R. S. Philips: *Functional Analysis and semi-groups. Colloq. Pub. Amer. Math. Soc. 1957.*

[3] T. Kotake—M. S. Narasimhan: *Fractional powers of a linear elliptic operator — Bull. Soc. Math. France, t. 90 (1962), p. 449—471.*

[4] J. L. Lions—E. Magenes: *Problemes aux limites non homogénes et applications — Vol. 1 and 2, Dunad, 1968. Vol. 3, to appear*

[5] J. L. Lions—E. Magenes: *Espaces de functions et distributions du type de Gevrey et problemes aux limites paraboliques. Annali di Mat. Pura ed Appl. Vol. LXVIII, (1965), p. 341—418.*

[6] J. L. Lions—E. Magenes: *Espaces du type Gevrey et problemés aux limites pour diverses classes d'équations d'évolution. Annali di Mat. Pura ed Appl. Vol. LXIX (1966).*

[7] S. Mandelbrojt: *Séries adhérentes, régularisation des suites, applications. Gauthier — Villars, Paris, 1952.*

[8] E. Nelson: *Analytic vectors. Annals of Maths. 70 (1959), p. 572—615.*

[9] C. Roumieu. *Sur quelques extensions de la notion de distributions. Ann. Sc. E. N. S. 77 (1960), p. 47—121.*

[10] K. Yosida: *Functional Analysis. Springer. t. 123. 1965.*

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
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ON THE EXISTENCE AND REGULARITY OF SOLUTIONS OF NON-LINEAR ELLIPTIC EQUATIONS

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Introduction. We shall consider boundary value problems for elliptic equations of order $2k$ in the divergent form

$$\sum_{|i|\leq k} (-1)^{|i|} D^i[a_i(x, D^j u)] = f(x)$$

where D^i is the well-known symbol for derivatives in Euclidean space $E_N : D^i = \partial^{|i|}/\partial x_1^{i_1} \ldots \partial x_N^{i_N}$. We shall deal with the problem of existence of weak solutions using direct variational methods and for them the regularity theorems will be derived. In the conclusion the converse process will be used for investigation of existence of regular solution.

Contents:

§1. Weak solution of the boundary value problem. Its determining by the variational method.

Let Ω be a bounded domain in E_N with Lipschitzian boundary $\partial\Omega$. Let us denote by $E(\overline{\Omega})$ the space of such real-valued infinitelly differentiable functions on Ω that can be continuously extended (with all their derivatives) to the closure of $\Omega : \overline{\Omega}$. $D(\Omega)$ is a subspace of $E(\overline{\Omega})$ which contains all functions with compact support.

Let $k \geq 1$ be an integer, $1 \leq m < \infty$. Let $W_m^{(k)}(\Omega)$ be a normed space of all real-valued functions which are integrable with m-th power over Ω and so do all their derivatives (in the sense of distributions) up to the k-th order. The norm of u is $\|u\|_{W_m^{(k)}} \equiv \left(\int_\Omega \sum_{|\alpha| \leq k} |D^\alpha u(x)|^m \, dx \right)^{1/m}$. Let us denote $\overset{0}{W}{}_m^{(k)}(\Omega) = \overline{D(\Omega)}$.

Let $C^{(k)}(\overline{\Omega})$ be a space of all real-valued functions which are continuous with all their derivatives up to k-th order on $\overline{\Omega}$ with usual norm and let $C^{(k),\mu}(\overline{\Omega})$ be subspace of $C^{(k)}(\overline{\Omega})$ of these functions whose k-th derivatives are μ-Hölder continuous.

We shall define functions $a_i(x, \zeta_j)$, $|i| \leq k$ for $x \in \Omega$, $-\infty < \zeta_j < \infty$, $|j| \leq k$ continuous in variables ζ_j for almost every x and measurable as functions of x for ζ_j being fixed. Each positive constant will be denoted by C. To distinguished the constants, if it is necessary we shall use indices. Let us assume

$$(1.1) \qquad |a_i(x, \zeta_j)| \leq C(1 + \sum_{|j| \leq k} |\zeta_j|^{m-1}), \qquad 1 < m < \infty$$

or less: we set $\dfrac{1}{q_{|i|}} = \dfrac{1}{m} - \dfrac{k - |i|}{N}$ if $(k - |i|)\, m < N$, $\dfrac{1}{q_{|i|}} = 0$ if $(k - |i|)\, m > N$, $\dfrac{1}{q_{|i|}} > 0$ if $(k - |i|)\, m = N$. For $1 \leq q \leq \infty$ let $q' = \dfrac{q}{q-1}$, $\varkappa_{|i|,|j|} = \dfrac{q_{|j|}}{q'_{|i|}}$ and let $C(s)$ be continuous non-negative function for $0 \leq s < \infty$. Let $g_i \in L_{q'_{|i|}}(\Omega)$, $g_{|i|}(x) \geq 0$. Let us suppose

$$(1.2) \qquad |a_i(x, \zeta_j)| \leq C\Big(\sum_{(j) < k - N/m} |\zeta_j| \Big) \Big(g_i(x) + \sum_{k - N/m \leq |j| \leq k} |\zeta_j|^{\varkappa_{|i|,|j|}}\Big).$$

The following assertion is valid: the operator $a_i(x, D^j u)$ is continuous from $W_m^{(k)}(\Omega)$ into $L_{q'_{|i|}}(\Omega)$. Its proof is based upon imbedding theorems for $W_m^{(k)}(\Omega)$ spaces. (See, for instance, E. Cagliardo [10] and also M. M. Vajnberg [28].)

Let now be $D(\Omega) \subset \mathfrak{Y} \subset E(\Omega)$, $V = \overline{\mathfrak{Y}}$ in $W_m^{(k)}(\Omega)$ and let Q be such Banach space that $D(\overline{\Omega}) = Q$ and that $\overset{0}{W}{}_m^{(k)}(\Omega) \subset Q$ algebraically and topologically. Let $u_0 \in W_m^{(k)}(\Omega)$ (stable boundary condition), $g \in V'$ such functional that $gv = 0$ for $v \in \overset{0}{W}{}_m^{(k)}(\Omega)$ (unstable boundary condition), and $f \in Q'$ (the right-hand side) be given. Let us denote $gv = \langle v, g\rangle_{\partial\Omega}$, $fv = \langle v, f\rangle_\Omega$.

Definition of the boundary value problem and of weak solution: We are looking for such $u \in W_m^{(k)}(\Omega)$ that

$$(1.3) \qquad u - u_0 \in \overset{0}{W}{}_m^{(k)}(\Omega),$$

$$(1.4) \qquad \text{for each } v \in V\text{:} \int_\Omega \sum_{|i| \leq k} D^i v a_i(x, D^j u)\, dx = \langle v, f\rangle_\Omega + \langle v, g\rangle_{\partial\Omega}.$$

Thus, boundary value problem (1.3), (1.4), we shall transfer to the problem of finding a minimum of certain functional $\Phi(v)$. There are many other aspects the problem can be approached. Thus, many authors have dealt with the existence of the solution of boundary value problem using the concept of "monotone operators" which we shall use further. (See, e. g. F. E. BROWDER [2], [3], M. I. VIŠIK [30], J. LERAY, J. L. LIONS [17]...) We shall obtain similar results; the difference is that we shall suppose certain additional condition concerning symmetry of the operator. But we shall know that certain functional has minimum in our solution. If the functional is a priori known then further considerations are analogic to those in papers: F. E. BROWDER [6], M. M. VAJNBERG, R. I. KAČUROVSKIJ [29]. See also the book by S. G. MICHLIN [18].

The condition of symmetry: Let d be the number of indices with lenght $|i| \leq k$, $\varphi \in D(E_d)$. Then (1.5) holds almost ewerywhere in Ω:

$$(1.5)\qquad (-1)^{|j|} \int_{E_d} \frac{\partial\varphi}{\partial\zeta_j} a_i(x, \zeta_\alpha)\, d\zeta = (-1)^{|i|} \int_{E_d} \frac{\partial\varphi}{\partial\zeta_i} a_j(x, \zeta)\, d\zeta.$$

There is proved in author's paper [20] (using the formula for integration of differential, see M. M. VAJNBERG [28]):

Theorem 1.1. *Let the conditions* (1.2) *and* (1.5) *be satisfied. Then*

$$(1.6)\qquad \Phi(v) = \int_0^1 dt \int_\Omega \sum_{|i|\leq k} D^i v a_i(x, D^j u_0 + tD^j v)\, dx - \langle v, f\rangle_\Omega - \langle v, g\rangle_{\partial\Omega}$$

is continuous functional on V; its Gateaux' differential is

$$(1.7)\qquad D\Phi(v, \tilde{v}) \equiv \lim_{\tau\to 0} \frac{\Phi(v + \tau\tilde{v}) - \Phi(v)}{\tau} = \int_\Omega \sum_{|i|\leq k} D^i \tilde{v} a_i(x, D^j u_0 + D^j v)\, dx - \\ - \langle\tilde{v}, f\rangle_\Omega - \langle\tilde{v}, g\rangle_{\partial\Omega}.$$

To prove the existence of minimum $\Phi(v)$ on V, we shall investigate the conditions under which the following relations hold:

$$(1.8)\qquad \lim_{\|v\|_{W_m^{(k)}}\to\infty} \Phi(v) = \infty$$

(1.9) $\Phi(v)$ is weakly lower-semicontinuous.

If v is the point of minimum of $\Phi(v)$, then $D\Phi(v, \tilde{v}) = 0$, which is (1.4). Differential (1.7) is said to be totally monotone (strictly totally monotone) if for all $v, w \in V$, $v \neq w$,

$$(1.10)\qquad \int_\Omega \sum_{|i|\leq k} D^i(w - v)\,[a_i(x, D^j u_0 + D^j w) - a_i(x, D^j u_0 + D^j v)]\, dx \geq 0,\ (> 0)$$

holds.

We shall say that the differential (1.7) is coercitive if for all $v \in V$

$$(1.11) \quad \int_{\Omega} \sum_{|i| \leq k} D^i v a_i(x, D^j u_0 + D^j v) \, \mathrm{d}x \geq \lambda(\|v\|_{W_m^{(k)}}) \qquad \text{holds}$$

where $\lambda(s)/s \in L_1(0, R)$ for every $R > 0$ and $\lim\limits_{R \to \infty} \frac{1}{R} \int_0^R \frac{\lambda(s)}{s} \, \mathrm{d}s = \infty$.

There is proved in author's paper [20].

Theorem 1.2. *Let* (1.2), (1.5), (1.10), (1.11) *be satisfied. Then there exists* min $\Phi(v)$ *(*$\Phi(v)$ *is defined by* (1.6)*), namely, in the point* v. *Function* $v + u_0$ *is the solution of problem. If the condition* (1.10) *of the strict monotony is satisfied, the solution is unique. In this case* $\Phi(v_n) \to \Phi(v) \Rightarrow v_n \rightharpoonup v$ *(weak convergence).*

Let us remark that (1.11) is satisfied, e.g. if $u_0 = 0$ and

$$\sum_{|i| \leq k} \zeta_i a_i(x, \zeta_j) \geq C \sum_{|i| = k} |\zeta_i|^m + C \cdot |\zeta_{(0,0\cdots 0)}|^m.$$

If

$$\sum_{|i| \leq k} (\xi_i - \eta_i) \cdot [a_i(x, \xi_j) - a_i(x, \eta_j)] \geq 0$$

then (1.10) is satisfied e.t.c. See author's paper [20].

Let us write the operators $a_i(x, D^j u)$ in the form $a_i(x, D^\alpha u, D^\beta u)$ where the symbol $D^\alpha u$ denotes a vector of derivatives with $|\alpha| = k$ and $D^\beta u$ a vector with $|\beta| < k$.

We say that the main part of the differential (1.7) is monotone if for v, w, $\omega \in V$

$$(1.12) \quad \int_{\Omega} \sum_{|i| = k} D^i(w - v) \, [a_i(x, D^\alpha u_0 + D^\alpha w, D^\beta u_0 + D^\beta \omega) - \\ - a_i(x, D^\alpha u_0 + D^\alpha v, D^\beta u_0 + D^\beta \omega)] \, \mathrm{d}x \geq 0$$

holds.

Let us investigate the conditions under which the functional (1.6) is weakly lower-semicontinuous. For this we need monotony of the highest derivatives [see condition (1.12)] and strengthened continuity which is to be locally uniform regardig the derivatives $D^\alpha u$.

Sufficient conditions for this are following:

Let $c(s)$, $d(s)$ be continuous functions for $0 \leq s < \infty$, non-negative, $d(0) = 0$ and assume

$$(1.13) \quad |i| = k : |a_i(x, \zeta_\alpha, \xi_\beta) - a_i(x, \zeta_\alpha, \eta_\beta)| \leq \\ \leq c(\max(\sum_{|\beta| < k - N/m} |\xi_\beta|, \sum_{|\beta| < k - N/m} |\eta_\beta|)) \cdot [d(\sum_{|\beta| < k - N/m} |\xi_\beta - \eta_\beta|) \cdot \\ \cdot (1 + \sum_{|\alpha| = k} |\zeta_\alpha|^{m-1}) + \sum_{|\alpha| = k, k - N/m \leq |\beta| < k} |\zeta_\alpha|^\lambda \, |\xi_\beta - \eta_\beta|^\mu \, |\beta|],$$

where $0 < \mu_{|\beta|} < q_{|\beta|} \cdot \frac{m - 1 - \lambda}{m}$. Let

(1.14) $\quad a_i(x, \zeta_\alpha, \zeta_\beta) = \sum_{|\alpha|=k} \zeta_\alpha a_{i\alpha}(x, \zeta_\beta) + a_i(x, \zeta_\beta)$

hold for $|i| < k$. Let $a_{i\alpha} \not\equiv 0$ at most when $q_{|i|} > \dfrac{m}{m-1}$. Let us suppose

(1.15) $\quad |a_{i\alpha}(x, \zeta_\beta)| \leq c(\sum_{|\beta|<k-N/m} |\zeta_\beta|) \cdot (1 + \sum_{k-N/m\leq|\beta|<k} |\zeta_\beta|^{\nu_{|\beta|}})$

where $0 \leq \nu_{|\beta|} < \dfrac{(m-1)\, q_{|i|} - m}{m \cdot q_{|i|}} \cdot q_{|\beta|}$ and

(1.16) $\quad |a_i(x, \zeta_\beta)| \leq c\,(\sum_{|\beta|<k-N/m} |\zeta_\beta|) \cdot (g_i(x) + \sum_{k-N/m\leq|\beta|\leq k} |\zeta_\beta|^{\varkappa^*_{|i|,|\beta|}})$,

where $g_i(x) \geq 0$, $g_i \in L_{q^*_{|i|}}$ and $q^*_{|i|} > q'_{|i|}$ if $k - N/m \leq |i|$; $q^*_{|i|} = 1$ if $|i| <$ $< k - N/m$. Further $\varkappa^*_{|i|,|\beta|} < \dfrac{q_{|\beta|}}{q'_{|i|}}$.

We can prove (see again [20]).

Theorem 1.3. *Let the conditions* (1.2), (1.5), (1.11), (1.12), (1.13), (1.14), (1.15), (1.16) *be satisfied. Then there exists a minimum of* (1.6); *let us denote it* v. *Function* $v + u_0$ *is the solution of problem.*

Let us remark, that

(1.17) $\quad \sum_{|i|=k} (\xi_i - \eta_i) \cdot [a_i(x, \xi_\alpha, \zeta_\beta) - a_i(x, \eta_\alpha, \zeta_\beta)] \geq 0$

is sufficient for the validity of (1.12).

§2. Regularity of the solution; application of differences method.

E. Hopf in his article [14] and many other authors have used this method to prove the regularity of solution of non-linear second order elliptic equations. Thus it is possible to obtain properties of $k + 1$-st derivatives. Author doesn't know how to apply this method, if it is possible, when investigating regularity of the derivatives of $k + 2$-nd and higher orders (as for the nonlinear elliptic equations in general form).

We shall assume, that functions $a_i(x, \zeta_j)$ are continuously differentiable for $x \in \overline{\Omega}$, $-\infty < \zeta_j < \infty$ and we denote $a_{ij}(x, \zeta_\alpha) = \dfrac{\partial a_i}{\partial \zeta_j}(x, \zeta_j)$. Assuming $m \geq 2$, we restrict ourselves to the following conditions (see [20]):

$$
(2.1)\quad \begin{cases}
|a_{ij}(x, \zeta_\alpha)| \leq c|\zeta_i|^{\frac{m}{2}-1} \cdot |\zeta_j|^{\frac{m}{2}-1}, \quad |i| = |j| = k, \\
|a_{ij}(x, \zeta_\alpha)| \leq c|\zeta_j|^{\frac{m}{2}-1} \cdot \Big(1 + \sum_{|\alpha| \leq k} |\zeta_\alpha|^{\frac{m}{2}-1}\Big), \quad |i| < k, \ |j| = k; \\
\text{analogically for } |i| = k, \ |j| < k, \\
|a_{ij}(x, \zeta_\alpha)| \leq c \cdot (1 + \sum_{|\alpha| \leq k} |\zeta_\alpha|^{m-2}), \quad |i| < k, \ |j| < k; \\
\sum_{|i|=|j|=k} a_{ij}(x, \zeta_\alpha)\, \xi_i \xi_j \geq c \sum_{|i|=k} |\zeta_i|^{m-2} \xi_i^2, \\
\left|\dfrac{\partial a_i}{\partial x_l}(x, \zeta_\alpha)\right| \leq c \cdot |\zeta_i|^{\frac{m}{2}-1} \Big(1 + \sum_{|\alpha| \leq k} |\zeta_\alpha|^{\frac{m}{2}}\Big) \quad \text{for } |i| = k, \\
\left|\dfrac{\partial a_i}{\partial x_l}(x, \zeta_\alpha)\right| \leq c(1 + \sum_{|\alpha| \leq k} |\zeta_\alpha|^{m-1})
\end{cases}
$$

or to the conditions

$$
(2.2)\quad \begin{cases}
|a_{ij}(x, \zeta_\alpha)| \leq c(d + \sum_{|\alpha|=k} \zeta_\alpha^2)^{\frac{m}{2}-1}, \quad |i| = |j| = k, \ d \geq 0, \\
|a_{ij}(x, \zeta_\alpha)| \leq c(d + \sum_{|\alpha|=k} \zeta_\alpha^2)^{\frac{m}{4}-\frac{1}{2}} \cdot (1 + \sum_{|\alpha| \leq k} \zeta_\alpha^2)^{\frac{m}{4}-\frac{1}{2}}, \quad |i| = k, \ |j| < k \\
\text{and analogically for } |i| < k, \ |j| = k, \\
|a_{ij}(x, \zeta_\alpha)| \leq c\,(1 + \sum_{|\alpha| \leq k} \zeta_\alpha^2)^{\frac{m}{2}-1} \quad \text{for } |i| < k, \ |j| < k, \\
c_1(d + \sum_{|\alpha|=k} \zeta_\alpha^2)^{\frac{m}{2}-1} |\xi|^2 \leq \sum_{|i|=|j|=k} a_{ij}(x, \zeta_\alpha)\, \xi_i \xi_j \leq c_2(d + \sum_{|\alpha|=k} \zeta_\alpha^2)^{\frac{m}{2}-1} |\xi|^2, \\
\left|\dfrac{\partial a_i}{\partial x_l}\right| \leq c\,(1 + \sum_{|\alpha| \leq k} \zeta_\alpha^2)^{\frac{m}{2}-\frac{1}{2}}, \quad |i| < k, \\
\left|\dfrac{\partial a_i}{\partial x_l}\right| \leq c(1 + \sum_{|\alpha|=k} \zeta_\alpha^2)^{\frac{m}{4}-\frac{1}{4}} (1 + \sum_{|\alpha| \leq k} \zeta_\alpha^2)^{\frac{m}{4}-\frac{1}{4}}, \quad |i| = k.
\end{cases}
$$

Let us denote by $\sigma(x)$ an infinitely differentiable function which is equivalent with $\operatorname{dist}(x, \partial\Omega)$ and which satisfies $|D^i\sigma| \leq c \cdot \sigma^{1-|i|}$. (Existence of such function is proved by author in [22].)

We shall consider smoothness of the solution in Ω, not in $\overline{\Omega}$. We shall assume that the right-hand side satisfies an inequality

$$
(2.3)\quad \sum_{l=1}^{N} \left\| \frac{\partial f}{\partial x_l} \sigma^k \right\|_{W_2^{(-k)}(\Omega)'} \leq c,
$$

where $W_2^{(-k)}(\Omega)$ is the dual space to $\mathring{W}_2^{(k)}(\Omega)$.

Applying the standard differences method (see e.g. J. NEČAS [21]) we obtain

Theorem 2.1. *Let $u \in W_m^{(k)}(\Omega)$, $m \geq 2$ be the solution of problem* (1.3), (1.4). *(Generally we do not suppose* (1.5).*) Let* (2.1), (2.3) *also be satisfied. Then*

$$\int_\Omega \sigma^{2k} \sum_{l=1}^{N} \sum_{|i|=k} \left(\frac{\partial}{\partial x_l} |D^i u|^{\frac{m}{2}} \right)^2 \mathrm{d}x < c$$

and thus ($N \geq 3$):

(2.4) $\int_\Omega \sum_{|i|\leq k} \sigma^{\frac{2kN}{N-2}} \,.\, |D^i u|^{\frac{mN}{N-2}} \,\mathrm{d}x \leq c < \infty,$

(2.5) $\int_\Omega \sum_{|i|\leq k} \sigma^{2kp} |D^i u|^p \,\mathrm{d}x \leq C_p < \infty, \quad 1 < p < \infty, \; N = 2.$

Similarly the next theorem is valid:

Theorem 2.2. *Let $u \in W_m^{(k)}(\Omega)$, $m \geq 2$ be the solution of problem* (1.3), (1.4) *(Generally we do not suppose* (1.5).*) Let* (2.2), (2.3) *also be satisfied. Then the inequalities*

$$\int_\Omega \sigma^{2k} \,.\, \sum_{l=1}^{N} \left(\frac{\partial}{\partial x_l} [\, d + \sum_{|\alpha|=k} (D^\alpha u)^2]^{\frac{m}{4}} \right)^2 \mathrm{d}x \leq c < \infty,$$

$$\int_\Omega \sigma^{2k} \,.\, [d + \sum_{|\alpha|=k} (D^\alpha u)^2]^{\frac{m}{2}-1} \,.\, \sum_{|i|=k+1} (D^i u)^2 \,\mathrm{d}x \leq c < \infty$$

and (2.4), (2.5) *hold.*

Analogical assertion is valid if we set $\sum_{|\alpha|\leq k} \zeta_\alpha^2$ instead of $\sum_{|\alpha|=k} \zeta_\alpha^2$ in (2.2).

If $k = 1$ (the equation of second order) we can weaken our requirements. Let us denote functions $a_i(x, \zeta_j)$ by symbols: $a_i(x, u, p)$, $i = 1, 2, \ldots, N$) $a(x, u, p)$, where $p = (p_1, \ldots, p_N)$, $p_i = \dfrac{\partial u}{\partial x_i}$ and let $\nu(s)$, $\mu(s)$, $\mu_1(s)$ be non-negative functions for $0 \leq s < \infty$. Let us denote $|p| = (\sum_{i=1}^{N} p_i^2)^{1/2}$. Let us assume

$$(2.6) \quad \begin{cases} \nu(|u|) \,.\, (1 + |p|)^{m-2} \,.\, \sum_{i=1}^{N} \xi^2 \leq \sum_{i,j=1}^{N} \dfrac{\partial a_i}{\partial p_j}(x, u, p)\, \xi_i \xi_j \leq \\ \leq \mu(|u|) \,.\, (1 + |p|)^{m-2} \sum_{i=1}^{N} \xi_i^2, \\ \sum_{i=1}^{N} \left(\left| \dfrac{\partial a_i}{\partial u} \right| + |a_i| \right) . (1 + |p|) + \sum_{i,j=1}^{N} \left| \dfrac{\partial a_i}{\partial x_j} \right| + |a| \leq \\ \leq \mu(|u|) \,.\, (1 + |p|)^m, \; 1 < m < \infty \end{cases}$$

$$(2.7)\quad \begin{cases} \sum_{i,k=1}^{N}\left|\frac{\partial a_i}{\partial x_k}(x,u,p)\right|.(1+|p|)+\sum_{i=1}^{N}\left|\frac{\partial a}{\partial p_i}(x,u,p)\right|.(1+|p|)+ \\ +\left|\frac{\partial a}{\partial u}(x,u,p)\right|+\sum_{i=1}^{N}\left|\frac{\partial a}{\partial x_i}(x,u,p)\right|\le \mu_1(|u|).(1+|p|)^m, \\ 1<m<\infty. \end{cases}$$

Let $u \in W_m^{(1)}(\Omega)$ be a weak solution satisfying the next condition: for each $\varphi \in D(\Omega)$ the equation

$$(2.8)\quad \int_{\Omega}\left(\sum_{i=1}^{N} a_i(x,u,p)\frac{\partial\varphi}{\partial x_i}+a(x,u,p)\,\varphi\right)\mathrm{d}x=0$$

holds. Then the next assertion holds (see O. A. LADYŽENSKAJA, N. N. URALCEVA [16]):

Theorem 2.3. *Let* $u \in W_m^{(1)}(\Omega)$, $1<m<\infty$ *be the weak solution satisfying* (2.8), *let* $\sup_{x\in\Omega}|u(x)|<\infty$. *Let* (2.6) *and* (2.7) *be valid. Then for* $\overline{\Omega}' \subset \Omega$

$$(2.8)'\quad \int_{\Omega'}(1+|p|)^{m-2}\sum_{i,j=1}^{N}\left(\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right)^2 \mathrm{d}x \le c(\Omega')<\infty \quad \textit{holds.}$$

If $k=1$, $u_0 \in C^2(\bar{\Omega})$ and if we consider the Dirichlet problem we can substitute Ω for Ω' in Theorem 2.3 when $\partial\Omega$ in sufficiently smooth. (See [16].)

Analogical results concerning the solution of the variational problem for the functional $\int_{\Omega} f(x,u,p)\,\mathrm{d}x$ (as Theorem 2.3 and following) proved C. B. MOREY [19]. Let $f(x,u,p)$ be a function which has two Hölder continuous derivatives according to each variable and let the inequality

$$(2.9)\quad C_1(1+u^2+|p|^2)^{\frac{m}{2}}-C_3\le f(x,u,p)\le C_2\,(1+u^2+|p|^2)^{\frac{m}{2}}$$

be satisfied for $1<m<\infty$.

Furthermore, let $u_0 \in W_m^{(1)}(\Omega)$. Let us look for such

$$(2.10)\quad u\in W_m^{(1)}(\Omega),\ \ u-u_0\in \overset{0}{W}{}_m^{(1)}(\Omega),$$

that

$$(2.11)\quad \int_{\Omega} f\left(x,u,\frac{\partial u}{\partial x}\right)\mathrm{d}x \text{ is minimal.}$$

The solution u satisfies Euler equation in the weak form: for $\varphi \in D(\Omega)$:

$$(2.12)\quad \int_{\Omega}\left(\sum_{i=1}^{N}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial f}{\partial p_i}(x,u,p)+\varphi\frac{\partial f}{\partial u}(x,u,p)\right)\mathrm{d}x=0.$$

Let us denote $\frac{\partial f}{\partial p_i}=a_i(x,u,p)$, $\frac{\partial f}{\partial u}(x,u,p)=a(x,u,p)$.

Let

$$(2.13)\quad \begin{cases} |a_i(x, u, p)| + \left|\dfrac{\partial a_i}{\partial x_l}(x, u, p)\right| + |a(x, u, p)| + \left|\dfrac{\partial a}{\partial x_l}(x, u, p)\right| \leq \\ \leq C(1 + u^2 + |p|^2)^{\frac{m}{2} - \frac{1}{2}}, \\ \left|\dfrac{\partial a_i}{\partial u}\right| + \left|\dfrac{\partial a}{\partial u}\right| \leq C(1 + u^2 + |p|^2)^{\frac{m}{2} - 1}, \\ C_1(1 + u^2 + |p|^2)^{\frac{m}{2} - 1} \sum_{i=1}^{N} \xi_i^2 \leq \sum_{i,j=1}^{N} \dfrac{\partial a_i}{\partial p_j}(x, u, p)\, \xi_i \xi_j \leq \\ \leq C_2(1 + u^2 + |p|^2)^{\frac{m}{2} - 1} \sum_{i=1}^{N} \xi_i^2 \end{cases}$$

be satisfied. (Comp. with (2.2).) Then (see C. B. Morey [19]):

Theorem 2.4. *If* $u \in W_m^{(1)}(\Omega)$, $m \geq 2$, *u satisfies* (2.12) *and if the conditions* (2.13) *are satisfied then* (2.8)$'$ *holds. If* $1 < m < 2$ *then there exists u satisfying* (2.12) *such that* (2.8)$'$ *holds again.*

See also E. R. Buley [6].

§3. Regularity of the solution; on the Hölder continuity of k—th derivatives.

Under the assumptions of the Theorems 2.1 or 2.2 we have (3.1) for the weak solution and $\varphi \in D(\Omega)$

$$(3.1)\quad \int_\Omega \sum_{|i|,|j| \leq k} a_{ij}(x, D^\alpha u)\, D^i\varphi D_j \frac{\partial u}{\partial x_l}\, dx =$$
$$= -\int_\Omega \sum_{|i| \leq k} \frac{\partial a_i}{\partial x_l}(x, D^\alpha u)\, D^i\varphi\, dx + \left\langle \varphi, \frac{\partial f}{\partial x_l} \right\rangle_\Omega, \quad l = 1, 2, \ldots, N.$$

Thus if we denote $\omega = \dfrac{\partial u}{\partial x_l}$ then ω is a weak solution of linear differential equation. The investigation of regularity of higher derivatives is based upon (3.1) and upon regularity theorems for the linear equations. In this section we restrict ourselves to the assumptions (2.2) with $d = 1$. Simple example can be given to exhibit that conditions (2.1) do not guarantee continuity of $k + 1$-st derivatives in Ω in spite of the analyticity of functions $a_i(x, \zeta_j)$, $f(x)$. (See J. Nečas [20].)

If $k = 1$ then (3.1) yields further information if we set $\varphi = \dfrac{\partial u}{\partial x_l} b_n^s\, \psi^2$, $\psi \in$

$\in D(\Omega)$, $s \geq 0$, $b_n(x) = \min(|p|^2, n)$, $n = 1, 2, \ldots$ (φ — the comparison function). See e.g. O. A. LADYŽENSKAJA, N. N. URALCEVA [16]. The comparison function

$$
\begin{aligned}
(3.2) \qquad & \varphi = d_n^s \frac{\partial u}{\partial x_l} \psi^2, \quad \psi \in D(\Omega), \quad s \geq 0, \\
& d_n = \min \{(1 + u^2 + |p|^2), n\}, \quad n = 1, 2, \ldots
\end{aligned}
$$

has been used in E. R. BULEY's paper [6] under assumptions (2.9), (2.13) and $m \geq 2$. The same function has been used by C. B. MOREY [19] but with $s < 0$. From this the boundedness of the first derivatives on every $\Omega' \subset \subset \bar{\Omega}' \subset \Omega$ can be obtained when $s \to \infty$. (See E. R. BULEY [6], J. NEČAS [21].) If

$$
(3.3) \qquad \sup_{\Omega'} |p(x)| \leq C(\Omega') < \infty
$$

is proved and if (2.8)$'$ holds then $\frac{\partial u}{\partial x_l} = \omega$ is a weak solution of linear equation with bounded and measurable coefficients on Ω' according to (2.1). When $k = 1$ we can use DE GIORGI's result $\left(\text{if } \frac{\partial f}{\partial x_l} = 0 \text{ see [12]}\right)$ or more general result of G. STAMPACCHIA $\left(\text{if } \frac{\partial f}{\partial x_l} \neq 0\right)$ [27]:

Theorem 3.1. *Let $u \in W_2^1(\Omega)$ be a weak solution of the equation: for $\varphi \in D(\Omega)$,*

$$
\int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial \varphi}{\partial x_i} \cdot \frac{\partial u}{\partial x_j} \cdot \mathrm{d}x = \int_\Omega \varphi f \,\mathrm{d}x + \int_\Omega \sum_{i=1}^N \frac{\partial \varphi}{\partial x_i} f_i,
$$

where $f \in L_p(\Omega)$, $f_i \in L_{2p}(\Omega)$, $p > \frac{N}{2}$, $a_{ij} \in L_\infty(\Omega)$, $\sum_{i,j=1}^N a_{ij}\xi_i\xi_j \geq C|\xi|^2$, then there exists such $0 < \mu < 1$ that

$$
\|u\|_{C^{(0),\mu}(\bar{\Omega}')} \leq C(\Omega') \left(\|f\|_{L_p(\Omega)} + \sum_{i=1}^N \|f_i\|_{L_{2p}(\Omega)} + \|u\|_{W_2^{(1)}(\Omega)}\right), \quad \bar{\Omega}' \subset \Omega
$$

holds.

The proof of Hölder continuity for higher derivatives and (for $k = 1$ of the analyticity of solution) follows e.g. by the result of A. DOUGLIS, L. NIRENBERG [9] (or by results of E. HOPF [14]). We shall formulate the results: E. R. BULEY [6]:

Theorem 3.2. *Let $k = 1$, $m \geq 2$, let u be the solution of* (2.10), (2.11) *and let* *the assumptions* (2.9), (2.13) *be satisfied. Then* (3.3) *holds and there exists* $0 < \mu < 1$ *that*

(3.4) $\quad ||u||_{C^{(1)\mu}(\Omega')} \leq C(\Omega') < \infty$ *holds.*

Applying C. B. MORREY's result the Theorem 3.2 can be obtained for such u which satisfies the condition (2.12). Furthermore, this author obtained:

Theorem 3.3. *Let $k = 1$, $1 < m < 2$ and otherwise let all assumptions of the preceding theorem be satisfied. Then there exists such solution of the problem* (2.10), (2.11), *that* (3.1), (3.4) *hold.*

O. A. LADYŽENSKAJA, N. N. URALCEVA:

Theorem 3.4. *Let $u \in W_m^{(1)}(\Omega)$, $1 < m < \infty$ be a weak solution which satisfies the condition* (2.8). *Let $\sup_{x \in \Omega} |u(x)| < \infty$ and let* (2.6), (2.7) *hold. Then* (3.3), (3.4) *hold.*

The inequality (3.3) was essential in proof of regularity of the solution for $k = 1$. The inequality (3.1) ($k = 1$) has been considered by many authors that generalized the result of T. RADO [26] under essentially weaken assumptions (supposing that $\Omega = \Omega'$, $\partial\Omega$ is smooth and Ω is strictly convex). (See e.g. P. HARTMAN, G. STAMPACCHIA [13], D. GILBARG [11].)

Now let us consider $k \geq 2$. The use of the comparison function of the type (3.2) does not lead to any result and the information

$$\sup_{x \in \Omega'} \sum_{|i| \leq k} |D^i u(x)| \leq C(\Omega') < \infty \tag{3.5}$$

is not available. Accordingly, we shall consider the case $m = 2$ or we shall suppose that (3.5) holds. Thus we transfer the problem of regularity of k-th derivatives to the linear problem.:

Let A_{ij} be a real matrix of bounded measurable functions in a domain O, $|i| = |j| = k$. We shall use the following assumptions:

$$C_1|\zeta|^2 \leq \sum_{|i|=|j|=k} A_{ij}\zeta_i\zeta_j \leq C_2|\zeta|^2, \tag{3.6}$$

$$A_{ij} = A_{ji}. \tag{3.7}$$

Function $w \in W_2^{(k)}(O)$ is a weak solution of the equation $\sum_{|i|=|j|=k} D^i(A_{ij}D^j w) = \sum_{|i|=k} D^i f_i$ with $f_i \in L_2(O)$, if for each $\varphi \in D(\Omega)$

$$\int_O \sum_{|i|=|j|=k} A_{ij} D^i\varphi D^j w \, dx = \int_O \sum_{|i|=k} D^i\varphi f_i \, dx. \tag{3.7'}$$

Further let us denote $O\ d = \{x \in O, \operatorname{dist}(x, \partial O) = d\}$, $B(x_0, r) = \{x, |x - x_0| < r\}$. For $0 < \lambda < N$ let $\mathscr{L}^{(2,\lambda)}(O)$ be such subspace of $L_2(O)$ that

$$\sup_{x_0 \in O, \varrho > 0} (\varrho^{-\lambda} \cdot \int_{B(x_0, \varrho) \cap O} f^2(x) \, dx)^{1/2} \equiv ||f||_{\mathscr{L}^{(2,\lambda)}(O)} < \infty.$$

For the properties of these spaces see, e.g. S. CAMPANATO [7].

Applying S. CAMPANATO's method [7] whose generalization for the equation of higher order has been given in the paper [15] of J. KADLEC and J. NEČAS, we obtain the following:

Theorem 3.5. *Let w be a weak solution satisfying* (3.7)′. *If* (3.6), (3.7) *and if*

$$\lambda = \frac{N \cdot \log \dfrac{1 - \frac{3}{4} \frac{C_1}{C_2}}{1 - \frac{C_1}{C_2}}}{\log \dfrac{2AC_2}{C_1} + \log \dfrac{1 - \frac{3}{4} \cdot \frac{C_1}{C_2}}{1 - \frac{C_1}{C_2}}} > N - 2, \tag{3.8}$$

$$f_i \in \mathscr{L}^{(2,\lambda)}(O_d), \quad d > 0 \tag{3.9}$$

are satisfied then we obtain

$$||W||_{C^{(k-1),\mu}(\bar{O}_d)} \leq C(d) \sum_{|i|=k} ||f_i||_{\mathscr{L}^{(2,\lambda)}(O_{d/2})}, \quad \mu = \frac{\lambda + 2 - N}{2}.$$

(3.8) *is always satisfied for* $N = 2$. *For* $N \geq 3$ *it holds when the positively-definite matrix* $\frac{1}{C_2} \cdot A_{ij}$ *is sufficiently near (uniformly on O) to the unit matrix in the sense of* (3.8). *The constant A is absolute.*

The Theorem 3.5 is — in certain sense — an analogy of the Theorem 3.1 for $k \geq 2$.

If

$$||A_{ij}||_{C(\bar{O})} \leq C < \infty \tag{3.10}$$

holds, then (see [15]):

Theorem 3.6. *Let w be a weak solution satisfying* (3.7)′ *and let the assumptions* (3.6), (3.10), (3.9) *with* $\lambda > N - 2$ *be satisfied. Then*

$$||w||_{C^{(k-1),\mu}(\bar{O}_d)} \leq C(d) \sum_{|i|=k} ||f_i||_{\mathscr{L}^{(2,\lambda)}(O_{d/2})}, \quad \mu = \frac{\lambda + 2 - N}{2} \quad \textit{holds.}$$

Replace $\left\langle \varphi_i, \dfrac{\partial f}{\partial x_l} \right\rangle_\Omega$ in (3.1) by the expression $\int_\Omega \sum_{|i|=k} D^i \varphi \dfrac{\partial f_i}{\partial x_l} dx$ where

$$\int_\Omega \sum_{|i|=k} \sum_{l=1}^{N} \left(\frac{\partial f_i}{\partial x_l}\right)^2 \sigma^{2k} \, dx < \infty. \tag{3.12}$$

Further let us suppose that (2.2) is valid and (for technical reason)

(3.13) $\quad a_i \equiv 0$ for $|i| < k,\ \dfrac{\partial a_i}{\partial \varphi_j} \equiv 0$ for $|j| < k,\ a_{ij} = a_{ji}.$

According to Theorems 2.2, 3.5, 3.6 we obtain (see J. NEČAS [23]):

Theorem 3.7. *Let $u \in W_m^{(k)}(\Omega)$, $m \geq 2$ be a solution of the problem* (1.3), (1.4) *and let the assumptions* (2.2), (3.12), (3.13) *be satisfied (the constants C_1, C_2 have the same meaning as before). Then we have*

(a) *if $m = N = 2$ and*

$$\sum_{|i|=k} \sum_{l=1}^{N} \left\| \frac{\partial f_i}{\partial x_l} \right\|_{\mathscr{L}^{(2,\lambda)}(\Omega_d)} \leq C d^{-k},\ d > 0 \tag{3.14}$$

then $\|u\|_{C^{(k),\frac{\lambda}{2}}(\bar{\Omega}_d)} \leq C d^{-k-\frac{\lambda}{2}}$, *($\lambda$ is taken of* (3.8))

(b) *if $m > 2$, $N = 2$,* (3.5) *has the form* $\sup\limits_{x \in \Omega_d} \sum\limits_{|i|=k} |D^i u(x)|^2 \equiv A_d \leq C_3 d^{-\alpha}$ *and if* (3.14) *with*

$$\gamma \geq 2\mu_d \equiv \frac{2 \,.\, \log \dfrac{1 - \dfrac{3}{4}\dfrac{C_1}{C_2}(1 + C_3 d^{-\alpha})^{1-\frac{m}{2}}}{1 - \dfrac{C_1}{C_2}(1 + C_3 d^{-\alpha})^{1-\frac{m}{2}}}}{\log 2A \dfrac{C_1}{C_2}(1 + C_3 d^{-\alpha})^{1-\frac{m}{2}} + \log \dfrac{1 - \dfrac{3}{4}\dfrac{C_1}{C_2}(1 + C_3 d^{-\alpha})^{1-\frac{m}{2}}}{1 - \dfrac{C_1}{C_2}(1 + C_3 d^{-\alpha})^{1-\frac{m}{2}}}}$$

is valid then $\|u\|_{C^{(k),\mu_d}(\bar{\Omega}_d)} \leq \dfrac{C}{\mu_d} d^{-k-\mu_d}$,

(c) *if $m = 2$, $N \geq 3$,* $\dfrac{\partial a_i}{\partial x_l} = 0$, (3.8) *is valid with the constants C_1, C_2 from* (2.2) *and if* (3.14) *with λ from* (3.8) *is satisfied then*

$$\|u\|_{C^{(k),\frac{\lambda - N + 2}{2}}(\bar{\Omega}_d)} \leq C d^{-k-\frac{\lambda}{2}},$$

(d) *if $m \geq 2$, $N \geq 3$ and* (3.5) *in the form* $\sup\limits_{x \in \Omega} \sum\limits_{|\alpha|=k} |D^\alpha u(x)|^2 \leq C_3$ *is satisfied, further if* (3.8), (3.14) *with*

$$\lambda = \frac{N \log \dfrac{1 - \dfrac{3}{4}\dfrac{C_1}{C_2}(1 + C_3)^{1-\frac{m}{2}}}{1 - \dfrac{C_1}{C_2}(1 + C_3)^{1-\frac{m}{2}}}}{\log 2A \dfrac{C_1}{C_2}(1 + C_3)^{1-\frac{m}{2}} + \log \dfrac{1 - \dfrac{3}{4}\dfrac{C_1}{C_2}(1 + C_3)^{1-\frac{m}{2}}}{1 - \dfrac{C_1}{C_2}(1 + C_3)^{1-\frac{m}{2}}}}$$

is valid then $||u||_{C^{(k)},\frac{\lambda-N+2}{2}}(\bar{\Omega}_d) \leq Cd^{-k-\frac{\lambda}{2}}$

(e) *if* $m \geq 2$, $N \geq 2$, $||u||_{C^{(k)}}(\bar{\Omega}) \leq C_3$ *and if* (3.14) *with* $\lambda > N - 2$ *is valid then*

$$||u||_{C^{(k)},\frac{\lambda-N+2}{2}}(\bar{\Omega}_d) \leq Cd^{-k-\frac{\lambda}{2}}.$$

§4. The existence of the regular solution. Application of the first differential.

Let Ω be a bounded domain with infinitely differentiable boundary $\partial\Omega$. Let $a_i(x, \zeta_j, t)$ be real functions with the same meaning as in section §1, defined for $|i| \leq k$ continuous on $\bar{\Omega} \times (-\infty < \zeta_j < \infty) \times (0 \leq t \leq 1)$ and continuously differentiable in ζ_j, t and let $a_i(x, 0, 0) = 0$. Using the same notation as above i.e. $a_{ij} = \dfrac{\partial a_i}{\partial \zeta_j}$ we suppose

$$(4.1)\quad \begin{cases} |a_{ij}(x, \zeta_\alpha, t) - a_{ij}(y, \eta_\alpha, t)| \leq \\ \leq C_2\Big(\sum\limits_{|\alpha|\leq k}(|\zeta_\alpha| + |\eta_\alpha|)\Big) \cdot \Big(|x-y|^\mu + \sum\limits_{|\alpha|\leq k}|\zeta_\alpha - \eta_\alpha|\Big) \\ \text{and the same for } \dfrac{\partial a_i}{\partial t}, \\ |a_{ij}(x, \zeta_\alpha, t_1) - a_{ij}(y, \eta_\alpha, t_1) + a_{ij}(y, \eta_\alpha, t_2) - a_{ij}(x, \zeta_\alpha, t_2)| \leq \\ \leq C_2\Big(\sum\limits_{|\alpha|\leq k}|\zeta_\alpha| + |\eta_\alpha|\Big)\,\omega(|t_1 - t_2|)\Big(|x-y|^\mu + \sum\limits_{|\alpha|\leq k}|\zeta_\alpha - \eta_\alpha|\Big), \\ \text{and the same for } \dfrac{\partial a_i}{\partial t} \end{cases}$$

where $C_2(s)$ is a non-negative continuous function for $0 \leq s < \infty$, $0 < \mu < 1$ and $\omega(s)$ is continuous function for $0 \leq s < \infty$, $\omega(0) = 0$.

Let us assume further

$$(4.2)\qquad C_1\Big(\sum_{|\alpha|\leq k}|\eta_\alpha|\Big)|\zeta|^2 \leq \sum_{|i|=|j|=k} a_{ij}(x, \eta_\alpha, t)\,\zeta_i\zeta_j$$

where $C_1(s)$ is a continuous positive function for $0 \leq s < \infty$. Further let $f_i \in C^{(0),\mu}(\bar{\Omega})$, $|i| \leq k$, $u_0 \in C^{(k),\mu}(\bar{\Omega})$. Let us denote by $\mathring{C}^{(k),\mu}(\bar{\Omega})$ the subspace of $C^{(k),\mu}(\bar{\Omega})$ whose elements are functions for which $\dfrac{\partial^l u}{\partial n^l} = 0$ on $\partial\Omega$, $l = 0, 1, \ldots k-1$. (The derivation in the direction of exterior normal.) We look for such weak solution of the Dirichlet problem $u \in C^{(k),\mu}(\bar{\Omega})$ that

(4.3) $\quad u - u_0 \in \mathring{C}^{(k),\mu}(\bar{\Omega})$

(4.4) $\quad$ for each $\varphi \in D(\Omega)$ $\displaystyle\int_\Omega \sum_{|i|\leq k} D^i\varphi a_i(x, D^j u, 1)\,dx = \int_\Omega \sum_{|i|\leq k} D^i\varphi f_i\,dx.$

Let the functions $b_i(x, D^j u, t)$, $|i| \leq k$, $|j| < k$ be continuous on $\bar{\Omega} \times -\infty <$

$< \zeta_j < \infty \times 0 \leq t \leq 1$ continuously differentiable in ζ_j, t, $b_i(x, 0, 0) = 0$. Let us denote $b_{ij} = \frac{\partial b_i}{\partial \zeta_j}$ and assume that b_{ij}, $\frac{\partial b_i}{\partial t}$ satisfy the conditions (4.1).

Roughly speaking, we shall solve the problem (4.3), (4.4) as follows: We shall look for such curve $u(t)$, $0 \leq t \leq 1$ with its values in $C^{(k),\mu}(\bar{\Omega})$ that $u(t)$ satisfies the problem (4.3), (4.4) with tu_0, tf_i. For this curve we shall obtain a differential equation $\frac{\mathrm{d}u}{\mathrm{d}t} = N[t, u(t)]$ and we shall look for such solution that $u(0) = 0$. See J. Nečas [24] see also F. E. Browder [4]. Thus instead of solving the problem (4.3), (4.4) we look for a mapping $u(t, \tau)$ with a domain $\tau = 0$, $0 \leq t \leq 1$, $t = 1$, $0 \leq \tau \leq 1$ and a range in $C^{(k),\mu}(\bar{\Omega})$ which is continuous with its derivative $\frac{\partial u}{\partial t}(t, 0)$ from $0 \leq t \leq 1$ to $C^{(k),\mu}(\bar{\Omega})$ for $\tau = 0$. (The case when $a_i(x, \zeta_j, t)$ does not depend on t is of great importance.) Further we require

$$(4.5) \qquad u(t, \tau) - tu_0 \in \mathring{C}^{(k),\mu}(\bar{\Omega}),$$

$$(4.6) \qquad \varphi \in D(\Omega) : \int_\Omega \sum_{|i| \leq k} D^i\varphi a_i(x, D^j u, t)\,\mathrm{d}x + (1 - \tau) \int_\Omega \sum_{|i| \leq k} D^i\varphi b_i(x, D^j u, t)\,\mathrm{d}x = t \int_\Omega \sum_{|i| \leq k} D^i\varphi f_i\,\mathrm{d}x.$$

Further let us assume that for $||u||_{C^{(k),\mu}(\bar{\Omega})} \leq R \leq \infty$ the following holds: if $w \in \mathring{W}_2^{(k)}(\Omega)$ and (4.7) holds for every $\varphi \in D(\Omega)$:

$$(4.7) \qquad \int_\Omega \sum_{|i|,|j| \leq k} a_{ij}(x, D^\alpha u, t)\, D^i\varphi D^j w\,\mathrm{d}x + \int_\Omega \sum_{|i|,|j| \leq k} b_{ij}(x, D^\alpha u, t)\, D^i\varphi D^j w\,\mathrm{d}x = 0$$

then $w \equiv 0$. This assumption implies the existence of only one element (for $||u||_{C^{(k),\mu}(\bar{\Omega})} \leq R$, *if* $R < \infty$) $w \in C^{(k),\mu}(\bar{\Omega})$ for which

$$(4.8) \qquad w - u_0 \in \mathring{C}^{(k),\mu}(\bar{\Omega})$$

$$(4.9) \qquad \text{for } \varphi \in D(\Omega) : \int_\Omega \sum_{|i|,|j| \leq k} a_{ij}(x, D^\alpha u, t)\, D^i\varphi D^j w\,\mathrm{d}x + \int_\Omega \sum_{|i|,|j| \leq k} b_{ij}(x, D^\alpha u, t)\, D^i\varphi D^j w\,\mathrm{d}x = -\int_\Omega \sum_{|i| \leq k} \left(\frac{\partial a_i}{\partial t}(x, D^\alpha u, t) + \frac{\partial b_i}{\partial t}(x, D^\alpha u, t)\right) D^i\varphi\,\mathrm{d}x + \int_\Omega \sum_{|i| \leq k} f_i D^i\varphi\,\mathrm{d}x \quad \text{is valid.}$$

It follows e.g. from the article by S. Agmon, A. Douglis, L. Nirenberg [1] or from J. Kadlec, J. Nečas [15].

Let us denote by $w = N(u, t, f_i, u_0)$ the mapping that assigns to a function $u \in C^{(k),\mu}(\bar{\Omega})$ from the sphere $||u||_{C^{(k),\mu}(\bar{\Omega})} \leq R$, to the parameter t from $\langle 0, 1\rangle$, to the elements f_i, $|i| \leq k$ and to the element u_0 the function w. Now, we have for a function $w \in \mathring{C}^{(k),\mu}(\overline{\Omega})$, which is a weak solution of the equation

$$\int_\Omega \sum_{|i|,|j|\le k} a_{ij}(x, D^\alpha u, t)\, D^i\varphi\, D^j w\, dx + \int_\Omega \sum_{|i|,|j|\le k} b_{ij}(x, D^\alpha u, t)\, D^i\varphi D^j w\, dx = \\ = \int_\Omega \sum_{|i|\le k} G_i D^i\varphi\, dx$$

that there holds:

$$(4.10)\quad \|\omega\|_{C^{(k),\mu}(\overline{\Omega})} \le C_3(\|u\|_{C^{(k),\mu}(\overline{\Omega})}, \mu) \sum_{|i|\le k} \|G_i\|_{C^{(0),\mu}(\overline{\Omega})}$$

where $C_3(\eta_1, \eta_2)$ is continuous and positive function for $0 \le \eta_1 < \infty$, $0 < < \eta_2 < 1$. According to this it follows:

$$(4.11)\ \begin{cases} \text{(a) The mapping } N(u, t, f_i, u_0) \text{ is locally Lipschitzian: for } \|u_l\|_{C^{(k),\mu}(\overline{\Omega})} \le \\ \le R_0 < \infty,\ l = 1,\ 2,\ R_0 \le R,\ 0 \le t \le 1,\ \|f_i\|_{C^{(k),\mu}(\overline{\Omega})} \le R_1 < \infty, \\ \|u_0\|_{C^{(k),\mu}(\overline{\Omega})} \le R_1 \text{ there is } \|w_1 - w_2\|_{C^{(k),\mu}(\overline{\Omega})} \le C(R_0, R_1)\, \|u_1 - u_2\|_{C^{(k),\mu}(\overline{\Omega})}, \\ \text{(b) } N \text{ is continuous as the mapping } u, t \to w, \\ \text{(c) } N \text{ is continuous in } f_i, u_0 \text{ uniformly with respect to } \|u\|_{C^{(k),\mu}(\overline{\Omega})} \le \\ \quad \le R_0,\ 0 \le t \le 1. \end{cases}$$

For $\tau = 0$ we have: if $u(t, 0)$ is a solution of the problem (4.5), (4.6) for $0 \le t < \varepsilon$ and if $0 < \varepsilon \le 1$, $\|u(t, 0)\|_{C^{(k),\mu}(\overline{\Omega})} \le R$ then

$$(4.12)\quad \frac{\partial u}{\partial t}(t, 0) = N(u(t), t, f_i, u_0),\ 0 \le t < \varepsilon,\ u(0, 0) = 0$$

holds and thus

$$(4.13)\quad u(t, 0) = \int_0^t N(u(s), s, f_i, u_0)\, ds,\ 0 \le t < \varepsilon.$$

Now, using the standart method based upon the theorem of contraction, owing to the validity of (4.11) we obtain the existence of the solution of (4.13) for some interval $\langle 0, \varepsilon\rangle$, $\varepsilon > 0$; if there is such solution for some interval $\langle 0, \varepsilon\rangle$, $\varepsilon < 1$ then it also exists for the interval

$$\langle 0, \varepsilon_1\rangle, \qquad 1 \ge \varepsilon_1 > \varepsilon.$$

We assume that $u(t, 0)$ is such solution on the interval $\langle 0, \varepsilon\rangle$ and that

$$(4.14)\quad \|N(u(t), t, f_i, u_0)\|_{C^{(k),\mu}(\overline{\Omega})} \le F(\|u(t)\|_{C^{(k),\mu}(\overline{\Omega})})$$

holds for $t \in \langle 0, \varepsilon\rangle$, where $F(s)$ is continuous and non-decreasing function for $s \in \langle 0, \infty)$, $F(0) > 0$. Let $y(t)$ be the solution of Cauchy problem $y(0) = 0$, $y'(t) = F(y(t))$. Evidently the following holds:

$$(4.15)\quad \|u(t)\|_{C^{(k),\mu}(\overline{\Omega})} \le y(t).$$

But (4.15) implies the existence of the solution of (4.13) wherever $y(t)$ is defined, i. e. for $0 \leq t \leq \varepsilon$, where

$$(4.16) \quad \varepsilon < \int_0^\infty \frac{\mathrm{d}z}{F(z)}.$$

According to this we have

Theorem 4.1. *Let the assumptions* (4.1), (4.2), (4.7) *with* $R = \infty$ *be satisfied and let* $b_i(x, \zeta_j, t) \equiv 0$. *Then there exists a solution of the problem* (4.3), (4.4) *if* $\int_0^\infty \frac{\mathrm{d}z}{F(z)} > 1$. *Otherwise there exists a solution of the problem* (4.5), (4.6) *for* εf_i, εu_0 *where* $\varepsilon < \int_0^\infty \frac{\mathrm{d}z}{F(z)}$. *If an a priori estimate* $\|u(t)\|_{C^{(k),\mu}(\bar{\Omega})} \leq \frac{R}{2} < \infty$ *is known (u is a solution of* (4.5), (4.6)), *where R is from* (4.11) *then there exists a solution of the problem because it is possible to set* $F(z) =$ const.

If there exists a function from (4.14) *with* $\int_0^\infty \frac{\mathrm{d}z}{F(z)} > 1$ *uniformly with respect to some neighbourhood of* f_i, u_0 *then the solution* $u(1,0)$ *is continuous in* f_i, u_0 *in this neighbourhood.*

Theorem 4.2. *Let the assumptions* (4.1), (4.2) *and the following condition* (4.17) *be satisfied:*

$$(4.17) \quad \left\{ \begin{array}{l} \text{If } \sum_{|i| \leq k} \|g_i\|_{C^{(0),\mu}(\bar{\Omega})} \leq C,\ u_0 \text{ being fixed, } u(t, 0) \text{ is an eventual solution for} \\ tu_0,\ tg_i, \text{ then there exists such continuous non-negative function } R(a) \\ \text{that } \|u(t, 0)\|_{C^{(k),\mu}(\bar{\Omega})} \leq R(a) \text{ and } (4.7) \text{ holds with } 2R(a). \end{array} \right.$$

Furthermore let the "a priori" estimate $\|u(1, \tau)\|_{C^{(k),\mu}(\bar{\Omega})} \leq \varrho$ *hold for* u_0, f_i *being fixed. Then there exists a solution of the problem* (4.3), (4.4).

Actually, according to the preceding theorem, our problem has a solution if $\tau = 0$ (for considered u_0 and arbitrary g_i) constructed above. (It is possible to guarantee the existence of this solution also under different assumptions, see the preceding theorem.) Let $A(g_i)$ be this solution. Let us consider the mapping $A(f_i - \tau b_i(x, D^j u, 1))$ from $\langle 0, 1 \rangle \times C^{(k),\mu}(\bar{\Omega})$ to $C^{(k),\mu}(\bar{\Omega})$ for $0 \leq \leq \tau \leq 1$. This mapping represents homotopy of compact transformations and the mapping $A - u$ is different from zero on the boundary of the sphere $B_{2\varrho} \equiv \|u\|_{C^{(k),\mu}(\bar{\Omega})} \leq 2\varrho$. Now, for the degree of mapping with respect to O and to the sphere in question we have

$$\mathrm{d}[A(f_i - b_i(x, D^j u, 1)) - u, 0, B_{2\varrho}] = \mathrm{d}[A(f_i) - u, 0, B_{2,\varrho}] = -1.$$

Hence there exists the solution of our problem. See J. Cronin [8].

BIBLIOGRAPHY

[1] S. AGMON, A. DOUGLIS, L. NIRENBERG: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Comm. Pure Appl. Math. XII, 1959, 623—727.*

[2] F. E. BROWDER: *Séminaire de mathématiques supérieures, Montréal, 1965.*

[3] F. E. BROWDER: *Variational boundary value problems for quasilinear elliptic equations of arbitrary order I, Proc. Nat. Amer. Acad. Sc., 1963, num. 1, II, ibid. num. 4, 1963, III, ibid. num. 5, 1963.*

[4] F. E. BROWDER: *Topological methods for nonlinear elliptic equations of arbitrary order, Pacif. J. Math. 17, num. 1, 1966.*

[5] F. E. BROWDER: *Variational methods for nonlinear elliptic eigenvalue problems, Bull. Amer. Math. Soc. 71, 1965, 176—183.*

[6] E. R. BULEY: *The differentiability of solutions of certain variational problems for multiple integrals, Technical Report 16, 1960, Univ. Berkeley.*

[7] S. CAMPANATO: *Equazioni ellitiche del II° ordine e spazi $\alpha^{(2,\lambda)}$, Annali Matem. pura ed appl. LXIX, 321—382.*

[8] J. CRONIN: *Fixed poits and topological degree in non-linear analysis, Amer. Math. Soc. 1964.*

[9] A. DOUGLIS, L. NIRENBERG: *Interiorestimates for elliptic systems of partial differential equations, Comm. Pure Appl. Math. 8, 1955, 503—538.*

[10] E. CAGLIARDO: *Proprietà di alcune classi in più variabili, Ricerche di Matem 7, 1958, 102—137.*

[11] D. GILBARG: *Boundary value problems for nonlinear elliptic equations in n variables, Proc. Symp. Madis. Wisc. 1962, Univ. Wisc. Press 1963.*

[12] E. DE GIORGI: *Sulla differenziabilità e l'analyticità delle estremali degli integrali multipli regulari, Mem. Accad. Sci. Torino, 3, 1957, 25—43.*

[13] P. HARTMAN, G. STAMPACCHIA: *On some nonlinear elliptic differential functional equations, Acta Math. Uppsala, 115, 3—4, 1966, 271—310.*

[14] E. HOPF: *Über den Funktionalen, insbesondere dem analytischen Charakter der Lösungen elliptischer Differentialgleichungen zweiter Ordnung, Math. Zeit. 34. 1931, 194—233.*

[15] J. KADLEC, J. NEČAS: *to be published in CMUC.*

[16] O. A. LADYŽENSKAJA, N. N. URALCEVA: *Linnějnyje i kvazilinějnyje uravněnija eliptičeskovo tipa, Izd. NAUKA, Moskva 1964.*

[17] J. LERAY, J. L. LIONS: *Quelques résultats de Vischik sur les problémes elliptiques non linéaires par les méthodes de Minty-Browder, Sém. équat. part., Coll. de France 1964*

[18] S. G. MICHLIN: *Čislennaja realizacija variacionnych metodov, Izd. NAUKA, Moskva 1966.*

[19] C. B. MORREY: *Quelques résultats récents du calcul des variations, Colloque CNRS Paris 1962.*

[20] J. NEČAS: *Sur la méthode variationnelle pour les équations aux dérivées partielles non linéaires du type elliptique; l'existence et la régularité des solutions, CMUC 7, 3, 1966.*

[21] J. NEČAS: *Les méthodes directes dans la théorie des équations elliptiques, ČSAV, Praha 1967.*

[22] J. NEČAS: *Sur une méthode pour resoudre les équations aux dérivées partielles du type elliptique voisine de la variationnelle, Ann. Scuola Norm. Sup. Pisa, XVI, 1962, 305—326.*

[23] J. NEČAS: *Sur la régularité des solutions variationnelles, to be published in CMUC.*

[24] J. NEČAS: *Sur une méthode générale pour la solution des problèmes aux limites non linéaires, to be published in Ann. Scuola Norm. Sup. Pisa 1966.*

[25] J. NEČAS: *Une remarque sur l'existence d'une solution régulière pour le problème de Dirichlet et l'équation elliptique non linéaire, CMUC 1967.*

[26] T. RADO: *Geometrische Betrachtungen über zweidimensionale reguläre Variationsprobleme, Acta Lit. Sci. Reg. Univ. Hungar. Francisco Josephine Sect. Sci. Math., 1924–1926, 228–253.*

[27] G. STAMPACCHIA: *Équations elliptiques du second ordre à coefficients discontinus, Séminaire sur les équat. part., Collège de France, 1963–1964.*

[28] M. M. VAJNBERG: *Variacionnyje metody issledovanija nělinějnych operatorov, Moskva, 1956.*

[29] M. M. VAJNBERG, R. I. KAČUROVSKIJ: *On the variational theory of nonlinear operators and equations (Russian), DAN SSSR, 129, 1959, 1199–1202.*

[30] M. I. VIŠIK: *Quasilinear strongly elliptic systems of differential equations having divergence form (Russian), Trudy Mosk. Mat. Obšč. 12, 1963, 125–184.*

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XVII — 1967

SOME BOUNDARY PROBLEMS FOR THE EQUATIONS WITH STRONG DEGENERATION

S. Nikolskij, Moskva

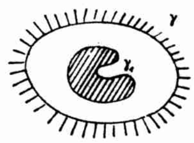
Let Ω be a bounded open set of the n-dimentional space R of the points $x = (x_1, \ldots, x_n)$ with enough smooth boundary Γ.

We shall consider two cases:

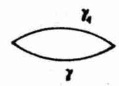
1) $\varrho = \varrho(x)$ is the distance from x to Γ,

2) $\varrho = \varrho(x)$ is the distance from x to γ, where γ is a part of Γ, $\gamma + \gamma_1 = \Gamma$, $\gamma\gamma_1 = 0$.

Here are two characteristic plots for the case 2)



Plot 1.



Plot 2.

But I must warn that in different problems under consideration it's necessary to propose some conditions on disposition γ to γ_1, as it will be seen below.

We shall use the following notation

$$||f||_{L_p(\varepsilon)} = \left(\int_\varepsilon |f(x)|^p \, dx\right)^{1/p} \qquad (1 \leq p \leq \infty, \ \varepsilon \subset R).$$

By definition $W^r_{p,\alpha}(\Omega)$ is the class of the functions defined on Ω, which have finite norm

$$||f||_{W^r_{p,\alpha}(\Omega)} = ||f||_{L_p(\Omega)} + \sum_{|K|=r} \left\|\frac{f^{(K)}}{\varrho^\alpha}\right\|_{L_p(\Omega)}.$$

Here $\sum$ extends on all derivatives of the order r. At first we shall mean that

$$r + \alpha - \frac{1}{p} > 0$$

and

$$s - 1 = \left[r + \alpha - \frac{1}{p}\right]$$

is its entier. Thus s is an integer depending on τ, α, p and satisfying inequalities

$$1 \leq s \leq \tau.$$

It is well known (see [1] theorem 38) that every function $f \in W_{p,\alpha}(\Omega)$ has traces on Γ.

They are in the case 1)

$$\left.\frac{\partial^K f}{\partial h^K}\right|_\Gamma = \varphi_K \qquad (K = 0, 1, \ldots, s-1)$$

and in the case 2)

$$\left.\frac{\partial^K f}{\partial h^K}\right|_\gamma = \varphi_K \qquad (K = 0, 1, \ldots, s-1),$$

$$\left.\frac{\partial^K f}{\partial h^K}\right|_{\gamma_1} = \varphi_K \qquad (K = 0, 1, \ldots, \tau-1).$$

If

$$\tau + \alpha - \frac{1}{p} \leq 0$$

well let $s = 0$. It is natural because in this case function f of the class $W^\tau_{p,\alpha}(\Omega)$ generally speaking has no traces on

1) Γ or 2) γ.

But in the case 2) f has still traces on γ_1, corresponding $K = 0, 1, \ldots, \tau - 1$.

If $s = \tau$ we shall say that the weak degeneration takes place and if $s < \tau$ — — the strong one.

With the class $W^\tau_{2,\alpha}(\Omega)$ $(p = 2)$ we relate a differential equation

$$(1) \qquad Lu = \sum_{|K|,|l| \leq \tau} (-1)^{|l|} \mathscr{D}^{(l)} (Q_{Kl} U^{(K)}) = f \qquad x \in \Omega,\ Q_{Kl} = Q_{lK}(x)$$

with conditions

$$\sum_{|K|,|l| \leq \tau} Q_{Kl}\, \xi_K\, \xi_l > \frac{\varkappa}{\varrho^{2\alpha}} \sum_{|K| = \tau} \xi_K^2,$$

$$|Q_{Kl}(x)| < \frac{M}{\varrho^{2\alpha_{Kl}}}, \qquad \alpha_{Kl} = \tau + \alpha - \max(|K|, |l|),$$

$$\alpha_{Kl} < \tfrac{1}{2}.$$

Here Q_{Kl} are functions of x and vector parameters $\mathbf{K}$, τ; ξ_K are variables related with considered vectors $\mathbf{K}$ and $\varkappa$, M don't depend on x, ξ_K, ξ_l.

As asualy to consider questions of the smootniss of the classical solution it is necessary to propose in addition the usual conditions on differentiability of Q_{Kl}.

Such restrietions on the coefficients are necessary in our considerations too, because we consider not only generalised solutions but classical ones (belonging to $W^{r}_{2,\alpha}(\Omega)$).

We consider here the problem:

To find solution of the equality (1) belonging to the class $W^{r}_{2,\alpha}(\Omega)$ with boundary conditions 1) or 2).

This problem includes at $\alpha = 0$ the usual. Dirichlet problem for differential equation of the elliptic type.

If $s = \tau$ we shall call our problem "the weak problem" and if $s < \tau$ — "the strong one".

The series of investigations has been devoted to different problems with degeneration; see [16] §6, [18] where are given the lists of literature and also [1—15], [19—23].

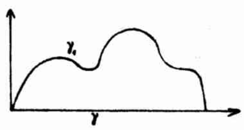
Now we are interested only in the mentioned formulated strong (boundary!) problem ($s < \tau$).

First investigations on this problem referred to the case 2) for the equation of the second order when therefore boundary values are given only on a part γ of Γ, (because $s = 0$ in this case).

M. B. Келдыш [10] has considered (in metric C) such a problem for the equations, which includes in particular the following one

$$(2) \qquad Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} y^m + cu = 0$$

on a two dimentional domain of such a kind (Plot 3). M. B. Келдыш has discovered in particular that for $m > 1$



Plot 3.

the Dirichlet problem for the equation (1) is not correct, but it is perfectly correct if to give boundary values only on γ_1.

It is possible to show that in the case 2) of the strong problem for the equations of the high order there exists the unique solution.

But the case 1) is quite different. In general in this case uniquess breaks for the strong problem.

For instance uniquess breaks for the equation (2), where $c = 0$, because every constant then satisfies corresponding homogeneous equation.

Recently (1964 [18]) П. И. Лизоркин and I have proved that the strong problem in case 1) has always the unique solution if $2s \geq \tau$, and in case $2s < \tau$ it is not right, generally speaking.

From the point of view of the variational method these questions may be explained as follows.

Uniquess of the generalised solution of the boundary problem depends essentialy on answer to the question: does the inequality of the Poincare type (for $p = q$):

$$(3) \qquad ||f||_{L_p(\Omega)} \leq c \left(\sum ||\varphi_K||_\Gamma + \sum_{|K|=\tau} \left\| \frac{f^{(K)}}{\varrho^\alpha} \right\|_{L_p(\Omega)} \right)$$

hold or not?

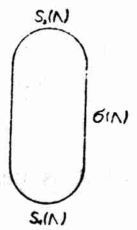
Here the norms $||\varphi_K||_\Gamma$ are taken in corresponding metric $W_2(\Gamma)$.

As for the uniquess of the classical solution this question in strong problem, to compare it with the weak one, has not principal differences.

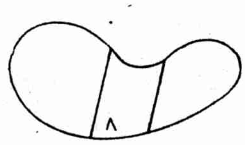
Let Λ be an open set, which is cut out from a circular cilinder by two not intersecting
smooth surfaces $S_1 = S_1(\Lambda)$, $S_2 = S_2(\Lambda)$. It is important that every line belonging to our closed cilinder intersects S_1, as well as S_2 only at one point being not tangent to S_1 (S_2) at this point. Such a domain Λ we shall name a regular one. We propose also that $\sigma = \sigma(\Lambda)$ is the side surface of Λ without points belonging to S_1, S_2 and name the $a \times e$ of the considered cilinder the $a \times e$ of our regular domain.

Let now as above Ω be an open bounded set with smooth boundary Γ.

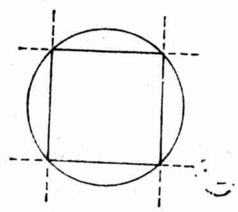
By definition Λ is a regular "bridge" of Ω if it satisfies the following conditions:



Plot 4.



Plot 5.


Plot 6.

1) Λ is a regular domain belonging to Ω,

2) $S_1(\Lambda)$, $S_2(\Lambda) \subset \Gamma$,

3) $\sigma(\Lambda) \not\subset \Gamma$.

We proved [17] the following

Lemma. *It is possible to cover Ω by a finite number of the regular bridges Λ.*

For istance, a two dimentional circle we can cover by two regular bridges, as on this plot

The proof of the Poincare inequality in the case 1) for $2s \geq \tau$ may be obtained by following steps.

At first we prove (П. И. Лизоркин and С. М. Никольский [18]) this inequality for the functions given on a one-dimentional segment $[a, b]$:

$$(4) \qquad \|f\|_{L_p(a,b)} < C_l \left\{ \sum_0^{S-1} (|f^{(K)}_{(a)}| + |f^{(K)}_{(b)}|) + \left\| \frac{f^{(k)}}{\varrho^\alpha} \right\|_{L_p(a,b)} \right\}.$$

It is right for $2s \geq \tau$, but it's not right for $2s < \tau$. Here C_l is a constant continuosly depending on $l = b - a > 0$.

The next step is the generalising of this inequality for regular bridges Λ:

$$(5) \qquad \|f\|_{L_p(\Lambda)} < c \left\{ \sum_0^{S-1} (\|\varphi_K\|_{L_p(S_1)} + \|\varphi_K\|_{L_p(S_2)}) + \left\| \frac{\frac{\partial^2 f}{\partial \xi^2}}{\varrho_\xi^\alpha} \right\|_{L_p(\Lambda)} \right\},$$

$$\varphi_K = \left. \frac{\partial^K f}{\partial \xi^K} \right|_{S_1, S_2}.$$

Here ϱ_ξ is the distance from x to Γ in direction of the axe ξ of the bridge. To obtain (5) we introduce the new coordinates $(\xi, \eta) = (\xi, \eta_1, \ldots, \eta_{n+1})$. The coordinate axe ξ is directed as the axe of the considered bridge and the other coordinate axes $\eta_1, \ldots, \eta_{n-1}$, are for instance ortogonal to it.

First we use the inequality (4) for $f = f(\xi, \eta)$, when η is fixed, then take (4) in power p and integrate on η. It leads to (5) if to take in account that the constant in (4) is bounded for $0 < l_1 \leq l \leq l_2$.

Lastly we substitute $\varrho(x)$ instead of $\varrho_\xi(x)$ in (5). It is possible because ϱ_ξ and ϱ have the same order $(c_1\varrho(x) < \varrho_\xi(x) < c_2\varrho(x))$ for all x belonging to a regular bridge Λ.

It is also possible to substitute normes $\|\varphi_K\|_\Gamma$ instead of the normes $\|\varphi_K\|_{L_p(S_i)}$ $(i = 1, 2)$, where already $\varphi_K = \left. \frac{\partial^K h}{\partial h^K} \right|_\Gamma$ and $\| \quad \|_\Gamma$ are understood in the corresponding metric $W_p^l(\Gamma)$ (instead $L_p(\Gamma)$). Finaly using the mentioned cover lemma we obtain the Poincare inequality (3).

To prove the inequality (3) in the case 2) one can begin from the following one dimentional inequality

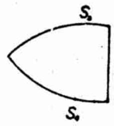
$$(6) \qquad \|f\|_{L_p(a,b)} < c_e \left\{ \sum_0^{s-1} |f^{(K)}_{(b)}| + \left\| \frac{f^{(K)}}{(x-a)^\alpha} \right\|_{L_p(a,b)} \right\}.$$

Here degeneration takes place only at one boondary point of $[a, b]$, namely at a. But there is no at all degeneration at other boundary point b.

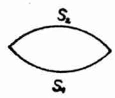
Let's now consider the same domain as on the plot 1. We propose also that our domain may be covered by bridges which connect either γ_1 with γ, or

γ_1 with γ_1. For the bridges of the first kind we use the inequality which generalises (6) and for the ones of the second kind the inequality (5) for $\alpha = 0$.

Pay attention that (6) differs from (4). In (4) C_l; is continuous only for $l > 0$, and in (6) for $l \geq 0$. The last gives possibility to generalise (6) for domains, more general than on the plot 4.



Plot 7.



Plot 8.

Now surfaces S_1 and S_2 can have common points.

Some remarks.

1) The mentioned method of the covering Ω by regular bridges may be used for transfer many other inequalities from one dimentional segment to the domains with enough smooth boundary, for instance, inequalities in the approximation theory by polinomials.

2) It is possible to extend the method on the domains with Lipshitz boundary.

3) Ю. Салманов received some development of the results. Namely he has obtained the corresponding inequality in the case of the strong degeneration on a domain with pricked out a point.

REFERENCES

[1] А. И. Ачальгиев, *Краевые задачи для эллиптических уравнений, испытывающих вырождение порядка на многообразиях. ДАН Тадж. ССР, 1964, 7, № 1, 3—7.*

[2] А. В. Бицадзе, *Уравнения смешанного типа, Москва 1959.*

[3] Н. Д. Введенская, *Об одной краевой задаче для уравнений эллиптического типа, вырождающихся на границе области, ДАН; 1953, 91, № 4, 711—714.*

[4] A. Weinstein, *Discontinous integrals and generalised potential theory. Trans. Amer. Math. Soc., 1948, 63, N 2.*

[5] A. Weinstein, *Some applications of generalised axially symmetric potential theory to continuos mechanics.*

[6] F. Wolf, *On singular partial differential boundary problems. Annali di mathematica pura and applieata, 1960, 49, 67—179.*

[7] В. П. Глушко, *Об уравнениях эллиптического типа, вырождающихся на многообразиях. Сборник „Функциональный анализ и его применение", Баку, АН Азерб. ССР 1961, 36—45.*

[8] В. К. Захаров, *Первая краевая задача для уравнений эллиптического типа второго и четвертого порядков, вырождающихся или имеющих особенности в конечном числе внутренних точек области. ДАН СССР, 1959, 124, № 4, 747—750.*

[9] А. М. Ильин, *Вырождающиеся эллиптические и параболические уравнения. Мат. сб. 1960, 50, 443—498.*

[10] М. В. Келдыш, *О некоторых случаях вырождения уравнений эллиптического типа на границе области. ДАН СССР, (1951), 77, № 2.*

[11] И. А. Киприянов, *О вариационном методе решения одного класса вырождающихся эллиптических уравнений. ДАН СССР, 1963, 152, № 1, 35—38.*

[12] Л. Д. Кудрявцев, *Теоремы вложения для классов функций, определенных на всем пространстве I, II.*

[13] A. Kufner, *Einige Eigenschaften der sobolevschen Räume mit Belegungsfunktion,* *Чехословаский Математический Журнал, 1965, 15(90), Прага, 597—620.*

[14] A. Kufner, *Lösungen der dirichletschen Problems für elliptische Differentialgleichungen in Räumen mit Belegungsfunktionen. Чехосл. Мат. Журнал, 1965, 15(90), Праха, 621—633.*

[15] J. Nečas, *Sur une methode pour re'soudre les equations aux derivees partielles du type elliptique voisine de la variationalle. Ann. Scuola Norm. Pisa ser. 3, 1962, 16, 4, 305—326.*

[16] С. М. Никольский, *О теоремах вложения, продолжения и приближения дифференцируемых функций многих переменных, Успехи Мат. Наук. 1961, 16, 5, 63—114.*

[17] С. М. Никольский, *Об одном методе покрытия области и неравенства для многочленов от многих переменных. Mathematica Soc. desciinte matematica si fisicendin.*

[18] С. М. Никольский и П. И. Лизоркин, *О некоторых неравенствах для функций из весовых классов и краевых задачах с сильным вырождением. ДАН СССР, 159, № 3.*

[19] С. М. Никольский, *К теории весовых классов дифференцируемых функций многих переменных и ее приложениям к краевым задачам для эллиптических уравнений. ДАН СССР, 1965, № 3, 510—512.*

[20] В. Р. Портнов, *Две теоремы вложения для пространств $L_{p\beta}$ и их приложения. ДАН СССР, 1964, 155, № 4, 762—764.*

[21] Л. Н. Слободецкий, И. А. Соломещ, *О первой краевой задаче для некоторых вырождающихся на границе эллиптических уравнений. Известия ВУЗов, Математика 1961, № 3.*

[22] С. А. Терсенов, *О задаче Римана-Гилберта для одной системы вырождающейся на границе, ДАН СССР, 1964, 155, № 3.*

[23] С. В. Успенский, *О теоремах вложения для весовых классов. Труды Матем. института им. В. Р. Стеклова АН СССР, 1961, 60, 282—303.*

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THE $\mathscr{L}^{p,\lambda}$ SPACES AND APPLICATIONS TO THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

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§ 1. The $\mathscr{L}^{p,\lambda}$ spaces.

In this lecture I propose to expose some results about the spaces $\mathscr{L}^{p,\lambda}$ and some of their applications to the theory of differential equations of elliptic type.

The theory of the $\mathscr{L}^{p,\lambda}$ spaces permits us to unify in a single family the spaces of Hölder continuous functions and the spaces L^p.

For some particular values of λ these spaces were already introduced some time ago by C. B. MORREY [16] and were used in the theory of differential equations of elliptic type both linear and non — linear.

Let $f(x)$ be a function defined, for simplicity on a cube Q_0 of R^n and belonging to $L^p(Q_0)$ ($p \geq 1$). The function $f(x)$ is said to belong to the space of Morrey $L^{p,\lambda}$ if there exists a constant K such that

$$\int_Q |fx)|^p \, dx \leq K|Q|^{1-\lambda/n} \tag{1.1}$$

for every subcube Q of Q_0 whose sides are parallel to those of Q_0.

We denote by $|Q|$ the n-dimensional measure of Q.

If $\lambda \geq 0$ one obtains a Banach space defining the norm as follows:

$$||f||^p_{L^{p,\lambda}} = \sup_{Q \subset Q_0} |Q|^{\lambda/n-1} \int_Q |f(x)|^p \, dx \, .$$

The condition that $\lambda \geq 0$ is essential because if $\lambda < 0$ then one would find that the only function belonging to $L^{p,\lambda}$ is the function 0. For $\lambda = n$ evidently we have $L^{p,n} \equiv L^p$ and for $\lambda = 0$ we have $L^{p,0} \equiv L^\infty$ for all $p \geq 1$.

More recently [13], [14], [1], [21] the spaces $\mathscr{L}^{p,\lambda}$ were introduced in the following manner: a function of $L^p(Q_0)$ is said to belong to $\mathscr{L}^{p,\lambda}$ if there exists a constant K, such that

$$\int_Q |f(x) - f_Q|^p \, dx \leq K^p |Q|^{1-\lambda/n}, \tag{1.2}$$

for every subcube Q of Q_0 with sides parallel to those of Q_0, where f_Q denotes the (integral) mean value of f on Q. Let us set

$$[f]^p_{\mathscr{L}^{p,\lambda}} = \sup_{Q \subset Q_0} |Q|^{\lambda/n-1} \int_Q |f(x) - f_Q|^p \, dx \tag{1.3}$$

and

$$||f||_{\mathscr{L}^{p,\lambda}} = ||f||_{L^p} + [f]_{\mathscr{L}^{p,\lambda}}. \tag{1.4}$$

In this manner $||f||_{\mathscr{L}^{p,\lambda}}$ will be a norm of the Banach space $\mathscr{L}^{p,\lambda}$ while $[f]_{\mathscr{L}^{p,\lambda}}$ is on the other hand a norm if we identify two functions which differ by a constant.

We observe that a function f belongs to $\mathscr{L}^{p,\lambda}$ if and only if there exists a constant K and for each subcube $Q \subset Q_0$ a constant $\bar{f}_Q$ such that

$$\int_Q |f(x) - \bar{f}_Q|^p \, dx \leq K^p |Q|^{1-\lambda/n} \tag{1.5}$$

for any subcube Q of Q_0 with sides parallel to those of Q_0. We obtain a semi-norm equivalent to $[f]_{\mathscr{L}^{p,\lambda}}$ if we take

$$\sup_{Q \subset Q_0} \inf |Q|^{\lambda/n-1} \int_Q |f(x) - \bar{f}_Q|^p \, dx$$

where the infinum is taken over all the constants $\bar{f}_Q$ associated to f and Q.

If $q \geq p$ and $\frac{\mu}{q} \leq \frac{\lambda}{p}$ then $\mathscr{L}^{q,\mu} \subset \mathscr{L}^{p,\lambda}$.

If $\lambda > 0$ the two spaces $\mathscr{L}^{p,\lambda}$ and $L^{p,\lambda}$ coincide and hence one can assume $\bar{f}_Q \equiv 0$ in (1.5). But the spaces $L^{p,0}$ and $\mathscr{L}^{p,0}$ are different. In fact, while the first coincides with the space of all (essentially) bounded functions the second coincides with a space studied by F. John and L. Nirenberg [13] which consists of functions of bounded mean oscillation and we denote this space by $\mathscr{E}_0$.

The space $\mathscr{E}_0$ consists of functions $f(x)$ for which there are two constants H and β such that

$$\text{meas}\,\{x; |f(x) - f_Q| > \sigma\} \leq H\, e^{-\beta\sigma} |Q|$$

for every subcube Q of Q_0.

This is equivalent to say that there exist two constants ϑ and K such that

$$\int_Q e^{\vartheta |f(x) - f_Q|} \, dx \leq K|Q|,$$

for every cube Q contained in Q_0.

For $p < \lambda < 0$ the space $\mathscr{L}^{p,\lambda}$ coincides with the space of Hölder continuous functions $C_{0,\alpha}$ where the exponent α is given by $\alpha = -\frac{\lambda}{p}$. In fact, setting

$$[f]_{0,\alpha} = \sup_{x',x'' \in Q_0} \frac{|u(x') - u(x'')|}{|x' - x''|^{\alpha}},$$

the two norms $[f]_{0,\alpha}$ and $[f]_{\mathscr{L}^{p,\lambda}}$, after identifying two functions which differ by a constant, are equivalent. This result was proved (independently) by S. CAMPANATO [1] and N. MEYERS [14].

It is important to observe that the role played by the cubes Q in the previous definitions can be substituted by any family of sets $\{E\}$ which are "regular" in the sense that for each set E of the family there exists two cubes $Q' \subset Q''$ such that

$$Q' \subset E \subset Q'', \qquad \nu^{-1} \leq \frac{|Q'|}{|Q|} \leq \nu$$

where ν is a constant independent of the particular set E considered.

Thus one can remark that the property that a function f belongs to a space $\mathscr{L}^{p,\lambda}$ is not altered by a change of variables which is bilipschitzian.

In a manner analogous to what one does in the case of the L^p spaces one can introduce also the weak $\mathscr{L}^{p,\lambda}$ spaces. A function $f(x)$ is said to belong to the space $\mathscr{L}^{p,\lambda}$ — weak if there exists a constant K such that for each cube $Q \subset Q_0$ with sides parallel to those of Q_0 we have

$$\operatorname{meas}\left\{x \in Q;\ |f(x) - f_Q| > \sigma\right\} \leq \left(\frac{K}{\sigma}\right)^p \cdot |Q|^{1-\lambda/n}.$$

The introduction of the spaces $\mathscr{L}^{p,\lambda}$ permits us to rediscover and to generalize a classical result of C. B. MORREY.

Let $u(x) \in H^{1,p}(Q_0)$(1) and suppose that for each subcube Q of Q we have

$$\int_Q |u_x|^p \, dx \leq K^p |Q|^{1-\lambda/n}, \qquad 0 \leq \lambda \leq n,$$

with a constant K independent of Q; that is to say $u_x \in L^{p,\lambda}$. Then, if $p < \lambda$ the function u belongs to $\mathscr{L}^{\tilde{p},\lambda}$ — weak where

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{\lambda}$$

and

$$\operatorname{meas}\{x \in Q;\ |u - u_Q| > \sigma\} \leq \left(\frac{K}{\sigma}\right)^{\tilde{p}} |Q|^{1-\lambda/n}.$$

1) We denote by $H^{1,p}(\Omega)$ the completion of the functions u which together with their first derivatives are continuous in Ω with respect to the norm

$$\|u\|_{H^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_i \|u_{x_i}\|_{L^p(\Omega)}$$

while $H_0^{1,p}(\Omega)$ denotes the closure in $H^{1,p}(\Omega)$ of the functions with compact support. We will write, in the following, H^1 and H_0^1 instead of $H^{1,2}$ and $H_0^{1,2}$.

If, instead, $p = \lambda$, then $u \in \mathscr{L}^{1,0} \equiv \mathscr{E}_0$ and

$$[u]_{\mathscr{L}^{1,0}} \leq K.$$

Finally if $p > \lambda$ then $u \in \mathscr{L}^{1,\mu}$ with $\mu = \dfrac{\lambda}{p} - 1$; that is $u \in C_{0,\beta}$ where $\beta = 1 - \dfrac{\lambda}{p}$.

These results for $\lambda = n$ take a weak form of the well known Sobolev inequality.

§ 2. Interpolation in the spaces $\mathscr{L}^{p,\lambda}$.

The problem of interpolation in the spaces $\mathscr{L}^{p,\lambda}$ presents itself in an nteresting manner. To this end we shall introduce the following definitions:

Definition (2.1) — *A linear operation T on functions f defined over Q_0 is said to be of strong type $\mathscr{L}[p, (q, \mu)]$ if there exists a constant K, indepedent of f, such that*

$$[Tf]_{\mathscr{L}^{q,\mu}} \leq K\|f\|_{L^p}; \tag{2.1}$$

the smallest of the constants K in (2.1) *is called the strong $\mathscr{L}[p, (q, \mu)]$ norm of T.*

We now introduce the following expression:

$$\Phi_\mu(u, \sigma) = \sup_{Q \subset Q_0} [|Q|^{\mu/n-1} \operatorname{meas} \{x \in Q;\, |u(x) - u_Q| > \sigma\}].$$

Definition (2.2) — *A linear operation T on functions defined over Q_0 is said to be of weak type $\mathscr{L}[p, (q, \mu)]$ if there exists a constant K, independent of f, such that*

$$\Phi_\mu(Tf, \sigma) \leq \left(\frac{K\|f\|_{L^p}}{\sigma}\right)^q; \tag{2.2}$$

the smallest of the constants K in (1.5) *is called the weak $\mathscr{L}[p, (q, \mu)]$ norm of T.*

Theorem (2.1) [21] — *Let $[p_i, q_i, \mu_i]$ be real numbers satisfying the conditions*

$$p_i \geq 1, \quad p_i \leq q_i \quad (i = 1, 2); \quad p_1 \neq p_2 \quad \text{and} \quad q_1 \neq q_2.$$

For $0 < t < 1$ let $[p(t), q(t), \mu(t)]$ be defined by the relations

$$\begin{cases} \dfrac{1}{p} = \dfrac{(1-t)}{p_1} + \dfrac{t}{p_2}, \qquad \dfrac{1}{q} = \dfrac{(1-t)}{q_1} + \dfrac{t}{q_2}, \\[2ex] \dfrac{\mu}{q} = (1-t)\dfrac{\mu_1}{q_1} + t\dfrac{\mu_2}{q_2} \end{cases} \tag{2.3}$$

If T is a linear operation which is simultaneously of weak types $\mathscr{L}[p_i, (q_i, \mu_i)]$ with respective norms K_i $(i = 1, 2)$ then T is of strong type $\mathscr{L}[p, (q, \mu)]$ for $0 < t < 1$ and

$$[Tf]_{\mathscr{L}^{(q,\mu)}} \leq \mathscr{K}\, K_1^{(1-t)}\, K_2^t ||f||_{L^p(Q_0)}$$

where $\mathscr{K}$ is a constant, independent of f, but depending on t, p_i, q_i, μ_i and it is bounded for t away from 0 and 1.

An useful corollary of theorem (2.1) is the following.

Corollary (2.1) — *Any time a linear operation T maps L^{p_1} into a space of Hölder continuous functions and L^{p_2} into a (weak) L^{q_2} — space, then exist there a special $\bar{p}$ such that T maps $L^{\bar{p}}$ into the space $\mathscr{E}_0$.*

For generalizations of this theorem see [8], [9], [18].

Theorem (2.2) [5] — *Let $[p_i, (q_i, \mu_i)]$ be real numbers such that p_i, $q_i \geq 1$ $(i = 1, 2)$. If T is a linear operation (in general on complex valued function on Q_0) which is simultaneously of strong types $\mathscr{L}[p_i, (q_i, \mu_i)]$ with respective norms K_i $(i = 1, 2)$ then T is of strong type $\mathscr{L}[p, (q, \mu)]$ where p, q, μ are defined for $0 \leq t \leq 1$ by (1.6) and further the following estimate holds*

$$[u]_{\mathscr{L}^{q,\mu}} \leq K_1^{(1-t)}\, K_2^t ||u||_{L^p}\,.$$

The previous theorems generalize respectively the theorems of interpolation of Marcinkiewicz and of Riesz—Thorin.

Another theorem of interpolation is found to be very useful; it completes the theorems above. For this purpose we shall introduce the spaces N^p.

We shall denote by $\bar{S}$ the family of systems S of a finite number of subcubes Q_i no two of which have an interior point in common and having their sides parallel to those of Q_0 $(\bigcup_i Q_i = Q_0)$.

For any (real or complex valued) function $u \in L^1(Q_0)$ and for any $1 < < p < +\infty$ we consider the expressions of the form

$$\sum_i \Big| \int_{Q_i} |u - u_{Q_i}|\, dx \Big|^p |Q_i|^{(1-p)}$$

where Q_i runs through a system $S \in \bar{S}$.

For $1 < p < +\infty$ set

$$[u]_{N^p} = \sup_{\{Q_i\} \equiv S \in \bar{S}} \Big\{ \sum_i \Big| \int_{Q_i} |u - u_{Q_i}|\, dx \Big|^p |Q_i|^{(1-p)} \Big\}^{1/p}$$

and the following.

Definition (2.3) — *A function u is said to belong to N^p $1 \leq p < +\infty$ if $[u]_{N^p} < +\infty$. We observe that $[u]_{N^p}$ defines a semi-norm in N^p and we obtain a Banach space by taking*

$$||u||_{N^p} = ||u||_{L^1} + [u]_{N^p}$$

as the norm in N^p.

If $q \geq p$, then $N^q \subset N^p$.

If $u \in L^1(Q_0)$ then we have

$$\lim_{p \to +\infty} [u]_{N^p} = [u]_{\mathscr{L}^{1,0}} = \mathscr{E}_0$$

i.e. *we may set $N^\infty = \mathscr{L}^{(1,0)} = \mathscr{E}_0$.*

In connection with these spaces N^p the following result due to F. JOHN and L. NIRENBERG holds [13].

If $u \in N^p$ with $p > 1$ then there exists a constant C such that, for any cube $Q \subset Q_0$, we have

$$\text{meas}\,\{x \in Q;\, |u(x) - u_Q| > \sigma\} \leq C \left(\frac{[u]_{N^p(Q)}}{\sigma}\right)^p.$$

Conversely, one can show that if u is a measurable function satisfying the condition

$$\text{meas}\,\{x \in Q;\, |u(x) - u_Q| > \sigma\} \leq C \left(\frac{K(Q)}{\sigma}\right)^p$$

for each cube $Q \subset Q_0$ where $K(Q)$ are constants with the following property:

for any system $\{Q_i\} \equiv S \in \bar{S}$, introduced above, and for some $r \leq p$ we have

$$\sum_i |K(Q_i)|^r \leq |K(Q)|^r,$$

then $u \in N^p$ and we have

$$[u]_{N^p} \leq \frac{2}{(p-1)^{1/p}} K.$$

In fact, we have

$$\int_Q |u(x) - u_Q|\, dx \leq \frac{2K(Q)}{(p-1)^{1/p}} |Q|^{1-1/p}$$

from which it follows that for $\{Q_i\} \equiv S \in \bar{S}$,

$$\sum |Q_i|^{1-p} \left| \int_{Q_i} |u(x) - u_{Q_i}|\, dx \right|^r \leq \frac{2^p}{p-1} |K(Q_i)|^r\, |K(Q_i)|^{p-r} \leq \frac{2^p}{p-1} |K(Q)|^p.$$

Admitting this result we have the following theorem of interpolation.

Theorem (2.3) [22] — *Let T be a linear operation defined on the class $\mathscr{F}$ of (real valued) simple functions on Q_0 such that*

$$[Tu]_{\mathscr{L}^{(1,0)}} \leq K_1 \|u\|_{L^{p_1}},$$

$$[Tu]_{N^{q_2}} \leq K_2 \|u\|_{L^{p_2}}$$

where p_1, p_2, $q_2 > 1$ *with* $q_2 \geq p_2$. *If* p, $q \geq 1$ *are defined by*

$$\frac{1}{p} = \frac{(1-t)}{p_1} + \frac{t}{p_2}, \qquad \frac{1}{q} = \frac{t}{q_2} \tag{2.4}$$

then

$$\|Tu - (Tu)_{Q_0}\|_{L^q} \leq \mathscr{K} K_1^{(1-t)} K_2^t \|u\|_{L^p} \quad \textit{for} \quad u \in \mathscr{F}$$

where $\mathscr{K}$ *is a constant which is bounded if* t *is away from* 0 *and* 1.

The theorem is valid also for $p_1 = +\infty$.

Before giving some applications of this theorem we observe that if $f \in L^p$ — weak and

$$\text{meas}\,\{x \in Q;\ |f(x)| > \sigma\} \leq \left(\frac{K(Q)}{\sigma}\right)^p$$

and if there exists an $r < p$ such that $\sum |K(Q_i)|^r \leq |K(Q)|^r$, then

$$[f]_{N^p} \leq \text{const}\ |K(Q)|.$$

In fact, then there exists a constant $C(p)$ such that

$$\text{meas}\,\{x \in Q;\ |f(x) - f_Q| > \sigma\} \leq C(p) \left(\frac{K(Q)}{\sigma}\right)^p.$$

In particular, the assumption is satisfied provided $f \in L^p$ with $K(Q) = \int_Q |f|^p\, dx$.

We deduce from theorem (2.3) the following results:

Theorem (2.4) — *Let* T *be a linear operation defined on the class* $\mathscr{F}$ *of simple functions on* Q_0 *such that*

$$[Tu]_{\mathscr{L}^{1,0}} \leq K_1 \|u\|_{L^{p_1}}; \qquad \|Tu\|_{L^{q_2}} \leq K_2 \|u\|_{L^{p_2}},$$

where p_1, p_2, $q_2 > 1$ *with* $q_2 \geq p_2$. *Then*

$$\|Tu\|_{L^q} \leq \mathscr{K} K_1^{(1-t)} K_2^t \|u\|_{L^p},$$

where $\mathscr{K}$ *is a constant which is bounded if* t *is away from* 0 *and* 1 *and* p *and* q *are given by* (2.4).

The theorem is valid also for $p_1 = +\infty$.

Theorem (2.4) can be extended in the following way

Theorem (2.5) — *Let* T *be a linear operation defined on the class* $\mathscr{F}$ *of simple functions on* Q_0 *such that*

$$[Tu]_{\mathscr{L}^{1,0}} \leq K_1 \|u\|_{L^{p_1}},$$

$$\text{meas}\ \{|Tu| > \sigma\} \leq \left(\frac{K_2||u||_{L^{p_2}}}{\sigma}\right)^{q_2}$$

where $p_1 \geq 1$, $p_2 \geq 1$, $q_2 > 1$. *Then*

$$||Tu||_{L^q} \leq \mathscr{K}\, K_1^{1-t}\,.\, K_2||u||_{L^p}$$

where $\mathscr{K}$ *is a constant which is bounded if* t *is away from* 0 *and* 1 *and* p *and* q *are given by* (2.3).

The theorem holds also for $p_1 = +\infty$.

We are going to sketch the proof of this theorem making use of a trick introduced by CAMPANATO in giving a new proof of theorem (2.4) [4].

Let S a fixed system of a finite number of subcubes Q_i no two of which have an interior point in common and having their sides parallel to those of Q_0. Set

$$\mathscr{T}(u) = \frac{1}{|Q_i|}\int_{Q_i} |Tu - (Tu)_{Q_i}|\, dx \quad \text{in} \quad Q_i.$$

The map $\mathscr{T}(u)$ is sub-linear and satisfy

$$||\mathscr{T}(u)||_{L^\infty} \leq K_1||u||_{L^{p_1}}$$

$$\text{meas}\ \{|\mathscr{T}(u)| > \sigma\} \leq \left(\frac{K_2'||u||_{L^{p_2}}}{\sigma}\right)^{q_2}.$$

The first inequality is obvious; the second one can be proved easily. In fact if we denote by Q_i' the cubes of S for which one has

$$\int_{Q_i'} |Tu - (Tu)_{Q_i'}|\, dx > \sigma|Q_i'|,$$

it follows

$$\sigma \sum |Q_i'| \leq 2 \int_{\cup Q_i'} |Tu|\, dx \leq 2\left(1 - \frac{1}{q_2 - 1}\right) K_2||u||_{L^{p_2}} \left(\sum |Q_i'|\right)^{1-1/q_2},$$

and then

$$\text{meas}\ \{|\mathscr{T}(u)| > \sigma\} = \sum |Q_i'| \leq \left\{2\left(1 - \frac{1}{q_2 - 1}\right) K_2||u||_{L^{p_2}}/\sigma\right\}^{q_2}.$$

Applying the theorem of MARCINKIEWICZ it follows that

$$||\mathscr{T}(u)||_{L^q} \leq \mathscr{K}\, K_1^{1-t}\, K_2^t||u||_{L^p}$$

where p and q are given by (2.3) and $\mathscr{K}$ is a constant which is bounded if t stay away from 0 and 1.

But, from the definition of $\mathscr{T}(u)$, we have

$$\{\sum_i | \int_{Q_i} |Tu - (Tu)_{Q_i}|\, dx|^q\, |Q_i|^{1-q}\}^{1/q} \leq \mathscr{K}\, K_1^{1-t}\,.\, K_2^t||u||_{L^p},$$

and thus, since S is arbitrary

$$[Tu]_{N^q} \leq \mathscr{K} K_1^{1-t} K_2^t ||u||_{L^p}$$

therefore, applying the lemma of F. JOHN and L. NIRENBERG,

$$\text{meas}\,\{|Tu - (Tu)_Q| > \sigma\} \leq \left(\frac{\mathscr{K}' K_1^{1-t} K_2^t ||u||_{L^p}}{\sigma}\right)^q .$$

Then making use again of the theorem of MARCINKIEWICZ one has

$$||Tu - (Tu)_Q||_{L^q} \leq \mathscr{K}'' \,.\, K_1^{1-t} \,.\, K_2^t ||u||_{L^p}$$

and from this the conclusion of the theorem follows easily.

It would be interesting to know whether the theorem (2.5) holds for $q_2 = 1$.

Theorem (2.5) can be considered as a generalization of the theorem of MARCINKIEWICZ where the space $\mathscr{E}_0$ replaces usefully the space L^∞.

From the corollary (2.1) and theorem (2.5) the theorem of interpolation follows:

Theorem (2.6) — *Let T be a linear mapping such that, continuously*

$$T \;:\; L^{p_1} \to C^{0,\alpha}$$
$$T \;:\; L^{p_2} \to L^{q_2} \qquad (weak),\ q_2 > 1,\ p_2 \leq q_2$$

then, for $\dfrac{1}{p} = \dfrac{1-t}{p_1} + \dfrac{t}{p_2}$, $0 < t < 1$, *set* $\vartheta = \alpha \Big/ \left(\alpha + \dfrac{n}{q_2}\right)$

$$T : L^p \to \begin{cases} C^{0,\beta}, & for \quad 0 \leq t < \vartheta, \qquad \beta = (1-t)\,\alpha - \dfrac{n}{q_2}\, t \\ \mathscr{E}_0, & for \quad t = \vartheta \\ L^q, & for \quad \vartheta < t < 1, \qquad \dfrac{1}{q} = \dfrac{1}{q_2}\left\{\left(1 + \dfrac{\alpha q_2}{n}\right) t - \dfrac{\alpha q_2}{n}\right\} \end{cases}$$

The previous results on interpolation show that the $\mathscr{L}^{p,\lambda}$ spaces form a family of spaces of interpolation with respect to special families of spaces, the L^p — spaces. There might be more general families of spaces than the L^p spaces with respect to which the spaces $\mathscr{L}^{p,\lambda}$ are spaces of interpolation (see [19]), but, on the other side, the spaces $\mathscr{L}^{p,\lambda}$ are not spaces of interpolation with respect to the family of the spaces $\mathscr{L}^{p,\lambda}$ themselves. E. M. STEIN and A. ZYGMUND [24] have indeed proved this fact adapting an example given by HARDY and LITTLEWOOD [11]. They have proved that there exists a linear mapping T which maps continuously $C^{0,\alpha}$ into $C^{0,\alpha}$, L^2 into L^2 but it does not map $\mathscr{E}_0$ into $\mathscr{E}_0$.

Thus, it is interesting to find families of operations which leave the spaces $\mathscr{L}^{p,\lambda}$ invariant. One of these families of operators has been found by J. PEETRE [17].

This family includes the singular integral transform of CALDERON—ZYGMUND.

A consequence of theorem (2.4) is the following.

Theorem (2.6) — *If the operator T leaves the spaces $\mathscr{L}^{p,\lambda}$ invariant for a fixed p and for $0 \leq \lambda < n$, then T leaves invariant the spaces L^q for all $q \geq p$.*

In fact, one has

$$T : L^\infty \to \mathscr{E}_0$$
$$T : L^p \to L^p$$

and, thus, from theorem (2.4), follows

$$T\colon\ L^q \to L^q \qquad \text{for} \qquad q \geq p.$$

Making use of the interpolation theorem (2.4) it is possible to give an easy proof of a theorem by HORMANDER [12], (see [23], [19]).

Consider the translation invariant mapping

$$Tf = \int K(x - y)\, f(y)\, dy$$

and assume that the Fourier transform $\widehat{K}$ of K, as distribution, satisfies: $|\widehat{K}(x)| \leq A$. Moreover assume that

$$\int_{|x| \geq 2|y|} |K(x - y) - K(x)|\, dx \leq A.$$

Then Tf maps L^2 into L^2 because of the first assumption. It can be proved that T maps L^∞ into $\mathscr{E}_0$ [23], [19].

It follows, from theorem (2.4) that Tf maps L^p into L^p for $p \geq 2$.

By a duality argument the same conclusion holds for $p > 1$.

The proof that T maps L^∞ into $\mathscr{E}_0$ is easy and we are going to sketch it here.

Let f be a bounded function ($|f(x)| \leq 1$) and write $u(x) = Tf$. Fix a cube Q, which we may assume centered at the origin. Let us split $f = f_1 + f_2$ where $f_1(x) = f(x)$ in the sphere S' of diameter twice that of Q and having the same center that Q; $f_1(x) = 0$ outside this sphere. Write $u_i(x) = T(f_i)$ $(i = 1,2)$; $u(x) = u_1(x) + u_2(x)$.

Now

$$\int_Q |u_1(x)|^2\, dx \leq A^2 \int_S |f_1(x)|^2\, dx \leq A^2\, c|Q|.$$

Next

$$u_2(x) = \int K(x - y)\, f_2(y)\, dy.$$

Let

$$u_Q = \int K(y)\, f_2(y)\, dy.$$

Therefore

$$|u_2(x) - u_Q| \leq \int_{y \notin S} |K(x - y) - K(y)| \leq A.$$

Combining the informations above we get

$$\frac{1}{|Q|}\int_Q |u(x) - u_Q|^2 \, dx \leq A^2(1 + c)$$

i.e.: $u \in \mathscr{E}_0$.

§ 3. Application to the theory of differential equations.

C. B. Morrey has extensively used the spaces $\mathscr{L}^{2,\lambda}$ for $0 < \lambda < n$ in the theory of differential equations of elliptic type linear and non-linear [16]. Some of his results can be extended making use of the spaces $\mathscr{L}^{2,\lambda}$ either for positive or negative values of λ. We mention the following theorem which generalizes a theorem by Morrey [15]. It can be proved essentially in the same way.

Let $a_{ij}(x)$ $(i, j = 1, 2, \ldots, n)$ be bounded measurable functions in an open set Ω, satisfying

$$\sum_{i,j}^{1\ldots n} a_{ij}(x)\, \xi_i \xi_j \geq \nu(\xi)^2 \qquad \nu = \text{const} > 0, \qquad \xi \in R^n$$

and let f_i be n functions of $L^2(\Omega)$. Let u be a function of $H^1(\Omega)$ which, with the usual convention on the sum, satisfies

$$\int_\Omega a_{ij}(x)\, u_{x_i} v_{x_j} \, dx = \int_\Omega f_i v_{x_i} \, dx \qquad \text{for all} \qquad v \in H_0^1(\Omega). \tag{3.1}$$

The following theorem holds

Theorem (3.1) — *There exists a constant λ_0, $0 < \lambda_0 < 2$ such that, for $f_i \in \mathscr{L}^{2,\lambda}$ with $\lambda_0 < \lambda \leq n$, one has, in any Ω' with $\overline{\Omega}' \subset \Omega$, $u_{x_i} \in L^{2,\lambda}$ and, consequently $u \in \mathscr{L}^{\tilde{2},\lambda} \subset \mathscr{L}^{2,\lambda-2}$ where $\dfrac{1}{\tilde{q}} = \dfrac{1}{2} - \dfrac{1}{\lambda}$ for $\lambda > 2$, and $u \in \mathscr{L}^{2,\lambda-2}$ for $\lambda \leq 2$.*

In [15] this theorem is proved assuming $\lambda_0 < \lambda < 2$; with such a limitation the function u is Hölder continuous.

From theorem (3.1) and using the interpolation theorem (2.4) it is possible to deduce some estimates found in [20]:

If $f_i \in L^p$, $p > 2$, then (i) $u \in L^{p^*}$ where $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$ for $p < n$ (ii) $u \in \mathscr{E}_0$ for $p = n$, (iii) u is Hölder continuous for $p > n$.

When in (3.1) the coefficients $a_{ij}(x)$ are assumed to be Hölder continuous more informations can be obtained for u.

Campanato [2] has proved the following theorem.

Theorem (3.2) — *Let f_i be in $\mathcal{L}^{2,\lambda}$, with $-2 < \lambda \le n$, and let Ω' be a open set such that $\overline{\Omega}' \subset \Omega$.*

(i) If the coefficients a_{ij} are continuous and $0 < \lambda \le n$, then, $u_{x_i} \in \mathcal{L}^{2,\lambda}$ in Ω'.
(ii) If a_{ij} are Hölder continuous in $\overline{\Omega}$ and $\lambda = 0$ then, in Ω', $u_{x_i} \in \mathcal{E}_0$.
(iii) If $a_{ij} \in C^{0,-\lambda/2}$ and $-2 < \lambda < 0$ then $u_{x_i} \in \mathcal{L}^{2,\lambda} \equiv C^{0,-\lambda/2}$.
If Ω is "smooth" and $u \in H_0^1(\Omega)$, then the same conclusions hold in $\overline{\Omega}$.

This theorem unifies CACCIOPPOLI—SCHAUDER estimates with MORREY'S estimates.

The proof of this theorem does not make use of the potential theory.

From theorem (3.2) and the interpolation theorem (2.4) it follows that when $f_i \in L^p(\Omega)$, $p > 1$ one has $u_{x_i} \in L^p(\Omega)$. This method has been used in [6].

It should be mentioned that a generalization of the spaces $\mathcal{L}^{p,\lambda}$, with respect to a different norm in R^n, has been considered. This generalization turns out to be useful in dealing with parabolic and quasi elliptic differential equations. See [7], [3], [10].

BIBLIOGRAPHY

[1] S. CAMPANATO — *Proprietà di holderianità di alcune classi di funzioni*, Ann. Scuola Norm. Sup. Pisa, Vol. 17, 1963, pp. 175 — 188.

[2] S. CAMPANATO — *Equazioni ellittiche del II° ordine e spazi $\mathcal{L}^{(2,\vartheta)}$*, Ann di. Matematica, Vol. 69, 1965, pp. 321—382.

[3] S. CAMPANATO — *Equazioni paraboliche del secondo ordine e spazi $\mathcal{L}^{2,\theta}(\Omega, \delta)$*, Ann .di Matematica, Vol. 73, 1966, pp. 55—102.

[4] S. CAMPANATO — *Su un teorema di G. Stampacchia*, Ann. Sc. Norm. Sup. Pisa, Vol. XX, 1966, pp. 649—652.

[5] S. CAMPANATO and M. K. V. MURTHY — *Una generalizzazione del teorema di Riesz-Thorin*, Ann. Sc. Norm. Sup. Pisa, Vol. 19, 1965, p. 87—100.

[6] S. CAMPANATO and G. STAMPACCHIA — *Sulle maggiorazioni in L^p nella teoria dell' equazioni ellittiche*, Boll. U. M. I., Vol, XX, 1965, pp. 393—399.

[7] G. DA PRATO — *Spazi $\mathcal{L}^{(p,\theta)}(\Omega, \delta)$ e loro proprietà*, Ann. di Matem. Vol. 69, 1965, pp. 383—392.

[8] P. GRISVARD — *Semi-groups faiblement continus et interpolation*, C. R. Acad. Sc. Paris, Vol. 259, 1964, pp. 27—29.

[9] P. GRISVARD — *Commutativité de deux foncteurs d'interpolation et applications*, Thèse.

[10] E. GIUSTI — *Equazioni quasi-ellittiche e spazi $\mathcal{L}^{(p,\vartheta)}(\Omega, \delta)$*, (I) to appear in Ann. di Matem., (II) Ann. Sc. Norm. Sup.

[11] G. H. HARDY and J. E. LITTLEWOOD — *Theorems concerning the mean values of analytic or harmonic functions*, Quart. J. of Math. Vol. 12, 1941, pp. 221—256.

[12] L. HÖRMANDER — *Estimates for translation invariant operators in L_p spaces*, Acta *Math., 104 (1960), pp. 93—140.*

[13] F. John and L. Nirenberg — *On functions of bounded mean oscillations*, Comm. Pure Appl. Math., Vol. 14, 1961, pp- 415—426.

[14] G. N. Meyers — *Mean oscillation over cubes and Hölder continuity*, Proc. Amer. Math. Soc. Vol. 15, 1964, p. 717—721.

[15] C. B. Morrey — *Second order elliptic equations in several variables and Hölder continuity*, Math. Z. *72* (1959), 146—164.

[16] C. B. Morrey — *On the solution of quasi-linear elliptic partial differential equations*, Trans. Am. Math. Soc. Vol. 43, 1938, pp. 126—166.

[17] J. Peetre — *On convolution operators leaving $\mathscr{L}^{p,\lambda}$ spaces invariant*, Ann. Matem. Vol. 72, 1966, pp. 295—304.

[18] J. Peetre — *Espaces d'interpolation et théorème de Soboleff*, Ann. Inst. Fourier, Grenoble 16, 1966, pp. 279—317.

[19] S. Spanne — *Sur l'interpolation entre les espaces $\mathscr{L}_k^{p,\Phi}$*. Ann. Sc. Norm. Sup. Pisa, Vol. 20, 1966, pp. 625—648.

[20] G. Stampacchia — *Régularisation des solutions de problèmes aux limites elliptiques à données discontinues*, Intern. Symp. on Lin. Spaces, Jerusalem, 1960

[21] G. Stampacchia — *$\mathscr{L}^{(p,\lambda)}$ spaces and interpolation*, Comm. Pure and Appl. Math. Vol. 17, 1964, pp. 293—306.

[22] G. Stampacchia — *The Spaces $\mathscr{L}^{(p,\lambda)}$, $N^{(p,\lambda)}$ and interpolation*, Ann. Sc. Norm. Sup. Pisa. Vol. 19, 1965, pp. 443—462.

[23] E. M. Stein — *Singular integrals, harmonic functions, and differentiability properties of functions of several variables*, to appear.

[24] E. M. Stein and A. Zygmund — *Boundedness of translation invariant operators on Hölder spaces and L^p spaces*, to appear.

[25] A. Zygmund — *Trigonometric Series*, Cambridge University Press, 1959.

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INVESTIGATION OF THE SOLUTIONS OF DIFFERENTIAL EQUATIONS ON AN INFINITE INTERVAL AND THE FIXED POINT THEOREMS

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The fundamental question which is to solve in the theory of the differential equations is the question of existence. It can be solved by various methods, chosen following the made assumptions and the expected properties of the solution. In the last time the methods based on the theorems of the fixed points seem to very efficient. Those theorems serve as a very important and convenient way, and we can state that they are the móst elegant, for the proof of the existence of the solution determined for instance by the initial conditions (not only for the proof of the local existence, but also of the existence in large). Then there are boundary-value problems, the linear problems, the problems of the existence of the periodic or almost periodic solutions, the existence of bounded solutions, of monotone solutions or of the solutions having other required properties. In all this problems the theorems of the fixed point have been used with a great succes. I could quote a very long list of the works concerning with those problems (to begin with the works of G. D. BIRKHOFF, O. D. KELLOG, R. CACCIOPOLI, SCHAUDER, LERAY to the last works of KRASNOSELSKI, BROWDER, CESARI, HALE, URABE, KNOBLOCH, CONTI, KAKUTANI, LASOTA, OPIAL, HAIMOVICI, BIELECKI, CORDUNEANU and others).

I will consider the theorem of SCHAUDER and indicate some of the variants which are very convenient especially in the case when the existence of solutions is to be proved with the required properties in an infinite interval.

We find the theorem of SCHAUDER quoted in the literature essentially in two forms of what the following form seems to be more convenient for application:

Let M be a convex and closed set of a Banach space. Let T be a continuous operator on M such that $TM \subset M$ and TM is (relatively) compact. Then T has at least one fixed point in M.

In utilizing this theorem one takes for M generally the closed sphere which is evidently a convex set. Then there are three things to prove: The continuity of T on M, the transformation of M by T in itself and the compactness of TM. It is chiefly the compactness which gives many difficulties. It can be proved often by use of the theorem of Arzela, but this theorem requires that the domain of the definition of the functions of TM be bounded. If this domain is not bounded, it is possible to use the theorem of HAUSDORF of the existence of the ε-net.

In the following lines I will consider the cases where the interval of the definition of the functions of TM is not bounded and I will show, in using the notion of the quasiconvergence, how to evade the difficulties which can arise. In the first place I shall try to explain the ideas on a concrete Banach space and to prepare everything in order to their application in the dofferential equations of the n-th order.

Let A_{n-1} be the set of all functions which have, on the interval J, the continuous derivatives till the order $n-1$ inclusively.

I must give some definitions.

D$_1$. *Let $f_k(x)$, $k = 1, 2, \ldots$, be the functions of A_{n-1}. We will say that the sequence $\{f_k(x)\}$ converges quasi-uniformly (or shortly q-converges) to the function $f(x)$ on J, if for every $x \in J$ and $i = 0, 1, \ldots, n-1$, $\lim\limits_{k\to\infty} f_k{}^{(i)}(x) = f^{(i)}(x)$. We write $f_k \xrightarrow{q} f$.*

It is evident that every subsequence of a sequence which q-converges to $f(x)$, q-converges to $f(x)$.

D$_2$. *Let S_{n-1} be the Banach space of all functions of A_{n-1}, which have the bounded derivatives till the order $n-1$ inclusively. The norm is given by the formula*

$$\|f(x)\| = \max_{0 \leq i \leq n-1} \{\sup_J |f^{(i)}(x)|\}.$$

It is easily to shown that the convergence in this norm implicates the q-convergence, and that is essential for us.

D$_3$. *The infinite set $M \subset S_{n-1}$ is said to be q-compact in S_{n-1} if every sequence extracted from M contains a subsequence q-convergent to a function of S_{n-1}.*

It is to be noted that the limit of a q-convergence sequence of S_{n-1} ought no to be of S_{n-1}.

D$_4$. *We will say that the set $M \subset S_{n-1}$ is q-closed if the following implication holds: $\{f_k \in M, f_k \xrightarrow{q} f\} \Rightarrow \{f \in M\}$.*

D$_5$. *We will say that the functions of the set $M \subset S_{n-1}$ are uniformly bounded on J by a number K, if $|f^{(i)}(x)| \leq K$ for every $x \in J$, $i = 0, 1, \ldots, n-1$ and for every $f(x) \in M$. We will say that the functions of M are equicontinuous on J if holds: for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such, that for every $f(x) \in M$, for $i = 0, 1, \ldots, n-1$ and for $|x - x'| < \delta(\varepsilon)$ holds: $|f^{(i)}(x) - f^{(i)}(x')| < \varepsilon$.*

It is easy to prove [2]:

Lemma 1. *If the functions of an infinite set $M \subset S_{n-1}$ are uniformly bounded and equicontinuous on J, then M is q-compact in S_{n-1}.*

With the help of the q-convergence we can define the q-continuity of an operator T on S_{n-1} (or on a set $M \subset S_{n-1}$).

D_6. *An operator T on S_{n-1} into S_{n-1} (on M into S_{n-1}) is q-continuous on S_{n-1} (on M) iff the following implication holds:* $\{f_k \xrightarrow{q} f,\ f_k,\ f \in S_{n-1}\} \Rightarrow \Rightarrow \{\|Tf_k - Tf\| \to 0$ *for* $k \to \infty\}$, $(\{f_k \xrightarrow{q} f,\ f_k,\ f \in M\} \Rightarrow \{\|Tf_k - Tf\| \to 0$ *for* $k \to \infty\})$.

The q-continuous operator has the following (for us very important) property):

The q-continuous operator is also continuous.

Lemma 2. *If $M \subset S_{n-1}$ is q-compact in S_{n-1} and if T is q-continuous operator on M into S_{n-1}, then $TM \subset S_{n-1}$ is compact in S_{n-1}.* (See [2]).

From this property follows immediately the first variant of the theorem of SCHAUDER [2].

Theorem 1. *Let T be an operator q-continuous on $M \subset S_{n-1}$, let M be convex, closed and q-compact, and let $TM \subset M$. Then T has at least one fixed point on M.*

The assumption of the theorem of SCHAUDER mentioned above that TM is compact, is substitued here by the assumption that T is q-continuous on M and that M is q-compact. For which follows the following lemmas are very important.

Lemma 3. *Let the functions of $M \subset S_{n-1}$ be uniformly bounded and equicontinuous on J. Then the functions of the convexe hull $\widehat{M}$ of M and also the functions of the closure $\overline{M}$ of M are uniformly bounded and equicontinuous on J* [2].

Lemma 4. *If $M \subset S_{n-1}$ is convex, then the closure $\overline{M}$ of M is also convexe.*

Now we are able to prove the

Theorem 2. *Let T be an operator q-continuous on S_{n-1}. Let $M \subset S_{n-1}$ be a convex and closed set. Let $TM \subset M$ and let the functions of TM be uniformly bounded and equicontinuous on J. Then T has at least one fixed point on M* [2].

The proof of this theorem is based on the fact that, following the lemmas 3 and 4, the closure of the convex hull $\overline{\widehat{TM}} = N$ of TM is a set of the functions which are uniformly bounded and equicontinuous on J. Following the lemma 1 N is q-compact. But this set is convexe and closed and $N \subset M$. From here we have: $TN \subset TM \subset \overline{\widehat{TM}} = N$. The application of the theorem 1 finishes the proof.

For the applications the following theorems are more convenient [2].

Theorem 3. *Let T be an operator on S_{n-1} such that the following implication takes place:*

$$\{f_k, f \in S_{n-1}, f_k \xrightarrow{q} f, \{\|f_k\|\} \text{ bounded}\} \Rightarrow \{\|Tf_k - Tf\| \to 0 \text{ for } k \to \infty\}.$$

Futher, let $M \subset S_{n-1}$ be a convex and bounded set and $TM \subset M$. Let the functions of TM be uniformly bounded and equicontinuous on J. Then T has at least one fixed point in M.

Theorem 4. *Let $M \subset S_{n-1}$ be a convex and bounded set, let T be an operator q-continuous on M such that $TM \subset M$. Let the functions of TM be uniformly bounded and equicontinuous on J. Then T has at least one fixed point in M.*

The proof of those two theorems is not different from the proof of the theorem 2.

We shall now proceed to the application of those theorems on the differential equations.

First of all we need the following lemma.

Lemma 5. *([1] and [2]). Let $Q(x)$ be a function, which is, on the interval (a, ∞), $-\infty \leqq a$, continuous and non-negative in such a way that it is not identically zero on none of the subintervals of the interval (a, ∞). Then the differential equation*

$$\text{(A)} \qquad u^{(n)} + (-1)^{n+1} Q(x)\, u = 0$$

has a solution $u(x)$ having the following properties:

$$(\mathrm{V}_1) \quad \begin{cases} (-1)^k u^{(k)}(x) > 0 \quad \text{or} \quad (-1)^{k+1} u^{(k)}(x) > 0, \qquad k = 0, 1, \ldots, n-1, \\ \lim\limits_{x \to \infty} u^{(k)}(x) = 0, \quad k = 1, 2, \ldots, n-1, \\ \lim\limits_{x \to \infty} u(x) \quad \text{exists and is finite.} \end{cases}$$

It holds yet that $\lim\limits_{x\to\infty} u(x) = 0$ iff $\int^{\infty} x^{n-1} Q(x)\, dx = \infty$. If $\int^{\infty} x^{n-1} Q(x)\, dx < \infty$, there is exactly one solution (excepted the linear dependence) having the properties (V_1). In this case $\lim\limits_{x\to\infty} u(x) \neq 0$, and we will say that $u(x)$ has the properties (V).

Let us now consider the differential equation

$$\text{(B)} \qquad y^{(n)} + (-1)^{n+1} B(x, y, y', \ldots, y^{(n-1)})\, y = 0.$$

Has this equation a solution having the properties (V) when the function B has similar properties as those of $Q(x)$? The following theorem gives an affirmative answer [2].

Theorem 5. *Let be the following conditions fulfilled:*

1. *The function* $B(x, \mathbf{u})$, $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ *(*$\mathbf{u}$ *being a vector with the terms* $u_0, u_1, \ldots, u_{n-1}$*) is in the domain*

$$\Omega : a < x < \infty, \qquad -\infty < u_i < \infty, \qquad i = 0, 1, \ldots, n-1,$$

continuous in $(x, \mathbf{u})$ *and non-negative such that for every point* $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \neq (0, 0, \ldots, 0)$ *the function* $B(x, \mathbf{c})$ *equals identicaly to zero in none of the subintervals of the interval* (a, ∞).

2. $B(x, \mathbf{u})$ *is monoton in every one of his variables* u_i, $i = 0,1, \ldots, n-1$, *for* $u_i \geqq 0$ *as well as for* $u_i < 0$ *(the monotony for* $u_i \geqq 0$ *can be different from that for* $u_i < 0$*).*

3. *For every point* $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ *is*

$$\int^{\infty} x^{n-1} B(x, \mathbf{c})\, \mathrm{d}x < \infty.$$

4. $$\lim_{k\to\infty} \frac{1}{k} \int^{\infty} x^{n-2} B(x, \mathbf{c})\, \mathrm{d}x = 0 \text{ for } |\mathbf{c}| = \sum_{i\,\|\,0}^{n-1} |c_i| \leqq k.$$

Then through every point (x_0, y_0), $x_0 \in (a, \infty)$, $y_0 \neq 0$, *passes at least one solution* $z(x)$ *of the equation* (B) *having the properties* (V) *on the interval of his existence (which is not smaller then* $\langle x_0, \infty)$*).*

We draw a sketch of the proof. Let be $J = \langle x_0, \infty)$, $x_0 > a$. We are looking for the solution $z(x)$ of (B) in the space S_{n-1}. Let be $G_k = \{f(x) \in S_{n-1} | \; ||f(x)|| \leqq \leqq k\}$ (the sphere closed). Then from the monotony of $B(x, \mathbf{u})$ in u_i follows that for every $f(x) \in G_k$
$B(x, f(x), f'(x), \ldots, f^{(n-1)}(x)) = B(x, \mathbf{f}(x)) \leqq B(x, \vartheta_0, \vartheta_1, \ldots, \vartheta_{n-1}) = B(x, \boldsymbol{\theta})$ where ϑ_i means one of the numbers 0, k, $-k$ according to the monotony of B in u_i. $B(x, \boldsymbol{\theta})$ is a majorante integrable.

In view of 3. we have

$$\int^{\infty} x^{n-1} B(x, \mathbf{f}(x))\, \mathrm{d}x \leqq \int^{\infty} x^{n-1} B(x, \boldsymbol{\theta})\, \mathrm{d}x < \infty. \tag{1}$$

Then for the equation

$$y^{(n)} + (-1)^{n+1} B(x, \mathbf{f}(x))\, y = 0 \tag{2}$$

holds the lemme 5. There exists just one solution $u(x)$ of this equation which passes through the point (x_0, y_0) having the properties (V) on J. With the help of this we can define an operator T on G_k in this way: If $f(x) \in G_k$, then $Tf(x) = u(x)$ is the unique solution of (2) having the properties (V) on J and which passes through the point (x_0, y_0). This solution is also a solution of the integral equation

$$u(x) = y_0 - (-1)^{n+1} \int_{x_0}^{\infty} \frac{(x_0 - t)^{n-1}}{(n-1)!} B(t, f(t))\, u(t)\, dt +$$

$$+ (-1)^{n+1} \int_{x}^{\infty} \frac{(x - t)^{n-1}}{(n-1)!} B(t, f(t))\, u(t)\, dt.$$

With the help of 3^0 we can prove that $TG_k \subset S_{n-1}$ and with the help of 4^0 we can prove the existence of such a number k_0 that $TG_{k_0} \subset G_{k_0}$. The sphere G_{k_0} is evidently closed and convexe. It is easy to prove that it is also q-closed. From 3^0, (1) and from the Lebesgue's theorem follows the q-continuity of T on G_{k_0}. Then we prove that the functions of TG_{k_0} are uniformly bounded and equicontinuous on J. The application of the theorem 4 gives the existence of a solution of (B) having the properties (V) on J. Next it can be easily proved that this solution can be extended to an interval (b, ∞), $a \leqq b < x_0$ and this extended solution has the properties (V) on (b, ∞).

Let us now return a little to the equation (A). If we suppose that $\int^{\infty} x^{n-1}Q(x)\, dx < \infty$, then there exists just one solution $u(x)$ of (A) having the properties (V) and such that $\lim_{x\to\infty} u(x)\ m_0 \neq 0$, m_0 beeing a real number choosen arbitrary. On the basis of this we can prove the

Theorem 6. *Let the conditions* 1^0, 2^0 *and* 3^0 *of the theorem* 5 *be fulfilled. Let* m_0 *be an arbitrary real number different from zero. Then there exists at least one solution* $z(x)$ *of* (B) *which has, on the interval of his existence, the propertiets* (V) *and for which* $\lim_{x\to\infty} z(x) = m_0$.

The proof is similar to that of the theorem 5. We define the operator T on the sphere G_k : if $f(x) \in G_k$, then $Tf(x) = u(x)$, where $u(x)$ is the solution of the equation

$$y^{(n)} + (-1)^{n+1} B(x, f(x))\, y = 0$$

having the properties (V) and such that $\lim_{x\to\infty} u(x) = m_0$. This solution is unique and it satisfies also the integral equation

$$u(x) = m_0 + (-1)^{n+1} \int_{x}^{\infty} \frac{(x - t)^{n-1}}{(n-1)!} B(t, f(t))\, u(t)\, dt.$$

The existence of a number $k_0 > m_0$ such that $TG_{k_0} \subset G_{k_0}$ is assured by a convinient choice of x_0. The rest of the proof is nearly the same as in the proof of the theorem 5.

Also the proof of the following theorem is analogous [2].

Theorem 7. 1° *Let $P(x, \mathbf{u})$ be a function defined and continuous on Ω and non-negative in such a way that for every point $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \neq (0, 0, \ldots, 0)$ the function $P(x, \mathbf{c})$ is identically zero on none of the subintervals of the interval (a, ∞).*

2° *Let be $P(x, \mathbf{u}) \leqq B(x, \mathbf{u})$ for every point $(x, \mathbf{u}) \in \Omega$ and let the function $B(x, \mathbf{u})$ fulfil all the conditions of the theorem 5 respectively 6.*

Then for the equation

$$\text{(P)} \qquad y^{(n)} + (-1)^{n+1} P(x, y, y', \ldots, y^{(n-1)})\, y = 0$$

hold all the statements of the theorem 5 respectively 6.

Let us return now to the theorems 1—4 which have established in the case of space S_{n-1}. But we can prove the validity of those theorems also in the case of other Banach spaces then S_{n-1} [2]:

Let $X \subset A_{n-1}$ be a Banach space with the norm $\| \ \|_X$ such that the convergence according to this norm implies also the q-convergence. Then the theorem 1 holds if we substitute S_{n-1} by X. If the q-compactness of the set $M \subset X$ follows from the properties that the functions of M are uniformly bounded and equicontinuous, the theorem 2 and 4 hold for X. If yet from the fact that the functions of the set M are uniformly bounded and equicontinuous follows that they are also bounded in the sense of the norm $\| \ \|_X$, the theorem 3 holds if we substitute S_{n-1} by X.

We are giving now some examples in which the above exposed ideas find their application.

Theorem 8. [7] *Let $B(x, \mathbf{u})$, $F(x, \mathbf{u})$, $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$, $n \geqq 1$, be the functions non-decreasing in every of his variables u_i, $i = 0, 1, \ldots, n-1$ and such that*

$$\text{(3)} \qquad |B(x, \mathbf{u})| \leqq F(x, \mathbf{u}) \quad \text{on} \quad \Omega.$$

Let K be a positive number, $x_0 > a$ and $0 \leqq k \leqq n-1$ an integer. Let

$$\text{(4)} \qquad \varphi(x) = K \sum_{s=0}^{k} \frac{(x - x_0)^s}{s\,!},$$

$$\text{(5)} \qquad \int^{\infty} x^{n-k-1} F(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(k)}(x), K, K, \ldots, K)\, dx < \infty$$

for every $K < 0$,

$$\text{(6)} \qquad \lim_{K \to \infty} \frac{1}{K} \int^{\infty} (x - x_0 + 1)^{n-k-1} F(x, \varphi(x), \ldots, \varphi^{(k)}(x), K, \ldots, K)\, dx = 0$$

Let be finaly $c_0, c_1, \ldots, c_k$ real arbitrary numbers. Then the differential equation

$$\text{(E)} \qquad y^{(n)} + B(x, y, y', \ldots, y^{(n-1)}) = 0$$

has at least one solution defined on $J = \langle x_0, \infty)$ and satisfaying the conditions

$$(7)\qquad y^{(i)}(x_0) = c_i, \qquad i = 0, 1, \ldots, k-1$$

$$\lim_{x\to\infty} y^{(k)}(x) = c_k,$$

$$\lim_{x\to\infty} y^{(i)}(x) = 0, \qquad i = k+1, \ldots, n-1.$$

I am going to scatch the proof. By a simple calculus we can see that the solution of the integral equation

$$(8)\quad y(x) = \sum_{s=0}^{k} c_s \frac{(x-x_0)^s}{s\,!} - \sum_{s=0}^{k-1} \frac{(x-x_0)^s}{s\,!} \int_{x_0}^{x} \frac{(x_0-t)^{n-s-1}}{(n-s-1)\,!} B(t, \mathbf{y}(t))\,dt + $$

$$+ \sum_{s=k}^{n-1} \frac{(x-x_0)^s}{s\,!} \int_{x}^{\infty} \frac{(x_0-t)^{n-s-1}}{(n-s-1)\,!} B(t, \mathbf{y}(t))\,dt$$

is also the solution of the equation (E) and fulfils the conditions (7).

We are looking for this solution in the Banach space $C_{n-1,k} \subset A_{n-1}$ of all functions which have the bounded derivatives of the orders k, $k+1$, ..., $n-1$ on the interval $J = \langle x_0, \infty)$. Let the norm in $C_{n-1,k}$ be

$$||f(x)|| = \max_{k\le i\le n-1} \{\sup_J |f^{(i)}(x)|\} + \sum_{i=0}^{k-1} |f^{(i)}(x_0).$$

It can be easily shown that the convergence according this norm implies the q-convergence.

Let be $G_K = \{f(x) \in C_{n-1,k}|\ ||f(x)|| \le K\}$. Then for every $f(x) \in G_K$ holds

$$|f^{(i)}(x)| \le \varphi^{(i)}(x), \qquad i = 0,1, \ldots, k,$$
$$|f^{(i)}(x)| \le K, \qquad i = k+1, \ldots, n-1.$$

If we respect (3), (5) and the monotony of $F(x, \mathbf{u})$ we have

$$(9)\qquad |B(x, \mathbf{f}(x))| \le F(x, \boldsymbol{\varphi}(x), \mathbf{K}),$$

where $F(x, \boldsymbol{\varphi}(x), \mathbf{K}) = F(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(k)}(x), K, \ldots, K))$ and

$$(10)\quad \left| \int_{x}^{\infty} \frac{(x_0-t)^{n-s-1}}{(n-s-1)\,!} B(t, \mathbf{f}(t))\,dt \right| \le \int_{x}^{\infty} \frac{(x_0-t)^{n-s-1}}{(n-s-1)\,!} F(t, \boldsymbol{\varphi}(t), \mathbf{K})\,dt < \infty$$

$$s = k, k+1, \ldots, n-1.$$

This allows us to define the operator T on G_K by the formula

$$(11)\qquad Tf(x) = v(x) = \sum_{s=0}^{k} c_s \frac{(x-x_0)^s}{s\,!} -$$

$$-\sum_{s=0}^{k-1}\frac{(x-x_0)^s}{s\,!}\int_{x_0}^{x}\frac{(x_0-t)^{n-s-1}}{(n-s-1)\,!}\,B(t,\mathbf{f}(t))\,\mathrm{d}t+$$
$$+\sum_{s=k}^{n-1}\frac{(x-x_0)^s}{s\,!}\int_{x}^{\infty}\frac{(x_0-t)^{n-s-1}}{(n-s-1)\,!}\,B(t,\mathbf{f}(t))\,\mathrm{d}t.$$

We can see immediately that $TG_K \subset C_{n-1,k}$. From the condition (6) follows the existence a certain number K_0 such that $TG_{K_0} \subset G_{K_0}$. The q-continuity of T on G_{K_0} follows from the conditions (3), (5), from the monotony of $F(x, \mathbf{u})$ and from the theorem of Lebesgue.

Note $TG_{K_0} = H$. Let $H^{(i)}$, $i = 0,1, \ldots, n-1$ be the set of the derivatives of the order i of all the functions of H. It can be proved that the functions of $H^{(i)}$, $i = k, k+1, \ldots, n-1$ are uniformly bounded and equicontinuous on J. Le us make the closure of the convexe hull $\hat{\hat{H}} = M$ of H. We can prove that the functions of $M^{(i)}$, $i = k, k+1, \ldots, n-1$ are uniformly bounded and equicontinuous on J. From this we can prove that M is q-compact. Then we now that M is convexe, closed, q-compact and $M \subset G_{K_0}$. From this last relation we obtain that $TM \subset TG_{K_0} = H \subset \hat{\hat{H}} = M$. The application of the theorem 1 finishes the proof.

If we take for $F(x, \mathbf{u})$ a linear expression,

$$F(x,\mathbf{u}) = a(x) + \sum_{i=0}^{n-1} a_{n-i}(x)\,|u_i|,$$

we obtain from the theorem 8 the

Theorem 9. *Let be* $|B(x,\mathbf{u})| \leqq a(x) + \sum_{i=0}^{n-1} a_{n-i}(x)\,|u_i| = F(x,\mathbf{u})$ *for every* $(x,\mathbf{u}) \in \Omega$ *and let be* $a(x) \geqq 0$, $a_{n-i}(x) \geqq 0$,

$\int^{\infty} x^{n-k-1}a(x)\,\mathrm{d}x < \infty$, $\int^{\infty} x^{n-k-1}a_{n-i}(x)\,\mathrm{d}x < \infty$, $i = k+1, k+2, \ldots, n-1$,

$\int^{\infty} x^{n-i-1}a_{n-i}(x)\,\mathrm{d}x < \infty$, $\quad i = 0,1, \ldots, k.$

Then for x_0 sufficiently large the affirmations of the theorem 8 hold.

The theorem 8 gives the results which are a generalization of the results of M. P. Waltman [4] for the equation $y^{(n)} + f(x, y) = 0$. He proves, by a different way, the existence of a solution $y(x)$ for which $\lim_{x\to\infty} y(x)/x^{n-1} = \beta \neq 0$ under the conditions: $|f(x, y)| \leqq a(x)\,y^{\alpha}$, $\alpha > 0$ and $\int^{\infty} x^{\alpha(n-1)}a(x)\,\mathrm{d}x < \infty$.

By the same method as above we can prove the following theorems (in the space $C_{n-1,n-1}$): [3].

Theorem 10. *Let $B(x, \mathbf{u})$, $F(x, \mathbf{u})$ be the continuous functions in the domain Ω. Let $F(x, \mathbf{u})$ be non-decreasing in every of the variables u_i, $i = 0,1, \ldots, n-1$. Let $K > 0$, $x_0 > a$ and*

$$\varphi(x) = K \sum_{s=0}^{n-1} \frac{1}{s!} (x - x_0)^s ,$$

$$\int^{\infty} x^{n-1} F(x, \boldsymbol{\varphi}(x)) \, dx < \infty \qquad \text{for every } K > 0, \tag{12}$$

$$\lim_{K\to\infty} \frac{1}{K} \int^{\infty} x^{n-1} F(x, \boldsymbol{\varphi}(x)) \, dx = 0, \tag{13}$$

$$|B(x, \mathbf{u})| \leqq F(x, \mathbf{u}) \qquad \text{for every } (x, u) \in \Omega. \tag{14}$$

Finaly, let $c_0, c_1, \ldots, c_{n-1}$ be arbitrary real numbers.

Then the equation (E) has at least one solution $u(x)$, which exists on $J = \langle x_0, \infty)$ and for which hold the formulae:

$$u^{(i)}(x) = \sum_{s=i}^{n-1} c_s \frac{(x - x_0)^{s-i}}{(s-1)!} + o(1), \qquad i = 0,1, \ldots, n-1. \tag{15}$$

Note. If we substitute the condition (12) by

$$\int^{\infty} x^{n-1+\varepsilon} F(x, \boldsymbol{\varphi}(x)) \, dx < \infty, \qquad \varepsilon > 0,$$

then in the formulae (15) it is possible to substitute $o(1)$ by $o(x^{-\varepsilon})$. And if we take for $F(x, u) = a(x) + \sum_{i=0}^{n-1} a_{n-i}(x) \, |u_i|$, theorem 10 gives a generalization of the results of M. Zlámal [5] found for the linear differential equations.

Theorem 11. [6] *Let fulfil all the conditions of the theorem 10 with the exception of the conditions (12) and (13), which will be substitued by*

$$\int^{\infty} F(x, \boldsymbol{\varphi}(x)) dx < \infty \qquad \text{for every } K > 0, \tag{12'}$$

$$\lim_{K\to\infty} \frac{1}{K} \int^{\infty} F(x, \boldsymbol{\varphi}(x)) \, dx = 0. \tag{13'}$$

Then by the initial conditions $(x_0; c_0, c_1, \ldots, c_{n-1})$, where c_i are real arbitrary numbers, is determined at least one solution $u(x)$ of (E) which exists on $J = \langle x_0, \infty)$.

If morover the condition (12) is satisfied, then for this solution $u(x)$ hold the formulae:

$$u^{(i)}(x) = \sum_{s=i}^{n-1} c_s \frac{(x - x_0)^{s-i}}{(s - i)\,!} + O(1), \qquad i = 0, 1, \ldots, n - 1. \tag{16}$$

Those are some exemples where *I* profit with success of the variants of the theorem of SCHAUDER mentioned above.

I wish yet to remark that one can utilise this method in many other cases, chiefly in the cases where the theorem of Arzela has been applied and that's why it has been limited on a finite interval. In the first place that are the problems of the global existence, the linear problems and the boundary-value problems.

The notions and the theorems of which *I* spoke, have been prepared in to their application to the problems of the differential equations of n-th order. There is no difficulty to adapt those for the systems of differential equations.

REFERENCES

[1] ŠVEC M.: *Monotone solutions of some differential equations. Colloqium mathematicum, Wroclaw XVIII, 1967, 7–21.*

[2] ŠVEC M.: *Fixpuntsatz und monotone Lösungen der Differentialgleichung* $y^{(n)} + B(x, y, y', \ldots, y^{(n-1)})\, y = 0$. *Archivum mathematicum (Brno), T. 2 (1966), 43–55.*

[3] ŠVEC M.: *Les propriétés asmptotiques des solutions d'une équation différentielle non-linéaire d'ordre n. Czechoslov. M. J. 17 (92), 1967, 550–557.*

[4] WALTMAN P.: *Asymptotic Behavior of solutions of an n-th order differential equation. Monath. Math., Osterr. (1965), 69, n°. 5, 427–430.*

[5] ZLÁMAL M.: *Asymptotische Eigenschaften der Lösungen linearer Differentialgleichungen. Math. Nachrichten, 10. Band, Heft 3/4, 1953.*

[6] ŠVEC M.: *L'existence globale et les propriétés asmptotiques des solutions d'une équation différentielle non-linéaire d'ordre n. Archivum mathematicum (Brno) T. 2 (1966), 141–151.*

[7] ŠVEC M.: *Sur un problème aux limites. Czech. mat. j. (to appear in 1969).*

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ARZELA — LIKE THEOREM WITH APPLICATIONS TO DIFFERENTIAL EQUATIONS AND CONTROL THEORY

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§ 1. In numereous problems concerning given differential equations or control systems the sequences of functions

$$y = g_i(x), \qquad (i = 1, 2, \ldots) \tag{1.1}$$

approximating solutions of respective equations for instance one equation

$$y' = f(x, y) \tag{1.2}$$

play an important role.

Usually one starts with equation (1.2) and then determine suitable sequences (1.1).

Our paper deals with the following opposite:

Problem Z. *Given the sequence* (1.1), *such an equation* (1.2) *called "asymptotically inducted by sequence* (1.1)" *is to be found, that the limits of all convergent subsequences* $y = k_i(x)$ *of* (1.1) *satisfy* (1.2). It is evident that the required function $f(x, y)$ can be determined exclusively on the accumulation set L for sequence of curves (1.1). It can happen that L reduces to a single arc.

Problem Z is considered globally.

We shall give conditions for subsequences k_i to converge towards an "*extensive solution*" of (1.1) i.e. solution tending to the boundary of open set W (containing L) at both ends.

Such conditions are given in Theorem A which proved to be very convenient for didactic purposes because of many applications (for instance global existence theorems, continuous dependence on and differentiability in respect to initial values, constructing of approximate solutions).

The curves (1.1) occuring in Theorem A are arcs.

In some applications of the classical Arzela's theorem on equicontinuous sequences (1.1) is of no use because one is obliged to use functions (1.1) with graphs consisting of finite number of points. In this case we replace the notion of equicontinuity by notion of asymptotic smoothness and extensivity (Theorem B).

Theorem C gives a construction of a control system "asymptotically induced" by the sequence (1.1).

§ 2. Theorem A. *Let $W \in R^2$ be an open non-empty set. Consider a sequence of real functions $g_i(x)$ defined, continuous and having the right-hand (finite) derivative D_+g_i on open intervals J_i, $(i = 1,2, \dots)$.*

Denote by g_i the graph of $g_i(x)$ and put

$$(2.1) \qquad L = \{(x, y) : (x, y) \in W,\ \liminf_{i\to\infty} r((x, y), g_i) = 0\}$$

where $r((x, y), B)$ denotes the distance of point (x, y) from set B.

Suppose the following implication:

if for any subsequence $g_{k(i)}$

$$(2.2) \qquad (x_i, g_{k(i)}(x_i)) \to (x, y) \in W,$$

then two following conditions (2.3), (2.4) *hold:*

(2.3) *there exist a neighbourhood of x contained for large i in $J_{k(i)}$,*

(2.4) $\lim D_+ g_{k(i)}(x_i)$ *exists and is finite.*

Under above assumptions:

the limit (2.4) *depends on (x, y) only and is independent of the particular choice of the subsequence $g_{k(i)}$.*

This limit denoted by $F(x, y)$ is defined and continuous on L [see (2.1)*].*

(2.5) *For any point $(x_0, y_0) \in L$ there exists such a subsequence $g_{m(i)}$ that* $\lim g_{m(i)}(x) = h(x)$,

where $h(x)$ is an "extensive" solution of the equation

$$(2.6) \qquad y' = F(x, y),$$

i.e. *$h(x)$ is an open arc "tending" at both ends to the boundary of W.*

Moreover if through each point of L passes a unique solution of (2.6) *and*

$$\lim r((x_0, y_0), g_i) = 0$$

then the original sequence g_i is convergent to the extensive solution passing through (x_0, y_0).

§ 3. Remark 1. **The sequence g_i satisfying the implication (2.2) ⇒ (2.3) will be called expansive on W.**

The differential equation (2.6) will be called to be "induced by sequence g_i".

This notions will be generalized in the following.

Remark 2. The proof of this theorem can be based on the classical Arzela's Lemma and on the theorem on differentiability of the limit.

Theorem A can serve as the starting point for generalizations. For this purpose we introduce some definitions.

§ 4. Limits restreint, complete and exact in Hausdorff-like sense.

Let $A = \{A_i\}$ be a sequence of sets $A_i \subset P = R^2$. We define

(4.1) $\zeta(A) = \lim \text{restr}\, A_i = \{z\colon z \in P, \lim r(z, A_i) = 0\}$,

(4.2) $\eta(A) = \lim \text{compl}\, A_i = \{z\colon z \in P, \lim\inf r(z, A_i) = 0\}$.

If $\zeta(A) = \eta(A)$ we say that A is *H-convergent* and we define

(4.3) $\lambda(A) = \lim \text{exact}\, A_i = \zeta(A) = \eta(A)$.

(4.4) Proposition. For $A_i \neq 0$ there exists H-convergent subsequence of $\{A_i\}$.

§ 5. Univalent sets and smooth functions.

For $B \subset P$ we define by

B^{I} = projection of B on x-axis,

B^{II} = projection of B on y-axis.

We say that set B is *univalent* if

$$p \in B, \qquad q \in B, \qquad p^{\mathrm{I}} = q^{\mathrm{I}} \Rightarrow p = q.$$

The univalent sets will be considered as functions of variable x. The whole of such functions (or sets) will be denoted by Unival.

Let $f \in$ Unival and put $g = f \cap W$.

f is called *extensive* if g is continuous, g^{I} is open and $(x, g(x))$ tends to the boundary of W, as x tends to the boundary of g^{I}.

f is called smooth if g is continuous and closed in W.

§ 6. Suppose that

1) $g = \{g_i\}$, $g_i \in$ Unival,

2) (x_0, y_0) is an arbitrary point of W,

3) $k = \{k_i\}$ is an arbitrary H-convergent subsequence of g for which $(x_0, y_0) \in \lambda(k)$.

(6.1) g is called *asymptotically smooth in W* if
$[x_i \in k_i^{\mathrm{I}},\ x_i \to x_0] \Rightarrow [k_i(x_i) \to y_0]$.

(6.2) g is called *asymptotically extensive in W* if x_0 is an interior point of $[\eta(k)]^{\mathrm{I}}$.

(6.3) g is called *asymptotically lipschitzian in* W if
$[x_i \in k_i^{\mathrm{I}},\ s_i \in k_i^{\mathrm{I}},\ x_i \to x_0,\ s_i \to x_0,\ k_i(x_i) \to y_0] \Rightarrow$
$\Rightarrow [\limsup |(k_i(s_i) - k_i(x_i))/(s_i - x_i)| < +\infty]$.

(6.4) Obviously if g is asymptotically lipschitzian it is asymptotically smooth.

Theorem B. *If* $g_i \in$ **Unival**, $g = \{g_i\}$ *is asymptotically smooth and asymptotically extensive (in* W*),* $(x_0, y_0) \in \eta(g)$ *then there exists such an* H*-convergent subsequence* k *of* g *that* $(x_0, y_0) \in \lambda(k)$ *and* $\lambda(k)$ *is univalent, extensive and smooth in* W.

Remark 3. If moreover g is asymptotically lipschitzian then $\lambda(k)$ is locally lipschitzian.

Remark 4. The theorem B generalizes the classical lemma of Arzela on equicontinuous sequences of functions. It is used in the following.

§ 7. For $G \subset P$ we define
Convex G = closed convex hull of G.
For $a \in P$, $b \in P$, $a^{\mathrm{I}} \neq b^{\mathrm{I}}$ we put

$$\text{slope}\,(a, b) = (b^{\mathrm{II}} - a^{\mathrm{II}})/(b^{\mathrm{I}} - a^{\mathrm{I}}).$$

If $f \in$ **Unival,** Q is open, $Q \subset P$, we define

$$\text{slope}\,(f, Q) = \bigcup_{a,b} \text{slope}\,(a, b), \text{ for } a \in f \cap Q,\ b \in f \cap Q.$$

For $g = \{g_i\}$, where $g_i \in$ Unival, g asymptotically lipschitzian in W, $\varnothing \neq Q$ open, we put

$$B(i, Q) = \bigcup_{j/i}^{\infty} \text{slope}\,(g_j, Q).$$

For $(x, y) \in \zeta(g) \cap W$ we define

(7.1) $$C(x, y) = \textbf{Convex}\,[\lim_{i,Q \to (\infty,x,y)} \text{exact}\, B(i, Q)].$$

The relation (7.1) means that for any such sequence of open non-empty sets Q_i that $(x, y) \in Q_i$, diameter $Q_i \to 0$, we have:

$$C(x, y) = \text{Convex}\,[\lim_{i \to \infty} \text{exact}\, B(i, Q_i)].$$

(7.2) Definition. *The contingent condition*

(7.3) $$D^* y(x) \in C(x, y(x)),$$

where D^* *denotes the contingent derivative is called contingent equation asymptotically induced by sequence* g.

Theorem C. *Suppose that* g *is asymptotically lipschitzian and asymptotically extensive in* W.

Then $C(x, y)$ is upper semicontinuous (in respect to inclusion) in $\eta(g) \cap W$.

If $(x_0, y_0) \in \eta(g) \cap W$ then there exists such a H-convergent subsequence k of g, that $\lambda(k) \cap W$ is an extensive solution of contingent equation (7.3) *passing through* (x_0, y_0).

(7.4) Remark 5. In our case the classical assumption of Zaremba—Marchaud theory on contingent and paratingent equations are satisfied.

(7.5) Remark 6. The condition (7.3) can be considered as a control system with eliminated control variables.

§ 8. Remark 7. Theorem B of § 6 can be easily reformulated for the case $P = R^m \times R^n$, where m, n are arbitrary positive integers. It can be even generalized for the case $P = H \times V$, H and V being suitable topological spaces.

COMPREHENSIVE LECTURES PRESENTED IN SECTIONS

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XVII — 1967

1. Ordinary Differential Equations

AXIOMATIZATION OF DIFFERENTIAL EQUATION THEORY

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This lecture is an attempt to motivate, describe and justify an axiomatic treatment of several basic portions of differential equation theory, or more precisely, of the initial value problem for ordinary differential equations.

1. It seems that a situation in mathematics is judged ripe for axiomatization (non-categorial, i.e. possessing non-isomorphic realizations) if, in loose terms, there is a number of independent subjects which exhibit common or similar properties; and second, if it is also recognized, explicitly or not, that significant portions of the development of these subjects stem from these common properties rather than from the individual specific nature of the subjects themselves.

I claim that such a situation has evolved in connection with differential equations. The basic subject there is the theory of ordinary differential equations in the classical sense,

$$\frac{\mathrm{d}x}{\mathrm{d}\vartheta} = f(x, \vartheta) \qquad \text{with } x \in R^n,\ \vartheta \in R^1, \tag{1}$$

and with $f : R^{n+1} \to R^n$ continuous. However, one frequently meets with similar equations in which the right-hand term f exhibits various types of discontinuity (e.g. a discontinuous forcing term or feed-back or coefficients); and also with the less closely related concepts of difference- and functional-differential equations, differential inequalities and equations in contingents. Next, significant generalizations are obtained by relaxing the requirement on the euclidean structure of the phase space in which the equations are to act; e.g., on replacing euclidean n-space R^n in (1) by a differential manifold (cf. differential equations on the torus, etc.), or even by various abstract spaces familiar from functional analysis (cf. ordinary differential equations in function spaces, to treat some partial differential equations). As slightly less important members of this family, one may mention the implicit differential equations, some integro-differential equations, and the finite difference equations.

Separately, each of these theories is, of course, perfectly adequate to its own professed main problem; however, they are intimately but informally related, using a similar terminology and arsenal of primitive notions. Thus in each case, a fundamental concept is that of an appropriately defined solution to an initial value problem; and in each case it is felt necessary to carry out, to some extent at least, a programme of development on the lines of classical ordinary differential equation theory in R^n. As a trivial example, in the case of difference-differential equations one is not surprised at finding an existence theorem proved via the Banach contraction mapping theorem; indeed, rather the opposite situation would be surprising.

To proceed one step further, I believe that hypothetical further theories would be held to belong to differential equation theory only if they conform in a similar sense as do those listed above, i.e. if they exhibit a reasonable recurrence of the fundamental properties and results. To express this even more strongly, I wish to suggest that most differential equationists actually possess an informal — and possibly unrecognized — concept of a general theory of differential equations, of which the theories mentioned previously are special cases.

The advantages to be gained from an axiomatic approach are then exactly those which apply to the axiomatic treatment of any informal theory: generality, perspicuity and economy of results and methods, and, as a secondary effect, in a number of cases even significant simplification or extension.

All this is, in my opinion, sufficient motive to attempt the explicit formulation of a general theory.

2. The first task then is to select a suitable general concept, capable of representing all the objects studied in differential equation theory; the term chosen was that of a process, [3]. As is often the case, this concept was not arrived at in a single stage, but represents the final step of what now appear to be partial axiomatizations of the notion of a differential equation. These include the dynamical systems (A. A. Markov, **1931**; [4, chap. *V*]), the flows (origin unknown), the "general systems" of Zubov (**1957**; [6, chap. *IV*]), and the local dynamical systems (Hájek, **1964**; [1]). These correspond to, or rather generalize, differential equations under various combinations of requirements on autonomness, unicity and prolongability of solutions; and in this sense, the processes correspond to differential equations, without any extraneous assumptions.

To introduce the concept of a process, first consider the basic model, viz. a classical ordinary differential equation (1). Explicitly, the assumptions are that f is a continuous partial map $R^{n+1} \to R^n$ with $D =$ **domain** f open in R^{n+1}; and the solutions of (1) are defined as those partial maps $s : R^1 \to R^n$ with **domain** s an interval, which satisfy (1) in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta} s(\vartheta) = f(s(\vartheta), \vartheta) \qquad \text{for all} \qquad \vartheta \in \textbf{domain}\ s.$$

Of course, all this is easily carried over to differential equation on differentiable n-manifolds. With this differential equation we shall associate a process p. This is the relation in R^{n+1} determined as follows: $(x, \alpha) \in R^n \times R^1$ is to be in the relation p to an $(y, \beta) \in R^n \times R^1$, and this is written as $(x, \alpha)\, p(y, \beta)$, if and only if $\alpha \geq \beta$ and there exists a solution s of (1) with

$$x = s(\alpha), \qquad y = s(\beta)$$

(this includes the requirement that the interval domain of s contains both α, β).

It can be shown rather easily that the relation p describes the originally given équation (1) completely. This established a general method of assigning a process — to be called a differential process — to a differential equation. Similarly, there is a canonic method of assigning processes to discontinuous differential equations, to functional-differential equations, etc. (two further cases are discussed below). The processes obtained in this manner are all special cases of a single general concept which will now be described explicitly.

It will be said that p is a *process* in P over R iff P is a set (the phase space), R is a subset of R^1 (the set of admissible time instants), and p is a relation in $P \times R$ with the following three properties:

0^0 If $(x, \alpha)\, p\, (y, \beta)$ then $\alpha \geq \beta$.

1^0 If $(x, \alpha)\, p\, (y, \beta)$ and $\alpha = \beta$ then also $x = y$ (the inital value property).

2^0 p is a transitive relation, i.e.

(2) $$(x, \alpha)\, p\, (y, \beta) \qquad \text{and} \qquad (y, \beta)\, p\, (z, \gamma)$$

imply $(x, \alpha)\, p\, (z, \gamma)$; also, in partial converse, whenever $(x, \alpha)\, p(z, \gamma)$ and $\alpha \geq \beta \geq \gamma$ in R, there exists an $y \in P$ with (2) (the compositivity property).

Occasionally a minor modification of this notation is more useful. Given objects p, P, R with property 0^0 as above, for each $\alpha \geq \beta$ in R define a relation ${}_\alpha p_\beta$ on P by letting

(3) $$x\, {}_\alpha p_\beta\, y \qquad \text{iff} \qquad (x, \alpha)\, p(y, \beta).$$

Evidently p is completely determined by the indexed system of relations $\{{}_\alpha p_\beta \mid \alpha \geq \beta \text{ in } R\}$. Then 1^0 and 2^0 may be formulated more concisely:

1^0 ${}_\alpha p_\beta \subset I$ (the identity relation on P) for all $\alpha \in R$.

2^0 ${}_\alpha p_\beta \circ {}_\beta p_\gamma = {}_\alpha p_\gamma$ for all $\alpha \geq \beta \geq \gamma$ in R.

Both these descriptions, using p and the ${}_{\alpha}p_{\beta}$, will be used, always invoking definition (3) automatically.

Returning to the (differential) processes associated with differential equations as described above, it is easily seen that 0^0 and 1^0 are satisfied automatically; and 2^0 follows from obvious properties of solutions of (1), namely from the fact that any interval-partialization of a solution is again a solution, and that the concatenation of (concatenable) solutions is a solution. Thus p is a process in R^n over R^1.

As a less immediate interpretation, consider a difference-differential equation with constant time lag

$$\frac{\mathrm{d}x}{\mathrm{d}\vartheta} = f(x(\vartheta - \tau), x(\vartheta), \vartheta), \tag{4}$$

given continuous $f : R^3 \to R^1$ and $\tau > 0$. For definiteness, the solutions of (4) are continuous maps $s : [\beta - \tau, \alpha] \to R^1$ for given $-\infty \leq \beta \leq \alpha \leq +\infty$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta} s(\vartheta) = f(s(\vartheta - \tau), s(\vartheta), \vartheta) \qquad \text{for} \qquad \beta < \vartheta \leq \alpha$$

(with obvious modifications for the case of non-closed domains). It will be convenient to write x_λ for the λ-translate of a partial map $x : R^1 \to R^1$, so that $x_\lambda(\vartheta) = x(\vartheta + \lambda)$ whenever defined. The inital value problem for (4) is to find, to given $\beta \in R^1$ and continuous $y : [-\tau, 0] \to R^1$, a solution s of (4) as above, and satisfying $y \subset s_\beta$, i. e. such that $s(\vartheta) = y(\vartheta - \beta)$ for $\beta - \tau \leq \vartheta \leq \beta$. This situation may be usefully described by a process p in the function space $C^1[-\tau, 0]$ over R^1: For x, y in $C^1[-\tau, 0]$ and $\alpha \geq \beta$ in R^1 let $(x, \alpha)\, p(y, \beta)$ iff $x \subset s_\alpha$, $y \subset s_\beta$ for some solution s of (4). Again it is easily verified that this relation p satisfies axioms 0^0 to 2^0 and hence defines a process $C^1[-\tau, 0]$ over R^1; and that this process characterizes the original equation completely. Very similar constructions may be carried out more generally for functional-differential equations; not necessarily of retarded type, in n-space.

The final example concerns a one-dimensional partial differential equation

$$\frac{\partial u}{\partial \vartheta} = f\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial^2 u}{\partial \xi^2}, \xi, \vartheta\right) \tag{5}$$

with continuous $f : R^5 \to R^1$; consider the corresponding homogeneous boundary value problem in the strip $\{(\xi, \vartheta) \in R^2 : |\xi| \leq 1, \vartheta \geq 0\}$. The associated process p will act in the set P of all continuous functions on $[-1, 1]$ with zero end values. For x, $y \in P$ and $\alpha \geq \beta$ in R^1 one defines that $(x, \alpha)\, p(y, \beta)$ iff

$\beta \geq 0$ and there exists a solution u of (5) with zero boundary values and such that

$$u(\xi, \alpha) = x(\xi), \qquad u(\xi, \beta) = y(\xi) \qquad \text{for} \qquad |\xi| \leq 1.$$

Again, a similar construction may be carried out for higher orders, for more complicated domains and boundary conditions, and for systems of such equations.

3. It is now appropriate to show how several fundamental concepts may be carried over from differential equations to processes. Thus, assume given a process p in P over R. (In the envisaged applications, the set R of admissible time instants is either the real axis R^1, or the set C^1 of integers for processes with discrete time; the present formulation was designed to cover both situations.) A *solution* of p is defined as any partial map $s : R \to P$ with domain s an interval in R and such that $(s(\alpha), \alpha)\, p(s(\beta), \beta)$ for all $\alpha \geq \beta$ in domain s. For differential processes these are precisely the solutions of the equation in the usual sense.

The set of all pairs $(x, \alpha) \in P \times R$ such that $(x, \alpha)\, p(x, \alpha)$ will be denoted by D and termed the *domain* of p. Directly from the axioms, $(x, \alpha)\, p(y, \beta)$ implies that both (x, α), (y, β) are in D; thus essentially p concerns only the elements of $D \subset P \times R$. For the differential process associated with (1) this set D coincides with **domain** f.

The process p is said to have *unicity* iff $u_\vartheta p_\alpha x$ and $u'_\vartheta p_\alpha x$ always imply $u' = u$. The process p is termed global or said to have global existence (or indefinite prolongability) iff to any $(x, \alpha) \in D$ and $\vartheta \geq \alpha$ in R there exists an $u \in P$ with $u_\vartheta p_\alpha x$. Slightly more generally, to any $(x, \alpha) \in D$ one may assign a numerical characteristic $\varepsilon(x, \alpha)$, the *extent of existence* of p at (x, α), defined as

$$\varepsilon(x, \alpha) = \sup \{\vartheta \in R : u_\vartheta p_\alpha x \qquad \text{for some} \qquad u \in P\}.$$

Easily, $\alpha \leq \varepsilon(x, \alpha) \leq +\infty$. If $\alpha < \varepsilon(x, \alpha)$ one says that *local existence* obtains at (x, α) and in the opposite case (x, α) is called an *end-pair*. If $\varepsilon(x, \alpha) = +\infty$ one says that global existence obtains at (x, α), and in the opposite case (x, α) is said to have *finite escape time*.

The process p will be termed *stationary* (or autonomous) iff R is an additive subgroup of R^1 and, for all $\alpha \geq \beta$ and ϑ in R, ${}_\alpha p_\beta = {}_{\alpha+\vartheta}p_{\beta+\vartheta}$. In this case a point $x \in P$ is called *critical* iff $x_\vartheta p_\alpha x$ for all $\vartheta \geq \alpha$ in R. In the obvious manner one may define *cycles* with given primitive period, invariant sets, etc.

A real-valued function λ on $P \times R$ is called a Liapunov function for p if $(x, \alpha)\, p(y, \beta)$ implies $0 \leq \lambda(x, \alpha) \leq \lambda(y, \beta)$. (This definition can be generalized extensively.)

For differential processes, all these concepts assume their classical meaning;

thus they are the corresponding generalizations. Having determined the appropriate formulations of these concepts in the general situation, one may apply them automatically in the various special cases. Thus one has, e. g. the concept of critical points for stationary difference-differential and functional-differential equations. As a matter of fact, in the former case these had already been introduced, and agree with the present; to my knowledge, in the latter case these have not been studied.

There is one exception to this rule, concerning the concept of solutions. Thus a solution of the partial differential equation (5) in the customary sense is a real-valued function u of two real variables ξ, ϑ; and a solution of the associated process is a function-valued map s with the variable ϑ. However, one has an obvious one-to-one correspondence determined by

$$u(\xi, \vartheta) = (s(\vartheta))(\xi).$$

In the case of the difference-differential equation (4) the divergence is even more marked, but once again there is a one-to-one correspondence between the corresponding solutions.

This illustrates the assertion that the fundamental concepts from differential equation theory find adequate and natural generalizations within process theory. As concerns the methods, I have space only for an elementary example. It is well known that every differential equation (1) in R^n may be "made stationary" by passing to a different equation in R^{n+1}, namely the system

$$\text{(6)} \qquad \frac{dx}{d\vartheta} = f(x, \lambda), \qquad \frac{d\lambda}{d\vartheta} = 1.$$

The relation between these is that the first n coordinates of any solution of (6) constitute a solution of (1), and conversely. This stationarization procedure appears in process theory also. Thus, let p be a process in P over $R = R^1$, say. Define a new process q in $Q = P \times R$ over R by setting, for (x, ξ), $(y, \eta) \in Q$ and $\alpha \geq \beta$ in R,

$$(x, \xi)\, {}_\alpha q_\beta (y, \eta) \quad \text{iff} \quad x\, {}_\xi p_\eta\, y \quad \text{and} \quad \xi - \alpha = \eta - \beta.$$

It is then easily verified that q is indeed a stationary process in Q over R, and that it has to p a relation corresponding precisely to that obtaining between (6) and (1).

In this example, to carry over the method from differential equations to processes, it was not necessary to assume anything concerning the nature of the phase space P; indeed, it could be any abstract set. However, in other cases one must introduce further requirements. Thus, e.g. in attempting to introduce the concept of limit points or of orbital stability for processes, it is necessary to employ notions describing the nearness of a set to a point;

slightly more precisely, to assume that some structure such as a topology for P has been given in advance. Then it may (but need not) be necessary to require that the process p itself be compatible in some sense with the given structure on the phase space (that p be a "continuous" process). As reasonable candidates for interesting structures, the following seem to present themselves:

structure for P	compatible p
topology (or uniformity, metric, differential, etc.)	continuous
group	additive
linear space	linear
differential	differential

(combinations of these are also interesting; e.g. BANACH spaces and continuous linear processes).

As an example of this group of definitions, a process p in a linear space L over R is termed *linear* iff

$$x_\alpha p_\beta y, \qquad x'_\alpha p_\beta y', \qquad \lambda \in R^1 \qquad \text{imply} \qquad (x + \lambda x')_\alpha p_\beta (y + \lambda y').$$

Two linear processes p and p' in a HILBERT space H over R are called *adjoint* iff

$$x_\alpha p_\beta y, \qquad x'_\alpha p_\beta y' \qquad \text{imply} \qquad (x, x') = (y, y')$$

with (x, x') denoting the scalar product.

The definition of continuity of a process (in a topological space) is considerably more involved. However, in the not too special case of processes with global existence and unicity, this is quite straightforward. Assume given such a process p in a topological space T over R (the latter is to inherit the natural topology from $R^1 \supset R$). Unicity then yields that in any relation $(x, \alpha)\, p(y, \beta)$, the point $x \in T$ is uniquely determined by (α, y, β), thereby defining a partial map

$$t : R \times T \times R \to T, \qquad x = t(\alpha, y, \beta) \qquad \text{iff} \qquad (x, \alpha)\, p(y, \beta).$$

(This map t is called the global *flow* associated with p.) Then the process p is called *continuous*, or compatible with the given topology for T, iff the corresponding partial map t is continuous in the customary sense. In greater detail, the requirement is that

$(x_t, \alpha_t)\ p\ (y_t, \beta_t), \quad (\alpha_t, y_t, \beta_t) \longrightarrow (\alpha, y, \beta)$ in $R \times T \times R$, $\quad (x, \alpha)\ p\ (y, \beta)$ imply $x_t \to x$ in T.

To define stability or recursive motions, a topology on the phase space is insufficient, since one must treat the nearness of two sets rather than that of a set to a point; and it is necessary to assume that the phase space is endowed with some structure such as a proximity or uniformity or metric. As concerns the process studied, in the differential case it is not necessary to assume that it be uniformly continuous (or distance preserving, etc.), but only continuous. Therefore one does not require compatibility between the process and e.g. the metric structure, but it still may be useful to impose compatibility with the topology induced by the metric. Thus one studies continuous processes on uniform spaces, on differential manifolds, etc.

This concept of continuity of processes is surprisingly versatile, allowing many classical results to be carried over to the more general situation. Thus e.g. MASSERA's first theorem on periodic solutions in R^1 [5, p. 445] can be transferred bodily, including its proof. To illustrate a more complicated case with a definitely non-trivial transfer to processes, the POINCARÉ—BENDIXSON theory of limit points and cycles for autonomous differential equations in the plane can be extended to stationary processes with unicity and local existence (the dynamical systems) on a large class of 2-manifolds [2].

Perhaps it is not surprising that the axioms 0^0 to 2^0 still permit some rather pathological objects as processes. Thus, consider the following very reasonable property: a process p is called *solution-complete* if all p-related pairs can be joined by a solution, i.e. iff $(x, \alpha)\, p(y, \beta)$ implies that there is a solution s of p with $x = s(\alpha)$, $y = s(\beta)$. Evidently the differential processes, etc., are all solution-complete. However, there do exist otherwise reasonable processes which are not solution-complete. Indeed, let P be the set of all real rationals, and for (x, α), $(y, \beta) \in P \times R^1$ with $\alpha \geq \beta$ let

$$(x, \alpha)\, p(y, \beta) \quad \text{iff} \quad \begin{array}{ll} 0 < x - y < \alpha - \beta & \text{in case } \alpha > \beta, \\ x = y & \text{in case } \alpha = \beta. \end{array}$$

Then p is a process in P over R^1 (in verifying the second part of requirement 2^0 use the fact that P is dense in R^1) indeet, p is closely related to the differential inequality $0 < \mathrm{d}x/\mathrm{d}\vartheta < 1$. Second, all solutions of p are continuous, since they have LIPSCHITZ constant 1; thus they are rational-valued continuous functions with interval domains, and hence all solutions are constant. But evidently the process has no (non degenerate) constant solutions at all. Therefore no distinct p-related pairs (x, α), (y, β) can be joined by any solution, i.e. p is not solution-complete.

4. The preceding section suggests that much of differential equation theory can be adequately represented within the wider setting of process theory. However, to justify the introduction and further study of processes, there are two further questions which should be answered satisfactorily. First, is

process theory capable of an autonomous development, of obtaining interesting results within itself, or is it merely an arid generalization or medium of reformulation, in which all the impetus is due to the classical underlying theories. And second, does process theory yield new results outside itself, i.e. can one obtain, via the processes, hereto unknown results formulatable in terms of only, e.g., differential equations.

Naturally, a decisive answer will not be available until much later; but even at this early stage of development, I have the impression that the answer to both these questions is affirmative. As examples to the first, consider the following two results (the formulations are somewhat loose):

Theorem. *Every solution-complete process can be represented, in a certain minimal and canonic fashion, as having been obtained from a process with unicity by identifying some elements in its domain; indeed, this representability characterizes solution-complete processes.*

Incidentally, the construction of the corresponding process with unicity seems closely related to that for difference-differential equations described earlier.

Differentiable representation theorem. *Every continuous process on a differentiable manifold P over R^1 with unicity and local existence and with domain open in $P \times R^1$ can be homeomorphically represented as corresponding to a differential equation.*

This shows, in particular, that at least for the indicated type of process, the axioms 0^0 to 2^0 exactly adequate, that no further independent axioms can be added. As concerns the second question, of the direct effect of process theory on differential equation theory, the results obtained are far less decisive and spectacular. However, one has the following

Proposition. *Let p and q be adjoint linear processes (in a* Hilbert *space). If p has global existence, then q has unicity in the negative direction; in particular, if p has global existence in both directions, then q has unicity in both directions.*

This has some interesting applications. Some time ago Dr. Karták studied linear homogeneous equations in n-space,

$$\frac{\mathrm{d}x}{\mathrm{d}\vartheta} = A(\vartheta)\, x \tag{7}$$

with continuity of the matrix A weakened to Newton-integrability (i.e. $A(\vartheta) = \frac{\mathrm{d}}{\mathrm{d}\vartheta} B(\vartheta)$ pointwise for some matrix B). Recently, he solved positively the general existence problem. Since change of orientation and passage to the adjoint equation in (7) yield equations which again have Newton-integrable coefficients (to which this existence theorem applies), the proposition above answers the general unicity problem positively. Obviously one even has a more general assertion: For every class of linear equations closed with respect to orientation change and formation of adjoints, global existence implies unicity.

REFERENCES

[1] HÁJEK O., *Critical points of abstract dynamical systems, Comment. Math. Univ. Carol. 5, 3 (1964), 121–124.*

[2] HÁJEK O., *Transversal theory in abstract dynamical systems, Collections ČVUT, IV, 7 (1965), 11–46.*

[3] HÁJEK O., *Theory of processes I–V, to appear in Czech. Math. Journ.*

[4] NĚMYCKIJ V. V., STĚPANOV V. V., *Kačestvennaja těorija differencial'nych uravněnij (2nd ed.), Gostechizdat, Moscow–Leningrad, 1949.*

[5] SANSONE G., CONTI R., *Equazioni differenziali non lineari, Edizioni Cremonese, Roma, 1956.*

[6] ZUBOV V. I., *Metody A. M. Ljapunova i ich priměněnijě, Izdat. Leningrad, Univ., Leningrad, 1957.*

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INVARIANT MANIFOLDS FOR DISCRETE SYSTEMS

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In this paper a general theorem on the existence of invariant manifolds for discrete systems is obtained using the general method of J. Kurzweil [1]. As this theorem is true for discrete systems in a Banach space it may be applied to prove the existence of invariant manifolds for some systems with time lag.

I. General theory

Consider a discrete system $x_{n+1} = f_n(x_n)$; $f_n : G_n \subset X \to X$, n is an integer, X is a Banach space and G_n is a domain in X. If $\tilde{x} \in G_{\tilde{n}}$ we may define the solution $x_n(\tilde{n}, \tilde{x})$ for $n \geq \tilde{n}$ such that $x_{\tilde{n}}(\tilde{n}, \tilde{x}) = \tilde{x}$; if $f_n(x_n(\tilde{n}, \tilde{x})) \in G_{n+1}$ this solution is defined for all $n \geq \tilde{n}$. Suppose this is the case; then obviously $x_n(n_1, x_{n_1}(\tilde{n}, \tilde{x})) = x_n(\tilde{n}, \tilde{x})$ for all $n \geq n_1 \geq \tilde{n}$.

The general theorem we shall prove concerns discrete systems in a product space $\mathfrak{X} = C \times \mathfrak{C}$; these systems will be described by two functions $c_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$ and $\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$ defined for $n \geq \tilde{n}$, $\tilde{c} \in C$, $\tilde{\gamma} \in \mathfrak{C}$, $c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) \in C$, $\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) \in \mathfrak{C}$, and such that $c_n(n_1, c_{n_1}(\tilde{n}, \tilde{c}, \tilde{\gamma}), \gamma_{n_1}(\tilde{n}, \tilde{c}, \tilde{\gamma})) = c_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$, $\gamma_n(n_1, c_{n_1}(\tilde{n}, \tilde{c}, \tilde{\gamma}), \gamma_{n_1}(\tilde{n}, \tilde{c}, \tilde{\gamma})) = \gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$.

Theorem 1. *Consider a discrete system in the product space $C \times \mathfrak{C}$. Suppose there exist positive constants l, L, N, α_1, α_2, $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$, k_1, k_2 such that:* 1^0 *$||\tilde{c}|| \leq l$ imply that $c_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$ is defined for all $n \geq \tilde{n}$ and $||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma})|| \leq l$ for $n \geq \tilde{n} + N$.*

2^0. *$||\tilde{c}_1|| \leq l$, $||\tilde{c}_2|| \leq l$, $\tilde{n} + N \leq n \leq \tilde{n} + 2N$ imply*
$$||c_n(\tilde{n}, \tilde{c}_1, \tilde{\gamma}) - c_n(\tilde{n}, \tilde{c}_2, \tilde{\gamma})|| + L\, ||\gamma_n(\tilde{n}, \tilde{c}_1, \tilde{\gamma}) - \gamma_n(\tilde{n}, \tilde{c}_2, \tilde{\gamma})|| \leq \alpha_1 ||\tilde{c}_1 - \tilde{c}_2||.$$

3^0. *$||\tilde{c}_1|| \leq l$, $||\tilde{c}_2|| \leq l$, $||\tilde{c}_1 - \tilde{c}_2|| \leq L\, ||\tilde{\gamma}_1 - \tilde{\gamma}_2||$ imply*

a) $||\gamma_n(\tilde{n}, \tilde{c}_1, \tilde{\gamma}_1) - \gamma_n(\tilde{n}, \tilde{c}_2, \tilde{\gamma}_2) - \tilde{\gamma}_1 + \tilde{\gamma}_2|| \leq \alpha_2 ||\tilde{\gamma}_1 - \tilde{\gamma}_2||$
for $\tilde{n} \leq n \leq \tilde{n} + 2N$

b) $||c_n(\tilde{n}, \tilde{c}_1, \tilde{\gamma}_1) - c_n(\tilde{n}, \tilde{c}_2, \tilde{\gamma}_2)|| \leq (1 - \alpha_2) L ||\tilde{\gamma}_1 - \tilde{\gamma}_2||$
for $\tilde{n} + N \leq n \leq \tilde{n} + 2N$

4^0. $||c_n(\tilde{n}, \tilde{c}_1, \tilde{\gamma}_1) - c_n(\tilde{n}, \tilde{c}_2, \tilde{\gamma}_2)|| + ||\gamma_n(\tilde{n}, \tilde{c}_1, \tilde{\gamma}_1) - \gamma_n(\tilde{n}, \tilde{c}_2, \tilde{\gamma}_2)|| \leq$ $\leq k_1 k^{n-\tilde{n}}(||\tilde{c}_1 - \tilde{c}_2|| + ||\tilde{\gamma}_1 - \tilde{\gamma}_2||)$ *for all* $n \geq \tilde{n}$ *for which the functions are defined. Then for each integer* n *there exist a function* $p_n\colon \mathfrak{C} \to C$ *and positive constants* K, $0 < \alpha < 1$ *such that*

a) $||p_n(\gamma)|| \leq l$;

b) $||p_n(\gamma_1) - p_n(\gamma_2)|| \leq L ||\gamma_1 - \gamma_2||$;

c) $||\tilde{c}|| \leq l$ *implies* $||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - p_n(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma})) \leq K\alpha^{n-\tilde{n}} ||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})||$;

d) $\tilde{c} = p_{\tilde{n}}(\tilde{\gamma})$ *implies* $c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) = p_n(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))$ *for all* n;

e) p_n *is uniquely determined by the above properties;*

f) 1^0. *If* $c_{n+\nu}(\tilde{n} + \nu, \tilde{c}, \tilde{\gamma}) \equiv c_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$, $\gamma_{n+\nu}(\tilde{n} + \nu, \tilde{c}, \tilde{\gamma}) \equiv \gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$ *for all* n, $\tilde{n}$, $\tilde{c}$, $\tilde{\gamma}$ *for which the functions are defined, then* $p_{n+\nu}(\gamma) \equiv p_n(\gamma)$.
2^0. *If* $c_n(\tilde{n}, \tilde{c}, \tilde{\gamma} + \omega) = c_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$, $\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma} + \omega) \equiv \gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) + \omega$ *for all* n, $\tilde{n}$, $\tilde{c}$, $\tilde{\gamma}$ *for which the functions are defined, then* $p_n(\gamma + \omega) \equiv p_n(\gamma)$.

g) *If each sequence* $n_k \to \infty$ *contains a subsequence* n_{kl} *such that* $c_{n+n_{kl}}(\tilde{n} + n_{kl}, \tilde{c}, \tilde{\gamma})$, $\gamma_{n+n_{kl}}(\tilde{n} + n_{kl}, \tilde{c}, \tilde{\gamma})$ *are convergent for* $l \to \infty$, *uniformly on each finite set of values* $n \geq \tilde{n}$ *and uniformly with respect to* $\tilde{n}$, $\tilde{c}$, $\tilde{\gamma}$, *then the sequence* p_n *is almost periodic uniformly with respect to* γ.

Proof. A. Denote by $Q(l, L)$ the set of functions $q\colon \mathfrak{C} \to C$ such that $||q(\gamma_1) - q(\gamma_2)|| \leq L ||\gamma_1 - \gamma_2||$, $||q(\gamma)|| \leq l$. Let $\vartheta^q_{n,\tilde{n}} : \mathfrak{C} \to \mathfrak{C}$ be defined by $\vartheta^q_{n,\tilde{n}}(\tilde{\gamma}) = \gamma_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})$. From condition 3^0 a) follows for $\tilde{n} \leq n \leq \tilde{n} + 2N$ that $||\vartheta^q_{n,\tilde{n}}(\tilde{\gamma}_1) - \vartheta^q_{n,\tilde{n}}(\tilde{\gamma}_2) - \tilde{\gamma}_1 + \tilde{\gamma}_2|| \leq \alpha_2 ||\tilde{\gamma}_1 - \tilde{\gamma}_2||$ hence $(1 - \alpha_2) ||\tilde{\gamma}_1 - \tilde{\gamma}_2|| \leq ||\vartheta^q_{n,\tilde{n}}(\tilde{\gamma}_1) - \vartheta^q_{n,\tilde{n}}(\tilde{\gamma}_2)|| \leq (1 + \alpha_2) ||\tilde{\gamma}_1 - \tilde{\gamma}_2||$.

It is proved then by a lemma of Kurzweil that for $\tilde{n} \leq n \leq \tilde{n} + 2N$ $\vartheta^q_{n,\tilde{n}}$ is a one-to-one mapping of $\mathfrak{C}$ onto $\mathfrak{C}$; let $\sigma^q_{n,\tilde{n}}\colon \mathfrak{C} \to \mathfrak{C}$ be the inverse mapping.

B. Define the mapping $P_{n,\tilde{n}}q\colon \mathfrak{C} \to C$ by

$$[P_{n,\tilde{n}}q] (\tilde{\gamma}) = c_n(\tilde{n}, q[\sigma^q_{n,\tilde{n}}(\tilde{\gamma})], \sigma^q_{n,\tilde{n}}(\tilde{\gamma})), \qquad \tilde{n} \leq n \leq \tilde{n} + 2N.$$

For $\tilde{n} + N \leq n \leq \tilde{n} + 2N$ we have $||[P_{n,\tilde{n}}q] (\tilde{\gamma})|| \leq l$ from condition 1^0 and $||[P_{n,\tilde{n}}q](\tilde{\gamma}_1) - [P_{n,\tilde{n}}q](\tilde{\gamma}_2)|| \leq (1 - \alpha_2) L ||\sigma^q_{n,\tilde{n}}(\tilde{\gamma}_1) - \sigma^q_{n,\tilde{n}}(\tilde{\gamma}_2)|| \leq L ||\tilde{\gamma}_1 - \tilde{\gamma}_2||$ from conditions 3^0 a) and b).

It follows that for $\tilde{n} + N \leq n \leq \tilde{n} + 2N$ we have $P_{n,\tilde{n}}q \in Q(l, L)$, hence $P_{n,\tilde{n}} : Q(l, L) \to Q(l, L)$.

C. Let $\tilde{n} + N \leq \tilde{n}_1 \leq \tilde{n} + 2N$, $\tilde{n}_1 + N \leq \tilde{n}_2 \leq \tilde{n}_1 + 2N$, $q_1 = P_{\tilde{n}_1,\tilde{n}}q$. We have $\vartheta^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma}) = \gamma_{\tilde{n}_2}(\tilde{n}_1, c_{\tilde{n}_1}(\tilde{n}, q[\sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})), \tilde{\gamma}) =$

$$= \gamma_{\tilde{n}_2}(\tilde{n}_1, c_{\tilde{n}_1} (\tilde{n}, q[\sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})), \vartheta^q_{\tilde{n}_1,\tilde{n}}[\sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})]) =$$
$$= \gamma_{\tilde{n}_2}(\tilde{n}, q[\sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})) = \vartheta^q_{\tilde{n}_2,\tilde{n}}[\sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})].$$

From here we deduce $\vartheta^{q_1}_{\tilde{n}_2,\tilde{n}_1}[\vartheta^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})] = \vartheta^q_{\tilde{n}_2,\tilde{n}}(\gamma)$. The mapping $\vartheta^q_{\tilde{n}_2,\tilde{n}}$ is the product of two mappings which are one-to-one and onto hence $\vartheta^q_{\tilde{n}_2,\tilde{n}}$ has an inverse $\sigma^q_{\tilde{n}_2,\tilde{n}}$ defined on $\mathfrak{C}$. It follows that $\vartheta^q_{n_1,\tilde{n}}$ has an inverse defined on $\mathfrak{C}$ for all $\tilde{n} \leq n \leq \tilde{n} + 4N$; the reasoning may be repeated and we deduce that $\vartheta^q_{n,\tilde{n}}$ has for all $n \geq \tilde{n}$ an inverse defined on $\mathfrak{C}$. In our proof we used the fact that $q_1 = P_{\tilde{n}_1,\tilde{n}}q$ belongs to $Q(l, L)$; hence we must prove that $P_{n,\tilde{n}}q \in Q(l, L)$ for all $n \geq \tilde{n} + N$.

We have $[P_{\tilde{n}_2,\tilde{n}}q]\,(\tilde{\gamma}) = c_{\tilde{n}_2}(\tilde{n}, q[\sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})) =$

$= c_{\tilde{n}_2}(\tilde{n}_1, c_{\tilde{n}_1}(\tilde{n}, q[\sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})), \gamma_{\tilde{n}_1}(\tilde{n}, q[\sigma^q_{n_2,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma}))).$

But $\sigma^q_{\tilde{n}_2,\tilde{n}} = \sigma^q_{\tilde{n}_1,\tilde{n}}(\sigma^{q_1}_{\tilde{n}_2,n_1})$, hence $c_{\tilde{n}_1}(\tilde{n}, q[\sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})) =$

$= c_{\tilde{n}_1}(\tilde{n}, q[\sigma^q_{\tilde{n}_1,\tilde{n}}(\sigma^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma}))], \sigma^q_{\tilde{n}_1,\tilde{n}}(\sigma^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma}))) =$

$= [P_{\tilde{n}_1,\tilde{n}}q]\,(\sigma^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma})) = q_1[\sigma^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma})]$ and

$\gamma_{\tilde{n}_1}(\tilde{n}, q[\sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})) = \vartheta^q_{\tilde{n}_1,\tilde{n}}(\sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})) = \sigma^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma}).$

It follows that

$[P_{\tilde{n}_2,\tilde{n}}q]\,(\tilde{\gamma}) = c_{\tilde{n}_2}(\tilde{n}_1, q_1[\sigma^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma})], \sigma^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma})) = [P_{\tilde{n}_2,\tilde{n}_1}q_1]\,(\tilde{\gamma})$

hence $P_{\tilde{n}_2,\tilde{n}}q \in Q(l, L)$ and $P_{\tilde{n}_2,\tilde{n}}\,q = P_{\tilde{n}_2,\tilde{n}_1}P_{\tilde{n}_1,\tilde{n}}q$.

The reasoning may be repeated and we deduce that $P_{n,\tilde{n}}q \in Q(l, L)$ for all $n \geq \tilde{n} + N$ and that $P_{\tilde{n}_2,\tilde{n}} = P_{\tilde{n}_2,\tilde{n}_1}P_{\tilde{n}_1,\tilde{n}}$ for all $\tilde{n}_2 \geq \tilde{n}_1 \geq \tilde{n} + N$.

Let us remark the most important relation

$[P_{n,\tilde{n}}q]\,(\gamma_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})) = [P_{n,\tilde{n}}q]\,(\vartheta^q_{n,\tilde{n}}(\tilde{\gamma})) =$

$= c_n(\tilde{n}, q[\sigma^q_{n,\tilde{n}}\vartheta^q_{n,\tilde{n}}(\tilde{\gamma})], \sigma^q_{n,\tilde{n}}\vartheta^q_{n,\tilde{n}}(\tilde{\gamma})) = c_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})$ for all $n \geq \tilde{n}$.

D. We have $||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - [P_{n,\tilde{n}}q]\,(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq$

$\leq ||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - c_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})|| +$

$+ ||[P_{n,\tilde{n}}q]\,(\gamma_n\,(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})) - [P_{n,\tilde{n}}q]\,(\gamma_n\,(\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq$

$\leq ||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - c_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})|| + L\,||\gamma_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) - \gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma})|| \leq$

$\leq \alpha_1||q(\tilde{\gamma}) - \tilde{c}||$ for $||\tilde{c}|| \leq l$, $\tilde{n} + N \leq n \leq \tilde{n} + 2N$.

From here follows that

$||[P_{n,\tilde{n}}q_2]\,(\gamma_n(\tilde{n}, q_2(\tilde{\gamma}), \tilde{\gamma})) - [P_{n,\tilde{n}}q_1]\,(\gamma_n(\tilde{n}, q_2(\tilde{\gamma}), \tilde{\gamma}))|| \leq \alpha_1\,||q_2(\tilde{\gamma}) - q_1(\tilde{\gamma})||$

hence

$||[P_{n,\tilde{n}}q_2]\,(\tilde{\gamma}) - [P_{n,\tilde{n}}q_1]\,(\tilde{\gamma})|| \leq \alpha_1||q_2[\sigma^{q_1}_{n,\tilde{n}}(\tilde{\gamma})] - q_1[\sigma^{q_2}_{n,\tilde{n}}(\tilde{\gamma})]||,$

$\tilde{n} + N \leq n \leq \tilde{n} + 2N.$

Let now $q_i \in Q(l, L)$, $\lim\limits_{i\to\infty} q_i(\tilde{\gamma}) = q(\tilde{\gamma})$ uniformly with respect to $\tilde{\gamma} \in \mathfrak{C}$. Let $\tilde{\gamma} \in \mathfrak{C}$, $\tilde{\gamma}_i = \sigma^{q_i}_{n,\tilde{n}}(\tilde{\gamma})$, hence $\tilde{\gamma} = \gamma_n(\tilde{n}, q_i(\tilde{\gamma}_i), \tilde{\gamma}_i)$. We have $||q_i(\tilde{\gamma}_i) - q_i(\tilde{\gamma}_j)|| \leq$ $\leq L\,||\tilde{\gamma}_i - \tilde{\gamma}_j||$ hence from condition 3^0 a) we deduce

$(1 - \alpha_2)\,||\tilde{\gamma}_i - \tilde{\gamma}_j|| \leq ||\gamma_n(\tilde{n}, q_i(\tilde{\gamma}_i), \tilde{\gamma}_i) - \gamma_n(\tilde{n}, q_i(\tilde{\gamma}_j), \tilde{\gamma}_j)|| \leq$

$\leq ||\gamma_n(\tilde{n}, q_i(\tilde{\gamma}_i), \tilde{\gamma}_i) - \gamma_n(\tilde{n}, q_j(\tilde{\gamma}_j), \tilde{\gamma}_j)|| +$

$+ ||\gamma_n(\tilde{n}, q_j(\tilde{\gamma}_j), \tilde{\gamma}_j) - \gamma_n(\tilde{n}, q_i(\tilde{\gamma}_j), \tilde{\gamma}_j)|| =$

$= ||\gamma_n(\tilde{n}, q_j(\tilde{\gamma}_j), \tilde{\gamma}_j) - \gamma_n(\tilde{n}, q_i(\tilde{\gamma}_j), \tilde{\gamma}_j)|| \leq k_1k_2^{n-\tilde{n}}||q_j(\tilde{\gamma}_j) - q_i(\tilde{\gamma}_j)||.$

For $\varepsilon > 0$ let $N_\varepsilon > 0$ be such that $n \geq N_\varepsilon$ implies $||q_{n+p}(\gamma) - q_n(\gamma)|| \leq \varepsilon$ for all $\gamma \in \mathfrak{C}$; then $||q_{n+p}(\tilde{\gamma}_n) - q_n(\tilde{\gamma}_n)|| \leq \varepsilon$ for $n \geq N_\varepsilon$ and $||\tilde{\gamma}_{j+p} - \tilde{\gamma}_j|| \leq$

$\leq \frac{1}{1-\alpha_2} k_1 k_2^{n-\tilde{n}} \| q_j(\tilde{\gamma}_j) - q_{j+p}(\tilde{\gamma}_j)\| \leq \frac{\varepsilon}{1-\alpha_2} k_1 k_2^{n-\tilde{n}}$ hence $\tilde{\gamma}_j$ is a Cauchy sequence. Let $\tilde{\gamma}_0 = \lim_{j\to\infty} \tilde{\gamma}_j$. We have $\lim_{j\to\infty} q_j(\tilde{\gamma}_j) = q(\tilde{\gamma}_0)$, $\lim_{j\to\infty} c_n(\tilde{n}, q_j(\tilde{\gamma}_j), \tilde{\gamma}_j) =$ $= c_n(\tilde{n}, q(\tilde{\gamma}_0), \tilde{\gamma}_0)$, $\lim_{j\to\infty} \gamma_n(\tilde{n}, q_j(\tilde{\gamma}_j), \tilde{\gamma}_j) = \gamma_n(\tilde{n}, q(\tilde{\gamma}_0), \tilde{\gamma}_0)$, hence

$$\gamma_n(\tilde{n}, q(\tilde{\gamma}_0), \tilde{\gamma}_0) = \tilde{\gamma}, \quad \lim_{j\to\infty} [P_{n,\tilde{n}} q_j](\tilde{\gamma}) = \lim_{j\to\infty} [P_{n,\tilde{n}} q_j](\gamma_n(\tilde{n}, q_j(\tilde{\gamma}_j), \tilde{\gamma}_j)) =$$
$$= \lim_{j\to\infty} c_n(\tilde{n}, q_j(\tilde{\gamma}_j), \tilde{\gamma}_j) = c_n(\tilde{n}, q(\tilde{\gamma}_0), \tilde{\gamma}_0) = [P_{n,\tilde{n}} q](\gamma_n(\tilde{n}, q(\tilde{\gamma}_0), \tilde{\gamma}_0)) =$$
$$= [P_{n,\tilde{n}} q](\tilde{\gamma}),$$

the convergence being uniform with respect to $\tilde{\gamma} \in \mathfrak{C}$.

We have thus proved that for all $n \geq \tilde{n}$ from $q_i \overset{u}{\to} q$ follows that $P_{n,\tilde{n}} q_i \overset{u}{\to}$ $\overset{u}{\to} P_{n,n} q$.

E. We have $\lim_{\tilde{n}\to-\infty} P_{n,\tilde{n}} q = p_n$, $p_n \in Q(l, L)$, $P_{n_2,\tilde{n}_1} p_{n_1} = p_{n_2}$ for $n_2 \geq n_1$. Let $\tilde{n}_1 = n$, $\tilde{n}_i - 2N \leq \tilde{n}_{i+1} \leq \tilde{n}_i - N$, $j > i > 1$; we have $P_{n,\tilde{n}_j} q =$ $= P_{n,\tilde{n}_i}(P_{\tilde{n}_i,\tilde{n}_j} q)$ and

$$\|[P_{n,\tilde{n}_i} q](\tilde{\gamma}) - [P_{n,\tilde{n}_j} q](\tilde{\gamma})\| =$$
$$= \|[P_{\tilde{n}_1,\tilde{n}_2}, \ldots, P_{\tilde{n}_{i-1},\tilde{n}_i} q](\tilde{\gamma}) - [P_{\tilde{n}_1,\tilde{n}_2}, \ldots, P_{\tilde{n}_{i-1},\tilde{n}_i}(P_{\tilde{n}_{i-1},\tilde{n}_j} q)](\tilde{\gamma})\| \leq$$
$$\leq \alpha_1^{i-1} \sup_{\gamma} \|[P_{\tilde{n}_i,\tilde{n}_j} q](\tilde{\gamma}) - q(\tilde{\gamma})\| \leq \alpha_1^{i-1} . 2l, \quad \text{hence}$$

$\lim_{i\to\infty} [P_{n,\tilde{n}_i} q](\tilde{\gamma})$ exists, uniformly with respect to $\tilde{\gamma} \in \mathfrak{C}$.

Moreover

$\|[P_{n,\tilde{n}_i} q_2](\tilde{\gamma}) - [P_{n,\tilde{n}_i} q_1](\tilde{\gamma})\| \leq \alpha_1^i \sup \|q_1(\tilde{\gamma}) - q_2(\tilde{\gamma})\|$, hence $\lim_{i\to\infty} [P_{n,\tilde{n}_i} q_2](\tilde{\gamma}) = \lim_{i\to\infty} [P_{n,\tilde{n}_i} q_1](\tilde{\gamma})$ and p_n does not depend on q. From $P_{n_2,n_1} P_{n_1,\tilde{n}} q = P_{n_2,\tilde{n}} q$ it follows for $\tilde{n} \to -\infty$ that $P_{n_2,n_1} p_{n_1} = p_{n_2}$ $(n_2 \geq n_1)$. Let indeed $\tilde{n}_i \to -\infty$; then $P_{n_1,\tilde{n}_i} q \overset{u}{\to} p_{n_1}$ hence $P_{n_2,n_1} P_{n_1,\tilde{n}_i} q \overset{u}{\to} P_{n_2,n_1} p_{n_1}$ and $P_{n_2,\tilde{n}_i} q \overset{u}{\to} p_{n_2}$.

F. The functions p_n have all properties stated in theorem 1. It is obvious that $p_n \in Q(l, L)$ hence a), b) are verified. We have further

$[P_{n,\tilde{n}} q](\gamma_n(\tilde{n}, q(\tilde{\gamma},) \tilde{\gamma})) = c_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})$ and $P_{n,\tilde{n}} p_{\tilde{n}} = p_n$ for all $n \geq \tilde{n}$. Let in the first relation $q = p_{\tilde{n}}$; we get

$$c_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma}) = [P_{n,\tilde{n}} p_{\tilde{n}}](\gamma_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma})) = p_n(\gamma_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma}))$$

for $n \geq \tilde{n}$. We shall prove that the relation holds for all n. We have

$$p_{\tilde{n}}(\tilde{\gamma}) = [P_{\tilde{n},\tilde{n}-i} p_{\tilde{n}-i}](\tilde{\gamma}) = c_{\tilde{n}}(\tilde{n} - i, p_{\tilde{n}-i}[\sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma})], \sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma}))$$
$$\gamma_{\tilde{n}}(\tilde{n} - i, p_{\tilde{n}-i}[\sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma})], \sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma})) = \vartheta^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i} \sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma}) = \tilde{\gamma}$$

hence

$$c_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma}) = c_n(\tilde{n} - i, p_{\tilde{n}-i}[\sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma})], \sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma})) =$$
$$= p_n(\gamma_n(\tilde{n} - i, p_{\tilde{n}-i}[\sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma})], \sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma})))$$

and this relation is true for $n \geq \tilde{n} - i$. We get from here

$$c_n(\tilde{n} - i, p_{\tilde{n}-i}(\tilde{\gamma}), \tilde{\gamma}) = p_n(\gamma_n(\tilde{n} - i, p_{\tilde{n}-i}(\tilde{\gamma}), \tilde{\gamma}))$$

and relation d) is proved for all n.

To establish c) we start from $||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - [P_{n,\tilde{n}}q]\, \gamma_n\, (\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq \leq \alpha_1||q(\tilde{\gamma}) - \tilde{c}||$ for $\tilde{n} + N \leq n \leq \tilde{n} + 2N$, $||\tilde{c}|| \leq l$; let in this relation $q = p_{\tilde{n}}$. We get $||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - [P_{n,\tilde{n}}p_{\tilde{n}}]\, (\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq \alpha_1||p_{\tilde{n}}(\tilde{\gamma}) - \tilde{c}||$, hence $||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - p_n(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq \alpha_1||p_n(\tilde{\gamma}) - \tilde{c}||$ for $\tilde{n} + N \leq n \leq \tilde{n} + 2N$. By induction it is then proved that

$$||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - p_n(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq \alpha_1^k||p_{\tilde{n}}(\tilde{\gamma}) - \tilde{c}|| \qquad \text{for} \qquad \tilde{n} + kN \leq n \leq \tilde{n} + (k + 1)\, N.$$

For $\tilde{n} \leq n \leq \tilde{n} + N$ we have

$$||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - c_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma})|| + ||\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - \gamma_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma})|| \leq \leq k_1k_2^N||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})||$$

$$||p_n(\gamma_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma})) - p_n(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq L\, ||\gamma_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma}) - \gamma_{\tilde{n}}(\tilde{n}, \tilde{c}, \tilde{\gamma})|| \leq \leq Lk_1k_2^N||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})||,$$ hence

$$||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - p_n(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq (1 + L)\, k_1k_2^N||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})||.$$

Let $K = (1 + L)\, k_1 \left(\dfrac{k_2}{\alpha}\right)^N$, $\alpha = \alpha_1^{\frac{1}{N}}$; we have

$$||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - p_n(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq K\alpha^N||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})|| \leq K\alpha^{n-\tilde{n}}||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})||, \quad \tilde{n} \leq n \leq \tilde{n} + N$$

$$||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - p_n(\gamma_n\, (\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq \alpha^{kN}||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})|| \leq \frac{1}{\alpha^N}\, \alpha^{n-\tilde{n}}||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})|| \leq \leq K\alpha^{n-\tilde{n}}||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})||$$

for $\tilde{n} + kN \leq n \leq \tilde{n} + (k + 1)\, N$, hence for all $n \geq \tilde{n}$, and property c) is established. Let us prove property e). Let p'_n with properties a), b), c), d), $\tilde{\gamma} \in \mathfrak{C}$, $\tilde{n}' = \tilde{n} - N$, $\tilde{\gamma}' = \sigma^{p'\tilde{n}'}_{\tilde{n},\tilde{n}'}(\tilde{\gamma})$; we have $p'_{\tilde{n}}(\tilde{\gamma}) = p'_{\tilde{n}}(\gamma_{\tilde{n}}(\tilde{n}', p'_{\tilde{n}'}(\tilde{\gamma}'), \tilde{\gamma}')) = c_{\tilde{n}}(\tilde{n}', p'_{\tilde{n}'}(\tilde{\gamma}'), \tilde{\gamma}')$ (by d)) and $||c_{\tilde{n}}(\tilde{n}', p'_{\tilde{n}'}(\tilde{\gamma}'), \tilde{\gamma}') - p_{\tilde{n}}(\gamma_{\tilde{n}}(\tilde{n}', p'_{\tilde{n}'}(\tilde{\gamma}'), \tilde{\gamma}')|| \leq \leq K\alpha^{\tilde{n}-\tilde{n}'}||p'_{\tilde{n}'}(\tilde{\gamma}') - p_{\tilde{n}'}(\tilde{\gamma}')||$ (by c)). It follows that $||p'_{\tilde{n}}(\tilde{\gamma}) - p_{\tilde{n}}(\tilde{\gamma})|| \leq \leq K\alpha^N||p'_{\tilde{n}'}(\tilde{\gamma}') - p_{\tilde{n}'}(\tilde{\gamma}')||$ and by induction $||p'_{\tilde{n}}(\tilde{\gamma}) - p_{\tilde{n}}(\tilde{\gamma})|| \leq \leq K\alpha^{jN}||p'_{\tilde{n}-jN}(\tilde{\gamma}^{(j)}) - p_{\tilde{n}-jN}(\tilde{\gamma}^{(j)})|| \leq 2lK\alpha^{jN}$ and for $j \to \infty$ we get $p'_{\tilde{n}}(\tilde{\gamma}) = = p_{\tilde{n}}(\tilde{\gamma})$.

G. To obtain property f) 1° we remark that

$$[P_{n,\tilde{n}}q]\, (\gamma_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})) = c_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) = c_{n+\nu}(\tilde{n} + \nu, q(\tilde{\gamma}), \tilde{\gamma}) = = [P_{n+\nu,\tilde{n}+\nu}q]\, (\gamma_{n+\nu}(\tilde{n} + \nu, q(\tilde{\gamma}), \tilde{\gamma})) = [P_{n+\nu,\tilde{n}+\nu}q]\, (\gamma_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}))$$

hence $P_{n,\tilde{n}}q = P_{n+\nu,\tilde{n}+\nu}q$ and for $\tilde{n} \to -\infty$ we get $p_n = p_{n+\nu}$.

Let then in the conditions of f) 2° q be periodic of period ω; we have

$$[P_{n,\tilde{n}}q]\, (\gamma_n(\tilde{n}, q(\tilde{\gamma} + \omega), \tilde{\gamma} + \omega)) = c_n(\tilde{n}, q(\tilde{\gamma} + \omega), \tilde{\gamma} + \omega) = = c_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) = [P_{n,\tilde{n}}q]\, (\gamma_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}))$$

hence $[P_{n,\tilde{n}}q]\, (\vartheta^q_{n,\tilde{n}}(\tilde{\gamma}) + \omega) = [P_{n,\tilde{n}}q]\, (\vartheta^q_{n,\tilde{n}}(\tilde{\gamma}))$ and for $\tilde{\gamma} = \sigma^q_{n,\tilde{n}}(\gamma)$ we get

$[P_{n,\tilde{n}}q](\gamma + \omega) = [P_{n,\tilde{n}}q](\gamma)$. For $\tilde{n} \to -\infty$ we get $p_n(\gamma + \omega) = p_n(\gamma)$. We shall now prove g).

Let $\lim\limits_{i\to\infty} c^i_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) = c^*_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$, $\lim\limits_{i\to\infty} \gamma^i_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) = \gamma^*_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$ the convergence being for $n = \tilde{n} + N$ uniform with respect to $\tilde{n}, \tilde{c}, \tilde{\gamma}$. We have

$c_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) = [P^{(i)}_{\tilde{n}+N,\tilde{n}}q](\gamma^{(i)}_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}))$

$c^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) = [P^*_{\tilde{n}+N,\tilde{n}}q](\gamma^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}))$

since if systems $c^i_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$, $\gamma^i_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$ have all the properties 1^0, 2^0, 3^0, 4^0, the same is true for the limit system $c^*_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$, $\gamma^*_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$.

We deduce

$$\begin{aligned}
&\|[P^{(i)}_{\tilde{n}+N,\tilde{n}}q](\gamma^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})) - [P^*_{\tilde{n}+N,\tilde{n}}q](\gamma^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}))\| \leq \\
&\quad \leq \|c^i_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) - [P^{(i)}_{\tilde{n}+N,\tilde{n}}q](\gamma^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}))\| + \\
&\quad + \|c^i_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) - c^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})\| \leq \\
&\quad \leq L\,\|\gamma^i_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) - \gamma^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})\| + \\
&\quad + \|c^i_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) - c^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})\| \leq k'\varepsilon_i, \quad \lim_{i\to\infty} \varepsilon_i = 0,
\end{aligned}$$

$\|[P^{(i)}_{\tilde{n}+N,\tilde{n}}q](\gamma) - [P^*_{\tilde{n}+N,\tilde{n}}q](\gamma)\| \leq k'\varepsilon_i$. We have further

$$\begin{aligned}
&\|[P^{(i)}_{\tilde{n}+N,\tilde{n}}q_1](\tilde{\gamma}) - [P^*_{\tilde{n}+N,\tilde{n}}q_2](\tilde{\gamma})\| \leq \|[P^{(i)}_{\tilde{n}+N,\tilde{n}}q_1](\tilde{\gamma}) - [P^*_{\tilde{n}+N,\tilde{n}}q_1](\tilde{\gamma})\| + \\
&\quad + \|[P^*_{\tilde{n}+N,\tilde{n}}q_1](\tilde{\gamma}) - [P^*_{\tilde{n}+N,\tilde{n}}q_2](\tilde{\gamma})\| \leq k'\varepsilon_i + \alpha_1 \sup_{\tilde{\gamma}} \|q_1(\tilde{\gamma}) - q_2(\tilde{\gamma})\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|[P^{(i)}_{n,n-jN}q](\tilde{\gamma}) - [P^*_{n,n-jN}q](\tilde{\gamma})\| = \\
&\quad = \|[P^{(i)}_{n,n-N}P^{(i)}_{n-N,n-2N}\dots P^{(i)}_{n-(j-1)N,n-jN}q](\tilde{\gamma}) - \\
&\quad - [P^*_{n,n-N}\dots P^*_{n-(j-1)N,n-jN}q](\tilde{\gamma})\| \leq k'\varepsilon_i(1 + \alpha_1 + \alpha_1^2 + \dots + \alpha_1^j) \leq \\
&\quad \leq \frac{k'\varepsilon_i}{1 - \alpha_1}
\end{aligned}$$

and for $j \to \infty$ we get

$$\|p^{(i)}_n(\tilde{\gamma}) - p^*_n(\tilde{\gamma})\| \leq \frac{k'\varepsilon_i}{1 - \alpha_1}$$

hence $\lim\limits_{i\to\infty} p^i_n(\tilde{\gamma}) = p^*_n(\tilde{\gamma})$ uniformly with respect to n and $\tilde{\gamma} \in \mathfrak{C}$.

Let now $n_k \to \infty$, n_{k_l} the subsequence from the statement of g); denote $c^l_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) = c_{n+n_{k_l}}(n + n_{k_l}, \tilde{c}, \tilde{\gamma})$, $\gamma^l_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) = \gamma_{n+n_{k_l}}(\tilde{n} + n_{k_l}, \tilde{c}, \tilde{\gamma})$. The systems $c^l_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$, $\gamma^l_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$ have all properties 1^0, 2^0, 3^0, 4^0 from the statement since these properties depend uniquely on the difference $n - \tilde{n}$. Hence $\lim\limits_{l\to\infty} p^{(l)}_n(\tilde{\gamma}) = p^*_n(\tilde{\gamma})$, the convergence being uniform with respect to n and $\tilde{\gamma}$. But $P^{(l)}_{n,\tilde{n}} = P_{n+n_{k_l},\tilde{n}+n_{k_l}}$ hence for $\tilde{n} \to -\infty$ we get $p^{(l)}_n = p_{n+n_{k_l}}$ and $p_{n+n_{k_l}}$ converges to p^*_n uniformly with respect to n and $\tilde{\gamma}$. The almost periodicity of p_n is thus proved.

Remarks. 1^0. If the system has the property of periodicity from f) 1^0 we can get p_n by proving that the mapping $P_{\nu,0}\colon Q(l, L) \to Q(l, L)$ has a unique fixed-point. We may organize $Q(l, L)$ as a metric space in the usual way with the distance $\varrho(q_1, q_2) = \sup\limits_{\gamma} ||q_1(\gamma) - q_2(\gamma)||$. Let h be such that $N \leq h\nu \leq \leq 2N$. We have $||[P_{h\nu,0}q_1](\gamma) - [P_{h\nu,0}q_2](\gamma)|| \leq \alpha_1 \sup\limits_{\gamma} ||q_1(\tilde{\gamma}) - q_2(\tilde{\gamma})|| = \alpha_1\varrho(q_1, q_2)$ hence $\varrho(P_{h\nu,0}q_1, P_{h\nu,0}q_2) \leq \alpha_1\varrho(q_1, q_2)$ and $P_{h\nu,0}$ is a contraction in $Q(l, L)$. It follows that $P_{h\nu,0}$ admits a unique fixed point q_0. But $P_{h\nu,0} = P_{h\nu,(h-1)\nu}P_{(h-1)\nu,0} = P_{\nu,0}P_{(h-1)\nu,0}$ and by induction $P_{h\nu,0} = (P_{\nu,0})^h$ which shows that q_0 is a fixed point for $P_{\nu,0}$.

For this proof to be complete we must show that $P_{n_3,n_2}P_{n_2,n_1} = P_{n_3,n_1}$ holds for all $n_1 \leq n_2 \leq n_3$ (and not only for $n_2 \geq n_1 + N$).

From the fact that the fundamental relation $[P_{n,\tilde{n}}q](\gamma_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})) = c_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})$ holds for all $n \geq \tilde{n}$ we deduce

$$
\begin{aligned}
&[P_{n_3,n_1}q](\gamma_{n_3}(n_1, q(\tilde{\gamma}), \tilde{\gamma})) = c_{n_3}(n_1, q(\tilde{\gamma}), \tilde{\gamma}) = \\
&= c_{n_3}(n_2, c_{n_2}(n_1, q(\tilde{\gamma}), \tilde{\gamma}), \gamma_{n_2}(n_1, q(\tilde{\gamma}), \tilde{\gamma})) = \\
&= c_{n_3}(n_2, [P_{n_2,n_1}q](\gamma_{n_2}(n_1, q(\tilde{\gamma}), \tilde{\gamma})), \gamma_{n_2}(n_1, q(\tilde{\gamma}), \tilde{\gamma})) = \\
&= [P_{n_3,n_2}P_{n_2,n_1}q][\gamma_{n_3}(n_2, c_{n_2}(n_1, q(\tilde{\gamma}), \tilde{\gamma}), \gamma_{n_2}(n_1, q(\tilde{\gamma}), \tilde{\gamma}))] = \\
&= [P_{n_3,n_2}P_{n_2,n_1}q][\gamma_{n_3}(n_1, q(\tilde{\gamma}), \tilde{\gamma})];
\end{aligned}
$$

if we set in this relation $\tilde{\gamma} = \sigma^q_{n_3,n_1}(\gamma)$ we get $[P_{n_3,n_1}q](\gamma) = [P_{n_3,n_2}P_{n_2,n_1}q](\gamma)$.

Let then q_0 the fixed point of $P_{\nu,0}$ and $p_n = P_{n,0}q_0$. We have $P_{n,\tilde{n}}p_{\tilde{n}} = P_{n,\tilde{n}}P_{\tilde{n},0}q_0 = P_{n,0}q_0 = p_n$. Observe that $p_n \in Q(l, L)$; indeed $P_{n,0}q_0 = P_{n+h\nu,h\nu}q_0 = P_{n+h\nu,h\nu}P_{h\nu,0}q_0 = P_{n+h\nu,0}q_0 \in Q(l, L)$ since $n + h\nu \geq N$. Properties a), b), c), d), e) are eastly verified since in proving them we used only $p_n \in Q(l, L)$ and $P_{n,\tilde{n}}p_{\tilde{n}} = p_n$. We have then $p_{n+\nu} = P_{n+\nu,0}q_0 = P_{n+\nu,\nu}P_{\nu,0}q_0 = P_{n+\nu,0}q_0 = P_{n,0}q_0 = p_n$ and if we observe that $P_{h\nu,0}$ maps the set of periodic functions of period ω from $Q(l, L)$ in itself when condition f) 2° is verified, it is seen that q_0 is periodic and $p_n(\gamma + \omega) = [P_{n,0}q_0](\gamma + \omega) = [P_{n,0}q_0](\gamma) = p_n(\gamma)$.

2°. We can use the above method for discrete systems of the form $c_n(\tilde{n}, \tilde{c})$ and obtain conclusions about the existence of an exponentially stable bounded solution which is periodic in the case of periodic systems and almost-periodic in the case of almost-periodic systems. The proof for this case is much simpler.

We state the following *proposition*.

Let a discrete system have the properties:

1°. $||\tilde{c}|| \leq l$, $\tilde{n} + N \leq n \leq \tilde{n} + 2N$ implies $||c_n(\tilde{n}, \tilde{c})|| \leq l$.

2°. $||\tilde{c}_1|| \leq l$, $||\tilde{c}_2|| \leq l$, $\tilde{n} + N \leq n \leq \tilde{n} + 2N$ imply
$||c_n(\tilde{n}, \tilde{c}_1) - c_n(\tilde{n}, \tilde{c}_2)|| \leq \alpha_1||\tilde{c}_1 - \tilde{c}_2||$.

3°. $||c_n(\tilde{n}, \tilde{c}_1) - c_n(\tilde{n}, \tilde{c}_2)|| \leq k_1k_2^{n-\tilde{n}}||\tilde{c}_1 - \tilde{c}_2||$ for all $n \geq n$, $||\tilde{c}_i|| \leq H$.

Then there exists a sequence $p_n \in C$ such that

a) $||p_n|| \leq l$,

b) $p_n = c_n(n_1, p_{n_1})$ hence p_n is a solution,

c) $||c_n(\tilde{n}, \tilde{c}) - p_n|| \leq k\alpha^{n-\tilde{n}}||\tilde{c} - p_{\tilde{n}}||$ for $||\tilde{c}|| \leq l$, $n \geq \tilde{n}$,

d) if $c_{n+\nu}(\tilde{n}+\nu, \tilde{c}) = c_n(\tilde{n}, \tilde{c})$ then $p_{n+\nu} = p_n$,

e) for almost periodic systems p_n is almost periodic.

We prove this proposition in the same way as we proved the theorem. Let $||\tilde{c}|| \leq l$, $P_{n,\tilde{n}}\tilde{c} = c_n(\tilde{n}, \tilde{c})$; $P_{\tilde{n}_2,\tilde{n}_1}\tilde{c} = P_{\tilde{n}_2,\tilde{n}_1}P_{\tilde{n}_1,\tilde{n}}\tilde{c}$ is obvious. Let $n = \tilde{n}_1$, $\tilde{n}_i - 2N \leq \tilde{n}_{i+1} \leq \tilde{n}_i - N$, $j > i > 1$; we have
$||c_n(\tilde{n}_i, \tilde{c}) - c_n(\tilde{n}_j, \tilde{c})|| = ||c_{\tilde{n}_1}(\tilde{n}_2, c_{\tilde{n}_2}(\tilde{n}_i, \tilde{c})) - c_{\tilde{n}_1}(\tilde{n}_2, c_{\tilde{n}_2}(\tilde{n}_i, \tilde{c}'))||$
where $\tilde{c}' = c_{\tilde{n}_i}(\tilde{n}_j, \tilde{c})$. We get $||c_n(\tilde{n}_i, \tilde{c}) - c_n(\tilde{n}_j, c)|| \leq \alpha_1^{i-1}||\tilde{c} - \tilde{c}'|| \leq 2l\alpha_1^{i-1}$ hence $\lim\limits_{\tilde{n}\to-\infty} c_n(\tilde{n}, \tilde{c})$ exists for $||\tilde{c}|| \leq l$. We define $p_n = \lim\limits_{\tilde{n}\to-\infty} c_n(\tilde{n}, \tilde{c})$ and the proof of properties b), c), d), e) is as in the general case.

II. The theorem on continuous dependence on parameters and the stability theorem.

In order to get a system for which the conditions from the general theorem are verified we have to prove a theorem on the continuous dependence on parameters and a stability theorem.

Theorem 2. *Consider the discrete systems* $x_{n+1} = f_n(x_n)$, $x_{n+1} = f_n^\circ(x_n)$ *and suppose that* $||f_n(x) - f_n^\circ(x)|| \leq \xi$, $\left|\left|\frac{\partial f_n}{\partial x}(x) - \frac{\partial f_n^\circ}{\partial x}(x)\right|\right| \leq \xi$ *for all* n *and for all* $x \in G_n$, $\left|\left|\frac{\partial f_n}{\partial x}\right|\right| \leq K_1$, $\left|\left|\frac{\partial f_n{}^0}{\partial x}\right|\right| \leq K_1$.

Suppose that

$$\left|\left|\frac{\partial f_n}{\partial x}(x_1) - \frac{\partial f_n}{\partial x}(x_2)\right|\right| \leq \omega(||x_1 - x_2||), \quad \left|\left|\frac{\partial f_n^\circ}{\partial x}(x_1) - \frac{\partial f_n^\circ}{\partial x}(x_2)\right|\right| \leq \omega(||x_1 - x_2||)$$

$\lim\limits_{\varrho\to 0} \omega(\varrho) = 0$, ω *increasing*.

Then $||x_n(\tilde{n}, \tilde{x}) - x_n^\circ(\tilde{n}, \tilde{x})|| \leq \dfrac{K_1^N - 1}{K_1 - 1}\xi$

$||x_n(\tilde{n}, \tilde{x}_2) - x_n(\tilde{n}, \tilde{x}_1) - x_n^\circ(\tilde{n}, \tilde{x}_2) + x_n^\circ(\tilde{n}, \tilde{x}_1)|| \leq \alpha_N(\xi)\,||\tilde{x}_2 - \tilde{x}_1||$

for $\tilde{n} \leq n \leq \tilde{n} + N$, $\lim\limits_{\xi\to\infty} \alpha_N(\xi) = 0$.

Proof. We have $||x_{\tilde{n}+1}(\tilde{n}, \tilde{x}) - x_{\tilde{n}+1}^\circ(\tilde{n}, \tilde{x})|| = ||f_{\tilde{n}}(\tilde{x}) - f_{\tilde{n}}^\circ(\tilde{x})|| < \xi$.

Suppose $||x_{\tilde{n}+p}(\tilde{n}, \tilde{x}) - x_{\tilde{n}+p}^\circ(\tilde{n}, \tilde{x})|| \leq (1 + K_1 + \ldots + K_1^{p-1})\,\xi$. Then

$$\begin{aligned}
&||x_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}) - x_{\tilde{n}+p+1}^\circ(\tilde{n}, \tilde{x})|| = ||f_{\tilde{n}+p}(x_{\tilde{n}+p}(\tilde{n}, \tilde{x})) - f_{\tilde{n}+p}^\circ(x_{\tilde{n}+p}^\circ(\tilde{n}, \tilde{x}))|| \leq \\
&\quad \leq f_{\tilde{n}+p}(x_{\tilde{n}+p}(\tilde{n}, \tilde{x})) - f_{\tilde{n}+p}(x_{\tilde{n}+p}^\circ(\tilde{n}, \tilde{x}))|| + \\
&\quad + ||f_{\tilde{n}+p}(x_{\tilde{n}+p}^\circ(\tilde{n}, \tilde{x})) - f_{\tilde{n}+p}^\circ(x_{\tilde{n}+p}^\circ(\tilde{n}, \tilde{x}))|| \leq \\
&\quad \leq K_1||x_{\tilde{n}+p}(\tilde{n}, \tilde{x}) - x_{\tilde{n}+p}^\circ(\tilde{n}, \tilde{x})|| + \xi \leq (1 + K_1 + \ldots + K_1^p)\,\xi
\end{aligned}$$

and the first assertion is proved. Let us remark that from this assertion it follows that if the solution of system $x_{n+1} = \overset{\circ}{f}_n(x_n)$ is defined for $\tilde{n} \leq n \leq$ $\leq \tilde{n} + N$ then if ξ is small enough the solution of the system $x_{n+1} = f_n(x_n)$ will be also defined for such n.

To prove the second assertion we start from

$$x_{\tilde{n}+1}(\tilde{n}, \tilde{x}_2) - x_{\tilde{n}+1}(\tilde{n}, \tilde{x}_1) - \overset{\circ}{x}_{\tilde{n}+1}(\tilde{n}, \tilde{x}_{\tilde{n}2}) + x_{\tilde{n}+1}(\tilde{n}, \tilde{x}_1) =$$
$$= f_{\tilde{n}}(\tilde{x}_2) - f_{\tilde{n}}(\tilde{x}_1) - \overset{\circ}{f}_{\tilde{n}}(\tilde{x}_2) + \overset{\circ}{f}_{\tilde{n}}(\tilde{x}_1) =$$
$$= \int_0^1 \left[\frac{\partial f_{\tilde{n}}}{\partial x}(\tilde{x}_1 + \lambda(\tilde{x}_2 - \tilde{x}_1))(\tilde{x}_2 - \tilde{x}_1) - \frac{\partial \overset{\circ}{f}_{\tilde{n}}}{\partial x}(\tilde{x}_1 + \lambda(\tilde{x}_2 - \tilde{x}_1))(\tilde{x}_2 - \tilde{x}_1)\right] d\lambda.$$

We get

$$||x_{\tilde{n}+1}(\tilde{n}, \tilde{x}_2) - x_{\tilde{n}+1}(\tilde{n}, \tilde{x}_1) - \overset{\circ}{x}_{\tilde{n}+1}(\tilde{n}, \tilde{x}_2) + \overset{\circ}{x}_{\tilde{n}+1}(\tilde{n}, \tilde{x}_1)|| \leq \xi||\tilde{x}_2 - \tilde{x}_1||.$$

We have then

$$x_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_2) - x_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_1) - \overset{\circ}{x}_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_2) + \overset{\circ}{x}_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_1) =$$
$$= f_{\tilde{n}+p}(x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) - f_{\tilde{n}+p}(x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)) - \overset{\circ}{f}_{\tilde{n}+p}(\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) +$$
$$+ \overset{\circ}{f}_{\tilde{n}+p}(\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) = \overset{\circ}{f}_{\tilde{n}+p}(x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) -$$
$$- f_{\tilde{n}+p}[x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) + \overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - \overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)] +$$
$$+ f_{\tilde{n}+p}[x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) + \overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - \overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)] -$$
$$- f_{\tilde{n}+p}(x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) + f_{\tilde{n}+p}(\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)) -$$
$$- f_{\tilde{n}+p}(x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)) + f_{\tilde{n}+p}(\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) -$$
$$- f_{\tilde{n}+p}(\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)) - \overset{\circ}{f}_{\tilde{n}+p}(\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) + \overset{\circ}{f}_{\tilde{n}+p}(\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)) =$$
$$= \int_0^1 \frac{\partial f_{\tilde{n}+p}}{\partial x}[x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) +$$
$$+ \lambda(x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - \overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) + \overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1))]\, d\lambda(x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) -$$
$$- x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - \overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) + \overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)) -$$
$$- \int_0^1 \frac{\partial f_{\tilde{n}+p}}{\partial x}[\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) +$$
$$+ \lambda(\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - \overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2))]\, d\lambda(\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - \overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) +$$
$$+ \int_0^1 \frac{\partial f_{\tilde{n}+p}}{\partial x}[x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) +$$
$$+ \lambda(\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - \overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2))]\, d\lambda(\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) +$$
$$+ \int_0^1 \frac{\partial f_{\tilde{n}+p}}{\partial x}[\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) +$$
$$+ \lambda(\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - \overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1))]\, d\lambda(\overset{\circ}{x}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)) -$$

$$- \int_0^1 \frac{\partial f^{\circ}_{\tilde{n}+p}}{\partial x} [x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) +$$

$$+ \lambda(x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1))] \, d\lambda(x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)).$$

It follows that

$$\begin{aligned} v_{p+1} = {} & ||x_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_2) - x_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_1) - x^{\circ}_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_2) + x^{\circ}_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_1)|| \leq \\ & \leq K_1||x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) + x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)|| + \\ & + \omega(||x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)||) \, ||x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)|| + \\ & + \xi||x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)|| \leq \\ & \leq K_1 v_p + \omega\left(\frac{K_1^N - 1}{K_1 - 1}\xi\right) K_1^p||\tilde{x}_1 - \tilde{x}_2|| + \xi K_1^p||\tilde{x}_1 - \tilde{x}_2|| \end{aligned}$$

hence $v_{p+1} \leq K_1 v_p + \beta_N(\xi) \, ||\tilde{x}_2 - \tilde{x}_1||$.

From here we get

$$v_p \leq K_1^{p-1}\xi \, ||\tilde{x}_2 - \tilde{x}_1|| + K_1^N\left(\omega\left(\frac{K_1^N - 1}{K_1 - 1}\xi\right) + \xi\right) \frac{K_1^{p-1}}{K_1 - 1} \, ||\tilde{x}_2 - \tilde{x}_1||$$

hence $v_p \leq \alpha_N(\xi) \, ||\tilde{x}_2 - \tilde{x}_1||$ for $0 \leq p \leq N$, $\lim\limits_{\xi \to 0} \alpha_N(\xi) = 0$ and the theorem is proved.

Theorem 3. *Consider the system*

$$\begin{aligned} y_{n+1} &= Y_n(y_n, \vartheta_n) \\ \vartheta_{n+1} - \vartheta_n &= \Theta_n(y_n, \vartheta_n) \end{aligned}$$

and suppose that:

a) Y_n, Θ_n *are defined for* $||y|| \leq H$, $\vartheta \in \mathfrak{C}$,

$$\left\|\frac{\partial Y_n}{\partial y}(y, \vartheta)\right\| \leq K_1, \quad \left\|\frac{\partial Y_n}{\partial \vartheta}(y, \vartheta)\right\| \leq K_1, \quad \left\|\frac{\partial \Theta_n}{\partial y}(y, \vartheta)\right\| \leq K_1,$$

$$\left\|\frac{\partial \Theta_n}{\partial \vartheta}(y, \vartheta)\right\| \leq K_1,$$

$$\left\|\frac{\partial Y_n}{\partial y}(y', \vartheta') - \frac{\partial Y_n}{\partial y}(y'', \vartheta'')\right\| \leq K_1(||y' - y''||^{\mu} + ||\vartheta' - \vartheta''||^{\mu}),$$

$$\left\|\frac{\partial Y_n}{\partial \vartheta}(y', \vartheta') - \frac{\partial Y_n}{\partial \vartheta}(y'', \vartheta'')\right\| \leq K_1(||y' - y''||^{\mu} + ||\vartheta' - \vartheta''||^{\mu}),$$

$$\left\|\frac{\partial \Theta_n}{\partial y}(y', \vartheta') - \frac{\partial \Theta_n}{\partial y}(y'', \vartheta'')\right\| \leq K_1(||y' - y''||^{\mu} + ||\vartheta' - \vartheta''||^{\mu}),$$

$$\left\|\frac{\partial \Theta_n}{\partial \vartheta}(y', \vartheta') - \frac{\partial \Theta_n}{\partial \vartheta}(y'', \vartheta'')\right\| \leq K_1(||y' - y''||^{\mu} + ||\vartheta' - \vartheta''||^{\mu});$$

b) $Y_n(0, \vartheta) \equiv 0$, $\quad \Theta_n(0, \vartheta) \equiv \alpha_n$.

c) *Let* $A_n(\vartheta) = \dfrac{\partial Y_n}{\partial y}(0, \vartheta)$, $\quad \delta_n = \tilde{\vartheta} + \sum\limits_{k=\tilde{n}}^{n-1} \alpha_k$; *then there exists* $0 < q < 1$

and K such that $||z_n(\tilde{n}, \tilde{z})|| \leq Kq^{n-\tilde{n}}||\tilde{z}||$ for all solutions of the system $z_{n+1} = A_n(\delta_n) z_n$.

If all these conditions are fulfilled there exist q', K', l such that

1^0 $||\tilde{y}|| \leq l$ *implies* $||y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})|| \leq K'q'^{n-\tilde{n}}||\tilde{y}||$ *for* $n \geq \tilde{n}$

2^0 $||\tilde{y}'|| \leq l$, $||\tilde{y}''|| \leq l$ *imply*

$$||y_n(\tilde{n}, \tilde{y}', \tilde{\vartheta}') - y_n(\tilde{n}, \tilde{y}'', \tilde{\vartheta}'')|| \leq K'q'^{n-\tilde{n}}(||\tilde{y}' - \tilde{y}''|| + \varrho||\tilde{\vartheta}' - \tilde{\vartheta}''||)$$

$$||\vartheta_n(\tilde{n}, \tilde{y}', \tilde{\vartheta}') - \vartheta_n(\tilde{n}, \tilde{y}'', \tilde{\vartheta}'') - \tilde{\vartheta}' + \tilde{\vartheta}''|| \leq K'(||\tilde{y}' - \tilde{y}''|| + \varrho||\tilde{\vartheta}' - \tilde{\vartheta}''||)$$

(l depends on ϱ).

Proof. A. Let $V_n(z) = \sup_{p\geq 0} ||z_{n+p}(n, z)|| \dfrac{1}{q^p}$; we have $||z|| \leq V_n(z) \leq K||z||$.

Let $V_n^* = V_n[z_n(\tilde{n}, \tilde{z})] = \sup_{p\geq 0} ||z_{n+p}(n, z_n(\tilde{n}, \tilde{z}))|| \dfrac{1}{q^p} = \sup_{p\geq 0} ||z_{n+p}(\tilde{n}, \tilde{z})|| \dfrac{1}{q^p}$;

it follows $V_{n+1}^* = \sup_{p\geq 0} ||z_{n+p+1}(\tilde{n}, \tilde{z})|| \dfrac{1}{q^p} = \sup_{p\geq 1} ||z_{n+p}(\tilde{n}, \tilde{z})|| \dfrac{1}{q^{p-1}} \leq$

$\leq \sup_{p\geq 0} ||z_{n+p}(\tilde{n}, \tilde{z})|| \dfrac{1}{q^{p-1}}$

hence

$$V_{n+1}^* - V_n^* \leq (q-1) \sup_{p\geq 0} ||z_{n+p}(\tilde{n}, \tilde{z})|| \frac{1}{q^p} = -(1-q)\, V_n^*.$$

We have further $V_n(z') - V_n(z'') = \sup_{p\geq 0} ||z_{n+p}(n, z')|| \dfrac{1}{q^p} - \sup_{p\geq 0} ||z_{n+p}(n, z'')|| \dfrac{1}{q^p} \leq$

$\leq \sup_{p\geq 0} ||z_{n+p}(n, z') - z_{n+p}(n, z'')|| \dfrac{1}{q^p} = \sup_{p\geq 0} ||z_{n+p}(n, z' - z'')|| \dfrac{1}{q^p} =$

$= V_n(z' - z'') \leq K||z' - z''||$

hence $|V_n(z') - V_n(z'')| \leq K||z' - z''||$.

B. We put the first equation of the system in the form

$$y_{n+1} = A_n(\delta_n)\, y_n + B_n(y_n, \vartheta_n);$$

$$B_n(y_n, \vartheta_n) = Y_n(y_n, \vartheta_n) - A_n(\delta_n)\, y_n = Y_n(y_n, \vartheta_n) - Y_n(0, \vartheta_n) - A_n(\delta_n)\, y_n =$$

$$= \int_0^1 \frac{\partial Y_n}{\partial y}(\lambda y_n, \vartheta_n)\, y_n \, d\lambda - \frac{\partial Y_n}{\partial y}(0, \delta_n)\, y_n =$$

$$= \int_0^1 \left[\frac{\partial Y_n}{\partial y}(\lambda y_n, \vartheta_n) - \frac{\partial Y_n}{\partial y}(0, \vartheta_n)\right] d\lambda y_n + \left(\frac{\partial Y_n}{\partial y}(0, \vartheta_n) - \frac{\partial Y_n}{\partial y}(0, \delta_n)\right) y_n;$$

hence $||B_n(y_n, \vartheta_n)|| \leq K_1||y_n||^{\mu+1} + K_1||y_n||\, ||\vartheta_n - \delta_n||^{\mu}$.

Let $\beta_n = \vartheta_n - \delta_n$; we have $\beta_{n+1} - \beta_n = \vartheta_{n+1} - \vartheta_n - (\delta_{n+1} - \delta_n) = \Theta_n(y_n, \vartheta_n) - \alpha_n = \Theta_n(y_n, \vartheta_n) - \Theta_n(0, \vartheta_n)$, hence $||\beta_{n+1} - \beta_n|| \leq K_1||y_n||$.

If $\delta_{\tilde{n}} = \vartheta_{\tilde{n}}$ we get $\|\beta_n\| \leq \sum_{k=\tilde{n}}^{n-1} \|y_k\|$ for $n > \tilde{n}$, $\beta_{\tilde{n}} = 0$.

C. Let $y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})$, $\Theta_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})$ a solution of the system,

$\tilde{V}_n^* = V_n[y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})]$. We have $\tilde{V}_{n+1}^* - \tilde{V}_n^* = \overset{\mu}{V}_{n+1}[z_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))] -$

$$- V_n[y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})] + V_{n+1}[y_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))] - V_{n+1}[z_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))] \leq$$
$$\leq -(1-q)\, \tilde{V}_n^* + K\|y_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})) - z_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))\|.$$

But $y_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})) = A_n(\delta_n)\, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}) + B_n(y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}), \vartheta_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))$

$$z_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})) = A_n(\delta_n)\, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})$$

hence

$$\|y_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})) - z_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))\| = \|B_n(y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}), \vartheta_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))\| \leq$$
$$\leq K_1\|y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})\|^{\mu+1} + K_1^{1+\mu}\|y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})\|\Big(\sum_{k=\tilde{n}}^{n-1}\|y_k(\tilde{n}, \tilde{y}, \tilde{\vartheta})\|\Big)^{\mu}.$$

It follows that

$$\tilde{V}_{n+1}^* - \tilde{V}_n^* \leq -(1-q)\, \tilde{V}_n^* + K_2\|y_n(\tilde{n}, \tilde{y}, \vartheta)\|^{\mu+1} +$$
$$+ K_3\|y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})\|\Big(\sum_{k=\tilde{n}}^{n-1}\|y_k(\tilde{n}, \tilde{y}, \tilde{\vartheta})\|\Big)^{\mu} \leq$$
$$\leq -(1-q)\, \tilde{V}_n^* + K_2 \tilde{V}_n^{*1+\mu} + K_3 \tilde{V}_n^* \Big(\sum_{k=\tilde{n}}^{n-1} \tilde{V}_k^*\Big)^{\mu}.$$

Let $q < q' < 1$, $W_n = \dfrac{1}{q'^{n-\tilde{n}}}\tilde{V}_n^*$; we have

$$W_{n+1} - W_n = \frac{1}{q'^{n-\tilde{n}+1}}\tilde{V}_{n+1}^* - \frac{1}{q'^{n-\tilde{n}}}\tilde{V}_n^* = \frac{1}{q'^{n-\tilde{n}+1}}(\tilde{V}_{n+1}^* - \tilde{V}_n^*) +$$
$$+ \frac{1}{q'^{n-\tilde{n}}}\tilde{V}_n^*\left(\frac{1}{q'} - 1\right) \leq -\frac{1}{q'}W_n\Big[1 - q - K_2\tilde{V}_n^{*\mu} - K_3\Big(\sum_{k=\tilde{n}}^{n-1}\tilde{V}_n^*\Big)^{\mu}\Big] +$$
$$+ W_n\left(\frac{1}{q'} - 1\right) = -W_n\left[1 - \frac{q}{q'} - \frac{K_2}{q'}\tilde{V}_n^{*\mu} - \frac{K_3}{q'}\Big(\sum_{k=\tilde{n}}^{n-1}\tilde{V}_k^*\Big)^{\mu}\right].$$

We deduce that

$$W_{n+1} - W_n \leq -W_n\left[1 - \frac{q}{q'} - \frac{K_2}{q'}q'^{\mu(n-\tilde{n})}W_n^{\mu} - \frac{K_3}{q'}\Big(\sum_{k=\tilde{n}}^{n-1} q'^{k-\tilde{n}}W_k\Big)^{\mu}\right];$$

$W_{\tilde{n}} = \tilde{V}_{\tilde{n}}^* = V_{\tilde{n}}(\tilde{y}) \leq K\|\tilde{y}\|$.

Suppose $W_k \leq l'$ for $k \leq n$; then

$$\left(\sum_{k=\tilde{n}}^{n-1} q'^{k-\tilde{n}}W_k\right)^{\mu} \leq l'^{\mu}\left(\sum_{k=\tilde{n}}^{n-1} q'^{k-\tilde{n}}\right)^{\mu} < l'^{\mu}\frac{1}{(1-q')^{\mu}}, \qquad \text{and}$$

$$1 - \frac{q}{q'} - \frac{K_2}{q'} q'^{\mu(n-\tilde{n})} W_n^{\mu} - \frac{K_3}{q'} \left(\sum_{k=\tilde{n}}^{n-1} q'^{k-\tilde{n}} W_k \right)^{\mu} \geq$$

$$\geq 1 - \frac{q}{q'} - \frac{K_2}{q'} q'^{\mu(n-\tilde{n})} l'^{\mu} - \frac{K_3}{q'} \frac{l'^{\mu}}{(1-q')^{\mu}} > 0$$

if $1 - \dfrac{q}{q'} > l'^{\mu} \left(\dfrac{K_2}{q'} + \dfrac{K_3}{q'(1-q')^{\mu}} \right)$ hence if $l' < \dfrac{(q'-q)^{\frac{1}{\mu}}}{\left[K_2 + \dfrac{K_3}{(1-q')^{\mu}} \right]^{\frac{1}{\mu}}}$.

For such l' and for $W_k \leq l'$, $k \leq n$ we get $W_{n+1} - W_n < 0$ hence $W_{n+1} < < W_n \leq l'$ and the inequality is proved by induction if it is true for $k = \tilde{n}$,

i.e. if $||\tilde{y}|| \leq \dfrac{1}{K} \dfrac{(q'-q)^{\frac{1}{\mu}}}{\left[K_2 + \dfrac{K_3}{(1-q')^{\mu}} \right]^{\frac{1}{\mu}}}$.

For such $\tilde{y}$ we have $W_n \leq l'$ for all $n \geq \tilde{n}$ hence

$W_{n+1} - W_n \leq -\alpha W_n$, $\quad W_{n+1} \leq (1-\alpha) W_n$,
$W_n \leq (1-\alpha)^{n-\tilde{n}} W_{\tilde{n}} = K(1-\alpha)^{n-\tilde{n}} ||\tilde{y}||$;
it follows that $\tilde{V}_n^* \leq K[q'(1-\alpha)]^{n-\tilde{n}} ||\tilde{y}||$ hence
$||y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})|| \leq K[q'(1-\alpha)]^{n-\tilde{n}} ||\tilde{y}||$
and the first assertion of the theorem is proved.

D. Let now $y'_n = y_n(\tilde{n}, \tilde{y}', \tilde{\vartheta}')$, $y''_n = y_n(\tilde{n}, \tilde{y}'', \tilde{\vartheta}''_n)$, $\vartheta'_n = \vartheta_n(\tilde{n}, \tilde{y}', \tilde{\vartheta}')$, $\vartheta''_n = \vartheta_n(\tilde{n}, \tilde{y}''_n, \tilde{\vartheta}''_n)$; suppose $||\tilde{y}'|| \geq ||\tilde{y}''||$, $\vartheta_{\tilde{n}} = \tilde{\vartheta}'$.

Denote $V_n^{**} = V_n(y'_n - y''_n)$; we have

$$V_{n+1}^{**} - V_{n_\mu}^{**} = V_{n+1}[z_{n+1}(n, y'_n - y''_n)] - V_n[z_n(n, y'_n - y''_n)] +$$
$$+ V_{n+1}(y'_{n+1} - y''_{n+1}) - V_{n+1}[z_{n+1}(n, y'_n - y''_n)] \leq$$
$$\leq -(1-q) V_{n\mu}^{**} + K||y'_{n+1} - y''_{n+1} - z_{n+1}(n, y'_n - y''_n)||.$$

But

$$||y'_{n+1} - y''_{n+1} - z_{n+1}(n, y'_n - y''_n)|| =$$
$$= ||Y_n(y'_n, \vartheta'_n) - Y_n(y''_n, \vartheta''_n) - A_n(\delta_n)(y'_n - y''_n)|| \leq$$
$$\leq ||Y_n(y''_n, \vartheta'_n) - Y_n(y''_n, \vartheta''_n)|| +$$
$$+ ||Y_n(y'_n, \vartheta'_n) - Y_n(y''_n, \vartheta'_n) - A_n(\delta_n)(y'_n - y''_n)|| \leq$$

$$\leq \left\| \int_0^1 \frac{\partial Y_n}{\partial \vartheta} (y''_n, \vartheta^{\lambda}_n)(\vartheta'_n - \vartheta''_n) \, d\lambda \right\| +$$

$$+ \left\| \int_0^1 \left(\frac{\partial Y_n}{\partial y} (y^{\lambda}_n, \vartheta'_n) - \frac{\partial Y_n}{\partial y} (0, \delta_n) \right) (y'_n - y''_n) \, d\lambda \right\| =$$

$$= \left\| \int_0^1 \left(\frac{\partial Y_n}{\partial \vartheta} (y''_n, \vartheta^\lambda_n) - \frac{\partial Y_n}{\partial \vartheta} (0, \vartheta^\lambda_n) \right) \mathrm{d}\lambda (\vartheta'_n - \vartheta''_n) \right\| +$$

$$+ \left\| \int_0^1 \left(\frac{\partial Y_n}{\partial y} (y^\lambda_n, \vartheta'_n) - \frac{\partial Y_n}{\partial y} (0, \vartheta'_n) + \frac{\partial Y_n}{\partial y} (0, \delta'_n) - \right.\right.$$

$$\left.\left. - \frac{\partial Y_n}{\partial y} (0, \delta_n) \right) \mathrm{d}\lambda (y'_n - y''_n) \right\| \leq$$

$$\leq K_1 ||y''_n||^\mu ||\vartheta'_n - \vartheta''_n|| + K_1 \sup ||y^\lambda_n||^\mu ||y'_n - y''_n|| +$$
$$+ K_1 ||\vartheta'_n - \delta_n||^\mu ||y'_n - y''_n|| \leq$$
$$\leq K_1 K'^\mu [q'(1 - \alpha)]^{\mu(n - \tilde{n})} ||\tilde{y}'|| \, (||y'_n - y''_n|| + ||\vartheta'_n - \vartheta''_n||) +$$
$$+ K_1 ||\vartheta'_n - \delta_n||^\mu ||y'_n - y''_n||.$$

We know that

$$||\beta_n|| = ||\vartheta'_n - \delta_n|| \leq K_1 \sum_{k=\tilde{n}}^{n-1} ||y'_k|| \leq K_1 K' ||\tilde{y}'|| \sum_{k=\tilde{n}}^{n-1} \left[q'(1 - \alpha) \right]^{k - \tilde{n}},$$

hence $||\Theta'_n - \delta_n||^\mu \leq K_4 ||\tilde{y}'||^\mu$. We have further

$$\vartheta''_{n+1} - \vartheta'_{n+1} - (\vartheta''_n - \vartheta'_n) = \Theta_n(y''_n, \vartheta''_n) - \Theta_n(y'_n, \vartheta'_n) =$$

$$= \int_0^1 \left[\frac{\partial \Theta_n}{\partial y} (y^\lambda_n, \vartheta^\lambda_n) (y''_n - y'_n) + \frac{\partial \Theta_n}{\partial \vartheta} (y^\lambda_n, \vartheta^\lambda_n) (\vartheta''_n - \vartheta'_n) \right] \mathrm{d}\lambda =$$

$$= \int_0^1 \frac{\partial \Theta_n}{\partial y} (y^\lambda_n, \vartheta^\lambda_n) (y''_n - y'_n) \, \mathrm{d}\lambda +$$

$$+ \int_0^1 \left[\frac{\partial \Theta_n}{\partial \vartheta} (y^\lambda_n, \vartheta^\lambda_n) - \frac{\partial \Theta_n}{\partial \vartheta} (0, \vartheta^\lambda_n) \right] (\vartheta''_n - \vartheta'_n) \, \mathrm{d}\lambda$$

hence setting $\gamma_n = \vartheta''_n - \vartheta'_n$ we get

$$||\gamma_{n+1} - \gamma_n|| \leq K_1 ||y''_n - y'_n|| + K_1 K'^\mu [q'(1 - \alpha)]^{\mu(n - \tilde{n})} ||\tilde{y}'||^\mu ||\gamma_n||.$$

It follows that

$$||\gamma_n|| \leq ||\gamma_{\tilde{n}}|| + \sum_{k=\tilde{n}}^{n-1} (K_1 ||y''_k - y'_k|| + K_1 K'^\mu [q'(1 - \alpha)]^{\mu(k - \tilde{n})} ||\tilde{y}'||^\mu ||\gamma_k||)$$

hence

$$||\vartheta''_n - \vartheta'_n|| \leq ||\tilde{\vartheta}'' - \tilde{\vartheta}'|| + K_1 \sum_{k=\tilde{n}}^{n-1} ||y''_k - y'_k|| +$$

$$+ K_1 K'^\mu ||y'||^\mu \sum_{k=\tilde{n}}^{n-1} [q'(1 - \alpha)]^{\mu(k - \tilde{n})} ||\vartheta''_k - \vartheta'_k||$$

for $n \geq \tilde{n} + 1$.

By a discrete analogue of the Gronwall lemma this inequality yields

$$\|\vartheta''_n - \vartheta'_n\| \leq K_5\|\tilde{\vartheta}'' - \tilde{\vartheta}'\| + \sum_{k=\bar{n}}^{n-1} \|y''_k - y'_k\|.$$

Let us estimate $\|\vartheta''_n - \vartheta'_n - \tilde{\vartheta}'' + \tilde{\vartheta}'\| = \|\gamma_n - \gamma_{\bar{n}}\|$. We have

$$\|\gamma_n - \gamma_{\bar{n}}\| \leq \sum_{k=\bar{n}}^{n-1} (K_1\|y''_k - y'_k\| + K_1K'^{\mu}[q'(1-\alpha)]^{\mu(k-\bar{n})}\|\bar{y}'\|^{\mu}\|\gamma_k\|) \leq$$

$$\leq \sum_{k=\bar{n}}^{n-1} (K_1\|y''_k - y_k'\| + K_1K'^{\mu}[q'(1-\alpha)]^{\mu(k-\bar{n})}\|\bar{y}'\|^{\mu}\|\gamma_{\bar{n}}\|) +$$

$$+ K_1K'^{\mu}\|\bar{y}'\|^{\mu} \sum_{k=\bar{n}}^{n-1} [q'(1-\alpha)]^{\mu(k-\bar{n})}\|\gamma_k - \gamma_{\bar{n}}\|$$

which yields the inequality

$$\|\vartheta''_n - \vartheta'_n - \tilde{\vartheta}'' + \tilde{\vartheta}'\| \leq K_6\left(\sum_{k=\bar{n}}^{n-1} \|y'' - y'_k\| + \|\bar{y}'\|^{\mu}\|\vartheta'' - \vartheta'\|\right).$$

Using these inequalities we have

$$V^{**}_{n+1} - V^{**}_n \leq -(1-q)\, V^{**}_n + K_7\|\bar{y}'\|^{\mu}[q'(1-\alpha)]^{\mu(n-\bar{n})}\|y'_n - y''_n\| +$$

$$+ K_8\|\bar{y}'\|^{\mu}\|y'_n - y''_n\| + K_9[q'(1-\alpha)]^{\mu(n-\bar{n})}\|\bar{y}'\|^{\mu} \sum_{k=\bar{n}}^{n-1} \|y''_k - y'_k\| +$$

$$+ K_9[q'(1-\alpha)]^{\mu(n-\bar{n})}\|\bar{y}'\|^{\mu}\|\tilde{\vartheta}'' - \tilde{\vartheta}'\|.$$

Let $q'' = q + K_8 l^{\mu}$ and choose l small enough in order that $q'' < q_1$, i.e. $l < \left(\dfrac{q_1 - q}{K_8}\right)^{\frac{1}{\mu}}$. Suppose $\|y'\| < l$; it follows

$$V^{**}_{n+1} - V^{**}_n \leq -(1-q'')\, V^{**}_n + K_{10}[q'(1-\alpha)]^{\mu(n-\bar{n})}\|\bar{y}'\|^{\mu} \sum_{k=\bar{n}}^{n-1} \|y''_k - y'_k\| +$$

$$+ K_9[q'(1-\alpha)]^{\mu(n-\bar{n})}\|\bar{y}'\|^{\mu}\|\tilde{\vartheta}'' - \tilde{\vartheta}'\|.$$

Let $W^*_n = \dfrac{1}{q''^{n-\bar{n}}} V^{**}_n$; we have

$$W^*_{n+1} = \frac{1}{q''^{n+1-\bar{n}}} V^{**}_{n+1} \leq$$

$$\leq \frac{1}{q''^{n+1-\bar{n}}} (q'' V^{**}_n + K_{10}[q'(1-\alpha)]^{\mu(n-\bar{n})}\|\bar{y}'\| \sum_{k=\bar{n}}^{n-1} \|y''_k - y'_k\| +$$

$$+ K_9[q'(1-\alpha)]^{\mu(n-\bar{n})}\|y'\|^{\mu}\|\tilde{\vartheta}'' - \tilde{\vartheta}'\|) =$$

$$= W^*_n + \frac{K_{10}}{q''}\|\bar{y}'\|^{\mu}\left(\frac{[q'(1-\alpha)]^{\mu}}{q''}\right)^{n-\bar{n}} \sum_{k=\bar{n}}^{n-1} \|y''_k - y'_k\| +$$

$$+ \frac{K_9}{q''}\|\bar{y}'\|^{\mu}\left(\frac{[q'(1-\alpha)]^{\mu}}{q''}\right)^{n-\bar{n}}\|\tilde{\vartheta}'' - \tilde{\vartheta}'\|,$$

hence

$$W_n^* \leq W_{\tilde n}^* + \sum_{k=\tilde n}^{n-1} \frac{K_{10}}{q''} ||\tilde y'||^\mu \left(\frac{[q'(1-\alpha)]^\mu}{q''}\right)^{k-\tilde n} \sum_{j=\tilde n}^{k} ||y_j'' - y_j'|| +$$

$$+ \frac{K_9}{q''} ||\tilde y'||^\mu ||\tilde\vartheta'' - \tilde\vartheta'|| \sum_{k=\tilde n}^{n-1} \left(\frac{[q'(1-\alpha)]^\mu}{q''}\right)^{k-\tilde n} \leq$$

$$\leq V_{\tilde n}^{**} + \frac{K_9}{q''} \cdot \frac{||\tilde y'||^\mu}{1 - \dfrac{[q'(1-\alpha)]^\mu}{q''}} ||\tilde\vartheta'' - \tilde\vartheta'|| +$$

$$+ \frac{K_{10}}{q''} ||\tilde y'||^\mu \sum_{j=\tilde n}^{n-1} \left(\sum_{k=j}^{n-1} \left(\frac{[q'(1-\alpha)]^\mu}{q''}\right)^{k-\tilde n} \right) ||y_j'' - y_j'|| \leq$$

$$\leq K\, ||\tilde y' - \tilde y''|| + \frac{K_9 ||\tilde y'||^\mu}{q'' - [q'(1-\alpha)]^\mu} ||\tilde\vartheta'' - \tilde\vartheta'|| +$$

$$+ \frac{K_{10}}{q''} ||\tilde y'^\mu|| \sum_{j=\tilde n}^{n-1} \left(\frac{[q'(1-\alpha)]^\mu}{q''}\right)^{j-\tilde n} \frac{||y_j'' - y_j'||}{1 - \dfrac{[q'(1-\alpha)]^\mu}{q''}}$$

hence

$$||y_n' - y_n''|| \leq V_n^{**} = q''^{n-\tilde n} W_n^* \leq$$

$$\leq K q''^{n-\tilde n} ||\tilde y' - \tilde y''|| + K_{11} ||\tilde y'||^\mu ||\vartheta'' - \vartheta'||\, q''^{n-\tilde n} +$$

$$+ K_{12} ||\tilde y'||^\mu q''^{n-\tilde n} \sum_{j=\tilde n}^{n-1} \left(\frac{[q'(1-\alpha)]^\mu}{q''}\right)^{j-\tilde n} ||y_j'' - y_j'||.$$

Let $u_n = \dfrac{1}{q_1^{n-\tilde n}} ||y_n' - y_n''||$; we have

$$q_1^{n-\tilde n} u_n \leq K q''^{n-\tilde n} ||\tilde y' - \tilde y''|| + K_{11} ||\tilde y'||^\mu ||\tilde\vartheta'' - \tilde\vartheta'||\, q''^{n-\tilde n} +$$

$$+ K_{12} ||\tilde y'||^\mu q''^{n-\tilde n} \sum_{j=\tilde n}^{n-1} \left(\frac{[q'(1-\alpha)]^\mu}{q''}\right)^{j-\tilde n} q_1^{j-\tilde n} u_j,$$

$$u_n \leq K\, ||\tilde y' - \tilde y''|| + K_{11} ||\tilde y'||^\mu ||\tilde\vartheta'' - \tilde\vartheta'|| +$$

$$+ K_{12} ||\tilde y'||^\mu \sum_{j=\tilde n}^{n-1} \left(\frac{[q'(1-\alpha)]^\mu q_1}{q''}\right)^{j-\tilde n} u_j$$

hence $u_n \leq K_{13}(||\tilde y' - \tilde y''|| + ||\tilde y'||^\mu ||\tilde\vartheta'' - \tilde\vartheta'||)$.

It follows that

$$||y_n' - y_n''|| \leq K_{13} q_1^{n-\tilde n}(||\tilde y' - \tilde y''|| + ||\tilde y'||^\mu ||\tilde\vartheta'' - \tilde\vartheta'||)$$

$$||\vartheta_n'' - \vartheta_n' - \tilde\vartheta'' + \tilde\vartheta'|| \leq K_{14}(||\tilde y' - \tilde y''|| + ||\tilde y'||^\mu ||\tilde\vartheta'' - \tilde\vartheta'||)$$

and the theorem is proved.

III. The theorem on invariant manifolds.

We may now prove the following theorem on the existence of exponentially stable invariant manifolds.

Theorem 4. *Consider the discrete system*

$$y_{n+1} = Y_n^\circ(y_n, \vartheta_n) + \varepsilon Y_n^1(y_n, \vartheta_n, \varepsilon)$$
$$\vartheta_{n+1} = \vartheta_n + \Theta_n^\circ(y_n, \vartheta_n) + \varepsilon \Theta_n^1(y_n, \vartheta_n, \varepsilon)$$

Suppose that Y_n°, Θ_n° *verify all the conditions of theorem* 3 *and* Y_n^1, Θ_n^1 *verify the regularity conditions of theorem* 2. *Then for* $|\varepsilon|$ *small enough there exist* $p_n : \mathfrak{C} \to C$ *such that*

a) $||p_n(\vartheta)|| \leq l(\varepsilon)$,

b) $||p_n(\vartheta_1) - p_n(\vartheta_2)|| \leq L(\varepsilon)\,||\vartheta_1 - \vartheta_2||$, $\lim\limits_{\varepsilon\to 0} l(\varepsilon) = \lim\limits_{\varepsilon \to 0} \mathrm{L}(\varepsilon) = 0$;

c) $||\tilde{y}|| \leq l$ *implies* $||y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}) - p_n(\vartheta_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))|| \leq K' q^{n-\tilde{n}} ||\tilde{y} - p_{\tilde{n}}(\tilde{\vartheta})||$,

d) $\tilde{y} = p_{\tilde{n}}(\tilde{\vartheta})$ *implies* $y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}) = p_n(\vartheta_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))$

and the solution is defined for all integers n.

e) p_n *is unique with the above properties,*

f) 1⁰. *If* $Y_{n+\nu}^\circ(y, \vartheta) = Y_n^\circ(y, \vartheta)$, $Y_{n+\nu}^1(y, \vartheta, \varepsilon) = Y_n^1(y, \vartheta, \varepsilon)$,
$\Theta_{n+\nu}^\circ(y, \vartheta) = \Theta_n^\circ(y, \vartheta)$, $\Theta_{n+\nu}^1(y, \vartheta, \varepsilon) = \Theta_n^1(y, \vartheta, \varepsilon)$
then $p_{n+\nu} \equiv p_n$.

2⁰. *If* $Y_n^\circ(y, \vartheta + \omega) = Y_n^\circ(y, \vartheta)$, $Y_n^1(y, \vartheta + \omega, \varepsilon) = Y_n^1(y, \vartheta, \varepsilon)$,
$\Theta_n^\circ(y, \vartheta + \omega) = \Theta_n^\circ(y, \vartheta)$, $\Theta_n^1(y, \vartheta + \omega, \varepsilon) = \Theta_n^1(y, \vartheta, \varepsilon)$,
then $p_n(\vartheta + \omega) = p_n(\vartheta)$.

g) *If* Y_n°, Y_n^1, Θ_n°, Θ_n^1 *are almost periodic sequences (uniformly with respect to* y, ϑ, ε*) then* p_n *is an almost-periodic sequence.*

Proof. We have to verify that the discrete system considered verifies all conditions of theorem 1. Let y_n°, ϑ_n° be defined by the system for $\varepsilon = 0$. From theorem 3 we have $||y_n^\circ(\tilde{n}, \tilde{y}, \vartheta)|| \leq K' q'^{n-\tilde{n}} ||\tilde{y}||$ for $n \geq \tilde{n}$, $||\tilde{y}|| \leq l$. Let N be such that $K' q'^N < \frac{1}{3}$; we have for $\tilde{n} \leq n \leq \tilde{n} + 2N$ using theorem 2

$||y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})|| \leq ||y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}) - y_n^\circ(\tilde{n}, \tilde{y}, \tilde{\vartheta})|| + ||y_n^\circ(\tilde{n}, \tilde{y}, \tilde{\vartheta})|| \leq \beta_N |\varepsilon| + K' l \leq$
$\leq \frac{3}{4} H$ for $|\varepsilon|$ and l small enough and the solution is defined for such n. Further, for $n \geq \tilde{n} + N$ we have $||y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})|| \leq \beta_N |\varepsilon| + K' q'^N l < \beta_N |\varepsilon| +$
$+ \frac{1}{3} l < l$, for $|\varepsilon|$ small enough. Condition 1⁰ of theorem 1 is verified.

We have then by theorem 3

$$||y_n^\circ(\tilde{n}, \tilde{y}', \vartheta') - y_n^\circ(\tilde{n}, \tilde{y}'', \vartheta'')|| \leq K' q'^{n-\tilde{n}} (||\tilde{y}' - \tilde{y}''|| + \varrho ||\vartheta' - \vartheta''||)$$
$$||\vartheta_n^\circ(\tilde{n}, \tilde{y}', \vartheta') - \vartheta_n^\circ(\tilde{n}, \tilde{y}'', \vartheta'') - \vartheta' + \vartheta''|| \leq K' (||\tilde{y}' - \tilde{y}''|| + \varrho ||\vartheta' - \vartheta''||).$$

It follows using theorem 2 that

$$\begin{aligned}
&||y_n(\tilde n, \bar y') - y_n(\tilde n, \bar y'', \vartheta)|| + L\,||\vartheta_n(\tilde n, \bar y', \vartheta) - \vartheta_n(\tilde n, \bar y'', \vartheta)|| \leq \\
&\quad \leq ||y_n(\tilde n, \bar y', \vartheta) - y_n(\tilde n, \bar y'', \vartheta) - y_n^\circ(\tilde n, \bar y', \vartheta) + y_n^\circ(\tilde n, \bar y'', \vartheta)|| + \\
&\quad + L\,||\vartheta_n(\tilde n, \bar y', \vartheta) - \vartheta_n(\tilde n, \bar y'', \vartheta) - \vartheta_n^\circ(\tilde n, \bar y', \vartheta) + \vartheta_n^\circ(\tilde n, \bar y'', \vartheta)|| + \\
&\quad + ||y_n^\circ(\tilde n, \bar y', \vartheta) - y_n^\circ(\tilde n, \bar y'', \vartheta)|| + L\,||\vartheta_n^\circ(\tilde n, \bar y', \vartheta) - \vartheta_n^\circ(\tilde n, \bar y'', \vartheta)|| \leq \\
&\quad \leq \alpha_{2N}(\varepsilon)\,||\bar y' - \bar y''|| + L\alpha_N(\varepsilon)\,||\bar y' - \bar y''|| + K'q'^N||\bar y' - \bar y''|| + \\
&\quad + LK'||\bar y' - \bar y''||
\end{aligned}$$

for $\tilde n + N \leq n \leq \tilde n + 2N$, hence for $|\varepsilon|$, L small enough we get

$$\begin{aligned}
&||y_n(\tilde n, \bar y', \vartheta) - y_n(\tilde n, \bar y'', \vartheta)|| + L\,||\vartheta_n(\tilde n, \bar y', \vartheta) - \vartheta_n(\tilde n, \bar y'', \vartheta)|| \leq \\
&\quad \leq \alpha_1||y' - y''||, \quad \alpha_1 < 1,
\end{aligned}$$

and condition 2⁰ of theorem 1 is verified.

In order to verify condition 3⁰ a) we see that for $\tilde n \leq n \leq \tilde n + 2N$,

$||\bar y' - \bar y''|| \leq L\,||\vartheta' - \vartheta''||$ we have

$$\begin{aligned}
&||\vartheta_n(\tilde n, \bar y', \vartheta') - \vartheta_n(\tilde n, \bar y'', \vartheta'') - \vartheta' + \vartheta''|| \leq \\
&\quad \leq ||\vartheta_n(\tilde n, \bar y', \vartheta') - \vartheta_n(\tilde n, \bar y'', \vartheta'') - \vartheta_n^\circ(\tilde n, \bar y', \vartheta') + \vartheta_n^\circ(\tilde n, \bar y'', \vartheta'')|| + \\
&\quad + ||\vartheta_n^\circ(\tilde n, \bar y', \vartheta') - \vartheta_n^\circ(\tilde n, \bar y'', \vartheta'') - \vartheta' + \vartheta''|| \leq \\
&\quad \leq \alpha_{2N}(\varepsilon)\,(||\bar y' - \bar y''|| + ||\vartheta' - \vartheta''||) + K'(||\bar y' - \bar y''|| + \varrho||\vartheta' - \vartheta''||) \leq \\
&\quad \leq \alpha_{2N}(\varepsilon)\,(1 + L)\,||\vartheta' - \vartheta''|| + K'(L + \varrho)\,||\vartheta' - \vartheta''|| \leq \alpha_2||\vartheta' - \vartheta''||,
\end{aligned}$$

$\alpha_2 < \dfrac{1}{3}$ if $|\varepsilon|$, L and ϱ are small enough.

We have then for $\tilde n + N \leq n \leq \tilde n + 2N$, $||\bar y' - \bar y''|| \leq L\,||\vartheta' - \vartheta''||$ the estimation

$$\begin{aligned}
&||y_n(\tilde n, \bar y', \vartheta') - y_n(\tilde n, \bar y'', \vartheta'')|| \leq ||y_n^\circ(\tilde n, \bar y', \vartheta') - y_n^\circ(\tilde n, \bar y'', \vartheta'')|| + \\
&\quad + ||y_n(\tilde n, \bar y', \vartheta') - y_n(\tilde n, \bar y'', \vartheta'') - y_n^\circ(\tilde n, \bar y', \vartheta') + y_n^\circ(\tilde n, \bar y'', \vartheta'')|| \leq \\
&\quad \leq K'q'^N(||\bar y' - \bar y''|| + \varrho\,||\vartheta' - \vartheta''||) + \alpha_{2N}(\varepsilon)\,(||\bar y' - \bar y''|| + ||\vartheta' - \vartheta''||) \leq \\
&\quad \leq \frac{1}{3}\,(L + \varrho)\,||\vartheta' - \vartheta''|| + \alpha_{2N}(\varepsilon)\,(L + 1)\,||\vartheta' - \vartheta''|| \leq \\
&\quad \leq (1 - \alpha_2)\,L\,||\vartheta' - \vartheta''||
\end{aligned}$$

if $|\varepsilon|$ and ϱ are small enough.

Condition 4⁰ is obvious from the regularity conditions.

It is easy to see that conditions in f) and g) theorem 4 imply conditions in f) and g) theorem 1.

Theorem 4 is thus proved.

It is useful to consider the "autonomous" case

$$\begin{aligned}
y_{n+1} &= Y^\circ(y_n, \vartheta_n) + \varepsilon Y^1(y_n, \vartheta_n, \varepsilon) \\
\vartheta_{n+1} &= \vartheta_n + \Theta^\circ(y_n, \vartheta_n) + \varepsilon\Theta^1(y_n, \vartheta_n, \varepsilon)
\end{aligned}$$

An invariant manifold for such system will be a function $p\colon \mathfrak{C} \to C$ such that if $\tilde{y} = p(\vartheta)$ then $Y^\circ(\tilde{y}, \vartheta) + \varepsilon Y^1(\tilde{y}, \vartheta) = p(\vartheta + \Theta^\circ(\tilde{y}, \vartheta) + \varepsilon\Theta^1(y, \vartheta, \varepsilon))$, i.e. an invariant manifold for the mapping defined by the system.

BIBLIOGRAPHY

[1] J. Kurzweil, *Exponentially Stable Integral oranifolds. Averaying Principle and Continuous dependence on a Parameter. Czech. Math. J. 16 (91), 1966.*

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ON ORDINARY LINEAR DIFFERENTIAL EQUATIONS OF HIGH ORDER

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We shall briefly report here on the ordinary linear differential equations of high order whereby we suppose that the differential equations are of the normal form, and that their coefficients are integrable and bounded, respectively.

If a function satisfies a sequence of such differential equations with increasing order, then this is the property of the function, which is, apart from the differentiability of all orders connected in a certain way with the regularity of the function and it is so in the case of real as well as in the case of complex functions.

Let us begin with the real case and let us suppose the differential equation is in its normal form:

$$y^{(n)} + \sum_{i=0}^{n-1} \sigma_i y^{(i)} = \varphi \tag{1}$$

where the σ_i and φ are L-integrable, for example on the interval $[0, L]$, $L > 0$.

Without loss of generality, we can now limit ourselves to the solution y with

$$y(0) = y'(0) = \ldots = y^{(n-1)}(0) = 0, \tag{2}$$

otherwise there is only the right side of (1) correspondingly to modify.

We begin with an estimate:

Let on the interval $[0, L]$

$$S_i = \sup |\sigma_i|, \qquad \Phi = \sup |\varphi|$$

and let L *be so small that*

$$\sum_{i=0}^{n-1} S_i \frac{L^{n-i}}{(n-i)!} < 1.$$

Then for the solution y with (2) *holds the estimate*[1])

$$(3) \qquad |y(x)| \leq \frac{\Phi L \dfrac{x^{n-1}}{(n-1)!}}{1 - \sum\limits_{i=0}^{n-1} S_i \dfrac{L^{n-i}}{(n-i)!}} \, .$$

Now for a fixed X is the quotient $\dfrac{X^{n-1}}{(n-1)!} \to 0$ for $n \to \infty$; thus if the σ_i are equibounded for all $i = 1, 2, \ldots,$

$$|\sigma_i| < K$$

and if

$$K(e^L - 1) < 1$$

then for sufficiently great n the solution y with (2) is arbitrarily small. Therefore is holds:

Any differentiable function $f \in C^\infty$ on $[0, L]$ with $f^{(i)}(0) = 0$ for $i = 0, \ldots,$ which is not identically 0 *can not be represented as the solution of linear differential equations of the corresponding high order and of coefficients $< K$.*[2])

Naturally such a function is not regular.

We can extend this theorem for the functions $f \in C^\infty$ on $[0, L]$ which are not regular with arbitrary initial values $f^{(i)}(0)$.

There are functions $f \in C^\infty$ with $f^{(i)}(0)$ groowing so that

$$\overline{\lim} \sqrt[n]{\left|\frac{f^{(n)}(0)}{n!}\right|} = \infty.$$

For such (surely not regular) functions it can be easily shown that for an increasing sequence of natural numbers $(n_1, n_2, \ldots)$ they can not satisfy a sequence of linear differential equations of the order n_k if k is sufficiently great and the coefficients remain bounded for $x = 0$.

Let us consider therefore only the case of a function $f \in C^\infty$ on $[0, L]$ for which

$$\overline{\lim} \sqrt[n]{\left|\frac{f^{(n)}(0)}{n!}\right|} < \infty.$$

[1]) Proc. Amer. Math. Soc. 17 (1966), 321—324.

[2]) l. c.

Let f satisfy the differential equation

$$y^{(n)} + \sum_{i=0}^{n-1} \sigma_i y^{(i)} = \varphi.$$

If we put

$$P(x) = \sum_{i=0}^{n-1} f^{(i)}(0) \frac{x^{(i)}}{i!}$$

then $\bar{f}(x) = f(x) - P(x)$ satisfies the differential equation

$$\bar{y}^{(n)} + \sum_{i=0}^{n-1} \sigma_i \bar{y}^{(i)} = \varphi - \sum_{i=0}^{n-1} \sigma_i P^{(i)} = \bar{\varphi}$$

with $\bar{y}(0) = \bar{y}'(0) = \ldots = \bar{y}^{(n-1)}(0) = 0$.

Now we can apply to our differential equation with $\bar{y}, \bar{f}$ our above mentioned estimation and we see that $\bar{y}$ by a corresponding $L > 0$ becomes arbitrarily small for all x and sufficiently great n.

We have supposed f to be non-regular at $x = 0$, therefore is for every natural $m > 0$

$$\sup_x \left| f(x) - \sum_0^{m-1} f^{(i)}(0) \frac{x^i}{i!} \right| = h_m > 0$$

and naturally also

$$\inf h_m = h > 0$$

(otherwise f would be represented as a polynomial or a power series).

Then there exists for every m a point x_m with

$$\left| f(x_m) - \sum_0^{m-1} f^{(i)}(0) \frac{x_m^i}{i!} \right| \geq h.$$

On the other hand $\bar{f}$ is arbitrarily small for $n \to \infty$.

So the inequalities

$$|\sigma_i| < K, \qquad |\varphi| < K \qquad \text{and} \qquad n > n(K)$$

can not exist simultaneously for a $n(K)$.

Then f surely does not satisfy a differential equation with bounded coefficients for a sufficiently great order n.[3]

A simple consequence follows from this:

[3] l. c. 1.) S. 321.

If $f \in C^\infty$ is on the interval $[0, L]$ *and if there exists such a sequence of linear differential equations for every sufficiently great order with equibounded coefficients that f satisfies every one of these differential equation, then f is regular at every point.*

We will now deal with the regular functions f and we shall consider them either in the real domain or in the complex plane.

There are simple examples of regular functions which can be represented as the solutions of differential equations of an arbitrary high order with bounded coefficients, and there are other examples of such functions which can not be represented as the solutions of such differential equations.

It is known now, that for the regular functions a number can be determined upon which the behaviour relative to the representation as a solution of differential equations of high order depends. This number depends — likewise as the order of an entire function — only on the behaviour of the power series which represents this function,

First we introduce this number:[4]

Let f be regular at the point 0 and let

$$\sup_{n\geq 1} \sqrt[n]{|f^{(n)}(0)|} < \infty,$$

then

$$f(x) = \sum_{0}^{\infty} f^{(n)}(0) \frac{z^n}{n!}$$

is the entire function of an order ≤ 1.

If we put for $n > 0$

$$\sup_{i\geq n} \sqrt[i]{|f^{(i)}(0)|} = A_n(0),$$

then $A_n(0) \geq 0$ is decreasing and thus the sequence $\{A_n(0)\}$ converges.

For $i > 0$ and an arbitrary z

$$\sqrt[i]{|f^{(i)}(z)|} \leq \sqrt[i]{\sum_{n=0}^{\infty} [A_i(0)]^{n+i} \frac{|z|^n}{n!}} = A_i(0) \,.\, e^{\frac{1}{i} \,.\, A_i(0) \,.\, |z|}.$$

Then exists also

$$\sup_{i\geq n} \sqrt[i]{|f^{(i)}(z)|} = A_n(z)$$

[4] Monatshefte für Math. 70 (1966), S. 330—336.

for an arbitrary z and $n > 0$ and

$A_n(z) \leq A_n(0) e^{\frac{1}{n} A_n(0) \cdot |z|}$ holds.

For every z exists also

$$\lim A_n(z) = A(z)$$

and following the above mentioned

$$A(0) = A(z)$$

is constant and we denote the constant

$$\lim_n \sup_{i \geq n} \sqrt[i]{|f^{(i)}(z)|} = ||f||$$

as the degree of convergence of f.

For an arbitrary function f regular at the point 0

$$\sup_n \sqrt[n]{|f^{(n)}(0)|} = \infty$$

(this expression is then for every z infinit!)
we put correspondingly $||f|| = \infty$.

Now our problem is: *when can a regular function be represented as a solution of a linear ordinary differential equation*

$$y^{(n)} + \sum_{i=0}^{n-1} \sigma_i y^{(i)} = \varphi$$

of a high order.

We put for the function f

$$S_i = \sqrt[i]{|f^{(i)}(0)|} \qquad i = 1, 2, \ldots \qquad \text{and} \qquad S_0 = |f(0)|$$

and

$$\frac{S_n^n}{1 + S_0 + S_1^1 + \ldots + S_{n-1}^{n-1}} = q_n$$

We see at once: when the coefficients σ_i and φ at the point $z = 0$ are all absolute $< q_n$

$$|\sigma_i| < q_n, \qquad |\varphi| < q_n \qquad \text{for } z = 0$$

then f does not satisfv our differentialbe equation, because there would be for $z = 0$

$$S_n^n = q_n[1 + S_0 + S_1^1 + \ldots + S_{n-1}^{n-1}] > |\varphi| + \sum_{i=0}^{n-1} |\sigma_i| \, S_i^i .$$

Now is for our function f, resp. for the sequence of numbers S_i is

$$\overline{\lim}\, q_n = q > 0 \cdot$$

then there exists a sequence of natural numbers $(n_1, n_2, \ldots)$ with

$$q_{n_k} \to q$$

and f is surely not the solution of a differential equation

$$y^{(n_k)} + \sum_{i=0}^{n_k-1} \sigma_i^{(n_k)} y^{(i)} = \varphi^{(n_k)}$$

if the coefficients

$$|\sigma_i^{(n_k)}| < \frac{q}{2}, \qquad |\varphi^{(n_k)}| < \frac{q}{2} \qquad \text{for } z = 0$$

for sufficiently great k.

And now we can show:

If the degree of convergence is for our function f, $||f|| > 1$ (also for $||f|| = \infty$) then it is always

$$q = \overline{\lim}\, q_n = \overline{\lim} \frac{S_n^n}{1 + S_1^1 + \ldots + S_{n-1}^{n-1}} > 0.$$

If this would not be the case, then the

$$\frac{S_n^n}{1 + S_1^1 + \ldots + S_{n-1}^{n-1}} = \varepsilon_n \to 0$$

would converge, but then would be for an n

$$S_n^n = \varepsilon_n(1 + S_0 + S_1^1 + \ldots + S_{n-1}^{n-1})$$
$$S_{n+1}^{n+1} = \varepsilon_{n+1}(1 + S_0 + S_1^1 + \ldots + S_{n-1}^{n-1})(1 + \varepsilon_n)$$
$$S_{n+2}^{n+2} = \varepsilon_{n+2}(1 + S_0 + S_1^1 + \ldots + S_{n-1}^{n-1})(1 + \varepsilon_n)(1 + \varepsilon_{n+1})$$
$$S_{n+m}^{n+m} = \varepsilon_{n+m}(1 + S_0 + S_1^1 + \ldots + S_{n-1}^{n-1})(1 + \varepsilon_n) \ldots (1 + \varepsilon_{n+m-1})$$

Because of $\varepsilon_K \to 0$ then it would be for a sufficiently great n

$$S_{n+m}^{n+m} < (1 + S_1^1 + \ldots + S_{n-1}^{n-1})(1 + \eta)^m \cdot \varepsilon_{n+m}$$

with an arbitrary small $\eta > 0$.

On account of $||f|| > 1$ is at least for one sequence $(m_1, m_2, \ldots)$ with an $\varepsilon > 0$ arbitrarily small:

$$S_{n+m_k}^{n+m_k} > (||f|| - \varepsilon)^{m_k}$$

which can not hold with $||f|| > 1$ after what has been mentioned above.

Then it holds: *If $||f|| > 1$, then there exists a sequence of natural numbers $(n_1, n_2, \ldots)$ so that f is surely not the solution of a linear differential equation, of which the coefficients for $z = 0$ absolute $< \frac{q}{2}$ are with an order n_k for every sufficiently great k*[5].

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[5] l. c. S. 336.

If finally $||f|| < 1$, *then for a given and sufficiently great n*

$$\sqrt[n]{|f^{(n)}(z)|} < q < 1 \qquad \textit{hence}$$
$$f^{(n)}(z) \to 0 \qquad \text{for} \qquad n \to \infty.$$

Then f always satisfies a differential equation with arbitrarily small coefficients of sufficiently high order.

We have for $||f|| = 1$ simple examples as $f(z) = e^z$, for which f satisfies a differential equation of any high order with bounded coefficients, but I do not know an example with $||f|| = 1$, in which this would not be the case.

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ON THE TRANSFORMATION OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE n^{th} ORDER

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We call the equation of the following form

$$(0.1)\quad \sum_{i=0}^{n} \binom{n}{i} a_i(x)\, y^{(n-i)}(x) = 0, \quad a_i \in C_0(I_1), \quad i = 0, 1, \ldots, n, \quad a_0 \not\equiv 0 \text{ in } I_1,$$

a *general* homogeneous linear differential equation of the n^{th} order. Instead of "homogeneous linear differential equation" we shall call it simply "equation".

The equation (0.1) is *normal (semi-canonical)* [*canonical*] if $a_0 \equiv 1$ ($a_1 \equiv 0$) [$a_1 \equiv a_2 \equiv 0$]. If $a_i/a_0 \in C_0(I_1)$, $i = 1, 2, \ldots, n$, then we call the equation

$$(0.2)\quad y^{(n)} + \sum_{i=1}^{n} \binom{n}{i} (a_i/a_0)\, y^{(n-i)} = 0$$

the normal form of the equation (0.1).

We call two equations *quasi-identical* if they have the identical range of definition and the same fundamental system of solution. We denote the quasi-identical equations by the sign $\doteq$. F.i. (0.1) $\doteq$ (0.2).

1. Perturbated equations.

Let us have the equation

$$(\mathrm{a})\quad y^{(n)}(x) + \sum_{i=1}^{n} \binom{n}{i} a_i(x)\, y^{(n-i)}(x) = 0, \quad a_i \in C_{n-i}(I_1), \quad i = 1, 2, \ldots, n$$

Let $u(x)$ be an arbitrary solution of the equation

$$(\mathrm{u})\qquad u'' + \frac{3}{n+1}(a_2 - a_1' - a_1^2)\, u = 0.$$

We call the equation (u) the accompanying equation to the equation (a).

By $(n-1)$ fold iteration of the equation of the first order

$$(1.1) \qquad P_1(y) = u^2y' + [a_1u^2 - (n-1)\,uu']\,y = 0$$

we obtain an equation of the n^{th} order

$$(1.2) \qquad P_n(y) = P_1[P_{n-1}(y)] = u^{2n} \sum_{i=0}^{n} \binom{n}{i} f_i^n(a_1, a_2)\, y^{(n-i)} = \\ = u^{2n} I_n(y; a_1, a_2) = 0,$$

where the function

$$(1.3) \qquad f_i^n(a_1, a_2), \qquad i = 0, 1, \ldots, n$$

is for the given n a polynomial of the elements a_1, a_2 of the dimension i, which we obtain as a solution of a certain difference equation of the first order — see [1; pp. 39—48]. For instance there is

$$f_0^n(a_1, a_2) = 1, \qquad f_1^n(a_1, a_2) = a_1, \qquad f_2^n(a_1, a_2) = a_2,$$

$$f_3^n(a_1, a_2) = \frac{3}{2}\, a_2' - \frac{1}{2}\, a_1'' + 3a_1a_2 - 3a_1'a_1 - 2a_1^3.$$

We call the polynomial $f_i^n(a_1, a_2)$ the *iterated polynomial* of the dimension i, the equation (1.2) we call an *iterated equation*. Let us note yet, that we take for an iterated equation every equation, which is quasi-identical with the equation (1.2).

Put

$$(1.4) \qquad \omega_i^n = a_i - f_i^n(a_1, a_2), \qquad i = 3, 4, \ldots, n.$$

With the aid of (1.4) we can write this in the form

$$(\omega) \quad I_n(y; a_1, a_2) + \sum_{i=3}^{n} \binom{n}{i} \omega_i^n\, y^{(n-i)} = 0,$$

where $I_n(y; a_1, a_2) = 0$ is the normal form of the equation (1.2). We call the function ω_i^n the coefficient of perturbation of the dimension i of the equation (a), the equation (ω) we call the perturbated form of the equation (a) or the perturbated equation of the equation (a), briefly *the perturbated equation.*

The following can be proved — see [1; pp.50]

Theorem 1. *The equation* (a) *is iterated just then when its fundamental system is the function*

$$(1.5) \quad u^{n-k}v^{k-1} \exp\left\{- \int_{x_0}^{x} a_1\, ds\right\}, \qquad x_0 \in I_1, \qquad k = 1, 2, \ldots, n,$$

where u and v are linearly independent solutions of the equation (u).

The perturbated equation (ω) comes in handy for the study of the asymptotic and oscillatory properties of the equation (a). We give at least two examples on the understanding that the equation (a) is semi-canonical in the interval $I_1 \equiv \equiv \langle x_0, \infty)$, i.e. $a_1 \equiv 0$ in I_1. Let us put for the sake of simplicity $A = \frac{3}{n+1} a_2$.

Example 1. Let the following assumptions hold:

$$(1.6) \qquad A^{(r)} = 0(1), \qquad r = 0, 1, \ldots, n-5,$$

$$\int_{x_0}^{\infty} x^{-2s}|Ax^{4s} + \varepsilon c^2| \, dx < \infty, \qquad c > 0, \qquad s < \frac{1}{2}, \qquad (c, s \in E_1), \qquad \varepsilon = \pm 1.$$

$$\int_{x_0}^{\infty} x^{2s(k+j-1)}|\omega_k^n| \, dx < \infty, \qquad k = 3, 4, \ldots, n; \qquad j = 0, 2, 3, \ldots, n-k.$$

Hence the equation (a) has in the case of $\varepsilon = 1$ the fundamental system

$$\exp \{\beta(n - 2\nu + 1) x^{1-2s}\} [1 + o(1)], \qquad \nu = 1, 2, \ldots, n,$$

in the case $\varepsilon = -1$

$$[\sin (\beta x^{1-2s})]^{n-\nu}[\cos (\beta x^{1-2s})]^{\nu-1} + o(1), \qquad \nu = 1, 2, \ldots, n,$$

where $\beta = \frac{c}{1-2s}$ — see [2; pp. 184].

For $s = 0$ we obtain the following statement:
Let the following hold: formula (1.6),

$$\int_{x_0}^{\infty} |A + \varepsilon c^2| \, dx < \infty, \qquad \int_{x_0}^{\infty} |\omega_k^n| \, dx < \infty, \qquad k = 3, 4, \ldots, n.$$

Then the equation (a) has in the case $\varepsilon = 1$ the fundamental system

$$e^{c(n-2\nu+1)x}[1 + o(1)], \qquad \nu = 1, 2, \ldots, n,$$

and in the case of $\varepsilon = -1$

$$[\sin cx]^{n-\nu}[\cos cx]^{\nu-1} + o(1), \qquad \nu = 1, 2, \ldots, n.$$

Example 2. Let $\omega_n^n \geqq 0$. If the equation (u) is oscillatory, then every solution of the equation

$$(1.7) \qquad I_n(y; 0, a_2) + \omega_n^n y = 0,$$

which has at least one zero point oscillates, too. If n is even, then the equation (1.7) is strictly oscillatory.

We note yet that M. GREGUŠ dealt in his paper with the properties of the integrals of the equation

$$(1.8) \qquad I_n(y; 0, 0) + n\omega_{n-1}^n y' + \omega_n^n y = 0$$

— see [21].

2. Transformation

We denote by the symbol $m(I_{1x})$ where $\emptyset \neq I_{1x} \subset I_1$ the set of the elements which are defined as follows: The ordered pair of functions $\{T(x), u(x)\}$ is an element of the set $m(I_{1x})$ if

$$T(x) \in C_{n+1}(I_{1x}), \qquad u(x) \in C_n(I_{1x}), \qquad T'(x) \,.\, u(x) \neq 0 \text{ in } I_{1x}.$$

Let us choose an arbitrary element $\{T(x), u(x)\} \in m(I_{1x})$. If we put into the equation (a)

$$y(x) = u(x)\, Z(x), \qquad t = T(x),$$

we have the equation

$$(\bar{\text{a}}) \quad u(x)\,[T'(x)]^n [z^{(n)}(t) + \sum_{i=1}^{n} \tbinom{n}{i}\, \bar{a}_i(t)\, z^{(n-i)}(t)] = 0, \qquad t \in I_{2t} = T(I_{1x}),$$

where we put $x = T_{-1}(t)$ $[T_{-1}(t)$ is the inverse function to the function $T(x)]$, $z(t) = Z[T_{-1}(t)]$. We call the equation $(\bar{\text{a}})$ *the image of the equation* (a) *in the interval* I_{1x} *with the coordinates* $T(x)$, $u(x)$ and we denote it by the sign $(\bar{\text{a}})$ $\{T(x), u(x)\}$. It can be proved that in the interval I_{2t} the following relations hold

$$\bar{a}_i(t) = [T'(x)]^{-i} \sum_{k=0}^{i} \tbinom{i}{k}\, a_k(x)\, \Phi_{i-k}^{n,i}[\eta(x), \zeta(x)], \qquad x = T_{-1}(t), \qquad i = 0, 1, \ldots, n,$$

where $\eta = T''/T'$, $\zeta = u'/u$ are the transformed coordinates of the image $(\bar{\text{a}})$ $\{T(x), u(x)\}$ and

$$\Phi_{i-k}^{n,i}(\eta, \zeta) = \sum_{j=k}^{i} \tbinom{i-k}{j-k}\, \varphi_{i-j}^{n-j}(\eta)\, \chi_{j-k}(\zeta),$$

see [3; 3,1.10]. The function φ_{i-j}^{n-j} resp. χ_{j-k} is the polynomial of the element η resp. ζ of the dimension $i - j$ resp. $j - k$. We obtain both functions as a solution of certain linear difference equations of the first order — see [3; (2,1.6), (2,2.3)]. The difference equation which satisfies the polynomial χ is specially simple and therefore we write it here:

$$\chi_k(\zeta) = \zeta\chi_{k-1}(\zeta) + [\chi_{k-1}(\zeta)]', \qquad \chi_0(\zeta) = 1.$$

From this follows f.i. $\chi_1(\zeta) = \zeta$, $\chi_2(\zeta) = \zeta^2 + \zeta'$, and so on.

We introduce yet some explicit polynomials: $\varphi_0^m(\eta) = 1$,

$$\varphi_1^m(\eta) = \frac{m-1}{2}\,\eta\,, \qquad \varphi_2^m(\eta) = \frac{m-2}{3}\left(\frac{3m-5}{4}\,\eta^2 + \eta'\right),$$

$$\Phi_0^{n,i}(\eta, \zeta) = 1, \qquad \Phi_1^{n,i}(\eta, \zeta) = \frac{n-i}{2}\,\eta + \zeta,$$

$$\Phi_2^{n,i}(\eta, \zeta) = (n-i)\left[\frac{3n-3i+1}{12}\,\eta^2 + \frac{1}{3}\,\eta' + \eta\zeta\right] + \zeta^2 + \zeta'.$$

By the symbol $o_a(I_{1x})$ $[p_a(I_{1x})]$ $\{k_a(I_{1x})\}$ we denote the set of all images [semi-canonical images] {canonical images} of the equation (a) in the interval I_{1x}.

If we choose

$$U(x) = c|T'(x)|^{\frac{1-n}{2}} \exp\left\{-\int_{x_0}^{x} a_1 \, ds\right\}, \qquad 0 \neq c \in E_1, \tag{2.1}$$

then the image $(\bar{a})$ $\{T(x),\ U(x)\} \in o_a(I_{1x})$ is semicanonic. As the semicanonical image is following (2.1) determined by the coordinate $T(x)$ we write instead of $(\bar{a})$ $\{T(x),\ U(x)\} \in p_a(I_{1x})$ in a shorter way $(\bar{a})$ $\{T(x)\} \in p_a(I_{1x})$.

We call the image (A) $\{x\} \in p_a(I_1)$ the *fundamental semicanonic image* or also the *semi-canonical fundamental form* of the equation (a).

If we put

$$A_i = \sum_{k=0}^{i} \binom{i}{k} a_k \chi_{i-k}(-a_1), \qquad i = 2, 3, \ldots, n, \qquad x \in I_1, \tag{2.2}$$

then we can write the semicanonical image in the form

$$\text{(A)} \quad U_1(x) \left[Z^{(n)}(x) + \sum_{i=2}^{n} \binom{n}{i} A_i(x)\, Z^{(n-i)}(x)\right] = 0,$$

$$\text{where } U_1(x) = c \,.\, \exp\left\{-\int_{x_0}^{x} a_1 \, ds\right\}, \ 0 \neq c \in E_1.$$

We call the function (2.2) the *fundamental coefficients* of the equation (a).

Let us put

$$f_i^n(A_2) = f_i^n(0, A_2), \qquad i = 0, 1, \ldots, n, \tag{2.3}$$

$$I_n(Z;\ A_2) = \sum_{i=0}^{n} \binom{n}{i} f_i^n(A_2)\, Z^{(n-i)}, \tag{2.4}$$

$$\Omega_i^n = A_i - f_i^n(A_2), \qquad i = 3, 4, \ldots, n.$$

There is for instance

$$f_0^n(A_2) = 1, \qquad f_1^n(A_2) = 0, \qquad f_2^n(A_2) = A_2,$$

$$f_3^n(A_2) = \frac{3}{2} A_2', \qquad f_4^n(A_2) = \frac{9}{5} A_2'' + \frac{3(5n+7)}{5(n+1)} A_2^2. \tag{2.5}$$

Let us introduce yet the formula (2.4) for $n = 3, 4$:

$$I_3(y;\ A_2) = y''' + 3A_2 y' + \frac{3}{2} A_2' y. \tag{2.6}$$

$$I_4(y;\ A_2) = y^{(4)} + 6A_2 y'' + 6A_2' y' + \frac{9}{5}\left(A_2'' + \frac{9}{5} A_2^2\right) y. \tag{2.7}$$

Then we can write the normal form of the equation (A) in the perturbated form

$(\Omega)\quad I_n(Z; A_2) + \sum_{i=3}^{n} \binom{n}{i} \Omega_i^n Z^{(n-i)} = 0.$

We call the function $f_i^n(A_2)$ resp. Ω_i^n the fundamental iterated polynomial — briefly the *fundamental polynomial* — resp. the *fundamental semiinvariant* of the dimension i of the equation (a). We call the equation (Ω) the perturbated fundamental semicanonical form of the equation (a) or briefly the *perturbated fundamental equation.*

We introduce the perturbated fundamental equations of the order 3 and 4 in their most often occurring arrangements. If we put $A = \frac{3}{2} A_2$, $\omega_3 = \omega_3^3 = A_3 - \frac{3}{2} A_2'$ we obtain with the aid of (2.5), (2.6) the perturbated fundamental equation of the 3^d order in the form

$$y''' + 2Ay' + (A' + \omega_3)\, y = 0.$$

If we put $A = \frac{3}{5} A_2$, $\omega_3 = 4\omega_3^4 = 4\left(A_3 - \frac{3}{2} A_2'\right)$, $\omega_4 = \omega_4^4 = A_4 - \frac{9}{5}\left(A_2'' + \frac{9}{5} A_2^2\right)$, we obtain with the aid of (2.5), (2.7) a perturbated fundamental equation of the 4^{th} order in the form

$$y^{(4)} + 10Ay'' + (10A' + \omega_3)\, y' + [3(A'' + 3A^2) + \omega_4]\, y = 0,$$

see f.i. [17; pp. 511—3·26, pp. 528—4·11], [20], [11], [7].

Between the functions (2.3) and (1.3) resp. (2.5) and (1.4) hold the following relations:

$$f_i^n(A_2) = \sum_{k=0}^{i} \binom{i}{k} f_k^n(a_1, a_2)\, \chi_{i-k}(-a_1), \qquad i = 3, 4, \ldots, n,$$

$$\Omega_i^n = \sum_{k=0}^{i} \binom{i}{k} \omega_k^n \chi_{i-k}(-a_1), \qquad i = 3, 4, \ldots, n,$$

see [6; (2.5)].

The semicanonic image $(\bar{\mathrm{A}})\,\{T(x)\} \in p_a(I_{1x})$ can be written in the form

$(\bar{\mathrm{A}})\quad U(x)\, [T'(x)]^n [z^{(n)}(t) + \sum_{i=2}^{n} \binom{n}{i} \bar{A}_i(t)\, z^{(n-i)}(t)] = 0, \quad x = T_{-1}(t),$ where

(2.8) $\bar{A}_i(t) = [T'(x)]^{-i} \sum_{k=0}^{i} \binom{i}{k} A_k(x)\, \Phi_{i-k}^{n,i}(\eta), \; x = T_{-1}(t), \qquad i = 2, 3, \ldots, n,$

where the functions A_k, $k = 2, 3, \ldots, n$ are the fundamental coefficients of the equation (a), $A_0 \equiv 1$, $A_1 \equiv 0$, $\Phi_{i-k}^{n,i}(\eta) = \Phi_{i-k}^{n,i}\left(\eta, -\frac{n-1}{2}\eta\right)$ and (2.1) holds, see [3; 3, 2.15].

If the function $T(x)$ is in the interval I_{1x} the solution of the equation $\{T, x\} = (3/n + 1) A_2$ [the symbol $\{T, x\}$ stands for the SCHWARZ derivative of the function $T(x)$], then the image $(\bar{\mathrm{A}})\ \{T(x)\} \in p_a(I_{1x})$ is canonical and it can be written in the form $(\bar{\mathrm{A}})$ where (2.1), (2.8), $A_2 \equiv 0$ hold and

$$(2.9) \qquad \Phi^{n,i}_{i-k}(\eta) = \frac{(n-i)!}{(n-k)!\,(i-k)!} \sum_{\varrho=0}^{i-k} \eta^{i-k-\varrho} F^{n,i,k}_{\varrho}(A_2)$$

— see [3; 3, 3.5]. The function $F^{n,i,k}_{\varrho}$ is the polynomial of the element A_2 of the dimension ϱ, which is defined like the polynomial $\Phi^{n,i}_{i-k}$ — see [3; 2, 3,4].

If the equation (a) is canonical, i.e. if $a_1 \equiv a_2 \equiv 0$, then the canonical image $(\bar{\alpha})\ \{T(x)\} \in k_a(I_{1x})$ is of the form

$$(\bar{\alpha}) \quad U_2(x)\,[T'(x)]^n [z^{(n)}(t) + \sum_{i=3}^{n} \binom{n}{i} (\bar{\alpha}_i(t)\, z^{(n-i)}(t)] = 0, \qquad x = T_{-1}(t),$$

where

$$U_2(x) = c|T'(x)|^{\frac{1-n}{2}}, \qquad 0 \neq c \in E_1,$$

$$(2.10) \qquad \bar{\alpha}_i(t) = [T'(x)]^{-i} \sum_{\nu=0}^{i-3} [\eta(x)]^{\nu} \left(-\frac{1}{2}\right)^{\nu} \binom{i}{\nu} \binom{i-1}{\nu} \nu!\, a_{i-\nu}(x),$$

$$i = 3, 4, \ldots, n, \qquad x = T_{-1}(t).$$

The function $T(x)$ is in the interval I_{1x} the solution of the equation $\{T, x\} = 0$, see [3; 3, 3.6].

We shall introduce yet the perturbated forms of the images of the equation (a).

The equation

$$(\bar{\omega}) \quad uT'^n[I_n(z; \bar{a}_1, \bar{a}_2) + \sum_{i=3}^{n} \binom{n}{i} \bar{\omega}^n_i(t)\, z^{(n-i)}(t)] = 0,$$

where

$$\bar{a}_1 = (T')^{-1}[\Phi^{n,1}_1(\eta, \zeta) + a_1],$$

$$\bar{a}_2 = (T')^{-2}[\Phi^{n,2}_2(\eta, \zeta) + 2\Phi^{n,2}_1(\eta, \zeta)\, a_1 + a_2],$$

$$I_n(z; \bar{a}_1, \bar{a}_2) = \sum_{i=0}^{n} \binom{n}{i} f^n_i(\bar{a}_1, \bar{a}_2)\, z^{(n-i)},$$

$$f^n_i(\bar{a}_1, \bar{a}_2) = (T')^{-i} \sum_{k=0}^{i} \binom{i}{k} f^n_k(a_1, a_2)\, \Phi^{n,i}_{i-k}(\eta, \zeta), \qquad i = 0, 1, \ldots, n,$$

$$\bar{\omega}^n_i = (T')^{-i} \sum_{k=3}^{i} \binom{i}{k} \omega^n_k \Phi^{n,i}_{i-k}(\eta, \zeta), \qquad i = 3, 4, \ldots, n,$$

is the perturbated form of the image $(\bar{a})\ \{T(x), u(x)\} \in o_a(I_{1x})$. The equation

$$(\bar{\Omega}) \qquad UT'^n[I_n(z; \bar{A}_2) + \sum_{i=3}^{n} \binom{n}{i} \bar{\Omega}_i^n z^{(n-i)} = 0,$$

where

$$\bar{A}_2 = (T')^{-2} \left[\frac{n+1}{6}\left(\frac{1}{2}\eta^2 - \eta'\right) + A_2\right],$$

$$I_n(z; \bar{A}_2) = z^{(n)} + \sum_{i=2}^{n} \binom{n}{i} f_i^n(\bar{A}_2) z^{(n-i)},$$

$$f_i^n(\bar{A}_2) = (T')^{-i} \sum_{k=0}^{i} f_k^n(A_2) \Phi_{i-k}^{n,i}(\eta),$$

$$\bar{\Omega}_i^n = (T')^{-i} \sum_{k=3}^{i} \binom{i}{k} \Omega_k^n \Phi_{i-k}^{n,i}(\eta),$$

is the perturbated form of the image $(\bar{\mathrm{A}})$ $\{T(x)\} \in p_a(I_{1x})$.

If the function $T(x)$ is the solution of the equation $\{T, x\} = \dfrac{3}{n+1} A_2$, then the equation $(\bar{\Omega})$ is the perturbated form of the canonical image $(\bar{\alpha})$ $\{T(x)\} \in k_a(I_{1x})$, where we put $\bar{A}_2 \equiv 0$, $I_n(z; 0) = z^{(n)}$,

$$\bar{\Omega}_i^n = i!(n-i)!\,(T')^{-i} \sum_{k=3}^{i} \frac{1}{k!\,(n-k)!} \Omega_k^n \sum_{\varrho=0}^{i-k} \eta^{i-k-\varrho} F_\varrho^{n,i,k}(A_2).$$

Between the polynomials (2.3) and $F_\varrho^{n,i,k}(A_2)$ hold the relations

$$\sum_{k=0}^{\nu} \frac{1}{k!\,(n-k)!} f_k^n(A_2) F_{\nu-k}^{n,i,k}(A_2) = 0, \qquad \nu = 0, 1, \ldots, i, \qquad i = 3, 4, \ldots, n.$$

3. Equivalence

The notion of equivalence is an important notion in the theory of linear differential equations.

Let us have the equation

$$\text{(b)} \qquad z^{(n)}(t) + \sum_{i=1}^{n} \binom{n}{i} b_i(t) z^{(n-i)}(t) = 0, \qquad b_i \in C_{n-i}(I_2), \qquad i = 1, 2, \ldots, n$$

and let $o_b(I_{2t})$ be the set of the images of the equation (b) in the interval $\emptyset \neq I_{2t} \subset I_2$.

We say that the sets $o_a(I_{1x})$, $o_b(I_{2t})$ are *quasi-identical* denoted by the sign

$$\text{(3.1)} \qquad o_a(I_{1x}) \doteq o_b(I_{2t}),$$

if every element of the set $o_a(I_{1x})$ is quasi-identical with one of the elements of the set $o_b(I_{2t})$. The relation (3.1) is reflexive, symmetrical and transitive and holds just when at least one element of the set $o_a(I_{1x})$ is quasi-identical with some of the elements of the set $o_b(I_{2t})$.

If (3.1) holds, then we say that the equations (a), (b) are in the intervals I_{1x}, I_{2t} equivalent and we denote it by

$$(\mathrm{a})\ I_{1x} \sim (b)\ I_{2t}\{T(x)\},$$

where $T(x)$ is the first coordinate of the image $(\bar{\mathrm{a}}) \in o_a(I_{1x})$, which is in the interval I_{2t} quasi-identical with the equation (b) so that $T(I_{1x}) = I_{2t}$ holds. We call the function $T(x)$ the carrier of the equivalence of the equation (b) to the equation (a).

The second coordinate $u(x)$ of the image $(\bar{\mathrm{a}})$ is given by the formula

$$u(x) = c|T'|^{\frac{1-n}{2}} \exp\{\int_{x_0}^{x} (b_1[T(s)]\, T'(s) - a_1(s))\, ds\}.$$

With the aid of the relations (2.10) the necessary and sufficient conditions for the equivalence of the canonical equations can be proved.

$$(\alpha) \qquad y^{(n)}(x) + \sum_{i=3}^{n} \binom{n}{i} \alpha_i(x)\, y^{(n-i)}(x) = 0, \quad \alpha_i \in C_{n-i}(I_1), \quad i = 3, 4, \ldots, n,$$

$$(\beta) \qquad z^{(n)}(t) + \sum_{i=3}^{n} \binom{n}{i} \beta_i(t)\, z^{(n-i)}(t) = 0, \quad \beta_i \in C_{n-i}(I_2), \quad i = 3, 4, \ldots, n.$$

Let us denote

$$\vartheta_3(\alpha_3) = \alpha_3$$

$$(3.2) \qquad \vartheta_i(\alpha_3, \ldots, \alpha_i) = \sum_{r=3}^{i} (-1)^{i-r} C_r^i \alpha_r^{(i-r)}, \qquad i = 4, 5, \ldots, n,$$

where

$$(3.3) \qquad C_r^i = \binom{i+r-2}{i-1}\binom{i}{r} \Big/ \binom{2i-2}{i-1}, \qquad r = 3, 4, \ldots, i.$$

The formula (3.2) is quoted in the literature as the formula of Brioschi, see [16; p. 197], [18; p. 35], [5; 3.7]. Then holds

Theorem 2. $(\alpha)\ I_{1x} \sim (\beta)\ I_{2t}\{T(x)\} \Leftrightarrow \vartheta_i\{\beta_3[T(x)], \ldots, \beta_i[T(x)]\}\, [T'(x)]^i = \vartheta_i(\alpha_3, \ldots, \alpha_i)$, $i = 3, 4, \ldots, n$, $x \in I_{1x}$, *where* $T(x)$ *is the solution of the equation* $\{T, x\} = 0$. See [5; 2.1].

The function $\vartheta_i(\alpha_3, \ldots, \alpha_i)$ *is the canonical invariant of the equation* (α) *of the dimension and weight* i. *As it is a polynomial of the first order it is also called the linear invariant.*

With the aid of the theorem 2 and with the aid of the relations (2.8), (2.9) the necessary and sufficient conditions for the equivalence of the equations (a), (b) can be proved.

Let us denote

$$(3.4) \qquad \Theta_i^n(A_2, \ldots, A_i) = \sum_{r=3}^{i} C_r^i \Psi_i^{n,r,i-r}(A_2, \ldots, A_i), \qquad i = 3, 4, \ldots, i,$$

where the constants C_r^i are determined by the formula (3.3) and $\Psi_i^{n,r,i-r}$ is the polynomial of the elements $A_2, \ldots, A_i$ of the dimension i, which satisfies a certain linear difference equation of the first order — see [5; (1.6)]. Then the theorem 3 holds.

Theorem 3. (a) $I_{1x} \sim$ (b) $I_{2t}\{T(x)\} \Leftrightarrow \Theta_i^n\{B_2[T(x)], \ldots, B_i[T(x)]\} \,.\, [T'(x)]^i = \Theta_i^n(A_2, \ldots, A_i)$, $i = 3, 4, \ldots, n$, $x \in I_{1x}$, *where* A_i *resp.* B_i, $i = 2, 3, \ldots, n$ *are the fundamental coefficients of the equation* (a) *resp.* (b) *and the function* $T(x)$ *is the solution of the equation*

$$\{T, x\} + \frac{3}{n+1} B_2[T(x)] \,.\, [T'(x)]^2 = \frac{3}{n+1} A_2.$$

See [5; 3.3].

The function $\Theta_i^n(A_2, \ldots, A_i)$ *is the fundamental invariant of the equation* (a) *of the dimension and weight* i.

The theorem 3 is stated without proof and inexactly in [16; p. 191].

Between the functions ω_i^n and Θ_i^n hold the following relations:

$$(3.5.) \qquad \Theta_j^n \equiv 0, \quad j = 3, 4, \ldots, i \Leftrightarrow \omega_j^n \equiv 0, \quad j = 3,4, \ldots, i; \quad i = 3,4, \ldots, n.$$

From these relations follows

Theorem 4. *The equation* (a) *is iterated just then when all its fundamental invariants are identical to zero.*

In [16; p. 204—205) is quoted without proof the theorem of F. Brioschi which is a special case of the theorems 1 and 4: If all the fundamental invariants of the equation (a) are identical to zero, then the equation (a) has a fundamental system of the form (1.5).

The first non-zero coefficient of the perturbation of the equation (a) is a fundamental invariant, which means that if (3.5) holds, then

$$\Theta_{i+1}^n \not\equiv 0 \Leftrightarrow \omega_{i+1}^n \not\equiv 0 \qquad \text{and at the same time} \qquad \Theta_{i+1}^n \equiv \omega_{i+1}^n.$$

Theorem 5. *Let* $I_1 \equiv I_2$. *The equations* (a), (b) *are mutually adjoint if and only if the relations*

$$\Theta_i^n(B_2, \ldots, B_i) = (-1)^i \Theta_i^n(A_2, \ldots, A_i), \qquad i = 3, 4, \ldots, n.$$

See [10; 1.18].

Corollary: *Let* ($\bar{A}$) $\{T(x)\} \in p_a(I_{1x})$. *The normal form of the equation* ($\bar{A}$) *is a self-adjoint equation just when*

$$\Theta^n_{2\nu+1}(A_2, \ldots, A_i) \equiv 0, \qquad \nu = 1, 2, \ldots, \left[\frac{n-1}{2}\right], \qquad x \in I_{1x}.$$

See [10; 2.9].

From the corollary of the theorem 5. follows this statement: If all the fundamental invariants with odd indices of the equation

$$y^{(n)} + \sum_{i=2}^{n} \binom{n}{i} a_i y^{(n-i)} = 0$$

are identical to zero, then this equation is self-adjoint. This theorem is mentioned without proof in [16; p. 224] and [18; p. 235].

It seems that it is convenient to introduce the notion of the genus of homogeneous linear differential equations.

Let $2 \leqq k \leqq n$ be a natural number. If

$$\Theta^n_j(A_2, \ldots, A_i) \equiv 0, \qquad j = 3, 4, \ldots, n + 2 - k, \qquad \Theta^n_{n+3-k} \not\equiv 0$$

(for $k = 2$ we put $\Theta^n_{n+1} \equiv 0$), we say then that the equation (a) is of the genus k.

The theorem 6 holds.

Theorem 6. *The equation* (a) *is of the same genus k if and only if the equation*

$$I_n(y; a_1 a_2) + \sum_{i=n+3-k}^{n} \binom{n}{i} \omega^n_i y^{(n-i)} = 0, \qquad \omega_{n+3-k} \not\equiv 0, \quad \sum_{i=n+1}^{n} \equiv 0$$

is the perturbated form of the equation (a). See [6; (3.1)].

We take note that under the assumption $\omega^n_n \not\equiv 0$ resp. $\omega^n_{n-1} \not\equiv 0$ is the equation (1.6) resp. (1.7) of the genus 3 resp. 4.

From the theorem 5 (corollary) follows that the self-adjoint equation of the n^{th} order can be of the genus not higher than $(n - 1)$. The iterated equations are of the genus 2. The equation (a) is not of a higher genus than 3 if its canonical image is a binomial equation.

Many of the properties which hold for the equations of the second order hold also for the equations of the n^{th} order of the genus 2. It can be expected that some of the properties of the k^{th} order will hold also for the equations of the n^{th} order of the genus k, f.i. see [8].

REFERENCES

[1] Z. Hustý: *Die Iteration homogener linearer Differentialgleichungen. Publ. Fac. Sci. Univ. J. E. Purkyně Brno 449 (1964), 23—56.*

[2] Z. Hustý: *Asymptotische Eigenschaften von Lösungen homogener linearer Differentialgleichungen n-ter Ordnung. Math. Nachr. 32, 173—185 (1966).*

[3] Z. HUSTÝ: *Über die Transformation und Äquivalenz homogener linearer Differentialgleichungen von höherer als der zweiten Ordnung.*
I. Teil: Transformation regulärer Gleichungen. Czechoslovak Math. J. 15 (90) (1965), 479—502.
[4] *II. Teil: Äquivalenz regulärer Gleichungen mit Dimension. Czechoslovak Math. J. 16 (91) (1966), 1—13.*
[5] *III. Teil: Äquivalenz regulärer Gleichungen mit stetiger Dimension, Czechoslovak Math. J. 16 (91) (1966), 161—185.*
[6] Z. HUSTÝ: *Perturbierte homogene lineare Differentialgleichungen. Časopis Pěst. Mat. 91 (1965), 154—169.*
[7] Z. HUSTÝ: *Колебательные свойства однородного линейного дифференциального уравного уравнения n-ого порядка (n. $\geqq$ 3). Czechoslovak Math. J. 8 (83) (1958), 62—75.*
[8] Z. HUSTÝ: *Некоторые колеблательные свойства однородного линейного дифференциального уравнения n-ого порядка (n $\geqq$ 3). Czechoslovak Math. J. 14 (89) (1964), 27—38.*
[9] Z. HUSTÝ: *Asymptotische Formeln für die Lösungen homogener linearer Differentialgleichungen n-ter Ordnung im oszillatorischen Fall. Časopis Pěst. Mat. 90 (1965), 79—86.*
[10] Z. HUSTÝ: *Adjungierte und selbstadjungierte lineare Differentialgleichun gen. Archivum mathematicum (Brno) T 1 (1965), 21—34.*
[11] G. SANSONE: *Studi sulle equazioni differenziali lineari omogenee di terzo ordine nel campo reale. Revista Matem. y Fisica Teorica, Serie A, 1948, 195—253.*
[12] G. SANSONE: *Le equazioni differenziali lineari, omogenee, del quarto ordine, nel campo reale. Annali della R. Scuola Normale Superiore di Pisa, Serie II, Vol. XI. Fasc. III—IV, 1942, 151—196.*
[13] G. SANSONE: *Equazioni differenziali nel campo reale I. Bologna 1948.*
[14] O. BORŮVKA: *Sur la transformation des integrales des équations differentielles linéaires ordinaires du second ordre. Ann. di Mat. p. ed. app. (1956), 325—342.*
[15] O. BORŮVKA: *Lineare Diferentialtransformationen 2. Ordnung. VEB Deutscher Verlag der Wissenschaften, Berlin (1967).*
[16] L. SCHLESINGER: *Handbuch der Theorie der linearen Differentialgleichungen. Band II/1, Leipzig 1897.*
[17] E. KAMKE: *Differentialgleichungen, Lösungen und Lösungsmethoden, Band I, Leipzig 1951.*
[18] E. BRIOSCHI: *Les invariants des équations differentielles linèaires. Acta Mathematica 14 (1890—91), 233—248.*
[19] V. ŠEDA: *Über die Transformation der linearen Differentialgleichungen n-ter Ordnung. Časopis Pěst. Mat. 90 (1965), 385—412.*
[20] M. GREGUŠ: *Über die lineare homogene Differentialgleichung dritter Ordnung. Wissenschaftliche Zeitschrift der Martin Luther-Universität Halle—Wittenberg. XII/3, 265—286.*

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XVII — 1967

ON BOUNDED SOLUTIONS OF A CERTAIN DIFFERENTIAL EQUATION

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We shall deal with the second-order linear differential equation

$$\text{(q)} \qquad y'' = q(t)\, y$$

where the function $q(t)$ is a continuous function on the interval $(-\infty, \infty)$, and periodic with period π. The well-known Floquet theory gives all possible types of behaviour of solutions of this differential equation. For the second-order differential equation (q) the characteristic equation

$$(1) \qquad \lambda^2 - \Delta\lambda + 1 = 0$$

is of special interest. Here, the coefficient (so called discriminant) $\Delta = u'(\pi) + v(\pi)$, where u and v are solutions of (q) determined by the initial conditions $u(0) = v'(0) = 0$, $u'(0) = v(0) = 1$. The following cases may occur:

1) If $|\Delta| > 2$ then no non-trivial solution of (q) is bounded on $(-\infty, \infty)$.
2) If $|\Delta| < 2$ then every solution of (q) is bounded on $(-\infty, \infty)$ In this case 2) the differential equation (q) is called stable.
3) If $|\Delta| = 2$ then either all solutions of (q) are bounded on $(-\infty, \infty)$ or a solution of (q) is bounded on $(-\infty, \infty)$ and every solution independent of it is unbounded on $(-\infty, \infty)$.

First, let us deal with case 2). In this case we can obtain the general solution of the differential equation (q) and at the same time the necessary and sufficient condition establishing all the stable differential equations (q). Let us restrict ourselves on the theorems only:

Theorem 1. *In every stable differential equation (q) there is the function $q(t)$ of the form*

$$q(t) = -\{\operatorname{tg} \alpha, t\}$$

where $\{\operatorname{tg} \alpha, t\}$ is Schwarz's derivative, i.e. $\frac{1}{2}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)' - \frac{1}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 + \alpha'^2(t)$

and $\alpha(t) = P(t) + (2n + a)\,t$ where n is integer, a is a number in the interval $(0, 1)$, $P(t)$ is a periodic function with period π such that it has continuous derivatives up to and including the order 3 and $P'(t) + 2n + a \neq 0$. The function $P(t)$, number a and integer n are uniquely determined by the stable differential equation (q).

Moreover, every differential equation $y'' = q(t)\,y$ with the function $q(t)$ constructed in this way is a stable differential equation (q).

Theorem 2. *The general solution of the stable differential equation (q)* (i.e. *with $|\Delta| < 2$) is of the form*

$$y(t; k_1, k_2) = k_1 \frac{\sin\,[P(t) + (2n + a)\,t + k_2]}{\sqrt{|P'(t) + 2n + a|}}$$

where $a \in (0, 1)$ and $e^{\pm a\pi i}$ are roots of the characteristic equation (1), n is a suitable integer and $P(t)$ satisfies the above conditions.

Let us note that the integer n gives the density of zeros of the solutions.

Now, let us deal with case 3) (i.e. $|\Delta| = 2$). We I shall introduce the necessary and sufficient condition under which we may state whether all the solutions of a given differential equation (q) are bounded or not. This condition is based on the behaviour of one bounded solution of equation (q) which, in this case 3), must exist.

Theorem 3. *Let $y(t)$ be a non-trivial bounded solution of equation* (q) *(with $|\Delta| = 2$). Let $a_1 < \ldots < a_n$ be all zeros of $y(t)$ on $[0, \pi)$. Set*

$$r(t) = \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{1}{y'^2(a_i)\sin^2(t - a_i)} \\ 0 \text{ if there is no zero of } y(t). \end{cases}$$

Then every solution of (q) is bounded on $(-\infty, \infty)$ if and only if

$$\int_0^\pi \left[\frac{1}{y^2(t)} - r(t)\right] \mathrm{d}t = 0.$$

By means of this result we may construct all the differential equations of a prescribed type. Especially, we may obtain all such second-order linear differential equations every solution of which has zeros in the same distances π. The well-known representative of such differential equations is the equation $y'' = -y$. One result of this sort:

There are $2^{\aleph_0}$ differential equations of this property. Another result:

The set of all differential equations (q) with $\Delta = -2$ all solutions of which are bounded on $(-\infty, \infty)$ and such that a solution has just one zero in the interval $[0, \pi)$ is exactly the same set as the set of all second-order linear differential equations every solution of which has zeros in the same distances equal to π.

This is the simplest result of several results establishing a close relation between disposition of zeros of solutions of a differential equation (q) and the boundedness of these solutions.

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XVII — 1967

ASYMPTOTIC FORMULAS FOR THE SOLUTIONS OF THE EQUATION $(py')' + qy = 0$

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In the course of preceding EQUADIFF I had the honour of holding the lecture on asymptotic formulas for the solutions of the equation

$$y'' + q(x)\, y = 0 \tag{1}$$

in $J = \langle a, \infty)$ derived by means of the transformation of (1) into

$$Y'' + Q(x)\, Y = 0, \tag{2}$$

and by means of the method of perturbation. The formulas under discussion were of the form

$$y = \varphi(x)\, \{[c_1 + \varepsilon_1(x)]\, U[\Phi(x)] + [c_2 + \varepsilon_2(x)]\, V[\Phi(x)]\}, \tag{3}$$

where U, V denote linear independent solutions of (2), φ, Φ are functions which satisfy certain conditions and ε_1, ε_2 are continuous functions converging to zero for $x \to \infty$.

From the numerical point of view it is necessary to estimate the speed with which the functions ε_1, ε_2 converge to zero or, in particular, to deduce asymptotic formulas concerning these functions or, at last, to approximate the solution y with a given exactness on the entire interval J. One can find in the literature the estimations of functions taken into consideration in very special cases, especially in the case $U = \sin x$, $V = \cos x$. E.g. G. ASCOLI dealt with the majorisation of functions ε_i [1], [2], V. RICHARD derived asymptotic formulas of these functions; a very fine result has also been introduced in the book by G. SANSONE, where the solutions of the equation $y'' + [1 + q(x)]\, y = 0$ were given under suppositions $\int_x^\infty |q(x)|\, dx < \infty$, $\lim_{x \to \infty} q(x) = 0$ by means of infinite series that make possible to approximate the solution with an arbitrary exactness on J. The same problem can be solved under more general suppositions for the equation

$$(p(x)\, y')' + q(x)\, y = 0 \tag{4}$$

when using FUBINI's and PEANO—BAKER's method [8].

(*) $\begin{cases} \textit{Assume } q, Q \in C_0, p, P \in C_1, p > 0, P > 0 \text{ on } J. \textit{ Let } \varphi, \Phi \textit{ be functions} \\ \textit{satisfying the conditions } \varphi, \Phi \in C_2, \varphi > 0, \Phi > 0, \Phi' > 0. \end{cases}$

Let be U, V two independent solutions of the equation

$$(P(x)\, Y')' + Q(x)\, Y = 0 \tag{5}$$

$W = UV' - U'V$. *Put*

$$k = \frac{1}{W(\Phi)} \left[\log \frac{P(\Phi)}{p\varphi^2\Phi'}\right]',$$

$$l = \frac{1}{p\varphi\Phi' W(\Phi)} [(p\varphi')' + q\varphi - p\varphi\Phi'^2 Q(\Phi)\, P^{-1}(\Phi)],$$

$$A = \begin{pmatrix} V(\Phi)\,[l\dot{U}(\Phi) - kU(\Phi)], & V(\Phi)\,[l\dot{V}(\Phi) - k\dot{V}(\Phi)] \\ U(\Phi)\,[k\dot{U}(\Phi) - lU(\Phi)], & U(\Phi)\,[k\dot{V}(\Phi) - lV(\Phi)] \end{pmatrix}$$

and assume

$$\int_a^\infty |k|\, \{|U(\Phi)| + |V(\Phi)|\}\, \{|\dot{U}(\Phi)| + |\dot{V}(\Phi)|\} < \infty,$$

$$\int_a^\infty |l|\, \{U^2(\Phi) + V^2(\Phi)\} < \infty.$$

Then Equation (4) *has the general solution of the form*

$$y = \varphi(x)\, (U[\Phi(x)],\ V[\Phi(x)])\, \sum_0^\infty (-1)^n R^n(x)\, c, \tag{6}$$

$$y' = (\{\varphi(x)\, U[\Phi(x)]\}',\ \{\varphi(x)\, V[\Phi(x)]\}')\, \sum_0^\infty (-1)^n R^n(x)\, c$$

where c denotes a constant vector and

$$R^0(x) = \begin{pmatrix} 1,0 \\ 0,1 \end{pmatrix}, \qquad R^n(x) = \int_x^\infty AR^{n-1}.$$

A suitable choice of functions φ, Φ enables us to find not only asymptotic formulas derived by means of these methods, but also to make full use of formulas (6) for the solution of numerical problems mentioned above, and to study asymptotic properties in connexion with the transformation of equation (4).

Equations (4) and (5) being given, one can find a constant $\xi \geq a$ and functions φ, Φ in such a way that all the elements of the matrix A are equal to zero, supposing only, these equations are simultaneously either oscillatory or non-oscillotory on J. yn this case the functions φ, Φ are solutions of the non-linear system

$$\varphi^2\Phi' = \frac{P(\Phi)}{p},$$

$$(p\varphi')' + \left[q - \frac{Q(\Phi)}{P(\Phi)}\,p\Phi'^2\right]\varphi = 0$$

and formulas (6) have, for $x \geq \xi$, a simple form

$$y = \varphi(x)\,\{c_1 U[\Phi(x)] + c_2 V[\Phi(x)]\},$$
$$y' = c_1\{\varphi(x)\,U[\Phi(x)]\}' + c_2\{\varphi(x)\,V[\Phi(x)]\}'.$$

In concluding *I* would like to introduce some applications of the preceding theorem when substituing equation (5) by the equation $y'' + \varepsilon y = 0, \varepsilon = 0, \pm 1$ and omitting in formulas (6) all the members from $n = 2$ up to infinity.

In what follows, we suppose that the conditions (*) are valid.

1. *Put*

$$k = (\log p\varphi^2\Phi')', \qquad l = \frac{1}{p\varphi\Phi'}\,[p\varphi\Phi'^2 - (p\varphi')' - q\varphi], \qquad h = \sqrt{k^2 + l^2}.$$

Let ψ *be a function defined by means of the equations*

$$\sin\psi = \frac{k}{h}, \qquad \cos\psi = \frac{l}{h} \quad \textit{for } h \neq 0, \qquad \psi = 0 \quad \textit{for } h = 0.$$

Assume furthermore $\int\limits_a^\infty |h| < \infty$. *Then equation* (5) *has the general solution of the form*

$$y = \lambda\varphi(x)\,\{\sin[\Phi(x) + \alpha] - \int\limits_x^\infty h(t)\sin[\Phi(x) - \Phi(t)]\sin[\Phi(t) - \psi(t) + \alpha]\,dt\} + \eta_1(x),$$

$$|\eta_1(x)| \leq |\lambda|\,\varphi(x)\,\varkappa^2(x)\,e^{\varkappa(x)}, \qquad \varkappa(x) = 4\int\limits_x^\infty h.$$

2. *Put*

$$k = \frac{1}{2}(\log p\varphi^2\Phi')', \qquad l = -\frac{1}{2p\varphi\Phi'}\,[(p\varphi')' + q\varphi + p\varphi\Phi'^2],$$

and suppose

$$\int\limits_a^\infty (|k| + |l|)e^{2\Phi} < \infty.$$

Then equation (4) *has the general solution of the form*

$$y = c_1\varphi(x)\,\{e^{\Phi(x)} + \int\limits_x^\infty e^{-\Phi(x)}[e^{2\Phi(x)} - e^{2\Phi(t)}]\,[k(t) - l(t)]\,dt\} +$$

$$+\, c_2\varphi(x)\,\{e^{-\Phi(x)} + \int\limits_x^\infty e^{\Phi(x)}[e^{-2\Phi(x)} - e^{-2\Phi(t)}]\,[k(t) + l(t)\,dt\} + \eta_2(x),$$

$$|\eta_2(x)| \leq 4\,||c||\,\varphi(x)\,\varkappa^2(x)\,e^{2\varkappa(x)-3\Phi(x)}, \qquad \varkappa(x) = \int_x^\infty (|k| + |l|)\,e^{2\Phi}.$$

3. *Put*

$$k = (\log p\varphi^2\Phi')', \qquad l = -\frac{1}{p\varphi\Phi'}\,[(p\varphi')' + q\varphi],$$

and suppose

$$\int_a^\infty (|k|\,\Phi + |l|\,\Phi^2) < \infty.$$

Then equation (4) *has the general solution of the form*

$$y = c_1\varphi(x)\,\{\Phi(x) + \int_x^\infty [\Phi(x) - \Phi(t)]\,[k(t) - l(t)\,\Phi(t)]\,\mathrm{d}t\} +$$

$$+ c_2\varphi(x)\,\{1 - \int_x^\infty [\Phi(x) - \Phi(t)]\,l(t)\,\mathrm{d}t\} + \eta_3(x),$$

$$|\eta_3(x)| \leq 2\,||c||\,\varphi(x)\,\Phi^{-1}(x)\,\varkappa^2(x)\exp\{2\varkappa(x)\,\Phi^{-1}(x)\}\,[1 + \Phi^{-1}(x)],$$

$$\varkappa(x) = \int_x^\infty (|k|\,\Phi + |l|\,\Phi^2).$$

In all the cases **1.**, **2.**, **3.** analogues formulas for the derivatives of solutions are valid.

REFERENCES

[1] ASCOLI G., *Sulla forma degli integrali dell'equazione differenziale $y'' + A(x)\,y = 0$ in un caco notevole di stabilità, Revista Universidad nacional del Tucumán. Facultad de ciencias exactas y aplicadas. Tucumán. Ser. A. Matemáticas y fisica teórica 2 (1941), p. 131—140.*

[2] ASCOLI G., *Sopra un caso di stabilità per la equazione differenziale $y'' + A(x)\,y = 0$, Annali di Matematica Pura ed Applicata (4) 26 (1947), p. 199—206.*

[3] RICHARD U., *Sulla risoluzione asintotico-numerica dell'equazione differenziale $(py')' + qy = 0$, Atti della Accademia delle Scienze di Torino. Classe di Scienze Fisiche, Matematiche e Naturali 97 (1962/63), p. 857—890.*

[4] SANSONE G., *Equazioni differenziali nel campo reale, Bologna 1949.*

[5] FUBINI G., *Studi asintotici per alcune equazioni differenziali, Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali, 6 (26) (1937), p. 253—259.*

[6] PEANO G., *Integrazione per serie delle equazioni differenziali lineari, Atti della Accademia delle Scienze di Torino. Classe di Scienze Fisiche, Matematiche e Naturali 22 (1887), p. 293—302.*

[7] BAKER H. F., *On the integration of linear differential equations, Proceedings of the London Mathematical Society 35 (1903), p. 333/378.*

[8] RÁB M., *Les développements asymptotiques des solutions de l'équation $(py')' + qy = 0$, Arch. Math. (Brno), T2 (1966), 1—17.*

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AN APPLICATION OF GREEN'S FUNCTION IN THE DIFFERENTIAL EQUATIONS

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In solving of various types of problems in the theory of ordinary and partial differential equations, difference equations, there occurs a notion of Green's function. With help of it many problems of various character from the theory of ordinary and partial differential equations, especially from nonlinear equations, can be reduced to an integral equation of Hammerstein's type and thus can be studied from a uniform standpoint. This enables us to carry over the methods and the results from a one group of the problems to another group and, of course, to use the results of the theory of integral equations and in the main, of functional analysis.

The aim of this lecture is to show some methods for obtaining the sufficient conditions for the existence and partly for the uniqueness of the solution of a nonlinear boundary value problem, using the fixed point theorems. The methods may be used in solving of related problems too.

Notations and Assumptions.

Let R^n mean the n-dimensional real Euclidean space and if $x, y \in R^n$, let $|x, y|$ be their distance. $|x, S|$ will mean the distance between the point x and the set $S \subset R^n$. If $x \in R^n$, $\delta > 0$, then $B(x, \delta) = \{y : y \in R^n, |x, y| < \delta\}$. $j \in R^m$ denotes the vector with all its components equal to 1.

Let $D \subset R^n$ be a region, $\bar{D}$ the closure of D, $\emptyset \neq S \subset \bar{D}$ a set. These sets will satisfy

Assumption 1. *Let $D \cup S$ be compact.*

From this assumption it follows that $D \cup S = \bar{D}$ and hence D is bounded and S contains the boundary of D.

Denote $E = (D \cup S) \times R^m$, $E^0 = D \times R^m$ and if $b \geq 0$, let

$$E_b = (D \cup S) \times \underbrace{\langle -b, b \rangle \times \ldots \times \langle -b, b \rangle}_{m\text{-times}}.$$

Further, let U be partially ordered Banach space of all real $m \times 1$ vector functions $u(x) = (u_1(x), \ldots, u_m(x))$, $u(x) \in C_0(D \cup S)$, $x = (x_1, \ldots, x_n)$, with the norm $||u|| = \max\limits_{k=1,\ldots,m} \max\limits_{x \in D \cup S} |u_k(x)|$. If $u, v \in U$, then $u \leqq v$ if and only if for every $k = 1, \ldots, m$, $x \in D \cup S$, $u_k(x) \leqq v_k(x)$ holds. Similarly the sharp inequality is valid and also the inequality in R^m. Denote $|u(x)| = (|u_1(x)|, \ldots, |u_m(x)|)$ and analogically, if $u \in R^m$, then $|u| = (|u_1|, \ldots, |u_m|)$. As usual, if $v_1 \leqq v_2$, then $\langle v_1, v_2 \rangle = \{u : u \in U, v_1 \leqq u \leqq v_2\}$. $\langle v_1, v_2 \rangle$ is a closed, convex and bounded set in U. $E_{v_1,v_2} = \{(x, u) : (x, u) \in E, v_{1k}(x) \leqq u_k \leqq v_{2k}(x), k = 1, \ldots, m\}$, where $v_1(x) = (v_{11}(x), \ldots, v_{1m}(x)) \leqq v_2(x) = (v_{21}(x), \ldots, v_{2m}(x)) \in U$, $u = (u_1, \ldots, u_m)$. $U_b = \{u : u \in U, ||u|| \leqq b\}$.

Similarly as for the vector functions, the matrix function $|G(x, t)|$ is defined by $|G(x, t)| = (|G_{kl}(x, t)|)$ if $G(x, t) = (G_{kl}(x, t))$, $k, l = 1, \ldots, m$. For $H(x) = (H_{kl}(x)) \in C_0(D \cup S)$, $k, l = 1, \ldots, m$, it is $||H(x)|| = \max\limits_{k,l=1,\ldots,m} \max\limits_{x \in D \cup S} |H_{kl}(x)|$. $G(x, t) < H(x, t)$ if and only if $G_{kl}(x, t) < H_{kl}(x, t)$ for every (x, t) of their common domain and all $k, l = 1, \ldots, m$. J (εJ) is the $m \times m$ matrix whose all elements are equal to 1 (are equal to ε). J_0 is the unit $m \times m$ matrix.

In what follows, the matrices and the vectors will be supposed to be of the type $m \times m$ and $m \times 1$, respectively.

Consider the (boundary-value) problem

$$L(u) = f(x, u), \qquad x \in D, \tag{1}$$

$$M(u) = g(x), \qquad x \in S, \tag{2}$$

where $f(x, u) = (f_1(x, u), \ldots, f_m(x, u))$ is a real vector function of the variables $x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_m)$ defined in E, $g(x)$ is a real vector function defined in S, L is a linear differential operator, and M is a linear operator. These functions and operators will be supposed to satisfy an assumption. By a solution of the problem (1), (2) will be meant every $u \in U$ satisfying the equations (1), (2) and possessing as many continuous derivatives as one usually requires from the solution of the problem (1), (2).

Assumption 2. *Let the problem*

$$L(v) = 0j, \qquad x \in D \tag{3}$$

$$M(v) = 0j, \qquad x \in S \tag{4}$$

have only the trivial solution, let there exist a solution $v(x)$ of the problem

$$L(v) = 0j, \qquad x \in D$$

$$M(v) = g(x), \qquad x \in S$$

and the matrix function $G(x, t)$, so called Green's function of the problem (3), (4), *with the following properties:*

1. $\int_D |G(x, t)|\, dt$ *exists for each* $x \in D \cup S$.

2. *Given any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $\int_D |G(x, t) - G(y, t)| \, dt <$ $< \varepsilon J$ *whenever* $|x, y| < \delta$, $x, y \in D \cup S$.

3. *The alternative holds: Either for every* $r(x) \in U$ *the function*

$$w(x) = v(x) + \int_D G(x, t) \, r(t) \, dt, \qquad x \in D \cup S \tag{5}$$

is a solution of the problem

$$L(w) = r(x), \qquad x \in D \tag{6}$$
$$M(w) = g(x), \qquad x \in S \tag{7}$$

or for every $r(x) \in U$ *the function* (5) *satisfies a Hölder's condition and for every* $r(x) \in U$ *satisfying a Hölder's condition the function* (5) *is a solution of the problem* (6), (7).

Remark 1. By the assumption on the problem (3), (4), the solutions $v(x)$, $w(x)$, as well as $G(x, t)$, are uniquely determined ($G(x, t)$ except on a set of Lebesguemeasure zero).

Lemma 1. *Let Assumption* 1 *be fulfilled and let the matrix function* $G(x, t)$ *possess the following properties:*

1. $G(x, t)$ *is defined and continuous for every* $x \in D \cup S$, $t \in D$, $t \neq x$.

2. *For* $x \neq t$ *the function* $G(x, t)$ *is almost uniformly bounded in the sense that, for any* $\delta > 0$, *there exists an* $N = N(\delta) > 0$ *such that* $|G(x, t)| < NJ$ *for all* $x \in D \cup S$, $t \in D$, $|x, t| \geqq \delta$.

3. $\int_D |G(x, t)| \, dt$ *is uniformly convergent for every* $x \in D \cup S$, *that is, given any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $\int_{D \cap B(x, \delta)} |G(x, t)| \, dt < \varepsilon J$ *for all* $x \in D \cup S$.

Then the function $G(x, t)$ *possesses the properties* 1 *and* 2 *from Assumption* 2.

Proof. Obviously $G(x, t)$ has the property 1 from Assumption 2. The property 2 can be shown in this way. By the property 3, there exists $\delta > 0$ such that $\int_{D \cap B(x, \delta)} |G(x, t)| \, dt < \frac{\varepsilon}{3} J$. Suppose $y \in B\left(x, \frac{\delta}{4}\right) \cap (D \cup S)$. Then $\int_{D \cap B(x, \delta)} |G(x, t) - G(y, t)| \, dt \leqq \int_{D \cap B\left(x, \frac{\delta}{2}\right)} |G(x, t)| \, dt + \int_{D \cap B\left(x, \frac{\delta}{2}\right)} |G(y, t)| \, dt <$ $< \frac{2\varepsilon}{3} J$. With respect to the property 1 of $G(x, t)$ $\lim_{y \to x} |G(x, t) - G(y, t)| =$ $= 0J$ for all $t \in D - B\left(x, \frac{\delta}{2}\right)$ $\left(\text{that is, for all } t \in D \text{ such that } |t, x| \geqq \frac{\delta}{2}\right)$. Further the function $|G(x, t)| + N\left(\frac{\delta}{4}\right) J$ is an integrable majorant for $|G(x, t) - G(y, t)|$. By the Lebesgue theorem there exists $0 < \delta_1 = \delta_1(x, \varepsilon) <$

$< \frac{\delta}{4}$ such that $\int\limits_{D-B\left(x,\frac{\delta}{2}\right)} |G(x, t) - G(y, t)|\, dt < \frac{\varepsilon}{3} J$ for $|x, y| < \delta_1$ and hence, $\int\limits_D |G(x, t) - G(y, t)|\, dt < \varepsilon J$. Finally Assumption 1 implies that δ_1 does not depend on x.

Remark 2. If Assumptions 1 and 2 hold, then the function $H(x) = \int\limits_D |G(x, t)|\, dt$ is continuous on $D \cup S$ and $C = ||H(x)|| < \infty$.

Assumption 3. *Let $f(x, u) \in C_0(E)$ and if in Assumption 2 the second part of the alternative is true, let $f(x, u)$ satisfy on every bounded subset $Z \subset E$ a Hölder's condition with constants which may depend on Z.*

Study of the problem (1), (2).

First, the equivalence of this problem to an integral equation will be shown.

Lemma 2. *Let Assumptions 1, 2 and 3 be satisfied. Then, if the first part of alternative in Assumption 2 holds, the boundary-value problem (1), (2) is equivalent to the integral equation*

(8) $$u(x) = v(x) + \int\limits_D G(x, t)\, f(t, u(t))\, dt, \qquad x \in D \cup S.$$

If the second part of alternative is valid, every solution of (8) is a solution of the problem (1), (2) satisfying a Hölder's condition and conversely, every solution of the problem (1), (2) which satisfies a Hölder's condition is a solution of the equation (8), too. Here the only request on the solution of (8) is to be of U.

The equation (8) is a functional equation of the type

(9) $$u = Tu.$$

The properties of the operator T defined for every $u \in U$ by

(10) $$Tu = v + \int\limits_D G(x, t)\, f(t, u(t))\, dt$$

will now be considered.

Lemma 3. *If Assumptions 1, 2 and 3 hold, the operator T given by (10) is continuous, compact and $TU \subset U$.*

Proof. Let $\varepsilon > 0$ and $b > 0$ be arbitrary numbers. From the inequality $|Tu_1 - Tu_2| \leq \int\limits_D |G(x, t)|\, |f(t, u_1(t)) - f(t, u_2(t))|\, dt$ and from the uniform continuity of $f(x, u)$ on E_b follows the existence of such a $\delta = \delta(b, \varepsilon)$ that $||Tu_1 - Tu_2|| < \varepsilon mC$ for $u_1, u_2 \in U_b$, $||u_1 - u_2|| < \delta$. Thus T is continuous on U_b. Denote $K_b = \max\limits_{k=1,\dots,m} \max\limits_{(x,u)\in E_b} |f_k(x, u)|$. If $u \in U_b$, then $|Tu(x) -$

$- Tu(y)| \leqq |v(x) - v(y)| + K_b \int_D |G(x, t) - G(y, t)| j \, dt$. Hence for a sufficiently small $\delta > 0$, on the basis of Assumption 2, $|Tu(x) - Tu(y)| < \varepsilon(1 + mK_b) j$ follows from $|x, y| < \delta$, $x, y \in D \cup S$. Finally, $|Tu(x)| \leqq (\|v\| + K_b Cm) j$. By the Ascoli theorem one gets that TU_b is relatively compact. At the same time $TU \subset U$ was proved.

Consider the interval $\langle v - bj, v + bj \rangle$, $b > 0$. Let $K_{v,b} = \max |f_k(x, u)|$ for $k = 1, \ldots, m$, $(x, u) \in E_{v-bj, v+bj}$. For $u \in \langle v - bj, v + bj \rangle$ the inequality $|Tu - v| \leqq mC. \ K_{v,b} j$ is valid. From it, using Lemmas 2, 3 and Schauder's fixed point theorem ([1], p. 355) one obtains

Theorem 1. *Let Assumptions 1, 2 and 3 be satisfied. Let $b > 0$ exist, for which*

$$mCK_{v,b} \leqq b.$$

Then there exists at least one solution of the problem (1), (2) *contained in the interval $\langle v - bj, v + bj \rangle$ (which satisfies a Hölder's condition if in Assumption 2 the second part of the alternative holds).*

With help of the Schauder theorem a generalization of the first Fredholm theorem was proved by another Polish mathematician A. Lasota. This affirms that a nonlinear equation has at least one solution if a certain system of homogeneous linear equations possesses only the trivial solution.

Let R be a Banach space. Let $L_s(R, R)$ be the space of all linear (additive and homogeneous) operators on R into R. In the space $L_s(R, R)$ the simple convergence is defined as follows: The sequence $\{A_n\} \subset L_s(R, R)$ converges simply to $A \in L_s(R, R)$ $(A_n \xrightarrow[s]{} A)$ if for each $z \in R$ $A_n z \to Az$.

Lasota's Theorem ([2], p. 89—91). *Let $Q \subset L_s(R, R)$ be a set satisfying the following conditions:*

1. *Each sequence $\{A_n\} \subset Q$ contains a subsequence $A_{n_k} \xrightarrow[s]{} A \in Q$.*

2. *The set $\bigcup_{A \in Q, \|z\| = 1} A \, z$ is relatively compact in R.*

Suppose that for each $A \in Q$ the equation

$$z = A \, z$$

has only the trivial solution.

Further let $A = A(z)$ be the operator on R into Q such that

3. *$z_n \to z$ implies $A(z_n) \xrightarrow[s]{} A(z)$.*

Finally, let $b(z)$ be the operator which maps R into R and satisfies the conditions:

4. *$b(z)$ is compact.*

5. $\lim_{\|z\| \to \infty} (\|z\|^{-1} \|b(z)\|) = 0.$

Under these assumptions there exists at least one solution of the equation

$$z = A(z)\,z + b(z).$$

From this theorem, by method used in the paper [3], one gets

Theorem 2. *Let Assumptions* 1, 2 *and* 3 *be fulfilled. Let* $f(x, u)$ *satisfy the inequality*

$$|f(x, u)| \leqq (N + \sum_{l=1}^{m} L_l |u_l|)\, j \tag{11}$$

on E, *where* $N \geqq 0$, $L_l \geqq 0$, $l = 1, \ldots, m$, *are arbitrary constants. Let the equation*

$$u(x) = \int_D G(x, t)\, F(t)\, u(t)\, dt$$

have only the trivial solution for every matrix function $F(x) = (F_{kl}(x))$, *where* $F_{kl}(x) = a_l(x)$, $k, l = 1, \ldots, m$, $a_l(x)$ *are measurable on* D *and satisfy the inequality*

$$|a_l(x) \leqq L_l, \qquad l = 1, \ldots, m.$$

Then the problem (1), (2) *has at least one solution.*

Proof. Defining the vector functions

$$p_k(x, u) = f(x, u)\,(N + \sum_{l=1}^{m} L_l |u_l|)^{-1} L_k \eta(u_k), \qquad k = 1, \ldots, m,$$

$$q(x, u) = f(x, u) - \sum_{l=1}^{m} p_l(x, u)\, u_l,$$

where the scalar function $\eta(u) = u$ for $|u| \leqq 1$, $\eta(u) = \operatorname{sgn} u$, $|u| > 1$, (here u is scalar variable) the equation (8) can be rewritten in the form

$$u(x) = \int_D G(x, t) [\sum_{l=1}^{m} p_l(t, u(t))\, u_l(t)]\, dt + \tag{12}$$

$$+ \int_D G(x, t)\, q(t, u(t))\, dt + v(x).$$

The functions $p_k(x, u)$, $q(x, u) \in C_0(E)$ and, by (11), they satisfy

$$|p_k(x, u)| \leqq L_k j, \qquad |q(x, u)| \leqq (N + \sum_{l=1}^{m} L_l)\, j. \tag{13}$$

Denote the set of all matrix functions F satisfying the assumption of Theorem 2, by M_F. Let Q be the set of all operators A from U into U defined by the relation

$$w = Au = \int_D G(x, t)\, F(t)\, u(t)\, dt, \qquad F(x) \in M_F. \tag{14}$$

By the assumption the equation $u = Au$ has for each $A \in Q$ only the trivial solution. Further for $||u|| = 1$, $u \in U$ and $L_0 = L_1 + \ldots + L_m$ $|Au(x) - Au(y)| \leqq m\, \varepsilon\, L_0 j$ for $|x, y| < \delta$ by Assumption 2. Moreover $|Au(x)| \leqq mCL_0 j$, $x \in D \cup S$, and thus by Ascoli's Theorem the set $\bigcup_{A \in Q,\ ||u||=1} Au$ is relatively compact.

Denote $A_n u = \int_D G(x, t)\ F_n(t)\ u(t)\, \mathrm{d}t = \int_D G(x, t) \sum_{l=1}^{m} a_{l,n}(t)\ u_l(t)\, j\, \mathrm{d}t$. In view of $|a_{l,n}(x)| \leqq L_l$, for each $l = 1, \ldots, m$ the set $\{a_{l,n}(x)\}$ is weakly compact in $L_1(D)$ and therefore there exists $a_l(x) \in L_1(D)$ and a subsequence $\{a_{l,n_k}(x)\}$ such that for every $g(x) \in L_1(D)$

$$\lim_{k \to \infty} \int_D g(t)\, a_{l,n_k}(t)\, \mathrm{d}t = \int_D g(t)\, a_l(t)\, \mathrm{d}t$$

holds. Obviously $|a_l(x)| \leqq L_l$ and besides, we can reach that $\{n_k\}$ is the same for all $l = 1, \ldots, m$. Thus for each $x \in D \cup S$ and $u \in U$ there exists

$$\lim_{k \to \infty} \int_D G(x, t) \sum_{l=1}^{m} a_{l,n_k}(t)\ u_l(t)\, j\, \mathrm{d}t = \int_D G(x, t) \sum_{l=1}^{m} a_l(t)\ u_l(t)\, j\, \mathrm{d}t. \tag{15}$$

The functions (14) being equicontinuous on $D \cup S$, the convergence (15) is uniform.

For each $u \in U$ define the operator $A(u) \in Q$ by the relation

$$w = A(u)\, y = \int_D G(x, t) \sum_{l=1}^{m} p_l(t, u(t))\, y_l(t)\, \mathrm{d}t.$$

If $u_n \to u$, then $A(u_n) \xrightarrow[s]{} A(u)$. In fact, denoting $w_n = A(u_n)\, y$, $w = A(u)\, y$, the inequality

$$|w_n - w| \leqq ||y|| \int_D |G(x, t)| \sum_{l=1}^{m} ||p_l(t, u_n(t)) - p_l(t, u(t))||\, j\, \mathrm{d}t \leqq$$
$$\leqq ||y||\, m^2 \max_{l=1,\ldots,m} ||p_l(t, u_n(t)) - p_l(t, u(t))||\, Cj$$

holds, which implies $||w_n - w|| \to 0$ from $||u_n - u|| \to 0$.

Consider now the operator

$$w = bu = \int_D G(x, t)\, q(t, u(t))\, \mathrm{d}t + v(x).$$

From (13) follows $||bu|| \leqq (N + L_0)\, mC + ||v||$, so that the operator b is bounded. Obviously it is also continuous. Finally, from the inequality $|bu(x) - bu(y)| \leqq (N + L_0 + 1)\, m\, \varepsilon\, j$ for $|x, y| < \delta$, δ is sufficiently small, follows the relative compactness of bU in U.

Thus, all assumptions of Lasota's Theorem being satisfied, the equation (12) has at least one solution in U.

In the following, some theorems will be proved, where the properties of the partially ordered space U will be used. The first ones will be the theorems of a comparison character. Examples of such theorems can be found in the paper [4]. Here the following definition will be of use.

The function $h(x, u)$ defined on $E_{v_1 v_1}$ will be said to be nondecreasing (nonincreasing) in u on $E_{v_1 v_2}$ if for each $x \in D \cup S$ $h(x, u_1) \leqq h(x, u_2)$ $(h(x, u_1 \geqq \geqq h(x, u_2)$ whenever $v_1(x) \leqq u_1 \leqq u_2 \leqq v_2(x)$.

Theorem 3. *Let Assumptions* 1, 2 *and* 3 *hold. Let the Green's function* $G(x, t) \geqq 0J$ $(\leqq 0J)$ *for all points of its domain. Let there exist the vector functions* $h_j(x, t)$, $j = 1, 2$, *with the following properties*:

a. $h_j(x, t)$ *satisfy Assumption* 3.

b. The problem
$$L(u) = h_j(x, u), \quad x \in D$$
$$M(u) = g(x), \quad x \in S$$

has a solution $v_j(x)$ *and* $v_1 \leqq v_2$. *If the second part of the alternative in Assumption* 2 *is valid, then* $v_j(x)$ *satisfy a Hölder's condition.*

c. If $G(x, t) \geqq 0J$ $(\leqq 0J)$. *the functions* $h_j(x, u)$ *are nondecreasing (nonincreasing) in* u *on* E_{v_1, v_2} *and satisfy the inequalities*

$$h_1(x, u) \leqq f(x, u) \leqq h_2(x, u)$$
$$(h_1(x, u) \geqq f(x, u) \geqq h_2(x, u))$$

there. Then the problem (1), (2) *has at least one solution in* $\langle v_1, v_2 \rangle$.

Proof. With respect to Lemma 3 it suffices to prove that $T\langle v_1, v_2 \rangle \subset \subset \langle v_1, v_2 \rangle$. Assume $G(x, t) \geqq 0J$. If $u \in \langle v_1, v_2 \rangle$, then $G(x, t)\, h_1(t, v_1(t)) \leqq \leqq G(x, t)\, h_1(t, u(t)) \leqq G(x, t) f(t, u(t)) \leqq G(x, t)\, h_2(t, u(t)) \leqq G(x, t)\, h_2(t, v_2(t))$. From these inequalities the assertion of the theorem follows. The case $G(x, t) \leqq 0J$ is proved analogically.

Theorem 4. *Let Assumptions* 1, 2 *and* 3 *hold. Let in Assumption* 2 *mentioned Green's function* $G(x, t) \geqq 0J$ $(\leqq 0J)$ *and the solution* $v(x) \geqq 0j$ $(\leqq 0j)$ *for all points of their domain. Let there exist a vector function* $h(x, u)$ *with the properties*

a. $h(x, t) \geqq 0j$.

b. $h(x, t)$ *satisfies Assumption* 3.

c. The problem
$$L(u) = h(x, u), \quad x \in D$$
$$M(u) = g(x), \quad x \in S$$

has a solution $v_0(x)$ *(satisfying a Hölder's condition if the second part of the alternative in Assumption* 2 *holds).*

d. If $G(x, t) \geqq 0J$ $(\leqq 0J)$, *then* $h(x, u)$ *is nondecreasing (nonincreasing) in* u *on* E_{-v_0, v_0} $(E_{v_0, -v_0})$ *and the inequality*

$$|f(x, u)| \leqq h(x, u)$$

holds there.

Then the problem (1), (2) *has at least one solution contained in the interval* $\langle -v_0, v_0\rangle$ $(\langle v_0, -v_0\rangle)$.

Proof. Let $G(x, t) \geqq 0J$, $v(x) \geqq 0j$. Then for $-v_0 \leqq u \leqq v_0$ the inequalities $-G(x, t)\, h(t, v_0(t)) \leqq -G(x, t)\, h(t, u(t)) \leqq G(x, t)\, f(t, u(t)) \leqq G(x, t)\, h(t, u(t)) \leqq$ $\leqq G(x, t)\, h(t, v_0(t))$ hold, whence it follows that $-v_0 + 2v \leqq Tu \leqq v_0$.

The case $G(x, t) \leqq 0J$, $v(x) \leqq 0j$ is proved analogically.

A further result can be obtained by using the method developed in [1], p. 277—280. This method is based on the assumption that the operator T given by (10) is decomposable into a sum of an isotone operator T_1 and an antitone operator T_2, $T_1U \subset U$, $T_2U \subset U$.

If two elements v_0, $w_0 \in U$ are chosen, by the relations

$$\begin{aligned} v_{n+1} &= T_1v_n + T_2w_n \\ w_{n+1} &= T_1w_n + T_2v_n, \qquad n = 0, 1, \ldots, \end{aligned}$$

the sequences $\{v_n\}$, $\{w_n\}$ are defined. If

$$v_0 \leqq w_0, \qquad v_0 \leqq v_1, \qquad w_1 \leqq w_0$$

hold, then for all $n = 0, 1, 2, \ldots,$

$$v_n \leqq w_n, \qquad v_n \leqq v_{n+1}, \qquad w_{n+1} \leqq w_n \tag{16}$$

and $T\langle v_n, w_n\rangle \subset \langle v_{n+1}, w_{n+1}\rangle$. Assuming T is continuous and compact there exists $\lim\limits_{n\to\infty} v_n = \bar{v}$, $\lim\limits_{n\to\infty} w_n = \bar{w}$, $\bar{v} \leqq \bar{w}$. The operator T has at least one fixed point in the interval $\langle \bar{v}, \bar{w}\rangle$. Each fixed point of T, belonging to $\langle v_0, w_0\rangle$, is contained in $\langle \bar{v}, \bar{w}\rangle$. Moreover, if T is isotone, then both points v, w are its fixed points.

With help of this consideration the following theorem will be proved. For the sake of simplicity denote $G^+(x, t) = \dfrac{1}{2}(G(x, t) + |G(x, t)|)$, $G^-(x, t) =$ $= \dfrac{1}{2}(G(x, t) - |G(x, t)|)$. Then $G(x, t) = G^+(x, t) + G^-(x, t)$.

Theorem 5. *Let Assumption* 1 *hold. Let there exist a matrix function* $P_1(x)$ $(P_2(x))$ *defined on* $D \cup S$ *with the properties:*

a. The operator $L_1(u) = L(u) - P_1(x)\, u$ $(L_2(u) = L(u) - P_2(x)\, u)$, *as well as* $M(u)$, *satisfies Assumption* 2 *with the Green's function* $G_1(x, t)$ $(G_2(x, t))$.

b. The function $f_1(x, u) = f(x, u) - P_1(x)\, u$ $(f_2(x, u) = f(x, u) - P_2(x)\, u)$ *satisfies Assumption* 3.

c. The function $f_1(x, u)$ $(f_2(x, u))$ *is nondecreasing in* u *on* E *(nonincreasing in* u *on* E*).*

d. There exists a pair of functions v_0, $w_0 \in U$, $v_0 \leqq w_0$, such that for $n = 0$ the functions v_{n+1}, w_{n+1} defined by the relations

$$(17) \qquad v_{n+1}(x) = v(x) + \int_D G_1^+(x, t) f_1(t, v_n(t))\, dt + \int_D G_1^-(x, t) f_1(t, w_n(t))\, dt$$

$$w_{n+1}(x) = v(x) + \int_D G_1^+(x, t) f_1(t, w_n(t))\, dt + \int_D G_1^-(x, t) f_1(t, v_n(t))\, dt$$

$$(17') \qquad v_{n+1}(x) = v(x) + \int_D G_2^-(x, t) f_2(t, v_n(t))\, dt + \int_D G_2^-(x, t) f_2(t, w_n(t))\, dt$$

$$w_{n+1}(x) = v(x) + \int_D G_2^-(x, t) f_2(t, w_n(t))\, dt + \int_D G_2^-(x, t) f_2(t, v_n(t))\, dt$$

satisfy the inequaiity (16).

Then the following assertions are true:

1. *The functions $v_n(x)$, $w_n(x)$ given by the recirsive relations* (17) ((17')) *fulfil the inequalities* (16) *for every $n \geqq 0$ and there exists* $\lim_{n\to\infty} v_n(x) = \bar{v}(x)$ $\lim_{n\to\infty} w_n(x) = \bar{w}(x)$, $\bar{v}(x) \leqq \bar{w}(x)$.

2. *The problem* (1), (2) *has at least one solution in the interval* $\langle \bar{v}, \bar{w} \rangle$.

3. *Each solution of the problem* (1), (2) *belonging to* $\langle v_0, w_0 \rangle$ *is contained in* $\langle \bar{v}, \bar{w} \rangle$.

4. *If $G_1(x, t) \geqq 0J$ ($G_2(x, t) \leqq 0J$), then both functions $\bar{v}(x)$, $\bar{w}(x)$ are solutions of* (1), (2).

Proof. In the sense of Lemma 2 the problem (1), (2) is equivalent to the equation

$$u(x) = (v(x) + \int_D G_1^+(x, t) f_1(t, u(t))\, dt) + \int_D G_1^-(x, t) f_1(t, u(t)) dt = T_1 u + T_2 u,$$

where T_1 is an isotone and T_2 an antitone operator. Analogous result is obtained in the second case.

Remark 3. Theorem 5 represents a generalization of Theorem 1 in the paper [5].

Remark 4. More general results could be obtained using a Schröder's theorem ([1], p. 293).

The theory of pseudometric spaces yields great consequences for the theorems on existence and uniqueness of fixed points of functional operators. The basic facts of that theory are mentioned in [1], p. 40—44. A very general theorem on existence and uniqueness of the solutions of operator equations in pseudometric space was proved by a German mathematician J. Schröder ([1], p. 164—269). This theorem comprises Banach's Theorem and, slightly modified, the Kantorovič fixed point theorem ([6], p. 358). For the sake of simplicity, it will be mentioned here in a weaker form (the operator P will be supposed to be linear).

Schröder's fixed point theorem in a weaker form. *Let equation* (9) *be given and assume the following conditions hold:*

1. *The domain X of the operator T is contained in a complete pseudometric space R with the associated partially ordered linear space H. $TX \subset R$.*

2. *The operator T is bounded, that is, there exists a linear, continuous, and positive operator P defined on H, $PH \subset H$, with the property*

$$\varrho(Tu, Tw) \leqq P\varrho(u, w) \qquad \text{for each pair } u, w \in X. \tag{18}$$

3. *If $u_0 \in X$ is given, then the sequence σ_n defined by*

$$\sigma_n = P\sigma_{n-1} + \varrho(u_0, Tu_0), \qquad \sigma_0 = 0$$

converges. Its limit will be denoted by σ.

4. *The sphere γ of elements $w \in R$ satisfying the inequality*

$$\varrho(w, Tu_0) \leqq \sigma - \varrho(u_0, Tu_0) \tag{19}$$

is contained in X, or

4.' *X is complete and all u_n given recursively by*

$$u_n = Tu_{n-1}, \qquad n = 1, 2, \ldots, \tag{20}$$

are contained in X.

Then there exists at least one solution of the equation (9) *and the sequence u_n, given by* (20), *converges to such a solution. All u_n and u are contained in γ and the following estimate*

$$\varrho(u, u_n) \leqq \sigma - \sigma_n$$

holds.

Remark 5. The conditions **3** any **4** can be replaced by stronger conditions

3.' $\sum_{j=0}^{\infty} p^j f$ exists for each $f \in H$. ($P^0 = I$ means the identity operator.) (It suffices to consider only $f \geqq 0$.)

4.'' The sphere γ of elements $w \in R$ satisfying the inequality

$$\varrho(w, Tu_0) \leqq (I - P)^{-1}\varrho(u_0, Tu_0) - \varrho(u_0, Tu_0)$$

as well as u_0, are contained in X.

Theorem on uniqueness. *Under the assumptions* 1 *through* 4 *of the last theorem the sphere γ given by* (19) *contains at most one solution of the equation* (9).

Lemma 4. *If the assumptions* 1, 2 *and* 3' *of the weakened Schröder's fixed point theorem hold, whereby R need not be complete and P continuous, then there exists at most one solution of* (9) *in X.*

Proof. Obviously P is isotone. Assume $w_1 = Tw_1$, $w_2 = Tw_2$. By (18), then it is $\varrho(w_1, w_2) = f \leqq Pf$ and further, $f \leqq Pf \leqq P^2 f \leqq \ldots$. Hence ($0 \leqq$) $f \leqq \frac{1}{n+1} \sum_{j=0}^{n} P^j f$. Since $\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} P^j f = 0$, it follows that $f = 0$.

As an application of the Schröder's theorem the following theorem will be mentioned here (compare with an analogous Schröder's theorem in [1], p. 202).

Theorem 6. *Let Assumptions* 1, 2 *and* 3 *hold. Let there exist a matrix function* $N(x)$, *bounded and measurable on* D *such that for every* (x, u_1) *and* (x, u_2) *in* E^0

$$|f(x, u_1) - f(x, u_2)| \leqq N(x)\,|u_1 - u_2|.$$

Let the greatest positive eigenvalue λ *(provided positive eigenvalues exist) of the operator* P *defined by*

$$Pu = \int_D |G(x, t)|\, N(t)\, u(t)\, dt$$

satisfy the inequality $\lambda < 1$.

Then the following is true:

1. *There exists at most one solution of the problem* (1), (2).
2. *If* $u_0 \in U$ *is chosen, the sequence* u_n *defined for* $n = 1, 2, \ldots,$ *by*

$$\begin{aligned} L(u_n) &= f(x, u_{n-1}(x)), \qquad && x \in D \\ M(u_n) &= g(x), && x \in S \end{aligned}$$

converges to the solution u *of the problem* (1), (2), *whereby all* u_n *and* u *are contained in the sphere* γ *of the elements* w *satisfying the inequality*

$$|w(x) - u_1(x)| \leqq \sigma(x) - |u_0(x) - u_1(x)|, \qquad x \in D \cup S$$

where $\sigma(x)$ *is a solution of the equation*

$$\sigma(x) = |u_0(x) - u_1(x)| + P\sigma(x).$$

Here the solution of the problem (1), (2) *satisfying a Hölder's condition is dealt with if in Assumption* 2 *the second part of the alternative holds.*

Proof. Consider the pseudometric space V of all vector functions $f \in C_0(D \cup S)$ with the pseudometric $\varrho(f, g) = |f(x) - g(x)|$. By the convergence in this space is understood the uniform convergence on $D \cup S$. V, as well as each interval contained in it, are complete. The operator P is linear, positive and compact. From the inequality $\lambda < 1$, by the Theorem on alternative ([1], p. 244), it follows that for every $f \geqq 0j$, $f \in V$, there exists a unique solution $u_f \geqq 0j$ of $u = Pu + f$. Define the sequence σ_n by $\sigma_n = P\sigma_{n-1} + f$, $\sigma_0 = 0$. Then $\sigma_n = \sum_{j=0}^{n-1} P^j f$ and $\sigma_{n-1} \leqq \sigma_n$, $\sigma_n \leqq u_f$ for every $n \geqq 1$. At the same time σ_n form an equicontinuous set of functions. By Ascoli's Theorem there exists their uniform limit σ. From Schröder's theorem and Lemma 4 the assertion of the theorem follows.

As an illustration of possibilities of this theory the Rozenblatt—Nagumo theorem will be generalized. By the PERRON method ([7], p. 216—217) the following theorem can be proved.

Theorem 7. *Let Assumptions* 1, 2 *and* 3 *be satisfied, whereby let* S *be the* *boundary of D and $G(x, t)$ need not have the property 2 from Assumption 2. Further assume that:*

a. There exists a constant $N > 0$ such that

$$|f(x, u_1) - f(x, u_2)| \leqq \frac{N}{|x, S|} |u_1 - u_2|$$

for every $(x,, u_1)$, $(x, u_2) \in E^0$.

$$b.\ N \int_D |G(x, t)|\, j\, dt \leqq |x, S|\, j.$$

c. For any two solutions u_1, u_2 of the problem (1), (2) there exists

$$\lim_{y \to x} \frac{|u_1(y) - u_2(y)|}{|y, S|} = 0, \qquad \textit{for each } x \in S.$$

Then there exists at most one solution of the problem (1), (2) (satisfying a Hölder's condition if the second part of alternative in Assumption 2 is valid).

Proof. For any two solutions u_1, u_2 of the problem (1), (2) (satisfying a Hölder's condition if need be) the inequality

$$|u_1(x) - u_2(x)| \leqq N \int_D |G(x, t)| \frac{|u_1(t) - u_2(t)|}{|t, S|} dt$$

holds. The function $p(x) = \dfrac{|u_1(x) - u_2(x)|}{|x, S|}$, $x \in D$, $p(x) = 0$, $x \in S$, is continuous on $D \cup S$. If $p(x) \not\equiv 0j$, then $||p(x)|| = p > 0$. Moreover,

$$N \int_D |G(x, t)|\, p(t)\, dt < Np \int_D |G(x, t)|\, j\, dt \leqq p\, |x, S|\, j.$$

Combining the last inequality with the foregoing one, there results finally $p(x) < pj$ for each $x \in D$ but this leads to a contradiction.

Remark 6. The assertion of the theorem remains valid if the points b. and c. are replaced by the points:

b.' $$N \int_D |G(x, t)|\, j\, dt < |x, S|\, j.$$

c.' For any two solutions u_1, u_2 of the problem (1), (2) there exists a finite $\lim\limits_{\substack{y \to x \\ y \in D}} \dfrac{|u_1(y) - u_2(y)}{|y, S|}$ for each $x \in S$, which is continuous on S.

The developed theory will be illustrated on the following example.

Let $f(x, u) = (f_1(x, u), \ldots, f_m(x, u)) \in C_0(\langle 0, 1\rangle \times R^m)$ be a vector function of the variables x, $u = (u_1, \ldots, u_m)$, let it be periodic in x of period 1, $f(x, u) \equiv \equiv f(x + 1, u)$. Consider the periodic boundary-value problem

(21) $$u' = f(x, u)$$
(22) $$u(0) - u(1) = 0.$$
By [8], p. 718, the problem is equivalent to the integral equation

$$u(x) = \int_0^1 G(x, t)\,[f(t, u(t)) - u(t)]\,\mathrm{d}t$$

where the matrix function $G(x, t)$ is of the form

$$G(x, t) = \begin{cases} \dfrac{1}{1-e}\, e^{x-t} J_0, & 0 \leqq t \leqq x \leqq 1 \\[2ex] \dfrac{e}{1-e}\, e^{x-t} J_0, & 0 \leqq x < t \leqq 1 \end{cases}$$

It is easy to see that Assumptions 1, 2 and 3, as well as the assumptions of Lemma 1, are satisfied. Further $G(x, t) \leqq 0J$. The function $H(x) = \int_0^1 |G(x, t)|\,\mathrm{d}t = J_0$. Hence $C = \|H(x)\| = 1$. Let $K_b = \max\limits_{k=1,\ldots,m} \max\limits_{\substack{0 \leqq x \leqq 1 \\ |u| \leqq bj}} |f_k(x, u) - u_k|$.

Consider the operator P_j, $j = 1, 2$, given by the relation

$$P_1 u = \int_0^1 G(x, t)\, F(t)\, u(t)\,\mathrm{d}t, \qquad P_2 u = \int_0^1 |G(x, t)|\, N(t)\, u(t)\,\mathrm{d}t,$$

where $F(x)$ is a matrix function satisfying the conditions mentioned in Theorem 2 on $\langle 0, 1\rangle$ and $N(x) \in C_0(\langle 0, 1\rangle)$ is a matrix function, $u(x) \in C_0(\langle 0, 1\rangle)$ is any vector function. Then $\|P_1\| \leqq (L_1 + \ldots + L_m)$, $\|P_2\| \leqq \|N\|\, m$.

From Theorems 1, 2 and 6 these sufficient conditions for the existence of the solution of the problem (21), (22) follow.

Theorem 8. *The following statements hold:*

1. *If there exists a $b > 0$, for which $m\, K_b \leqq b$ (especially, if $f(x, u) - u$ is bounded on $\langle 0, 1\rangle \times R^m$), then there exists at least one solution of the problem* (21), (22) *in the interval $\langle -bj, bj\rangle$.*

2. *If $|f_k(x, u) - u_k| \leqq (N + \sum\limits_{l=1}^{m} + L_l|u_l|)$, $k = 1, \ldots, m$, $x \in \langle 0, 1\rangle$, $u \in R^m$, $N \geqq 0$, $L_l \geqq 0$, $l = 1, \ldots, m$ are constants and*

$$\sum_{l=1}^{m} L_l < 1,$$

then there exists at least one solution of the problem (21), (22).

3. *If $|f(x, u_1) - u_1 - f(x, u_2) + u_2| \leqq N(x)\, |u_1 - u_2|$, where for the matrix function $N(x) \in C_0(\langle 0, 1\rangle)$ the inequality $\|N(x)\| < \dfrac{1}{m}$ holds, then there exists a unique solution of* (21), (22).

BIBLIOGRAPHY

[1] L. Collatz: *Funktionalanalysis und numerische Mathematik, Springer Verlag, 1964.*

[2] A. Lasota: *Sur une généralisation du premier théorème de Fredholm, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., 11 (1963), 89—94.*

[3] A. Lasota: *Sur l'existence de solutions des problèmes aux limites de Neumann et de Dirichlet pour l'équation différentielle elliptique de second ordre, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. 11 (1963), 441—446.*

[4] Т. А. Бессмертных: *Несколько замечаний к вопросу о существовании решения у сингулярных систем обыкновенных дифференциальных уравнений, Приближенные методы решения дифференциальных уравнений, вып. 2, Киев 1964, 23—32.*

[5] С. А. Пак: *Теоремы о дифференциальных неравенствах для многоточечных краевых задач, Приближенные методы решения дифференциальных уравнений, вып. 2, Киев 1964, 106—114.*

[6] Б. З. Вулих: *Введение в теорию полупорядоченных пространств, Москва 1961.*

[7] O. Perron: *Eine hinreichende Bedingung für die Unität der Lösung von Differentialgleichungen erster Ordnung, Math. Zeitschr., 28 (1928), 216—219.*

[8] A. Lasota, Z. Opial: *Linear Problems for Ordinary Non-linear Differential Equations and Integral Equations of Hammerstein's Type, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., 12 (1965), 715—718.*

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
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EXTENSION OF THE AVERAGING METHOD TO STOCHASTIC EQUATIONS

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This lecture was devoted to Ito's stochastic equations. These equations are usually written in the integral form

$$(1)\qquad x(t, \omega) = x_0(\omega) + \int_{t_0}^{t} a(\tau, x(\tau, \omega))\,\mathrm{d}\tau + \int_{t_0}^{t} B(\tau, x(\tau, \omega))\,\mathrm{d}w(\tau, \omega)$$

or in the equivalent differential form

$$(1')\qquad \mathrm{d}x(t, \omega) = a(t, x(t, \omega))\,\mathrm{d}t + B(t, x(t, \omega))\,\mathrm{d}w(t, \omega)$$

The expressions $w(t, \omega)$ and $x(t, \omega)$ are random processes, i.e. there is given a triplet $(\Omega, \mathscr{F}, P)$ where Ω is a space, $\mathscr{F}$ is a σ-field of subsets of Ω and P is a probability measure which is defined on $\mathscr{F}$. All random processes or random values are $\mathscr{F}$-measurable functions of the parameter ω. Let R_n denote the n-dimensional Euclidean space. First the conditions are given under which the existence theorem holds:

1) Let $w(t, \omega)$ be a vector random process with stochastically independent increments and $F(t)$ a continuous function such that

$$E\,||w(t_2, \omega) - w(t_1, \omega)||^2 = F(t_2) - F(t_1), \qquad E(w(t_2, \omega) - w(t_1, \omega)) = 0$$

where E means the expectation.

2) There are given a vector function $a(t, x)$ and a matrix function $B(t, x)$ where x is also an n-dimensional vector. $a(t, x)$, $B(t, x)$ are continuous in both arguments and Lipschitz continuous in x:

$$||a(t, x) - a(t, y)|| \leqq K\,||x - y||, \qquad ||B(t, x) - B(t, y)|| \leqq K\,||x - y||.$$

3) There is given a random value $x_0(\omega)$ which is stochastically independent of all increments of $w(t, \omega)$ and $E||x_0(\omega)||^2 < \infty$.

Under these assumptions we can find the solution of (1) in the space $R_n \times \Omega$ of random processes $z(t, \omega)$ with the norm $\sqrt{E \sup_{\tau\in\langle 0,t\rangle} ||z(\tau, \omega)||^2}$. It is possible to prove this statement by means of the method of successive approximations, which converge in this space.

Now we can already pass to the average theory. Let us assume that the process $w_\varepsilon(t, \omega)$ and the function $F_\varepsilon(t)$ depend on a „small" parameter ε for $\varepsilon \in \langle 0, \delta \rangle$ and that the following assumptions are fulfilled:

4) $w_\varepsilon^*(t, \omega) = w_\varepsilon(t, \omega) - w_0(t, \omega)$ is a process with stochasically independent increments again and $\lim\limits_{\varepsilon \to 0} E\, ||w_\varepsilon^*(t_2, \omega) - w_\varepsilon^*(t_1, \omega)||^2 = 0$ uniformly on every compact set of t_1, t_2

5) $\mathscr{F}_\varepsilon(t) \subset \mathscr{F}_0(t), \quad \mathscr{F}_\varepsilon(t) \subset \mathscr{F}_\varepsilon^*(t)$ or

5') $\mathscr{F}_0(t) \subset \mathscr{F}_\varepsilon(t), \quad \mathscr{F}_0(t) \subset \mathscr{F}_\varepsilon^*(t)$

where $\mathscr{F}_0(t)$, $\mathscr{F}_\varepsilon(t)$, $\mathscr{F}_\varepsilon^*(t)$ are the smallest σ-fields corresponding to $w_0(t, \omega)$, $w_\varepsilon(t, \omega)$, $w_\varepsilon^*(t, \omega)$.

6) $a(t, x, \varepsilon)$ depends on ε for $\varepsilon \in \langle 0, \delta \rangle$ such that K in 2) is independent of ε, there exists a continuous function $\psi(t)$ such that $\int\limits_{t_1}^{t_2} ||a(t, 0, \varepsilon)||\, dt \leqq \psi(t_2) - \psi(t_1)$ and a function $\varphi(\varepsilon) > 0$, $\varphi(\varepsilon) \to 0$ for $\varepsilon \to 0$ such that

$$|| \int\limits_{t_1}^{t_2} (a(\tau, x, \varepsilon) - a(\tau, x, 0))\, d\tau || \leqq \varphi(\varepsilon)\, (1 + ||x||) \qquad \text{for } t_1 \leqq t_2 \leqq t_1 + 1.$$

7) $B(t, x, \varepsilon)$ depends on ε for $\varepsilon \in \langle 0, \delta \rangle$. The constant K in 2) is independent of ε,

$$\int\limits_{t_1}^{t_2} ||B(t, 0, \varepsilon)||^2\, dF_\varepsilon(t) \leqq \psi(t_2) - \psi(t_1)$$

and

$$\int\limits_{t_1}^{t_2} ||B(\tau, x, \varepsilon) - B(\tau, x, 0)||^2\, dF_\varepsilon(\tau) \leqq \varphi(\varepsilon)\, (1 + ||x||^2)$$

for $t_1 \leqq t_2 \leqq t_1 + 1$, the functions $\varphi(\varepsilon)$, $\psi(t)$ being the same as in 6).

8) $x_0^{(\varepsilon)}(\omega)$ depends on ε for $\varepsilon \in \langle 0, \delta \rangle$ such that $x_0^{(\varepsilon)}(\omega)$ is stochastically independent of all increments of the processes $w_\varepsilon(t, \omega)$ and $w_0(t, \omega)$. The initial value $x_0^{(0)}(\omega)$ is stochastically independent of all increments of all the processes $w_\varepsilon(t, \omega)$ and $E||x_0^{(\varepsilon)}(\omega) - x_0^{(0)}(\omega)||^2 \to 0$ for $\varepsilon \to 0$.

Now ewerything is prepared to formulate the

Theorem 1. *Let the stochastic equations*

$$(2) \qquad x_\varepsilon(t, \omega) = x_0^{(\varepsilon)}(\omega) + \int\limits_0^t a(\tau, x_\varepsilon(\tau, \omega), \varepsilon)\, d\tau + \int\limits_0^t B(\tau, x_\varepsilon(\tau, \omega), \varepsilon)\, dw_\varepsilon(\tau, \omega)$$

be given and assumptions 1) *to* 8) *be fulfilled, then to every* $L > 0$ *and* $\eta > 0$ *there is* $\varepsilon_0 > 0$ *such that*

$$E \sup_{\tau \in \langle 0, L \rangle} ||x_\varepsilon(\tau, \omega) - x_0(\tau, \omega)||^2 \leqq \eta \qquad \textit{for } 0 \leqq \varepsilon \leqq \varepsilon_0.$$

This result is very similar to a result of Gichman I. I. [1] which was unknown to me for a long time, since his work was not available. But Gichman's result was derived under different assumptions about the processes $w(t, \omega)$ pu the statement itself is also slightly different.

If we put $B(t, x) \equiv 0$ or $w(t, \omega) =$ const. in (1) then we obtain an ordinary differential equation and Theorem 1 is then the well-known theorem where the right-hand side of (2) fulfils condition 6). The stochastic part of (2) that is $B(t, x, \varepsilon)$ must fulfil condition 7) and that is stronger than a condition analogous to 6). The following example shows that the condition on B analogous to 6) would not be sufficient. Let x be a scalar and $w(t, \omega)$ the scalar Wiener proces i.e. the almost everywhere continuous process with stochastically independent increments for which $F(t) = t$. We shall consider the equation $x_\varepsilon(t, \omega) = \int_0^t \sin \frac{\tau}{\varepsilon} \, dw(\tau, \omega)$. By the well known theorem it holds $E|x_\varepsilon(t, \omega)|^2 = \int_0^t \sin^2 \frac{\tau}{\varepsilon} \, d\tau = \frac{t}{2} - \frac{\varepsilon}{4} \sin \frac{2t}{\varepsilon}$ and $\lim_{\varepsilon \to 0} E|x_\varepsilon(t, \omega)|^2 = \frac{t}{2}$ while $\int_0^t \sin \frac{\tau}{\varepsilon} \, d\tau \to 0$ for $\varepsilon \to 0$.

The number ε_0 in Theorem 1 depends on L. However, if we add some stability properties of solution of unperturbed equation i.e. of equation (2) for $\varepsilon = 0$ and if we omit the least upper bound in statement of Theorem 1 we can choose ε_0 independent of L. We shall usé the concept of stability in average.

Definition 1. *The solution $\bar{x}(t, \omega)$ of (1′) is stable in average, if there is a function $\sigma(\eta) > 0$ such that $E||\bar{x}(t_0, \omega) - x(t_0, \omega)||^2 < \sigma(\eta)$ implies $E||\bar{x}(t, \omega) - x(t, \omega)||^2 <$* $< \eta$ for all $t \geq t_0$.

Definition 2. *The solution $\bar{x}(t, \omega)$ of (1′) is asymptotically stable in average if it is stable in average and if there exist a number $A > 0$ and a function $T(\sigma, \eta)$ defined for $\sigma < A$, $\eta < A$ such that $E||\bar{x}(t_0, \omega) - x(t_0, \omega)||^2 < \sigma$ implies $E||\bar{x}(t, \omega) - x(t, \omega)||^2 < \eta$ for all $t \geq t_0 + T(\sigma, \eta)$.*

Definition 3. *We say that the process $w(t, \omega)$ is homogeneous if all distributions $F_{t_1+h, t_2+h}(A) = P(w(t_2 + h) - w(t_1 + h) \in A)$ are independent of h (A is an arbitrary n-dimensional Borel set).*

Theorem 2. *Let the conditions 1) to 8) be fulfilled, let the convergence $E||w_\varepsilon^*(t_2, \omega) - w_\varepsilon^*(t_1, \omega)||^2 \Rightarrow 0$ in 4) hold uniformly with respect to all t_1, t_2, let $\psi(t)$ (cf. 6) and 7)) be estimated by a continuous functions $\psi^*(t)$: $\psi(t_2) - \psi(t_1) \leq \psi^*(t_2 - t_1)$. Let the processes $w_\varepsilon(t, \omega)$ be homogeneous and let the equation*

$$(2') \qquad dx_\varepsilon(t, \omega) = a(t, x_\varepsilon(t, \omega), \varepsilon)\, dt + B(t, x_\varepsilon(t, \omega), \varepsilon)\, dw_\varepsilon(t, \omega)$$

for $\varepsilon = 0$ have a constant solution $x_0(t, \omega) = x_0(\omega)$ for $t \geqq t_0$ which is asymptotically stable in average, then to every $\eta > 0$ there are $\varepsilon_0 > 0$, $\sigma > 0$ such that

$$\sup_{\langle t_0, \infty)} E\|x_\varepsilon(t, \omega) - x_0(t, \omega)\|^2 < \eta \qquad \text{for } 0 \leqq \varepsilon \leqq \varepsilon_0$$

where $x_\varepsilon(t, \omega)$ is an arbitrary solution of (2′) with the initial condition $E\|x_\varepsilon(t, \omega) - x_0(\omega)\|^2 < \sigma$.

We can formulate sufficient conditions for the stability and the asymptotic stability in average by means of LYAPUNOV functions.

Let the function $F(t)$ from 1) be absolutely continuous, then there are absolutely continuous functions $F_{ij}(t)$ such that

$$E[(w_i(t_2, \omega) - w_i(t_1, \omega))\,(w_j(t_2, \omega) - w_j(t_1, \omega))] = F_{ij}(t_2) - F_{ij}(t_1)$$

where $w_i(t, \omega)$ is an i-compenent of the vector process $w(t, \omega)$. Denote by $f(t)$ and $f_{ij}(t)$ derivatives of $F(t)$ and $F_{ij}(t)$, respectively.

Theorem 3. *Let assumptions 1) to 3) be fulfilled where $F(t)$ is absolutely continuous and let equation (1′) have the solution $x(t, \omega) \equiv 0$. If there exists a quadratic form $V(t, x) = \Sigma\, c_{ij}(t)\, x_i x_j$ which fulfils the conditions that the $c_{ij}(t)$ have continuous second derivatives and that there are constants $d_1 > 0$, $d_2 > 0$ such that*

$$d_1\|x\|^2 \leqq V(t, x) \leqq d_2\|x\|^2,$$

$$W(t, x) = \frac{\partial V}{\partial t} + \sum \frac{\partial V}{\partial x_i}\, a_i(t, x) + \sum_{i,j,k,l} c_{ij}(t)\, B_{ik}(t, x)\, B_{jl}(t, x)\, f_{kl}(t) \leqq 0 \tag{3}$$

for almost all $t \geqq 0$, then the solution $x(t, 0) \equiv 0$ is stable in average.

Theorem 4. *Let the assumptions from Theorem 3 be fulfilled with (3) replaced by $W(t, x) \leqq -d_3\|x\|^2$ for almost all t, $d_3 > 0$, then $x(t, \omega) \equiv 0$ is asymptotically stable in average.*

The following question is of interest in the averaging theory. Under what conditions the stability of the unperturbed equation (i.e. equation (2′) for $\varepsilon = 0$) implies the stability of (2′) for small $\varepsilon > 0$ and under what conditions the existence of a periodic solution of the unperturbed equation implies the existence of such solution of (2′) for small $\varepsilon > 0$. Considering this problem we compare equation (2′) with the deterministic equation

$$dy = a(t, y, 0)\, dt \tag{4′}$$

with random initial values. Conditions 4) to 8) must be now reformulated:

4*) The processes $w_\varepsilon(t, \omega)$ are now defined only for $\varepsilon \in (0, \delta)$, they are processes with stochastically independent increments and there is a continuous function $F(t)$ (independent of ε) such that

$$E\|w_\varepsilon(t_2, \omega) - w_\varepsilon(t_1, \omega)\|^2 \leqq F(t_2) - F(t_1), \qquad E(w_\varepsilon(t_2, \omega) - w_\varepsilon(t_1, \omega)) = 0.$$

Assumptions 5) and 5′) are not necessary.

6*) $a(t, x, \varepsilon)$ is defined for $\varepsilon \in \langle 0, \delta \rangle$ and fulfils condition 2) where the constant K is independent of ε and

$$\int_0^t (a(\tau, y(\tau), \varepsilon) - a(\tau, y(\tau), 0))\, \mathrm{d}\tau \to 0 \qquad \text{for } \varepsilon \to 0$$

uniformly with respect to constant vectors y and $t \in \langle 0\ L \rangle$ for every $L > 0$, where $y(t)$ are solutions of (4′) with the initial conditions $y(0) = y$.

7*) $B(t, x, \varepsilon)$ is defined for $\varepsilon \in (0, \delta)$ and fulfils condition 2) where the constant K is independent of ε and

$$\int_0^t \|B(\tau, y(\tau), \varepsilon)\|^2\, \mathrm{d}F_\varepsilon(\tau) \to 0 \qquad \text{for } \varepsilon \to 0$$

uniformly with respect to constant vectors y and $t \in \langle 0, L \rangle$ for every $L > 0$, where $y(t)$ have the same meaning as in 6*).

8) The partial derivatives $\dfrac{\partial a}{\partial x}, \dfrac{\partial B}{\partial x}$ exist and they are LIPSCHITZ continuous in x.

The asymptotic stability in average will be replaced by exponential stability in average, too.

Definition 4. *The solutions of* (1′) *are uniformly exponentially stable in average, if they are stable in average and there exist positive constants* $K > 0$ *and* $0 < \beta < 1$ *such that*

$$E\|x^{(1)}(t, \omega) - x^{(2)}(t, \omega)\|^2 \leqq \beta E\|x^{(1)}(t_0, \omega) - x^{(2)}(t_0, \omega)\|^2 \qquad \text{for } t \geqq t_0 + K$$

for all the solutions of (1′).

Definition 5. *A process* $z(t, \omega)$ *is periodic with period* T, *if*

$$P(z(t_1, \omega) \in A_1, z(t_2, \omega) \in A_2, \ldots, z(t_s, \omega) \in A_s) =$$
$$= P(z(t_1 + kT, \omega) \in A_1, z(t_2 + kT, \omega) \in A_2, \ldots, z(t_s + kT, \omega) \in A_s)$$

for all n-dimensional Borel sets A_i, *all* $t_1 < t_2 < \ldots t_s$ *and for all integers* k.

Theorem 5. *Let the assumptions* 4*), 6*) *to* 8*) *be fulfilled, let* $a(t, x, \varepsilon)$, $B(t, x, \varepsilon)$ *be periodic functions in* t *with the period* T *and let* $w_\varepsilon(t + h, \omega) - w_\varepsilon(t, \omega)$ *be periodic processes with the same period* T. *If the solutions of equation* (4′) *are uniformly exponentially stable in average, then there is an* $\varepsilon_0 > 0$ *such that the solutions of* (2′) *are uniformly exponentially stable in average for* $0 < \varepsilon \leqq \varepsilon_0$ *and there exist periodic solutions* $x^*_\varepsilon(t, \omega)$ *of* (2′) *for* $0 < \varepsilon \leqq \varepsilon_0$ *and a deterministic periodic solution* $y^*(t)$ *of* (4′) *and*

$$\lim_{\varepsilon \to 0} \sup_{t \geqq 0} E\|x^*_\varepsilon(t, \omega) - y^*(t)\|^2 = 0$$

holds.

This Theorem has an interesting consequence for parabolic differential equations. If $a(t, x, \varepsilon)$, $B(t, x, \varepsilon)$ fulfil the assumptions of Theorem 5 and if we add some assumptions which are used in the theory of parabolic equations (e.g. $B^T B$ is positive definite for all positive ε, B^T is the transpose matrix, that B are Hölder continuous in t and there are $\frac{\partial a_i}{\partial x_i}, \frac{\partial B_{ij}}{\partial x_j}, \frac{\partial^2 B_{ij}}{\partial x_i \partial x_j}$ which are continuous and bounded), then for small $\varepsilon > 0$ the parabolic equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j} \frac{\partial^2 \left(\sum_k B_{ik}(t, x, \varepsilon) B_{jk}(t, x, \varepsilon) u\right)}{\partial x_i \partial x_j} - \sum_i \frac{\partial (a_i(t, x, \varepsilon) u)}{\partial x_i}$$

has periodic solutions with the initial values $\alpha f_0(x)$ where α is an arbitrary real number and $\int f_0(x) \, dx = 1, f_0 \geqq 0$. These solutions are relatively asymptotically stable in the sense that

$$\lim_{t \to \infty} \int_{\lambda_1}^{\mu_1} \ldots \int_{\lambda_n}^{\mu_n} (u(t, x; f_1) - \alpha u(t, x; f_0)) \, dx = 0$$

uniformly with respect to $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$ where $u(t, x; f_i)$ is the solution of the parabolic equation with the initial condition $u(0, x; f_i) = f_i$, if $\int |f_1(x)| \, dx < \infty$ and $\int f_1(x) \, dx = \alpha$ holds.

LITERATURE

[1] Гихман И. И.: *Дифференциальные уравнения со случайными функциями. Зимняя школа по теории вероятностей и математической статистике. Киев 1964.*

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2. PARTIAL DIFFERENTIAL EQUATIONS

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A GENERAL METHOD OF MAJORATING OF DIRICHLET PROBLEM SOLUTIONS

1. Let $u(x)$ be a function in a domain G in the Euclidean n-space E_n. We say that $x_0 \in G$ is its convexity point if the surface $S : z = u(x)$ in $(n + 1)$-space has at the point x_0, $u(x_0)$ a supporting plane from below, i.e. $z = p_i x^i + q \leq u(x)$, $p_i x_0^i + q = u(x_0)$. To such a plane we make to correspond the point $(p_1, \ldots, p_n)$ in E_n. Let $\Psi_u(M)$, $M \subset E_n$, be the set of all such points corresponding to all points $x \in M$ (if M includes no convexity points of u, $\Psi_u(M)$ is empty). It is „the lower supporting image of M by u". mes $\Psi_u(M)$ is a totally additive set functions. One can obviously define the upper supporting image $\overline{\Psi}_u(M)$.

We consider functions u subject to the following conditions;

(A) u is continuous in $G + \partial G$,

(B) the set function mes Ψ_u (M_u) is absolutely continuous: this is fulfilled, in particular, if $u \in W_n^2(D)$ for every D, $D + \partial D \subset G$.

Suppose that u satisfies at almost all its convexity points the inequality

$$w \leq X(x, u)\, \mathrm{U}(\nabla u), \qquad w = \det(u_{ij}), \qquad X, U \geq 0. \tag{1}$$

(Note: any function is twice approximatively differentiable at almost all its convexity points. Thus no special differentiability conditions are necessary as soon as we understand u_i, u_{ij} as the coefficients of the approximative differentials $\mathrm{d}u$, d^2u).

In order to formulate our basic theorem introduce the following notations: $h(x, \nu)$ be the distance from a point $x \in G$ to the supporting plane to ∂G with the external normal ν; Ω be the unite sphere — the set of all unite vectors ν; we put $\nabla u = p\nu$, $p = |\nabla u|$.

Theorem 1. *If a function u with above conditions* (A), (B) *satisfies* (1) *at almost all convexity points, then for any $x \in G$ where $u(x) < 0$ the following inequality takes place*

$$\int_{\Omega}\int_{0}^{\frac{|u(x)|}{h(x,\nu)}} U^{-1}(p\nu)\, p^{n-1}\, dp\, d\nu < \int_{G} X(x, u(x))\, dx. \tag{2}$$

This implies an estimation for $|u(x)|$ provided $X(x, u(x))$ is summable and the left integral grows to the infinity with the upper limit of integration.

The proof of our theorem runs as follows. Let M be the set of convexity points of u. Owing to (1)

$$\int_{M} U^{-1}w\, dx \leq \int_{M} X\, dx. \tag{3}$$

But $w = \dfrac{\partial(u_1, \ldots, u_n)}{\partial(x^1, \ldots, x^n)}$ is the Jacobian of the supporting mapping $(x^1, \ldots, x^n) \to (p_1, \ldots, p_n)$ for almost everywhere $p_i = u_i$. Thus owing to the condition (B)

$$\int_{M} U^{-1}w\, dx = \int_{\Psi u(M)} U^{-1}(p\nu)\, dp_1 \ldots dp_n. \tag{4}$$

Obviously $\Psi_u(M) = \Psi_u(G)$ and $\int_{M} X\, dx \leq \int_{G} X\, dx$. Therefore (3) and (4) imply

$$\int_{\Psi_u(G)} U^{-1}(p\nu)\, dp_1 \ldots dp_n \leq \int_{G} X(x, u(x))\, dx. \tag{5}$$

Now take a point $x \in G$ where $u(x) < 0$ and construct in $(n + 1)$-space the cone C that projects ∂G from the point x, $u(x)$. One can easily observe, from direct geometrical consideration, that to every supporting plane to the cone C there corresponds a parallel supporting plane to the surface $S : z = u(x)$. It means that the supporting image of S includes that of C; i.e. $\Psi_u(G) \supset \Psi_C$; and moreover $\operatorname{mes} \Psi_u > \operatorname{mes} \Psi_C$. Hence (5) implies

$$\int_{\Psi_C} U^{-1} dp_1 \ldots dp_n < \int_{G} X\, dx. \tag{6}$$

Now, elementary geometrical consideration show that the supporting image of the cone C is a convex domain bounded by the surface with the equation (in spherical coordinates p, ν)

$$p = \frac{|u(x)|}{h(x, \nu)}.$$

Thus, if we transform the left integral (6) to the spherical coordinates p, ν, we shall see that it is the left integral in (2). Hence (6) implies (2) and our theorem is proved.

2. Suppose u satisfies an equation

$$F(u_{ij}, u_i, u, x) = 0 \tag{7}$$

where F is such that (7) implies (1) at almost all convexity points of u. Then we can apply our Theorem 1 which will give the estimations of the values $u(x)$.

One can observe that the inequality $F \leq 0$ imlies $w \leq K(x, u, \nabla u)$, when $\overline{d^2 u} \geq 0$, for every strictly elliptic F and even for wider class of F. The estimation $K(x, u, \nabla u) \leq X(x, u)\, U(\nabla u)$ usually takes place. Thus Theorem 1 proves to be applicable to a very wide class of equations.

The simplest case is the linear equation

$$a^{ij}u_{ij} + b\,\nabla u = g, \qquad g = f - cu, \qquad a^{ij}\xi_i\xi_j \geq 0. \tag{8}$$

Because of $a^{ij}\xi_i\xi_j \geq 0$ we have at the point where $d^2u > 0$

$$a^{ij}u_{ij} \geq n(aw)^{\frac{1}{n}}, \qquad a = \det\,(a^{ij}). \tag{9}$$

Hence $n(aw)^{\frac{1}{n}} \leq g - b\,\nabla u$ which easily leads to the inequality of the form (1). The results got for linear equations will be given somewhat further.

3. Under certain conditions on the function U in (1) the inequality (2) can be transformed into a simpler form. Introduce the functions $h_K(x)$ — the mean values of the distances $h(x, \nu)$:

$$h_K(x) = \left[\frac{1}{\varkappa_n}\int\limits_{\Omega} h^{-K}(x, \nu)\, d\nu,\right]^{-\frac{1}{K}} \quad K \neq 0; \quad h_0(x) = \exp \frac{1}{\varkappa_n}\int\limits_{\Omega} \ln h(x, \nu)\, d\nu \tag{10}$$

where $\varkappa_n = \operatorname{mes} \Omega$.

Theorem 2. *If* $U(p\nu) \leq \overline{U}(p)$ *and* $\overline{U}(p)\, p^{K-n}$ *is a non-increasing function, then* (2) *implies*

$$\varkappa_n \int\limits_0^{\frac{|u(x)|}{h_K(x)}} \overline{U}^{-1}(p)\, p^{n-1}\, dp < \int\limits_G X(x, u(x))\, dx. \tag{11}$$

4. For the linear equation (8) we get the following results.

Theorem 3. *If in* (8) $\det\,(a^{ij}) = 1$ *then at every point* x *where* $u(x) < 0$

$$|u(x)| < \alpha_n \|g_+\|\, F_n(\|b\|)\, h_0(x) \tag{12}$$

where the norms are those in $L_n(G)$, $\alpha_n = n^{-1}\tau_n^{-\frac{1}{n}}$, $\tau_n = \varkappa_n n^{-1}$ *is the volume of the unite sphere,*

$$F_n(\xi) = e^{\frac{\xi^n}{n^n \varkappa_n}} + \varphi_n(\xi), \; (\xi \geq 0), \tag{13}$$

$\varphi_n(\xi)$, *for* $n > 1$, *being a bounded increasing function,* $\varphi_n(0) = 0$, *and* $\varphi_1(\xi) \equiv 0$. *Presize definition of the function* F_n *can be given as a convers to an explicitely represented elementary function.*

Theorem 3 leads to the following corollaries.

Theorem 4. *The homogeneous equation* (8) *with* det $(a^{ij}) = 1$ *has no non-zero solution if* $||c_+|| < \infty$ *and*

$$\alpha_n ||c_+ h_0|| \, F_n(||b||) \leq 1. \tag{14}$$

If the strict inequality takes place here, then at every x *where* $u(x) < 0$

$$|u(x)| < \frac{||f_+|| \, h_0(x)}{\alpha_n^{-1} F_n^{-1}(||b||) - ||c_+ h_0||} \,. \tag{15}$$

5. The inequalities of Theorems 3,4 are presize and no general estimations nor general uniqueness conditions are possible in terms of norms weaker than those in $L_n(G)$. The presize meaning of this statement is given by the following theorems in which we speak on elliptic equations (8) with smooth coefficients, det $(a^{ij}) = 1$ and on theri smooth solutions u with $u/\partial G = 0$.

Theorem 5. *Let the domain* G *be convex.*

(1) *Consider in* G *equations with a given value of the magnitude* $\alpha_n ||g|| \, F_n(||b||) = = H$. *The lower upper bound of the values* $|u(x)|$ *of their solutions, for every* x, *is* sup $|u(x)| = Hh_0(x)$. *(If* G *is a sphere.* x_0 *is its center,* A, B, ε *positive numbers, there exist in* G *equations with* $||g|| = A$, $||b|| = B$ *and the solution,* u *for which,* $|u(x_0)|$ *differs from the right part of* (12) *less than by* ε.*)*

(2) *For any* $\varepsilon > 0$ *such a homogeneous equation can be given that*

$$\alpha_n ||c_+ h_0|| \, F_n(||b||) < 1 + \varepsilon,$$

but it has non-zero solution.

(3) *The estimation* (15) *is presize in the sense analogous to* (1).

Theorem 6. *Let* G *be a sphere; let* $\varphi(\xi)$ *be such a function,* $\xi \in [0, \infty)$, *that* $\varphi(\xi) \, \xi^{-1} \to 0$ *when* $\xi \to \infty$. *Put for a function* g *in* G

$$N(g) = \int_G \varphi(g^n) \, dx. \tag{17}$$

(1) *Such a sequence of equations* $a^{ij} u_{ij} = f$ *can be given in* G *that* $N(f) \to 0$, *but* $|u(x)| \to \infty$ *for every* $x \in G$.

(2) *For any* $\varepsilon > 0$ *such equations*

$$a^{ij} u_{ij} + b \, \nabla u = 0, \qquad \bar{a}^{ij} u_{ij} + cu = 0 \tag{18}$$

can be given in G *that* $N(b) < \varepsilon$, $N(c) < \varepsilon$, *but the equations have non-zero solutions.*

6. Let $r = r(x)$ be the distance from $x \in G$ to the boundary of the convex hull of G in the direction of the vector $-b = -b(x)$. Put $\bar{c} = c + |b| \, r^{-1}$, $\bar{g} = f - \bar{c}u$.

Theorem 7. *Under the conditions of Theorem* 3

$$|u(x)| < \alpha_n ||\bar{g}_+|| \, h_n(x). \tag{19}$$

The condition of nonexistance of non-zero solution is

$$\alpha_n||\bar{c}_+h_n|| \leq 1, \qquad ||\bar{c}_+|| < \infty, \tag{20}$$

and if here the strict inequality takes place,

$$|u(x)| < \frac{||f_+||\, h_n(x)}{\alpha_n^{-1} - ||\bar{c}_+h_n||}\,. \tag{21}$$

These inequalities are presize in a sense analogous to that of Theorem 5; we have but to consider in this Theorem the equations with $b \equiv 0$.

The estimation (19) is formally always true but it has a meaning if $||\bar{g}_+|| < < \infty$ which is ensured if $||br^{-1}|| < \infty$. This implies certain conditions on b. Let G be convex and $x \to \partial G$. Then, if roughly speaking $b(x)$ is directed from ∂G, $r(x) \to 0$ and the condition $||br^{-1}|| < \infty$ gives a comparatively strong limitation on $|b(x)|$; but if $b(x)$ is directed towards ∂G, $r(x) > \text{const} > 0$, and $||br^{-1}|| < \infty$ if $||b|| < \infty$.

The advantage of the inequalities of Theorem 7 in comparison to those of Theorems 3,4 consists in the properties of the function $h_n(x)$. Owing to well known properties of meanvalues, $h_n(x) < h_0(x)$ with the only exception when G is a sphere and x is its center. Moreover, if G is convex and $\varrho(x)$ denotes the distance of x from ∂G, we have the estimation $h_n(x) < \text{Const}\, \varrho^{\frac{1}{n}}(x)$. On the contrary, at every point $x \in \partial G$ which is the vertex of a paraboloid (of any degree > 1) included in G, $h_0(x) > 0$.

7. All above results allow of an essential generalization which, shortly speaking, consists in application of the some considerations to the projections of the solution u on various planes E of any dimensionality m, $1 \leq m \leq n$. We may suppose that E is $(x^1, \ldots, x^m)$ — plane. Then the lower projection of a function $\varphi(x) \equiv \varphi(x^1, \ldots, x^n)$, $x \in G$, is

$$\varphi_E(x^1, \ldots, x^m) = \inf_{(x^{m+1}, \ldots, x^n)} \varphi(x^n, \ldots, x^n), \tag{22}$$

and the upper projection is $\varphi_E(x^1, \ldots, x^m) = \sup \varphi(x^1, \ldots, x^n)$; they are defined in the projection G_E of G.

The results for linear equation (8) imply the norms $||\varphi||_E$ defined as follows. Let $a_E = \det (a^{ij})$, $i, j \leq m$, provided E is $(x^1, \ldots, x^m)$-plane. We define

$$||\varphi||_E = ||a_E^{-\frac{1}{m}} \cdot |\varphi|^E||_{L_m(G_E)}. \tag{23}$$

We define the functions $h_{KE}(x)$ by the same formula (10) with the only difference that we integrate over the set Ω_E of the unite vectors in E and pivide by $\varkappa_m = \text{mes}\, \Omega_E$.

Theorem 8. *Under the conditions of the Theorem* 3, *for almost all planes E of any bundle there takes place the inequalities*

$$|u(x)| < \alpha_m \|g_+\|_E F_m(\|b\|_E)\, h_{0E}(x). \tag{24}$$

Theorems 4, 5 admit corresponding generalizations, too.

8. The methods and results given here are expounded with proofs in a series of my papers published in

Сибирский математический журнал. 1966, № 3; Вестник Ленинградского унивeрзитета 1966, NN№ 1, 7, 13; Доклады Академии наук СССР, 1966, v. 169, № 4,

and partly in a course of lectures "*The method of normal map in uniqueness problems and estimations for elliptic equations*", *Seminari dell' Instituto Nazionale di Alta Matematica* 1962—1963, *vol.* 2, *Roma* 1965.

By a different method under different conditions the problem of majorating the Dirichlet problem solutions has been studied by C. Pucci and M. Fraxa; cf. in particular C. Pucci, *Operatori ellittici estremanti, Annali di Mat., vol.* 72, *pp.* 141—170 (1966).

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STRUCTURE OF GREEN'S OPERATORS AND ESTIMATES FOR THE CORRESPONDING EIGENVALUES.(1)

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It is well-known that one of the most difficult problems which mathematical physics poses to quantitative analysis is the *rigorous approximation* of the eigenvalues of certain boundary value problems which arise in applied mathematics. By *rigorous approximation* we mean giving lower and upper bounds for any particular eigenvalue such that these bounds approach this eigenvalue to any prescribed degree of accuracy. It is convenient to consider these problems in an abstract Hilbert space setting. To this end we consider a complex separable Hilbert space S and a linear operator T which maps S into itself. We suppose that T is a positive compact operator (PCO), i.e., T is such that $(Tu, u) > 0$ for $u \neq 0$ [$(.,.)$ is the scalar product in S] and T maps weakly convergent sequences onto strongly convergent sequences. It is well-known (and very easy to prove) that positiveness implies that T is hermitian [i.e., $(Tu, v) = (u, Tv)$ for any u and v in S] and compactness implies that T is bounded.

Let us consider the eigenvalue problem

$$Tu - \mu u = 0. \tag{1}$$

A fundamental theorem in Hilbert space theory states that the eigenvalues of problem (1) constitute a sequence of positive real numbers converging to zero if — as we shall suppose — S is infinite dimensional. Each eigenvalue has finite multiplicity, i.e., the kernel of the linear operator $T_\mu = T - \mu I$ has finite dimension (multiplicity of μ). Let

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq \cdots$$

be the sequence of the eigenvalues of T, each repeated as many times as its multiplicity. From now on when we mention the sequence of eigenvalues of

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a PCO, it shall be understood that this sequence is ordered according to the criterion just specified; thus the statement: *μ is the k-th eigenvalue of a certain PCO is precise.*

The problem of the rigorous approximation of the eigenvalues of T is the following.

For any given k we want to construct two sequences $\{\mu_k^{\nu}\}$ and $\{\sigma_k^{(\nu)}\}$, $\nu = 1, 2, \ldots,$ *such that*:

(2) $\qquad \mu_k^{(\nu)} \leq \mu_k^{(\nu+1)},\qquad$ (3) $\qquad \lim\limits_{\nu\to\infty} \mu_k^{(\nu)} = \mu_k,$

(4) $\qquad \sigma_k^{(\nu)} \geq \sigma_k^{(\nu+1)},\qquad$ (5) $\qquad \lim\limits_{\nu\to\infty} \sigma_k^{(\nu)} = \mu_k.$

The sequence $\{\mu_k^{(\nu)}\}$ (called lower bounds) can be constructed in a rather simple way by means of the classical *Rayleigh-Ritz method.* Let $\{w_i\}$ be a sequence of linearly independent vectors complete in the space S. Let us denote by

(6)
$$\mu_1^{(\nu)} \geq \mu_2^{(\nu)} \geq \ldots \geq \mu_\nu^{(\nu)}$$

the roots of the determinantal equation

(7)
$$\det\{(Tw_i, w_j) - \mu(w_i, w_j)\} = 0, \qquad (i, j = 1, \ldots, \nu).$$

The following theorem, which goes back to Plancherel [11], states that $\{\mu_k^{(\nu)}\}$ is a sequence of the lower bounds.

Theorem 1. *The sequence $\{\mu_k^{(\nu)}\}$ obtained through the Rayleigh-Ritz method satisfies conditions* (2), (3).

It is of interest to remark, in view of applications to partial differential equations, that, instead of using Eq. (7), we may obtain the Rayleigh-Ritz approximations from the equation

(8)
$$\det\{(Tw_i, Tw_j) - \mu(Tw_i, w_j)\} = 0, \qquad (i, j = 1, \ldots, \nu).$$

Theorem I. still holds if we substitute Eq. (8) for Eq. (7).

The construction of the sequence $\{\sigma_k^{(\nu)}\}$ (called *upper bounds*) is a much more difficult problem. The first approach to this problem is due to A. Weinstein [16], who considered the eigenvalue problems connected with the classical boundary value problems of elastic plates. The Weinstein method, known now as the *method of intermediate problems,* was later reformulated in terms of a PCO in a Hilbert space and deeply investigated (and generalized) by Aronszajn (see [1], [2] and [10]). Further important results have been obtained by the Weinstein school, especially by Weinberger [15], Bazley [3] and Bazley—Fox [4].

It has been proved (see [6], [7]) that the Weinstein method can be included in the following formulation of the theory of intermediate problems, due to

Aronszajn. Let T_0 be a PCO such that $T_0 > T$. Assume that *the sequence of eigenvalues* $\{\mu_k^{(0)}\}$ *of* T_0 *and the corresponding sequence of eigenvectors are known.* T_0 is called a *base operator*. Then it is possible to construct a sequence $\{T_\nu\}$ of PCO's such that

i) $T_0 \geq T_\nu \geq T_{\nu+1} \geq T$;

ii) $T_0 - T_\nu$ is a degenerate operator, i.e., its range is finite dimensional;

iii) T_ν converges uniformly to T for $\nu \to \infty$, i.e. $\lim_{\nu \to \infty} ||T_\nu - T|| = 0$. If

we let $\sigma_k^{(\nu)}$ be the k-th eigenvalue of T_ν, conditions i), iii) assure that $\sigma_k^{(\nu)}$ satisfies conditions (4), (5). Of course the problem of the actual computation of $\sigma_k^{(\nu)}$ must still be solved. To this end one takes advantage of condition ii) and, by using standard techniques of finite rank perturbation theory, it is possible to find eigenvalues of T_ν as zeros of certain meromorphic functions, which Weinstein introduced for the first time. One then uses two different procedures: one for the eigenvalues of T_ν which are not eigenvalues of the base operator; and another for the eigenvalues which are eigenvalues both for T_0 and T_ν. The main numerical difficulty occurs in finding the zeros of the above mentioned meromorphic functions. Some procedures have been given by Weinstein, Weinberger, Bazley and Fox in order to avoid this difficulty. As a matter of fact in many important applications the eigenvalues of T_ν can be found as zeros of very simple functions.

The method of intermediate problems has led to the solution of many interesting eigenvalue problems since Weinstein published his important paper [16] on 1937. Let us mention, among all these, the outstanding result obtained on 1961 by Bazley [3], who was able to give remarkable lower bounds for the first two eigenvalues of the helium atom.

However, one of the theoretical restriction in the method of the intermediate problems is the assumption that a base operator T_0 must be known. For example, if we consider the very simple and classical eigenvalue problem in the space $S \equiv \mathrm{L}^2(0, 1)$, for the Fredholm operator

$$Tu = \int_0^1 K(x, y)\, u(y)\, dy,$$

($K(x, y)$ continuous and $K(x, y) = \overline{K(y, x)}$), we do not know, in general, how to construct a base operator.

Therefore in the last two years a different method has been developed by the author. This new method applies to a class of operators smaller than those considered in the theory of intermediate problems, but its application requires less (e.g., base operator) information. The resulting application of this method to eigenvalue problems for elliptic linear differential systems has led to new investigations in the theory of those systems, which, in the opinion of the author, are of interest on their own.

In the present paper we wish to expose the main results of this new method, together with some numerical applications which have been carried out at the Computing Center of the Faculty of Sciences at the University of Rome. For these numerical calculations the author wishes to express his sincere thanks to L. de Vito, director of the Computing Center, and to A. Fusciardi, F. Scarpini and M. Schaerf. A complete account of the theory of orthogonal invariants and their applications to partial differential equations can be found in [6] or in [7].

1. Method of orthogonal invariants.

Let T be the above considered PCO. Let us denote by $\Gamma^{(n)}(w_1, \ldots, w_s)$ the Gram determinant of s given vectors $w_1, \ldots, w_s$ in the space S, with respect to the scalar product $(T^n u, v)$; i.e., $\Gamma(w_1, \ldots, w_s) = \det \{(T^n w_i, w_j)\}$, $(i, j = 1, \ldots, s)$. Let $\{v_k\}$ $(k = 1, 2, \ldots)$ be an orthonormal complete system in the space S. We set

$$\mathscr{Y}^n_0(T) = 1$$

and, for $s > 0$,

$$\mathscr{Y}^n_s(T) = \frac{1}{s!} \sum_{k_1 \ldots k_s} \Gamma^{(n)}(v_{k_1}, \ldots, v_{k_s}). \tag{9}$$

The summation $\sum_{k_1 \ldots k_s}$ must be understood to be over any set of s positive integers. Since the multiple series on the right hand side of (9) has non-negative terms, its sum — finite or not — is independent of the summation procedure. It is evident that

$$\mathscr{Y}^n_s(T^m) = \mathscr{Y}^{nm}_s(T).$$

The following theorems hold.

Theorem II. *$\mathscr{Y}^n_s(T)$ is independent of the particular orthonormal complete system used in its definition, i.e., $\mathscr{Y}^n_s(T)$ is an orthogonal invariant for the operator T.*

The index s will be called the *order* of the orthogonal invariant $\mathscr{Y}^n_s(T)$ and the index n the *degree* of this invariant.

Theorem III. *We have $\mathscr{Y}^n_s(T) < +\infty$ if and only if $\mathscr{Y}^n_1(T) < +\infty$.*

We denote by $\mathfrak{C}^n$ the class of all the PCO's such that $\mathscr{Y}^n_1(T) < +\infty$. We then have $\mathfrak{C}^m \subset \mathfrak{C}^n$ if $m < n$. There exists PCO's such that they do not belong to any $\mathfrak{C}^n$. However PCO's which are encountered in mathematical physics generally belong to some $\mathfrak{C}^n$ for n large enough.

Theorem IV. *The sequence* $\mathscr{Y}^n_s(T)$, $(s = 1, 2, \ldots)$ *is a complete system of invariants for unitary equivalence of two operators of the class* $\mathfrak{C}^n$.

This means that if T, $R \in \mathfrak{C}^n$, we have $T = U^{-1}RU$, with U an unitary operator, if and only if $\mathscr{Y}^n_s(T) = \mathfrak{I}^n_s(R)$, $s = 1, 2, \ldots$

Theorem V. *If* $T_2 \geq T_1$, $(T_i \in \mathfrak{C}^n, i = 1, 2)$, *then* $\mathscr{Y}^n_s(T_2) \geq \mathscr{Y}^n_s(T_1)$.

Theorem VI. *If* T_k *converges uniformly to* T, $(T_k, T \in \mathfrak{C}^n)$, *then* $\lim_{k\to\infty} \mathscr{Y}^n_s(T_k) = \mathscr{Y}^n_s(T)$.

Let $\{w_i\}$ be the above introduced complete sequence of linearly independent vectors. Let W^ν be the ν-dimensional subspace spanned by the vectors $w_1, \ldots, \omega_\nu$ and P_ν the orthogonal projector which maps S onto W^ν. Let us consider the positive eigenvalues of the operator $P_\nu T P_\nu$, that is to say, the roots (6) of the equation (7). Let $\tilde{\omega}^{(\nu)}_k$ $(k \leq \nu)$ be an eigenvector corresponding to the eigenvalue $\mu^{(\nu)}_k$ of $P_\nu T P_\nu$. We denote by $\tilde{W}^\nu_k$ the one-dimensional subspace spanned by $\tilde{\omega}^{(\nu)}_k$. Let $P^{(\nu)}_k$ be the orthogonal projector of S onto $\tilde{W}^\nu \ominus W^\nu_k$. Let us remark that $P_\nu T P_\nu$ and $P^{(k)}_\nu T P^{(k)}_\nu$ are PCO's when considered in the spaces W^ν and $W^\nu \ominus \tilde{W}^\nu_k$ respectively.

Theorem VII. *Let* $T \in \mathfrak{C}^n$. *Given* $s > 0$, *for* $\nu \geq s$ *let*

$$\sigma^{(\nu)}_k = \left\{ \frac{\mathscr{Y}^n_s(T) - \mathscr{Y}^n_s(P_\nu T P_\nu)}{\mathfrak{I}^n_{s-1}(P^{(k)}_\nu T P^{(k)}_\nu)} + [\mu^{(\nu)}_k]^n \right\}^{\frac{1}{n}}. \tag{10}$$

Then the sequence $\{\sigma^{(\nu)}_k\}$ *satisfies conditions* (4), (5).[2]

The two orthogonal invariants $\mathscr{Y}^n_s(P_\nu T P_\nu)$ and $\mathscr{Y}^n_{s-1}(P^{(k)}_\nu T P^{(k)}_\nu)$ must be considered as numerically known since they are expressed as follows through the Rayleigh-Ritz approximations

$$\mathscr{Y}^n_s(P_\nu T P_\nu) = \sum_{h_1 < \ldots < h_s}^{1,\ldots,\nu} [\mu^{(\nu)}_{h_1} \cdots \mu^{(\nu)}_{h_s}]^n,$$

$$\mathscr{Y}^n_{s-1}(P^{(k)}_\nu T P^{(k)}_\nu) = \sum_{h_1 < \ldots < h_{s-1}}^{1,\ldots,\nu(k)} [\mu^{(\nu)}_{h_1} \cdots \mu^{(\nu)}_{h_{s-1}}]^n.$$

The symbol $\sum\limits_{h_1 < \ldots < h_{s-1}}^{1,\ldots,\nu(k)}$ means that the indices $h_1 \ldots h_{s-1}$ are always chosen among the integers $1, \ldots, k-1, k+1, \ldots, \nu$.

Theorem VII solves the problem of the upper approximation of μ_k, *provided that one of the orthogonal invariants* $\mathscr{Y}^n_s(T)$ of T *is known.*

Theoretically, because of the definition (9), $\mathscr{Y}^n_s(T)$ can be considered as known. However from the numerical point of view, we can only obtain a lower bound for $\mathscr{Y}^n_s(T)$ since it is expressed as a sum of a series with non-negative terms. On the other hand, formula (10) requires an upper bound for $\mathscr{Y}^n_s(T)$ if we wish $\sigma^{(\nu)}_k$ to be an upper bound for μ_k. In conclusion, we are allowed to use formula (10) for giving an upper bound for μ_k if we are able to estimate the remainder of the series which defines $\mathscr{Y}_s^n(T)$, or if we can compute this invariant by some different procedure. We shall see in the following how to overcome this difficulty.[2] We believe that theorem VII must be considered as a remarkable advance in the problem of obtaining upper approximations for the μ_k's, since *the problem of finding upper bounds for a sequence of numbers* $\mu_1, \mu_2, \ldots, \mu_k, \ldots$ *has been reduced to the problem of giving an upper bound to a single number: one of the orthogonal invariants of* T.

Let us now consider a measure space with a non-negative measure μ and let A be a measurable set in this space. Denote by $L^2(A, \mu)$ the Hilbert space of complex valued functions $u(x)$ on A, with $|u(x)|^2$ summable on A with respect to the measure μ. The scalar product in $L^2(A, \mu)$ is the following

$$(u, v) = \int_A u(x)\, \bar{v}(x)\, d\mu_x.$$

Suppose that S is Hilbert-isomorphic to $L^2(A, \mu)$. It is well-known that it is always possible to choose the measure-space, μ and A in such a way that this is true. For instance, we may take as A any bounded open set of an euclidean space and μ the classical Lebesgue measure.

Theorem VIII. *Let* T *belong to* $\mathfrak{C}^n$. *Then there exists a kernel* $K^{(n)}(x, y)$ *belonging to* $L^2(A \times A, \mu \times \mu)$ *such that* T^n *admits the following representation in the space* $L^2(A, \mu)$

$$T^n u = \int_A K^{(n)}(x, y)\, u(y)\, d\mu_y. \tag{11}$$

From this theorem we can deduce the following one which provides an integral representation for the orthogonal invariants of $T \in \mathfrak{C}^n$.

Theorem IX. *Consider the function*

$$f(x_1, \ldots, x_s) = \begin{vmatrix} K^{(n)}(x_1, x_1) & \ldots & K^{(n)}(x_1, x_s) \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ K^{(n)}(x_s, x_1) & \ldots & K^{(n)}(x_s, x_s) \end{vmatrix}.$$

It is summable on the cartesian product $\underset{1}{A} \times \underset{2}{A} \times \ldots \times \underset{s}{A}$ *with respect to the product-measure* $\mu_{x_1} \times \mu_{x_2} \times \ldots \times \mu_{x_s}$. *For* $T \in \mathfrak{C}^n$ *we have*

$$\mathscr{Y}_s^n(T) = \frac{1}{s!} \int_A \ldots \int_A f(x_1, \ldots, x_s)\, d\mu_{x_1} \ldots d\mu_{x_s}. \tag{12}$$

Representation (12) solves the problem of the computation of $\mathscr{Y}_s^n(T)$ when

[2] $\sigma_k^{(n)}$ obviously depends on s and n. However we don't want to indicate this dependence explicitly, since we assume s and n fixed and wish to avoid cumbersome notation.

the kernel $K^{(n)}(x, y)$ corresponding to the operator T^n is known. It follows that (10) furnishes a double sequence of formulas, each of them (for any fixed s and n) solving the problem of upper approximation of eigenvalues for Fredholm (hermitian) integral operators. We would like to observe that using the integral representation (12) of $\mathcal{Y}_s^n(T)$ and letting $n = 2$ and $s = 1$ in the general formula (10), we arrive at a particular formula already known to Trefftz [14].

2. Structure of Green's operators.

If we wish to use representation (12) for eigenvalue problems for differential equations, we must face the main difficulty consisting in the actual knowledge of the Green's function for the associated boundary value problem. It is well-known that only in a very few cases — especially for partial differential equations — is the Green's function explicitly known.

We now want to show to overcome this difficulty for linear elliptic differential systems by means of a new approach to boundary value problems for these systems; the latter will lead us to an explicit construction of the Green's operator. This construction is particularly suitable for using formula (10) or the slight generalization of this formula, given by the following theorem.

Theorem X. *Let $\{T_\varrho\}$ be a decreasing sequence of PCO's uniformly converging to T. Let T_ϱ, T belong to $\mathfrak{C}^n$. Set*

$$\sigma_k^{(\varrho,\nu)} = \left\{\frac{\mathcal{Y}_s^n(T_\varrho) - \mathcal{Y}_s^n(P_\nu T P_\nu)}{\mathcal{Y}_{s-1}^n(P_\nu^{(k)} T P_\nu^{(k)})} + [\mu_k^{(\nu)}]^n\right\}^{\frac{1}{n}}.$$

Then

$$\sigma_k^{(\varrho,\nu)} \geq \sigma_k^{(\tilde\varrho,\tilde\nu)} \qquad \text{for} \qquad \varrho \leq \tilde\varrho,\ \ \nu \leq \tilde\nu,$$

and

$$\lim_{\substack{\varrho\to\infty\\ \nu\to\infty}} \sigma_k^{(\varrho,\nu)} = \mu_k.$$

Let X^r be the r-dimensional real cartesian space; we denote by $x \equiv (x_1, \ldots, x_r)$ a variable point in X^r. If u and v are n-vectors with complex components, their scalar product $u_i \bar v_i$ will be denoted by uv. If $a \equiv \{a_{ij}\}$ is an $l \times l$ matrix with complex entries, the l-vector whose components are $a_{ij} u_j$ $(i = 1, \ldots, l)$, will be indicated by au, the adjoint matrix of a, i.e., the matrix $\{\alpha_{ij}\}$ with $\alpha_{ij} = \bar a_{ji}$ will be indicated by $\bar a$.

Let A be a bounded domain (connected open set) of X^r. We suppose that A is a properly regular domain[3]. Let us consider the following linear differential matrix-operator of order $2m$

$$L(x, D) \equiv D^p a_{pq}(x)\, D^q, \qquad (0 \leq p \leq m,\ \ 0 \leq q \leq m),$$

where $a_{pq}(x)$ are $l \times l$ matrices, which — for simplicity — we assume to be of class C^∞ in the whole space X^r; if p is the multi-index $(p_1, \ldots, p_r)$, D^p denotes — as usual — the partial derivative

$$D^p = \frac{\partial^{|p|}}{\partial x_1^{p_1} \ldots \partial x_r^{p_r}} .$$

We make the following hypotheses.

1) $L(x, D)$ is elliptic for every $x \in X^r$, i. e., we have for any real non-zero n-vector ξ

$$\det a_{pq}(x)\, \xi^p \xi^q \neq 0 \qquad (|p| = |q| = m)$$
$$(\xi^p = \xi_1^{p_1} \ldots \xi_r^{p_r}, \qquad \xi_i^{p_i} = 1 \qquad \text{if } \xi_i = p_i = 0);$$

2) $L(x, D)$ is formally self-adjoint, i.e.,

$$a_{pq}(x) \equiv (-1)^{|p|+|q|} \bar{a}_{qp}(x);$$

3) The bilinear integro-differential form

$$B(u, v) = \sum_{pq} (-1)^{|p|} \int_{\mathring{A}} (a_{pq} D^q u)\, D^p v\, dx$$

is such that for any function[4] u of class C^∞ in X^r, we have

$$(-1)^m B(u, u) \geq c \sum_{|p|=m} \int_{\mathring{A}} |D^p u|^2\, dx,$$

where c is a positive constant independent of u.

4) There exists a linear operator R which enjoys the following properties: i) R is a bounded operator with domain $L^2(A')$ (A' is a domain such that $A' \supset \mathring{A}$) and range in the Hilbert space $H_m(A')$ of functions with weak derivatives of order $\leq m$ belonging to $L^2(A')$; ii) R is hermitian on $L^2(A')$; iii) for any $f \in L^2(A')$ we have $LRf = f$.

Hypothesis 4) is satisfied when there exists a *fundamental solution* for the operator R. Hypotheses 1), 2), 3), 4) are satisfied by the classical differential operators encountered in eigenvalue theory.

Let us consider the space $C^\infty(\mathring{A})$ of C^∞ functions in $\mathring{A}$ and the finite dimensional manifold Γ of all the functions w such that $B(w, w) = 0$. Let us denote by $\mathscr{H}(A)$ the Hilbert space obtained through functional completion from the quotient space $C^\infty(\mathring{A})/\Gamma$ by means of the norm introduced by the scalar product

$$((u, v)) = (-1)^m B(u, v).$$

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[3] For the precise definition of properly regular domain see [6] p. 21. Roughly speaking, a properly regular domain is a domain with a piece-wise regular boundary such that $\partial A = \partial \bar{A}$ and which satisfies a cone-hypothesis.

[4] The term "function" must be understood as "vector-valued function", since the values of the function are l-vectors with complex components.

Let $(.,.)_{A'}$ denote the scalar product in the space $L^2(A')$. Let R^* be the bounded linear operator with domain $\mathscr{H}(A)$ and range, in $L^2(A')$, defined by the equations

$$((Rf, g)) = (f, R^*g)_{A'} \qquad [f \in L^2(A'), \ g \in \mathscr{H}(A)].$$

Let P be the orthogonal projector of $\mathscr{H}(A)$ onto its subspace $\Omega(A)$ determined by the solution of the homogeneous equations $Lu = 0$.

Theorem XI. *Let $U(A)$ be the class of all functions belonging to $H_m(A) \cap \cap H_{2m}(A_0)$ for every domain A_0 such that $\bar{A}_0 \subset A$. Then for every $f \in L^2(A)$ there exists in the class $U(A)$ one and only one solution u of the boundary value problem*

$$(13) \qquad \begin{cases} Lu = (-1)^m f & \text{in } A \\ D^p u = 0 \qquad (0 \leq |p| \leq m-1) & \text{on } \partial A. \end{cases}$$

Set

$$G = R^*R - R^*PR.$$

Then the solution u of problem (13) is given by $u = Gf$. Thus G is the Green operator for the boundary value problem (13).

Let $\{\omega_k\}$ be a complete system in the space $\Omega(A)$ and $\Omega_\varrho(A)$ be the ϱ-dimensional manifold spanned by $\omega_1, \ldots, \omega_\varrho$. Let P_ϱ be the orthogonal projector of $\mathscr{H}(A)$ onto $\Omega_\varrho(A)$.

Theorem XII. *Set $G_\varrho = R^*R - R^*P_\varrho R$. Then both operators G and G_ϱ, as operators on the Hilbert space $L^2(A)$, belong to $\mathfrak{C}^n$ for any $n > r/2m$. Moreover $\lim_{\varrho\to\infty} ||G - G_\varrho|| = 0$ and $G_\varrho > G_{\varrho+1}$.*

The following eigenvalue problem, considered in the space $U(A)$

$$(14) \qquad \begin{cases} Lu - (-1)^m \lambda u = 0 & \text{in } A, \\ D^p u = 0 \qquad (0 \leq p \leq m-1) & \text{on } \partial A \end{cases}$$

has only positive eigenvalues. Letting $\lambda^{-1} = \mu$, problem (14) is equivalent to the following one in the space $L^2(A)$:

$$(15) \qquad Gu - \mu u = 0.$$

For the upper approximation of the eigenvalues of (15) [i.e., the lower approximation of the eigenvalues of (14)] we can apply theorem X with $T = G$ and $T_\varrho = G_\varrho$. This is possible by theorem XII. For the computation of $I_s^n(G_\varrho)$ we may use theorem IX if an integral representation of R is known; i.e., if a fundamental solution of L is available.

On some other cases the explicit representation of G_ϱ, which we have given, can be used in order to give upper bounds to the remainder of the series which defines $I_s^n(G_\varrho)$.

In the following sections we shall consider as examples some classical eigenvalue problems of mathematical physics.

3. Two or three-dimensional elasticity.

Let us consider the differential operator of classical elasticity, which we shall write as follows in the space X^r ($r = 2, 3$):

$$L_i u = u_{i/hh} + \alpha u_{h/ih}, \qquad (i = 1, \ldots, r);$$

α is a given real constant (depending on the elastic material) such that $\alpha > -1$.

From now on we shall consider only vector-valued functions with real components.

Let us consider the eigenvalue problem

$$(16) \qquad \begin{cases} Lu + \lambda u = 0 & \text{in } A \\ u = 0 & \text{on } \partial A. \end{cases}$$

We can use the following bilinear form

$$B(u, v) = -\int_A (u_{i/h} v_{i/h} + \alpha u_{i/i} v_{h/h})\, dx.$$

Set

$$\varphi\ (t) \begin{cases} = \log t^{-1} & \text{for } r = 2 \\ = t^{-1} & \text{for } r = 3 \end{cases}$$

$$F_{ij}\,(x-y) = \frac{\alpha}{8\pi(1+\alpha)} \frac{\partial^2 |x-y|^2\, \varphi(|x-y|)}{\partial x_i\, \partial x_j} - \frac{\delta_{ij}}{2(r-1)\pi}\, \varphi(|x-y|);$$

$$\gamma_{ij}(x, y) = -\int_A \{F_{ik/h}(x-t)\, F_{jk/h}(t-y) + \alpha F_{ik/k}(x-t)\, F_{jh/h}(t-y)\}\, dt.$$

Let $\{\omega^s\}$ be a complete system of solutions of the homogeneous equations $Lu = 0$, such that $-B(\omega^s, \omega^l) = \delta_{sl}$[(5)]. Set

$$\varrho_i^s(x) = \int_A \{F_{ik/h}(x-t)\, \omega_{k/h}^s(t) + \alpha F_{ik/k}(x-t)\, \omega_{h/h}^s(t)\}\, dt.$$

Let $\{w^i\}$ be any system of linearly independent functions such that $w^i = 0$ on ∂A and such that $\{Lw_i\}$ be complete in the space $L^2(A)$. Let $\mu_1^{(\nu)} \geq \geq \ldots \geq \mu_k^{(\nu)} \geq \ldots \geq \mu_\nu^{(\nu)}$ be the roots of the determinantal equation

$$\det \{\int_A w_h^i w_h^j\, dx + \mu \int_A w_h^i L_h w^j\, dx\} = 0 \qquad (i, j = 1, \ldots, \nu)$$

(Rayleigh-Ritz approximations). Set

(5) For the construction of a complete system of solutions for $Lu = 0$ see [8] chap. III. The orthonormality condition $-B(\omega^s, \omega^l) = \delta_{sl}$ is assumed here only for the sake of simplicity. It is not necessary in numerical applications.

$$\tau_k^{(\nu)} = \{\sum_{i,j}^{1,r}[\int_A\int_A |\gamma_{ij}(x, y)|^2\, dx\, dy + \sum_{s,l}^{1,\nu(k)}\int_A \varrho_i^s(x)\, \varrho_i^l(x)\, dx \int_A \varrho_j^s(x)\, \varrho_j^l(x)\, dx -$$

$$- 2\sum_s^{1,\nu}\int_A\int_A \gamma_{ij}(x, y)\, \varrho_i^s(x)\, \varrho_j^s(y)\, dx\, dy] - \sum_i^{1,\nu(k)} [\mu_i^{(\nu)}]^2\}^{-\frac{1}{2}},$$

with the usual meaning for the symbol $\sum_i^{1,\nu}$. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \ldots$ be the eigenvalues of problem (16). Then we have

$$\tau_k^{(\nu)} \leq \lambda_k \leq \frac{1}{\mu_k^{(\nu)}}, \qquad (k \leq \nu), \tag{17}$$

and

$$\lim_{\nu\to\infty} \tau_k^{(\nu)} = \lim_{\nu\to\infty} \frac{1}{\mu_k^{(\nu)}} = \lambda_k. \tag{18}$$

4. Vibrations of a clamped plate.

We assume $r = 2$. The eigenvalue problem is the following

$$\begin{cases} \Delta_2\Delta_2 u - \lambda u = 0 & \text{in } A, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial A. \end{cases}$$

($\Delta_2 \equiv$ Laplace operator, $\dfrac{\partial}{\partial n} \equiv$ differentiation along the normal); u is a real valued function. The $\mu_k^{(\nu)}$ are now the roots of the equations

$$\det \left\{\int_A w_i\, w_j\, dx - \mu \int_A \Delta_2\, w_i\, \Delta_2\, w_j\, dx\right\} = 0,\ (i, j = 1, \ldots\ldots, \nu),$$

where the sequence $\{w_i\}$ satiesfies the usual completeness condition and $w_i = \dfrac{\partial w_i}{\partial n} = 0$ on ∂A. Inequalities (17) and the limit relations (18) hold also in this case with

$$\tau_k^{(\nu)} = \left\{\frac{1}{4\pi^2}\int_A\int_A |\log|x - y||^2\, dx\, dy -\right.$$

$$\left. - \frac{1}{4\pi^2}\sum_i^{1,\nu}\int_A\left[\int_A \omega_i(t)\log|x - t|\, dt\right]^2 dx - \sum_i^{1,\nu(k)} \mu_i^{(\nu)}\right\}^{-1};$$

($\{\omega_i\}$ is a complete system of harmonic functions (harmonic polynomials if A is simply connected) orthonormalised in $L^2(A)$.

5. Buckling of a clamped plate.

The eigenvalue problem is now the following

$$\begin{cases} \Delta_2\Delta_2 u + \lambda\Delta_2 u = 0 & \text{in } A, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial A. \end{cases}$$

In this case the lower bounds for λ_k are given by

$$\tau_k^{(\nu)} = \left\{\frac{1}{4\pi^2}\left[\int_A\int_A |\log|x-y||^2\,dx\,dy - 2\sum_i^{1,\nu}\int_A[\int_A \omega_i(t)\log|x-t|\,dt]^2\,dx + \right.\right.$$
$$\left.\left. + \sum_{h,j}^{1,\nu}(\int_A\int_A \log|x-y|\,\omega_h(x)\,\omega_j(y)\,dx\,dy)^2\right] - \sum_i^{1,\nu}{}^{(k)}[\mu_i^{(\nu)}]^2\right\}^{-\frac{1}{2}}.$$

The upper bounds $[\mu_k^{(\nu)}]^{-1}$ are obtained from the equation

$$\det\{\int_A w_i\Delta_2 w_j\,dx + \mu\int_A \Delta_2 w_i\,\Delta_2 w_j\,dx\} = 0, \qquad (i, j = 1, \ldots, \nu).$$

The systems $\{\omega_k\}$ and $\{w_i\}$ are the same as in the preceding example.

6. Numerical examples.

We have included in this paper numerical results concerning eigenvalue problems for elastic plates. The upper bounds (i.e., the inverses of the lower bounds for the eigenvalues of the Green operator) have been obtained by the Rayleigh-Ritz method, wherein we have used systems of polynomials. The lower bounds have been obtained by the method of orthogonal invariants and the representation, of the Green operator described in the paper.

For numerical examples concerning ordinary differential equations see [9], [12], [13].

I) Square plate clamped along its boundary.

$$\Delta_2\Delta_2 u - \lambda u = 0 \qquad \text{in } A \equiv \left(-\frac{\pi}{2} < x_1 < \frac{\pi}{2},\; -\frac{\pi}{2} < x_2 < \frac{\pi}{2}\right)^{(6)},$$

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial A.$$

Let r_i be the x_i-axis ($i = 1, 2$). Let r_3 be the line $x_1 = x_2$. By $H^{(\alpha_1\alpha_2)}$, ($\alpha_1 = 0, 1$) we denote the subspace of $L^2(A)$ consisting of all functions which are symmetric with respect to r_i if $\alpha_i = 0$, anti-symmetric if $\alpha_i = 1$. By $H^{(\alpha_1\alpha_2\alpha_3)}$, ($\alpha_i = 0, 1$), we denote the subspace of $H^{(\alpha_1\alpha_2)}$ of all functions belonging to $H^{(\alpha_1\alpha_2)}$ which are symmetric (anti-symmetric) with respect to r_3 if $\alpha_3 = 0$

($\alpha_3 = 1$). The space $L^2(A)$ can then be decomposed into subspaces (which are invariant for the given problem) as follows:

$$L^2(A) = H^{(000)}(A) \oplus H^{(001)}(A) \oplus H^{(110)}(A) \oplus H^{(111)}(A) \oplus H^{(01)}(A) \oplus H^{(10)}(A).$$

	000		001	
	lower bound	upper bound	lower bound	upper bound
λ_1	13.29376	13.29378	177.7193	177.7401
λ_2	179.408	179.431	976.13	979.59
λ_3	496.55	497.03	1569	1584
λ_4	977.64	981.25	3158	3282
λ_5	1577	1593	4038	4306
λ_6	3120	3244	5865	6791
λ_7	3155	3284	6774	8330
λ_8	4037	4317	7555	9931
λ_9	5853	6817		
λ_{10}	6701	8276		

(6) For the analytical and numerical investigation of this problem see [5]. References concerning numerical work on the same problem can be found in [5].

	110		111	
	lower bound	upper bound	lower bound	upper bound
λ_1	120.2143	120.2143	601.488	601.983
λ_2	605.792	606.920	2133.1	2155.6
λ_3	1401.5	1415.7	3398	3491
λ_4	2111.8	2161.2	5429	5834
λ_5	3306	3506	6970	7894
λ_6	5037	5842	9366	12071
λ_7	5412	6451		
λ_8	6200	7931		

	lower bound	upper bound
λ_1	55.2982	55.2994
λ_2	279.35	279.50
λ_3	454.37	454.99
λ_4	896.8	901.6
λ_5	1180	1191
λ_6	1833	1875
λ_7	2171	2242
λ_8	2560	2677
λ_9	3371	3652
λ_{10}	4154	4716
λ_{11}	4556	5329
λ_{12}	4582	5372

II) Circular plate clamped along its boundary.

$$\Delta_2\Delta_2 u - \lambda u = 0 \qquad \text{in } A \equiv \{x_1^2 + x_2^2 < 1\}, \qquad u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial A. \text{[7]}$$

The space $L^2(A)$ can be decomposed into the direct sum of a sequence of subspaces (which are invariant for the problem) as follows:

$$L^2(A) = H^{(0)} \oplus H^{(1)} \oplus \ldots \oplus H^{(k)} \oplus \ldots,$$

where $H^{(k)}$ is the subspace spanned by all functions of the type $f(\varrho)\cos k\vartheta$ and $g(\varrho)\sin k\vartheta$ where $x_1 = \varrho\cos\vartheta$, $x_2 = \varrho\sin\vartheta$, k is a non-negative integer and f and g are arbitrary functions. Each subspace $H^{(k)}$ (for $k > 0$) is itself decomposable into two invariant subspaces

$$H_1^{(k)} \equiv \{f(\varrho)\cos k\vartheta\}, \qquad H_2^{(k)} \equiv \{g(\varrho)\sin k\vartheta\}.$$

It is obvious that the eigenvalues in $H_1^{(k)}$ coincide with those of $H_2^{(k)}$. Therefore the eigenvalues included in the tables with index $k > 0$ must be considered as double eigenvalues.

[7] Application of the general method to this problem is due to M. Schaerf and will appear in a forthcoming paper. The numerical results exhibited in the present paper are due to this author.

	$k = 0$		$k = 1$	
	lower bound	**upper bound**	**lower bound**	**upper bound**
λ_1	104.36311051	104.3631056	452.00448	452.00452
λ_2	1581.742	1581.745	3700.11	370013
λ_3	7939.38	7939.55	144418.2	14419.1
λ_4	25017.2	25022.3	39606.2	39622.3
λ_5	60939.5	61012.2	88482.2	88661.1
λ_6	125786	126430	171901	173225
λ_7	230123	234133	300129	307340
λ_8	380355	399323	476778	507392
λ_9	569823	640349	689901	794004

	$k = 2$		$k = 3$	
	lower bound	**upper bound**	**lower bound**	**upper bound**
λ_1	1216.4072	1216.4076	2604.061	2604.065
λ_2	7154.14	7154.23	12325.4	12325.8
λ_3	23656.3	23659.1	36207.4	36215.6
λ_4	58870.7	58913.3	83526.1	83625.1
λ_5	123047	123437	165470	166244
λ_6	227594	230089	293711	298098
λ_7	381914	394063	476150	495553
λ_8	585981	632954	708346	777466
λ_9	822673	970669		

	$k = 4$		$k = 5$	
	lower bound	**upper bound**	**lower bound**	**upper bound**
λ_1	4853.31	4853.33	8233.49	8233.57
λ_2	19629.1	19630.3	29513.3	29516.3
λ_3	52658.5	52678.8	73627.7	73673.3
λ_4	114314	114523	152001	152404
λ_5	216597	218019	277274	279738
λ_6	371076	378366	460483	472040
λ_7	583460	613097	704421	748019
λ_8	844252	942444		

	$k = 6$ lower bound	$k = 6$ upper bound	$k = 7$ lower bound	$k = 7$ upper bound
λ_1	13044.2	13044.5	19615.1	19615.8
λ_2	42457.8	42465.1	58973.7	58989.9
λ_3	99763.2	99857.0	131741	131922
λ_4	197374	198104	251235	252489
λ_5	348349	352407	430663	437070
λ_6	562700	580302	678459	704371
λ_7	839575	901677		

	$k = 8$ lower bound	$k = 8$ upper bound	$k = 9$ lower bound	$k = 9$ upper bound
λ_1	28304.7	28306.3	39500.6	39504.1
λ_2	79602.4	79635.6	104914	104979
λ_3	170265	170590	216062	216620
λ_4	314402	316460	387704	390951
λ_5	525050	534802	632331	646714
λ_6	808467	845496		

	$k = 10$ lower bound	$k = 10$ upper bound	$k = 11$ lower bound	$k = 11$ upper bound
λ_1	53618.9	53626.1	71103.5	71117.8
λ_2	135509	135626	172014	172215
λ_3	269882	270798	332495	333947
λ_4	471976	476928	568059	575391
λ_5	753313	773948	888788	917682

$k = 12$ $k = 13$

	lower bound	upper bound	lower bound	upper bound
λ_1	92426.2	92452.6	118085	118133
λ_2	215080	215414	265387	265921
λ_3	404689	406917	487270	490593
λ_4	676795	687371	799027	813933

$k = 14$ $k = 15$

	lower bound	upper bound	lower bound	upper bound
λ_1	148607	148687	184542	184674
λ_2	323636	324465	390553	391804
λ_3	581056	585887	686877	693747
λ_4	935594	956172		

$k = 16$ $k = 17$

	lower bound	upper bound	lower bound	upper bound
λ_1	226468	226678	274986	275311
λ_2	466883	468724	553390	556044
λ_3	805574	815148	937997	951097

$k = 18$ $k = 19$

	lower bound	upper bound	lower bound	upper bound
λ_1	330725	331214	394333	395054
λ_2	650861	654609	760097	765295

$k = 20$

	lower bound	upper bound
λ_1	466485	467526
λ_2	881916	889004

Several text-books exhibit the following numerical table due to H. Carrington (London—Edinburgh Phil. Mag., vol. 55 pp. 1261—64, 1925), for $\mu = \sqrt[4]{\lambda}$. It was obtained by computing the zeros of a well-known transcendental function expressed by means of Bessel functions.

	$k = 0$	$k = 1$	$k = 2$	$k = 3$
μ_1	3.1961	4.6110	5.9056	7.1433
μ_2	6.3064	7.7993	9.1967	10.537
μ_3	9.4395	10.958	12.402	13.795
μ_4	12.577	14.108	15.579	
μ_5	15.716			

Dr. Schaerf gets the following results.

	$k = 0$		$k = 1$	
	lower bound	upper bound	lower bound	upper bound
μ_1	3.19622	3.19623	4.61089	4.61090
μ_2	6.30643	6.30644	7.79926	7.79928
μ_3	9.43945	9.43950	10.9579	10.9581
μ_4	12.5764	12.5772	14.1072	14.1087
μ_5	15.7117	15.7165	17.2470	17.2558

It is interesting to observe that the numerical application of the methods described in this paper proves that some of the classical numerical results are incorrect in the fifth digit. On the other hand the numerical application of our method is simpler than the numerical solution of the classical above mentioned, transcendental equation.

	$k=2$		$k=3$	
	lower bound	upper bound	lower bound	upper bound
μ_1	5.90567	5.90568	7.14352	7.14354
μ_2	9.19685	9.19689	10.5366	10.5367
μ_3	12.4018	12.4023	13.7942	13.7951
μ_4	15.5766	15.5795	17.0002	17.0053

7. The problem of estimating eigenvalues when estimates for invariant subspaces are known.

Let us consider the linear operator L with domain the linear variety $\mathscr{D}_L$ of the Hilbert space S. Let V be a linear subvariety of $\mathscr{D}_L$. The following hypothesis be satisfied:

There exists a PCO G of the space S such that: i) *the range $G(S)$ of G is contained in V*; ii) $GL = LG = I$.

Let us consider the eigenvalue problem

$$(19) \qquad Lv - \lambda v = 0, \qquad v \in V.$$

This problem is equivalent to the following one

$$(20) \qquad Gu - \mu u = 0, \qquad u \in S$$

where $\mu = \lambda^{-1}$. It follows that all the eigenvalues of (19) constitute a non-decreasing sequence tending to $+\infty$

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$$

Each eigenvalue appears — as usual — in the above sequence as many times as its multiplicity.

Let us suppose that we can decompose the space S as direct sum of a finite or a countable set of mutually orthogonal subspaces, each of them being an invariant subspace for G.

$$S = H_1 \oplus H_2 \oplus \ldots \oplus H_s \oplus \ldots$$

Problem (20) is equivalent to the following system of eigenvalue problems:

$$(20_s) \qquad Gu - \mu^{(s)} u = 0, \qquad u \in H_s \qquad (s = 1, 2, \ldots)$$

Set $V_s = G(H_s)$. It is easy to prove that

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_s \oplus \ldots$$

and that problem (19) is equivalent to the following system of eigenvalue problems:

$$(19_s) \qquad Lv - \lambda^{(s)}v = 0, \qquad v \in V_s \qquad (s = 1, 2, \ldots)$$

Let $\lambda_1^{(s)} \leq \lambda_2^{(s)} \leq \ldots \leq \lambda_k^{(s)} \leq \ldots$ be the sequence of the eigenvalues of problem (19_s). Suppose we have obtained for the first $p_s \geq 1$ eigenvalues of problem (19_s) the following *table of estimates* (t_s)

$$(t_s) \qquad \begin{array}{c} \delta_1^{(s)} \leq \lambda_1^{(s)} \leq \varepsilon_1^{(s)} \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ \delta_{p_s}^{(s)} \leq \lambda_{p_s}^{(s)} \leq \varepsilon_{p_s}^{(s)} \end{array}$$

We suppose that the upper bounds $\varepsilon_k^{(s)}$ have been computed by the Rayleigh-Ritz method. That means that the $\varepsilon_k^{(s)}$ are the roots of the determinantal equation

$$\det \{(Lw_h^{(s)}, w_k^{(s)}) - \lambda(w_h^{(s)}, w_k^{(s)})\} = 0 \qquad (h, k = 1, \ldots, p_s),$$

where $w_1^{(s)}, \ldots, w_s^{(s)}$ are p_s linearly independent vectors of V_s.

The problem now arises. *From estimates of the tables* (t_s) *is it possible to deduce estimates for the k-th eigenvalue* λ_k *of problem* (19)?

In solving this problem we shall not make any assumption on the method used for computing the lower bounds $\delta_k^{(s)}$. We only assume — without any loss — that $\delta_1^{(s)} \leq \delta_2^{(s)} \leq \ldots \leq \delta_{p_s}^{(s)}$.

In order to consider a more concrete situation we shall assume that the tables (t_s) are given only for $s = 1, \ldots, q$ $(q \geq 1)$. Assuming that is necessary if the spaces H_s are infinitely many. For $s > q$ we shall only suppose that we know a positive real number c_s such that for any eigenvalue of (19_s) with $s > q$ we have $\lambda_k^{(s)} > c_s$. Moreover $\lim_{s\to\infty} c_s = +\infty$, if the H_s are infinitely many.

For instance, in the case of the example II, considered in section 6 (circular clamped plate), it is possible to show that we may assume

$$c_s = 16(s + 1)(s + 2)(s + 3).$$

Let us consider the two sequences $\{\delta_k^{(s)}\}$, $\{\varepsilon_k^{(s)}\}$ $(s = 1, \ldots, q;\ k = 1, \ldots, p_s)$. We shall denote by $\{\delta_h\}$, $\{\varepsilon_h\}$ $(h = 1, \ldots, m,\ m = p_1 + \ldots + p_q)$ the sequences obtained from $\{\delta_k^{(s)}\}$ and $\{\varepsilon_k^{(s)}\}$, respectively, by disposing all their elements in non-decreasing order.

It will be useful to introduce the function $l = l(s, k)$ $(s = 1, \ldots, q;\ k = 1, \ldots, p_s)$, whose range is the set $1, \ldots, m$ such that

$$\delta_{l(s,k)} = \delta_k^{(s)}.$$

This function is not unique if some of the numbers $\delta_k^{(s)}$ coincide. However we suppose to have chosen, amongst the possible ones, a well determined function $l = l(s, k)$.

Let us first consider the following lemma.

Lemma XIII. *Let $h \to \delta_h$ and $k \to \lambda_k$ be two real valued functions, the first defined for $h = 1, \ldots, m$ and the second for $k = 1, \ldots, n$. Assume that $m \geq n$ and*

$$\delta_1 \leq \delta_2 \leq \cdots \leq \delta_m$$
$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Let us suppose that there exists a function $k \to q_k$ defined for $k = 1, \ldots, n$ such that

I) *q_k is a positive integer and $1 \leq q_k \leq m$,*

II) *$q_i = q_j$ for $i \neq j$ implies $q_i \geq n$,*

III) *$\lambda_k \geq \delta_{q_k}$ $(k = 1, \ldots, n)$.*

Under the above hypotheses we have

$$\lambda_k \geq \delta_k. \tag{21}$$

Inequality (21) is obvious if $q_k \geq k$. Let us suppose $q_k < k$. It must exist an index s such that $1 \leq s \leq k - 1$, $q_s \geq k$. In fact $q_s < k$ for any $s \leq$ $\leq k - 1$ implies that there exist two indeces i, j such that $i \leq k$, $j \leq k$, $i \neq j$, $q_i = q_j < k \leq n$. That contradicts hypothesis II). Existence of $q_s \geq k$ with $s \leq k - 1$ implies $\lambda_k \geq \lambda_s \geq \delta_{q_s} \geq \delta_k$.

Theorem XIV. *Let $\delta_{p_r}^{(r)}$ be such that $\delta_{p_s}^{(s)} \geq \delta_{p_r}^{(r)}$ for $s = 1, \ldots, q$. We suppose that, if the spaces H_s decomposing S are more than q, then $c_s \geq \delta_{p_r}^{(r)}$ for every $s > q$. Let n be the smallest integer such that $\delta_n = \delta_{p_r}^{(r)}$. We have the following estimates for the first n eigenvectors of problem* (19).

$$\delta_k \leq \lambda_k \leq \varepsilon_k \qquad (k = 1, \ldots, n).$$

Let us associate to every eigenvalue λ_k of problem (19) a unit vector v_k such that

$$Lv_k - \lambda_k v_k = 0, \qquad v_k \in V, \qquad (v_h, v_k) = \delta_{hk}.$$

The sequence $\{v_k\}$ may be considered as the union of the subsequences $\{v_i^{(s)}\}$ such that
$$Lv_i^{(s)} - \lambda_i^{(s)} v_i^{(s)} = 0, \qquad v_i^{(s)} \in V_s;$$
$\{\lambda_i^{(s)}\}$ is the sequence of the eigenvalues of problem (19_s).

Let us consider for $1 \leq k \leq n$ the eigenvalue λ_k. Suppose that $v_k = v_i^{(s)}$. We have $\lambda_k = \lambda_i^{(s)} \geq \delta_i^{(s)}$ if $1 \leq s \leq q$ and $i \leq p_s$. We have $\lambda_k = \lambda_i^{(s)} \geq \delta_n$ either if $1 \leq s \leq q$, $i > p_s$ or if $s > q$. Set

$$q_k \begin{cases} = l(s, i) & \text{if } 1 \leq s \leq q, \ i \leq p_s \\ = n & \text{if } 1 \leq s \leq q, \ i > p_s \text{ or } s > q. \end{cases}$$

The functions $h \to \delta_h$, $k \to \lambda_k$, $k \to q_k$ satisfy hypotheses of lemma XIII. It follows that inequality (21) holds.

Let $w_1, \ldots, w_m$ be the m vectors of V obtained by disposing in a unique sequence the vectors of the q sequences $\{w_i^{(s)}\}$ $(s = 1, \ldots, q;\ i = 1, \ldots, p_i)$. The m roots of the determinantal equation

$$\det \{(Lw_i, w_j) - \lambda(w_i, w_j)\} = 0$$
$$(i, j = 1, \ldots, m)$$

are $\varepsilon_1 \leq \varepsilon_2 \leq \ldots \leq \varepsilon_m$. From the theory of the Rayleigh-Ritz method it follows $\lambda_k \leq \varepsilon_k$ $(k = 1, \ldots, m)$.

The following tables show the estimates, which is possible to deduce (for the eigenvalues of a square plate and of a circular plate) from the estimates already known for invariant subspaces. In both cases the lower bounds and the upper bounds have been compared with the asymptotic values given by a formula due to R. Courant and A. Pleijel (Comm. on Pure and Applied Math. III, 1, 1950, p. 1—10). These numerical results suggest that the use of asymptotic formulas for the numerical evaluation of eigenvalues, even of rather high index, could be misleading.

Square plate

	lower bound	upper bound	asymptotic value		lower bound	upper bound	asymptotic value
λ_1	13.29376	13.29378	1.6211	λ_{24}	1833	1875	933.77
λ_2	55.2982	55.29934	6.4845	λ_{25}	2111.8	2155.6	1013.2
λ_3	55.2982	55.29934	14.590	λ_{26}	2133.1	2161.2	1095.8
λ_4	120.2143	120.2232	25.938	λ_{27}	2171	2242	1181.8
λ_5	177.7113	177.7401	40.528	λ_{28}	2171	2242	1270.9
λ_6	179.408	179.431	58.361	λ_{29}	2560	2677	1363.3
λ_7	279.35	279.50	79.435	λ_{30}	2560	2677	1459.0
λ_8	279.35	279.50	103.75	λ_{31}	3120	3244	1557.9
λ_9	454.37	454.99	131.31	λ_{32}	3155	3282	1660.0
λ_{10}	454.37	454.99	162.11	λ_{33}	3158	3284	1765.4
λ_{11}	496.55	497.03	196.15	λ_{34}	3306	3491	1874.0
λ_{12}	601.488	601.983	233.44	λ_{35}	3371	3506	1985.8
λ_{13}	605.792	606.920	273.97	λ_{36}	3371	3652	2100.9
λ_{14}	896.8	901.6	317.74	λ_{37}	3398	3652	2219.3
λ_{15}	896.8	901.6	364.75	λ_{38}	4037	4306	2340.9
λ_{16}	976.13	979.59	415.01	λ_{39}	4038	4317	2465.7
λ_{17}	977.64	981.25	468.50	λ_{40}	4154	4716	2593.8
λ_{18}	1180	1191	525.24	λ_{41}	4154	4716	2725.1
λ_{19}	1180	1191	585.23	λ_{42}	4556	5329	2859.6
λ_{20}	1401	1415	648.45	λ_{43}	4556	5329	2997.4
λ_{21}	1569	1584	714.92	λ_{44}	4582	5372	3138.5
λ_{22}	1577	1593	784.63	λ_{45}	4582	5372	3282.8
λ_{23}	1833	1875	857.58				

Circular plate

	lower bound	upper bound	asymptotic value		lower bound	upper bound	asymptotic value
λ_1	104.363	104.364	16	λ_{47}	58870.7	58913.3	35344
λ_2	452.004	452.005	64	λ_{48}	58870.7	58913.3	36864
λ_3	452.004	452.005	144	λ_{49}	58973.7	58989.9	38416
λ_4	1216.40	1216.41	256	λ_{50}	58973.7	58989.9	40000
λ_5	1216.40	1216.41	400	λ_{51}	60939.5	61012.2	41616
λ_6	1581.74	1581.75	576	λ_{52}	71103.5	71117.8	43264
λ_7	2604.06	2604.07	784	λ_{53}	71103.5	71117.8	44944
λ_8	2604.06	2604.07	1024	λ_{54}	73627.7	73673.3	46656
λ_9	3700.11	3700.13	1296	λ_{55}	73627.7	73673.3	48400
λ_{10}	3700.11	3700.13	1600	λ_{56}	79602.4	79635.6	50176
λ_{11}	4853.31	4853.33	1936	λ_{57}	79602.4	79635.6	51984
λ_{12}	4853.31	4953.33	2304	λ_{58}	83526.1	83625.1	53824
λ_{13}	7154.14	7154.23	2704	λ_{59}	83526.1	83625.1	55696
λ_{14}	7154.14	7154.23	3136	λ_{60}	88482.2	88661.1	57600
λ_{15}	7939.38	7939.55	3600	λ_{61}	88482.2	88661.1	59536
λ_{16}	8233.49	8233.57	4096	λ_{62}	92426.2	92452.6	61504
λ_{17}	8233.49	8233.57	4624	λ_{63}	92426.2	92452.6	63504
λ_{18}	12325.4	12325.76	5184	λ_{64}	99763.2	99857.0	65536
λ_{19}	12325.4	12325.76	5776	λ_{65}	99763.2	99857.0	67600
λ_{20}	13044.2	13044.5	6400	λ_{66}	104914	104979	69696
λ_{21}	13044.2	13044.5	7056	λ_{67}	104914	104979	71824
λ_{22}	14418.2	14420.0	7744	λ_{68}	114314	114523	73984
λ_{23}	14418.2	14420.0	8464	λ_{69}	114314	114523	76176
λ_{24}	19615.1	19615.8	9216	λ_{70}	118085	118113	78400
λ_{25}	19615.1	19615.8	10000	λ_{71}	118085	118133	80656
λ_{26}	19629.1	19630.3	10816	λ_{72}	123047	123437	82944
λ_{27}	19629.1	19630.3	11664	λ_{73}	123047	123437	85264
λ_{28}	23656.3	23659.1	12544	λ_{74}	125786	126430	87616
λ_{29}	23656.3	23659.1	13456	λ_{75}	131741	131921	90000
λ_{30}	25017.2	25022.3	14400	λ_{76}	131741	131921	92416
λ_{31}	28304.7	28306.3	15376	λ_{77}	135509	135625	94864
λ_{32}	28304.7	28306.3	16384	λ_{78}	135509	135625	97344
λ_{33}	29513.3	29516.3	17424	λ_{79}	148607	148686	99856
λ_{34}	29513.3	29516.3	18496	λ_{80}	148607	148686	102400
λ_{35}	36207.4	36215.6	19600	λ_{81}	152001	152403	104976
λ_{36}	36207.4	36215.6	20736	λ_{82}	152001	152403	107584
λ_{37}	39500.6	39504.1	21904	λ_{83}	165470	166243	110224
λ_{38}	39500.6	39504.1	23104	λ_{84}	165470	166243	112896
λ_{39}	39606.2	39622.3	24336	λ_{85}	170265	170589	115600
λ_{40}	39606.2	39622.3	25600	λ_{86}	170265	170589	118336
λ_{41}	42457.8	42465.1	26896	λ_{87}	171901	172214	121104
λ_{42}	42457.8	42465.1	28224	λ_{88}	171901	172214	123904
λ_{43}	52658.5	52678.8	29584	λ_{89}	172014	173224	126736
λ_{44}	52658.5	52678.8	30976	λ_{90}	172014	173224	129600
λ_{45}	53618.9	53626.1	32400	λ_{91}	184542	184673	132496
λ_{46}	53618.9	53626.1	33856	λ_{92}	184542	184673	135424

REFERENCES

[1] N. Aronszajn — *The Rayleigh-Ritz and the Weinstein Methods for approximation of Eigenvalues — (1. Operators in Hilbert Space) — Dept. of Math. Oklahoma Agricul. and Mechanical College, Stillwater Oklahoma, Tech. Rep. No. 1, 1949.*

[2] N. Aronszajn — *Approximation methods for eigenvalues of completely continuous symmetric operators — Proc. Symp. Spectral Theory and Diff., Problems, Stillwater, Okla. 1951.*

[3] N. W. Bazley — *Lower Bounds for Eigenvalues — Jour. Math. & Mech. Indiana Univ. vol. 10, 1961.*

[4] N. W. Bazley—D. W. Fox — *Truncations in the Method of Intermediate Problems for Lower Bounds to Eigenvalues — Jour. of Research of Nat. Bureau of Standards, Math. & Math. Phy., vol. 56 B, 1961.*

[5] L. de Vito—G. Fichera—A. Fusciardi—M. Schaerf — *Sugli autovalori della piastra quadrata incastrata lungo il bordo — Rend. Accademia Naz. Lincei, Classe Fis. Mat. Nat., vol. XL, 1966.*

[6] G. Fichera — *Linear elliptic differential systems and eigenvalue problems — Lecture Notes in Mathematics, vol. 8, Springer Verlag, 1965.*

[7] G. Fichera — *Approximation and Estimates for Eigenvalues — Proc. of the Conference on Numerical Analysis, Univ. of Maryland, 1965, Academic Press.*

[8] G. Fichera — *Sull' esistenza e sul calcolo delle soluzioni dei problemi al contorno, relativi all' equilibrio di un corpo elastico — Annali Scuola Norm. Sup. Pisa, s. III, v. IV, 1950.*

[9] G. Fichera — *Sul calcolo degli autovalori — Atti del Simposio „Applicazioni della Analisi alla Fisica Matematica", Cagliari-Sassari 1964, Ed. Cremonese, Roma 1965.*

[10] S. H. Gould — *Variational Methods in Eigenvalue Problems — Univ. of Toronto Press, Toronto, 1957.*

[11] M. Plancherel — *Sur la méthode d'integration de Ritz — Bullet. des Sciences Mathem., s. II, vol. 47, 1923 and vol. 48, 1924.*

[12] F. Scarpina — *Sul calcolo degli autovalori relativi ad una equazione differenciale ineren te ad un problema di conduzione del calore — (Nota I e II) — Rend. Acc. Naz. Lincei, s. VIII, v. XXXVIII, 1965.*

[13] F. Scarpini — *Sul calcolo degli autovalori connessi ad un problema di Genetica — Rend. di Matem., vol. 24, 1965.*

[14] E. Trefftz — *Ueber Fehlerschätzung bei Berechnung von Eigenwerten — Math. Ann. Bd. 108, 1933.*

[15] H. F. Weinberger — *A theory of lower bounds for eigenvalues — Tech. Not. BN 183. Inst. Fluid Dynam. & Appl. Math. Univ. of Maryland, 1959.*

[16] A. Weinstein — *Études des spectres des équations aux dérivées partielles — Mémorial des Sci. Mathem. n. 88, Gauthier-Villars, Paris, 1937.*

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
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ON THE LINEAR AND QUASILINEAR PARABOLIC EQUATIONS

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1. If the coefficients in equations are smooth then it take place the unique solvability and the strong estimates for solutions of large classes of parabolic systems with very general boundary conditions. The first estimate of the same type was the inequality.

$$
\begin{aligned}
(1) \qquad & \int_{Q_T} (u_t^2 + u_{xx}^2)\,\mathrm{d}x\,\mathrm{d}t + \max_{0 \leq t \leq T} \int_{\Omega} u^2(x, t)\,\mathrm{d}x \leq \\
& \leq c \int_{Q_T} (u_t - a_{ij}(x, t)\, u_{x_i x_j} + \ldots)^2\,\mathrm{d}x\,\mathrm{d}t + c_1 \int_{\Omega} u^2(x, 0)\,\mathrm{d}x \equiv \\
& \equiv c \int_{Q_T} (Zu)^2\,\mathrm{d}x\,\mathrm{d}t + c_1 \int_{\Omega} u^2(x, 0)\,\mathrm{d}x, \\
& Q_T = \{(x, t) \colon x \in \Omega,\ t \in [0, T]\},
\end{aligned}
$$

which is hold for any arbitrary function $u(x, t)$, satisfying one of classical homogenous boundary conditions. After that had been proved the strong estimates for the parabolic operator in the space $L_p(Q_T)$, $p > 1$. (SOLONNIKOV) and in the HÖLDER space (A. FRIEDMAN). These estimates were generalised for parabolic systemes. The most wide class of systemes and boundary conditions was considered by SOLONNIKOV. The unique solvability had been proved with these estimates together.

The kernel of all these estimates [exepte my prove of the inequality (1)] is provided by the Korn—Schauder "gluiing" idea, which permits one to reduce the general estimation problem to some canonical estimation problems for systems with constant coefficients. For this approach to be feasible, however it is necessary that the coefficients of the leading termes of system be continuous.

2. Investigations of parabolic equations and systems with discontinuous coefficients have bean based principally on the energy inequality (see the papers LADYZENSKAYA, VISHIK, LIONS, BROWDER and others). This inequality holds for a harrower class of systems than that mentioned above — for so-called strongly-parabolic systems, under some simplest boundary

conditions. This class of parabolic systems belong the parabolic equations of the form

$$u_t - Z^{(2m)}u = f, \tag{2}$$

where $Z^{(2m)}u$ is an elliptic operator of the order $2m$ with principal part in divergence form: $Z^{(2m)}u = D_x^m(a(x, t)\, D_x^m u)\, t$.

The energy inequality enables one to prove the existence of generalised solutions, which for the equations (2) have derivatives $D_x^m u$ belonging to $L_2(Q_T)$ and which are continuous with respect to t in the norm of $L_2(\Omega)$ (we require that $u_0 = u(x, 0) \in L_2(\Omega)$, the inhomogeneous term f may be of rather general nature). But it gives no additional information about the solution even when u_0 and f are very smooth.

It was not possible on the basis of the methods available up to 1956—57 to derive any conclusions about the improvements of the differentiability properties of the solutions of parabolic equations with discontinuous coefficients when u_0 and f (but not coefficients) become smoother. This situation took place even for equations of the second order.

Some ten years ago, however, new methods began to develop following upon the pioneering work of Nash and De Georgi. By means of these methods had been established a series of new principles (relations) for linear equations of second order and they proved to be usfull for the study of quasilinear equations as well.

In joint papers by Uraltzeva and myself were investigated equations of the form

$$u_t - \frac{\partial}{\partial x_i}\left(a_{ij}(x, t)\, u_{x_j} + a_i(x, t)\, u\right) + b_i u_{x_i} + au = f + \frac{\partial f_i}{\partial x_i} \tag{3}$$

under the conditions that the coefficients a_{ij} be arbitrary measureable functions satisfying the inequality

$$\nu\xi^2 \leq a_{ij}\xi_i\xi_j \leq \mu\xi^2, \qquad \nu, \mu = \text{const} > 0, \tag{4}$$

and that the coefficients of the lower terms, as well as inhomogeneous terms f and f_i in (3), belong to the spaces $L_{q_k,\infty}(Q_T)$ with appropriate q_k (the norm in $L_{q_k,\infty}(Q_T)$ is defined by $||u||_{q,\infty,Q_T} = \operatorname{vrai}\max\limits_{0\leq t\leq T} (\int\limits_\Omega |u|^q\, dx)^{\frac{1}{q}}$.

Some examples were also given demonstrating that the dependence of the degree of smoothness of solutions of equation (3) on the parameters q_k established in the above mentioned joint papers is best possible.

Slight modifications of the methods developed by stadying these equations (3) and of the corresponding elliptic equations of the second order enable one to investigate the cases in which the coefficients and inhomogeneous terms are elements of $L_{q_k,r_k}(Q_T)$. [The norm in $L_{q,r}(Q_T)$ being defined by

$||u||_{q,r,Q_T} = \{\int\limits_0^T (\int\limits_\Omega |u|^q \, dx)^{\frac{r}{q}} \, dt\}^{\frac{1}{r}}]$. In a joint paper [*] by N. N. Uraltzeva, A. V. Ivanov, A. L. Treskunov and myself the study was made of the ranges of q_k and r_k in which solutions u of equation (3):

a) belong to the spaces $L_{q,r}$

b) give a finite value to the integral $\int \exp\{\lambda u(x, t)\} \, dx \, dt$, $\lambda > 0$;

c) have a bounded vraimaximum $|u|$;

d) have a finite Hölder norm $|u|_{x,t}^{(\alpha, \frac{\alpha}{2})}$.

The above has all been carried out for generalised solutions of equation (3) belonging to the space $V_2^{1,0}(Q_T)$. This space is the completion in the norm

$$|u|_{Q_T} = \max_{0 \leq t \leq T} (\int\limits_\Omega u^2(x, t) \, dx)^{\frac{1}{2}} + (\int\limits_{Q_T} |u_k|^2 \, dx \, dt)^{\frac{1}{2}}. \tag{5}$$

of the set of smooth functions. Examples were constructed showing that the regularity conditions given in [*] are exact (in the sence that the indices q_k, r_k cannot be reduced). I remark also that as it often takes place passage from the spaces $L_{q,\infty}$ to the full scale of spaces $L_{q,r}$ enable us to make the results obtained more transparent (observeable) and more complete.

As an example of the results obtained in [*] I shall formulate the following theorem:

Theorem I. *Assume that for equation* (3) *the inequality* (4) *and the conditions*

$$\sum_{i=1}^n a_i^2, \qquad \sum_{i=1}^n b_i^2, \qquad a \in L_{q,r}(Q_T), \qquad \frac{1}{r} + \frac{n}{2q} \leq 1, \tag{6}$$

hold. Then if

$$f_i \in L_2(Q_T), \qquad f \in L_{q,r_1}(Q_T), \qquad \frac{1}{r_1} + \frac{n}{2q_2} \leq 1 + \frac{n}{4},$$

$$u|_{t=0} = u_0(x) \in L_2(\Omega) \qquad \text{and } u|_S = 0$$

the first initial boundary value problem for equation (3) *is uniquely solvable in the space* $V_2^{1,0}(Q_T)$. *(To be more precise, uniquely solvable in* $V_2^{1,\frac{1}{2}}(Q_T)$*).*

If f and f_i satisfy the more restrictive conditions

$$\begin{aligned} & f \in L_{q_2,r_2}(Q_T), && \frac{1}{r_2} + \frac{n}{2q_2} \leq 1 + \frac{n}{4}\,\Theta, \\ & \sum_{i=1}^n f_i^2 \in L_{q_3,r_3}(Q_T), && \frac{1}{r_3} + \frac{n}{2q_3} \leq 1 + \frac{n}{2}\,\Theta, \end{aligned} \tag{7}$$

where $\Theta \in (0, 1)$, then the solution has the sharper properties:

$$u \in L_{\hat{q},\hat{r}}, \qquad \frac{1}{\hat{r}} + \frac{n}{2\hat{q}} = \frac{n}{4}\,\Theta.$$

(**Remark:** *from the assumption* $u \in V_2^{1,0}(Q_T)$ *one can only conclude that* $u \in L_{q,r}(Q_T)$, *where* $\left.\frac{1}{r}+\frac{n}{2q}=\frac{n}{4}\right)$. *If the constants* q *and* r *in* (6) *satisfy the inequality*

$$(8) \qquad \frac{1}{r}+\frac{n}{2q}<1$$

and the parameter Θ *in* (7) *vanishes, then* $\int\limits_{Q_T} \exp\{\lambda u(x,t)\}\,dx\,dt$ *must be finite for some* $\lambda > 0$. *If* $\sum\limits_{i=1}^{n} a_i^2, \sum\limits_{i=1}^{n} b_i^2, a, \sum\limits_{i=1}^{n} f_i^2, f \in L_{q,r}(Q_T)$ *and* q, r *satisfy* (8) *then* u *is Hölder-continuous in* (x, t).

In a particular case, when $a_i \equiv f_i \equiv f \equiv 0$ *and* $a(x,t) \geq 0$, *for weak solutions* $u(x,t)$ *takes place the maximum principle, that is*:

$$\min\{0; \operatorname*{vraimin}_{\Gamma_T} u\} \leq u(x,t) \leq \max\{0; \operatorname*{vraimax}_{\Gamma_T} u\}.$$

If $a(x,t) \equiv 0$, *then* $\operatorname*{vraimin}_{\Gamma_T} u \leq u(x,t) \leq \operatorname*{vraimax}_{\Gamma_T} u$.

In formulating theorem I have not indicated the allowable ranges of variation of q and r — these depend on the dimension n in (6).

3. Now I should like to give an example demonstrating the necessity of the restrictions of type (6) imposed on the degree of singularity of coefficients in (3). The function $u = e^{\frac{-|x|^2}{4t}}$ which vanishes when $t = 0$ presents the solution of the Cauchy problem in half space $\{(x,t): x \in E_n, t \geq 0\}$ for the equation

$$(9) \qquad u_t - \Delta u + n\sum_{i=1}^{n} \frac{x_i}{|x|^2} u_{x_i} = 0,$$

and for the equation

$$(10) \qquad u_t - \Delta u - \frac{n}{4t} u = 0$$

as well.

This solution evidently belongs to $V_2^{1,0}$. Moreover by smoothing u with respect to t or x one may construct almost-classical solutions of (9) and (10). This shows that if one of the most important properties of parabolic initial value problems, namely their deterministic nature, is to be retained, singularities of the coefficients $b_i(x,t)$ of order $\frac{1}{|x|}$ and singularities of the coeficients $a(x,t)$ of order $\frac{1}{t}$ must be excluded. Conditions like (6) take care of it.

4. All the above-mentioned relations have been established only for single parabolic equations and for certain limited classes of parabolic systems of second order. In the heart of these considerations lies the maximum principle (albeit in a disquised and unusual form). It would be interesting to know whether similar relationships hold for equations of higher order.

All the results mentioned above apply only to the equation with principal part of divergent form. For the equation

$$u_t - Mu \equiv u_t - a_{ij}u_{x_i x_j} + a_i u_{x_i} + au = f \tag{11}$$

of non-divergence form with arbitrary discontinuous bounded measurable coefficients $a_{ij}(x, t)$ in more than 2 dimensions the information available is extremly scarce. In a book by Uraltzeva and myself are the examples showing the pathological properties of the equation (11) in which the operator M has the form

$$M_0 u = a_{ij}u_{x_i x_j}, \qquad a_{ij} = \delta_i^j + \mu \frac{x_i x_j}{|x|^2}. \tag{12}$$

Another such pathological property is the following: the operator M_0 does not admit the closure in $L_2(\Omega)$ if $n > 2$. In fact, when $\mu = \dfrac{n}{n-2}$ for the sequence $u_{\varepsilon,\eta}(x) = (|x|^2 + \eta)^{1-\frac{n}{4}+\varepsilon} - (|x|^2 + \eta)^{1-\frac{n}{4}+\frac{\varepsilon}{2}}$ of functions we have

$$\frac{||u_{\varepsilon,\eta}||_{L_2}(|x| \leq 1)}{||M_0 u_{\varepsilon,\eta}||_{L_2}(|x| \leq 1)} \to 0 \text{ as } \varepsilon \to 0 \text{ and } \eta = \eta(\varepsilon) \to 0.$$

5. In the joint work Uraltzeva and myself quasilinear equations of the form

$$u_t - \frac{\partial}{\partial x_i} a_i(x, t, u, u_x) + a(x, t, u, u_x) = 0, \tag{13}$$

$$u_t - a_{ij}(x, t, u, u_x)\, u_{x_i x_j} + a(x, t, u, u_x) = 0, \tag{14}$$

have also been considered. For equations (13) generalised and classical solutions were both studied. For equations (14) principally classical solutions were studied.

We have analysed the smoothness properties of the full set of solutions of the above equations and the unique solvability „in the large" in the spaces of smooth functions of the classical boundary value problems for these equations. In these investigations we assume that the functions $a_i(x, t, u, p)$, $a(x, t, u, p)$ and $a_{ij}(x, t, u, p)$ are smooth with respect to u and p and that as functions of x and t they belong to the spaces $L_{q,r}$. I shall not give a detailed account of these results, as they together with the above-mentioned results on linear parabolic equations form the principal part of the forthcoming book on parabolic equations by Uraltzeva, Solonnikov and myself which should be appeared by the end of the present year. To indicate the nature of these results however I will state one result concerning the general class (14) of parabolic equations:

Theorem 2. *Let u be an arbitrary generalised solutions of the equation* (14) *belonging to class M, i.e. suppose that u is bounded, has generalised derivatives u_t and u_{xx} belonging to $L_2(Q_T)$, that the derivatives u_x are bounded and depend continuosly on t in the norm of $L_2(\Omega)$ and that u satisfies equation* (14) *almost everywhere.*

Suppose that the functions $a_{ij}(x, t, u, p)$ are differentiable with respect to x, u and p in a neighborhood of the surface $u = u(x, t)$, $p = u_x(x, t)$ aud that on this surface they satisfy the conditions

$$\nu\xi^2 \leq a_{ij}(x, t, u, p)\, \xi_i\xi_j \leq \mu\xi^2, \qquad \nu, \mu = const > 0,$$

$$\operatorname*{vrai\,max}_{Q_T} \left| \frac{\partial a_{ij}(x, t, u, p)}{\partial p_k} \right| \leq \mu_1, \qquad \left| \frac{\partial a_{ij}}{\partial u}, \frac{\partial a_{ij}}{\partial x_k}, a \right| \leq \varphi(x, t),$$

where $||\varphi||_{2q,2r} \leq \mu_1$ and where $\dfrac{1}{r} + \dfrac{n}{2q} < 1$. Then u_x will be Hölder-continuous in (x, t) with a Hölder constant

$$\langle u_x \rangle_{Q'}^{(\alpha, \frac{\alpha}{2})} \leq c\, (\operatorname*{vrai\,max}_{Q_T} |u|, n, \nu, \mu, \mu_1, q, r, d)$$

where d is the minimum distance from the subdomain Q' on Q_T to the base and laternal surface of the cylinder Q_T, and

$$\alpha = \alpha(\operatorname*{vrai\,max}_{Q_T} |u_x|, n, \nu, \mu, \mu_1, q, r).$$

This result as has been mentioned above takes place for the hole class of parabolic equations of second order. The restictions imposed on the functions a_{ij}, a and u being in the nature of the problem t_{00} (in particular, as has been shown in the book on elliptic equations by Uraltzeva and myself, it is impossible to eliminate the requirement that u_x be bounded). This result together with known results on linear equation with smooth coefficients reduce all a priori estimation problems for quasilinear equations of second order to the problem of estimating $\max\limits_{Q_T} |u|$ and $\max\limits_{Q_T} |u_x|$.

I have no time to explain our approach to all these problems. Instead this I'll mention some unsolved problems.

We studied the equations (13, 14) under the following conditions:

1) uniform ellipticity; for the equation (14) this means the restriction

(15) $\qquad \nu(|u|)\,(1 + |p|)^m\xi^2 \leq a_{ij}(x, t, u, p)\, \xi_i\xi_j \leq \mu(|u|)\,(1 + |p|)^m\xi^2;$

2) continuity (and mostly differentiability) of the function $a_{ij}(x, t, u, p)$, $a_i(x, t, u, p)$ and $a(x, t, u, p)$ in u and p.

These equations have been the object of most works of nonlinear problems up to now. They are well understood now. The methods and results developed in these studies made it possible to investigate certain problems of mechanics in which conditions 1) and 2) do not hold precisely. For example, problems of nonsteady flow through filters, problems connected with Prandtle's equations of boundary-layer flow (for which the form $a_{ij}\xi_i\xi_j$ degenerates, i.e. for which condition 1) not hold), the Stefan problem (in which condition 2) is not satisfied: the functions $a_{ij}(x, t, u, p)$ are discontinuous in u). The other various hydrodynamic problems in which unknown boundaries separate different phases (or flows), may be as the Stefan problem reformulated in terms of equations of the form (13) in which the functions $a_i(x, t, u, p)$ depend discontinuously on u. Attempts to weaken conditions 1) and 2) therefore seems to be of interest. Such attempts might lead to the discovery of new phenomena for parabolic equations and require new methods in addition to the present methods.

Generalisation of results established for equations of the second order to equations of higher order and to systems of equations would also be of interest. We have some results on the system of second order but I shall not formulate them for lack of time.

Show only on the one simple system

$$\vec{v}_t - \Delta\vec{v} + \frac{\partial}{\partial x_k}(v_k\vec{v}) = \vec{f} \tag{16}$$

$\vec{v} = (v_1(x_1, x_2, x_3, t), v_2, v_3)$ for which the solvability „in the large”, is not established. This system may be considered as model for Navie-Stokes system. For system (16) we have no energetic inequality, no maximum principle, in other words, we haven't the first step for the studying of system (16). It is interesting even the question on uniqueness of weak Hopf's solutions for (16) that is the solutions with finite energetic norm $|\vec{v}|Q_T$. I think that the uniqueness not takes place and it seems me that the Hopf's solutions for Navier-Stokes system inself is not unique too. The examples of nonuniqueness mentioned above support my conjecture.

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SOME APPLICATIONS OF THE SECOND METHOD OF LIAPUNOV TO DYNAMICAL SYSTEMS DESCRIBED BY PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

The second method of Liapunov is applied to the stability of dynamical systems described by partial differential equations. This extension of the well-known technique for ordinary differential equations is illustrated by two examples drawn from the field of aeroelasticity — the torsional divergence of a wing and the supersonic flutter of a panel. Reference is made to the work of other authors working in various promising fields of application.

Introduction

When applying the second method of Liapunov to stability problems of ordinary differential equations we generally wish to show that the Euclidean state space norm $\bar{S} \equiv (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$ tends to zero as time tends to infinity, and the stability definitions, Liapunov theorems and their proofs are expressed in terms of $\bar{S}$ — for example in KALMAN and BERTRAM [1]. When considering systems of partial differential equations it may be possible to choose a new norm, which will involve an integral of the system dependent variables and their space and time derivatives, which provides a measure of the disturbed system, from its undisturbed state. We may then take over all the definitions and theorems for ordinary differential equations, replacing Liapunov functions by "Liapunov functionals".

This idea has been put forward by ZUBOV [2], VOLKOV [3] and MOVCHAN [4] in the U.S.S.R., but only recently have applications of this theory been seen, PARKS [5] and WANG [6].

Basic Theorems

The following theorems were given by MOVCHAN [4], but are expressed here in the language of KALMAN and BERTRAM [1].

Stability Theorem

Suppose there exists a Liapunov functional V such that when $\bar{\varrho} \neq 0$, $0 < \alpha(\bar{\varrho}) \leq V \leq \beta(\bar{\varrho})$, $V = 0$ when $\bar{\varrho} = 0$, where $\alpha(\bar{\varrho})$ and $\beta(\bar{\varrho})$ are continuous non-decreasing scalar functions of ϱ, and that $\frac{dV}{dt}$, making use of the partial diff. equation and its boundary conditions, is such that $\frac{dV}{dt} \leq -\gamma(\bar{\varrho}) < 0$, $\bar{\varrho} \neq 0$, $\gamma(\bar{\varrho}) = 0$, $\bar{\varrho} = 0$, then the system is asymptotically stable. If $\alpha(\bar{\varrho}) \to \infty$ as $\bar{\varrho} \to \infty$ then the system is asymptotically stable in the large.

Instability Theorem

Suppose there exists a Liapunov functional V such that V is bounded above in terms of $\bar{\varrho}$ and that where $V > 0$, $\frac{dV}{dt}$ is also positive. Suppose further that given δ however small there always exists an initial motion at time t_0 with $\bar{\varrho}(t_0) < \delta$ such that at this time $V > 0$ then the undisturbed motion is unstable.

The stability theorem is stated in a general way and provides conditions for uniform asymptotic stability. Certain relaxations may be possible, for example when considering autonomous systems.

Applications

The important aeronautical engineering field known as „aeroelasticity" provides some interesting examples of the Liapunov functional technique.

(1) Torsional Divergence of a wing

For simplicity let us consider a uniform wing (Fig. 1) in torsion under the influence of aerodynamic loads which depend on the local incidence Θ and local angular velocity $\frac{\partial \Theta}{\partial t}$.

The equation of motion for a strip element will yield

(1) $$I \frac{\partial^2 \Theta}{\partial t^2} - \frac{\partial}{\partial y}\left(GJ \frac{\partial \Theta}{\partial y}\right) = k_\theta \Theta + k_{\dot{\theta}} \frac{\partial \Theta}{\partial t}$$

where k_θ and $k_{\dot{\theta}}$ are the aerodynamic strip „derivatives".

Consider now a norm $\bar{\varrho} = \left\{ \int_0^l \Theta^2 + \left(\frac{\partial \Theta}{\partial t}\right)^2 \mathrm{d}y \right\}^{1/2}$ and a tentative Liapunov functional

$$V = \frac{1}{2} \int_0^l GJ \left(\frac{\partial \Theta}{\partial y}\right)^2 + I \left(\frac{\partial \Theta}{\partial t}\right)^2 - k_\Theta \Theta^2 \, \mathrm{d}y \tag{2}$$

for which, on substituting for $I \dfrac{\partial^2 \Theta}{\partial t^2}$ from (1),

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \int_0^l GJ \frac{\partial \Theta}{\partial y} \frac{\partial^2 \Theta}{\partial t \, \partial y} + \frac{\partial \Theta}{\partial t} \left(\frac{\partial}{\partial y} \left[GJ \frac{\partial \Theta}{\partial y} \right] + k_\Theta \Theta + k_{\dot\Theta} \frac{\partial \Theta}{\partial t} \right) - \tag{3}$$

$$- k_\Theta \Theta \frac{\partial \Theta}{\partial t} \mathrm{d}y = \int_0^l k_{\dot\Theta} \left(\frac{\partial \Theta}{\partial t}\right)^2 \mathrm{d}y$$

on integrating the second term by parts and using the boundary conditions, which are that $\Theta = 0$ at $y = 0$ and $\dfrac{\partial \Theta}{\partial y} = 0$ at $y = l$.

Now $k_{\dot\Theta}$ will be negative and so we have stability (but not, without further argument, asymptotic stability) if V is positive definite in terms of $\bar{\varrho}$. Using the Schwarz inequality that

$$\left\{ \int_a^b fg \, \mathrm{d}x \right\}^2 \leq \int_a^b f^2 \, \mathrm{d}x \int_a^b g^2 \, \mathrm{d}x \tag{4}$$

we have

$$\left\{ \int_0^y \left(\frac{\partial \Theta}{\partial y}\right)^2 \mathrm{d}y \right\}^2 \leq \int_0^y \left(\frac{\partial \Theta}{\partial y}\right)^2 \mathrm{d}y \int_0^y \mathrm{d}y$$

or

$$\Theta^2(y) \leq \left[\int_0^y \left(\frac{\partial \Theta}{\partial y}\right)^2 \mathrm{d}y \right] y \leq \left[\int_0^l \left(\frac{\partial \Theta}{\partial y}\right)^2 \mathrm{d}y \right] y$$

and so

$$\int_0^l \Theta^2(y) \, \mathrm{d}y \leq \left[\int_0^l \left(\frac{\partial \Theta}{\partial y}\right)^2 \mathrm{d}y \right] \int_0^l y \, \mathrm{d}y = \left[\int_0^l \left(\frac{\partial \Theta}{\partial y}\right)^2 \mathrm{d}y \right] \frac{l^2}{2} \tag{5}$$

Thus V is positive definite in terms of $\bar{\varrho}$ if

$$GJ > \frac{1}{2} l^2 k_\Theta \tag{6}$$

Now, for a uniform wing there is an exact theory of torsional divergence found by solving the equation

$$-GJ \frac{\mathrm{d}^2\Theta}{\mathrm{d}y^2} = k_\Theta(\Theta + \alpha) \tag{7}$$

for $\Theta(y)$ when the wing root ($y = 0$) is at incidence α. The solution is

$$\Theta(y) = \alpha(\tan pl \sin py + \cos py - 1) \tag{8}$$

where $p^2 = k_\Theta/GJ$, and torsional divergence occurs when $pl \to \pi/2$. Thus the exact criterion is

$$GJ > \frac{4}{\pi^2} l^2 k_\Theta \tag{9}$$

Galerkin energy methods may also be applied to yield for an assumed mode $\Theta(y) = y/l$

$$GJ > \frac{1}{3} l^2 k_\Theta \tag{10}$$

and for an assumed mode $\Theta(y) = 2(y/l) - (y/l)^2$

$$GJ > \frac{2}{5} l^2 k_\Theta \tag{11}$$

We notice that the Liapunov criterion is conservative while the Galerkin methods underestimate the exact torsional stiffness required to prevent divergence as

$$\frac{1}{2} > \frac{4}{\pi^2} > \frac{2}{5} > \frac{1}{3} \tag{12}$$

(2) Panel flutter

Fig. 2 shows a pin jointed two dimensional panel. The equation of motion of this panel in supersonic flow is

$$D\frac{\partial^4 z}{\partial x^4} + m\frac{\partial^2 z}{\partial t^2} - F\frac{\partial^2 z}{\partial x^2} + \varrho a_\infty \left(U\frac{\partial z}{\partial x} + \frac{\partial z}{\partial t}\right) = 0 \tag{13}$$

where „piston theory" has been employed in calculating the aerodynamic force on a panel element. [ϱ in (13) is air density.]

Consider a norm

$$\bar{\varrho} = \left\{ \int\limits_{x=0}^{l} z^2 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2 \mathrm{d}x \right\}^{1/2}$$

and a tentative Liapunov functional

(14) $$V_1 = \frac{1}{2}\int_0^l m\left(\frac{\partial z}{\partial t}\right)^2 + F\left(\frac{\partial z}{\partial x}\right)^2 + D\left(\frac{\partial^2 z}{\partial x^2}\right)^2 \mathrm{d}x$$

for which

(15) $$\frac{\mathrm{d}V_1}{\mathrm{d}t} = -\int_0^l \varrho a_\infty \left(\frac{\partial z}{\partial t}\right)^2 - \varrho a_\infty\, U\,\frac{\partial z}{\partial t}\,\frac{\partial z}{\partial x}\,\mathrm{d}x$$

We should like a term in $\left(\frac{\partial z}{\partial x}\right)^2$ in $\frac{\mathrm{d}V}{\partial t}$ so we try

(16) $$V = V_1 + \lambda V_2$$

where

(17) $$V_2 = \frac{1}{2}\int_0 \varrho a_\infty z^2 + 2mz\frac{\partial z}{\partial t}\,\mathrm{d}x$$

with

(18) $$\frac{\mathrm{d}V_2}{\mathrm{d}t} = \int_0^l m\left(\frac{\partial z}{\partial t}\right)^2 - F\left(\frac{\partial z}{\partial x}\right)^2 - D\left(\frac{\partial^2 z}{\partial x^2}\right)^2 \mathrm{d}x$$

Now using a lemma due to LORD RAYLEIGH, employed also by MOVCHAN [4] that

(19) $$\int_0^l \left(\frac{\partial^2 z}{\partial x^2}\right)^2 \mathrm{d}x \geq \frac{\pi^2}{l^2}\int_0^l \left(\frac{\partial z}{\partial x}\right)^2 \mathrm{d}x \geq \frac{\pi^4}{l^4}\int_0^l z^2\,\mathrm{d}x$$

we shall have a positive definite V and $-\frac{\mathrm{d}V}{\mathrm{d}t}$ if

$$\begin{bmatrix} \frac{\pi^4 D}{l^4} + \frac{\pi^2}{l^2}F + \lambda\varrho a_\infty & \lambda m \\ \lambda m & m \end{bmatrix} \text{ and } \begin{bmatrix} \left(\frac{\pi^2 D}{l^2} + F\right)\lambda & \frac{1}{2}\varrho a_\infty U \\ \frac{1}{2}\varrho a_\infty U & \varrho a_\infty - \lambda m \end{bmatrix}$$

are positive definite matrices. An optimum choice of λ is $\lambda = \frac{1}{2}\frac{\varrho a_\infty}{m}$ when we obtain conditions

(20) $$\begin{cases} F > -\dfrac{\pi^2 D}{l^2} \\ \text{and} \quad U^2 < \left(F + \dfrac{\pi^2 D}{l^2}\right)/m \end{cases}$$

The first condition is precisely the Euler buckling criterion for the panel and the second condition, for long panels under tension, says that the air speed must be less than the speed of waves travelling in the stretched panel: this is a well known criterion, but obtained here by an unconventional method.

(3) Other aeroelasticity problems

We note the non-linear structural damping treated by PARKS [5], and the bending torsion flutter of a non-uniform wing considered by WANG [6] (but note the comments by PARKS [7]), and the body bending-tail flutter of WANG [8]. Most of these papers look at old problems using the new Liapunov technique.

(4) Other fields of applications

We note papers on a chemical reacter problem by BLODGETT [9], on plasma stability by McNAMARA and ROWLANDS [10], on instabilities in Shid dynamies by PRITCHARD [11], and on stability in elastic bodies by SHIELD [12].

There is an urgent need for further research into the construction of Liapunov functionals for these problems — physical quantities such as total energy are useful and other functionals may be generated by multiplying the differential equations through by suitable dependent variables and integrating by parts.

There are likely to be important advances in these directions before long.

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REFERENCES

[1] KALMAN, R. E. and BERTRAM, J. E., *"Control system analysis and design via the second method of Liapunov" Trans. ASME J. Basic Eng., 82D, 371—393, (June, 1960).*

[2] ZUBOV, V. I., *"The methods of Liapunov and their application" U. S. Atomic Energy Commission AEC-TR-4439 (1957).*

[3] VOLKOV, D. M., *"An analogue of the second method of Liapunov for nonlinear boundary value problems of hyperbolic equations" Leningrad. Gos. Univ. Uch. Zapiski, Ser. Math. Nauk 33, 90—96, (1956).*

[4] Movchan, A. A., "*The direct method of Liapunov in stability problems of elastic systems*", *Prik. Mat. Mekh. 23, 483—493, (1959).*

[5] Parks, P. C., "*A stability criterion for panel flutter via the second method of Liapunov*", *AIAA Journal, 4, 175—177, (January 1966).*

[6] Wang, P. K. C., "*Stability analysis of elastic and aeroelastic systems via Liapunov's direct method*", *J. Franklin Inst. 281, 51—72, (January 1966).*

[7] Parks, P. C .,"*Liapunov functionals for aeroelastic problems*", *J. Franklin Inst. 283, 426—432, (May 1967)*

[8] Wang, P. K. C., "*Stability analysis of a simplified flexible vehicle via Liapunov's direct method*", *AIAA Journal 3, 1764—1766 (September 1965).*

[9] Blodgett, R. E., "*Stability conditions for a class of distributed-parameter systems*", *Trans. ASME J. Basic Eng. 88D, 475—479, (June 1966).*

[10] McNamara, B., and Rowlands, G., "*Plasma stability and the Liapunov method*", *Proc. Instn. Mech. Eng. (London) 178 Part 3M, 47—50, (September 1965).*

[11] Pritchard A. T. "*A study of tvo classical problems of hydrodynamic stability usny the Liapunov method*", *J. Math. and its Applications.*

[12] Shield, R. T., "*On the stability of linear continuous systems*" *Zeit. für angew. Math. and Phys. 16, 649—686 (1965).*

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
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NON-EXPANSIVE MAPPINGS IN CONVEX LINEAR TOPOLOGICAL SPACES

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A self-mapping f of a metric space (X, d) is said to be non-expansive iff for all $x, y \in X$, $d[f(x), f(y)] \leq d(x, y)$. In [1] and [2], F. Browder proved that if X is a closed bounded convex subset of a uniformly convex Banach space B on which the metric d is that induced from the norm on B, then every non-expansive self-mapping of X has a fixed-point. Browder's result is included in a similar result by Kirk [3] who showed that, if X is a closed bounded convex subset of a reflexive Banach space possessing "normal structure", then the self-mapping f has a fixed-point in X. A convex subset K of a Banach space has normal structure iff, for each non-trivial bounded convex subset C of K, there is an $x \in C$ such that

$$\operatorname{diam}(C) > \sup_{y \in C} ||x - y||.$$

It is a simple matter to see that if B is a uniformly convex Banach space then every non-trivial convex set K in B has normal structure.

In proving these theorems the authors rely heavily upon the special properties of reflexive Banach spaces. We observe, however, that these results among others, may be proved directly from a rather simple, but general principle about semi-continuous mappings on locally convex spaces. In the sequel V will denote a locally convex linear topological space (over the reals or complexes) and ϱ_K a lower semi-continuous non-negative convex function defined on a convex subset K of V. ϱ_K is said to be normal if in addition ϱ_K is non-constant on each non-trivial closed convex subset C of K. We then have the following.

Proposition. *Let K be a weakly compact convex subset of the locally convex space V and let ϱ_K be a normal function on K. If f is a self-mapping of K such that*

$$(1) \qquad \varrho_K(f(x)) \leq \varrho_K(x), \qquad x \in K,$$

then f has a fixed-point in K.

Proof: ϱ_K is lower semi-continuous if and only if the sets, defined for $t \geq$),

$$A_t = \{x \in K | \varrho_K(x) \leq t\}$$

are closed. In view of the convexity of ϱ_K, the A_t are closed *convex* sets and, consequently, are weakly closed. Hence ϱ_K is weakly lower semi-continuous. Because of the weak compactnes of K,

$$M = \{x \in K | \varrho_K(x) = \inf \varrho_K(y)\}$$

s then a non-void closed convex subset of K. Indeed, since ϱ_K is normal M consists of exactly one point x_0. On the other hand, $f(x_0) \in K$ so that $\varrho_K(x_0) \leq \leq \varrho_K(f(x_0))$. But assumption (1) yields $\varrho(x_0) = \varrho(f(x_0))$, thus $f(x_0) \in M$ and hence $f(x_0) = x_0$.

A simple generalization of Kirk's result now follows immediately. Let p be a continuous semi-norm on V. We shall call a convex subset K of V p-normal iff for each non-trivial weakly compact convex subset C of K it is true that $\sup\limits_{y \in C} p(x - y)$ is non-constant on C.

Theorem 1. *Let V be a locally convex linear topological space, K a weakly compact convex subset of V, p a continuous semi-norm on V. If K is p-normal and f is a self mapping of V such that, for all $x, y \in K$*

$$p(f(x) - f(y)) \leq p(x - y), \tag{2}$$

then f has a fixed-point in K.

Proof: p is a continuous semi-norm on V, in particular p is convex. Hence p is weakly lower semi-continuous on V and, consequently, $\{x | p(x) \leq 1\}$ is a barrel relative to the weak topology on V. Thus, by [4] (Lemma 1, page 66), $\varrho_0(x) = \sup\limits_{y \in K_0} p(x - y)$ is a finite-valued non-negative convex function on K_0 which is weakly lower semi-continuous. Here K_0 is any weakly compact convex subset of V.

We now determine $K_0 \subset K$ such that $f(K_0) \subset K_0$ on which ϱ_0 satisfies (1). To this end, let $\mathscr{K}$ be the collection of all closed convex subsets C of K such that $f(C) \subset C$. These sets are, therefore, weakly closed and, thus, weakly compact, so that K possesses the finite intersection property. Since $K \in \mathscr{K}$, Zorn's lemma is applicable and, hence, there is a minimal weakly compact $K_0 \in \mathscr{K}$ such that $f(K_0) \subset K_0$. Since $\overline{C_0} f(K_0)$, the closed convex hull of $f(K_0)$, belongs to $\mathscr{K}$, and since $\overline{C_0} f(K_0) \subset K_0$, the minimality of K_0 allows us to conclude that $K_0 = \overline{C_0} f(K_0)$. This is the convex set K_0 that we use to apply the proposition.

To this end, for $x \in K_0$ we set

$$\varrho_0(x) = \sup_{y \in K_0} p(x - y).$$

Since $K_0 = \overline{C}_0 f(K_0)$ and since p is both continuous and convex,

$$\varrho_0(f(x)) = \sup_{y \in K_0} p(f(x) - f(y)), \qquad x \in K_0,$$

so that $\varrho_0(f(x)) \leq \varrho_0(x)$ for $x \in K_0$. Moreover, ϱ_0 is lower-semi continuous non-negative convex on K_0. That ϱ_0 is, indeed, normal follows from the minimality of K_0; for if $C \subset K_0$ is closed convex and non-trivial such that $\varrho_0(x) = k =$ constant on C, then actually $C = \{x \in K_0 | \varrho(x) \leq k\}$ and is a closed convex subset of K_0 on which $\varrho_0(f(z)) \leq \varrho_0(z)$. Thus $f(C) \subset C$ and, hence $C = K_0$, which is impossible by the P-normality of K. Hence the proposition is applicable and yields a fixed-point for f in K.

We may now state the KIRK and BROWDER results as immediate corollaries of Theorem 1.

Corollary 1. (KIRK). *Let B be a reflexive Banach space and K a closed bounded convex subset of B which possesses normal structure. If f is a non-expansive self-mapping of K, f has a fixed-point in K.*

Corollary 2. (F. BROWDER). *Let B be a uniformly convex Banach space and K a closed bounded convex subset of B. A non-expansive self-mapping of K has a fixed-point in K.*

The proof of Corollary 1 follows from the following facts: K is weakly compact because of the reflexivity of B; the norm is continuous, and K is norm-normal. As noted earlier Corollary 2 is a consequence of Corollary 1, since a uniformly convex Banach space is reflexive and has the property that any convex set has normal structure.

For the sake of completeness, we state two other results which follow from Theorem 1, or rather Corollary 1. The first is stated and proved in [3]. The second is due to BROWDER [2] under the more restrictive condition that B is uniformly convex. It is a partial generalization of the MARKOV—KAKUTANI result [5], [6] on commuting families of linear contractions.

Corollary 3. *Let B be a reflexive Banach space and K a closed convex subset of B possessing normal structure. A non-expansive self-mapping f of K has a fixed-point if and only if there exists an $x_0 \in K$ such that the sequence of iterates $\{f^{(n)}(x_0)\}$ is bounded.*

Corollary 4. *Let B be a reflexive, strictly convex Banach space and K a bounded closed convex subset of B possessing normal structure. If $\{f_\lambda\}$, $\lambda \in \Lambda$ is a commuting set of self-mappings of K, then there is an $x_0 \in K$ such that $f_\lambda(x_0) = x_0$ for all $\lambda \in \Lambda$.*

The proof of Corollary 3 is obtained from Corollary 1 as follows: Let $r = \sup ||x_0 - f^{(n)}(x_0)||$ and let K_n be the intersection of the closed ball of radius r about $f^{(n)}(x_0)$ with K. The K_n are closed, bounded and convex.

Because of the non-expansiveness of f, it follows that $f^{(m)}(x_0) \in K_n$ for all $m \geq n$. Thus the collection of weakly compact convex sets $\{K_n\}$ have the finite intersection property and, hence, $C = \bigcup_{s=1}^{\infty} \bigcap_{n=s}^{\infty} K_n$ is non-void convex and weakly compact. Moreover $f(C) \subset C$. Taking the closure $\overline{C}$ of C, we observe that $\overline{C}$ is a closed, bounded, convex subset of K on which f is a self-mapping. Since K has normal structure so does $\overline{C}$ and hence Corollary 1 is applicable to yield the desired result.

With respect to Corollary 4, we first note that, in view of the strict convexity of B, the fixed-point set of a non-expansive self-mapping of a convex subset of B is itself convex. If $C_\lambda = \{x \in K | f_\lambda(x) = x\}$, $\lambda \in \Lambda$, then these sets are closed convex subsets of K which by Corollary 1 are non-void. Taking account of the commutivity of the f_λ it is readily seen that the $\{C_\lambda\}$ have the finite intersection property. Since B is reflexive, $\bigcap_{\lambda \in \Lambda} C_\lambda \neq \emptyset$.

REFERENCES

[1] F. Browder: "*Fixed-point theorems for noncompact mappings in hilbert space*", *Proc. Nat. Ac. Sc. 53 (1965), pp. 1272—1276.*

[2] F. Browder: "*Nonexpansive nonlinear operators in a Banach space*", *Proc. Nat. Ac. Sc. 54 (1965), pp. 1041—1044.*

[3] W. A. Kirk: "*A fixed-point theorem for mappings which do not increase distances*", *Am. Math. Monthly 72 (1965), pp. 1004—1006.*

[4] A. P. and W. J. Robertson: "*Topological Vector Spaces*", *Cambridge University Press, 1964.*

[5] A. Markov: "*Quelques théorèmes sur les ensembles Abeliens*", *Dokl. Ak. Nauk SSSR (N. S.) 10 (1936), pp. 311—314.*

[6] S. Kakutani: "*Two fixed-point theorems concerning bicompact convex sets*", *Proc. Im. Ac. Tokyo, 14 (1938), pp. 242—245.*

[7] R. DeMarr: "*Common fixed-points for commuting contraction mappings*", *Pac. J. Math. 13 (1963), pp. 1139—1141.*

[8] D. Göhde: „*Zum Prinzip der Kontractie-Abbildung,*" *Math. Nach. (30) 1065 pp. 251—258.*

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PERIODIC SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF EVOLUTION.

The first paper on the subject is perhaps [23] A. VITT in 1934. As it is seen from the list of papers known to the author and quoted below, there are now about sixty published papers on the theory considered. Therefore, it is impossible to cover the topic in the whole in this survey. Thus, only papers on partial differential equations of hyperbolic type, which appeared (or are in press) after my expository talk at Equadiff I (see [20]) in 1962, will be mentioned.

Let us start with the author's paper [21]. Here, the existence of ω-periodic solutions of a perturbed wave equation

$$u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x, \varepsilon) \tag{1}$$

with boundary conditions

$$u(t, 0) = u(t, \pi) = 0 \tag{2}$$

where f is ω-periodic in t, is investigated by means of the Poincaré method. It is neccessary to distinguish three cases: (α) $\omega = 2\pi n$, n a natural number, (β), $\omega = 2\pi \frac{p}{q}$, p, q natural numbers, (γ) $\omega = 2\pi\alpha$, α an irrational number.

In the case $\omega = 2\pi n$ it is shown that the bifurcation equation of the problem reads either

$$\int_0^{2\pi n} \int_0^{\pi} f(t, x, u(t, x), u_t(t, x), u_x(t, x))\, v(t, x)\, dx\, dt = 0, \tag{3}$$

$v(t, x) = \varphi(x + t) - \varphi(-x + t)$ for any 2π-periodic function φ of class C^2,

or

$$f(t, x, u(t, x - t), u_t(t, x - t), u_x(t, x - t)\, dt \equiv 0. \tag{3'}$$

Using the latter form of it, it is proved that there exists a classical $2\pi n$-periodic solution for sufficiently small ε, if (i) f is sufficiently smooth and

$$f(t, 0, 0, 0, w, \varepsilon) = f(t, \pi, 0, 0, w, \varepsilon) = 0,$$

(ii) the equation

(4) $$G(s)(x) \equiv$$
$$\equiv \int_0^{2\pi n} f(\vartheta, x-\vartheta, s(x)-s(-x+2\vartheta), s'(x)-s'(-x+2\vartheta), s'(x)+s'(-x+2\vartheta))\, d\vartheta = 0$$

has a solution $s^*(x)$ in a subspace $\tilde{C}_2$ of the space C^2, (iii) there exists the inverse operator

$$[G_s'(s^*)]^{-1} \in L[D \to \tilde{C}_2], \qquad \text{where } D = G(\tilde{C}_2)$$

A similar result is obtained for $\omega = 2\pi \frac{p}{q}$, p, q natural numbers. (A paper on a similar problem with nonhomogeneous boundary conditions has just been finished.)

J. Kurzweil [10] applying his theory of integral manifolds of ordinary differential equations in the Banach space to the problem (1), (2) (with $\omega = 2\pi$) gets an analogous result assuming that besides (i), (ii) quoted above, the condition

(iii') $s^*(x)$ is an exponentially stable stationary solution of a certain ordinary differential equation in a Banach space holds. Then the found 2π-periodic solution of (1), (2) is also asymptotically stable.

In both the papers some particular cases are discussed, for which all assumptions take place. (E.g. in [21] $f = h(t, x) + \alpha u + \beta u^3$, or $f = h(t, x) + (1 - \alpha u^2)\, u_t$; in [10] an autonomous case is treated successfully, too.)

Usually, the verification of conditions (ii), (iii) is rather difficult. Therefore, the results assuring the two conditions to be fulfilled under some assumptions which may be verified easier, are desirable. Besides some older results [22], [24]—[28], [30] for $\omega = 2\pi \dfrac{2k+1}{2l}$, k, l natural numbers, P. Rabinowitz ([17]) assured the existence of a 2π-periodic solution of (1), (2) under the conditions that $f = f(t, x, u)$, f is sufficiently smooth and $\frac{\partial f}{\partial u}(t, x, u) < \beta < 0$ (β being a constant). His method is based on the fact that the bifurcation equation in the form (3) is an Euler equation of an apropriate variational problem, namely

$$\underset{u \in N}{\text{minimize}} \int_0^{2\pi} \int_0^{\pi} F(t, x, u)\, dx\, dt,$$

where $F(t, x, u) = \int^u f(t, x, v)\, dv$ and N is the subspace in L_2 of functions of the form $\varphi(x + t) - \varphi(-x + t)$, φ being 2π-periodic.

In general, the case $\omega = 2\pi\alpha$, α an irrational number, seems to be rather difficult. Recently, G. T. Sokolov in [29] has shown the existence of an

ω-periodic solution of the problem (1), (2) (he writes it in a somewhat different way) for $\omega = 2\pi\sqrt{n}$, n a natural number, and $f = f(t, x, u)$.

L. CESARI investigates the problem given by

$$u_{tx} = f(t, x, u, u_t, u_x) \tag{5}$$

$$u(t, 0) = u_0(t) \tag{6}$$

f and u_0 being ω-periodic in t and he asks when it is possible to choose the function $u(0, x) \equiv u_0(0) + \nu(x)$, $\nu(0) = 0$, on a sufficiently narrow strip $-a \leq x \leq a$ so that the solution of the problem (5), (6) be ω-periodic in t. He makes use of the fact that the modified problem given by

$$u_{tx} = f(t, x, u, u_t, u_x) - \tag{5'}$$

$$-\frac{1}{\omega}\int_0^\omega f(\vartheta, x, u(\vartheta, x), u_t(\vartheta, x), u_x(\vartheta, x))\,\mathrm{d}\vartheta$$

and by the condition (6) has always an ω-periodic solution if a is sufficiently small and f and u_0 are sufficiently smooth. After some anticipatory results in [2], [3] he proves in [4] that there exists an ω-periodic solution of (5), (6) for a sufficiently small a if the following assumptions are fulfilled: (i) f is sufficiently smooth, (ii) the equation

$$\int_0^\omega f(\vartheta, 0, u_0(\vartheta), \dot{u}_0(\vartheta), q(\vartheta)(\mu))\,\mathrm{d}\vartheta = 0, \tag{7}$$

where $q(t)(\mu)$ is the solution of the problem

$$\frac{\mathrm{d}q}{\mathrm{d}t} = f(t, 0, u_0(t), \dot{u}_0(t), q(t)), \qquad q(0) = \mu,$$

has at least one solution $\mu = \mu^*$, (iii) the Jacobian of the equation (7) at the point $\mu = \mu^*$ is nonvanishing. (In CESARI's papers the quantities μ, f, u_0 etc. are supposed to be vectors.)

Besides this CESARI ([5], [6]) studies the problem (5), (6) for

$$f = \varepsilon[\psi(t, x) + Cu + \alpha(x)\, u_t + \beta(t)\, u_x + \varepsilon g(t, x, u, u_t, u_x)],$$

where ψ, α, β and g and u_0 and ν are ω-periodic in t and x and he seeks a solution ω-periodic in both variables. Making use of the successive approximation method and the Fourier method he proves that under the condition $C \neq 0$ and some other less fundamental conditions an ω-periodic solution exists.

(Let us note right now that for a perturbed telegraph equation of a similar type i.e.

$$u_{tx} = \psi(t, x) + Cu + \alpha(x)\, u_t + \beta(t)\, u_x + \varepsilon g(t, x, u, u_t, u_x)$$

he also derives an existence theorem for an ω-periodic solution adding to the conditions above the requirement that the limit equation ($\varepsilon = 0$) have an ω-periodic solution of the form $u_0(t) + v_0(x)$.)

In [1] A. K. AZIZ investigates the existence of an ω-periodic solution of the modified problem (5′) under more general assumptions than CESARI does.

In [7] F. A. FICKEN and B. A. FLEISHMAN investigated the problem

$$u_{tt} - u_{xx} + 2au_t + 2bu_x + cu = h(t, x) + \varepsilon f(t, x, u, u_t, u_x) \tag{8}$$

either for $-\infty < x < +\infty$ or for $0 \leq x \leq \pi$ with the boundary conditions

$$u(t, 0) = u(t, \pi) = 0 \tag{9}$$

They suppose $a > 0$, $b = 0$, $c > 0$, $f = -u^3$ and h ω-periodic in t and sufficiently smooth. Then they prove the existence of an ω-periodic solution for sufficiently small ε by examining the transition operator $U(t, x)$ at points $t = \tau + n\omega$, for $n \to \infty$, n a natural number.

Making use of the same method J. HAVLOVÁ in [9] generalized their result to the case $a \neq 0$, b arbitrary, $\frac{b^2}{4} + c > 0$, f sufficiently smooth. (V. VÍTEK is preparing a paper on a similar equation in two spatial variables.)

A more general equation

$$u_{tt} + 2\gamma u_t + Au + F(t, u) = f(t) , \tag{10}$$

where A is a positive selfadjoint operator, $F(t, u)$ is a nonlinear sufficiently smooth operator with $F(t, 0) = 0$ is treated also by the same method in [11] by K. MASUDA and the existence of a generalized ω-periodic solution for sufficiently small f is assured.

Recently, the equation (10) was attacked by J. HAVLOVÁ under somewhat different assumptions by means of the Fourier method and the existence of a classical ω-periodic solution was shown.

Lately, a special case of the problem (8), (9), namely $a = b = 0$, $c \neq 0$ was studied by the author and for $\left|k^2 + c - m^2\left(\frac{2\pi}{\omega}\right)^2\right| > \delta > 0$ ($k \neq 0$, m-natural number), and $f = f(t, x, u)$ the existence of a classical ω-periodic solution was proved.

As early as in 1956 G. PRODI ([14]) treated successfully the strongly nonlinear equation

$$u_{tt} - \Delta u + g(t, x, u_t) = f(t, x, \operatorname{grad} u) \tag{11}$$

with $u = 0$ on the boundary of a domain G. In the last two years, another strongly nonlinear differential equation

$$u_{tt} - \Delta u + g(u_t) = f(t, x) \tag{12}$$

was studied by G. PROUSE ([16]) in the case $g(v) = v + |v|\, v$ and by G. PRODI

([15]) in the case $g(v) \approx |v|^{\varrho-1}v$ ($\varrho \geq 1$) for $v \to \pm\infty$, g being continuous and monotone. They derive some apriori estimates for periodic solutions and make use of the Galerkin method.

In all these cases the existence of a generalized ω-periodic solutions, only, is assured; it would be difficult, however, to describe the Banach spaces in which the solutions lie.

BIBLIOGRAPHY

Equations of hyperbolic type:

[1] Aziz A. K.: *Periodic solutions of hyperbolic partial differential equations. Proc. AMS 17, 3, 1966, 557–566.*

[2] Cesari L.: *Periodic solutions of hyperbolic partial differential equations. International Symp. Nonlinear Diff. Eq. and Nonlinear Mech., Colorado Springs 1961; Acad. Press, New York 1963, 33–57.*

[3] Cesari L.: *Periodic solutions of partial differential equations. Trudy Meždunar. simpoz. po nelin. kolebanijam, Kiev 1961, 440–457.*

[4] Cesari L.: *A criterion for the existence in a strip of periodic solutions of hyperbolic partial differential equations. Rend. Circ. Mat. Palermo II–XIV, 1, 1965, 95–118.*

[5] Cesari L.: *Existence in the large of periodic solutions of hyperbolic partial differential equations. Arch. Rat. Mech. Anal. 20, 3, 1965, 170–190.*

[6] Cesari L.: *Smoothness properties of periodic solutions in the large of nonlinear hyperbolic differential systems. Report No. 3, US-AFOSR Research project 942–65, Univ. Michigan, Dept. of Math.*

[7] Ficken F. A., Fleishman B. A.: *Initial value problems and time-periodic solutions for a nonlinear wave equation. Comm. Pure Appl. Math. 10, 1957, 331–356.*

**[8] Grammel R.: *Nichtlineare Schwingungen mit unendlich vielen Freiheitsgraden. Actes du Colloque International des Vibr. non linéaires, Ile de Porquerolle 1951, Publ. Sci. Tech. Ministère de l'Air Paris 281, 1953, 45–59.*

[9] Havlová J.: *Periodic solutions of a nonlinear telegraph equation. Čas. Pěst. Mat. 90, 1965, 273–289.*

[10] Kurzweil J.: *Exponentially stable integral manifolds, averaging principle and continuous dependence on a parameter. II. Czech. Math. J., 1967. See also* [54].

[11] Masuda K.: *On the existence of periodic solutions of nonlinear differential equations. Journal of the Faculty of Sci., Univ. Tokyo, XII, 2, 1966, 247–257.*

**[12] Petiau G.: *Sur des fonctions d'ondes d'un type nouveau, solutions d'équations non linéaires généralisant l'équation des ondes de la Mécanique ondulatoire. Compt. Rend. Ac. Sci. Paris 244, 1957, 1890–1893.*

[13] Petiau G.: *Sur une généralisation non linéaire de la mécanique ondulatoire et les propriétés des fonctions d'ondes correspondantes. Nuovo Cimento. Suppl. 9 (10), 1958, 542–568.*

[14] Prodi G.: *Soluzioni periodiche di equazioni a derivate parziali di tipo iperbolico non lineari. Ann. Mat. Pura ed Appl., ser. IV, XLII, 1956, 25–49.*

[15] PRODI G.: *Soluzioni periodiche dell' equazione delle onde con termine dissipativo non lineare. Rend. Sem. Mat. Padova XXXVI, 1, 1966, 37–49.*

[16] PROUSE G.: *Soluzioni periodiche dell' equazione delle onde non omogenea con termine dissipativo quadratico. Ricerche di Mat., XIII, 2, 1964, 261–280.*

[17] RABINOWITZ P. H.: *Periodic solutions of a nonlinear non-dissipative wave equation. Research Report IMM 343, Courant Inst. Math. Sci., New York Univ., 1965.*

*[18] RABINOWITZ P. H.: *Periodic solutions of nonlinear hyperbolic differential equations. New York Univ. thesis, 1965.*

**[19] STEWART F. M.: *Periodic solutions of a nonlinear wave equations. Bull. AMS 60, 1954, 344.*

**[20] VEJVODA O.: *Nonlinear boundary-value problems for differential equations. Differential Eq. and Their Appl., Proc. of the Con. held in Prague Sept. 1962, 199–215.*

[21] VEJVODA O.: *Periodic solutions of a linear and weakly nonlinear wave equation in one dimension., I. Czech. Math. J. 14 (89), 1964, 341–382.*

** [22] Артемьев Н. А.: *Периодические решения одного класса уравнений в частных производных. Изв. АН СССР, сер. мат. 1, 1937, 15–50.*

[23] Витт А.: *Распределенные автоколебательные системы. Журнал Техн. Физ. IV, 1, 1934, 144–157.*

* [24] Карп В. Н.: *О периодических решениях одного нелинейного уравнения гиперболического типа. ДАН Уз ССР XXX, 5, 1953, 8–13.*

[25] Карп В. Н.: *Применение метода волновых областей к решению задачи о вынужденных нелинейных периодических колебаниях струны. Изв. ВУЗ, мат. 6(25), 1961, 51–59.*

[26] Карп В. Н.: *О существовании и единственности периодического решения одного нелинейного уравнения гиперболического типа. Изв. ВУЗ, мат. 5, 1963, 43–50.*

[27] Митряков А. П.: *О периодических решениях нелинейного уравнения гиперболического типа. Труды ИММ АН УзССР 7, 1949, 137–149.*

* [28] Соколов Г. Т.: *О периодических решениях одного класса уравнений в частных производных. ДАН УзССР 12, 1953, 3–7.*

* [29] Соколов Г. Т.: *О периодических решениях волнового уравнения. Уч. записки Ферганского гос. педагог. инст., сер. мат. 1, 1965, 17–25.*

[30] Соловьев П. В.: *Некоторые замечания о периодических решениях нелинейных уравнений гиперболического типа. Изв. АН СССР, сер. мат. 2, 1939, 149–164.*

** [31] Жаботинский М. Е.: *О периодических решениях нелинейных уравнений в частных производных. ДАН СССР 56, 1947, 469–472.*

Equations of parabolic type:

*[32] PRODI G.: *Soluzioni periodiche di equazioni alle derivate parziali di tipo parabolico e non lineari. Riv. di Mat. della Univ. di Parma 3, 1952, 265–290.*

**[33] PRODI G.: *Soluzioni periodiche di equazioni non lineari di tipo parabolico. Atti del Quarto Congresso dell' Unione Matematica Ital. 2, 1953, 193–196.*

[34] PRODI G.: *Problemi al contorno non lineari per equazioni di tipo parabolico non lineari in due variabili — soluzioni periodiche. Rend. Sem. Mat. Univ. Padova 23, 1954, 25–85.*

[35] Горьков Ю. П.: О периодических решениях параболических уравнений. *Дифференциальные уравнения II, 7, 1966, 943–952.*

** [36] Каримов Д. Х.: О периодических решениях нелинейных дифференциальных уравнений параболического типа. *ДАН СССР 25, 1939, 3–6.*

** [37] Каримов Д. Х.: *О периодических решениях нелинейных дифференциальных уравнений параболического типа. ДАН СССР, 28, 1940, 403—406.*

** [38] Каримов Д. Х.: *О периодических решениях нелинейных дифференциальных уравнений параболического типа. ДАН СССР 46, 1945, 175—178.*

** [39] Каримов Д. Х.: *О периодических решениях нелинейных дифференциальных уравнений параболического типа. ДАН СССР 54, 1946, 293—295.*

** [40] Каримов Д. Х.: *О периодических решениях нелинейных дифференциальных уравнений параболического типа. ДАН СССР 56, 1947, 119—121.*

** [41] Каримов Д. Х.: *О периодических решениях нелинейных дифференциальных уравнений параболического типа. ДАН СССР 58, 1947, 969—972.*

* [42] Каримов Д. Х.: *О периодических решениях неплинейных дифференциальных уравнений параболического типа. Труды ИММ АН УзСССР 5, 1949, 30—53.*

* [43] Каримов Д. Х.: *О некоторых уравнениях задачи теплопроводности. Труды ИММ АН УзССР 6, 1950, 67—106.*

[44] Каримов Д. Х.: *О периодических решениях нелинейных дифференциальных уравнений параболического типа. Труды Инст. мат. и техн. АН УзССР 6, 1950, 30—53.*

* [45] Каримов Д. Х.: *Об одном уравнении параболического типа. Труды ИММ АН УзСССР 8, 1951, 128—137.*

** [46] Колесов Ю. С.: *О некоторых признаках существования периодических пешений у квазилинейных параболических уравнений. ДАН СССР 157(6), 1964, 1288—1290.*

** [47] Красносельский М. А.: *К теории периодических решений неавтономных дифференциальных уравнений. УМН 21, 3(129), 1966, 53—74.*

** [48] Красносельский М. А., Соболевский П. Е.: *On some non-linear problems for the partial differential equations. Outlines of the Joint Soviet-American Symposium on Partial Diff. Eq. 1963, Novosibirsk, 129-133.*

** [49] Шмулев И. И.: *О периодических решениях краевых задач без начальных условий для квазилинейных параболических уравнений. ДАН СССР 139(6), 1961, 1318—1321.*

[50] Шмулев И. И.: *Периодические решения первой краевой задачи для параболических уравнений. Мат. сборник 66(108), 3, 1965, 398—410.*

Equations of hydrodynamics:

[51] Prodi G.: *Résultats récents et problèmes anciens dans la théorie des équations de Navier-Stokes. Les Équations aux Dérivées Partielles. Paris, 1962, 181—196. Éditions du Centre Nat. de la Rech. Sci., Paris 1963.*

[52] Serrin J.: *A note on the existence of periodic solutions of the Navier-Stokes qeuation. Arch. Rat. Mech. Anal. 3, 1, 1959, 120—122.*

** [53] Юдович В. И.: *Периодические движения вязкой несжимаемой жидкости. ДАН СССР 130(6), 1960, 1214—1217*

Equations of a beam:

**[54] Grammel R.: *See* [8].

**[55] Kurzweil J.: *Problems which lead to a generalization of the concept of an ordinary nonlinear differential equation. Diff. Eq. and their Appl., Proc. of the Conf. held in Prague in Sept. 1962, 65—76.*

[56] **Каримов Д. Х.**: *О периодических решениях нелинейных уравнений четвертого порядка. ДАН СССР, 1945, 49(9), 618—621.*

[57] **Каримов Д. Х.**: *О периодических решениях нелинейных уравнений четвертого порядка. ДАН СССР 57, 1947, 651—653.*

* [58] **Каримов Д. Х.**: *О периодических решениях нелинейных уравнений четвертого порядка. ДАН УзССР, 8, 1949, 3—7.*

* [59] **Митряков А. П.**: *О периодических решениях нелинейных уравнений в частных производных высшего порядка. Труды Узб. унив. 65, 1956, 31—44.*

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DISCRETISATION AND ERROR ESTIMATES FOR ELLIPTIC BOUNDARY VALUE PROBLEMS OF THE FOURTH ORDER

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1. One of the problems arising in the application of the finite difference method in solving elliptic boundary value problems is the estimation of the discretization error. There exists an extensive literature for second order elliptic differential equations while there are only few papers dealing with higher order equations. The reason is that we have a very useful and simple tool for second order equations, the maximum principle, which holds both for differential equations and their finite difference analogs. There does not exist such a simple tool for higher order equations. In [1] I dealt with an elliptic equation of the fourth order a special case of which are the biharmonic equation and the equation for the deflection of orthotropic plates. In the paper there is described an $O(h^2)$ finite difference analog of the Dirichlet problem for this equation and an error estimate is proved but only for domains consisting of a finite number of rectangles the boundaries of which are a part of the mesh lines. In this lecture I will describe an $O(h^2)$ finite difference analog for domains of a general shape and will give estimates of the discretization error.

2. The equation considered is

$$(1) \qquad Lu \equiv \frac{\partial^2}{\partial x^2}\left(a(x, y)\frac{\partial^2 u}{\partial x^2}\right) + 2\frac{\partial^2}{\partial x\, \partial y}\left(b(x, y)\frac{\partial^2 u}{\partial x\, \partial y}\right) + \frac{\partial^2}{\partial y^2}\left(c(x, y)\frac{\partial^2 u}{\partial y^2}\right) = F(x, y)$$

(in fact, the method applies and the results remain true if we add to Lu an operator of the second order $Mu = -\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + qu$ where $x_1 = x$, $x_2 = y$ and $\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \geq 0$, $q \geq 0$). We assume that

(2) $\qquad a(x, y) > 0, \quad c(x, y) > 0, \quad 0 \leq b(x, y) \leq 2 \min [a(x, y), c(x, y)].$

Let $\mathscr{D}$ be the domain in which the equation is considered and let $\dot{\mathscr{D}}$ be its boundary. The boundary conditions have the form

(3) $$D^p u = D^p f \qquad (p = 0, 1) \text{ on } \dot{\mathscr{D}},$$

where $D^0 u = u$, $D^1 u$ means any of the first derivatives and f is a given function defined in a domain $\Omega \supset \overline{\mathscr{D}}$. The coefficients $a(x, y)$, $b(x, y)$, $c(x, y)$, the functions $f(x, y)$, $F(x, y)$ and the boundary $\dot{\mathscr{D}}$ are supposed so smooth that the solution $u(x, y)$ of the Dirichlet problem (1), (3) has bounded derivatives up to the sixth order inclusive.

To formulate the finite difference analog of the Dirichlet problem (1), (3) we cover the (x, y) plane in the usual manner by a square net formed by lines parallel to the axes. Let h be the corresponding mesh size. The mesh points will be denoted by (x, y) as any point in the (x, y) plane. The mesh functions, i.e. functions defined at mesh points will be denoted by $U(x, x)$, $E(x, y)$ etc. We use the usual notations

$$U_x(x, y) = h^{-1}[U(x + h, y) - U(x, y)], \quad U_{\bar{x}}(x, y) = h^{-1}[U(x, y) - U(x - h, y)],$$
$$U_{x\bar{x}} = h^{-2}[U(x + h, y) - 2U(x, y) + U(x - h, y)], \ldots$$

The operator Lu will be replaced by the difference operator

(4) $$L_h U = (aU_{x\bar{x}})_{x\bar{x}} + (bU_{xy})_{\bar{x}\bar{y}} + (bU_{\bar{x}\bar{y}})_{xy} + (cU_{y\bar{y}})_{y\bar{y}}$$

which represents an $O(h^2)$ approximation of Lu, i.e.

$$Lu - L_h u = O(h^2) \qquad \text{for } u \in C^6.$$

Let us introduce the sets $\mathscr{D}_h$, $\dot{\mathscr{D}}_h$ and $\dot{\mathscr{D}}_h^*$. By neighbors of a mesh point (x, y) we call 12 mesh points $(x + ih, y + jh)$ with $i, j = 0, \pm 1, \pm 2$, $1 \leq \leq i^2 + j^2 \leq 4$. Now $\mathscr{D}_h$ is the set of all mesh points from $\mathscr{D}$. $\dot{\mathscr{D}}_h$ is the set of neighbors of the mesh points from $\mathscr{D}_h$ which do not belong to $\mathscr{D}_h$, i.e. which do not lie in $\mathscr{D}$, $\dot{\mathscr{D}}_h^*$ is the set of mesh points from $\mathscr{D}_h$ such that at least one of their neighbors lies in $\dot{\mathscr{D}}_h$.

The discrete analog will be a mesh function defined on $\mathscr{D}_h$. First we set

(5) $$L_h U(x, y) = F(x, y), \qquad (x, y) \in \mathscr{D}_h - \dot{\mathscr{D}}_h^*.$$

To get the equations for the points $(x, y) \in \dot{\mathscr{D}}_h^*$ we will extrapolate the values $U(x, y)$, $(x, y) \in \dot{\mathscr{D}}_h$, by means of the boundary condition (3) and the values $U(x, y)$, $(x, y) \in \dot{\mathscr{D}}_h^*$ and we will insert these extrapolated values in the expression $L_h U$ formed formally. Consider first the point $(x - 2h, y)$. If it lies in $\dot{\mathscr{D}}_h$ the boundary $\dot{\mathscr{D}}$ intersects the segment $(x - 2h, y)$, (x, y) in a point $(x - \alpha h, y)$ with $0 < \alpha \leq 2$ and we set

$$U(x-2h,y)=\left(\frac{2-\alpha}{\alpha}\right)^2 U(x,y)+4\frac{\alpha-1}{\alpha^2}f(x-\alpha h,y)-$$
$$-2\frac{2-\alpha}{\alpha}h\frac{\partial F(x-\alpha h,y)}{\partial x}.$$

This is nothing else than an extrapolation of the second degree by means of the parabola assuming the value $U(x, y)$ in (x, y) and $u(x - \alpha h, y)$ in $(x - \alpha h, y)$ and having the derivate equal to $\frac{\partial u(x-\alpha h, y)}{\partial x}$ in $(x - \alpha h, y)$. If the point $(x - h, y)$ also belongs to $\dot{\mathscr{D}}_h$ then $0 < \alpha \leq 1$ and we set

$$U(x-h,y)=\left(\frac{1-\alpha}{1+\alpha}\right)^2 U(x+h,y)+\frac{4\alpha}{(1+\alpha)^2}f(x-\alpha h,y)-$$
$$-2\frac{1-\alpha^2}{(1+\alpha)^2}h\frac{\partial F(x-\alpha h,y)}{\partial x}.$$

This time we use to the extrapolation the value of U in the point $(x + h, y)$ and again the given values of u and $\frac{\partial u}{\partial x}$ in $(x - \alpha h, y)$. It can happen that the point $(x + h, y)$ does not belong to $\mathscr{D}_h$. In this case the boundary $\dot{\mathscr{D}}$ intersects the segment $(x - 2h, y)$, $(x + h, y)$ at least twice and we extrapolate $U(x - 2h, y)$ and $U(x - h, y)$ by means of the values of u and $\frac{\partial u}{\partial x}$ in these intersections. Futher if the point $(x - h, y + h)$ belongs to $\dot{\mathscr{D}}_h$ we extrapolate the value $U(x - h, y + h)$ in the same way as in the first case, namely by means of the values u, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ in the intersection $(x - \beta h, y + \beta h)$, $0 < \beta \leq 1$, of the boundary $\dot{\mathscr{D}}$ with the segment $(x - h, y + h)$, (x, y) and by means of $U(x, y)$. We have

$$U(x-h,y+h)=\left(\frac{1-\beta}{\beta}\right)^2 U(x,y)+\frac{2\beta-1}{\beta^2}F(x-\beta h,y+\beta h)+$$
$$+\frac{1-\beta}{\beta^2}h\left[-\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right]_{(x-\beta h,y+\beta h)}.$$

In this way we extrapolate all remaining values of $U(x, y)$ for $(x, y) \in \dot{\mathscr{D}}_h$. After inserting these values in the expression $L_h U(x, y)$ formed formally we get an expression of the form $\bar{L}_h U - l_h(f)$ where the operator $\bar{L}_h U$ contains the terms with $U(x, y)$, $(x, y) \in \mathscr{D}_h$, only and $l_h(f)$ consists of terms containing f, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ ($l_h(f) = 0$ if $f \equiv 0$). We set

(6) $$L_h(U) = F(x, y) + l_h(f), \qquad (x, y) \in \dot{\mathscr{D}}_h^*.$$

3. Following [2] let us introduce the L_2 norms $||E||_0$, $||E||_1$, $||E||_2$. We set

$$||E||_0^2 = h^2 \sum_{S_h} E^2,$$
$$||E||_1^2 = ||E||_0^2 + ||E_x||_0^2 + ||E_y||_0^2,$$
$$||E||_2^2 = ||E||_0^2 + ||E_x||_1^2 + ||E_y||_1^2.$$

S_h means the set of all mesh points in the plane (x, y) and E is any mesh function defined on $\mathscr{D}_h$ and extended on S_h by setting $E(x, y) = 0$ for $(x, y) \notin \mathscr{D}_h$.

The main result is given by the following estimate: If E is the discretization error, i.e. $E(x, y) = u(x, y) - U(x, y)$ for $(x, y) \in \mathscr{D}_h$, $E(x, y) = 0$ for $(x, y) \notin \mathscr{D}_h$ then

$$||E||_2 = O(h^{\frac{3}{2}}). \tag{7}$$

There are good reasons to believe that this estimate cannot be improved, i.e. the exponent $\frac{3}{2}$ is the best though we use an $O(h^2)$ approximation. By means of the discrete SOBOLEV inequality it follows from (7)

$$\max_{(x,y)\in D_h} |E(x, y)| = O(h^{\frac{3}{2}}).$$

Futher by means of an inequality due to BRAMBLE (see [3], lemma 3.2) we get

$$\max_{(x,y)\in\mathscr{D}_h} (|E_x(x, y)| + |E_y(x, y)| = O\left(h^{\frac{3}{2}} . \lg \frac{1}{h}\right)$$

By means of another inequality due to BRAMBLE (see [3], lemma 3.3) it is easy to show that for $Lu = \Delta^2 u$ it follows from (7)

$$||E||_1 = O(h^2), \qquad \max_{(x,y)\in\mathscr{D}_h} |E(x, y)| = O\left(h^2 . \lg \frac{1}{h}\right).$$

4. For domains consisting of a finite sum of rectangles the boundaries of which are a part of the mesh lines it is possible to formulate the discrete analog in such a way that the discretization error satisfies

$$||E||_2 = O(h^2). \tag{8}$$

The asumptions are the same as in the general case with the exception of (2). It is sufficient to assume the uniform ellipticity. For simplicity let us consider a rectangle. The set of the mesh points lying inside the rectangle will be denoted by $\mathscr{D}_h$. Γ_h is the set of the mesh points lying on the boundary of the rectangle, $\overline{\Gamma}_h$ is the set of the mesh points lying outside of the rectangle at a distance h from the boundary. The mesh function U will be defined on the set $\mathscr{D}_h \cup \overline{\Gamma}_h \cup \Gamma_h$. In $\mathscr{D}_h$ we set

$$L_h U(x, y) = F(x, y), \qquad (x, y) \in \mathscr{D}_h.$$

On Γ_h we set $U = f$ and on $\overline{\Gamma}_h$ we extrapolate the value of U by means of the boundary values and two neighbors lying inside the rectangle. If, for instance, $(\xi, \eta) \in \Gamma_h$ and $(\xi - h, \eta) \in \overline{\Gamma}_h$ we set

$$U(\xi - h, \eta) = 3U(\xi + h, \eta) - \frac{1}{2} U(\xi + 2h, \eta) - \frac{3}{2} f(\xi, \eta) - 3h \frac{\partial f(\xi, \eta)}{\partial x}.$$

The estimate of the discretization error is given by (8) from which it follows

$$\max_{x,y \in \mathcal{D}_h} |E(x, y)| = O(h^2)$$

$$\max (|E_x(x, y)| + |E_y(x, y)| = O\left(h^2 . \lg \frac{1}{h}\right).$$

REFERENCES

[1] M. Zlámal: *Asymptotic Error Estimates in Solving Elliptic Equations of the Fourth Order by the Method of Finite Differences, J. SIAM, Numer. Anal. Ser. B, Vol. 2 (1965), 337–344.*

[2] V. Thomée: *Elliptic Difference Operators and Dirichlet's Problem, Contributions to Dif. Eq. Vol. III (1964), 301–324.*

[3] J. H. Bramble: *A Second Order Finite Difference Analog of the First Biharmonic Boundary Value Problem, Technical Note BN-402, 1965, The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.*

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
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3. APPLICATIONS AND NUMERICAL METHODS

ON SOME FUNCTIONS WHICH VERIFY DIFFERENTIAL INEQUALITIES

T. Popoviciu, Cluj

1. In the very first chapters of the mathematical analysis we find differential inequalities which characterize important classes of functions with a given comportement.

Thus it is well-known that the differential inequality $f'(x) \geqq 0$ characterizes the derivable and nondecreasing functions, i.e. the functions whose divided difference $[x_1, x_2; f] = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ remains $\geqq 0$ on every set of two distinct points x_1, x_2 of the definition set of the function.

2. For the consideration of a more general case we introduce the notion of convex function of higher order.

In the first place the *divided difference* $[x_1, x_2, \ldots, x_{n+1}; f]$ (of order n) of the function f on the *knots* $x_1, x_2, \ldots, x_{n+1}$ may be defined by the recurrence relation

$$[x_1, x_2, \ldots, x_{n+1}; f] = \frac{[x_2, x_3, \ldots, x_{n+1}; f] - [x_1, x_2, \ldots, x_n; f]}{x_{n+1} - x_1},$$

$$[x_1; f] = f(x_1)$$

or in another way, as the coefficient of x^n of the polynomial of interpolation (of degree n) which takes the same values as the function f on the knots x_α, $\alpha = 1, 2, \ldots, n + 1$. Another definition is with the quotient of the form (7) of two determinants (for the functions (11) in this case).

For the sake of simplification and in the first stage ours considerations we may suppose that the knots x_α are distinct. The case of knots which are not distinct is obtained by a convenient limite process.

A function f is called *nonconcave of order* n ($n \geqq -1$) if we have

$$[x_1, x_2, \ldots, x_{n+2}; f] \geqq 0 \tag{1}$$

on any set of $n + 2$ distinct points $x_1, x_2, \ldots, x_{n+2}$ of the set of definition of the function. If the strict inequality (with the sign $>$) is valid in (1), the function is called *convex of order n*. In a similar way we can introduce the nonconvex functions respective the concave ones of order n, on condition that the divided difference in the first member of the formula (1) should remain $\leqq$ respective < 0.

3. If we suppose that function f is definite and has a derivative of order $n + 1$ ($n \geqq 0$) on the interval $[a, b]$, the condition

$$f^{(n+1)}(x) \geqq 0 \qquad (\text{on } [a, b]) \tag{2}$$

is necessary and sufficient for the nonconcavity of order n, and the condition $f^{(n+1)}(x) > 0$ (on $[a, b]$) is sufficient for the convexity of order n of the function. An analogous property exists for the nonconvexes and the concave functions of order n; respectively.

The demonstration of these properties is based on the mean-value formula ($n \geqq 0$)

$$[x_1, x_2, \ldots, x_{n+2}; f] = \frac{f^{(n+1)}(\xi)}{(n+1)!} \tag{3}$$

where ξ is within the smallest interval which contains all the knots x_α. The formula (3) is due to CAUCHY [2] but it can be perfected and admits various generalizations [11].

4. The *global* characterization through the inequality (1) of the nonconcave functions of order n, corresponds to the *local* characterization (2). There is also a more general, local characterization which does not require existence of the derivative of order $n + 1$ of the function. A function is nonconcave of order n on the point x if there is a neighbourhood of x, on which it is nonconcave of order n. If a function is nonconcave of order n on any point of the finite and closed interval $[a, b]$, it is nonconcave of order n on $[a, b]$. The demonstration of this property results from the general mean-value formula which we give below and also from the application of the well-known Borel—Lebesgue's covering lemma.

The general mean-value formula referred to is

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_{n+2}}; f] = \sum_{\alpha=1}^{m-n-1} A_\alpha [x_\alpha, x_{\alpha+1}, \ldots, x_{\alpha+n+1}; f] \tag{4}$$

where $x_1 < x_2 < \ldots < x_m$, $m \geqq n + 2$, $1 = i_1 < i_2 < \ldots < i_{n+1} < i_{n+2} = m$ and the A_α, $\alpha = 1, 2, \ldots, m - n - 1$ are coefficients independent of the function f and $A_\alpha \geqq 0$, $\alpha = 1, 2, \ldots, m - n - 1$, $A_1 > 0$, $A_{m-n-1} > 0$, $\sum_{\alpha=1}^{m-n-1} A_\alpha = 1$.

Thus the divided difference in the first member of the formula (4) is comprised between the smallest and the greatest of the divided differences $[x_\alpha, x_{\alpha+1}, \ldots, x_{\alpha+n+1}; f]$, $\alpha = 1, 2, \ldots, m - n - 1$. The mean-value formula (3) is deduced from a limit case of the general formula (4).

5. We mention by the way that the global characterization through the inequality (1) of the nonconcave (convexe, etc.) functions of order n may be replaced by a weaker characterization if we suppose that the function is continuous.

A necessary and sufficient condition for function f which is defined and continuous on the interval $[a, b]$ to be nonconcave of order n, is that the inequality (1) should be valid whenever the knots $x_1, x_2, \ldots, x_{n+2}$ are *equally spaced* [7].

It seems that this last condition may be replaced by another one which implies that the mutual ratios of the distances between the knots should have certain given values, so that the inequality (1) may be replaced by

$$[x + \lambda_1 h, x + \lambda_2 h, \ldots, x + \lambda_{n+2} h; f] \geqq 0$$

where λ_α, $\alpha = 1, 2, \ldots, n + 2$ are distinct given numbers and $x + \lambda_\alpha h \in [a, b]$, $\alpha = 1, 2, \ldots, n + 2$. This is certainly true if $n = 1$, $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = p =$ given natural number > 1. The cas of a given p comes to that of $p - 1$, as we have shown for $p = 3$ [12].

6. We come to a generalization of the differential inequality (2) through a convenient generalization of the notion of non-concave function of order n. Let us write

$$V\begin{pmatrix} g_1, g_2, \ldots, g_m \\ x_1, x_2, \ldots, x_m \end{pmatrix} = |g_\beta(x_\alpha)|_{\alpha,\beta} = 1, 2, \ldots, m \tag{5}$$

the determinant of the values of the functions $g_1, g_2, \ldots, g_m$ on the points x_α, $\alpha = 1, 2, \ldots, m$.

Let us consider now a sequence of $n + 2$ functions

$$f_0, f_1, \ldots, f_{n+1} \tag{6}$$

checking certain conditions of regularity [13], with which we shall deal below.

The quotient

$$V\begin{pmatrix} f_0, f_1, \ldots, f_n, f \\ x_1, x_2, \ldots, x_{n+2} \end{pmatrix} : V\begin{pmatrix} f_0, f_1, \ldots, f_n, f_{n+1} \\ x_1, x_2, \ldots, x_{n+2} \end{pmatrix}, \tag{7}$$

which we write $[x_1, x_2, \ldots, x_{n+2}; f]$, is the divided difference of the function f on the knots x_α, $\alpha = 1, 2, \ldots, n + 2$ with respect to the sequence of (given) functions (6).

We can generally consider the knots distinct. The divided difference may be defined also on knots which are not distinct, by limite process. By this limiting process we can get the divided difference on all knots coinciding with the same point x which can be written $[\underbrace{x, x, \ldots, x}_{n+2}; f]$ and which has the forme

$$(8)\quad \frac{[\underbrace{x, x, \ldots, x}; f]}{n+2} = \varphi_0(x) f^{(n+1)}(x) + \varphi_1(x) f^{(n)}(x) + \ldots + \varphi_{n+1}(x) f(x) = L[f]$$

where by virtue of the admitted conditions of regularity, the function $\varphi_0(x)$ does not vanish and the function f admits derivatives required by the existence of the formula.

The conditions which are fulfilled by the functions (6) are of such a kind that the quotient (7) should have sense (the denominator is $\neq 0$) and the divided difference taken into consideration should exist.

7. The definition of the nonconcave functions with respect to the sequence of functions (6) is given also by the inequality (1), when the divided difference in the first member has the value (7). The convexity, concavity and non-convexity with respect to the given sequence of functions may be similarly defined.

In the condition of regularity mentioned the corresponding mean-value formula (3) has the following form

$$[x_1, x_2, \ldots, x_{n+2}; f] = \frac{[\underbrace{\xi, \xi, \ldots, \xi}; f]}{n+2}$$

and the general mean-value formula (4) subsists.

It come out that the inequality

$$(9)\qquad L[f] \geqq 0$$

which on the basis of (8) is a differential inequality, characterizes the non-concave functions with respect to the sequence (6) and if they admit a $n+1$ order derivative.

The differential equation

$$(10)\qquad L[f] = 0$$

in the already mentioned conditions of regularity has just the functions $f_0, f_1, \ldots, f_n$, as a system of liniarly independent solutions.

The connection between the differential equation (10) and the differential inequality (9) is thus made by means of the divided difference (8) and of the notion of non-concave function with respect to a system of liniarly independent solutions of the equation (10) which checks certain supplementary regularity conditions.

The above mentioned remarks are to be found generally in the case $n = 1$, in a remarcable note of H. Poincaré [5]. Various authors have completed the results previously achieved. In this respect one may consult one of G. Pólya works [6] and some bibliographical indications given in my monograph on convex functions [9]. Some properties of continuity and derivability of the convex (nonconcave, etc.) functions with respect to a sequence (6) were studied in a previous work of mine [8].

The case of inequality (2) with the usual divided difference is the „polynomial" case when

(11) $$f_\alpha = x^\alpha, \quad \alpha = 0, 1, \ldots, n + 1.$$

In this case all the conditions of regularity, of the existence of the divided differences, etc. are satisfied, whatever the definition interval of the considered functions may be. Another important, peculiar case will be examinad below.

8. The convexity notion with respect to a given sequence of functions may be applied to the study of the structure of the remainder of some linear approximation formulae of the analysis.

Consider a linear functional (additive and homogeneous) $R[f]$ definite on a vectorial space S formed by continuous functions on the interval E. Suppose the functions (6) belong to S and check the regularity conditions required. In this case we say that the linear functional $R[f]$ is of *simple form* if for any $f \in S$ it is of the form

(12) $$R[f] = K \,.\, [\xi_1, \xi_2, \ldots, \xi_{n+2}; f]$$

where $K \neq 0$ is a constant independent of function f, and ξ_α, $\alpha = 1, 2, \ldots, n + 2$ are $n + 2$ distinct points from E and generally dependent on function f.

Supposing that the linear functional $R[f]$ vanishes in the first $n + 1$ functions (6), a necessary and sufficient condition for it to be of simple form is that we should have $R[f] \neq 0$ on any convex (not non-concave!) $f \in S$ with respect to functions (6). Then the universal constant K from the formula (12) may be easily calculated and is equal to $R[f_{n+1}]$.

9. The previous result may be applied to the remainder of various numerical derivation and numerical integration formulae. [10, 13, 14, 15]. The conclusion may be applied to the remainder of the well known quadrature formula

(13) $$\int_0^{2\pi} f(x)\, dx = \frac{2\pi}{m+1} \sum_{\alpha=0}^{m} f\left(\frac{2\alpha\pi}{m+1}\right) + R[f]$$

where m is a natural number and f a continuous function on $[a, b]$.

We have here a „trigonometric" case in which for functions (6) we can take ($n = 2m$)

$$f_0 = 1, \quad f_{2m+1} = x, \quad f_{2\alpha-1} = \cos \alpha x, \quad f_{2\alpha} = \sin \alpha x, \quad \alpha = 1, 2, \ldots, m.$$

In this case the conditions of regularity required are satisfied on the interval $[0, 2\pi)$ (closed on the left and open on the right) [13].

The remainder of the formula (13) is of simple form [13] and in this case we have

$$R[f] = \frac{2\pi^2}{m+1} [\xi_1, \xi_2, \ldots, \xi_{2m+2}; f].$$

We find J. RADON's formula if a derivative of order $2m + 1$ exists [16],

$$R[f] = \frac{2\pi^2}{(m+1)(m!)^2}\left[\frac{d}{dx}\left(\frac{d^2}{dx^2}+1^2\right)\left(\frac{d^2}{dx^2}+2^2\right)\cdots\left(\frac{d^2}{dx^2}+m^2\right)f\right]_{x=\xi},$$

$$\xi \in (0,2\pi).$$

In the demonstration we can initialy suppose that the function has a continuous $(2m+1)^{th}$ derivative on the interval $[0, 2\pi]$. Later on we can omit this supposition.

10. The conditions of regularity to which we referred and which are cheeked by sequence (6), or by the subsequence of its first $n + 1$ terms, is the same with interpolator property of the set of all linear combinations of these functions.

The sequence $g_1, g_2, \ldots, g_m$ is an interpolator sequence (system), or a system of TSCHEBYSCHEFF, if the determinant (5) is $\neq 0$ for any system of m distinct points x_α, $\alpha = 1, 2, \ldots, m$ of the definition set of the functions g_α, $\alpha = 1, 2, \ldots, m$.

The regularity conditions of the sequence (6) also contain limit properties which ensure the existence of divided differences wich knots not all different.

We can study much more general interpolator systems.

In order to be more precise, let's consider a set F of functions, definite and continuous on the finite and closed interval $[a, b]$. We shall say that the set F is *interpolator of order* $n + 1$ ($n \geqq 0$) if for any system of $n + 1$ distinct point x_α, $\alpha = 1, 2, \ldots, n + 1$ of $[a, b]$ and for any continuous function f, definite on the interval $[a, b]$, there is an element and only one $\varphi \in F$ which checks the equalities $\varphi(x_\alpha) = f(x_\alpha)$, $\alpha = 1, 2, \ldots, n + 1$. We can write $\varphi(x) = L(x_1, x_2, \ldots, x_{n+1}; f|x)$ this element of F. In this definition the continuity hypothesis of f is not essential. Still we take into consideration this hypothesis because in what we are going to present all the functions will be supposed to be continuous.

The difference ($n \geqq 0$)

$$[x_1, x_2, \ldots, x_{n+2}; f]_D = f(x_{n+2}) - L(x_1, x_2, \ldots, x_{n+1}; f|x_{n+2}) \tag{14}$$

where $x_1 < x_2 < \ldots < x_{n+2}$, plays the part of divided difference (7) in the linear case.

We can extend the mean-value theorems of the divided difference [3, 4]. Thus we get the following property: if f is a continuous function on $[a, b]$ and $x_1 < x_2 < \ldots < x_{n+2}$ are $n + 2$ points on this interval, there is a point $\xi \in (x_1, x_{n+2})$ (open interval) so that in all neighbouhood of ξ should exist the points $\xi_1 < \xi_2 < \ldots < \xi_{n+2}$ for wich we have

$$\operatorname{sg}[x_1, x_2, \ldots, x_{n+2}; f]_D = \operatorname{sg}[\xi_1, \xi_2, \ldots, \xi_{n+2}; f]_D$$

There are various other mean-value formulae which are closely connected with the given formula and which all come to mean-value formulae of the divided difference when F is the set of linear combinations of the first $n + 1$ terms of the sequence (6) checking the respective interpolator conditions.

A function f definite on $[a, b]$ is called nonconcave (convex, concave, nonconvex) with respect to the interpolator set F, if we have $[x_1, x_2, \ldots, x_{n+2}; f]_D \geqq$ $\geqq (>, <, \leqq)\ 0$, for any system of $n + 2$ points $x_1 < x_2 < \ldots < x_{n+2}$ of $[a, b]$.

The continuous convex (concave) functions have an equivalent definition because they cannot coincide in more than $n + 1$ points with a certain element of F.

The mean-value formulae already mentioned or such properties of intersection allon us to study the properties of the nonconcave (convex, concave, nonconvex) functions with respect to the interpolator set F [4, 17].

11. By a convenient particularizing of our researches we come to differential inequalities more general than the inequality (9) studied above.

Consider the differential equation $(n \geqq 0)$

$$y^{(n+1)} - G(x, y, y', \ldots, y^{(n)}) = 0 \tag{15}$$

where the function $G(z_1, z_2, \ldots, z_{n+2})$ is continuous for $z_1 \in [a, b]$, $z_\alpha \in (-\infty, +\infty)$, $\alpha = 2, 3, \ldots, n + 2$.

Suppose that the set of the solutions of the equation (15) is interpolator of order $n + 1$ and that Cauchy's problem has always a unique solution on any point of $[a, b]$.

If f is a function which has a continuous derivative of order $n + 1$ $(n \geqq 0)$ on $[a, b]$ and if it coincide with a solution of the equation (15) on the points $x_1 < x_2 < \ldots < x_{n+2}$ of $[a, b]$ there is a point $\xi \in (x_1, x_{n+2})$ so that

$$f^{(n+1)}(\xi) - G(\xi, f(\xi), f'(\xi), \ldots, f^{(n)}(\xi)) = 0$$

The differential inequality

$$f^{(n+1)}(x) - G(x, f(x), f'(x), \ldots, f^{(n)}(x)) \geqq 0 \text{ (respectively } >) \ 0$$

for $x \in [a, b]$ characterizes the functions f which have a continuous derivative of order $n + 1$ and which are nonconcave (convex) with respect to the set of solutions of the equation (15), [3].

As regards the hypotheses according to which the set of solutions of a differential equation (15) is interpolator of order $n + 1$, on a given interval, they constitute the so-called plurilocal problem of the differential equations. This problem was studied by many authors and has been for many years the preoccupation of the researchers of the Institut of Calculus from, CLUJ, of the Academy of the Socialist Republic of Romania [1].

BIBLIOGRAPHY

[1] ARAMA, O. RIPEANU, D. *Asupra problemei polilocale pentru ecuatii diferentiale liniare cu coeficienti constanti. Studii si Cerc. de matem. Cluj, VIII, 37—74, 211—265 (1957).*

[2] CAUCHY, A. *Sur les fonctions interpolaires. C. R. Acad. Sc. Paris, 11, 775—789 (1840).*

[3] MOLDOVAN, E. *Proprietati ale functiilor convexe generalizate. Studii si Cerc. de matem. Cluj, VIII, 21—35 (1957).*

[4] MOLDOVAN, E. *Sur une généralisation des fonctions convexes. Mathematica, 1 (24), 49—80 (1959).*

[5] POINCARÉ, H. *Réponse à la question 128. L'Intermédiaire des mathématiciens, I, 141—144 (1894).*

[6] PÓLYA, G. *On the mean-value theorem corresponding to a given linear homogeneous differential equation. Transac. of the Amer. Math. Soc., 24, 312—324 (1923/24).*

[7] POPOVICIU, T. *Sur quelques propriétés des fonctions d'une ou de deux variables réelles. Mathematica, 8, 1—85 (1934).*

[8] POPOVICIU, T. *Notes sur les fonctions convexes d'ordre supérieur (I.). Mathematica, 12, 81—92 (1936).*

[9] POPOVICIU, T. *Les fonctions convexes. Actualités scientifiques et industrielles, no. 992, Paris, 76 p. (1945).*

[10] POPOVICIU, T. *Asupra restului in unele formule de derivare numerica. Studii si Cerc. de matem., III, 53—122 (1952).*

[11] POPOVICIU, T. *Folytonos függvények középértéktételeiröl. A Magyar Tud. Akad. III (mat. és fiz.) osztályának közleményeiröl. IV, 353—356 (1954).*

[12] POPOVICIU, T. W
Mathematica, 1 (24), 81—93 (1959).

[13] POPOVICIU, T. *Sur le reste dans certaines formules linéaires d'approximation de l'analyse. Mathematica, 1 (24), 95—142 (1959).*

[14] POPOVICIU, T. *Différences divisées et dérivées. Mathematica, 1 (24), 297—319 (1959).*

[15] POPOVICIU, T. *La simplicité du reste dans certaines formules de quadrature. Mathematica 6 (29), 157—184 (1964).*

[16] RADON, J. *Restausdrücke bei Interpolation und Quadraturformeln durch bestimmte Integrale. Monatshefte f. Math. u. Phys., 42, 389—396 (1935).*

[17] TORNHEIM, L. *On n-parametr familles of functions and associated convexe functions. Transac. of the Amer. Math. Soc., 69, 457—467 (1950).*

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STABILITY OF NUMERICAL PROCESSES

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The high speed of modern computers and the corresponding employment of ever increasing number of arithmetic operations have brought to the foreground in recent years the importance of the problems of numerical stability. What does it mean numerical stability? It is the sensitivity of a computation on the errors, which are during the computation on the computer necessarily performed. The source of these errors are the round-off errors in single arithmetic operations, substitution of general analytic expressions by rational ones, etc. This problem and the necessity to solve it is now generally accepted. There are different ways of treatment of the problem. In the first place it is the estimation of the accumulated round-off error, which is e.g. done in the well known books of WILKINSON [1] and HENRICI [2, 3], further the method of closure of processes suggested by SOBOLEV [4], recently the techniques of the interval arithmetic by R. MOORE [5] or simply the intuitive use of multiple arithmetic.

In this paper we will consider the problem of the numerical stability for problems of mathematical analysis, particularly for some problems for differential equations. A characteristic feature of such problems is that the arithmetical operations predominate over the logical ones (the influence of the lasts will be therefore neglected in what follows) and that numerical methods for solution of these problems involve always a certain parameter (e.g. the step of the mesh, the number of approximating functions) and the exact solution of the given problem is then obtained by passing to the limit with this parameter.

This paper is a development of problems of numerical stability for initial-value problems for differential equations which was reported by the Authors on the first Equadiff conference [6]. The introducing of the β_s-solution of the sequence of numerical processes will here be essentially new and will be utilized for the study of numerical stability of some other problems. The main results are published in the book [7].

Now we can pass on the exact formulation of concepts and results. The basic concept is the concept of a numerical process. The numerical process (it is possible to say the computing algorithm, too) is a sequence of arithmetic operations, which transforms the set of the initial data in the set of results. It has been just mentioned that in problems of mathematical analysis we have always a sequence of such processes. We introduce consequently.

Definition 1. *Let there be given a sequence of normed vector spaces*

$$X_0^{(j)}, X_1^{(j)}, \ldots, X_{N_j}^{(j)}, \qquad j = 1, 2, \ldots$$

and a sequence of continuous operators

$$A_i^{(j)}, \qquad i = 0, 1, \ldots, N_j - 1; \qquad j = 1, 2, \ldots,$$

which map the Cartesian product $X_0^{(j)} \times \ldots \times X_i^{(j)}$ *into* X_{i+1}.

Then the sequence of equations

$$x_{i+1}^{(j)} = A_i^{(j)}(x_0^{(j)}, \ldots, x_i^{(j)}), \qquad i = 0, 1, \ldots, N_j - 1, \tag{1}$$

where $x_0^{(j)} \in X_0^{(j)}$ *is given and* $x_i^{(j)} \in X_i^{(j)}$ *is called a numerical process.*

The sequence of elements $x_i^{(j)}$ *is called the solution of the numerical process with the initial value* $x_0^{(j)}$.

Thus, by Definition 1 we have introduced the sequence of numerical processes, the results of which converge for $j \to \infty$ (j is the parameter of the sequence) in some or other sense to the exact solution of the given problem. In practical computations, we have obviously $X_i^{(j)} \in R_1$, $i = 1, 2, \ldots$ and $X_0^{(j)} = R_n$, where R_n is the n-dimensional Euclidean space and $A_i^{(j)}$ are the operators of elementary arithmetic operations. However, in order to simplify the study in many cases, it is convenient to introduce more general objects such as vectors, matrices or others.

The numerical process has an algorithmic, i.e., explicit character. For example, Euler's method for the solution of an initial-value problem for the differential equation $y' = f(x, y)$, i.e., the formula

$$y_{n+1} = y_n + hf(x_n, y_n),$$

where y_0 is given, represents a sequence of numerical processes in dependence of the number of subintervals as parameter. On the contrary, the method of finite differences for solution of the boundary-value problem $y'' = f$, $y(0) = y(1) = 0$ i.e.

$$y_{n+1} - 2y_n + y_{n-1} = h^2 f_n$$

with $y(0) = y(1) = 0$ is not a sequence of numerical processes since it is not indicated any method for solution of the obtained system of algebraic equations. If we add that this system will be solved, e.g., by elimination, then we have defined a sequence of numerical processes. In detail, it will be shown later.

Numerical process (in sense of Def. 1) cannot be in any case realized exactly on the computer because of the errors resulting of the finite character of the work of the computer. We introduce therefore

Definition 2. *Let $X_i^{(j)}$ and $A_i^{(j)}$ satisfy assumptions of Definition 1, let $\delta_i^{(j)} \in X_i^{(j)}$ and let there be given an initial element $x_0^{(j)}$. Then the sequence of equations*

$$\tilde{x}_{i+1}^{(j)} = A_i^{(j)}(\tilde{x}_0^{(j)}, \ldots, \tilde{x}_i^{(j)}) + \delta_{i+1}^{(j)}, \qquad i = 0, 1, \ldots, N_j - 1 \tag{2}$$

with $\tilde{x}_0^{(j)} = \tilde{x}_0^{(j)} + \delta_0^{(j)}$ will be called the perturbed numerical process (1).

The behaviour of the solution of the numerical process (1) with respect to numerical stability will be tested on the basis of comparison with perturbed numerical processes.

Definition 3. *We shall say that the solution of the numerical process* (1) *corresponding to the initial value $x_0^{(j)}$ is a β_s-solution, if*

$$\limsup_{\Delta \to 0} \frac{1}{\Delta} \sup_{\|\delta_i^{(j)}\| \leq \Delta} \sup_{i=1,\ldots,N_j} \|\tilde{x}_i^{(j)} - x_i^{(j)}\| \leq C_j^s$$

where C is a constant independent of j. We say that the given solution is a B_{s_0}-solution if $s_0 = \inf s$.

The subscript s in the concept of a B_s-solution indicates the character of the stability of the given numerical process. It is useful to note here that in accordance with practical experience the constant C in Definition 3 depends on the type of computer used whereas the constant s is independent on the special type of computer and therefore, it is a universal characterization of the given numerical process. It is quite obvious from Def. 3 that such numerical processes which have B_s-solutions with smallest possible s are most favourable from the point of view of numerical stability.

After this short survey of the general theory of numerical stability we shall pay our attention to concrete examples.

Let us investigate the numerical stability of the method of finite differences for a boundary-value problem for a second-order ordinary differential equation. In this connection, we shall also utilize the method of closure of processes.

Thus, let there be given the differential equation

$$y'' - qy = f$$

with the boundary conditions $y'(a) = \alpha$, $y'(b) = \beta$. All what will be said holds also for general self-adjoint equation and other types of boundary conditions and can be used even for fourth-order equations. For the sake of simplicity, we restrict us to this very simple case. By utilizing of the most simple method of finite differences, we obtain for the unknown approximate values y_n the following system of equations, written in matrix form

$$(3) \qquad \begin{pmatrix} 1, & -1, & 0, \dots \\ -1, & 2+h^2q_1, & -1, \dots \\ & \ddots & \\ & \dots, 0, & -1, 1 \end{pmatrix} \begin{pmatrix} y_0 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} -\alpha h \\ -h^2 f_1 \\ \vdots \\ \beta h \end{pmatrix}$$

where $q_i = q(x_i)$ etc, N is the number of subintervals and $h = (b - a)/N$.

We have already mentioned that this system is not a numerical process until we indicate some method for its solution. Thus, we shall connect it here with the elimination method.

This method leads in our case of tridiagonal matrix to three recurrence relations:

$$(4) \qquad \begin{aligned} &d_{i+1} = 2 + h^2 q_{i+1} - \frac{1}{d_i}, \qquad \text{for } i = 0, 1, \dots, N-2, \\ &d_0 = 1, \quad d_N = 1 - \frac{1}{d_{N-1}} \end{aligned}$$

$$(5) \qquad \begin{aligned} &C_{i+1} = -h^2 f_{i+1} + \frac{C_i}{d_i}, \qquad \text{for } i = 0, 1, \dots, N-2, \\ &C_0 = -\alpha h, \qquad C_N = \beta h + \frac{C_{N-1}}{d_{N-1}}. \end{aligned}$$

The backward substitution for computing the unknown y_n yields

$$(6) \qquad y_i = \frac{C_i + y_{i+1}}{d_i}, \qquad i = N-1, \dots, 0, \qquad y_N = \frac{C_N}{d_N}.$$

Before investigate these reccurence relations, we shall demonstrate their connection with the factorization method by the method of closure of processes. Thus, let there be φ_i and z_i defined by the following relations

$$(7) \qquad d_i = 1 + h\varphi_i, \qquad C_i = -hz_i.$$

Then, for the quantities φ_i and z_i we obtain

$$(8) \qquad \begin{aligned} &\varphi_{i+1} = \varphi_i + h\left(\frac{-\varphi_i^2}{1 + h\varphi_i} + q_{i+1}\right), \qquad i = 0, 1, \dots, N-2, \\ &\varphi_0 = 0, \end{aligned}$$

$$(9) \qquad \begin{aligned} &z_{i+1} = z_i + h\left(-\frac{z_i\varphi_i}{1 + h\varphi_i} + f_{i+1}\right), \qquad i = 0, 1, \dots, N-2, \\ &z_0 = \alpha, \end{aligned}$$

and, after an analogical arrangement, for the y_i

$$(10)\qquad \begin{aligned} y_i &= y_{i+1} - h\left(\frac{z_i + \varphi_i y_{i+1}}{1 + h\varphi_i}\right), \qquad i = N-1, \ldots, 0, \\ y_N &= \frac{\beta - z_{N-1} + h\beta\varphi_{N-1}}{\varphi_{N-1}}. \end{aligned}$$

From here it is seen that formulae (8) to (10) represent an approximate method for solution of following initial-value problems

$$\begin{aligned} \varphi' + \varphi^2 &= q, & \varphi(a) &= 0, \\ z' + \varphi z &= f, & z(a) &= \alpha, \\ y' - \varphi y &= z, & y(b) &= \frac{\beta - z(b)}{\varphi(b)}. \end{aligned}$$

The last equation is solved from right to left. It can be easily shown that the solution of the last equation is the solution of the original equation with corresponding boundary conditions. Thus we have obtained the so-called factorization method which represents a transformation of the boundary-value problem into numerically stable initial-value problems.

From here it is obvious that the quantities φ_i and z_i are bounded independently of h, that consequently d_i is also bounded and that the quantity C_i is of order h. Analysis of equations (4) to (6) yields then easily that the finite-difference method in connection with the elimination method gives β_2-solution of numerical processes in dependence on the number of subintervals used. Obviously, it is also B_2-solution.

The performed analysis suggests a possibility of a convenient modification of the process of elimination, namely so that we replace the recurrence relation (4) by (8) and then we use the relation (10). By this we obtain for the process even a B_1-solution.

Further, let us consider the stability of the solution of parabolic equation by the method of finite differences. For the sake of simplicity, we shall consider the equation

$$(11)\qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - q(x, t)\, u + f(x, t)$$

with the initial condition $u(x, 0) = g(x)$, $x \in \langle 0, 1\rangle$ and boundary conditions $u(0, t) = \gamma^0(t)$, $u(1, t) = \gamma^1(t)$, $t \in (0, T)$ and we shall investigate the stability of the Crank-Nicholson formula for the case that the space- and time-steps are related by $\tau = \omega h$, where ω is a constant.

The corresponding system will be written in the matrix form

$$A^{(l)}u^{(l)} = B^{(l)}u^{(l-1)} + (f^{(l)} + f^{(l-1)}), \qquad l = 1, 2, \ldots, r,$$

where the tridiagonal matrices $A^{(l)}$ and $B^{(l)}$ are given as follows

$$A^{(l)} = \begin{bmatrix} \frac{2h}{\omega} + 2 + h^2 q_{1,l}, & -1, & 0, \ldots & \\ -1, & \frac{2h}{\omega} + 2 + h^2 q_{2,l}, & -1, \ldots & \\ & & \ddots & \\ & & \ldots, 0, -1, & \frac{2h}{\omega} + 2 + h^2 q_{n-1,l} \end{bmatrix}$$

$$B^{(l)} = \begin{bmatrix} \frac{2h}{\omega} - 2 - h^2 q_{1,l-1}, & 1, & 0, \ldots & \\ 1, & \frac{2h}{\omega} - 2 - h^2 q_{2,l-1}, & 1, \ldots & \\ & & \ddots & \\ & & \ldots, 0, 1, & \frac{2h}{\omega} - 2 - h^2 q_{n-1,l-1} \end{bmatrix}$$

the vector $u^{(l)}$ is the vector of the unknown solution at the l-th time level and $f^{(l)}$ is the right-hand side vector

$$f^{(l)} = \{h^2 f_{1,l} + \gamma^0(t_l), h^2 f_{2,l}, \ldots, h^2 f_{n-1,l} + \gamma^1(t_l)\}$$

The numerical process by which $u^{(l)}$ is computed consists in the following recurrence procedure. Assuming that $u^{(l-1)}$ is known we compute

$$v^{(l)} = B^{(l)} u^{(l-1)} + f^{(l)} + f^{(l-1)}$$

and then the equation

(12) $$A^{(l)} u^{(l)} = v^{(l)}$$

is solved. The method for solution of the last system (not yet indicated) is an essential part of the numerical process.

The perturbed process is given by

$$\tilde{u}^{(l)} = B^{(l)} \tilde{u}^{(l-1)} + f^{(l)} + f^{(l-1)} + \delta^{(l)},$$

where $|\delta^{(l)}| < K\delta$, δ is the error of the elementary operation and by the solution of the system

$$A^{(l)} u^{(l)} = \tilde{v}^l.$$

We shall assume that the actual solution of the system $u^{(l)}$ fulfils the equation

$$A^{(l)}\tilde{u}^{(l)} = \tilde{v}^{(l)} + r^{(l)}.$$

In order to be able to say something about the stability of the method in question, it is not necessary to specify the method for solution (12) completely. It is sufficient to make certain assumptions about the magnitude of the residue $r^{(l)}$. We have the theorem

The numerical process of solution of the equation (11) *by the Crank-Nicholson formula is a $B_{2+\varrho}$-solution (with respect to the parameter $1/h$) under the assumption that the method for solution of* (12) *is such that the residual vector fulfils the estimate*

$$\max_k |r_k^{(l)}| \leqq K \frac{\delta}{h^\varrho}.$$

This theorem describes the stability of the Crank-Nicholson formula under the assumption that for the chosen method the asymptotic behaviour of residues arising by solving (12) is known. Analogically, the stability of a general parabolic equation with general boundary conditions may be investigated.

In the case, when for the solution of (12) the elimination method is used, one can prove by considering the concrete form of the matrix $A^{(l)}$ and the vector $v^{(l)}$ that the residue fulfils

$$\max_k |r^{(l)}| \leqq K\delta,$$

where K is independent on h. In this case, we obtain a B_2-solution for the entire process. And this is a rather favourable result.

We have investigated in some cases the numerical stability by the concept of β_s-solution. This assesment of numerical stability is of an asymptotic, essentially qualitative character. Our approach, maximalistic in essence, shows the trend of accumulated errors rather than their accurate bounds. The characterization of methods by β_s-solutions may be utilized in different ways. First of all a comparison of different methods will most conveniently be based not only on computer time and memory capacity required, but also their numerical stability. Another example is in some cases utilized combination of methods. It is no sense in utilizing some, e.g., iterative method in order to get a more accurate solution, if its numerical stability is equal or even worse than that of the original method. Occasionally, we may use the conclusions concerning β_s-solutions even in a quantitative way, for example by comparison with some simple case, where the error is known.

Naturally there are many other such possibilities which depend on a person's experience, intuition and skill.

REFERENCES

[1] J. H. WILKINSON: *Rounding errors in algebraic processes, H. M. S. O., London, 1963.*

[2] P. HENRICI: *Discrete variable methods in ordinary differential equations, J. Wiley & Sons, Inc., New York, London, 1962.*

[3] P. HENRICI: *Error propagation for difference methods, J. Wiley & Sons, New York, London, 1963.*

[4] С. Л. Соболев: *Некоторые замечания о численном решении интегральных уравнений, Изв. АН СССР, Сер. матем., 20 (1956), 413—436.*

[5] *Error in digital computation, T. I, II, edited by L. B. Rall, J. Wiley & Sons, New York, London, Sydney, 1965.*

[6] *Differential equations and their applications, Proceedings of the conference held in Prague in September 1962, NČSAV, Praha, 1963.*

[7] I. BABUŠKA, M. PRÁGER and E. VITÁSEK: *Numerical processes in differential equations, J. Wiley & Sons, New York, London, Sydney and SNTL, Praha, 1966.*

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