Bulletin de la Société des mathématiciens
et physiciens de la R. P. de Serbie
Vol. IX 1—2 (1957), Beograd
Yougoslavie

NOTE ON A FORMULA ON FACTORIALS

by ZLATKO MAMUZIĆ, BEOGRAD

Consider s kinds of objects there being α_j objects of the j-th kind, $1 \leqslant j \leqslant s$, so that $\alpha_1+\alpha_2+\ldots+\alpha_s=n$, where n is a natural. Then the number of all permutations of these objects amounts to

$$x=\frac{(\alpha_1+\alpha_2+\ldots+\alpha_s)!}{\alpha_1!\,\alpha_2!\ldots\alpha_s!}=\frac{n!}{\alpha_1!\,\alpha_2!\ldots\alpha_s!}.$$

If an arbitrary object of the j-th kind is denoted simply by j and if we take the permutation

$$p_1=\underbrace{1\ldots1}_{\alpha_1}\underbrace{2\ldots2}_{\alpha_2}\underbrace{3\ldots3}_{\alpha_3}\ldots\underbrace{j\ldots j}_{\alpha_j}\ldots\underbrace{s\ldots s}_{\alpha_s}$$

as a first one in the lexicographic order of all permutations, then p_i is the permutation standing on the i-th place from left to right.

Obviously we have

$$p_x=\underbrace{s\ldots s}_{\alpha_s}\ldots\underbrace{j\ldots j}_{\alpha_j}\ldots\underbrace{3\ldots3}_{\alpha_3}\underbrace{2\ldots2}_{\alpha_2}\underbrace{1\ldots1}_{\alpha_1}.$$

Applying the general method of finding the index i, $1 \leqslant i \leqslant x$, of a given permutation (s. for example [1], pp. 16—27) a formula on factorials can be derived by which the number x is decomposed in a sum in the following manner.

It is evident that before the permutation

$$s\,\underbrace{1\ldots1}_{\alpha_1}\underbrace{2\ldots2}_{\alpha_2}\underbrace{3\ldots3}_{\alpha_3}\ldots\underbrace{j\ldots j}_{\alpha_j}\ldots\underbrace{s\ldots s}_{\alpha_s-1}$$

there is

$$(\alpha_1+\alpha_2+\ldots+\alpha_{s-1})\cdot\frac{(\alpha_1+\alpha_2+\ldots+\alpha_s-1)!}{\alpha_1!\,\alpha_2!\ldots\alpha_s!}=x_1$$

permutations. Continuing this process we have: before the permutation

$$s\,s\,\underbrace{1\ldots1}_{\alpha_1}\,\underbrace{2\ldots2}_{\alpha_2}\,\underbrace{3\ldots3}_{\alpha_3}\ldots\underbrace{j\ldots j}_{\alpha_j}\ldots\underbrace{s\ldots s}_{\alpha_s-2}$$

there is

$$x_1+(\alpha_1+\alpha_2+\ldots+\alpha_{s-1})\frac{(\alpha_1+\alpha_2+\ldots+\alpha_s-2)!}{\alpha_1!\,\alpha_2!\ldots(\alpha_s-1)!}=x_1+x_2$$

permutations, and so on; before the permutation

$$\underbrace{s\ldots s}_{\alpha_s}\,\underbrace{1\ldots1}_{\alpha_1}\,\underbrace{2\ldots2}_{\alpha_2}\,\underbrace{3\ldots3}_{\alpha_3}\ldots\underbrace{j\ldots j}_{\alpha_j}\ldots\underbrace{\overline{s-1}\,\overline{s-1}\ldots\overline{s-1}}_{\alpha_{s-1}}$$

there is

$$x_1+x_2+\ldots+x_{\alpha_s-1}+(\alpha_1+\alpha_2+\ldots+\alpha_{s-1})\frac{(\alpha_1+\alpha_2+\ldots+\alpha_{s-1})!}{\alpha_1!\,\alpha_2!\ldots\alpha_{s-1}!\,1!}=$$

$$=x_1+x_2+\ldots+x_{\alpha_s-1}+x_{\alpha_s}$$

permutations; before the permutation

$$\underbrace{s\ldots s}_{\alpha_s}\,\overline{s-1}\,\underbrace{1\ldots1}_{\alpha_1}\,\underbrace{2\ldots2}_{\alpha_2}\,\underbrace{3\ldots3}_{\alpha_3}\ldots\underbrace{j\ldots j}_{\alpha_j}\ldots\underbrace{\overline{s-1}\ldots\overline{s-1}}_{\alpha_{s-1}-1}$$

there is

$$x_1+x_2+\ldots x_{\alpha_s}+(\alpha_1+\alpha_2+\ldots+\alpha_{s-2})\frac{(\alpha_1+\alpha_2+\ldots+\alpha_{s-1}-1)!}{\alpha_1!\,\alpha_2!\ldots\alpha_{s-1}!}=$$

$$=x_1+x_2+\ldots+x_{\alpha_s}+x_{\alpha_s+1}$$

permutations, and so on; before the permutation

$$\underbrace{s\ldots s}_{\alpha_s}\,\underbrace{\overline{s-1}\ldots\overline{s-1}}_{\alpha_{s-1}}\,\underbrace{1\ldots1}_{\alpha_1}\,\underbrace{2\ldots2}_{\alpha_2}\,\underbrace{3\,3\ldots3}_{\alpha_3}\ldots\underbrace{j\ldots j}_{\alpha_j}\ldots$$

there is

$$x_1+x_2+\ldots+x_{\alpha_s+\alpha_{s-1}-1}+(\alpha_1+\alpha_2+\ldots+\alpha_{s-2})\frac{(\alpha_1+\alpha_2+\ldots+\alpha_{s-2})!}{\alpha_1!\,\alpha_2!\ldots\alpha_{s-2}!\,1!}=$$

$$=x_1+x_2+\ldots+x_{\alpha_s+\alpha_{s-1}-1}+x_{\alpha_s+\alpha_{s-1}}$$

permutations and so on; before the permutation

$$\underbrace{s\ldots s}_{\alpha_s}\,\underbrace{\overline{s-1}\ldots\overline{s-1}}_{\alpha_{s-1}}\ldots\underbrace{j\ldots j}_{\alpha_j}\ldots\underbrace{3\ldots3}_{\alpha_3}\,\underbrace{1\ldots1}_{\alpha_1}\,\underbrace{2\ldots2}_{\alpha_2}$$

there is

$$x_1+x_1+\ldots+x_{\alpha_s+\alpha_{s-1}+\ldots+\alpha_3-1}+(\alpha_1+\alpha_2)\frac{(\alpha_1+\alpha_2)!}{\alpha_1!\,\alpha_2!}=$$

$$=x_1+x_2+\ldots+x_{\alpha_s+\alpha_{s-1}+\ldots+\alpha_3-1}+x_{\alpha_s+\alpha_{s-1}+\ldots+\alpha_3}$$

permutations; at the and we have: before the permutation

$$\underbrace{s\ldots s}_{\alpha_s}\ \underbrace{\overline{s-1}\ldots\overline{s-1}}_{\alpha_{s-1}}\ldots\underbrace{j\ldots j}_{\alpha_j}\ldots\underbrace{3\ldots 3}_{\alpha_3}\underbrace{2\ldots 2}_{\alpha_2}\underbrace{1\ldots 1}_{\alpha_1}$$

there is

$$x_1+x_2+\ldots+x_{\alpha_s+\alpha_{s-1}+\ldots+\alpha_2-1}+\alpha_1\cdot\frac{\alpha_1!}{\alpha_1!}=x_1+x_2+\ldots+x_{\alpha_s+\alpha_{s-1}+\ldots+\alpha_2}$$

permutations.

But it is evident that

$$x=1+x_{\alpha_s+\alpha_{s-1}+\ldots+\alpha_3+\alpha_2}+x_{\alpha_s+\alpha_{s-1}+\ldots+\alpha_3+\alpha_2-1}+\ldots+x_3+x_2+x_1$$

and, in view of the preceding results, we obtain[1]

$$\frac{(\alpha_1+\alpha_2+\ldots+\alpha_s)!}{\alpha_1!\,\alpha_2!\ldots\alpha_s!}=1+\alpha_1\cdot\sum_{\nu=1}^{\nu=\alpha_2}\frac{(\alpha_1+\alpha_2-\nu)!}{\alpha_1!\,(\alpha_2-\nu+1)!}+ \tag{1}$$

$$+(\alpha_1+\alpha_2)\cdot\sum_{\nu=1}^{\nu=\alpha_3}\frac{(\alpha_1+\alpha_2+\alpha_3-\nu)!}{\alpha_1!\,\alpha_2!\,(\alpha_3-\nu+1)!}+\ldots$$

$$\ldots+(\alpha_1+\alpha_2+\ldots+\alpha_{s-1})\cdot\sum_{\nu=1}^{\nu=\alpha_s}\frac{(\alpha_1+\alpha_2+\ldots+\alpha_s-\nu)!}{\alpha_1!\,\alpha_2!\ldots\alpha_{s-1}!\,(\alpha_s-\nu+1)!}.$$

In the special case when $\alpha_1=\alpha_2=\ldots=\alpha_s=1$ therefrom we obtain the well-known formula (s. [2])

$$n!=1+1\cdot 1!+2\cdot 2!+\ldots+(n-1)\cdot(n-1)!\,. \tag{2}$$

If we remark that using the gamma function Γ the formula (2) can be deduced by means of that function, it would be of some interest to see how could be also obtained the formula (1) by means either of the function Γ or of the function beta B. Moreover, it is obvious that $\Gamma(n+1)=n!$ so that $\Gamma(n+1)$ can be interpreted as an ordinal number according to the interpretation given to the formulae (1) and (2) in this note. Thus we have

---

[1] I dont know wether the formula (1) has been derived somewhere but, to the best of my knowledge, in publications about the subject I found only the special case (2), although (1) is a consequence of lexigraphic orderings of permutatuons in an analogous way as it is the case with the formula (2) too.