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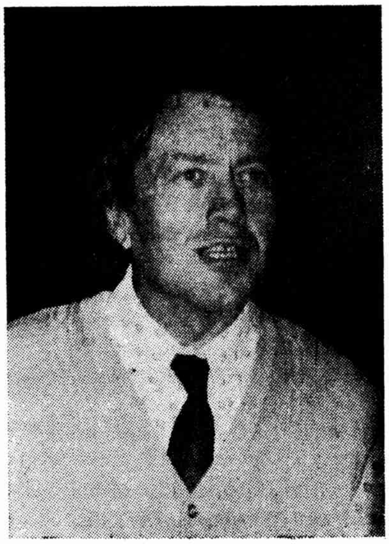
THIS ISSUE OF ARCHIVUM MATHEMATICUM
IS DEDICATED TO THE MEMORY OF

MILAN SEKANINA



Milan Sekanina, our distinguished fellow and many years' member of the editorial board of Archivum Mathematicum died on 21st October, 1987 at the age of fifty-six. He influenced considerably the development of mathematics in Czechoslovakia. Topology, the theory of ordered sets, algebra and the theory of graphs were the main fields of his interest. Widely there are known, for instance, his results on hamiltonian properties of powers of graphs. M. Sekanina is also an author of several textbooks of mathematics. The editors of Archivum Mathematicum dedicate this volume to the memory of Milan Sekanina.

V. Novák, J. Rosický

M. SEKANINA

THE LIST OF MILAN SEKANINA'S SCIENTIFIC PUBLICATIONS

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TRANSITIVE TERNARY RELATIONS AND QUASIORDERINGS

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Dedicated to the memory of Milan Sekanina

Abstract. Two operators are described which enable to construct a quasiordering from a transitive ternary structure and vice versa.

Key words. Relational structure, transitive and asymmetric ternary relation, quasiordering, strong homomorphism.
MS Classification. 06 A 10, 04 A 05.

0. INTRODUCTION

Some authors have studied cyclically ordered sets, e.g. E. Čech [4] who has used a cyclic order to define an orientation of a closed curve, G. Müller [6], N. Megiddo [5], P. Alles [1] and others. A cyclic order is a nontrivial example of a relation with arity greater than 2; thus a natural question arises, which problems of the theory of ordered sets can be posed for cyclically ordered sets (e.g. dimension theory [8], completion [10], representation theory [9] a.s.o.). A great disadvantage of these investigations is the fact that there is no simple realisation of a ternary relation. This paper is an attempt to construct ternary relations from binary relations and vice versa with preservation of transitivity. The relationship between binary and ternary relations were studied in literature. So G. Birkhoff [3] posed the problem of a connection of a partial order and corresponding relation betweeness; this problem was solved by M. Altwegg [2]. M. Sekanina studied the relation betweeness in graphs [11].

1. BASIC NOTIONS

Let $G \neq \emptyset$ be a set, $n \geqq 1$ an integer and R an n-ary relation on G. The pair $\boldsymbol{G} = (G, R)$ will be called an n-ary structure. If $\boldsymbol{G} = (G, R)$ is an n-ary structure, then the set G is called a carrier of the structure $\boldsymbol{G}$ and denoted $G = c(\boldsymbol{G})$, and the set R is called a relation of the structure $\boldsymbol{G}$ and denoted $R = r(\boldsymbol{G})$.

Let $\boldsymbol{G}$ be an n-ary structure, $x \in c(\boldsymbol{G})$. We call the element x isolated, if for any $(x_1, \ldots, x_n) \in r(\boldsymbol{G})$ we have $x \neq x_i$ for all $i = 1, \ldots, n$; otherwise it is nonisolated.

Let $\boldsymbol{G}, \boldsymbol{H}$ be n-ary structures, $f : c(\boldsymbol{G}) \to c(\boldsymbol{H})$ be a mapping. f is called a homomorphism of $\boldsymbol{G}$ into $\boldsymbol{H}$ iff

$$x_1, \ldots, x_n \in c(\boldsymbol{G}), (x_1, \ldots, x_n) \in r(\boldsymbol{G}) \Rightarrow (f(x_1), \ldots, f(x_n)) \in r(\boldsymbol{H}).$$

A homomorphism f of $\boldsymbol{G}$ into $\boldsymbol{H}$ is called strong, iff it is surjective and it holds

$$y_1, \ldots, y_n \in c(\boldsymbol{H}), (y_1, \ldots, y_n) \in r(\boldsymbol{H}) \Rightarrow \text{there exist}$$
$$x_1 \in f^{-1}(y_1), \ldots, x_n \in f^{-1}(y_n) \text{ with } (x_1, \ldots, x_n) \in r(\boldsymbol{G}).$$

A bijective strong homomorphism is an isomorphism. Two n-ary structures $\boldsymbol{G}, \boldsymbol{H}$ are called isomorphic iff there exists an isomorphism of $\boldsymbol{G}$ onto $\boldsymbol{H}$.

In the sequel we shall deal only with binary and ternary structures. Recall that a binary relation which is reflexive and transitive is a quasiordering; a binary structure $\boldsymbol{G}$ in which $r(\boldsymbol{G})$ is a quasiordering is a quasiordered set. A quasiordering which is antisymmetric is an ordering; a binary structure $\boldsymbol{G}$ in which $r(\boldsymbol{G})$ is an ordering is an ordered set.

Let R be a ternary relation on a set G. We shall call this relation

transitive, iff $(x, y, z) \in R$, $(z, y, u) \in R \Rightarrow (x, y, u) \in R$,
antisymmetric, iff $(x, y, z) \in R$, $(z, y, x) \in R \Rightarrow x = z$.

A ternary structure $\boldsymbol{G}$ is called transitive, resp. antisymmetric, iff $r(\boldsymbol{G})$ is transitive, resp. antisymmetric ternary relation.

Let $\boldsymbol{G}$ be a ternary structure. Put

$D(\boldsymbol{G}) = \{(x, y, x) \in (c(\boldsymbol{G}))^3$; there exists $z \in c(\boldsymbol{G})$ with either $(x, y, z) \in r(\boldsymbol{G})$ or $(z, y, x) \in r(\boldsymbol{G})\}$,

$A(\boldsymbol{G}) = r(\boldsymbol{G}) \cup D(\boldsymbol{G})$.

In the whole paper, the symbols $D(\boldsymbol{G})$, $A(\boldsymbol{G})$ will have just this meaning.

Trivially, it holds

1.1. Lemma. *Let $\boldsymbol{G}$ be a ternary structure, $x, y \in c(\boldsymbol{G})$. If $(x, y, x) \in r(\boldsymbol{G})$, then $(x, y, x) \in D(\boldsymbol{G})$.*

Further, we prove

1.2. Lemma. *Let $\boldsymbol{G}$ be a ternary structure. If the relation $r(\boldsymbol{G})$ is transitive, then $A(\boldsymbol{G})$ is transitive.*

Proof. Let $(x, y, z) \in A(\boldsymbol{G})$, $(z, y, u) \in A(\boldsymbol{G})$. If $z \neq x$, $z \neq u$, then $(x, y, z) \in r(\boldsymbol{G})$, $(z, y, u) \in r(\boldsymbol{G})$ and $(x, y, u) \in r(\boldsymbol{G}) \subseteq A(\boldsymbol{G})$ for $r(\boldsymbol{G})$ is transitive. If $z = x$, then $(x, y, u) \in A(\boldsymbol{G})$; similarly for $z = u$. Thus $A(\boldsymbol{G})$ is transitive.

2. OPERATOR $\mathcal{Q}$

Let G be a ternary structure. Put

$B(G) = \{((x, y, x), (z, y, z)) \in D(G) \times D(G); (x, y, z) \in A(G)\}$,
$\mathcal{Q}(G) = (D(G), B(G))$.
Thus, $\mathcal{Q}(G)$ is a binary structure with carrier $D(G)$.

2.1. Lemma. *Let G be a ternary structure. Then the binary structure $\mathcal{Q}(G)$ is reflexive.*

Proof. Let $(x, y, x) \in D(G)$. Then $(x, y, x) \in A(G)$, thus $((x, y, x), (x, y, x)) \in B(G)$ and $B(G) = r(\mathcal{Q}(G))$ is reflexive.

2.2. Lemma. *Let G be a ternary structure. Then it holds:*
(1) *If G is transitive, then $\mathcal{Q}(G)$ is a transitive binary structure,*
(2) *If $r(G) = A(G)$ and $\mathcal{Q}(G)$ is transitive, then G is transitive.*

Proof. (1) Let G be transitive and $(x, y, x), (z, y, z), (u, y, u) \in D(G) = c(\mathcal{Q}(G))$, $((x, y, x), (z, y, z)) \in B(G) = r(\mathcal{Q}(G))$, $((z, y, z), (u, y, u)) \in B(G)$. Then, by definition, $(x, y, z) \in A(G)$, $(z, y, u) \in A(G)$ and by 1.2. $(x, y, u) \in A(G)$. From this $((x, y, x), (u, y, u)) \in B(G)$ and $B(G)$ is transitive.

(2) Let $r(G) = A(G)$ and $\mathcal{Q}(G)$ be transitive. Let $x, y, z, u \in c(G)$, $(x, y, z) \in r(G)$, $(z, y, u) \in r(G)$. Then (x, y, x), (z, y, z), $(u, y, u) \in D(G) = c(\mathcal{Q}(G))$ and $((x, y, x), (z, y, z)) \in B(G) = r(\mathcal{Q}(G))$, $((z, y, z), (u, y, u)) \in B(G)$. The transitivity of $B(G)$ yields $((x, y, x), (u, y, u)) \in B(G)$ which means $(x, y, u) \in A(G) = r(G)$. Thus $r(G)$ is transitive.

From 2.1. and 2.2. it follows

2.3. Theorem. *Let G be a ternary structure. Then it holds:*
(1) *If G is transitive, then $\mathcal{Q}(G)$ is quasiordered set,*
(2) *If $r(G) = A(G)$, then G is transitive iff $\mathcal{Q}(G)$ is a quasiordered set.*

2.4. Lemma. *Let G be a ternary structure. Then it holds:*
(1) *If the binary structure $\mathcal{Q}(G)$ is antisymmetric, then G is antisymmetric.*
(2) *If $r(G) = A(G)$ and G is antisymmetric, then $\mathcal{Q}(G)$ is antisymmetric.*

Proof. (1) Let $\mathcal{Q}(G)$ be antisymmetric and $x, y, z \in c(G)$, $(x, y, z) \in r(G)$, $(z, y, x) \in r(G)$. Then $(x, y, x), (z, y, z) \in D(G) = c(\mathcal{Q}(G))$ and $((x, y, x), (z, y, z)) \in B(G) = r(\mathcal{Q}(G))$, $((z, y, z), (x, y, x)) \in B(G)$. The antisymmetry of $B(G)$ gives $(x, y, x) = (z, y, z)$, thus $x = z$ and $r(G)$ is antisymmetric.

(2) Let $r(G) = A(G)$ and G be antisymmetric. Let $(x, y, x), (z, y, z) \in D(G) = c(\mathcal{Q}(G))$, $((x, y, x), (z, y, z)) \in B(G) = r(\mathcal{Q}(G))$, $((z, y, z), (x, y, x)) \in B(G)$. Then $(x, y, z) \in A(G) = r(G)$, $(z, y, x) \in r(G)$ and antisymmetry of $r(G)$ yields $x = z$. Thus $(x, y, x) = (z, y, z)$ and $B(G) = r(\mathcal{Q}(G))$ is antisymmetric.

From 2.3. and 2.4. we get immediately

2.5. Theorem. *Let G be a ternary structure with the property $r(G) = A(G)$. Then G is transitive and antisymmetric if and only if $\mathcal{Q}(G)$ is an ordered set.*

3. OPERATOR $\mathcal{T}$

Let G be a binary structure. Let Θ be the least equivalence on $c(G)$, containing $r(G)$ and p be the natural projection of $c(G)$ onto $c(G)/_\Theta$. Put

$E(G)) = c(G) \cup c(G)/_\Theta$,

$F(G)) = \{(x, y, z); x, z \in c(G), y \in c(G)/_\Theta, (x, z) \in r(G), p(x) = p(z) = y\}$,

$\mathcal{T}(G) = (E(G), F(G))$.

Thus, $\mathcal{T}(G)$ is a ternary structure with carrier $E(G) = c(G) \cup c(G)/_\Theta$.

3.1. Lemma. *Let G be a binary structure. Then it holds:*

(1) *If G is reflexive, then the ternary structure $\mathcal{T}(G)$ satisfies $r(\mathcal{T}(G)) = A(\mathcal{T}(G))$,*

(2) *If G contains no isolated elements and if $r(\mathcal{T}(G)) = A(\mathcal{T}(G))$, then G is reflexive.*

Proof. (1) Assume that G is reflexive and that $A(\mathcal{T}(G)) - r(\mathcal{T}(G)) \neq \emptyset$. Let $m \in A(\mathcal{T}(G)) - r(\mathcal{T}(G))$ be any element. Then $m \in D(\mathcal{T}(G))$, thus $m = (x, y, x)$, where $x, y \in c(\mathcal{T}(G))$ and there exists $z \in c(\mathcal{T}(G))$ with either $(x, y, z) \in r(\mathcal{T}(G))$ or $(z, y, x) \in r(\mathcal{T}(G))$; say $(x, y, z) \in r(\mathcal{T}(G))$. This means $x \in c(G)$, $y = p(x)$ and as $r(G)$ is reflexive, we have $(x, x) \in r(G)$. From this it follows by definition $m = (x, y, x) \in F(G) = r(\mathcal{T}(G))$, a contradiction.

(2) Let G have no isolated elements, let $r(\mathcal{T}(G)) = A(\mathcal{T}(G))$ and assume that G is not reflexive. Then there exists an element $x \in c(G)$ with $(x, x) \bar{\in} r(G)$. Denote $p(x) = y$, thus $(x, y, x) \in F(G) = r(\mathcal{T}(G))$. As G has no isolated elements, there is an element $z \in c(G)$ satisfying either $(x, z) \in r(G)$ or $(z, x) \in r(G)$; let us say that $(x, z) \in r(G)$. Then $(x, y, z) \in F(G) = r(\mathcal{T}(G))$ and by definition it is $(x, y, x) \in D(\mathcal{T}(G)) \subseteq A(\mathcal{T}(G))$. Thus $(x, y, x) \in A(\mathcal{T}(G)) - r(\mathcal{T}(G))$, a contradiction.

3.2. Lemma. *Let G be a binary structure. Then G is transitive iff $\mathcal{T}(G)$ is a transitive ternary structure.*

Proof. 1. Let G be transitive and $x, y, z, u \in c(\mathcal{T}(G)) = E(G)$, $(x, y, z) \in r(\mathcal{T}(G)) = F(G)$, $(z, y, u) \in F(G)$. Then, by definition, $x, z, u \in c(G)$, $y \in c(G)/_\Theta$, and it holds $(x, z) \in r(G)$, $p(x) = p(z) = y$, $(z, u) \in r(G)$, $p(z) = p(u) = y$. As $r(G)$ is transitive, we have $(x, u) \in r(G)$ and $p(x) = p(u) = y$. Thus $(x, y, u) \in F(G)$ and $F(G) = r(\mathcal{T}(G))$ is transitive.

2. Let $F(G)$ be transitive ternary relation on $E(G)$ and let $x, y, z \in c(G)$, $(x, y) \in r(G)$, $(y, z) \in r(G)$. Then $(x, y) \in \Theta$, $(y, z) \in \Theta$, so that, if we denote $p(x) = u$, we have $p(y) = p(z) = u$. By definition of the relation $F(G)$ it is $(x, u, y) \in F(G)$, $(y, u, z) \in F(G)$ and transitivity of $F(G)$ yields $(x, u, z) \in F(G)$. This means $(x, z) \in r(G)$ and $r(G)$ is transitive.

From 3.1. and 3.2. we get

3.3. Theorem. *Let G be a binary structure. Then it holds:*

(1) *If G is a quasiordered set, then $\mathcal{T}(G)$ is a transitive ternary structure with the property $r(\mathcal{T}(G)) = A(\mathcal{T}(G))$.*

(2) *If G contains no isolated elements, then G is quasiordered set iff $\mathcal{T}(G)$ is a transitive ternary structure with the property $r(\mathcal{T}(G)) = A(\mathcal{T}(G))$.*

3.4. Lemma. *Let G be a binary structure. Then G is antisymmetric iff the ternary structure $\mathcal{T}(G)$ is antisymmetric.*

Proof. 1. Let G be antisymmetric and let $x, y, z \in c(\mathcal{T}(G)) = E(G)$, $(x, y, z) \in \in r(\mathcal{T}(G)) = F(G)$, $(z, y, x) \in F(G)$. Then $x, z \in c(G)$, $p(x) = p(z) = y$, $(x, z) \in r(G)$, $(z, x) \in r(G)$. The antisymmetry of $r(G)$ yields $x = z$ and thus $F(G) = r(\mathcal{T}(G))$ is antisymmetric.

2. Let $F(G)$ be antisymmetric and let $x, y \in c(G)$, $(x, y) \in r(G)$, $(y, x) \in r(G)$. Then $(x, y) \in \Theta$ and if we denote $p(x) = p(y) = u$, we have $(x, u, y) \in F(G)$, $(y, u, x) \in \in F(G)$. As $F(G)$ is antisymmetric, it is $x = y$ and thus $r(G)$ is antisymmetric.

From 3.3. and 3.4. we now get

3.5. Theorem. *Let G be a binary structure. Then it holds:*

(1) *If G is an ordered set, then $\mathcal{T}(G)$ is a transitive and antisymmetric ternary structure with the property $r(\mathcal{T}(G)) = A(\mathcal{T}(G))$,.*

(2) *If G contains no isolated elements, then G is an ordered set iff $\mathcal{T}(G)$ is a transitive and antisymmetric ternary structure with the property $r(\mathcal{T}(G)) = A(\mathcal{T}(G))$.*

4. OPERATORS $\mathcal{Q} \circ \mathcal{T}$ AND $\mathcal{T} \circ \mathcal{Q}$

4.1. Theorem. *Let G be a quasiordered set. Then the structures G and $\mathcal{Q}(\mathcal{T}(G))$ are isomorphic.*

Proof. By definition, it is $\mathcal{T}(G) = (E(G), F(G))$ where $E(G) = c(G) \cup c(G)/_\Theta$, and $\mathcal{Q}(\mathcal{T}(G)) = (D(\mathcal{T}(G)), B(\mathcal{T}(G)))$. Put for any $x \in c(G)$ $f(x) = (x, p(x), x)$. As $(x, x) \in r(G)$, it is $f(x) \in F(G) = r(\mathcal{T}(G))$ and by 1.1. $f(x) \in D(\mathcal{T}(G))$. Thus, f is a mapping of $c(G)$ into $D(\mathcal{T}(G)) = c(\mathcal{Q}(\mathcal{T}(G)))$. Let $w \in D(\mathcal{T}(G))$ be any element. Then $w = (x, y, x) \in (c(\mathcal{T}(G)))^3 = (E(G))^3$ and there exists an element $z \in E(G)$ such that either $(x, y, z) \in r(\mathcal{T}(G)) = F(G)$ or $(z, y, x) \in F(G)$. This means $x, z \in c(G)$, $y \in c(G)/_\Theta$, $p(x) = p(z) = y$ and either $(x, z) \in r(G)$ or $(z, x) \in r(G)$. But then $f(x) = (x, y, x) = w$ and the mapping f is surjective.

Let $x, y \in c(G)$, $f(x) = f(y)$. Then $(x, p(x), x) = (y, p(y), y)$, thus $x = y$. The mapping f is injective, hence a bijection of $c(G)$ onto $c(\mathcal{Q}(\mathcal{T}(G)))$. Let $x, y \in c(G)$, $(x, y) \in r(G)$. Then $(x, y) \in \Theta$, thus $p(x) = p(y) = u \in c(G)/_\Theta$ and $(x, u, y) \in F(G) = = r(\mathcal{T}(G))$. By 3.1. we have $r(\mathcal{T}(G)) = A(\mathcal{T}(G))$. Further, it is $f(x) = (x, u, x) \in \in D(\mathcal{T}(G))$, $f(y) = (y, u, y) \in D(\mathcal{T}(G))$ and by definition we have $((x, u, x), (y, u, y)) = = (f(x), f(y)) \in B(\mathcal{T}(G)) = r(\mathcal{Q}(\mathcal{T}(G)))$. Thus f is a bijective homomorphism of G onto $\mathcal{Q}(\mathcal{T}(G))$.

Let $x, y \in c(G)$ and $(f(x), f(y)) \in r(\mathcal{Q}(\mathcal{T}(G))) = B(\mathcal{T}(G))$. It is, of course, $f(x) = (x, u, x)$, $f(y) = (y, v, y)$ where $u = p(x)$, $v = p(y)$. By definition of the relation $B(\mathcal{T}(G))$ it is $u = v$, i.e. $p(x) = p(y)$, and $(x, u, y) \in A(\mathcal{T}(G))$. By 3.1. it is $A(\mathcal{T}(G)) = r((\mathcal{T}G)) = F(G)$ and this implies, by definition of the relation $F(G)$, $(x, y) \in r(G)$. Thus f is an isomorphism of G onto $\mathcal{Q}(\mathcal{T}(G))$.

4.2. Theorem. *Let G be a transitive ternary structure containing no isolated elements and such that $r(G) = A(G)$. Then there exists a strong homomorphism of $\mathcal{T}(\mathcal{Q}(G))$ onto G.*

Proof. By definition, it is $\mathcal{Q}(G) = (D(G), B(G))$, and $\mathcal{T}(\mathcal{Q}(G)) = (E(\mathcal{Q}(G)), F(\mathcal{Q}(G)))$, where $E(\mathcal{Q}(G)) = c(\mathcal{Q}(G)) \cup c(\mathcal{Q}(G))/_\Theta$; here Θ is the least equivalence on $c(\mathcal{Q}(G)) = D(G)$ containing $r(\mathcal{Q}(G)) = B(G)$.

Let $u \in E(\mathcal{Q}(G))$. If $u \in c(\mathcal{Q}(G)) = D(G)$, then $u = (x, y, x)$, where $x, y \in c(G)$ and there exists $z \in c(G)$ with either $(x, y, z) \in r(G)$ or $(z, y, x) \in r(G)$. In this case we put $f(u) = x$. Suppose that $u \in D(G)/_\Theta$. Then there exists $m \in D(G)$ such that $p(m) = u$ where p is a natural projection of $D(G)$ onto $D(G)/_\Theta$. Thus $m = (x, y, x)$ where $x, y \in c(G)$. We show that for any $n = (x', y', x') \in D(G)$ with the property $p(n) = u$ we have $y' = y$. Indeed, $p(m) = p(n)$ means $(m, n) \in \Theta$ and thus either $m = n$ or there exist a positive integer $k > 1$ and elements $m_1, \dots, m_k \in D(G)$ such that $m_1 = m$, $m_k = n$ and $(m_i, m_{i+1}) \in B(G) \cup (B(G))^{-1}$ for all $i = 1, \dots, k - 1$. Let $(m_i, m_{i+1}) \in B(G)$. Then $m_i = (x_i, y_i, x_i)$, $m_{i+1} = (x_{i+1}, y_{i+1}, x_{i+1})$ and by definition of the relation $B(G)$ it is $y_i = y_{i+1}$. If $(m_i, m_{i+1}) \in (B(G))^{-1}$, then $(m_{i+1}, m_i) \in B(G)$ and we have again $y_i = y_{i+1}$. Thus $y_1 = y_2 = \dots = y_k$ and for $m = m_1 = (x_1, y_1, x_1) = (x, y, x)$, $n = m_k = (x_k, y_k, x_k) = (x', y', x')$ we have $y = y'$. Thus, any element $u \in D(G)/_\Theta$ determines just one element $y \in c(G)$ such that for some $x \in c(G)$ there is $p(x, y, x) = u$. We put $f(u) = y$. Thus, we have defined a mapping $f: c(\mathcal{T}(\mathcal{Q}(G))) \to c(G)$..

Let $x \in c(G)$ be any element. As G contains no isolated elements, there are elements $y, z \in c(G)$ such that either $(x, y, z) \in r(G)$ or $(y, x, z) \in r(G)$ or $(z, y, x) \in r(G)$. In the first and third case it is $u = (x, y, x) \in D(G) \subseteq E(\mathcal{Q}(G))$ and by definition of the mapping f we have $f(u) = x$. In the second case it is $(y, x, y) \in D(G)$, $v = p(y, x, y) \in D(G)/_\theta \subseteq E(\mathcal{Q}(G))$ and by definition we have $f(v) = x$. Thus f is a surjective mapping of $c(\mathcal{T}(\mathcal{Q}(G)))$ onto $c(G)$.

Let $u, v, w \in c(\mathcal{T}(\mathcal{Q}(G))) = E(\mathcal{Q}(G))$ and $(u, v, w) \in r(\mathcal{T}(\mathcal{Q}(G))) = F(\mathcal{Q}(G))$. Then, by definition of the relation $F(\mathcal{Q}(G))$, there is $u, w \in c(\mathcal{Q}(G)) = D(G)$, $v \in c(\mathcal{Q}(G))/_\Theta = D(G)/_\Theta$ and it holds $(u, w) \in r(\mathcal{Q}(G)) = B(G)$, $p(u) = p(w) = v$. As $u, w \in D(G)$ and $(u, w) \in B(G)$, there is $u = (x, y, x)$, $w = (z, y, z)$ for suitable $x, y, z \in c(G)$, and $(x, y, z) \in A(G) = r(G)$. By definition of the mapping f then $f(u) = x$, $f(v) = y$, $f(w) = z$ so that $(f(u), f(v), f(w)) \in r(G)$. We have proved that $f: c(\mathcal{T}(\mathcal{Q}(G))) \to c(G)$ is a surjective homomorphism of the structure $\mathcal{T}(\mathcal{Q}(G))$ onto structure G.

Let, at the end, $x, y, z \in c(G)$, $(x, y, z) \in r(G)$. If we denote $(x, y, x) = u$,

$(z, y, z) = w$, then $u, w \in D(G) = c(\mathscr{Q}(G))$ and $(u, w) \in B(G) = r(\mathscr{Q}(G))$. Thus $(u, w) \in \in \Theta$ so that $p(u) = p(w)$. Denote $p(u) = p(w) = v$; then $u, v, w \in E(\mathscr{Q}(G))$ and $(u, v, w) \in F(\mathscr{Q}(G)) = r(\mathscr{T}(\mathscr{Q}(G)))$. At the same time, by definition of the mapping f, it is $f(u) = x$, $f(v) = y$, $f(w) = z$, i.e. $u \in f^{-1}(x)$, $v \in f^{-1}(y)$, $w \in f^{-1}(z)$. Thus the homomorphism f of $\mathscr{T}(\mathscr{Q}(G))$ onto G is strong.

In the last theorem, the structures G and $\mathscr{T}(\mathscr{Q}(G))$ need not be isomorphic, as the following example shows.

4.3. Example. Let $G = (c(G), r(G))$ be a ternary structure with $c(G) = \{0, 1, 2\}$ and $r(G) = \{(0, 1, 2), (1, 2, 0), (2, 0, 1), (0, 1, 0), (2, 1, 2), (1, 2, 1), (0, 2, 0), (2, 0, 2), (1, 0, 1)\}$. Evidently, G is transitive and $r(G) = A(G)$. Further, $D(G) = \{(0, 1, 0), (2, 1, 2), (1, 2, 1), (0, 2, 0), (2, 0, 2), (1, 0, 1)\}$, $B(G) = \{((0, 1, 0), (2, 1, 2)), ((1, 2, 1), (0, 2, 0)), ((2, 0, 2), (1, 0, 1)), ((0, 1, 0)), (0, 1, 0)), ((2, 1, 2), (2, 1, 2)), ((1, 2, 1), (1, 2, 1)), ((0, 2, 0), (0, 2, 0)), ((2, 0, 2), (2, 0, 2)), ((1, 0, 1), (1, 0, 1))\}$, and $\mathscr{Q}(G) = (D(G), B(G))$. The least equivalence on $D(G)$ containing $B(G)$ has blocks $B_0 = \{(1, 0, 1), (2, 0, 2)\}$, $B_1 = \{(0, 1, 0), (2, 1, 2)\}$, $B_2 = \{(0, 2, 0), (1, 2, 1)\}$ so that $E(\mathscr{Q}(G)) = D(G) \cup \{B_0, B_1, B_2\}$, and $F(\mathscr{Q}(G)) = \{((0, 1, 0), B_1, (2, 1, 2)), ((1, 2, 1), B_2, (0, 2, 0)), ((2, 0, 2), B_0, (1, 0, 1)), ((0, 1, 0), B_1, (0, 1, 0)), ((2, 1, 2), B_1, (2, 1, 2)), ((1, 2, 1), B_2, (1, 2, 1)), ((0, 2, 0,) B_2, (0, 2, 0)), ((2, 0, 2), B_0, (2, 0, 2)), ((1, 0, 1), B_0, (1, 0, 1))\}$. Thus $\mathscr{T}(\mathscr{Q}(G)) = (E(\mathscr{Q}(G)), F(\mathscr{Q}(G)))$ and as $c(G)$ has 3 elements, $c((\mathscr{T}\mathscr{Q}(G))) = E(\mathscr{Q}(G))$ has 9 elements the structures G and $\mathscr{T}(\mathscr{Q}(G))$ cannot be isomorphic. If we put $f((0, 1, 0)) = f((0, 2, 0)) = 0$, $f((1, 0, 1)) = f((1, 2, 1)) = 1$, $f((2, 0, 2)) = f((2, 1, 2)) = 2$, $f(B_0) = 0$, $f(B_1) = 1$, $f(B_2) = 2$, then f is a strong homomorphism of $\mathscr{T}(\mathscr{Q}(G))$ onto G.

4.4. Remark. Denote *Quas* the category of quasiordered sets with isotonic mappings as morphisms and *Tern* the category of transitive ternary structures without isolated elements and such that $r(G) = A(G)$ with obviously defined morphisms. For morphisms $h : G \to G'$ $(G, G') \in Tern$ and $k : Q \to Q'$ $(Q, Q' \in \in Quas)$ define $\mathscr{Q}(h) : \mathscr{Q}(G) \to \mathscr{Q}(G')$ and $\mathscr{T}(k) : \mathscr{T}(Q) \to \mathscr{T}(Q')$ in an expected way. Then $\mathscr{Q} : Tern \to Quas$ and $\mathscr{T} : Quas \to Tern$ are covariant functors.

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RETRACTS OF ABELIAN CYCLICALLY ORDERED GROUPS

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Abstract. In this paper it will be shown that a nonzero subgroup H of a cyclically ordered group G is a retract of G if and only if H is a large lexicographic factor of G.

Key words. Cyclically ordered group, retract, retract mapping, lexicographic product.
MS Classification. 06 F 20, 46 A 40.

Cyclically ordered groups were investigated in [1], [9], ..., [15]. The notion of cyclically ordered group is a generalization of the notion of linearly ordered group.

Retracts of partially ordered sets were studied in [2], ... ,[5].

Retracts of lattice ordered groups, and in particular, of linearly ordered groups, were investigated in [4]; cf. also [5].

All cyclically ordered groups dealt with in the present note are assumed to be abelian.

Let G be a cyclically ordered group. An endomorphism f of G will be said to be a retract mapping if $f(f(x) = f(x)$ for each $x \in G$. In such a case, the set $f(G)$ is called a retract of G.

It will be shown that to each retract of G there corresponds a two-factor lexicographic decomposition of G. More thoroughly, each retract mapping of G is a projection onto a large lexicographic factor of G, and conversely. This generalizes a result of [7] concerning retracts of linearly ordered groups.

1. PRELIMINARIES

For the sake of completeness we recall the definition of cyclically ordered group.

Let G be a group (the group operation will be denoted additively). Suppose that there is defined a ternary relation $[x, y, z]$ on G such that the following conditions are satisfied for each x, y, z, a, b a G:

I. If $[x, y, z]$ holds, then x, y and z are distinct; if x, y and z are distinct, then either $[x, y, z]$ or $[z, y, x]$.

II. $[x, y, z]$ implies $[y, z, x]$.
III. $[x, y, z]$ and $[y, u, z]$ imply $[x, u, z]$.
IV. $[x, y, z]$ implies $[a + x + b, a + y + b, a + z + b]$.

Under these assumptions G is said to be a cyclically ordered group; the ternary relation under consideration is said to be a cyclic order on G.

Each subgroup of G is considered as to be cyclically ordered under the induced cyclic order. The isomorphism of cyclically ordered groups is defined in the obvious way.

Let G and G' be cyclically ordered groups. A mapping $f: G \to G'$ is said to be a homomorphism if the following conditions are satisfied:

(i) f is a homomorphism with respect to the group operation;

(ii) whenever x, y and z are elements of G such that $[x, y, z]$ holds in G and the elements $f(x), f(y), f(z)$ are distinct, then the relation $[f(x), f(y), f(z)]$ is valid in G'.

Let L be a linearly ordered group. For distinct elements x, y and z of L we put $[x, y, z]$ if

$$x < y < z \quad \text{or} \quad y < z < x \quad \text{or} \quad z < y < x \tag{1}$$

is valid. Then G with the relation [ ] (which is said to be induced by the linear order) turns out to be a cyclically ordered group.

2. LEXICOGRAPHIC PRODUCTS

Let G_1 be a cyclically ordered group and let L be a linearly ordered group (each linearly ordered group is considered as to be cyclically ordered under the induced cyclic order).

Let $G_1 \times L$ be the (external) direct product of the groups G_1 and L. For distinct elements $u = (a, x)$, $v = (b, y)$ and $w = (c, z)$ of $G_1 \times L$ we put $[u, v, w]$ if some of the following conditions is satisfied:

(i) $[a, b, c]$;
(ii) $a = b \neq c$ and $x < y$;
(iii) $b = c \neq a$ and $y < z$;
(iv) $c = a \neq b$ and $z < x$;
(v) $a = b = c$ and $[x, y, z]$.

It is easy to verify that $G_1 \times L$ with this ternary relation is a cyclically ordered group; it will be denoted by $G_1 \oplus L$ and it is said to be a lexicographic product of G_1 and L. We call G_1 and L the large lexicographic factor or the small lexicographic factor of $G_1 \oplus L$, respectively.

If $G = G_1 \oplus L$, $g \in G$, $g = (u, x)$, then we denote $u = g(G_1)$ and $x = g(L)$.

Let us remark that if H_1 and H_2 are linearly ordered groups and if H is their lexicographic product $H_1 \circ H_2$ (cf., e.g., Fuchs [6]), then the cyclically ordered group H is a lexicographic product $H_1 \oplus H_2$ of the cyclically ordered groups H_1 and H_2, and conversely.

The following assertion is obvious.

2.1. Lemma. *Let G_1 be cyclically ordered groups and let L be a linearly ordered group. Put $G = G_1 \oplus L$ and for each $g \in G$ let $f(g) = g(G_1)$. Then f is a retract mapping of G.*

Let G_1 and L be as above and let φ be an isomorphism of a cyclically ordered group G onto $G_1 \oplus L$. Put

$$G_1^0 = \varphi^{-1}\{(a, 0) : a \in G_1\},$$

$$L^0 = \varphi^{-1}\{(0, x) : x \in L\}.$$

Then G_1^0 is isomorphic to G_1, and L^0 is isomorphic to L^0. The mapping

$$\varphi' : G \to G_1^0 \oplus L^0$$

defined by $\varphi'(g) = a^0 + x^0$, where $\varphi(g) = (a, x)$, $a^0 = \varphi^{-1}((a, 0))$ and $x^0 = \varphi^{-1}((0, x))$, is an isomorphism of G onto $G_1^0 \oplus L^0$. In such a case we write $G = G_1^0 \oplus_i L^0$ and G is said to be an internal lexicographic product of G_1^0 and L^0.

Analogously as above, G_1^0 and L^0 are called a large lexicographic factor and a small lexicographic factor of G, respectively.

In view of 2.1 we obtain:

2.2. Corollary. *Each large lexicographic factor of a cyclically ordered group G is a retract of G.*

Internal lexicographic product decompositions can be characterized intrinsically as follows.

2.3. Proposition. *Let G be a cyclically ordered group. Let G_1 and L be subgroups of G such that L is linearly ordered. Then the following conditions are equivalent:*

(a) $G = G_1 \oplus_i L$.

(b) The group G is an internal direct product of its subgroups G_1 and L. Whenever u, v and w are distinct elements of G with $u = a + x$, $v = b + y$, $w = x + z$ (where $a, b, c \in G_1$ and $x, y, z \in L$), then $[u, v, w]$ is valid if and only if some of the relations (i)–(v) above holds.

The proof can be perfoımed by a routine verification. (Cf. also [10].)

Let us denote by K the set of all real numbers x with $0 \leq x < 1$; the operation $+$ on K is defined to be the addition mod 1. For distinct elements x, y and z of K we put $[x, y, z]$ if the relation (1) above is valid. Then K is a cyclically ordered group.

2.4. Theorem. (Cf. [12].) *Let G be a cyclically ordered group. Then there exist a subgroup K_1 of K and a linearly ordered group L such that G is isomorphic to $K_1 \oplus L$.*

A subgroup H of a cyclically ordered group G is said to be c-convex (cf. [9]) if some of the following conditions is fulfilled:

(i) $H = G$;

(ii) for each $h \in H$ with $h \neq 0$ we have $2h \neq 0$; if $h \in H$, $g \in G$, $[-h, 0, h]$ and $[-h, g, h]$, then $g \in H$.

The following lemma is an easy consequence of 2.4.

2.5. Lemma. *Let f be an endomorphism of a cyclically ordered group G. Then the kernel of f is a c-convex subgroup of G.*

3. LARGE LEXICOGRAPHIC FACTOR CORRESPONDING TO A GIVEN NONZERO RETRACT MAPPING

Let G be a cyclically ordered group. In view of the consideration performed in Section 2 and according to 2.4 there exist subgroups G_1 and L of G such that

(i) G_1 is isomorphic to a subgroup of K;

(ii) L is linearly ordered;

(iii) $G = G_1 \oplus_i L$.

3.1. Lemma. *Let f be an endomorphism of G. Then either $f(G) = \{0\}$ or $f^{-1}(0) \subseteq L_1$.*

Proof. This is a consequence of 2.5, and [9] (3.5 and 4.6).

An endomorphism f of G is said to be nonzero if $f(G) \neq \{0\}$. In what follows we assume that f is a nonzero endomorphism of G.

3.2. Lemma. *Assume that f is a retract mapping of G. Then $f(x) \in L$ for each $x \in L$.*

Proof. By way of contradiction, assume that there exists an element $x \in L$ such that $f(x) \notin L$. Thus there are $a \in G_1$ and $y \in L$ with $f(x) = a + y$, $a \neq 0$. This yields that $f(a + y) = a + y$, hence $f(a + y - x) = 0$. The element $a + y - x$ does not belong to L, therefore the kernel of f fails to be a subset of L. In view of 3.1, $f(G) = \{0\}$, which is a contradiction.

Denote $f_2 = f \mid L$. According to 3.2 we have

3.3. Corollary. *Let f be as in 3.2. Then f_2 is a retract mapping of L.*

3.4. Lemma. *Let f be as in 3.2. Next let $f_1 = f \mid G_1$. Then f_1 is an isomorphism of G_1 onto $f(G_1)$.*

Proof. According to the definition, f_1 is a homomorphism of G_1 onto $f(G_1)$. Let $a \in G_1$, $a \neq 0$, $f(a) = a_1 + x$, $a_1 \in G_1$, $x \in L$. Hence $f(a_1 + x) = a_1 + x$, thus $f(-a + a_1 + x) = 0$. In view of 3.1, $-a + a_1 + x \in L$ and therefore $a = a_1$. Hence $f(a) \neq 0$. Thus f_1 is a monomorphism. By summarizing, f_1 is an isomorphism.

We have clearly $f(G_1) \cap L = \{0\}$. If $g \in G$ and $g = a + x$, $a \in G_1$, $x \in L$, $f(a) = a + x_1$, then $g = (a + x_1) + (-x_1 + x)$ with $a + x_1 \in f(G_1)$ and $-x_1 + + x \in L$. Hence we infer:

3.5. Lemma. *The group G is a direct product of the groups $f(G_1)$ and L.*

3.6. Lemma. *Let f be as in 3.2. Then $G = f(G_1) \oplus_i L$.*

The proof consists in a routine verification by applying 3.5 and 2.3.

3.7. Lemma. *Let f_2 be as above. There are subgroups L_1 and L_2 of L such that $f_2(L) = L_1$ and $L = L_1 \oplus_i L_2$.*

Proof. Since L is linearly ordered and since in view of 3.2, f_2 is a retract mapping of L as cyclically ordered group, it is also a retract mapping of L as linearly ordered group. Thus, according to [7], Theorem 3.4, there are l-subgroups L_1 and L_2 of L such that

(2) $$L = (i)\, L_1 \circ L_2,$$

(an internal lexicographic product of linearly ordered groups L_1 and L_2, cf. [7]). From (2) we obtain that the relation

$$L = L_1 \oplus_i L_2.$$

holds.

Put $L_3 = f(G_1) + L_1$. The relation $L_1 \subseteq L$ and Lemma 3.6 yield

(3) $$L_3 = f(G_1) \oplus_i L_1.$$

Next, from $f(L) = L_1$ we obtain

(4) $$f(G) = L_3.$$

Also, from 3.6 and 3.7 we infer that

(5) $$G = f(G_1) \oplus_i (L_1 \oplus_i L_2).$$

Clearly

$$f(G_1) \oplus_i (L_1 \oplus_i L_2) = (f(G_1) \oplus_i L_1) \oplus_i L_2 = f(G) \oplus_i L_2.$$

Thus in view of (5) we obtain

(6) $$G = f(G) \oplus_i L_2.$$

Let $g \in G$. In view of (6) there are uniquely determined elements $a \in g(G)$ and $x \in L_2$ such that $g = a + x$. Then $f(a) = a$. Next we have $f(x) \in f(G)$ and in view of 3.2, $f(x) \in L_2$. Hence $f(x) \in f(G) \cap L_2 = \{0\}$ and so $f(x) = 0$. We obtain

$$f(g) = f(a) + f(x) = a.$$

By summarizing, we have the following result:

3.8. Theorem. *Let f be a nonzero retract mapping of an abelian cyclically ordered group G. Then the retract f(G) is a large lexicographic factor of G and for each $g \in G$, f(g) is the component of the element g in the factor f(G).*

Theorem 3.8 and Lemma 3.1 yield:

3.9. Corollary. *Let G be an abelian cyclically ordered group and let $H \neq \{0\}$ be an l-subgroup of G. Then the following conditions are equivalent:*

(i) *H is a retract of G.*

(ii) *H is a large lexicographic factor of G.*

This generalizes Theorem 3.4, [7] concerning retracts of linearly ordered groups.

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ON EQUALITY OF EDGE-CONNECTIVITY AND MINIMUM DEGREE OF A GRAPH

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Dedicated to the memory of Milan Sekanina

Abstract. Sufficient conditions for the equality of edge-connectivity and minimum degree of a graph or a bipartite graph are presented. Also previously known conditions are surveyed.

Key words. Graph, bipartite graph, edge-connectivity, minimum degree, distance.
MS Classification. 05 C 40, 05 C 38.

Our terminology is based on [1]. Given a graph G, $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively; $n := V(G)$ is its order; $\lambda(G)$ is its edge-connectivity and $\delta(G)$ is the minimum degree of G. The distance between two vertices x and y is denoted $d(x, y)$ and diam (G) is the diameter of G. The vertex neighbourhood of a vertex x is denoted $V(x)$. For brevity, λ often stands for $\lambda(G)$ and δ for $\delta(G)$.

It is well known that $\lambda(G) \leqq \delta(G)$ and one may ask for conditions on G ensuring the equality $\lambda(G)$ and $\delta(G)$. In this paper we give first a survey of known sufficient conditions and then provide some new ones.

§ 1. A SURVEY OF KNOWN RESULTS

In this section we will give a series of known conditions ensuring $\lambda = \delta$ in terms of various parameters of a graph. Each of these conditions can be also referred to as a result, in which case it is meant the assertion that the condition yields $\lambda = \delta$.

The first such condition is due to Chartrand [3]:

(1) $$\delta(G) \geqq [n/2].$$

This was refined by Lesniak [6]:

(2) $$\deg(x) + \deg(y) \geqq n - 1$$

for any pair of nonadjacent vertices x, y.

The following result of Plesník [7] is based on the diameter and obviously implies results (1) and (2):

$$\text{diam}(G) \leqq 2. \tag{3}$$

Goldsmith and Entringer [5] observed: It is also sufficient that for each vertex x of minimum degree, the vertices in the neighbourhood $V(x)$ have large degree sums; more precisely:

$$\sum_{w \in V(x)} \deg(w) \geqq \begin{cases} [n/2]^2 - [n/2] & \text{for all even } n \text{ and} \\ & \text{for odd } n \leqq 15, \\ [n/2]^2 - 7 & \text{for odd } n \geqq 15. \end{cases} \tag{4}$$

This result implies (1) but is independent of (2) and (3). Indeed, the graph in Fig. 1 fulfils (2) and (3) but not (4); on the other hand the graph from Fig. 2 fulfils (4) but not (3) or (2).

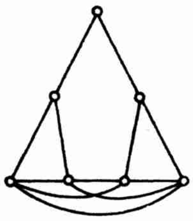

fig 1

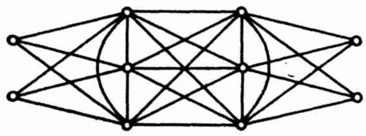

fig. 2

Bollobás [2] uses maximal graphs with $\delta > \lambda$ and derives several results. The following is a typical one and perhaps the most important of them: The degree sequence $d_1 \geqq d_2 \geqq \ldots \geqq d_n = \delta$ of G with $n \geqq 2$ fulfils

$$\sum_{i=1}^{k} (d_i + d_{n-i}) \geqq kn - 1 \tag{5}$$

for each k with $1 \leqq k \leqq \min\{[n/2] - 1, \delta\}$.

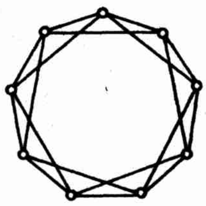

fig. 3

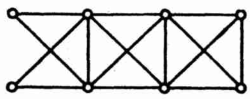

fig. 4

Although the result (5) implies (1) if n is even, in general (5) is independent of (1)–(4). This can be seen with aid of graphs in Figs. 3 and 4. The former fulfils (1)–(4) but not (5) and the latter works conversely.

Esfahanian [4] has given lower bounds on the edge-connectivity and, as a consequence, the following condition (Δ is the maximum degree of G and $D :=$ $:= \operatorname{diam}(G)$):

(6) $$n \geqq (\delta - 1)\frac{(\Delta - 1)^{D-1} + \Delta(\Delta - 2) - 1}{\Delta - 2} + 1.$$

The following similar condition is due to Soneoka, Nakada, Imase and Peyrat [8] and slightly improves (6):

(7) $$n > (\delta - 1)\frac{(\Delta - 1)^{D-1} + \Delta - 3}{\Delta - 2} + \Delta - 1.$$

As shown in [8] this bound is best possible (at least) for diameters $D = 3$ and 4. On the other hand, the graph of Fig. 3 does not fulfil (7) but fulfils (1)–(4).

Soneoka et al. [8] have established also the following generalization of (3) with g standing for the girth of G:

(8) $$D \leqq \begin{cases} g - 1 & \text{for } g \text{ odd,} \\ g - 2 & \text{for } g \text{ even.} \end{cases}$$

They show that this condition is best possible for an infinite number of values of δ when g is 4 or g is odd.

Figs. 2 and 4 provide examples of graphs fulfilling (4) and (5), respectively, and not fulfilling (8). Also there are examples in [8] where (7) works but (8) does not.

We conclude the survey by a result of Volkmann [9]:

(9) $$G \text{ is bipartite and } \delta \geqq \frac{n + 1}{4}.$$

Two disjoint copies of complete bipartite graph $K(n/4, n/4)$ provide an example demonstrating that this result is best possible. Moreover, it is not a corollary of (8), because there is a bipartite graph with $g = 4$ and $D > 2$ fulfilling (9) (e.g. with $n = 7$, $\delta = 2$). A generalization of (9) for p-partite graphs is given in [10].

§ 2. A NEW DISTANCE CONDITION

Here we show that the condition (3) can be slightly relaxed in sense that some distances can be greater than 2.

2.1. Theorem. *If in a connected graph no four vertices u_1, v_1, u_2, v_2 with*

(10) $$d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2) \geqq 3$$

exist, then $\lambda = \delta$.

Proof. For a contradiction consider a graph G fulfilling the distance condition with $\lambda < \delta$. Let E_0 be an edge cut of cardinality λ and let A and $\bar{A}$ be the vertex sets of the components arising after deleting E_0 from G. Further, let $A_1 \subseteq A$ and $\bar{A}_1 \subseteq \bar{A}$ be the sets of vertices incident with edges of E_0 and put $A_0 := A - A_1$ and $\bar{A}_0 := \bar{A} - \bar{A}_1$ (see Fig. 5). Denote the cardinalities of $A_0, A_1, \bar{A}_1$ and $\bar{A}_0$ by $a_0, a_1, \bar{a}_1$ and $\bar{a}_0$, respectively. Clearly $\lambda \geqq a_1$ and $\lambda \geqq \bar{a}_1$.

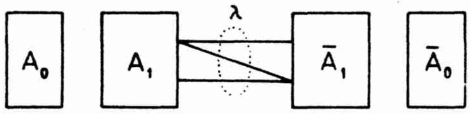

fig 5

The distance condition in our theorem implies that $a_0 \geqq 2$ and $\bar{a}_0 \geqq 2$ cannot hold simultaneously (otherwise there are $u_1, v_1 \in A_0$ and $u_2, v_2 \in \bar{A}_0$ fulfilling (10)). Thus owing to the reason of symmetry we can assume that $a_0 \leqq 1$. Each edge going from a vertex x of A ends in $A_0 \cup A_1$ or belongs to E_0. Since G has no loops or multiple edges, we have

$$\sum_{x \in A} \deg(x) \leqq \begin{cases} a_1(a_1 - 1) + \lambda \leqq \lambda(a_1 - 1) + \lambda = \lambda a_1 & \text{if} \quad a_0 = 0, \\ (a_1 + 1)\, a_1 + \lambda \leqq \lambda a_1 + a_1 + \lambda & \text{if} \quad a_0 = 1. \end{cases}$$

On the other hand

$$\sum_{x \in A} \deg(x) \geqq \begin{cases} a_1\delta \geqq a_1(\lambda + 1) = \lambda a_1 + a_1 & \text{if} \quad a_0 = 0, \\ (a_1 + 1)\,\delta \geqq (a_1 + 1)(\lambda + 1) = \lambda a_1 + a_1 + \lambda + 1 & \text{if} \quad a_0 = 1. \end{cases}$$

Being compared these inequalities give a contradiction in either case. ∎

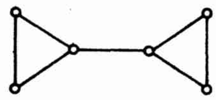

fig. 6

Fig. 6 shows that Theorem 2.1 is in a sense a best possible result. We have immediately:

2.2. Corollary. *If a connected graph G contains such a vertex v_0 that $d(x, y) \leqq 2$ for all $x, y \in V(G) - \{v_0\}$, then $\lambda = \delta$.*

§ 3. DISTANCE CONDITION FOR BIPARTITE GRAPHS

Now we will give an analog of Theorem 2.1 for bipartite graphs and show that it yields the result (9).

3.1. Theorem. *Let G be a bipartite graph with bipartition $[A, B]$. Then $\lambda = \delta$ whenever at least one of the following two conditions holds:*

(i) diam $(G) \leqq 4$ *and neither part contains four vertices u_1, v_1, u_2, v_2 such that*

$$d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2) = 4. \tag{11}$$

(ii) *There exists a part P with $d(x, y) \leqq 2$ for all $x, y \in P$.*

Proof. Suppose for a contradiction that there is an edge cut E_0 with cardinality $\lambda < \delta$. Clearly $\lambda > 0$. After deleting the edges of E_0 from G, we obtain two components with vertex sets S and $\bar{S} := V(G) - S$. In accordance with Fig. 7, $A_1, \bar{A}_1,$

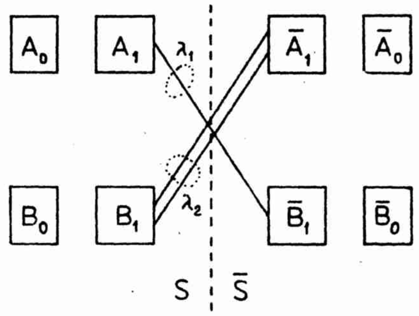

fig. 7

$B_1, \bar{B}_1$ denote the sets of vertices incident with some edge of the cut E_0 and lying in $A \cap S$, $A \cap \bar{S}$, $B \cap S$ and $B \cap \bar{S}$, respectively. The remaining vertices form the sets $A_0, \bar{A}_0, B_0$ and $\bar{B}_0$, i.e. $A_0 = A \cap S - A_1$, etc. Let the number of edges between A_1 and $\bar{B}_1$ be λ_1 and that between $\bar{A}_1$ and B_1 be λ_2. Thus $\lambda = \lambda_1 + \lambda_2$. Finally, let the cardinalities of the sets $A_0, \bar{A}_0, \ldots, \bar{B}_1$ be denoted by the corresponding small letters, i.e. $a_0, \bar{a}_0, \ldots, \bar{b}_1$. Clearly we have

$$a_1 \leqq \lambda_1, \ \bar{b}_1 \leqq \lambda_1, \ \bar{a}_1 \leqq \lambda_2, \ b_1 \leqq \lambda_2.$$

(i) First suppose that the condition (i) holds. We have to distinguish several cases, but owing to the reason of symmetry we can confine to the following:

Case 1: $a_0 \geqq 2$ and $\bar{a}_0 \geqq 2$. Then we can find $u_1, v_1 \in A_0$ and $u_2, v_2 \in \bar{A}_0$ fulfilling (11).

Thus without loss of generality in what follows we can suppose $a_0 \leqq 1$.

Case 2: $a_0 = b_0 = 0$. Then $a_1 + b_1 > 0$ and we can suppose that $A_1 \neq \emptyset$. For any $x \in A_1$ we have $\deg(x) \leqq \lambda_1 + b_1$. On the other hand $\deg(x) \geqq \delta \geqq \lambda + 1 = \lambda_1 + \lambda_2 + 1 \geqq \lambda_1 + b_1 + 1$, a contradiction.

Case 3: $a_0 = 0$, $b_0 \geqq 1$. Then for every $x \in B_0$ we have $\deg(x) \leqq a_1 \leqq \lambda_1 < \delta$, what is impossible.

Case 4: $a_0 = 1$, $b_0 = 1$. Then for $x \in A_0$ we have $\deg(x) \leqq b_1 + 1 \leqq \lambda_2 + 1$ and for $y \in B_0$ analogously $\deg(y) \leqq a_1 + 1 \leqq \lambda_1 + 1$. Thus we can write $2\delta \leqq$

$\leqq \deg(x) + \deg(y) \leqq \lambda_1 + \lambda_2 + 2 = \lambda + 2 \leqq \delta + 1$, which yields $\delta \leqq 1$, i.e. $\lambda = 0$, a contradiction.

Case 5: $a_0 = 1$ and $\bar{b}_0 = 1$. Then because of Cases 3 and 4 we have $b_0 \geqq 2$ and $\bar{a}_0 \geqq 2$ (use the symmetry). For any $x \in B_0$ we get $\deg(x) \leqq a_1 + 1 \leqq \lambda_1 + 1$ and for any $y \in \bar{A}_0$ we have $\deg(y) \leqq \bar{b}_1 + 1 \leqq \lambda_1 + 1$. Hence $2(\lambda_1 + \lambda_2 + 1) = 2(\lambda + 1) \leqq 2\delta \leqq \deg(x) + \deg(y) \leqq 2\lambda_1 + 2$, i.e. $\lambda_2 = 0$ and thus $\bar{a}_1 = b_1 = 0$. But for $u \in A_0$, $v \in \bar{B}_0$ we have $d(u, v) \geqq 5$, which contradicts our assumption (i).

Case 6: $a_0 = 1$, $b_0 \geqq 2$, $\bar{b}_0 \geqq 2$. This is excluded by Case 1 (use the symmetry).

Having covered all possibilities the proof is completed if (i) is assumed to hold.

(ii) Now let the condition (ii) hold. We can assume that $P = A$, i.e. $d(x, y) \leqq 2$ for all $x, y \in A$. This yields $d(u, v) \leqq 4$ for all $u, v \in B$ and $d(x, u) \leqq 3$ for all $x \in A$, $u \in B$. Hence diam $(G) \leqq 4$. However, $d(x, y) = 4$ for any $x \in A_0$, $y \in \bar{A}_0$ (see Fig. 7). Therefore $a_0 \cdot \bar{a}_0 = 0$ and we can assume that $a_0 = 0$. Then the considerations of above mentioned Cases 2 and 3 will work. ■

Fig. 8 shows that the assumption diam $(G) \leqq 4$ cannot be dropped; on the other hand this condition is not sufficient if the rest of (i) does not hold (see Fig. 9).

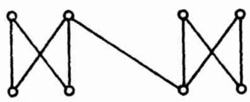

fig 8

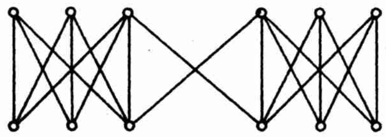

fig. 9

3.2. Corollary. *Let G be a bipartite graph with* diam $(G) \leqq 4$. *If in either part P there exists such a vertex v_0 that $d(x, y) \leqq 2$ for all $x, y \in P - \{v_0\}$, then $\lambda = \delta$.*

Proof. Immediately, since (i) is fulfilled. ■

3.3. Corollary. *If a bipartite graph G has* diam $(G) \leqq 3$, *then $\lambda = \delta$.*

Proof. Now the condition (ii) is fulfilled because the distances in the same part are even. ■

Our theorem implies also the above mentioned result (9) of Volkmann [9]:

3.4. Corollary. *If G is a bipartite graph with $\delta \geqq (n + 1)/4$, then $\lambda = \delta$.*

Proof. We will prove that the condition (ii) of Theorem 3.1 holds. Indeed, if it is not the case, then there exist vertices $x, y \in A$ with $d(x, y) > 2$ and so $V(x) \cap V(y) = \emptyset$. Consequently, B has at least $(n + 1)/4 + (n + 1)/4 = (n + 1)/2$ vertices. Symmetrically, A has at least $(n + 1)/2$ vertices too, what is impossible. ■

Examples from Figs. 10 and 11 show that there are no other relations between the conditions (i) and (ii) of Theorem 3.1 and (9). The graphs have $n = 11$, $\delta = 2$.

In Fig. 10 we have $d(1, 4) = d(1, 5) = 4$ and $d(x, y) \leqq 2$ for all $x, y \in A - \{1\}$. Also $d(6, 10) = d(6, 11) = 4$ and $d(x, y) \leqq 2$ for all $x, y \in B - \{6\}$. Thus (i) is fulfilled but neither (ii) nor (9) hold.

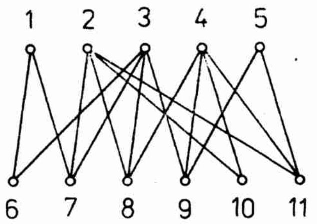

fig. 10

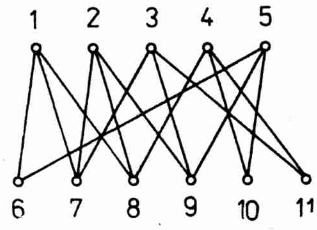

fig. 11

In Fig. 11 we see that $d(x, y) \leqq 2$ for all $x, y \in A$. Further $d(6, 11) = d(7, 10) = 4$. Thus (ii) holds but (i) and (9) do not.

Moreover, both these graphs have $g = 4$ and thus not even (8) is fulfilled.

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TOLERANCES, COVERING SYSTEMS, AND THE AXIOM OF CHOICE

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Dedicated to the memory of Milan Sekanina

Abstract. A *tolerance relation* of an algebra is a binary relation that is reflexive, symmetric, and has the Substitution Property. A number of authors (I. Chajda, G. Czédli, L. Klukovits, J. Niederle, I. Rosenberg, D. Schweigert, B. Zelinka, and the present authors) investigated how tolerances can be described by the system of *blocks* (maximal connected subsets). In this paper we show how to modify known results from idempotent algebras to arbitrary algebras. We prove the known characterization for lattices without the Axiom of Choice. For lattices with the Chain Condition, G. Czédli and L. Klukovits obtained a much better result. We generalize their result to arbitrary lattices, again avoiding the use of the Axiom of Choice. Finally, we show that for semilattices, the existence of a tolerance-block is equivalent to the Axiom of Choice.

Key words. Tolerance relation, covering system, Axiom of Choice, universal algebra, lattice.
MS Classification. Primary 08 A 30, 06 B 10; Secondary 08 A 05

1. INTRODUCTION

It is well-known that there is a one-to-one correspondence between congruence relations of an algebra $\langle A; F\rangle$ and partitions of A with the Substitution Property; in fact, in informal discussions, the congruence relation Θ is often identified with the corresponding partition. There is a similar one-to-one correspondence between tolerance relations of an algebra $\langle A; F\rangle$ and a certain type of covering systems of the set A.

For a binary relation ϱ on the set A, a subset B of A is a *ϱ-block* iff B is a *ϱ-connected set* (i.e., $a\varrho b$ for all $a, b \in B$) that is *maximal* (i.e., if $B \subseteq C$ and C is also ϱ-connected, then $B = C$). I. Chajda [1] observed that there is a one-to-one correspondence between tolerance relations and covering systems of blocks. For lattices, G. Czédli [5] proved that one can define a lattice on the blocks of a tolerance relation, generalizing the concept of *quotient lattice.*

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Various papers deal with the covering systems one obtains from blocks of a tolerance relation. A recent one is G. Czédli and L. Klukovits [6] in which they obtain a characterization of the covering system of blocks of a tolerance relation for an *idempotent algebra* (an algebra $\langle A; F\rangle$ is *idempotent*, if it has no nullary operations and $f(x, \ldots, x) = x$ for all $x \in A$) sharpening earlier results of I. Chajda [1] and I. Chajda, J. Niederle, and B. Zelinka [2]; see Theorem 1 in § 2.

For lattices, the covering system of blocks of a tolerance relation was characterized in G. Czédli [5] and in I. Rosenberg and D. Schweigert [13]; see Theorem 4 in § 2. A much more useful characterization was obtained for lattices with the Chain Condition in G. Czédli [5]. This result was proved again in G. Czédli and L. Klukovits [6], applying the characterization for idempotent algebras; see Theorem 5 in § 2.

In this paper we offer two generalizations of the result of G. Czédli and L. Klukovits [5] for idempotent algebras: To algebras in which the tolerance-blocks are subalgebras (Theorem 2 in § 2) and to arbitrary algebras (Theorem 3 in § 2).

For lattices, we make two contributions. First, we show that, using some results of G. Grätzer and G. H. Wenzel [11], the older characterization theorem, Theorem 4, can be proved without the Axiom of Choice. Secondly, we generalize Czédli's result to arbitrary lattices: Theorem 6 in § 2.

However, the lattice proof cannot be extended to algebras, in general, or idempotent algebras, in particular. We prove that the existence of a tolerance-block in a semilattice is equivalent to the Axiom of Choice; see Theorem 7 in § 5.

Tolerances have been used recently in universal algebra in an attempt to describe the variety generated by the product of two varieties in G. Grätzer and G. H. Wenzel [11] and E. Fried and G. Grätzer [7] and [8], and in lattice theory to describe monotone functionally complete finite lattices in M. Kindermann [12].

For the basic concepts of universal algebra and lattice theory, we refer the reader to G. Grätzer [9] and [10].

Conditions will be denoted by mnemonic names. We shall refer to condition (XX) of Theorem n as (n. XX); "n." is dropped if the context is clear.

2. RESULTS

The main result of G. Czédli and L. Klukovits [6] is as follows:

Theorem 1. *Let $\langle A; F\rangle$ be an idempotent algebra. A family $\mathscr{C}$ of nonempty subsets of A is the set of all blocks of a tolerance relation iff the following conditions hold:*

(Cov) $\mathscr{C}$ *is a* covering system, *i.e.*, $\bigcup \mathscr{C} = A$.

(AnC) $\mathscr{C}$ *is an* antichain, *i.e.* $X \subseteq Y$ *implies that* $X = Y$, *for* $X, Y \in \mathscr{C}$.

(SP) $\mathscr{C}$ *has the* Substitution Property, *i.e., for any n-ary operation* $f \in \mathscr{J}$ *and* $X_1, \ldots, X_n \in \mathscr{C}$ *there exists an* $X \in \mathscr{C}$ *such that* $f(X_1, \ldots, X_n) = \{f(x_1, \ldots, x_n) \mid x_1 \in X_1, \ldots, x_n \in X_n\} \subseteq X$.

(2-SA) *For any* 2-covered subalgebra *B of $\langle A; F\rangle$, i.e., any subalgebra B of $\langle A; F\rangle$ such that for any two elements $a, b \in B$ there is an $X \in \mathscr{C}$ satisfying $a, b \in X$, there exists an $E \in \mathscr{C}$ with $B \subseteq E$.*

We have two generalizations of Theorem 1:

Theorem 2. *Let $\langle A; F\rangle$ be an algebra, and let $\mathscr{C}$ be a family of subalgebras of $\langle A; F\rangle$. Then $\mathscr{C}$ is the set of all blocks of a tolerance relation iff the conditions* (1.Cov), (1.AnC), (1.SP), *and* (1.2-SA) *hold.*

Theorem 2 is a direct generalization of Theorem 1 since in an idempotent algebra all tolerance-blocks are subalgebras (I. Chajda and B. Zelinka [4]). Indeed, if D is a tolerance-block of the tolerance relation τ, f is an n-ary operation, $d_1, \ldots, d_n \in D$, then for every $d \in D$, $d\tau d_i$ for $i = 1, \ldots, n$, hence by the Substitution Property for τ and the idempotency of f,

$$d = f(d, \ldots, d) \equiv f(d_1, \ldots, d_n).$$

By the maximality of D, $f(d_1, \ldots, d_n) \in D$, i.e., D is a subalgebra.

Theorem 3. *Let $\langle A; F\rangle$ be an* arbitrary algebra. *A family $\mathscr{C}$ of nonempty subsets of A is the set of all blocks of a tolerance relation iff the conditions* (1.Cov), (1.AnC), (1.SP), *and the following condition hold:*

(2-SS) *For any* 2-covered subset *B of A, i.e., any subset B of A such that for any two elements $a, b \in B$ there is an $X \in \mathscr{C}$ satisfying $a, b \in X$, there exists an $E \in \mathscr{C}$ with $B \subseteq E$.*

The characterization for lattices is as follows (I. Rosenberg and D. Schweigert [13]; the result in G. Czédli [5] is somewhat different):

Theorem 4. *Let L be a lattice. A family $\mathscr{C}$ of nonempty subsets of L is the set of all blocks of a tolerance relation iff the following conditions hold:*

(Cov) $\mathscr{C}$ *is* a covering system, *i.e.,* $\bigcup \mathscr{C} = L$.

(AnC) $\mathscr{C}$ *is an* antichain, *i.e.* $X \subseteq Y$ *implies that* $X = Y$, *for* $X, Y \in \mathscr{C}$.

(SP) $\mathscr{C}$ *has the* Substitution Property, *i.e., for all* $X, Y \in \mathscr{C}$ *there exist* $U, V \in \mathscr{C}$ *such that* $X \mathbin{\curlyvee} Y \subseteq U$ *and* $X \mathbin{\curlywedge} Y \subseteq V$.

(2-SL) *For any* 2-covered sublattice *B* of *L, i.e., any sublattice B of L such that for any two elements $a, b \in B$ there is an $X \in \mathscr{C}$ satisfying $a, b \in X$, there exists an $E \in \mathscr{C}$ with $B \subseteq E$.*

In the condition (SP) above we use the notation:

$$X \mathbin{\curlyvee} Y = \{x \vee y \mid x \in X, y \in Y\},$$
$$X \mathbin{\curlywedge} Y = \{x \wedge y \mid x \in X, y \in Y\}.$$

Theorem 1 is applied by G. Czédli and L. Klukovits to lattices with the Chain Condition to obtain a much sharper form of Theorem 4, first proved in G. Czédli [5]:

Theorem 5. *Let L be a lattice satisfying the* Chain Condition, *i.e., all chains in L are finite. A family $\mathscr{C}$ of nonempty subsets of L is the set of all blocks of a tolerance relation iff $\mathscr{C}$ is a system of intervals of L and the following conditions hold:*

(Cov) $\mathscr{C}$ *is* a covering system, *i.e.,* $\bigcup \mathscr{C} = L$.

(SP) $\mathscr{C}$ *has the* Substitution Property, *i.e., for all* $[a_1, a_2]$, $[b_1, b_2] \in \mathscr{C}$ *there exist* $[u_1, u_2]$, $[v_1, v_2] \in \mathscr{C}$ *such that* $u_1 = a_1 \vee b_1$, $u_2 \geqq a_2 \vee b_2$ *and* $v_1 \leqq a_1 \wedge b_1$, $v_2 = a_2 \wedge b_2$.

(UE) *The intervals in* $\mathscr{C}$ *have* unique endpoints, *i.e.,* $a_1 = b_1$ *iff* $a_2 = b_2$, *for* $[a_1, a_2], [b_1, b_2] \in \mathscr{C}$.

The following theorem generalizes Theorem 5 to arbitrary lattices:

Theorem 6. *Let L be a lattice. A family $\mathscr{C}$ of nonempty subsets of L is the set of all blocks of a tolerance relation iff all $X \in \mathscr{C}$ are convex sublattices of L and the following conditions hold:*

(Cov) $\mathscr{C}$ *is a* covering system, *i.e.,* $\bigcup \mathscr{C} = L$.

[SP] $\mathscr{C}$ *has the* Substitution Property, *i.e., for all* $X, Y \in \mathscr{C}$ *there exist* $U, V \in \mathscr{C}$ *such that* $[U) = [X) \vee_d [Y)$, $(U] \supseteq (X] \vee_i (Y]$ *and* $(V] = (X] \wedge_i (Y]$, $[V) \supseteq [X) \wedge_d [Y)$.

(UE) *The convex sublattices in* $\mathscr{C}$ *have* unique endpoints, *i.e.,* $(X] = (Y]$ *iff* $[X) = [Y)$, *for* $X, Y \in \mathscr{C}$.

(2-CS) *For any* 2-covered convex sublattice *B of L, i.e., any convex sublattice B of L such that for any two elements $a, b \in B$ there is an $X \in \mathscr{C}$ satisfying $a, b \in X$, there exists an $E \in \mathscr{C}$ with $B \subseteq E$.*

Notation. For a nonempty subset X of a lattice L, the ideal and dual ideal generated by X is denoted $(X]$ and $[X)$, respectively. (Note that in [11], we used $(X]$ and $[X)$ for the order ideals generated by X.) We form three lattices from L: the lattice of ideals (ordered under $\subseteq$), the lattice of dual ideals (ordered under $\supseteq$), and the lattice L/τ of all τ-blocks (ordered by $\leqq$, see Lemma 2 below). To avoid confusion, the lattice operations will be denoted by $\vee_i$ and $\wedge_i$ in the ideal lattice, by $\vee_d$ and $\wedge_d$ in the dual ideal lattice, and by $\vee_b$ and $\wedge_b$ in L/τ (the lattice of τ-blocks). Recall that $\vee_d$ is intersection and $\wedge_d$ is the dual ideal generated by the union.

It is clear that Theorem 6 implies Theorem 5. Indeed, if the lattice L satisfies the Chain Condition, then convex sublattices are intervals, hence the first three conditions are equivalent. Condition (2-CS) trivially holds for L, since if B is a convex sublattice of L, then $B = [a, b]$ for some $a, b \in L$, $a \leqq b$, and the $X \in \mathscr{C}$ satisfying $a, b \in X$ also satisfies $B \subseteq X$. We shall show an example in Section 4 that (2-CS) cannot be dropped in general.

3. ALGEBRAS

For a tolerance relation τ, let $\mathscr{C}_\tau$ denote the set of all τ-blocks.

First, we prove Theorems 2 and 3.

Let τ be a tolerance relation, and $\mathscr{C} = \mathscr{C}_\tau$. Obviously, the conditions of Theorems 2 and 3 hold for $\mathscr{C}$; in particular, (2-SS) holds, since if B is 2-covered by $\mathscr{C}$, then B is τ-connected, hence (by the Axiom of Choice) it is contained in a maximal τ-connected set $E \in \mathscr{C}$.

Conversely, let the conditions of Theorem 2 or 3 hold for $\mathscr{C}$, and define the binary relation τ by $a\tau b$ iff $a, b \in B$ for some $B \in \mathscr{C}$. (1.Cov) and (1.SP) imply that τ is a tolerance relation. Let $\mathscr{C}_\tau$ denote the system of tolerance-blocks of τ. Since every $B \in \mathscr{C}$ is τ-connected by the definition of τ, there is a $B^* \in \mathscr{C}_\tau$ satisfying $B \subseteq B^*$.

We have to prove that $\mathscr{C} = \mathscr{C}_\tau$.

Let $B \in \mathscr{C}_\tau$. Then B is 2-covered by $\mathscr{C}$ by the definition of τ. In case of Theorem 2, B is a subalgebra, so condition (2-SA) can be applied; in case of Theorem 3, condition (2-SS) can be applied. In either case, $B \subseteq D$ for some $D \in \mathscr{C}$. Since D is τ-connected and B is a τ-block, therefore, $B = D$, proving that $B \in \mathscr{C}$.

Conversely, let $B \in \mathscr{C}$. Then $B \subseteq B^* \in \mathscr{C}_\tau \subseteq \mathscr{C}$ (the last containment by the previous paragraph), hence $B = B^*$ by (1.AnC). Thus $B \in \mathscr{C}_\tau$, proving Theorems 2 and 3.

4. LATTICES

We need some results from G. Grätzer and G. H. Wenzel [11]. These results are proved in [11] without the use of the Axiom of Choice.

Let L be a lattice and let τ be a tolerance relation on L. For a subset X of L, we define

$$X_\tau = \{y \mid y \in L, y \leqq x \textit{ for some } x \in X, \text{ and } y\tau x \textit{ for all } x \in X\}.$$

We define X^τ dually.

Lemma 1 (Lemma 5 of [11]). *If X is a τ-connected set, then $(X_\tau)^\tau$ is a τ-block.*

Let L/τ denote the lattice of tolerance-blocks. Lemma 1 shows that L/τ is non-empty. The lattice operations of L/τ can be described as follows:

Lemma 2 (Lemma 7 and Theorem 1 of [11]). *Let X and Y be τ-blocks. Then*

$$X \vee_b Y = (X \curlyvee Y)^\tau,$$

$$X \wedge_b Y = (X \curlywedge Y)_\tau,$$

and

$$X \leqq Y \textit{ iff for all } x \in X \textit{ there is a } y \in Y \textit{ satisfying } x \leqq y.$$

One should note the dual form of the last statement of Lemma 2:

$$X \leqq Y \text{ iff for all } y \in Y \text{ there is an } x \in X \text{ satisfying } x \leqq y.$$

First, we prove Theorem 4 without the Axiom of Choice. It is obvious that the conditions of Theorem 4 hold for the blocks of a tolerance relation. (To verify (4.Cov), use Lemma 1 with a singleton as the τ-connected set.)

Conversely, let $\mathscr{C}$ satisfy the four conditions of Theorem 4, and define the binary relation τ by

$$a\tau b \text{ iff } a, b \in B \text{ for some } B \in \mathscr{C}.$$

(4.Cov) and (4.SP) imply that τ is a tolerance relation. By the definition of τ, every $B \in \mathscr{C}$ is τ-connected. By Lemma 1, $B^* = (B_\tau)^\tau$ is a τ-block containing B.

Now let B be a τ-block. Then B is 2-covered by $\mathscr{C}$. By (2-SL), $B \subseteq D$ for some $D \in \mathscr{C}$. Thus $B \subseteq D$ and D is τ-connected. Since B is a τ-block, $B = D$, proving that $B \in \mathscr{C}$.

If $B \in \mathscr{C}$, then B^* is a τ-block with $B \subseteq B^*$, and by the previous paragraph, B^* is in $\mathscr{C}$. Again, by (AnC), $B = B^*$, so B is a τ-block. ∎

Now we prove Theorem 6. Let τ be a tolerance relation on the lattice L, let $\mathscr{C} = \mathscr{C}_\tau$. Conditions (6.Cov) and (2-CS) are obvious by Lemma 1. To verify (6.UE), let $X, Y \in \mathscr{C}$ and let $[X) = [Y)$. If $x \in X - Y$, then $x \in [X) = [Y)$, i.e., $x \geqq y$ for some $y \in Y$. Hence for every $x \in X$, $x \geqq y$ for some $y \in Y$. By Lemma 2, $X \geqq Y$ with respect to the ordering of the blocks. By symmetry, $Y \geqq X$, hence $X = Y$. Thus $(X] = (Y]$, verifying (6.UE).

Finally, we prove (6.SP). Let $X, Y \in \mathscr{C}$ and define $U = X \vee_b Y$. By Lemma 2 (and using that $X \curlyvee Y$ is downward directed),

$$\begin{aligned} U = X \vee_b Y &= (X \curlyvee Y)^\tau \\ &= \{z \mid z \geqq x \vee y \text{ for some } x \in X, y \in Y \\ &\quad \text{and } z\tau x \vee y \text{ for all } x \in X, y \in Y\} \\ &= [X \curlyvee Y) = [X) \vee_d [Y), \end{aligned}$$

and so $[U) \subseteq [X) \vee_d [Y)$. Conversely, if $u \in [X) \vee_d [Y)$, then $u \geqq x$ and $u \geqq y$ for some $x \in X$ and $y \in Y$. Thus $u \geqq x \vee y \in U$, proving $[U) \supseteq [X) \vee_d [Y)$. Thus $[U) = [X) \vee_d [Y)$. We also have

$$(X] \vee_d (Y] = (X \curlyvee Y] \subseteq ((X \curlyvee Y)^\tau] = (U],$$

completing, by duality, the proof of (6.SP).

Conversely, let $\mathscr{C}$ satisfy the four conditions of Theorem 6. We verify that the four conditions of Theorem 4 hold; then by Theorem 4, we obtain $\mathscr{C} = \mathscr{C}_\tau$ for some tolerance relation τ, proving Theorem 6.

(4.Cov) is the same as (6.Cov). To verify (2-SL), let B be a 2-covered sublattice of L. Then $C = \bigcup([a, b] \mid a, b \in B)$ is the convex hull of B, and C is also 2-covered by $\mathscr{C}$. By (2-CS), $C \in D$ for some $D \in \mathscr{C}$, and so $B \subseteq D \in \mathscr{C}$, proving (2-SL).

To prove (4.AnC), take $X, Y \in \mathscr{C}$ with $X \subseteq Y$. By (6.SP), there exists a $U \in \mathscr{C}$ such that $[U) = [X) \vee_d [Y)$ and $(U] \supseteq (X] \vee_i (Y]$. But $[X) \cap [Y) = [X) \vee_d [Y) = [X)$ since $X \subseteq Y$ implies that $[X) \subseteq [Y)$. Thus $[X) = [U)$ which by (6.UE) implies that $(X] = (U]$, and so $X = U$. Therefore, $(X] \supseteq (X] \vee_i (Y] = (Y]$, yielding $(X] = (Y]$. By (6.UE) again, $[X) = [Y)$, and so $X = Y$.

Finally, we prove (4.SP). Given $X, Y \in \mathscr{C}$, by (6.SP), there exist $U, V \in \mathscr{C}$ such that $[U) = [X) \vee_d [Y)$, $(U] \supseteq (X] \vee_i (Y]$ and $(V] = (X] \wedge_i (Y]$, $[V) \supseteq [X) \wedge_d Y)$. Thus $[X) \vee_d [Y) = [X \mathbin{\underline{\vee}} Y) = [U)$ and $(X] \vee_i (Y] = (X \mathbin{\underline{\vee}} Y] \subseteq (U]$. Now take $x \in X$ and $y \in Y$. Then $x \vee y \in (X] \vee_i (Y] \subseteq (U]$, thus $x \vee y \leqq u_1$, for some $u_1 \in U$. On the other hand, $x \vee y \in [X) \vee_d [Y) = [U)$, so $u_2 \leqq x \vee y$ for some $u_2 \in U$. Thus $u_1 \leqq x \vee y \leqq u_2$, and by the convexity of U, we conclude that $x \vee y \in U$, proving that $X \mathbin{\underline{\vee}} Y \subseteq U$. Similarly, $X \mathbin{\underline{\wedge}} Y \subseteq V$, verifying (4.SP). ∎

The following example shows that condition (2-CS) cannot be dropped from Theorem 6. Let L be the lattice of real numbers with the usual partial ordering. Let $\mathscr{C}$ be the system of all sets of the form $[r, r+1]$ for a rational number r. Then $\mathscr{C}$ satisfies the first three conditions of Theorem 6. However, $\mathscr{C}$ fails (2-CS): the set $B = [\sqrt{2}, 1 + \sqrt{2})$ is a convex sublattice which is 2-covered by $\mathscr{C}$; however, B is not contained in any member of $\mathscr{C}$.

5. SEMILATTICES

Our final result shows that the Axiom of Choice is needed to prove the existence of tolerance-blocks.

Theorem 7. *The Axiom of Choice is equivalent to the following statement:*

(TB) *For a semilattice $\langle S; \vee \rangle$ and a tolerance relation τ on $\langle S; \vee \rangle$, there exists a τ-block.*

Proof. Let $\mathscr{Y}$ be a nonempty collection of nonempty pairwise disjoint sets. Define $S = (\bigcup \mathscr{Y}) \cup \{u\}$, where $u \notin \bigcup \mathscr{Y}$. We define the semilattice $\langle S; \vee \rangle$ by $x \vee y = u$ for all $x \neq y$. Finally, we define the binary relation τ on S as follows:

$$x \tau y \text{ iff } \begin{cases} x = y, \\ x \in X \quad \text{and} \quad y \in Y \quad \text{for some} \quad X, Y \in \mathscr{Y}, X \neq Y, \\ x = u, \\ y = u. \end{cases}$$

It is trivial to check that τ is a tolerance relation.

Let B be a τ-block. We claim that for every $X \in \mathscr{Y}$, $B \cap X$ is a singleton $\{x_B\}$. Proof: if $x, y \in B \cap X$, then $x, y \in B$, hence $x \tau y$. On the other hand, $x, y \in X$, hence by the definition of τ, we obtain $x = y$. Therefore, $B \cap X$ contains at most one element. If $B \cap X$ is empty, define $B^* = B \cup \{x\}$, where $x \in X$. Observe that $x \tau y$

for every $y \in B$; indeed, either $x = u$ or $y \in Y$ for some $Y \in \mathscr{Y}$, $X \neq Y$; in both cases, $x\tau y$ holds. Thus B^* is τ-connected and $B \subset B^*$, a contradiction.

Now we can define the choice function f on $\mathscr{Y}$ by $f(X) = x_B$ for $X \in \mathscr{C}$. ∎

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SPECIAL INVARIANT SUBSPACES OF A VECTOR SPACE OVER $\mathbf{Z}/l\mathbf{Z}$

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Abstract. This article deals with a special linear operator S on the vector space $\mathbf{V}$ over the Galois field $\mathbf{Z}/l\mathbf{Z}$ of dimension $\frac{l-1}{2}$ (l an odd prime). All invariant subspaces are described in three ways. The background of this theme is found in the area of the ***Stickelberger ideal*** **mod** l. **It is** shown that the matrices of the ***Stickelberger ideals*** have a very convenient form for $l < 1{,}000$.

Key words. Invariant subspaces, Stickelberger ideal, group ring of a cyclic group over the Galois field, Bernoulli numbers, index of irregularity of a prime.

MS Classification. 10 M 20, 12 A 80

In this paper the vector space $\mathbf{V}$ over the Galois field $\mathbf{Z}/l\mathbf{Z}$ is considered (l is an odd prime) with dimension $\frac{l-1}{2}$. For this vector space special linear operators S_z $(1 \leqq z \leqq l-1)$ are defined. The main goal of this paper is to describe all invariant subspaces of $\mathbf{V}$ with respect to the operators S_z (Theorem 3.4).

There is defined a special isomorphism F from a group ring $\mathfrak{R}^-(l)$ (considered as a vector space) on $\mathbf{V}$ and the connection is shown between the ideals of $\mathfrak{R}^-(l)$ and invariant subspaces of $\mathbf{V}$ with respect to S_z (4.3.2).

The theme of this paper derives from the area of the ***Bernoulli numbers, index*** *of irregularity of the prime l* and the *Stickelberger ideal* mod l (4.3.3).

The final Section 5 deals with the normal matrix of a subspace of $\mathbf{V}$. Especially the normal matrix of an invariant subspace of $\mathbf{V}$ with respect to the operators S_z is investigated and it is mentioned that for each prime $l < 1{,}000$ the normal matrix of the subspace of $\mathbf{V}$ corresponding to the Stickelberger ideal has a very convenient form (5.9.1).

1. NOTATION

Throughout this paper it will be designated by
l an odd prime,

$$N = \frac{l-1}{2},$$

$\mathbf{V} = \{(a(1), a(2), \ldots, a(N)) : a(i) \in \mathbf{Z}/l\mathbf{Z}\} = (\mathbf{Z}/l\mathbf{Z})^{(N)}$ the vector space over the Galois field $\mathbf{Z}/l\mathbf{Z}$ (of residue classes mod l on the ring $\mathbf{Z}$ of integers) with dimension N and with componentwise operations,
$\mathbf{L} = \{1, 2, \ldots, N\}$.

For integers $1 \leqq x, z \leqq l - 1$ put

$$\varepsilon(x, z) = \begin{cases} 1 & \text{if} \quad xz \equiv y(\text{mod } l), 0 < y \leqq N, \\ -1 & \text{if} \quad xz \equiv y(\text{mod } l), N + 1 \leqq y < l, \end{cases}$$

$$f(x, z) \equiv \varepsilon(x, z)\, xz(\text{mod } l), \qquad f(x, z) \in \mathbf{L},$$

so $f(x, z) \equiv \pm xz(\text{mod } l)$.

For the vector $\mathbf{u} = (u(1), \ldots, u(N)) \in \mathbf{V}$ put

$$S_z(\mathbf{u}) = \mathbf{v} = (v(1), \ldots, v(N)) \in \mathbf{V},$$

where $v(x) = \varepsilon(x, z)\, u(f(x, z))$ $(x \in \mathbf{L})$. Sometimes an integer $x \in \mathbf{Z}$ will be considered as the residue class mod l containing x.

According to ([6], 3.4 and 3.5) it holds

1.1. Proposition. (*a*) *For each* $1 \leqq z \leqq l - 1$ $(z \in \mathbf{Z})$ *the mapping* S_z: $\mathbf{V} \to \mathbf{V}$ *is an automorphism of the vector space* $\mathbf{V}$.

(b) *For* $1 \leqq z, z' \leqq l - 1$ $(z, z. \in \mathbf{Z})$ *we have*

$$S_{z'} = S_z \qquad \textit{if and only if } z = z'.$$

(c) *If* $1 \leqq z, z', w \leqq l - 1$ $(z, z', w \in \mathbf{Z})$, $w \equiv z \,.\, z'(\text{mod } l)$, *then* $S_w = S_{z'} \circ S_z$.

(d) *The set* $\{S_z\colon 1 \leqq z \leqq l - 1,\ z \in \mathbf{Z}\}$ *with operation* e *forms a cyclic group of order* $l - 1$. *Generators of this group are the automorphisms* S_R, *where* $1 \leqq R \leqq$ $\leqq l - 1$ *are primitive roots* mod l.

(The operations $\circ$ means composition of mappings.)

The aim of this paper is to describe all invariant subspaces of the vector space $\mathbf{V}$ with respect to the group $(\{S_z : 1 \leqq z \leqq l - 1\}, \circ)$.

Choose a primitive root r mod l $(1 < r < l)$ and denote by S the mapping S_r. Then

$$\{S_z : 1 \leqq z \leqq l - 1, z \in \mathbf{Z}\} = \{S^n : 0 \leqq n \leqq l - 2, n \in \mathbf{Z}\}$$

and the S_z-invariant subspaces of $\mathbf{V}$ for each $1 \leqq z \leqq l - 1$, $z \in \mathbf{Z}$ are just the S-invariant subspaces of $\mathbf{V}$.

2. SOME S-INVARIANT SUBSPACES OF $\mathbf{V}$

2.1. Definition. For a subset $A \subseteq \mathbf{L}$ put

$$\mathscr{S}(A) = \{\alpha = (a(1), a(2), \ldots, a(N)) \in \mathbf{V}: \sum_{x=1}^{N} a(x)^{2a-1} = 0 \text{ for each } a \in A\}.$$

2.2. Proposition. (a) *For each subset $A \subseteq \mathbf{L}$ the set $\mathscr{S}(A)$ forms an S-invariant subspace of the vector space $\mathbf{V}$ and* $\dim \mathscr{S}(A) = N - |A|$. ($|A|$ means cardinal of A).

(b) *For $A \subseteq B \subseteq \mathbf{L}$ the relation $\mathscr{S}(A) \supseteq \mathscr{S}(B)$ holds.*

(c) $\mathscr{S}(\emptyset) = \mathbf{V}$, $\mathscr{S}(\mathbf{L}) = 0$. (0 means zero subspace.)

Proof. a) Clearly, $\mathscr{S}(A)$ is a subspace of the vector space $\mathbf{V}$. Let $\mathbf{u} = (u(1), \ldots, \ldots, u(N)) \in \mathscr{S}(A)$, $S(\mathbf{u}) = \mathbf{v} = (v(1), \ldots, v(N)) \in \mathbf{V}$. Then for $a \in A$ we have

$$\sum_{x=1}^{N} v(x)\, x^{2a-1} = \sum_{x=1}^{N} \varepsilon(x, r)\, u(f(x, r))\, x^{2a-1},$$

hence

$$r^{2a-1} \sum_{x=1}^{N} v(x)\, x^{2a-1} = \sum_{x=1}^{N} u(f(x, r)\, (rx)^{2a-1}\, (\varepsilon(x, r) = 1) +$$
$$+ \sum_{x=1}^{N} u(f(x, r))\, (-rx)^{2a-1}\, (\varepsilon(x, r) = -1) =$$
$$= \sum_{y=1}^{N} u(y)\, y^{2a-1}(\varepsilon(y, r_{-1}) = 1) + \sum_{y=1}^{N} u(y)\, y^{2a-1}(\varepsilon(y, r_{-1}) =$$
$$= -1) = \sum_{y=1}^{N} u(y)\, y^{2a-1} = 0,$$

where $r_{-1} \in \mathbf{Z}$, $0 < r_{-1} < l$, $r \,.\, r_{-1} \equiv 1 (\mathrm{mod}\, l)$. Therefore the subspace $\mathscr{S}(A)$ is S-invariant.

(b) The subspace $\mathscr{S}(A)$ is the space of solutions of the system of linear equations

$$\sum_{x=1}^{N} a(x)\, x^{2a-1} = 0 \qquad (a \in A),$$

over the field $\mathbf{Z}/l\mathbf{Z}$ with unknowns $a(1), \ldots, a(N)$. The matrix of this system equals the matrix

$$(x^{2a-1})\ (x \in \mathbf{L},\ a \in A),$$

which is of Vandermond's type, hence its rank is equal to $|A|$. It follows that $\dim \mathscr{S}(A) = N - |A|$.

(c) The assertions (b) and (c) are evident.

2.3. Definition. We denote by $\mathscr{N}$ the set of all non-quadratic residues $x \bmod l$ ($1 < x < l$). For $x \in \mathscr{N}$ put

$$\mathbf{u}(x) = (u(1), \ldots, u(N)) \in \mathbf{V},$$

where for $1 \leqq t \leqq N$ we have

$$u(t) = x^{\operatorname{ind} t},$$

(ind t denotes index of t relative to the primitive root r of l.)

The subspace of the space $\mathbf{V}$ generated by the vector $\mathbf{u}(x)$ will be denoted by $\mathbf{U}(x)$. Hence,

$$\mathbf{U}(x) = \{k \,.\, \mathbf{u}(x) : k \in \mathbf{Z}/l\mathbf{Z}\} \qquad \text{and} \qquad \dim \mathbf{U}(x) = 1.$$

Since $S(\mathbf{u}(x)) = x \,.\, \mathbf{u}(x)$, $\mathbf{U}(x)$ is an S-invariant subspace of the space $\mathbf{V}$ and $S(\mathbf{u}) = x \,.\, \mathbf{u}$ for each $\mathbf{u} \in \mathbf{U}(x)$.

2.4. Proposition. *The vectors* $\mathbf{u}(x)$ $(x \in N)$ *form a basis of the space* $\mathbf{V}$.

Proof. As $\dim \mathbf{V} = N$, it is enough to prove that the vectors $\mathbf{u}(x)$ $(x \in \mathcal{N})$ are linearly independent.

Let $c(x) \in \mathbf{Z}/l\mathbf{Z}$ for $x \in \mathcal{N}$ such that

$$\sum c(x)\, \mathbf{u}(x)\ (x \in \mathcal{N}) = \mathbf{o}.$$

($\mathbf{o}$ means zero vector.)

Then

$$\sum c(x)\, x^{\operatorname{ind} v} (x \in \mathcal{N}) = 0 \qquad \text{for each} \qquad 1 \leqq v \leqq N.$$

It follows

$$\sum c(x)\, x^{i} (x \in \mathcal{N}) = 0 \qquad \text{for each} \qquad 0 \leqq i \leqq N - 1.$$

The matrix (x^i) $(x \in \mathcal{N},\ 0 \leqq i \leqq N - 1)$ is of Vandermond's type, hence $c(x) = 0$ for each $x \in \mathcal{N}$. The proposition is proved.

2.5. Definition. For $X \subseteq \mathcal{N}$ let $\mathbf{U}(X)$ mean the subspace of the vector space $\mathbf{V}$ generated by the vectors $\mathbf{u}(x)$ $(x \in X)$, $\mathbf{U}(\emptyset)$ is defined as zero space. Hence $\mathbf{U}(X)$ is the direct sum of the subspaces $\mathbf{U}(x)$ $(x \in X)$:

$$\mathbf{U}(X) = \sum_{\oplus} \mathbf{U}(x)\ (x \in X)$$

and $\dim \mathbf{U}(X) = |X|$.

Since the subspace $\mathbf{U}(x)$ is S-invariant, the subspace $\mathbf{U}(X)$ is also S-invariant.

2.6. Proposition. *Let* $X, Y \subseteq \mathcal{N}$. *Then we have*

(*a*) $\mathbf{U}(X) \subseteq \mathbf{U}(Y)$ *if and only if* $X \subseteq Y$,

(*b*) $\mathbf{U}(X) = \mathbf{U}(Y)$ *if and only if* $X = Y$.

Proof. Clearly, (a) implies (b). Suppose $\mathbf{U}(X) \subseteq \mathbf{U}(Y)$ and $x \in X$. Then $\mathbf{u}(x) \in \mathbf{U}(Y)$ and hence $x \in Y$. Therefore (a) holds and hence (b) as well.

Between the subspaces $\mathbf{U}(X)$ $(X \subseteq \mathcal{N})$ and the subspaces $\mathcal{S}(A)$ $(A \subseteq \mathbf{L})$ the following relation holds.

2.7. Theorem. *Let* $X \subseteq \mathcal{N}$ *and* $A = \mathbf{L} - \left\{N - \frac{1}{2}(\text{ind}\, x - 1) : x \in X\right\}$. *Then*

$$\mathbf{U}(X) = \mathcal{S}(A).$$

Proof. I. We show that $\mathbf{U}(X) \subseteq \mathcal{S}(A)$. Let $x \in X$ and $\mathbf{u}(x) = (u(1), \ldots, u(N))$. Then $x^{\text{ind}\, v} = u(v)$ for each $1 \leqq v \leqq N$. For $a \in A$ the integer $\text{ind}\, x + 2a - 1$ is even and $\text{ind}\, x + 2a - 1 \not\equiv 0 (\text{mod}\, l - 1)$. Therefore we have

$$\sum_{v=1}^{N} x^{\text{ind}\, v} v^{2a-1} \equiv \sum_{v=1}^{N} (r^{\text{ind}\, x + 2a - 1})^{\text{ind}\, v} (\text{mod}\, l) \equiv$$

$$\equiv \sum_{u=0}^{\frac{l-3}{2}} (r^{\text{ind}\, x + 2a - 1})^{u} (\text{mod}\, l) \equiv 0 (\text{mod}\, l).$$

It follows that $\sum_{v=1}^{N} u(v)\, v^{2a-1} = 0$, hence $\mathbf{u}(x) \in \mathcal{S}(A)$.

II. Since $\dim \mathbf{U}(X) = |X| = N - |A| = \dim \mathcal{S}(A)$, we get $\mathbf{U}(X) = \mathcal{S}(A)$.

3. ALL S-INVARIANT SUBSPACES OF $\mathbf{V}$

In this Section we give description of all S-invariant subspaces of the vector space $\mathbf{V}$. The proofs use the known results concerning the structure of a linear operator in an n-dimensional vector spave over a number field that hold also for the field $\mathbf{Z}/l\mathbf{Z}$ as it is possibly easily to see. The notions and these results from this branch are taken from book [2] by *F. R. Gantmacher*, Chapter VII. Especially we use the notion of *minimal polynomial of a vector space* (with respect to a given linear operator) and „*The First Theorem on the Decomposition of a Space into Invariant Subspaces*" ([2], Chapter VII, Theorem 1).

3.1. Proposition. *The polynomial* $\Psi(\lambda) = \lambda^N + 1$ (considered over the field $\mathbf{Z}/l\mathbf{Z}$) *is the minimal polynomial of the space* $\mathbf{V}$ *with respect to the linear operator* S.

Proof. Recall that the minimal polynomial $\Psi(\lambda)$ is the non-zero monic polynomial over $\mathbf{Z}/l\mathbf{Z}$ of the least degree such that for each $\mathbf{u} \in \mathbf{V}$ we have $\Psi(S)(\mathbf{u}) = \mathbf{o}$.

If $\mathbf{u} \in \mathbf{V}$, then $S^N(\mathbf{u}) = S_r^N(\mathbf{u}) = S_{l-1}(\mathbf{u}) = -\mathbf{u}$, so $\Psi(S)(\mathbf{u}) = \mathbf{o}$.

Let $\mathbf{u}_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{V}$, where 1 is situated on the ith position. The vectors $\mathbf{u}_i$ $(1 \leqq i \leqq N)$ form a basis of $\mathbf{V}$.

For $0 \leqq n \leqq \frac{l-3}{2}$ let $x(n)$ be the integer, $1 \leqq x(n) \leqq N$, $e_n = \pm 1$ such that $e_n r^n x(n) \equiv 1 (\text{mod}\, l)$. Then $S^n = S_r^n = S_w$ according to 1.1 (c), where w is the integer, $1 \leqq w \leqq l - 1$, $w \equiv r^n (\text{mod}\, l)$. Hence $S^n(\mathbf{u}_1) = e_n \mathbf{u}_{x(n)}$. Since for $0 \leqq n$, $m \leqq \frac{l-3}{2}$ the equality $x(n) = x(m)$ follows $n = m$, the vectors $S^0(\mathbf{u}_1), S^1(\mathbf{u}_1), \ldots,$

$\dots, S^{\frac{l-3}{2}}(\mathbf{u}_1)$ are linearly independent hence $x(S)(\mathbf{u}_1) \neq \mathbf{o}$ for each non-zero polynomial $x(\lambda)$ over the field $\mathbf{Z}/l\mathbf{Z}$ of degree $< N$. The proposition follows.

3.2. Remark. Clearly

$$\Psi(\lambda) = \lambda^N + 1 = \Pi(\lambda - x) \qquad (x \in \mathcal{N})$$

over the field $\mathbf{Z}/l\mathbf{Z}$. The polynomial $\lambda - x$ is the minimal polynomial of the subspace $\mathbf{U}(x)$ with respect to the operator S for each $x \in \mathcal{N}$. The conversion of this assertion holds as well:

3.3. Proposition. *Let* $\mathbf{U}$ *be an invariant subspace of* $\mathbf{V}$ *with respect to the operator* S *with minimal polynomial* $\lambda - x$ $(x \in \mathcal{N})$ (over $\mathbf{Z}/l\mathbf{Z}$). *Then* $\mathbf{U} = \mathbf{U}(x)$.

Proof. Clearly, $\mathbf{U}$ is a non-zero space. Let $\mathbf{u} = (u(1), \dots, u(N)) \in \mathbf{U}$, $\mathbf{u} \neq \mathbf{o}$. There exists $1 \leqq i \leqq N$ such that $u(i) \neq 0$. For $1 \leqq j \leqq N$ let $1 \leqq z \leqq l - 1$ with the property $zi \equiv j (\text{mod } l)$. There exists $k \in \mathbf{Z}/l\mathbf{Z}$, $0 \neq k$ such that $k \,.\, \mathbf{u} = S_z(\mathbf{u})$, hence $0 \neq k \,.\, u(i) = \varepsilon(i, z)\, u(f(i, z)) = \pm u(j)$. Thus $u(j) \neq 0$ for each $1 \leqq j \leqq N$.

Put $\mathbf{v} = u(1)^{-1}\mathbf{u} = (v(1), \dots, v(N)) \in \mathbf{U}$. Then $v(j) \neq 0$ for each $1 \leqq j \leqq N$ and $v(1) = 1$.

a) For $1 \leqq a, b \leqq N$ we have $v(a) \,.\, v(b) = \varepsilon(a, b) \,.\, v(f(a, b))$. Namely, there exists $k \in \mathbf{Z}/l\mathbf{Z}$, $k \neq 0$ such that $k \,.\, \mathbf{v} = S_a(\mathbf{v}) = (w(1), \dots, w(N))$. Since $1 = v(1)$, we get $k = w(1) = \varepsilon(1, a)\, v(f(1, a)) = v(a)$, thus $v(a) \,.\, v(b) = k \,.\, v(b) = w(b) = \varepsilon(b, a) \,.\, v(f(b, a))$.

b) Let $1 \leqq c, d \leqq N$, $e = \pm 1$, n a positive integer and $c^n \equiv ed (\text{mod } l)$. Then $v(c)^n = ev(d)$.

We prove this assertion by mathematical induction with regard to n. The case $n = 1$ is clear. Let this assertion hold for $n \geqq 1$ and let $1 \leqq C, D \leqq N$, $E = \pm 1$ and let $C^{n+1} \equiv E \,.\, D (\text{mod } l)$.

There exist integers $\varepsilon, \delta, \varepsilon = \pm 1, 1 \leqq \delta \leqq N$ such that $C^n \equiv \varepsilon\delta (\text{mod } l)$. We have $v(C)^n = \varepsilon v(\delta)$ and according to a) $v(\delta) \,.\, v(c) = \varepsilon(\delta, c) \,.\, v(f(\delta, c))$. Further $\varepsilon(\delta, c)\, f(\delta, c) \equiv C\delta \equiv \varepsilon C^{n+1} \equiv \varepsilon E \,.\, D (\text{mod } l)$, hence $f(\delta, c) = D$ and $\varepsilon E = \varepsilon(\delta, c)$, thus $v(C)^{n+1} = \varepsilon v(\delta) \,.\, v(C) = Ev(D)$.

c) It holds $v(t) = x^{\text{ind}\, t}$ for each $1 \leqq t \leqq N$. Put $R = r$, $\varepsilon = 1$ in case $r < l/2$ and $R = l - r$, $\varepsilon = -1$ in case $r > l/2$. There holds $xv(j) = \varepsilon(j, r)\, v(f(j, r))$ $(1 \leqq j \leqq N)$, hence $x = xv(1) = \varepsilon(1, r)\, v(f(1, r)) = \varepsilon v(R)$, which follows $\varepsilon x = v(R)$. Let $1 \leqq t \leqq N$, $n = \text{ind}\, t$. According to b) $(c = R,\ d = t,\ e = \varepsilon^n)$ we get $v(t) = \varepsilon^n v(R)^n = x^n$, thus $x^{\text{ind}\, t} = v(t)$.

Assertion c) yields $\mathbf{v} = \mathbf{v}(x)$ and since each vector from $\mathbf{U}$ is a multiple of $\mathbf{v}$, we have $\mathbf{U} = \mathbf{U}(x)$.

3.4. Theorem. *Let* $\mathbf{U}$ *be a non-zero S-invariant subspace of the space* $\mathbf{V}$, $\dim \mathbf{U} = m$ $(1 \leqq m \leqq N)$. *Then there exists* $X \subseteq \mathcal{N}$, $|X| = m$ *such that* $\mathbf{U}(X) = \mathbf{U}$.

Proof. Let $G(\lambda)$ be the minimal polynomial of the space $\mathbf{U}$ with respect to S. Then $G(\lambda)$ divides the polynomial $\Psi(\lambda) = \lambda^N + 1$, hence there exists $X \subseteq \mathcal{N}$ with the property

$$G(\lambda) = \prod (\lambda - x) \qquad (x \in X),$$

(considered as a polynomial over the field $\mathbf{Z}/l\mathbf{Z}$). The First Theorem on the Decomposition *of a Space into Invariant Subspaces* then yields

$$\mathbf{U} = \sum_{\oplus} \mathbf{U}_x \qquad (x \in X),$$

where $\mathbf{U}_x$ is an S-invariant subspace of $\mathbf{V}$ with the minimal polynomial $\lambda - x$. Proposition 3.3 then implies Theorem.

4. CONNECTION WITH THE GROUP RING $(\mathbf{Z}/l\mathbf{Z})[G]$

4.1. Notation. Throughout this Section we shall use the following notation:

G a multiplicative cyclic group of order $l - 1$,

s a generator of G; thus $G = \{1 = s^0, s, \dots, s^{l-2}\}$,

$\mathfrak{R}(l)$ $= (\mathbf{Z}/l\mathbf{Z})[G]$ the group ring of G over the field $\mathbf{Z}/l\mathbf{Z}$; thus $\mathfrak{R}(l) = \{\sum_{i=0}^{l-2} a_i s^i : a_i \in \mathbf{Z}/l\mathbf{Z}\}$,

$\mathfrak{R}^-(l)$ $= \{\alpha = \sum_{i=0}^{l-2} a_i s^i \in \mathfrak{R}(l) : 0 = a_i + a_{i+N}$ for each $0 \leqq i \leqq N - 1\}$,

F the mapping of $\mathfrak{R}^-(l)$ onto $\mathbf{V}$ defined as follows: $F(\alpha) = \mathbf{u} = (u(1), \dots, u(N)) \in \mathbf{V}$, $\alpha = \sum_{i=0}^{l-2} a_i s^i \in \mathfrak{R}^-(l)$ and for $1 \leqq x \leqq N$, $u(x) = a_{l-1-\operatorname{ind} x} (a_{l-1} = a_0)$,

F_n the mapping of $\mathfrak{R}^-(l)$ onto $\mathfrak{R}^-(l)$ for an integer n defined by the formula $F_n(\alpha) = s^n . \alpha (\alpha \in \mathfrak{R}^-(l))$.

We consider the subring $\mathfrak{R}^-(l)$ of the ring $\mathfrak{R}(l)$ as the vector space over the field $\mathbf{Z}/l\mathbf{Z}$. Then F is an isomorphism of the vector space $\mathfrak{R}^-(l)$ onto the vector space $\mathbf{V}$ and the mappings F_n are automorphisms of the vector space $\mathfrak{R}^-(l)$.

4.2. Proposition. *Let* z *be an integer,* $1 \leqq z \leqq l - 1$, $n = \operatorname{ind} z$. *Then*

$$F \circ F_n \circ F^{-1} = S_z .$$

Thus *the following diagram is commutative*:

$$\begin{array}{ccc} \mathfrak{R}^-(l) & \xleftarrow{F^{-1}} & \mathbf{V} \\ \big\downarrow F_n & & \big\downarrow S_z \\ \mathfrak{R}^-(l) & \xrightarrow{F} & \mathbf{V} \end{array}$$

Proof. Let $\mathbf{u} = (u(1), \ldots, u(N)) \in \mathbf{V}$, $F^{-1}(\mathbf{u}) = \alpha = \sum_{i=0}^{l-2} a_i s^i \in \mathfrak{R}(l)$, $F_n(\alpha) = \beta = \sum_{i=0}^{l-2} b_i s^i \in \mathfrak{R}^-(l)$ and $F(\beta) = \mathbf{v} = (v(1), \ldots, v(N)) \in \mathbf{V}$. For each integer j let $a_j = a_i$, where $0 \leqq i \leqq l-2$, $i \equiv j (\mathrm{mod}\ l-1)$.

Then for $1 \leqq x \leqq N$ and $0 \leqq i \leqq l-2$ we have $u(x) = a_{-\mathrm{ind}\,x}$, $b_i = a_{i-n}$ and $v(x) = b_{l-1-\mathrm{ind}\,x} = a_{-\mathrm{ind}\,x-n} = a_{-\mathrm{ind}\,xz} = a_{-\mathrm{ind}\,\varepsilon(x,z)f(x,z)} = a_{-\mathrm{ind}\,\varepsilon(x,z)-\mathrm{ind}\,f(x,z)} = \varepsilon(x,z)\,u(f(x,z)) = u(x)$. It follows $S_z(\mathbf{u}) = \mathbf{v}$ and the proposition is proved.

4.3. Remark. The ideals of the ring $\mathfrak{R}^-(l)$ can also be characterized as follows:

4.3.1. *An additive subgroup I of the ring $\mathfrak{R}^-(l)$ is an ideal of the ring $\mathfrak{R}^-(l)$ if and only if $s \,.\, I \subseteq I$.*

Proof. Clearly, if I has the given property, then it is an ideal of $\mathfrak{R}^-(l)$. Let I be an ideal of $\mathfrak{R}^-(l)$ and let $\alpha \in I$. Denote by β the element $\frac{l+1}{2} s\,(1 - s^{\frac{l-1}{2}}) \in \mathfrak{R}^-(l)$, where 1 is considered as an element of $\mathbf{Z}/l\mathbf{Z}$. Since $\mathfrak{R}^-(l) = (1 - s^{\frac{l-1}{2}})\mathfrak{R}(l)$, there exists $\gamma \in \mathfrak{R}(l)$ such that $\alpha = (1 - s^{\frac{l+1}{2}})\,\gamma$. Then $\beta \,.\, \alpha = \frac{l+1}{2} s(1 - s^{\frac{l-1}{2}})^2\, \gamma = s \,.\, (1 - s^{\frac{l-1}{2}})\,\gamma = s \,.\, \alpha$, which implies $s \,.\, \alpha \in I$.

According to 4.3.1 there holds

4.3.2. *A subset I of $\mathfrak{R}^-(l)$ is an ideal of the ring $\mathfrak{R}^-(l)$ if and only if it forms an F_n-invariant subspace of the vector space $\mathfrak{R}^-(l)$ for each integer n.*

According to [5], Proposition 3.9 the ideals of the ring $\mathfrak{R}^-(l$ are in the one-to-one correspondence with the subsets X of $\mathcal{N}$ by the formula

$$X \subseteq \mathcal{N} \rightarrow \mathcal{J}(X) = \mathfrak{R}^-(l) \prod (s - x) \qquad (x \in X),$$

($s - x$ is considered as an element of $\mathfrak{R}(l)$). $\mathcal{J}(X)$ is a subspace of the vector space $\mathfrak{R}^-(l)$ and according to [5], Proposition 3.3 the system of elements α_L ($1 \leqq L \leqq$

$\leqq l-2$, L odd, $r_L \notin X$) $(1 \leqq r_n \leqq l-1,\ r_n \equiv r^n (\mathrm{mod}\ l)$ for an integer n) forms a basis of the subspace $\mathscr{J}(X)$, where $\alpha_L = \sum_{i=0}^{l-2} r_{-iL} s^i$. The image $F(\mathscr{J}(X)$ is then an S-invariant subspace of $\mathbf{V}$, whose basis is formed by the elements $F(\alpha_L) = \mathbf{u}(r_L)$, and then $F(\mathscr{J}(X)) = \mathbf{U}(\mathscr{N} - X)$.

We have got in this way another proof of Theorem 3.4.

The general situation looks like the following:

S-invariant subspaces of $\mathbf{V}$ $\leftrightarrow$ subsets of $\mathscr{N}$ $\leftrightarrow$ ideals of $\mathfrak{R}^-(l)$

$$\mathbf{U} = \mathbf{U}(X) = \mathscr{S}(A) =$$

$$= F(\mathscr{J}(\mathscr{N} - X)) \leftrightarrow X = \{r_{-2b+1} : b \in \mathbf{L} - A\} \leftrightarrow \mathscr{J}(\mathscr{N} - X) =$$

$$\updownarrow \qquad = \mathfrak{R}^-(l) \,.\, \prod (s - x)\ (x \in \mathscr{N} - X)$$

$$A = \mathbf{L} - \left\{N - \frac{1}{2}(\operatorname{ind} x - 1) : x \in X\right\},$$

subsets of $\mathbf{L}$

4.3.3. *Special case.* If we put $A = \{1 \leqq a \leqq \frac{l-3}{2};\ l / B_{2a}\}$ (B_n means the *Bernoulli number*), then $|A| = \mathrm{i}(l)$ the *index of irregularity of* l and according to [6], Theorem 2.4 (c) $\mathscr{J}(\mathscr{N} - X) = \mathfrak{J}(l)$ is the *Stickelberger ideal* mod l. The set X is then equal to the set $\{r_{-2b+1} : 1 \leqq b \leqq \frac{l-3}{2},\ l \dagger B_{2b}\} \cup \{r\}$.

The images of some concrete elements from the *Stickelberger ideal* $\mathfrak{J}^-(l)$ in the isomorphism F are described in Section 4 and 5 of [6].

5. THE NORMAL MATRIX OF A SUBSPACE OF V

All matrices are considered over the field $\mathbf{Z}/l\mathbf{Z}$.

5.1. Definition. A matrix $M = (m_{ij})$ of size $m \times n$ ($m \leqq n$) is said *to be in normal form* if there exist integers $1 \leqq j_1 < j_2 < \ldots < j_m \leqq n$ with the following property:

$$m_{ij} = \begin{cases} 1 & \text{for } j = j_i, \\ 0 & \text{for } j < j_i, \\ 0 & \text{for } j = j_k,\ 1 \leqq k \leqq m,\ k \neq i, \end{cases}$$

$1 \leqq i \leqq m$. Thus the columns with subscriptions $j_1, \ldots, j_m$ form the unit matrix of order m and the elements of M standing in the left of ones of this unit matrix are zeros. The number m is rank of M.

It is clear that any nonzero matrix C can be transformed in a matrix M in normal form by a sequence of elementary row operations (i.e. multiplication of a row by a nonzero element from $\mathbf{Z}/l\mathbf{Z}$ and addition to a row another one) and omitting rows containing only zeros.

This matrix M ir defined uniquely by this property and we will call it *the normal form of the matrix C.*

5.2. Definition. Let $0 \neq \mathbf{U}$ be a subspace of the vector space $\mathbf{V}$. The coordinates of vectors of a basis $\mathscr{B}$ of $\mathbf{U}$ form a nonzero matrix

$$U = (u(1), \dots, u(N)) \, (\mathbf{u} = (u(1), \dots, u(N)) \in \mathscr{B})$$

of size $\dim \mathbf{U} \times N$. We call the normal form M of the matrix U *the normal matrix of the subspace* $\mathbf{U}$.

Clearly, M doesn't depend on the basis $\mathscr{B}$, size of M equals $\dim \mathbf{U} \times N$ and the row vectors of M form a basis of $\mathbf{U}$. The normal matrix of the whole space $\mathbf{V}$ is the unit matrix of order N.

5.3. Let $\emptyset \neq \mathbf{U} \neq \mathbf{V}$ be an S-invariant subspace of $\mathbf{V}$, let $A \subseteq \mathbf{L}(\emptyset \neq A \neq \mathbf{L})$ and $\mathbf{U} = \mathscr{S}(A)$, and let $r = |A|$ $(0 < r < N)$.

There exist uniquely determined integers

$$0 = \xi_0 < 2 \leqq \xi_1 < \xi_2 < \dots < \xi_{r-1} < \xi_r = N,$$

such that for $x \in \mathbf{L}$, $\xi_k < x \leqq \xi_{k+1} (0 \leqq k < r - 1)$ rank of the matrix

$$(x^{2a-1}, \xi_{k+1}^{2a-1}, \xi_{k+2}^{2a-1}, \dots, \xi_r^{2a-1}) \qquad (a \in A)$$

of size $r \times (r - k + 1)$ equals $r - k$. (Since rank of the matrix (t^{2a-1}) $(a \in A, t \in \mathbf{L})$ of size r/N equals r (Vandermond's type)).

Let $1 \leqq i \leqq N$, $i \notin \{\xi_1, \xi_2, \dots, \xi_r\}$. Then there exists $0 \leqq k \leqq r - 1$ such that $\xi_k < i < \xi_{k+1}$. Since ranks of matrices

$$(i^{2a-1}, \xi_{k+1}^{2a-1}, \dots, \xi_r^{2a-1}) \qquad (a \in A),$$

$$(\xi_{k+1}^{2a-1}, \dots, \xi_r^{2a-1}) \qquad (a \in A)$$

equal one another and equal $r - k$, there exist uniquely determined integers $0 \leqq x_{i\gamma} < l (1 \leqq \gamma \leqq r - k)$ such that

$$(*) \qquad i^{2a-1} + \sum_{\gamma=1}^{r-k} \xi_{k+\gamma}^{2a-1} x_{i\gamma} \equiv 0 (\mathrm{mod}\, l).$$

Put for $1 \leqq j \leqq N (i \notin \{\xi_1, \dots, \xi_r\})$:

$$m_{ij} = \begin{cases} 1 & \text{for } j = i, \\ x_{i\gamma} & \text{for } j = \xi_{k+\gamma} (1 \leqq \gamma \leqq r - k), \\ 0 & \text{otherwise.} \end{cases}$$

5.3.1. Theorem. *The matrix* $M = (m_{ij})$ $(1 \leqq i \leqq N, i \in \{\xi_1, \xi_2, \dots, \xi_r\}, 1 \leqq j \leqq N)$ *is the normal matrix of the subspace* $\mathbf{U}$.

Proof. According to definition the matrix M is in normal form and has size $\dim \mathbf{U} \times N$ since $\dim \mathbf{U} = N - r$. It remains to prove that every row vector of M belongs to $\mathbf{U}$. Using (*) and the fact $\mathbf{U} = \mathscr{S}(A)$ we obtain the Theorem.

5.4. Definition. We call a subset $A \subseteq \mathbf{L}$ *normal* (*for the prime l*) if $A = \emptyset$ or $A = \mathbf{L}$ or $\emptyset \neq A \neq \mathbf{L}$ and the normal matrix M of the subspace $\mathscr{S}(A)$ of $\mathbf{V}$ has the form

$$M = (E, X),$$

where E is the unit matrix of order $N - |A|$ and X is a matrix of size $N - |A| \times \times |A|$.

The following two Propositions are immediate consequences of Theorem 5.3.1.

5.5. Proposition. *Each one-element subset of* $\mathbf{L}$ *is normal for the prime* l.

5.6. Proposition. *Let* $A \subseteq \mathbf{L}, \emptyset \neq A \neq \mathbf{L}, r = |A|$ *and* $B = \{a - a^* : a \in A\}$, *where* a^* *is the least integer in* A. *Then the following assertions are equivalent*:

(a) A *is normal for the prime* l,

(b) $\det (x^{2b})$ $(b \in B,\ N - r + 1 \leqq x \leqq N) \not\equiv 0 (\mathrm{mod}\ l)$,

(c) $\det ((2x - 1)^{2b})$ $(b \in B,\ 1 \leqq x \leqq r) \not\equiv 0 (\mathrm{mod}\ l)$.

We can see easily

5.7. Proposition. *Let* $3 \leqq l \leqq 11$. *Then each subset* $A \subseteq \mathbf{L}$ *is normal for the prime* l.

We also obtain by easy computation:

5.8. Proposition. *Let* $l = 13$. *Then each subset* $A \subseteq \{1, 2, \ldots, 6\}$ *is normal for* 13 *except*

(a) $A = \{1, 3, 5\}$ *or* $A = \{2, 4, 6\}$,

(b) $A = \{1, 4\}$ *or* $A = \{2, 5\}$ *or* $A = \{3, 6\}$.

In case (*a*) *the normal matrix* M *of* $\mathscr{S}(A)$ *has the form*

$$M = \begin{bmatrix} 1 & 0 & x_1 & 0 & y_1 & z_1 \\ 0 & 1 & x_2 & 0 & y_2 & z_2 \\ 0 & 0 & 0 & 1 & y_3 & z_3 \end{bmatrix}$$

and in case (*b*)

$$M = \begin{bmatrix} 1 & 0 & 0 & x_1 & 0 & y_1 \\ 0 & 1 & 0 & x_2 & 0 & y_2 \\ 0 & 0 & 1 & x_3 & 0 & y_3 \\ 0 & 0 & 0 & 0 & 1 & y_4 \end{bmatrix},$$

$(x_i, y_i, z_i \in \mathbf{Z})$.

The numbers x_i, y_i, z_i can be computed by means of the equalities (*). Thus e.g. for $A = \{1, 3, 5\}$ we have

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 5 & 0 \\ 0 & 1 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 8 \end{bmatrix}$$

and for $A = \{2, 5\}$

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 12 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 12 \end{bmatrix}.$$

5.9. Let $A = \left\{1 \leqq a \leqq \frac{l-3}{2} : l / B_{2a}\right\}$, $\bar{A} = A \cup \left\{\frac{l-1}{2}\right\}$. Using tables of indices ([3]) and tables of irregular primes ([4] , s. also [1], Table 9) we can derive:

5.9.1. Proposition. *For each prime l, $3 \leqq l < 1{,}000$ the sets A and $\bar{A}$ are normal for the prime l.*

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UNIVERSALITY OF DIRECTED GRAPHS OF A GIVEN HEIGHT

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(Received April 20, 1988)

Dedicated to the memory of Milan Sekanina

Abstract. We consider the classes of directed graphs which are determined by the existence of a homomorphism into (or from) a fixed graph. We completely answer the question when a class of this type is universal.

MS Classification. 05 C 20

1. INTRODUCTION

In this paper we deal with directed graphs (without loops and multiple arcs) Graphs may be infinite.

Given graphs $G = (V, E)$, $H = (W, F)$, a *homomorphism* $f: G \to H$ is a mapping $V \to W$ which satisfies $(f(x), f(y)) \in F$ for every $(x, y) \in E$. We also may say that *G maps into H* and we denote it by $G \to H$.

Denote by *GRA* the category of all graphs and all their homomorphism. Any category $\mathcal{K}$ for which there exists an embedding of *GRA* into $\mathcal{K}$ is said to be *universal* (binding), see [5], [3]. A universal category is very rich in the sense that every concrete category may be embedded into it.

One of the main streams in the study of universal categories is formed by efforts to find simple examples of universal categories, see [1], [2], [5], [8], [9] for numerous examples in various areas of mathematics.

In this context perhaps it is worth to mention the following. Some time ago M. Sekanina and the second author investigated the universality of classes of graphs related to Sekanina's characterization of Hamiltonian powers of graphs [12]:

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Let k be a positive integer, let $G = (V, E)$ be an undirected graph. Denote by $G^{(k)} = (V, E^{(k)})$ the graph defined by

$$[x, y] \in E^{(k)} \quad \text{iff} \quad x \neq y \quad \text{and} \quad d_G(x, y) \leqq k.$$

Here $d_G(x, y)$ is the distance of x and y in G. We call $G^{(k)}$ the *k-th power* of G. Among the result which Sekanina and Nešetřil obtained and which were not yet published is the following:

1.1. Theorem. *Let k be a positive integer. Then the class $Gra^{(k)}$ of all k-th powers is a universal category.*

In this note we consider the following classes of graphs from the point of view of their universality. Let A be a graph. We introduce the following special subclasses of the class *Gra*:

$$\begin{aligned} \rightarrow A &= \{G; G \rightarrow A\},\\ \not\rightarrow A &= \{G; G \not\rightarrow A\},\\ A \rightarrow &= \{G; A \rightarrow G\},\\ A \not\rightarrow &= \{G; A \not\rightarrow G\}. \end{aligned}$$

These classes were investigated previously in various context: in [10] from the point of view of algorithmic complexity and in [15] from the point of view of algebraic properties (such as the existence of products).

In [2] and [1] we considered the classes of undirected graphs which contain a given graph as a subgraph. As an easy modification we get from this the following:

1.2. Proposition. *For every graph A the classes $\not\rightarrow A$ and $A \rightarrow$ are universal.*

For the remaining two cases we do not get always an affirmative answer and we give a full solution in this paper. This is stated below as Theorem 3.1 and 3.2.

The motivation of this paper is two fold: First we want to complement the research for undirected graphs [1], [2]. Secondly the questions considered in this paper naturally arised in the study of directed rigid graphs, see our companion paper [4]. Our results support the common belief that the directed graphs although sometimes easier to construct are in the context of categorial representations mostly more difficult to analyse.

The key to our analysis is the study of balanced graphs. This is contained in Section 2 where we define invariants $\lambda(G)$ and $\Lambda(G)$; $\Lambda(G)$ is called the *height* of G. In Section 3 we state our main results. It appears that it suffices to consider the case $\rightarrow A$ as the case $A \not\rightarrow$ is a byproduct of our proof.

A bit surprisingly the universality of a class $\rightarrow A$ is fully characterized by a fact whether it contains (just) two mutually rigid graphs. A graph G is *rigid* if the identity is the only homomorphism $G \rightarrow G$. Two rigid graphs G and H are said to be *mutually rigid* if they are rigid and there are no homomorphisms $G \rightarrow H$ and $H \rightarrow G$.

2. BALANCED GRAPHS

Definition 2.1. *A cycle is balanced if it has the same number of arcs going one way as going the other way (with respect to a fixed transversal of the cycle). A directed graph $G = (U, E)$ is balanced if each of its cycles is balanced. The net length of a path is the number of arcs going forward minus the number going backwards. A directed path of length n (i.e. with $n + 1$ vertices) will be denoted by $\vec{P}_n$. Finally, $\vec{P}_\infty$ denotes the doubly infinite directed path.*

Proposition 2.2. *For a directed graph G the following two statements are equivalent;*

1. G is balanced,

2. there is a homomorphism $G \to \vec{P}_\infty$.

Proof. Since the homomorphic image of an unbalanced cycle must contain an unbalanced cycle, it suffices to prove that 1. implies 2. Without loss of generality let G be a connected balanced graph. Any two paths with a fixed beginning and a fixed end have the same net length. Let x be a fixed vertex of G and let $f(y)$ be the net length of any path from x to y. One can check that f is a homomorphism $G \to \vec{P}_\infty$.

□

This leads to the following:

Definition 2.3. *Let G be a balanced graph. Let $\Lambda(G)$ be the minimum n such that there exists a homomorphism $G \to \vec{P}_n$. (Possibly $n = \infty$). We call $\Lambda(G)$ the height of G. Denote also $\lambda(G)$ the maximum n such that there exists a homomorphism $\vec{P}_n \to G$. Clearly $\lambda(G) \leqq \Lambda(G)$.*

Let us remark that it follows from the above proof of Proposition 2.2 that for a connected graph G a homomorphism $f: G \to \vec{P}_\infty$ is uniquely determined by the value $f(x)$ for any one vertex x of G. It follows that for a connected balanced G with finite height Λ there exists unique homomorphism $f: G \to \vec{P}_\Lambda$. This homomorphism will also be denoted by Λ. By convention, we let Λ denote an arbitrary homomorphism $G \to \vec{P}_\infty$ if G has infinite height.

This has several corollaries. We want to mention the following results explicitely as we shall need them later:

Lemma 2.4. *Let G be a connected balanced graph with finite $\Lambda(G)$. Then $\Lambda(x) = \max \{\Lambda(P) \mid P$ is a path in G which terminates in $x\}$.*

□

Lemma 2.5. *Let G and H be balanced, $f: G \to H$ a homomorphism. Then $\Lambda(G) \leqq \Lambda(H)$.*

□

Lemma 2.6. *Let G and H be connected balanced graphs with $\Lambda(G) = \Lambda(H) < \infty$, and let $f: G \to H$ be a homomorphism. Then f preserves Λ. (Explicitely $\Lambda_H(f(x)) = \Lambda_G(x)$ for every $x \in V(G)$.)*

□

Finally we have

Proposition 2.7. *Let G be a rigid balanced graph with finite $\Lambda(G)$. Then G contains a rigid path P with $\Lambda(G) = \Lambda(P)$.*

Proof. Let P be a shortest path (i.e., having the fewest arcs) with $\lambda(P) = \lambda(G)$. (It exists by 2.4). Then P can be seen to be rigid by 2.6.

□

Remark. Of course 2.7 need not hold for infinite Λ.

An antidirected path is a path P with $\lambda(P) = 1$. Denote by $a(G)$ the maximal length (number of arcs) of an antidirected path in G. We put $a(G) = \infty$ if there are arbitrarily long antidirected paths. As we shall see below the numbers $a(G)$ may be used for testing the existence a homomorphism.

We begin our investigation of balanced rigid graphs of small height with an analysis of rigid trees.

Denote by T_a the path of length $2a + 3$ which contains an antidirected path of length $2a + 1$ and does not contain directed path of length 3. It is easy to see that T_a is uniquely determined (up to isomorphism). The path T_3 is depicted in Fig. 1 (where all arcs are directed upwards).


Fig. 1

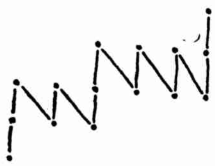
Fig. 2

Similarly, $T_{a,b}$ will denote a path of length $2a + 2b + 4$ and height 4, as illustrated in Fig. 2.

Proposition 2.8. *For a fixed Λ the following two statements are equivalent*

1. There are mutually rigid trees T, T', $\Lambda(T) = \Lambda(T') = \Lambda$,

2. $\Lambda \geqq 4$.

Proof. 2. ⇒ 1. Consider trees $T_{a,b}$.

Then, using 1.6 there exists a homomorphism $f: T_{ab} \to T_{a'b'}$ if and only if $a \leqq a'$, $b \leqq b'$ Thus $T_{1,2}$ and $T_{2,1}$ are mutually rigid. It is easy to extend these to $T'_{2,1}$ and $T'_{1,2}$ respectively, so that $T'_{2,1}$ and the $T'_{1,2}$ remain mutually rigid, and have the required Λ.

1. ⇒ 2. Exhausting a few cases one can check that all rigid trees with $\Lambda \leqq 3$ are

directed paths with $\Lambda \leqq 2$ and the graphs T_a, $a \geqq 1$. (In the only non-trivial case $\Lambda = 3$, this also follows from the next Proposition.)

□

The next result characterises rigid graphs with height $\leqq 3$. Recall that a *retract* of a graph G is a subgraph H of G such that there exists a homomorphism $G \to H$ with $f(H) = h$ for all $h \in V(H)$.

Proposition 2.8. (1) *Let G be connected and balanced,* $\Lambda(G) = 3$. *Then there exists an a such that G has a retract isomorphic to* T_a. (2) *Let G be connected and balanced,* $\Lambda(G) = i$, $i = 0, 1, 2$. *Then G has a retract isomorphic to* $\vec{P}_i$.

Proof. Let a be the minimal such that T_a is a subgraph of G. Put $V(T_a) = x_0$, $x_{1(0)}, x_{2(0)}, x_{1(1)}, \ldots, x_{1(a)}, x_{2(a)}, x_3$. We show that T_a is a retract of G. Define $r : G \to T_a$ by the following:

$r(z) =$ the unique vertex ξ of T_a with $\lambda(\xi) = \lambda(z)$ and with the distance (# arcs) to x_0 at least min $((2a + 3), d_z)$ (where d_z is the minimum distance between z and any vertex v with $\Lambda(v) = 0$ in G).

This r maps all z with $\Lambda(z) = 0$ to x_0, all z with $\Lambda(z) = 3$ to x_3 (by minimality of a) and all other vertices "as far away from x_0 as possible". It is easy to see that r is a homomorphism, and that $r(z) = z$ if $z \in T_a$.

The proof of (2) is easy. Since $\vec{P}_i \to G \to \vec{P}_i$ is rigid, $G \to \vec{P}_i$ must be a retraction.

□

3. MAIN RESULTS

Now we can formulate our main results:

Theorem 3.1. *For a directed graph A the following three statements are equivalent:*

1. Either A is unbalanced or $\Lambda(A) \geqq 4$;

2. There are two mutually rigid paths P_1 *and* P_2 *of height* **4** *which admit homomorphism into A;*

3. The class $\to A$ *is universal.*

Theorem 3.2. *For a directed graph A the following two statements are equivalent:*

1. Either A is unbalanced or $\Lambda(A) \geqq 3$;

2. The class $A \leftrightarrow$ *is universal.*

First, we shall prove Theorem 3.1, Theorem 3.2 will be proved similarly.

We shall make use of the following:

Lemma 3.3. *Let P be a rigid finite path,* $\lambda(P) \geqq 4$. *Then there are mutually rigid paths* P_1, P_2 *such that P is a homomorphic image of both* P_1 *and* P_2.

Proof. Put $a(P) = k$. An antidirected path in P is called of type 1 (type 2, respectively) if it contains only vertices x with $\Lambda(x) = 1$ and 2 ($\Lambda(x) = 2$ and 3, respectively). (Note that every antidirected path contains vertices with two values of Λ only.) Let P_1 (P_2 respectively) be the path which is obtained from P replacing every antidirected path of length a of type 1 (type 2 respectively) by an antidirected path of length $k + a$. It is easy to check (using 2.6) that P_1, P_2 are rigid, that there is no homomorphism $P_1 \to P_2$ and $P_2 \to P_1$, and that P is a homomorphic image of both P_1 and P_2.

□

Proof *of Theorem 3.1.* 1. ⇔ 2. is a combination of Lemma 3.3 and Proposition 2.8. Next, we prove 3 ⇒ 2, which is easier. Of course it follows from universality that there are 2 mutually rigid graphs G_1, G_2 which admit homomorphisms to H. Using Proposition 2.8 and Lemma 2.6 we get $\Lambda(H) \geqq 4$. Combining Proposition 2.7 with Lemma 3.3 yields 2.

Now we prove 2. ⇒ 3.

Let P_1, P_2 be two mutually-rigid paths of height 4. Explicitely, let $P_i = (V_i, E_i)$, $i = 1, 2$. Let $a_i^0, a_i \in V_i$ satisfy $\Lambda(a_i^0) = 0$, $\Lambda(a_i) = 3$, $i = 1, 2$. Let $k \geqq \max\{a(P_1)\, a(P_2)\}$ be a fixed odd number. Let $G = (V, E)$ be a given antisymmetric digraph (i.e. such that $(x, y) \in E \Rightarrow (y, x) \in E$).

We shall construct a directed graph $G^* = (V^*, E^*)$ as follows:

$$V^* = (V \times V_1) \cup (E \times V_2) \cup (E \times \{a_1, \ldots, a_k, b_1, \ldots, b_k\}).$$

The set of arcs consists of the following arcs:

$$((v, v_1), (v, v_1')) \quad \text{where} \quad (v_1, v_1') \in E_1,$$

$$((e, v_2), (e, v_2')) \quad \text{where} \quad (v_2, v_2') \in E_2.$$

Furthermore, for any $e = (v, v') \in E$, let the vertices $(v, a_1^0), (e, a_1), (e, a_2), \ldots, (e, a_k)$, (e, a_2^0) and the vertices $(e, a_2^3), (e, b_1), (e, b_2), \ldots, (e, b_k), (v', a_1^3)$ form an antidirected path of length $k + 1$ with $((v, a_1^0), (e, a_1)) \in E^*$ and $((e, a_2^3), (e, b_1)) \in E^*$.

Thus the graph G^* is obtained from G by replacing every vertex by a copy of P_1 and every edge of G by a copy of P_2 and by joining appropriate copies by "long" antidirected paths. Obviously G^* admits a homomorphism to H. See also Fig. 3 (again all arrows upwards):

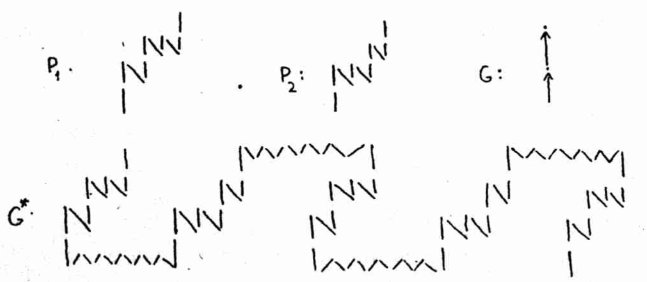

Fig.3

Now it should be clear that if $G = (V, E)$ and $G' = (V', E')$ are directed graphs and $f: G \to G'$ is a homomorphism then f induces a homomorphism $f^*: G^* \to G^*$. The mapping f^* may be defined by

$$f^*(v, v_1) = (f(v), v_1),$$
$$f^*((v, v'), x) = ((f(v), f(v')), x).$$

On the other hand, if $g: G^* \to G'^*$ is a homomorphism, then (using the mutual rigidity of P_1 and P_2 and the assumption on k) we have

$$g(\{v\} \times V_1) = \{v\} \times V_1,$$
$$g(\{e\} \times V_2) = \{e'\} \times V_2.$$

Put $\bar{v} = f(v)$. It is also clear from construction that $e' = (f(v), f(v'))$ if $e = (v, v')$. Thus $g = f^*$.

Consequently the homomorphisms between graphs G^* and G'^* are in $1 - 1$ correspondence with homomorphisms between G and G'. This correspondence establishes the desired embedding of the category of all antisymmetric graphs into the category of all digraphs which admit homomorphisms to H.

□

Proof *of Theorem 3.2.* We do not need to worry about homomorphic image. Thus let P be a path indicated on Fig. 4:

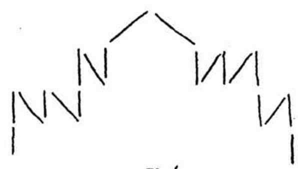

Fig. 4

It is easy to show that P is a rigid graph. For a given antisymmetric graph $G + (V, E)$ we can construct a directed graph $G^* = (V^*, E^*)$ by replacing every edge of G by a copy of the path P. It is a routine to check that every homomorphism between G^* and H^* is induced by a homomorphism between G and H. This is similar (in fact easier) to the above proof, we leave the details.

□

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D_0-FAVOURING EULERIAN TRAILS IN DIGRAPHS

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Dedicated to the memory of Milan Sekanina

Abstract. A characterization for a special class of Eulerian trails in digraphs which traverse a set of arcs of a subdigraph D_0 before any arc of $D_1 = D - D_0$ is traversed, is proved. The most general structure of a subdigraph D_1 to allow such a restricted Eulerian trail is given.

Key words. Directed graph, Eulerian trail, restricted Eulerian trail, spanning in-tree.

MS Classification. 05 C 139

PRELIMINARIES

For notation and terminology, see [2, 4]. Let D be a digraph with vertex set $V(D)$ and $A(D)$. In particular, $V(D)$ and $A(D)$ are always assumed to be finite, $A_v^+ \subset A(D)$ denotes the set of arcs, incident from v, for $v \in V(D)$. For a digraph D and a subdigraph D_1 let $D - D_1 \subseteq D - A(D_1)$ denote the uniquely determined digraph without isolated vertices. The following lemma is folklore.

Lemma 1. *Let D be a digraph and $\text{od}_D(v) \geqq 1$ for all $v \in V(D)$. Then there exists at least one non-trivial strongly connected component C with no arc of D incident from C (that is, $(a, b) \in A(D)$ implies either $a \notin V(C)$, or $b \notin V(D) - V(C)$).*

Lemma 2. *Let D be a digraph satisfying $\text{od}_D(v) \geqq 1$ for all $v \in V(D)$. Suppose D has precisely one (nontrivial) strongly connected component C with no arc of D incident from C. Then there exists a spanning in-tree with root v_0, where v_0 is an arbitrary vertex of C.*

Proof. Let v_0 be an arbitrary vertex of C, and let B_0 be an in-tree with root v_0 containing a maximum number of vertices. If $V(B_0) \neq V(D)$ then we consider $D_0 = \langle V(B_0) \rangle$, the digraph induced by $V(B_0)$. Because of the maximality of B_0 there does not exist an arc (x, y) with $x \in V(D) - V(D_0)$ and $y \in V(D_0)$; furthermore, one easily concludes that $C \subseteq D_0$. $D_1 = D - V(D_0)$ fulfills the assumptions of Lemma 1. Because of Lemma 1 there exists a strongly connected component $C' \subset D_1$ such that no arc of D_1 is incident from C'. By construction it follows that $C' \cap C = \emptyset$ which contradicts the uniqueness of C.

Definition. Let D be a weakly connected eulerian digraph, and let D_0 be a subdigraph of D. An eulerian trail T of D is called *D_0-favouring* if and only if for every $v \in V(D)$, T traverses every arc of D_0 incident from v before it traverses any arc of $D_1 = D - D_0$ incident from v.

Of course, every eulerian trail of D is a D_0-favouring eulerian trail for some D_0 (just take $D_0 = D$). For which subdigraph D_0 of exists a D_0-favouring eulerian trail? There are two known results on the existence of D_0-favouring eulerian trails depending on the structure of $D_1 = D - D_0$.

Theorem 1. *Let D be a weakly connected eulerian digraph, and for given $v \in V(D)$ let $D_0 \subset D$ be chosen such that $D_1 = D - D_0$ is a spanning in-tree of D with root v. Then there exists a D_0-favouring eulerian trail starting (and ending) at v. Conversely, if T is an eulerian trail of D starting (and ending) at v, and if we mark at every $w \in V(D)$, $w \neq v$, the last arc of T incident from w, then D_1, the subgraph of D induced by the marked arcs, is a spanning in-tree with root v (and hence T is a $(D - D_1)$-favouring eulerian trail of D).*

Theorem 1 plays an essential role in establishing the BEST-Theorem which gives a formula for the number of eulerian trails in an eulerian digraph. A proof of Theorem 1 can be found in [1].

Theorem 2. *Let D be an eulerian digraph. Let $D_1 \subseteq D$ be chosen such that* $\mathrm{od}_{D_1}(v) \geqq 1$ *for every $v \in V(D_1) \subset V(D)$, and let $D_0 = D - D_1$.* D has *a D_0-favouring eulerian trail if and only if D_1 has precisely one (nontrivial) strongly connected component C_1 with the property that no arc of D_1 is incident from C_1. Moreover, every D_0-favouring eulerian trail of D must start at some vertex of C_1, and for any vertex of C_1 there is a D_0-favouring eulerian trail of D starting at that vertex.*

Theorem 2 was proved by Berkowitz [3].

A GENERAL THEOREM

In view of Theorems 1 and 2, we ask the following question: What is the most general structure a subdigraph D_1 of an eulerian digraph D can have in order to imply the existence of a $(D - D_1)$-favouring eulerian trail T?

Theorem 2 implies that D_1 must not contain more than one nontrivial strongly connected component C_1 with the property that no arc of D_1 is incident from C_1. But this condition is not sufficient even if D_1 is weakly connected; this can be seen from the digraph D^* of Figure 1.

What if we go the other way round? That is, given an eulerian digraph D and $D_1 \subseteq D$, can we find $D_1^+ \subseteq D$ with $D_1 \subseteq D_1^+$ such that D has a $(D - D_1^+)$-favouring eulerian trail T^+ which induces a $(D - D_1)$-favouring eulerian trail T?

This approach and Theorem 1 and Theo:em 2 lead to the following theorem which answers our original question.

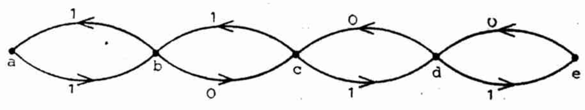

Figure 1

Figure 1. An eulerian digraph D^* having no D_0-favouring eulerian trail (the arcs of D_i are marked with i, $i = 0, 1$).

Theorem 3. *Let D be an eulerian digraph, and let D_1 be a subdigraph of D. Any two of the following statements are equivalent:*

1. *D has a $(D - D_1)$-favouring eulerian trail.*

2. *There exists a digraph D_1^+ with $D_1 \subseteq D_1^+ \subseteq D$ such that for every $v \in V(D)$*

a) $\mathrm{od}_{D_1^+}(v) = \mathrm{od}_{D_1}(v)$ *if and only if* $\mathrm{od}_{D_1}(v) \neq 0$;

b) $\mathrm{od}_{D_1^+}(v) = 1$ *otherwise.*

c) *D_1^+ has precisely one non-trivial strongly connected component C_1 with no arc of D_1^+ incident from C_1.*

3. *There exists a digraph D_1^+ with $D_1 \subseteq D_1^+ \subseteq D$ such that*

a) *D has a $(D - D_1^+)$-favouring eulerian trail;*

b) *for every D_1' with $D_1 \subseteq D_1' \subseteq D_1^+$, if $(x, y) \in A(D_1' - D_1)$, then* $\mathrm{od}_{D_1}(x) = 0$.

4. *D_1 contains a spanning in-forest D_1^- such that*

a) *for some v_0 and for every $x \in V(D_1) - v_0$,* $\mathrm{od}_{D_1^-}(x) = 0$ *if and only if* $\mathrm{od}_{D_1}(x) = 0$, *and* $\mathrm{od}_{D_1^-}(v_0) = 0$;

b) *D has an in-tree B with root v_0 and $D_1^- \subseteq B$.*

Proof. *1. implies 2.* Let T be a $(D - D_1)$-favouring eulerian trail starting at v_0. Define D_1^+ by $D_1^+ = D_1$ if $\mathrm{od}_{D_1}(v) \geqq 1$ for every $v \in V(D)$; otherwise, for every v with $\mathrm{od}_{D_1}(v) = 0$, mark the last arc of T which is incident from v, and let D_1^+ consist of D_1 plus the marked arcs. In any case, $D_1 \subseteq D_1^+$ and D_1^+ satisfies **2. a)**, **2. b)**. Moreover, T is a $(D - D_1^+)$-favouring eulerian trail because of the choice of the elements of $A(D_1^+) - A(D_1)$. It remains to show that D_1^+ has precisely one non-trivial strongly connected component C_1 with no arc of D_1^+ incident from C_1. Because of $\mathrm{od}_{D_1^+}(v) \geqq 1$ for every $v \in V(D_1^+)$ and the finiteness of D_1^+, D_1^+ has at least one non-trivial strongly connected component and, in particular, by Lemma 1 at least one non-trivial strongly connected component C_1^+ with no arc of D_1^+ incident from C_1^+.

T must start and end in a vertex of C_1^+. Otherwise, there exist one or more arcs (v, w) of D such that $v \in V(C_1^+)$ and $w \notin V(C_1^+)$; among these arcs let (v_1, w_1) be the last arc in T, such that $v_1, (v_1, w_1), w_1$ is a section of T. By definition of C_1^+, $(v_1, w_1) \notin A(D_1^+)$, and because of $\mathrm{od}_{D_1^+}(v_1) \geqq 1$ we get a contradiction to the fact that T is a $(D - D_1^+)$-favouring eulerian trail. It's clear now that there can be only one component C_1^+ with the desired property. The implication now follows.

2. implies 3. Take D_1^+ and C_1 as defined by **2 a). b),** and **c).** At first it will be proved that D has a $(D - D_1^+)$-favouring eulerian trail.

Properties **2. a), b),** imply that D_1^+ is a spanning subdigraph of D. Therefore and because of Lemma 1, and property **2. c)** there exists in D a spanning in-tree $B_1^+ \subset D_1$ with root $v_0 \in V(C_1)$ (sce Lemma 2).

Mark all the arcs of B^+. Construct T by starting at vertex v_0 with any arc (v_0, x), choose any unmarked arc incident from x, if such arc exists; otherwise, choose among the marked arcs one which does not belong to B_1^+ if such arc exists; otherwise, choose the arc of B_1^+. Continue this way until this procedure terminates at some $y \in V(D)$. Then $y = v_0$; otherwise, T contains more arcs incident to y than it contains arcs incident from y contradicting D being eulerian. Suppose T does not contain all arcs of D. Then let z be a vertex incident with arcs not contained in T. Since D is eulerian and T is a closed trail, $\mathrm{id}_{D-T}(z) = \mathrm{od}_{D-T}(z) \neq 0$. Moreover, $z \neq v_0$ by the very construction of T. By definition of B_1^+, there is a path $P(z, v_0) \subset \subset B_1^+$ joining z to v_0. Write

$$P(z, v_0) = z, (z, u_1), u_1, \ldots, u_k, (u_k, v_0), v_0;$$

possibly $z = u_k$ and $u_1 = v_0$ (i.e. $P(z, v_0)$ may contain just one arc). By the construction of T it follows that (z, u_1) is not contained in T; therefore, also (u_1, u_2) is not contained in T (note that (u_1, u_2) can be contained in T only if all arcs incident to u_1 are contained in T); a.s.o. In particular, (u_k, v_0) is not contained in T, contradicting the fact that $\mathrm{id}_T(v_0) = \mathrm{od}_T(v_0) = \mathrm{id}_D(v_0) = \mathrm{od}_D(v_0)$. Thus, T contains all arcs of D. This and the construction of T imply that T is a $(D - D_1^+)$-favouring eulerian trail of D.

Now consider any D_1' with $D_1 \subseteq D_1' \subseteq D_1^+$ and suppose $A(D_1' - D_1) \neq \emptyset$; let $(x, y) \in A(D_1' - D_1)$. By definition of D_1^+ in **2. a), b),** an arc of $D_1^+ - D_1$ is necessarily incident from a vertex z with $\mathrm{od}_{D_1}(z) = 0$. Hence $(x, y) \in A(D_1' - D_1)$ implies $\mathrm{od}_{D_1}(x) = 0$; thus **3. b)** holds as well.

3. implies 4. **Start with D_1^+ as described in 3., and consider a $(D - D_1^+)$-favouring eulerian trail T^+ of D. If there is $w \in V(D)$ different from the initial vertex v_0 of T^+ such that the last arc of T^+ incident from w is not in D_1^+, then mark this arc. Note that in this case none of the arcs incident from w lies in D_1^+.**

We define

$$D_1^{++} = D_1^+ \qquad \text{if no such } w \text{ exists;}$$

otherwise,

$$D_1^{++} = \langle A(D_1^+) \cup \{a \in A_w^+ / \mathrm{od}_{D_1^+}(w) = 0 \text{ and } a \text{ has been marked}\}\rangle.$$

In any case, by definition of D_1^{++}, T^+ is even a $(D - D_1^{++})$-favouring eulerian trail of D, and D_1^{++} satisfies **3. b)** as well. Moreover, $V(D_1^{++}) = V(D)$.

Marking for every $v \neq v_0$ the last arc of T^+ incident from v yields a spanning subdigraph B and $B \subset D_1^{++}$ follows from the very definition of D_1^{++}. Furthermore, $\mathrm{od}_B(v) = 1$ for all $v \neq v_0$ and $\mathrm{od}_B(v_0) = 0$. Suppose B is not connected; then there exists a weakly connected component B_1 of B which does not contain v_0 and $\mathrm{od}_{B_1}(w) = \mathrm{od}_B(w) = 1$ for all $w \in V(B_1)$. By Lemma 1 there exists at least one nontrivial strongly connected component $C_1 \subseteq B_1$ with no arc of B_1 incident from C_1. Now, if r is the last vertex of T in C_1, such that $r, (r, s), s$ is a section of T, then it follows from the construction of B that $(r, s) \in A(B)$; furthermore $s \in \in V(C_1)$ because of the definition of C_1. By the choice of r, T terminates in C_1 contradicting the fact, that T is an eulerian trail starting in $v_0 \notin V(B_1) \supset V(C_1)$. Thus B is connected, and $\mathrm{od}_B(v) = 1$ for all $v \neq v_0$, $\mathrm{od}(v_0) = 0$. This implies that B is a spanning in-tree of $D_1^{++} \subseteq D$ rooted at v_0.

Define D_1^- by $V(D_1^-) = V(D_1)$ and $A(D_1^-) = A(B) \cap A(D_1)$; thus D_1^- is a spanning in-forest of D_1 which satisfies *4. b)*. Let (x, y) be any arc of B not in D_1^-; then $x \neq v_0$. If $(x, y) \notin A(D_1^+)$, then it follows from the definition of D_1^{++} and $D_1^{++} \supset D_1$ that $\mathrm{od}_{D_1^+}(x) = 0 = \mathrm{od}_{D_1}(x)$. If $(x, y) \in A(D_1^+)$, then $(x, y) \notin A(D_1)$ by definition of D_1^-; and by **3. b)** with $D_1' = D_1^+$, $\mathrm{od}_{D_1}(x) = 0$ follows.

We summarize: D_1^- is a spanning in-forest of D_1, and if $x \neq v_0$ for some $v_0 \in \in V(D_1^-)$ (which is the root of B indeed) satisfies $\mathrm{od}_{D_1^-}(x) = 0$ then $\mathrm{od}_{D_1}(x) = 0$ (for, x not being the root of B implies $(x, y) \in A(B - D_1^-)$ for some y). Since $\mathrm{od}_{D_1}(x) = 0$ implies $\mathrm{od}_{D_1^-}(x) = 0$ anyway and $\mathrm{od}_{D_1^-}(v_0) = \mathrm{od}_B(v_0) = 0$, and because $D_1^- \subseteq B$ with $V(B) = V(D)$, the proof of the implication is finished.

4. implies 1. Let $D_1^- \subseteq D_1$ be chosen as described in **4. a)** and let B be a spanning in-forest of D with root v_0 and $D_1^- \subseteq B$. Marking all arcs of B we construct a trail T by starting at vertex v_0 with any arc (v_0, x). Choose any unmarked arc incident from x, if such arc exists; choose the marked arc incident from x, otherwise.

Continuing this way until this procedure terminates we get a $(D - B)$-favouring eulerian trail (for arguments see **2.** implies **3.**).

Because of the freedom to choose the order in which the arcs of $A_v^+ - A(B)$ appear in T for every $v \in V(D)$ we are even able to construct T in such a way that the arcs of $A_v^+ \cap (D - D_1)$ appear in T before any of the arcs of $A_v^+ \cap D_1$ are used. This is true even in the case where an arc $(x, y) \in B$ does not belong to D_1; for, in this case $\mathrm{od}_{D_1^-}(x) = \mathrm{od}_{D_1}(x) = 0$ by **4. a)**, i.e. $A_x^+ \cap A(D_1) = \emptyset$, i.e., $A_x^+ \subseteq \subseteq D - D_1$. In the case of v_0, if $A_{v_0}^+ \cap A(D_1) \neq \emptyset$, than we proceed in the construction of T by starting along an arc of $A_{v_0}^+ \cap A(D - D_1)$, and each time we arrive in v_0 we continue along an arc of $A_{v_0}^+ \cap A(D - D_1)$ not traversed before, as long as there is such an arc. Consequently, T is a $(D - D_1)$-favouring eulerian trail of D. This finishes the proof of the implication. Theorem 3 now follows.

It is easy to see that Theorem 3 is a generalization of Theorem 1 and Theorem 2. Both Theorems can be derived by using the equivalent statements of Theorem 3 and some details of their proof. We also note that in proving Theorem 3 we used ideas developed originally for the proofs of Theorem 1 and Theorem 2.

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REMARKS ON HAMILTONIAN PROPERTIES OF SQUARES OF GRAPHS

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In memory of Milan Sekanina

Abstract. This paper deals with problems concerning the existence of such Hamiltonian cycles or paths in squares of graphs containing some edges of the original graphs. Using a method due to Říha several results on blocks could be found generalizing previous ones.

Key words. Squares of graphs, blocks, Hamiltonian cycles, Hamiltonian paths.

MS Classification. 05 C

0. It has been M. Sekanina who 25 years ago posed the question for the structure of those graphs G the square of which has an open or a closed Hamiltonian line (i.e. G^2 is traceable or Hamiltonian, resp.), cf. [7]. Since that time many results concerning this problem could be obtained; to the most important and well-known ones among them certainly belong the Theorem of Fleischner [2], [3] verifying a conjecture of Plummer and Nash–Williams [4] (Every block G with at least 3 vertices has a Hamiltonian square) and its generalization by Chartrand, Hobbs, Jung, Kapoor, Nash–Williams [1] (For every block G its square is Hamiltonian-connected and, if G has at least 4 vertices, G^2 is 1-Hamiltonian as well). Recently, St. Říha, a young former co-worker of Sekanina's succeeded in finding an excellent proof of the following statement (cf. [6]) which implies Fleischner's theorem and its generalization mentioned above.

Theorem 0: *Let G be a block with at least* **3** *vertices and x any vertex of G. Then there are two different G-neighbours a, b of x and a Hamiltonian path in $G^2 - x$ joining a and b.* □

Using Říha's proof-method and his theorem, in the next sections of this paper we shall get several results on the existence of Hamiltonian cycles in G^2 containing some edges of G, especially a partial answer to the question, if the Hamiltonicity of G^2 always implies the existence of a Hamiltonian cycle in G^2 containing an edge of G. (In case that this conjecture were true it can be easily shown [8] that for any such G even there is a Hamiltonian cycle in G^2 containing at least two edges of G.)

All graphs considered here are supposed to be undirected, simple and finite (possibly empty). Let $G = (V, E)$ be a graph with the vertex-set $V(G) := V$ and the edge-set $E(G) := E$. If $x, y \in V$, $x \neq y$, are the end-vertices of an edge $l \in E$ we denote this edge l by the couple $\{x, y\}$. We say that $x \in V$ is a *G-neighbour* of $y \in V$ iff $\{x, y\} \in E$. The vertex x is called a G-neighbour of $M \subseteq V$ iff $x \notin M$ and x is a G-neighbour of some $y \in M$. If X is a vertex (a subgraph or a vertex-subset) of G then $N(X : G)$ denotes the set of all G-neighbours of the vertex X (of the set of all vertices belonging to X), and $G - X$ is defined to be the subgraph arising from G by deleting the vertex X (all vertices of X) and all edges incident with X (with some vertices of X). By $G(M)$ we denote the induced subgraph of G generated by $M \subseteq V$. The valency (degree) of the vertex $x \in V(H)$ in the subgraph H of G is denoted by $v(x : H)$. The *square* G^2 of G is the graph with $V(G^2) := V(G)$ and $\{x, y\} \in E(G^2)$ iff the distance of x and y in G is 1 or 2. A *block* is a graph which is 2-connected (*non-trivial block*) or a path of length 1 (*trivial block*). A block G is *minimal* iff there is no edge $l \in E(G)$ such that the graph arising from G by deleting l is a block. Paths and cycles w are comprehended to be special graphs (possibly subgraphs of a given graph); as usual they are represented by sequences of the vertices passed by w. Generally we shall not distinguish between a path (or a cycle) w and its representation by a vertex-sequence. A path of length 0 is called *trivial*. If $p = (x_0, x_1, \dots, x_{r-1}, x_r)$ is a vertex-sequence the inverse sequence $(x_r, x_{r-1}, \dots, x_1, x_0)$ is denoted by p^{-1}, and if $q = (y_0, y_1, \dots, y_s)$ is another vertex-sequence then (p, q) is defined to be the vertex-sequence $(x_0, x_1, \dots, x_{r-1}, x_r, y_0, y_1, \dots, y_s)$; analogously in similar cases. The number of elements of a set M is denoted by $|M|$.

1. Let G be a graph, w a non-trivial path in G and x an endvertex of w.

Definition. $\mathbf{S}$ is a $(G^2, w, x)-$*basic*$-$*set* iff $\mathbf{S}$ is a set of pairwise vertex-disjoint paths in $G^2 - w$ with $\bigcup_{p \in \mathbf{S}} V(p) = V(G) - V(w)$, and there is a mapping f from $\mathbf{S}$ into the power-set of $V(w)$ with the following properties:

(1) $\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2$ where $\mathbf{S}_i := \{p \in \mathbf{S} : |f(p)| = i\}$, $i = 1, 2$;

(2) for each $p \in \mathbf{S}_2$, if $\{a_1, a_2\} = f(p)$ and $\{e_1, e_2\}$ is the set of the endvertices of p (possibly $e_1 = e_2$), it holds: $\{a_1, e_1\}, \{a_2, e_2\} \in E(G)$ or $\{a_1, e_2\}, \{a_2, e_1\} \in E(G)$;

(3) for each $p \in \mathbf{S}_1$, if $\{a\} = f(p)$, then $\{a, e\} \in E(G)$ holds for every endvertex e of p;

(4) $f(p) \cap f(p') = \emptyset$ for any different $p, p' \in \mathbf{S}$;

(5) if $\mathbf{S}_2 = \emptyset$ then there is a $z \in V(w)$, $z \neq x$ such that $z \notin f(p)$ for each $p \in \mathbf{S}$.

The construction given by Řiha in [6] — it is the main point of his proof of Theorem 0 — verifies the following

Lemma. *Let G be a graph, w a non-trivial path in G, x an endvertex of w and* $\mathbf{S}$ *a (G^2, w, x)-basic-set. Then there is a G-neighbour x' of x and a Hamiltonian path in G^2 joining the endvertices of w and containing the edge $\{x, x'\}$ and each $p \in S$ as a subpath.* □

2. Using this Lemma we shall prove some generalizations of Řiha's Theorem 0. For this end we introduce the following notations. Let G be a non-trivial block (i.e. $|V(G)| \geqq 3$) and w a path in G. By $C_1(w)$ and $C_2(w)$ we denote the set of all components C of the graph $G - w$ with $|V(C)| = 1$ and $|V(C)| \geqq 2$, respectively, and we define $C(w) := C_1(w) \cup C_2(w)$. Let $C \in C(w)$. Then $N(C : G) \subseteq V(w)$ and $|N(C : G)| \geqq 2$; for $C \in C_2(w)$ at least two vertices of C have G-neighbours in $N(C : G)$ and therefore there are two different vertices in C having a pair of different G-neighbours in $N(C : G)$. For every $C \in C(w)$ we form the graph G_C arising from $G(V(C) \cup N(C : G))$ by contracting all vertices of $N(C : G)$ to a new vertex $0 \notin$ $\notin V(G)$ (the edges between C and $N(C : G)$ in G become edges between C and 0 in G_C, of course), where resulting multiple edges are replaced by a simple edge with the same endvertices and resulting loops are removed. Obviously, G_C is a block, and $|V(G_C)| \geqq 3$ if $C \in C_2(w)$. Let us suppose:

(6) For each $C \in C_2(w)$ there is given a Hamiltonian path h_C in $G_C^2 - 0$ joining two G_C-neighbours of 0.

Furthermore, for each $C \in C_1(w)$ we define $h_C := C$ (the trivial path consisting of the single vertex of C). Then it follows that h_C and $h_{C'}$ are vertex-disjoint if $C \neq C'$, $C, C' \in C(w)$. Denote by $\mathbf{P}$ the set of all subpaths arising from the family $(h_C : C \in C(w))$ by deleting, for each h_C, all edges in h_C which do not belong to $E(C^2)$. We remark that $\mathbf{P}$ consists of pairwise vertex-disjoint paths in $(G - w)^2$ the endvertices of which are G-neighbours of some vertices of w, that every edge belonging to $E(h_C) \cap E(G)$ for a $C \in C(w)$ is also an edge of some $p \in \mathbf{P}$, and that the (disjoint) union of all sets $V(p)$ with $p \in \mathbf{P}$ results in $V(G) - V(w)$. Now the following algorithm (*) is applied to $\mathbf{P}$ (see Řiha [6]):

(*) If there exist different paths $p, p' \in \mathbf{P}$ with the property that there is a $z \in V(w)$ which is a G-neighbour of an endvertex x of p as well as of an endvertex x' of p', we take such a pair $p = (a, \ldots, x)$, $p' = (x', \ldots, b)$ with $x, x' \in N(z : G)$ for a $z \in$ $\in V(w)$, form the path $p'' = (p, p') = (a, \ldots, x, x', \ldots, b)$ which is a path of $G^2 - w$ whose endvertices a, b are G-neighbours of some vertices of w (possibly of only one vertex of w), and replace the elements p, p' in $\mathbf{P}$ by p''. We obtain the set $\mathbf{P}' :=$ $:= (\mathbf{P} - \{p, p'\}) \cup \{p''\}$ and repeat this procedure with respect to $\mathbf{P}'$, and so on. After a finite number of steps – say r – this algorithm stops, and the resulting set $\mathbf{S} := \mathbf{P}^{(r)}$ has the properties:

(7) $\mathbf{S}$ consists of pairwise vertex-disjoint paths in $G^2 - w$ the endvertices of which are G-neighbours of some vertices of w;

(8) for any different elements $p = (x, \ldots, x')$ and $q = (y, \ldots, y')$ of $\mathbf{S}$, the end-

vertices of these paths satisfy

$$N(\{x, x'\} : G) \cap N(\{y, y'\} : G) \cap V(w) = \emptyset;$$

(9) $\bigcup_{p \in \mathbf{S}} V(p) = V(G) - V(w);$

(10) for any $C \in C_2(w)$ every $l \in E(h_C) \cap E(G)$ is also an edge of some $p \in \mathbf{S}$.

Let $\mathbf{S}(h_C : C \in C(w))$ denote the set of all such path-sets $\mathbf{S}$ which can be obtained if we apply algorithm (*) to $\mathbf{P}$ in any possible way. Then it is easy to see that every $\mathbf{S} \in \mathbf{S}(h_C : C \in C(w))$ fulfils (10) and all properties of a (G^2, w, x)-basic-set with the exception of (5), where x is either endvertex of w. The mapping f is chosen as follows: If for a $p \in \mathbf{S}$ with $|V(p)| \geqq 2$ the endvertices e_1, e_2 of p satisfy $m := |V(w) \cap (N(e_1 : G) \cup N(e_2 : G))| \geqq 2$, we take arbitrary $a_i \in N(e_i : G) \cap V(w)$, $i = 1, 2$, with $a_1 \neq a_2$ and define $f(p) := \{a_1, a_2\}$; if $m = 1$ we have to put $f(p) := \{a\} = N(e_1 : G) \cap V(w)$. For a $p \in \mathbf{S}$ with $|V(p)| = 1$ it follows $|N(e : G) \cap V(w)| \geqq 2$ for the vertex e of p if $p \in C_1(w)$, and we take $a_1, a_2 \in N(e : G) \cap V(w)$, $a_1 \neq a_2$, and $f(p) := \{a_1 a_2\}$; if $p \notin C_2(w)$ and $|N(e : G) \cap V(w)| \geqq 2$ we proceed as before; if $p \notin C_2(w)$ and $|N(e : G) \cap V(w)| = 1$ we define $f(p) := \{a\}$ with $\{a\} = N(e : G) \cap V(w)$.

3. Now we suppose G to be a minimal block with $(V(G)) \geqq 3$, and let x and y be different vertices. Then there exists a cycle in G containing x and y. Because this cycle has two different vertices a, b with $v(a : G) = v(b : G) = 2$ (see Plummer [5], Řiha [6]), at least one of the two independent paths joining x and y which form a separation of this cycle must contain a vertex $z \neq x$ with $v(z : G) = 2$. A path p satisfying this property (i.e. p joins x and y and contains a vertex $z \neq x$ with $v(z : G) = 2$) is called an *admissible* (x, y)-path in G and x its *initial* vertex. (Obviously, an admissible (x, y)-path is not necessarily an admissible (y, x)-path.) Note that for any $x \neq y$ there is an admissible (x, y)-path in the minimal block G; if $\{x, y\} \in E(G)$ then every path in G of length $\geqq 2$ joining x and y is an admissible (x, y)-path, and if $\{x, y\} \notin E(G)$ then there is an admissible (x, y)-path which is not a Hamiltonian path. Let w be an admissible (x, y)-path in G and assume (6) for this w. Then there is a $z \in V(w)$, $z \neq x$ with $v(z : G) = 2$. Assume that the family $(h_C : C \in C_2(w))$ satisfies the additional property:

(6a) If $v(y : G) = 2$ and y is not a G-neighbour of x and the (only) G-neighbour $y^* \notin V(w)$ of y belongs to a component C^* of $G - w$ fulfilling $C^* C_2(w)$, then $h_C{}^*$ contains an edge $\{y^*, z\} \in E(G)$ with some $z \in N(y^* : C^*)$. Now consider an $\mathbf{S} \in \mathbf{S}(h_C : C \in C(w))$ and a mapping f described at the end of section 2. Then it follows that the set $\mathbf{S}_2 = \{p \in \mathbf{S} : |f(p)| = 2\}$ is empty only in the case that for each p the premise $p \in \mathbf{S}$ implies $|V(p)| \geqq 2$ and $|V(w) \cap (N(e_1 : G) \cup N(e_2 : G)| = 1$, for the endvertices e_1, e_2 of p or $|V(p)| = 1$ and $|N(e : G) \cap V(p)| = 1$ with $(e) = p$. In the first case we conclude $f(p) = N(e_1 : G) \cap V(w) = N(e_2 : G) \cap$

$\cap V(w) = \{a_p\}$ because of (7); obviously, $v(a_p : G) \geqq v(a_p : w) + v(a_p : G(\{e_1, e_2, a_p\})) \geqq 3$, and thus we have $a_p \neq z$, i.e. $z \notin f(p)$.

In the second case we have $f(p) = N(e : G) \cap V(w) = \{a_p\}$ for the vertex e of p; further $v(a_p : G) \geqq v(a_p : w) + v(a_p : G(\{e, a_p\}) \geqq 2 + 1 = 3$ if a_p is an inner vertex of w, and if $a_p = y$ and $\{x, y\} \in E(G)$ then $v(y : G) \geqq v(y : G(w)) + v(y : G(\{e, y\}) \geqq 2 + 1 = 3$. Now let $a_p = y$, $\{x, y\} \notin E(G)$; if $v(y : G) = 2$ then because of (6a) and (10) it follows that $\{y^*, z\} \in E(p)$ for some $z \in N(y^* : G - w)$, where $y^* \in N(y : G) - V(w)$. This is a contradiction to $|V(p)| = 1$. Hence in every case $v(a_p : G) \geqq 3$, and thus $a_p \neq z$, i.e. $z \notin f(p)$. So we have proved $z \notin f(p)$ for each $p \in \mathbf{S}$ if $\mathbf{S}_2 = \emptyset$. Consequently, $\mathbf{S}$ and f fulfil (5); using the statements of section 2 and the notations introduced there we get

Corollary 1. *Let G be a minimal non-trivial block, $x, y \in V(G)$ with $x \neq y$, and w an admissible (x, y)-path in G. Furthermore, we assume that we are given a family $(h_C : C \in C_2(w))$ according to* (6) *and fulfilling* (6a). *Then every $S \in S(h_c$: $C \in C(w)$ is a (G^2, w, x)-basic-set satisfying property* (10). □

For any block H with $|V(H)| \geqq 3$ we define

$$s(H) := |V(H)| \sum_{x \in V(H)} (v(x : H) - 2) = 2|V(H)|| \; |C|(E(H)| - |V(H)|).$$

Obviously, $s(H) \geqq 0$ because of $v(x : H) \geqq 2$ for $x \in V(H)$, and $s(H) = 0$ iff H is a cycle. Referring to the notations of section 2 we can prove

Corollary 2. *Let G be a non-trivial block not being a cycle, and w a non-trivial path in G with the endvertices x, y. Then for every $C \in C_2(w)$ the graph G_C is a non-trivial block satisfying*

$$(11) \qquad s(G_C) < s(G).$$

Proof: $C \in C_2(w)$ implies (see section 2) $|V(C)| \geqq 2$, $|N(C : G)| \geqq 2$, $N(C : G) \subseteq V(w)$, and $V(C) \cap N(w : G) = N(0 : G_C)$. Hence, $|V(G_C)| < |V(G)|$, and G_C is a block with $(V(G_C)) \geqq 3$. Let $N(C : G) - \{x, y\} = \{e_1, \dots, e_k\}$, and write $e_0 = x$ and $e_{k+1} = y$. Obviously, for each $\bar{x} \in V(C)$ we have $v(\bar{x} : G_C) \leqq v(\bar{x} : G)$. If $x, y \notin N(C : G)$ we get $k \geqq 2$ and $2 \leqq v(0 : G_C) \leqq \sum_{i=1}^{k} (v(e_i : G) - 2)$; if $x \in N(C : G)$, $y \notin N(C : G)$ it follows $k \geqq 1$ and $2 \leqq v(0 : G_C) \leqq \sum_{i=0}^{k} (v(e_i : G) - 2) + 1$, analogously for $y \in N(C : G)$, $x \notin N(C : G)$; if $x, y \in N(C : G)$ we find $k \geqq 0$ and $2 \leqq v(0 : G_C) \leqq \sum_{i=0}^{k+1} (v(e_i : G) - 2) + 2$. In each of these cases we obtain

$$s(G_C) \leqq |V(G_C)| \sum_{\bar{x} \in V(G)} (v(\bar{x} : G) - 2) = \lambda s(G),$$

with $\lambda = \dfrac{|V(G_C)|}{|V(G)|} < 1$. This results in (11) because G is not a cycle and therefore $s(G) > 0$. □

Note that for blocks H, G with $|V(H)| \geqq 3$, where H is a subgraph of G and $H \neq G$, it follows $s(H) < s(G)$.

4. Generalizing Řiha's theorem (Theorem 0) we show

Theorem 1. *Let G be a block and x, y adjacent vertices. Then there is a G-neighbour x' of x and a Hamiltonian path in G^2 joining x and y and containing the edge $\{x, x'\}$.*

Proof: Obviously, the assertion is true if $|V(G)| = 2$ and also if G is Hamiltonian. Assume, Theorem 1 fails to hold. Let G be a block with the least value of $s(G)$ such that G does not fulfil the property stated in this theorem for some adjacent vertices $x \neq y$. Hence it follows, that G is a minimal block with $|V(G)| \geqq 3$ being not Hamiltonian, i.e. G is not a cycle, and therefore $s(G) > 0$. Because G is a minimal block there is an admissible (x, y)-path w. Obviously w is a non-Hamiltonian path. According to section 2 we form the set $C(w) = C_1(w) \cup C_2(w)$, and for each $C \in C_2(w)$ we consider the graph G_C which is a non-trivial block. Owing to Theorem 0 (cf. [6]) there exists a Hamiltonian path h_C in $G_C^2 - 0$ joining two G_C-neighbours of 0; therefore we can find a family $(h_C : C \in C_2(w))$ realizing (6). (Note that (6a) is trivial because of $\{x, y\} \in E(G)$.) Owing to Corollary 1 every $\mathbf{S} \in \mathbf{S}(h_C : C \in C(w))$ is a (G^2, w, x)-basic-set. Because w is a non-Hamiltonian path, $\mathbf{S}(h_C : C \in C(w)) \neq \emptyset$. Taking an $\mathbf{S} \in \mathbf{S}(h_C : C \in C(w))$ and using the Lemma of section 1 we get a Hamiltonian path in G^2 joining x and y and containing an edge $\{x, x'\}$ for some G-neighbour x' of x, which is a contradiction to the assumption on G. □

Theorem 2. *Let G be a non-trivial block, and x, y different vertices. Then there are different G-neighbours a, b of x, a G-neighbour z of y, and a Hamiltonian path in G^2-x joining a and b and containing the edge $\{y, z\}$.*

Proof: The assertion holds for Hamiltonian graphs, i.e. for all non-trivial blocks G with $s(G) = 0$. Assume Theorem 2 to be not true, and consider a block G with $|V(G)| \geqq 3$ and the least value of $s(G)$ such that the property stated in Theorem 2 is not fulfilled for some $x \neq y$. Then G is a minimal non-trivial block and not Hamiltonian (i.e. not a cycle), what implies $s(G) > 0$.

Case 1: Suppose that there is a cycle k in G with $x \in V(k)$ and $y \notin V(k)$. Let b be a k-neighbour of x. Deleting the edge $\{x, b\}$ in k we obtain a non-Hamiltonian path w which is an admissible (x, b)-path.

According to section 2 we form the set $C(w) = C_1(w) \cup C_2(w)$, and for each $C \in C_2(w)$ we consider the graph G_C which is a block with $|V(G_C)| \geqq 3$.

a) Let $y \in V(T)$ for some $T \in C_2(w)$. Then Corollary 2 yields $s(G_T) < s(G)$; hence it follows that there is a Hamiltonian path h_T in $G_T^2 - 0$ joining two G_T-neighbours of 0 and containing an edge $\{y, z\}$ with a suitable G_T-neighbour z of y. Then $y, z \neq 0$, and therefore z is a G-neighbour of y as well. Thus $\{y, z\} \in E(h_T) \cap E(G)$. For every $C \in C_2(w)$, $C \neq T$, Theorem 0 yields a Hamiltonian path h_C in $G_C^2 - 0$ joining two G_C-neighbours of 0. In this way we have succeeded in finding a family $(h_C : C \in C_2(w))$ realizing (6). Owing to Corollary 1 every $\mathbf{S} \in \mathbf{S}(h_C : C \in C(w)) \neq \emptyset$ (w is a non-Hamiltonian path) is a (G^2, w, x)-basic-set satisfying (10) and consequently, $\{y, z\} \in E(p)$ for some $p \in \mathbf{S}$. Using the Lemma of section 1 with such an $\mathbf{S}$ we obtain a G-neighbour a of x and a Hamiltonian path in G^2 joining x and b and containing the edges $\{x, a\}$ and $\{y, z\}$. Because of $|V(G)| \geqq 3$ we have $a \neq b$, and we have found a Hamiltonian path in $G^2 - x$, joining two different G-neighbours a, b of x and containing the edge $\{y, z\} \in E(G)$. This is a contradiction to the assumption on G.

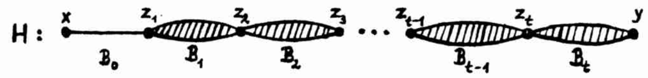
b) Let $y \in V(T)$ for some $T \in C_1(w)$. Then T consists of the vertex y, and y is a G-neighbour of exactly two vertices $z', z \in V(w) = V(k)$ which cannot be adjacent in G (note that k has not diagonals because G is a minimal block). We may assume $z \neq x$. Both paths w_1, w_2 joining z' and z and forming a separation of the cycle k must contain at least one inner vertex ($\neq z, z'$). Then it follows that $G - y$ is a block with $|V(G - y)| \geqq 4$ and $s(G - y) < s(G)$. Thus (because of $x \neq z$) there is a Hamiltonian path p in $(G - y)^2 - x$ joining two $(G - y)$-neighbours a, b' of x and containing an edge $\{z, t\} \in E(G - y) \subseteqq E(G)$. Replacing the subpath (z, t) (which corresponds to the edge $\{z, t\}$) in p by (z, y, t) which is a path of length 2 in G^2, we get a Hamiltonian path p' in $G^2 - x$ joining different G-neighbours a, b' of x and containing the edge $\{y, z\} \in E(G)$. But this is a contradiction to the assumption on G.

Case 2: We have to suppose that every cycle containing x must contain y as well. Note that at least one such cycle exists. Each of the components of the graph $G - \{x, y\}$ is adjacent with x and with y in G and contains exactly one G-neighbour of x. If x and y are adjacent in G, then $G - \{x, y\}$ has exactly one component (G is a minimal block), say T_1; otherwise $G - \{x, y\}$ has at least two components, say $T_1, T_2, \ldots, T_r$, $r \geqq 2$.

a) Let $\{x, y\} \notin E(G)$. By $\bar{H}_i$ we denote the graph arising from $H_i := G(V(T_i) \cup \cup \{x, y\})$ by adding the new edge $\{x, y\}$, $i = 1, \ldots, r$. Obviously, $\bar{H}_i$ is a block with $|V(\bar{H}_i)| \geqq 3$ and $s(\bar{H}_i) < s(G)$ (because of $r \geqq 2$), and, furthermore, $v(x : H_i) = 1$, $v(x : \bar{H}_i) = 2$, $i = 1, \ldots, r$. Consider any $i \in \{1, \ldots, r\}$ and write $\bar{H}$ and H instead of $\bar{H}_i$ and H_i, respectively. Note that H arises from $\bar{H}$ by deleting the edge $\{x, y\}$. Let z denote the $\bar{H}$ neighbour of x being different from y, and let p be any path in $\bar{H}$ joining x and y and not containing the edge $\{x, y\}$; such a path exists, for $\bar{H}$ is a block. Then p is a path in H which contains all cutpoints of H. (A cutpoint z' of H with $z' \notin V(p)$ would imply that both x, y belong to the same component C of $H - z'$, and that there is at least another component $C' \neq C$; therefore the edge $\{x, y\} \in E(\bar{H})$ joins vertices of the same component C of $H - z'$, and we get at least two components of $\bar{H} - z'$, in contradiction to the fact that $\bar{H}$ is a block.) Obviously, one cutpoint of H is z. Hence it follows that p can be represented by the sequence

$$p = (x, z = z_1, \ldots, z_2, \ldots, z_3, \ldots, z_t, \ldots, y) = \\ = (x, p_1', p_2', \ldots, p_{t-1}', p_t),$$

where $z_1, z_2, \ldots, z_t (t \geqq 1)$ are all the (different) cutpoints of H, and $p_0 = (x, z = z_1)$, $p_k = (z_k, \ldots, z_{k+1}) = (p_k', z_{k+1})$, $k = 1, \ldots, t - 1$, and $p_t = (z_t, \ldots, y)$ are non-trivial subpaths of p forming a separation of p. (Of course, $z_t \neq y$ holds because $\bar{H}$ is a block.) With $z_0 := x$, $z_{t+1} := y$ the couple $\{z_k, z_{k+1}\}$ of the endvertices of p_k determines a (maximal) block B_k of H (B_k is the maximal subgraph in H being a block and containing z_k and z_{k+1}), $k = 0, 1, \ldots, t$, and these $B_k's$ satisfy the properties: $V(B_k) \cap V(B_{k+1}) = \{z_{k+1}\}$, $k = 0, \ldots, t - 1$, $V(B_l) \cap V(B_k) = \emptyset$ for $0 \leqq l < k \leqq t$ with $k \neq l + 1$, and $B_0, B_1, \ldots, B_t$ are all the (maximal) blocks of H. (For otherwise we could find a path in H joining x and y and not containing every cutpoint of H, or we would get a cutpoint of $\bar{H}$, respectively, but we have seen that neither of these situations is possible. To put it concisely: The block-cutpoint-graph of H is a path, and x and y belong to its different end-blocks.) Of course, $B_0 = p_0 = (x, z)$.



Because of $s(\bar{H}) < s(G)$ we have $s(B_k) < s(G)$ if $|V(B_k)| \geqq 3$, $k = 0, 1, \ldots, t$. Hence it follows that for such a B_k there is a Hamiltonian path in $B_k^2 - z_{k+1}$ joining two suitable B_k-neighbours z_{k+1}' and z_{k+1}'' of z_{k+1} and containing some edge $\{z_k, \bar{z}_k\} \in E(B_k)$. We can write this path in the form $(z_{k+1}', \ldots, z_k, \bar{z}_k, \ldots, z_{k+1}'')$ and consider the two subpaths $q_k' := (z_{k+1}', \ldots, z_k)$ and $q_k'' := (\bar{z}_k, \ldots, z_{k+1}'')$; note that $\{z_k, \bar{z}_k\}$, $\{z_{k+1}', z_{k+1}\}$, $\{z_{k+1}'', z_{k+1}\} \in E(H)$. In case that $|V(B_k)| = 2$, $k \geqq 1$, we have $\{z_k, z_{k+1}\} \in E(H)$ and we consider the maximal sequence $B_k, B_{k+1}, \ldots, B_{k+l}$ with $|V(B_{k+j})| = 2$, $j = 0, 1, \ldots, l$, and $k \leqq k + l \leqq t$ (that is: Either $k + l = t$ or if $k + l < t$ then it holds $|V(B_{k+l+1})| \geqq 3$); now we define $q_k' := \emptyset$, $q_k'' := (z_k)$ if l is even, and $q_k' := (z_k)$, $q_k'' := \emptyset$ if l is odd. Then the sequence

$$q := (q_t', q_{t-1}', \ldots, q_1', q_1'', q_2'', \ldots, q_t'')$$

is a Hamiltonian path in $(H - x)^2 - y$ satisfying the following property:

If $|V(B_t)| \geqq 3$ then q joins two B_t-neighbours (and therefore H-neighbours) $y' := z_{t+1}'$ and $y'' := z_{t+1}''$ of $y = z_{t+1}$;

if $|V(B_t)| = 2$ and $t \geqq 2$ then q joins some B_{t-1}-neighbour y' of z_t (namely $y' := z_t'$ if $|V(B_{t-1})| \geqq 3$, and $y' := z_{t-1}$ if $|V(B_{t-1})| = 2$) with the B_t-neighbour $y'' := z_t$ of y;

if $|V(B_t)| = 2$ and $t = 1$ then $q = (z_1)$ consists of the only B_t-neighbour $y' := := y'' := z_1$ of y.

Thus we can write $q = (y', \dots, y'')$, where y'' is an H-neighbour of y and y' is an H^2-neighbour of y.

Furthermore, from Theorem 1 it follows, that for each block B_k, $k = 0, \dots, t$, there is a Hamiltonian path q_k^* in B_k^2 joining the vertices z_{k+1} and z_k and containing some edge $\{z_{k+1}, z_{k+1}^*\} \in E(B_k)$. (This is obvious if z_k and z_{k+1} are adjacent. If they are not adjacent we consider the block $\bar{B}$ consisting of B_k, a new vertex 0 and the edges $\{0, z_k\}$ and $\{0, z_{k+1}\}$. Then because of $r \geqq 2$ we have $s(\bar{B}) < s(G)$, and this remains valid also for $r = 1$ if $t \geqq 2$, i.e. in the next subcase **b)** only the situation for $t = 1$ must be considered separately. Hence it follows, that there is a Hamiltonian path in $\bar{B}^2 - 0$ joining the two $\bar{B}$-neighbours of 0 and containing some edge $\{z_{k+1}, z_{k+1}^*\} \in E(\bar{B})$.) We can write We can write $q_k^* = (z_{k+1}, z_{k+1}^*, \dots, z_k)$, $k = 0, 1, \dots, t$, and with $\bar{q}_k^* := (z_{k+1}^*, \dots, z_k)$—i.e. $q_k^* = (z_{k+1}, \bar{q}_k^*)$ —, $k = 0, 1, \dots, t$, it is obvious that the sequence

$$\bar{q} := (\bar{q}_t^*, \bar{q}_{t-1}^*, \dots, \bar{q}_0^*)$$

is a Hamiltonian path in $H^2 - y$ joining an H-neighbour $y^* := z_{t+1}^*$ of y and the vertex x and containing the edge $\{z, x\} \in E(H)$ (because z_1^* is a B_0-neighbour of $z_1 = z$ and therefore $z_1^* = x$).

Thus we have proved the following assertions for $i = 1, \dots, r$:

There is a Hamiltonian path $q_i = (y_i', \dots, y_i'')$ in $(H_i - x)^2 - y$ joining an H_i^2-neighbour y_i' of y and an H_i-neighbour y_i'' of y.

There is a Hamiltonian path $\bar{q}_i$ in $H_i^2 - y$ joining an H_i-neighbour y_i^* of y and the vertex x and containing the edge $\{z^i, x\} \in E(H_i)$, where z^i is the only H_i-neighbour of x; write $\bar{q}_i = (y_i^*, \dots, z^i, x)$ and $\tilde{q}_i := (y_i^*, \dots, z^i) = \bar{q}_i - x$.

Because $V(G - \{x, y\})$ and $E(G)$ are the disjoint unions of the sets $V(H_i - - \{x, y\})$ and $E(H_i)$, respectively, and $V(H_i) \cap V(H_j) = \{x, y\}$ if $i \neq j$ we obtain:

$$(\tilde{q}^{-1}, y, \tilde{q}_2, \tilde{q}_3^{-1}, \tilde{q}_4, \dots, \tilde{q}_{r-1}^{-1}, \tilde{q}_r) \qquad \text{if } r \text{ is even and}$$

$$(\tilde{q}_1^{-1}, y, q_2, \tilde{q}_3, \tilde{q}_4^{-1}, \dots, \tilde{q}_{r-1}^{-1}, \tilde{q}_r) \qquad \text{if } r \text{ is odd}$$

is a Hamiltonian path in $G^2 - x$ joining the two G-neighbours $a := z^1$and $b := z^r$ of x and containing the edge $\{y_1^*, y\} \in E(G)$ (and the edge $\{y, y_2^*\} \in E(G)$ if r is even as well). However, this is a contradiction to the assumption on G.

b) Let $\{x, y\} \in E(G)$. Then $G - \{x, y\}$ has exactly one component T_1. Write $\bar{H} := G$ and let H be the graph arising from $\bar{H}$ by deleting the edge $\{x, y\}$. Obviously, we have the same situation as considered in subcase **a)** with respect to the graphs H, $\bar{H}$ with the only exception that now $s(\bar{H}) < s(G)$ does not hold (because of $\bar{H} = G$). However, if $t \geqq 2$ (note that $t + 1$ is the number of the blocks of H) the construction of the path $\bar{q}$ remains valid. Now let $t = 1$. Then G consists of the block $B := B_1$ containing the two different vertices $z_1 = z$ and $z_2 = y$, of the vertex x and of the edges $\{x, z\}$ and $\{x, y\}$. Obviously, $s(B) < s(G)$ if B is a nontrivial block. To construct a path $\bar{q}$ wanted it suffices to construct a Hamiltonian path h in B^2 joining z and y and containing some edge $\{y, y^*\}$ with $y^* \in N(y : B)$. If $|V(B)| = 2$ or B is Hamiltonian (i.e. B is a cycle because G – and therefore B – is a minimal block) or B has a Hamiltonian (z, y)-path, the existence of such an h is obvious. So let B be a nontrivial block being not a cycle and therefore $0 < s(B) < s(G)$. Now consider an admissible (y, z)-path $\bar{w}$ in the minimal block B. Then $\{y, z\} \in E(B)$ is not possible because G is a minimal block. Thus $\{y, z\} \notin E(B)$, and we may suppose that $\bar{w}$ is not a Hamiltonian path in B. Then we proceed as in the proof of Theorem 1 (now for B instead of G, y instead of x, and z instead of y, of course) with the following modification: If $v(z : B) = 2$ and the only B-neighbour $z^* \notin V(\bar{w})$ of z belongs to a component C^* of $B - \bar{w}$ fulfilling $C^* \in C_2(\bar{w})$, we choose a Hamiltonian path h_{C^*} of $B_{C^*} - 0$ joining two B_{C^*} – neighbours of 0 and containing the edge $\{z^*, z'\} \in E(B)$ with some $z' \in N(z^* : B_{C^*})$; such an h_{C^*} exists because of $s(B_{C^*}) < s(B) < s(G)$. Hence, besides (6) also (6a) is fulfilled by the family $(h_C : C \in C_2(\bar{w}))$ having been chosen, and Corollary 1 and the Lemma of section 1 yield the required Hamiltonian path h.

So in every case there is a Hamiltonian path $\bar{q}$ in $H^2 - y$ joining an H-neighbour y^* of y and the vertex x and containing the edge $\{z, x\} \in E(H)$, where z denotes the only H-neighbour of x; write $\bar{q} = (y^*, \ldots, z, x)$ and $\tilde{q} := (y^*, \ldots, z) = \bar{q} - z$. Then $(\tilde{q}^{-1}, y)$ is a Hamiltonian path in $G^2 - x$ joining the G-neighbour $z \neq y$ of x with the G-neighbour y of x and containing an edge $\{y^*, y\} \in E(G)$. But this is a contradiction to the assumption on G. Thus Theorem 2 is proved. □

Now we can generalize Theorem 1 to

Theorem 1′ *Let G be a block and x, y, z vertices with $x \neq y$. Then there is a G-neighbour z' of z and a Hamiltonian path in G^2 joining x and y and containing the edge $\{z, z.\}$.*

Proof Form the graph H consisting of G, a new vertex 0 and the edges $\{0, x\}$ and $\{0, y\}$, and apply Theorem 2 to the nontrivial block H and the vertices 0 and z (instead of G and x and y, respectively). □

5. Let G be a connected graph, $z \in V(G)$ a cutpoint of G, further G_1 and G_2 two connected subgraphs of G forming a *non-trivial separation* of G with $V(G_1) \cap V(G_2) = \{z\}$ (that means: $V(G_1) \cup V(G_2) = V(G), E(G_1) \cap E(G_2) = E(G(\{z\})) = \emptyset$, $E(G_1) \cup E(G_2) = E(G)$, and $G_1, G_2 \neq G$) and h_1 and h_2 two paths in G_1^2 and G_2^2, respectively. Now we consider the following properties:

(12) h_1 is a Hamiltonian path in G_1^2 joining two different G-neighbours of z, and h_2 is a Hamiltonian path in $G_2^2 - z$ joining two different G-neighbours of z if $|V(G_2 - z)| \geqq 2$ and consisting of the only G-neighbour of z in G_2 if $|V(G_2 - z)| = 1$.

(13) h_1 is a Hamiltonian path in G_1^2 joining z with a G-neighbour of z, and h_2 is a Hamiltonian path in G_2^2 joining z with a G-neighbour of z.

Definition. $(h_1, G_1) \mapsto (h_2, G_2)$ iff property (12) is satisfied; $(h_1, G_1) \leftrightarrow (h_2, G_2)$ iff property (13) is satisfied.

Representing the paths h_1, h_2 by vertex-sequences we see immediately

Corollary 3. *If* $(h_1, G_1) \mapsto (h_2, G_2)$ *then*

$$h_1 + h_2 := (h_1, h_2, z'),$$

where z' is the initial vertex of h_1, is a Hamiltonian cycle in G^2. If $(h_1, G_1) \leftrightarrow (h_2, G_2)$ then

$$h_1 \cup h_2 := (h_1, h_2^{-1})$$

is a Hamiltonian cycle in G^2. □

(Of course, $(h_1, G_1) \leftrightarrow (h_2, G_2)$ holds iff $(h_2^{-1}, G_2) \leftrightarrow (h_1^{-1}, G_1)$; however, $(h_1, G_1) \mapsto (h_2, G_2)$ does not imply $(h_2, G_2) \mapsto (h_1, G_1)$.)

Corollary 4. If *G_1, G_2 form a non-trivial separation of a connected graph G with $V(G_1) \cap V(G_2) = \{z\}$ for some $z \in V(G)$, and if there exists a Hamiltonian cycle h in G^2, then there are paths h_1 and h_2 in G_1 and G_2, respectively, satisfying $(h_1, G_1) \mapsto (h_2, G_2)$ or $(h_2, G_2) \mapsto (h_1, G_1)$* or *$(h_1, G_1) \leftrightarrow (h_2, G_2)$.*

Corollary 4 can be easily proved by considering the maximal G_1-sections and the maximal G_2-sections of h. □

Note that the *block-cutpoint-graph* $bc(G)$ of a connected graph G with $|V(G)| \geqq 2$ is a tree and that its endvertices (i.e. vertices of valency $\leqq 1$) in every case are representing some (maximal) blocks of G. (If G is a block then $bc(G)$ is a one-vertex-tree, and this vertex is also considered to be an endvertex of $bc(G)$.) We define $bc(G) := \emptyset$ if $|V(G)| \leqq 1$.

Theorem 3. *Let G be a connected graph with $|V(G)| \geqq 3$ satisfying the property that G^2 is Hamiltonian. Suppose that $bc(G)$ has at least one endvertex representing a non-trivial (maximal) block of G. Then there is a Hamiltonian cycle in G^2 containing some edge $l \in E(G)$.*

Proof: If G is a block then we only need apply Theorem 2 to G.

If G is not a block consider an endvertex of $bc(G)$ representing a non-trivial block G_1 of G, and let z be the cutpoint of G belonging to G_1. Then G_1 and $G_2 := G - (V(G_1) - \{z\}) = G((V(G) - V(G_1)) \cup \{z\})$ form a non-trivial separation of G with $V(G_1) \cap V(G_2) = \{z\}$, and Corollary 4 implies the existence of some h_1, h_2 such that $(h_1, G_1) \mapsto (h_2, G_2) \vee (h_2, G_2) \mapsto (h_1, G_1) \vee (h_1, G_1) \leftrightarrow (h_2, G_2)$ holds Because G_1 is a non-trivial block according to Theorem 2 there is a Hamiltonian path h_1' in $G_1^2 - z$ joining two G_1-neighbours (i.e. G-neighbours) of z and containing an edge $l \in E(G_1)$.

If $(h_1, G_1) \mapsto (h_2, G_2)$ then $((z, h_1'), G_1) \leftrightarrow ((z, h_2), G_2)$, and $(z, h_1') \cup (z, h_2)$ is a Hamiltonian cycle in G^2 containing $l \in E(G)$. If $(h_2, G_2) \mapsto (h_1, G_1)$ then $(h_2, G_2) \mapsto (h_1', G_1)$, and $h_2 + h_1'$ is a Hamiltonian cycle in G^2 containing $l \in E(G)$.

If $(h_1, G_1) \leftrightarrow (h_2, G_2)$ then $((z, h_1'), G_1 \leftrightarrow (h_2, G_2)$, and $(z, h_1') \cup h_2$ is a Hamiltonian cycle in G^2 containing $l \in E(g)$. □

For a connected graph G with $|V(G)| \geqq 3$ we form $G^{(1)} := G - V_1(G)$, where $V_1(G) := \{x \in V(G): v(x : G) = 1\}$. Then it is easy to show

Corollary 5. *Let G be a connected graph with $|V(G)| \geqq 3$ satisfying the property that G^2 is Hamiltonian. Suppose that all endvertices of $bc(G)$ are representing trivial (maximal) blocks of G. If $bc(G^{(1)}) = \emptyset$, or if $bc(G^{(1)})$ has an endvertex representing a trivial (maximal) block of $G^{(1)}$ then there is a Hamiltonian cycle in G^2 containin an edge $l \in E(G)$.* □

Now it remains the case that all endvertices of $bc(G)$ are representing trivial (maximal) blocks of G and all endvertices of $bc(G^{(1)})$ are representing non-tiivial (maximal) blocks of $G^{(1)}$. It is rather obvious that this problem could be solved if the following statement were true.

Conjecture: For every connected graph G with $|V(G)| \geqq 3$ fulfilling (14) and every vertex $x \in V(G^{(1)})$ with $\iota(x : G^{(1)}) = v(x : G)$ the existence of a Hamiltonian path in $G^2 - x$ joining two G-neighbours of x implies the existence of a Hamiltonian path in $G^2 - x$ joining two suitable G-neighbours of x and containing some edge of G.

(14) $G^{(1)}$ is a non-trivial block $\wedge$ for any different veitices $x, y \in V_1(G)$ their G-neighbours are different (i.e. $N(x : G) \neq N(y : G)$).

We remark that this Conjecture holds in case that $|V_1(G)| \leqq 1$ because of Theorem 2.

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MEDIAN GROUPS

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Abstract. Some properties of groups endoved with a special ternary operation are investigated. Such groups are a natural generalization of lattice ordered groups.

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1. INTRODUCTION

By a *median algebra* is meant an algebra with one ternary operation satisfying the identities

(1) $$(a, a, b) = a,$$

(2) $$((a, d, c), b, c) = ((b, c, d), a, c).$$

Such algebras were investigated (under various names) by several authors. A survey of these algebras is e.g. in [1].

An important example of median algebras is derived from distributive lattices. Given a distributive lattice $\mathscr{L}$ and the operation

3) $$(a, b, c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$$

then $M(\mathscr{L}) = (L; (, ,))$ is a median algebra. According to [10] each median algebra is isomorphic to a subalgebra of an algebra $M(\mathscr{L})$.

In an l-group $\mathscr{G} = (G; +, -, 0, \wedge, \vee)$ the operations (3) and $+$ are related by the identity

(4) $$u + (a, b, c) + v = (u + a + v, u + b + v, u + c + v).$$

Definition. *By a median group (m-group) there is meant an algebra* $(G; +, -, 0, (, ,))$ *where* $(G; +, -, 0)$ *is a group,* $(G; (, ,))$ *is a median algebra and the identity* (4) *holds.*

If $\mathscr{G}$ is an l-group then the m-group $(G; +, -, 0, (, ,))$, where the ternary operation is given by (3), is said to be *associated with* $\mathscr{G}$.

The class of m-groups is much larger than that of m-groups associated with l-groups. Nevertheless some results which are valid for l-groups can be applied (possibly in a modified form) to m-groups. The present paper contains some examples of such results.

Some fundamental properties of m-groups were announced in [7]. Several interesting results on m-groups and their important classes are contained in [9].

2. SOME PROPERTIES OF MEDIAN ALGEBRAS

2.1. Fundamental notions and properties. Let $\mathscr{A} = (A; (, ,))$ be a median algebra. If $a, b, c \in A$ and $(a, b, c) = b$, we say that b is between a and c (in symbols, abc). If $a_1, \ldots, a_n \in A$ and $a_i a_j a_k$ holds for $1 \leqq i \leqq j \leqq k \leqq n$, we denote this by $a_1 a_2, \ldots, a_n$. (a, b) will denote the set $\{x \in A: axb\}$. $((a, b); \wedge, \vee)$ is a distributive lattice where $x \wedge y = (a, x, y)$ and $x \vee y = (b, x, y)$ [6]. aub, buc and cua imply $(a, b, c) = u$ [10]. Call a mapping $\varphi: A \to B$ between two median algebras betweenness-preserving if abc implies $(\varphi a)(\varphi b)(\varphi c)$. Let $\mathscr{C} = (C; \leqq)$ be a linearly ordered set and let abc mean that $a \leqq b \leqq c$ or $c \leqq b \leqq a$. If φ is a betweenness-preserving injective mapping from C to a median algebra $\mathscr{A}$, the set $\{\varphi c: c \in C\}$ will be called a *line* (in $\mathscr{A}$). A subset K of A is said to be convex if $a, b \in K$, $u \in A$ and aub imply $u \in K$. One can easily check that K forms a subalgebra of $\mathscr{A}$.

N will denote the set of positive integers.

2.2. The following identities hold in an m-algebra [8, Th. 2].

(5) $$(a, b, c) = (b, a, c) = (b, c, a),$$

(6) $$((a, b, c), d, e) = ((a, d, e), b, (c, d, e)).$$

The following relations are easy to prove.

(7) abc implies cba,

(8) $a(a, b, c)\, b$,

(9) [10] abc and buc imply $abuc$,

(10) abc and acb imply $b = c$.

These identities and relations are used freely in what follows.

2.3. We say that the elements a, b, c, d of a median algebra form a *cyclic quadruple* (a, b, c, d) whenever abc, bcd, cda and dab hold. It can be easily shown that the element d is uniquely determined by the elements a, b, c.

2.4 [3, Proposition 2]. A subset L of a median algebra with card $L \neq 4$ is a line iff for any $a, b, c \in L$ one of the relations abc, bca, cab holds. Obviously a subset of a line is a line. If a is an element of a line L such that for each $b, c \in L$ either abc or acb holds, we say that a is an end element of L.

2.5. *Let A be a line in a median algebra and* $0,\ a \in A,\ a \neq 0$. *Denote* $A' = \{x \in A\colon x0a\}$, $A_a = A - A'$. *Then* $A = A' \cup A_a$ *and* $x \in A'$ *together with* $y \in A_a$ *imply* $x0y$. Routine proof omitted.

2.6 [4]. A subset C of a median algebra $\mathscr{A}$ is called a *Čebyšev set* if for each $a \in A$ an element $a_C \in C$ exists such that aa_Ct holds for any $t \in C$. a_C will be said to be the *projection* of a into C. It can be easily shown that the element a_C is uniquely determined and that C is a convex subalgebra of $\mathscr{A}$. The mapping $x \to x_C$ is a homomorphism of $\mathscr{A}$ into C [4, 5.8].

2.7. *A maximal line in a median algebra $\mathscr{A}$, which is convex, is a Čebyšev set.*

Proof. Let C be such a line and $x \in A$. If for each $a, b \in C$ either abx or bax holds then $C \cup \{x\}$ is a line, hence $x \in C$ and $x_C = x$ is the projection. Consider the opposite case. If card $C = 4$ then x_C is the element (u, x, v) where u, v are the end elements of C. Suppose card $C \neq 4$. Then $u, v \in C$ exist such that neither uvx nor vux holds. We shall show that $t = (u, x, v)$ is the projection x_C. $t \in C$ and $t \notin \{u, v\}$. Let $a \in C$ and $s = (x, t, a)$. There are three possibilities: i) auv, ii) avu, iii) uav, and it suffices to consider i) and iii). The case i) yields $autv$. This together with ast implies that either $asut$ or $aust$ holds. In the first case xst implies xut which together with xtu yields $u = t$ — a contradiction. In the second case we have $ustx$ and tsx, hence $t = s = (x, t, a)$. In the case iii) either $uatv$ or $utav$ holds. In both cases this implies xta (e.g. the first possibility yields tau and xtu).

2.8. Let $\mathscr{A}$ be a median algebra and $0 \in A$. The algebra $(A; \wedge)$ where $a \wedge b = (a, 0, b)$, is a semilattice [10]. The corresponding order relation will be denoted by $\leqq$ (i.e. $a \leqq b$ means $0ab$). $a \vee b$ will denote sup $\{a, b\}$ if exists. In such a case $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for any $c \in A$ (see [10, 8] and [12, 3]).

3. ELEMENTARY PROPERTIES OF MEDIAN GROUPS

3.1. Examples of median groups

a) To any l-group $\mathscr{G}$ there is its associated m-group $\mathrm{M}(\mathscr{G})$. Such m-groups satisfy the identity

$$(x, 0, -x) = 0. \tag{*}$$

T. Marcisová [9] has shown that an m-group satisfies (*) iff it satisfies the identity

$$-(x, y, z) = (-x, -y, -z) \tag{**}$$

and it does not contain any non-zero element of a finite order.

The following examples show that there are finite m-groups.

b) Let $\mathscr{B} = (B; \wedge, \vee, ', 0, 1)$ be a Boolean algebra. Define $a + b = (a \wedge b')\vee \vee (a \wedge b)$, $-a = a$, and take the operation (3). Then $(B; +, -, 0, (, ,))$ is an m-group.

c) Let $\mathscr{C}_4$ be a (cyclic) group with the elements 0, 1, 2, 3, and addition mod 4. Take the distributive lattice with the same elements as $\mathscr{C}_4$, where 0 is the least element and 1, 3 are atoms. The group $\mathscr{C}_4$ with the operation (3) is an m-group different from that in b) with the four-element Boolean algebra $\mathscr{B}$.

In what follows $\mathscr{G}$ denotes an m-group.

3.2. *$a \leqq c$ and $b \leqq c$ imply $(a, b, c) = a\vee b$.*

The proof is straightforward.

3.3. *Let the elements $a, b \in G$ satisfy*

(i) $a\wedge b = 0 = a\wedge (-b) = (-a)\wedge b$.

Then a) $(-a)\wedge (-b) = 0$, b) $a + b = a\vee b = b + a$.

Proof. a) Denote $(-a)\wedge (-b) = (-a, 0, -b) = u$. Then $0u(-a)$ and $0u(-b)$ hold. Since $a0b$, $a0(-b)$ and $(-a)0b$, we get successively $(-a-b)(-b)0$, $(-a-b)(-b)u$, $(-a)0(u+b)$, $u0(u+b)$, $0(-u)b$, and symmetrically $0(-u)a$. This together with $b0a$ yields $b0(-u)$. Since $0(-u)b$, we get $u = 0$.

b) $(0, b, a + b) = (-b, 0, a) + b = b$, hence $b \leqq a + b$, and similarly $a \leqq a + b$. Using a) and 3.2 we get $a + b = a + (-b, -a, 0) + b = a\vee b$.

3.4. Elements $a, b \in G$ satisfying (i) in 3.3 will be said to be *orthogonal* (in symbols, $a \perp b$).

In l-groups the relation $a \perp b$ is defined to mean (ii) $|a|\wedge |b| = 0$, where $|a| = a\vee (-a)$ [2, 3.1]. It can be readily proved that in an m-group associated with an l-group, (i) and (ii) are equivalent.

3.5. *$a \perp b$ iff $(0, a, a + b, b)$ is a cyclic quadruple.*

We omit the easy proof.

3.6. *Given $a, b \in G$, $a + b = a\vee b$ iff $a \perp b$.*

Proof. $a+b = a\vee b$ implies $a=a\wedge (a+b)=(a, 0, a+b)=a+(0, -a, b)$, hence $(-a)\wedge b = 0$, and similarly, $a\wedge (-b) = 0$. Using 3.2 we get $a + b = (a, a + b, b) = a + (-b, 0, -a) + b$, hence $(-a)\wedge (-b) = 0$ and $a \perp b$. The converse implication was proved in 3.3b).

3.7. *Let $a, x, y \in G$ and $a \perp x$, $a \perp y$. Then $a\wedge (x + y) = 0$.*

Proof. First $(a, 0, x+y)=((a, 0, x), a, x+y)=((a, a, x+y), 0, (x, a, x+y))= (a, 0, (x, a, x + y))$. Now $(x, a, x + y) = x + (0, -x + a, y)$. Since $-x \perp a$, $-x + a = (-x)\vee a$ ((3.3b)), hence $(0, -x + a, y) = ((-x)\vee a)\wedge y = ((-x)\wedge y)\vee (a\wedge y) = (-x)\wedge y$ (see 2.8). Using this we get $(x, a, x + y) = x + (-x, 0, y) = (0, x, x + y)$ hence $(a, 0, x + y) = (a, 0, (0, x, x + y)) = ((a, 0, 0), (a, 0, x), x + y) = (0, 0, x + y) = 0$.

3.8. *$a \perp x$ and $a \perp y$ imply $a \perp x + y$.*

Proof. From the suppositions it follows $-a \perp x$, $-a \perp y$, $a \perp -x$ and $a \perp -y$. Using 3.7 we get $(-a) \wedge (x + y) = 0 = a \wedge (-(x + y))$.

3.9. *Let $a \in G$. The set $H = \{x \in G\colon a \perp x\}$ forms a subgroup of K.*

Proof. From the definition of the orthogonality it follows that $x \in H$ impliex $-x \in H$. This together with 3.8 proves the assertion.

4. CONVEX MAXIMAL LINES

If (i) $\varphi : \mathcal{G} \cong \mathcal{A} \times \mathcal{B}$ is a direct decomposition of an m-group $\mathcal{G}$ then the m-group $\mathcal{A}$ is isomorphic to the m-subgroup of $\mathcal{G}$, whose elements are $\varphi^{-1}(a, 0)$, $a \in A$. An analogous assertion holds for $\mathcal{B}$. In what follows we shall suppose that in the direct decomposition (i), $\mathcal{A}$ and $\mathcal{B}$ are m-subgroups of $\mathcal{G}$.

We shall deal with direct decompositions (i) in which $\mathcal{A}$ is a line. In this case convex maximal lines (i.e. maximal lines which are convex) prove to be important.

4.1. Theorem. *Let an m-group $\mathcal{G}$ satisfy the identity (*) and let A be a line in $\mathrm{M}(\mathcal{G})$ such that $0 \in A$. The following are equivalent.*

(a) *A forms a subgroup of $(G; +, -, 0)$ and a direct factor of $\mathcal{G}$.*

(b) *A is a convex maximal line in $\mathrm{M}(\mathcal{G})$.*

Corollary [5, Th. 1]. *Let $\mathcal{G}$ be an l-group. A maximal chain in $\mathcal{G}$ which is convex and contains 0, is a direct factor of $\mathcal{G}$.*

Remark. The following assertion is easy to prove. Let $\mathcal{G}$ be an l-group. A convex line in $\mathrm{M}(\mathcal{G})$ containing 0 is a (convex) chain.

The proof of Theorem 4.1 is divided into a sequence of lemmas. $\mathcal{G}$ is supposed to satisfy (*) unless other is said.

4.2. *Let $\mathcal{G} = \mathcal{A} \times \mathcal{B}$ where $\mathcal{A}$ is a non-singleton line. Then $A_1 = \{(a, 0)\colon a \in A\}$ is a maximal line in $\mathcal{G}$ and it is convex.*

Corollary. *Let $\mathcal{G}$ be an l-group and let $\mathcal{G} = \mathcal{A} \times \mathcal{B}$ where $\mathcal{A}$ is a non-singleton chain. Then $\{(a, 0)\colon a \in A\}$ is a convex maximal chain* (and contains $(0, 0)$).

Proof. Obviously A_1 is convex. Let $c = (u, v) \in A \times B$ and $A_1 \cup \{c\}$ be a line. Then either (i) c is an end element of $A_1 \cup \{c\}$ or (ii) c is between some two elements of A_1. The case (ii) yields $c \in A_1$ immediately. In the case (i) recall that 0 and $2u$ belong to A hence either $u0(2u)$ or $u(2u)\,0$ holds. Combining this with $0u(2u)$ (a consequence of $(-u)\,0u$) we get $u = 0$. Then for any $a \in A$ either $0a(-a)$ or $0(-a)\,a$ holds. Because of $a0(-a)$ this gives $a = 0$. This contradiction shows that the case (i) is not possible.

4.3. *Suppose* $\mathcal{G}$ *satisfies* (**). *Then* $0xy$ *implies* $0(y - x)\, y$ *and* $0(-x + y)\, y$.

Proof. From the supposition we get $0(-x)(-y)$, hence $y - x = y + (0, -x, -y) = (y, y - x, 0)$. The proof of the second relation is similar.

4.4. *Let* $\mathcal{G}$ *satisfy* (**). *Let* A, B *be convex lines in* $\mathcal{G}$ *with the end element* 0. *If* $p \in A - B$ *and* $q \in B - A$ *then* $p \wedge q = 0$.

Proof. Denote $p \wedge q = r$. $(r, p, q) = ((p, 0, q), p, q) = r$ and $0rp$, $0rq$. For the elements $p' = p - r$ and $q' = q - r$ we get $p' \wedge q' = (0, p - r, q - r) = (r, p, q) - - r = 0$. According to 4.3, $0p'p$ and $0q'q$. Hence there hold

either a1) $0p'r$ or a2) $rp'p$, and

either b1) $0q'r$ or b2) $rq'q$.

a1) and b1) yield $0p'q'$ or $0q'p'$, hence $p' = p' \wedge q' = 0$ i.e. $p = r$ or $q' = 0$ i.e. $q = r$. This is a contradiction, since $r \in A \cap B$. a1) and b2) yield $0p'q'$ – a contradiction as above. The case a2) and b1) is symmetric. If a2) and b2) hold then $0 = p' \wedge q' = r$ (since $r \leqq p' \leqq p$, $r \leqq q' \leqq q$ and $p \wedge q = r$).

4.5. *Let* $\mathcal{G}$, A, B *be as in* 4.4. *If neither* $A \subset B$ *nor* $B \subset A$ *then* $a \wedge b = 0$ *for each* $a \in A$ *and each* $b \in B$.

Proof. Let p, q have the same meaning as in 4.4. Then $p \wedge q = 0$. For $a \in A$, $b \in B$ there hold either a1) $0ap$ or a2) $0pa$, and either b1) $0bq$ or b2) $0qb$. a2) and b2) yield $a \in A - B$ and $b \in B - A$ hence $a \wedge b = 0$ according to 4.4. a1) and b2) yield $a \wedge b = a \wedge p \wedge b$. But $p \wedge b = 0$ by 4.4. The case a2) and b1) is symmetric and a1) and b1) give $a \wedge b = a \wedge p \wedge b \wedge q = 0$.

4.6. *Let* $\mathcal{G}$ *be arbitrary*, $u \in G$, *and let* A *be a line in* $\mathcal{G}$. *Then* $u + A = \{u + a : a \in A\}$ *is a line. If* A *is a maximal line (convex line) then so is* $u + A$.

Routine proof omitted.

4.7. (see [9]). *For each* $a \in G$ *and each* $n \in N$, $0a(na)$ *holds.*

4.8. *If* $a \in G$ *and* $m, n \in N$, $m < n$, *then* $0(ma)(na)$.

The proof proceeds by induction on m. For $m = 1$ the assertion holds by 4.7. Assume $0((m - 1)\, a)(na)$ $(m < n)$. According to 4.7 $0a((n + 1 - m)\, a)$ hence $((m - 1)\, a)(ma)(na)$ which, together with the assumption, yields $0(ma)(na)$.

4.9. *Let* $a \in G$. *If* $(0, a)$ *is a line then* $(-a, a)$ *is a line too.*

Proof. Since $(-a, 0) = -a + (0, a)$, $(-a, 0)$ is a line by 4.6. Because of $(-a)\, 0a$, $(-a, 0) \cup (0, a)$ is a line. It suffices to show:

(i) $t \in (-a, a)$ implies $t \in (-a, 0)$ or $t \in (0, a)$.

Denote $u = (0, t, -a)$, $v = (0, t, a)$. Then $-v = (0, -t, -a)$ and $(0, u, -v) = (0, (0, t, -a), (0, -t, -a)) = ((0, t, -t), 0, -a) = (0, 0, -a) = 0$. From $0va$ we get $0(-v)(-a)$ and, because of $0u(-a)$, either $0u(-v)$ or $0(-v)\, u$ holds. In the first case $u = (0, u, -v) = 0$, in the second $v = 0$. $u = 0$ implies $t0(-a)$ which together with $at(-a)$ yields $at0$ i.e. $t \in (0, a)$. Similarly, if $v = 0$ then $t \in (0, -a)$.

4.10. *If* $a \in G$ *and* $(0, a)$ *is a line then* $(-na, na)$ *is a line for each* $n \in N$.

Proof. For $n = 1$ this holds by 4.9. Suppose $(-(n-1)a, (n-1)a) = B$ is a line $(n > 1)$. According to 4.6, $a + B = ((-n+2)a, na)$ is a line. By 4.8 $0((n-2)a)(na)$ hence $0((-n+2)a)(-na)$ which together with $(na)0(-na)$ yields $0 \in (na, (-n+2)a)$ hence $(0, na)$ is a line. Now 4.9 is applicable.

4.11. *Let* A *be a convex maximal line in* $\mathscr{G}$, *containing* 0. *Then*

i) *For each* $a \in A$ *and* $n \in N$, na *belongs to* A.

ii) $a \in A$ *implies* $-a \in A$.

iii) $a, b \in A$ *imply* $a + b \in A$.

Proof. i) If $a = 0$ the assertion is trivial. Suppose $a \neq 0$. Let A' and A_a be as in 2.5. We apply 4.5 to the convex lines A_a and $(0, na)$ (see 4.10). There are three possibilities.

a) $(0, na) \subset A_a$, b) $A_a \subset (0, na)$,

c) $x \wedge y = 0$ for each $x \in A_a$ and each $y \in (0, na)$.

The case a) yields $na \in A$ immediately. In the case c) we get, using the relation $0a(na)$ (4.7), that $a = a \wedge na = 0$ – a contradiction. Consider the case b). Take $y \in A'$ and set $(y, 0, na) = u$. Then $yu0a$. Since a and u belong to $(0, na)$, either $0au$ or $0ua$ holds. The first case together with $a0u$ yields $a = 0$ – a contradiction. In the second case $u = 0$, so that $y0(na)$. Hence $y0(na)$ for each $y \in A'$ and $A' \cup (0, na)$ is a line and $A = A' \cup A_a \subset A' \cup (0, na)$. The maximality of A implies $na \in A$.

ii) The set $B = -a + A$ is a convex maximal line (4.6) and contains the elements 0 and $-a$. By i) $-2a \in B$, hence $-a \in a + B = A$.

iii) There are three possibilities: 1. $0ab$, 2. $0ba$, and 3. $a0b$. In the case 1. $b(a+b)(2b)$. Using i) we get $a + b \in A$. The case 2. is similar. In the third case $(-a)0(-b)$, hence $0a(a-b)$ and $b(a+b)a$ so that again $a + b \in A$.

4.12. *Let* A *be as in* 4.11 *and* $b \in G$. *If* $0 \neq a \in A$ *and* $b \perp a$ *then* $b_A = 0$.

Proof. $b \perp a$ implies $b0a$ and $b0(-a)$. This together with $a0(-a)$ yields $(b, a, -a) = 0$. For the element $t = b_A$ there hold $t \in A$, bta and $bt(-a)$. There are three possibilities: $a(-a)t$, $(-a)at$, and $(-a)ta$. In the first case we get $a(-a)tb$, hence $0 = (b, -a, a) = -a$ – a contradiction. Similarly the second case is not possible. In the last case we get $(b, a, -a) = t$ hence $t = 0$.

4.13. *Let* A *be as in* 4.11. *If* $0 \neq a \in A$ *and* $b \perp a$ *then* $b \perp x$ *for each* $x \in A$.

Proof. By 4.12 and 4.11, $b0x$ and $b0(-x)$. Since $-b \perp a$ too, $(-b)0x$. Hence $b \perp x$.

4.14. *Let* A *be as in* 4.11. *Denote* $B = \{-x_A + x\colon x \in G\}$, $x_1 = x_A$ *and* $x_2 = -x_1 + x$. *Then* $x = x_1 + x_2$, $x_1 \in A$, $x_2 \in B$ and $x_1 \perp x_2$.

Note that $x_1 + x_2 = x_2 + x_1$ (see 3.3).

Proof. The element $u = (x_1, 0, x_2)$ belongs to A. By 4.11 iii) $x_1 + u \in A$, hence $(x, x_1, x_1 + u) = x_1$, so that $0 = -x_1 + (x, x_1, x_1 + u) = (x_2, 0, u) = u$ i.e.

$(x_1, 0, x_2) = 0$. Further $(-x_1, 0, x_2) = -(x_1, 0, -x_1 + x) = -x_1 + (0, x_1, x) = = -x_1 + x_1 = 0$ and $(x_1, 0, -x_2) = 0$ by (**).

In what follows we shall use the notations from 4.14.

4.15. *For each* $x \in G$, $(-x)_A = -x_A$.

Proof. According to 4.14 $x_1 \perp x_2$. We have to show that $(-x)(-x_1)\, a$ for each $a \in A$. Since $(-x, -x_1, a) = (-x_2 - x_1, -x_1, a) = (-x_2, 0, a + x_1) - x_1$, it suffices to show

(i) $(-x_2, 0, a + x_1) = 0$.

If $x_1 = 0$ then $x0t$ for each $t \in A$, hence $x_2 0(-a)$ and, according to (**), $(-x_2)\,0a$, which yields (i). If $x_1 \neq 0$ then $x_2 \perp x_1$ by 4.14 and according to 4.12, $(x_2)_A = 0$, hence $x_2 0(-x_1 - a)$ and $(-x_2)\,0(a + x_1)$ i.e. (i) holds.

4.16. If $a_A = 0$ *then* $a \perp t$ *for each* $t \in A$.

Proof. By 4.15 $(-a)_A = 0$ hence for each $t \in A$ there holds $(-a)\,0t$ (and $a0t$, $a0(-t)$).

4.17. *The mapping* $x \to x_A$ *is a homomorphism from the group* $(G; +, -, 0)$ *onto its subgroup* A.

Proof. It suffices to show that for each $a \in A$ and $x, y \in G$, $(x + y, x_1 + y_1, a) = = x_1 + y_1$. Since $(x + y, x_1 + y_1, a) = x_1 + (x_2 + y_2, 0, -x_1 + a - y_1) + y_1$ it suffices to prove $x_2 + y_2 \perp -x_1 + a - y_1$, i.e., according to 3.9,

(i) $x_2 \perp -x_1 + a - y_1$,

(ii) $y_2 \perp -x_1 + a - y_1$.

If $x_1 = 0$, (i) holds by 4.16. If $x_1 \neq 0$, (i) follows from $x_2 \perp x_1$ by 4.13. The proof of (ii) is similar.

4.18. *The following holds for any* $x \in G$.

(i) *The representation* $x = a + b$, $a \in A$, $b \in B$, *is unique*.

(ii) *If* $a \in A$ *then* $a_1 = a$ *and* $a_2 = 0$. *If* $b \in B$ *then* $b_1 = 0$ *and* $b_2 = b$.

(iii) $a \in A$ *and* $b \in B$ *imply* $(a + b)_1 = a$, $(a + b)_2 = b$.

(iv) $a \in A$ *and* $b \in B$ *imply* $a \perp b$.

(v) $a + b = b + a$ *for each* $a \in A$ *and* $b \in B$.

Proof. (i) $b = -y_A + y$ for some $y \in G$ (see 4.14). $x = a + b$ hence $y - x = = y_A - a$, and using 4.11 and 4.17 we get $y_A - a = (y - x)_A = y_A - x_A$. Thus $a = x_A$ and $b = -x_A + x$.

(ii) The assertion on a is obvious. If $b \in B$, there is $y \in G$ such that $b = -y_1 + y$. According to 4.17, $b_1 = b_A = -y_1 + y_1 = 0$ aod $b_2 = b$. (iii) follows from 4.17 and (ii), (iv) follows from (ii) and 4.16, and (v) follows from (iv) and 3.3.

4.19. *If* $x = x_1 + x_2$ *and* $y = y_1 + y_2$ *are the representations in* 4.18 (i) *then* $x + y = (x_1 + y_1) + (x_2 + y_2)$, $-x = (-x_1) + (-x_2)$ *where* $x_1 + y_1$ *and* $-x_1$ *belong to* A *and* $x_2 + y_2$, $-x_2$ *belong to* B.

Proof. By 4.17, $(x + y)_1 = x_1 + y_1$. Further $(x + y)_2 = -(x + y)_1 + (x + y) = x_2 + y_2$. This proves the first assertion. The proof of the second assertion is similar.

4.20. *B forms a subgroup of the group $\mathcal{G}$.*

Proof. The assertion follows from 4.18 and 4.19.

4.21. *B is a Čebyšev set and $x \to -x_A + x$ is the corresponding projection.*

Proof. Let $x \in G$ and let $x = x_1 + x_2$ be the representation in 4.18 (i). For each $b \in B$ we get $(x, x_2, b) = (x_1 + x_2, x_2, b) = (x_1, 0, b - x_2) + x_2$. $b - x_2$ belongs to B (4.20) and by 4.18 (iv), $x_1 \perp b - x_2$, so that $(x_1, 0, b - x_2) = 0$. Hence x_2 is the desired projection.

The following theorem, together with 4.2, completes the proof of the theorem 4.1.

4.22. *The mapping φ: $x \to (x_1, x_2)$ is an isomorphism of m-groups $\mathcal{G}$ and $\mathcal{A} \times \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are m-subgroups of $\mathcal{G}$ with carriers A and B respectively.*

Proof. According to [4, 5.8] the projection into a Čebyšev subset is a homomorphism of median algebras. This together with 4.18 and 4.19 implies that φ is a homomorphism of m-groups. Consider the mapping ψ: $A \times B \to G$ with $\psi(a, b) = a + b$. From the definition of φ it follows that $\psi \circ \varphi = \mathrm{id}_G$ and by 4.18 $\varphi \circ \psi = \mathrm{id}_{A \times B}$, hence φ is a bijection.

4.23. **Theorem.** *Let $\mathcal{G}$ be an m-group satisfying (*) and let A be a convex maximal line in $\mathcal{G}$. If $a \in A$ then $-a + A$ is a direct factor of K.*

The theorem follows from 4.6 and 4.1.

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REALIZATIONS OF TOPOLOGIES AND CLOSURE OPERATORS BY SET SYSTEMS AND BY NEIGHBOURHOODS

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Dedicated to the memory of my friend Milan Sekanina

Abstract. Milan Sekanina and his collaborators have investigated the realizability of topologies and of closure operators by set systems. In particular they have shown that Top has precisely two [8] and Clos has no [3, 7, 2] realization by set systems. Moreover Top and Clos have precisely one realization by Conv [10]. In this paper it is shown that Top has a large (even illegitimate) collection of realizations by neighbourhoods, but Clos has only one. Moreover Clos has precisely two realizations by uniform neighbourhoods.

Key words: realizations of constructs, topological space, closure space, (uniform) neighbourhood space.

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TERMINOLOGY

Constructs are pairs (A, U) consisting of a category A and a faithful functor U: A $\to$ Set [1]. A realization of a construct (A, U) by a construct (B, V) is a full embedding E: A $\to$ B with $U = V \circ E$ [6].

Top is the construct of topological spaces and continuous maps.

Clos is the construct of closure spaces (sets with a closure operation satisfying Kuratowski's axioms except possibly the idempotency axiom) and continuous (= closure-preserving) maps.

Neigh has as objects all *neighbourhood spaces*, i.e. pairs (X, N) where $N: X \to \mathscr{P}\mathscr{P}X$ is a map, associating with any $x \in X$ a collection $N(x)$ of subsets U of X with $x \in U$; and has as morphisms $f: (X, N) \to (X', N')$ all maps $f: X \to X'$ such that $x \in X$ and $U \in N'(f(x))$ imply $f^{-1}[U] \in N(x)$.

UNeigh has as objects all *uniform neighbourhood spaces*, i.e., pairs $(X, <)$, where $<$ is a binary relation on $\mathscr{P}X$ satisfying the conditions (1) $A < B \to A \subset B$

and (2) $A \subset B < C \subset D \to A < D$, and has as morphisms $f\colon (X, <) \to (X', <')$ all maps $f\colon X \to X'$ such that $A <' B$ implies $f^{-1}[A] < f^{-1}[B]$.

SSet has as objects all pairs $(X, \mathcal{S})$ with $S \subset \mathcal{P}X$ and as morphisms $f\colon (X, \mathcal{S}) \to$ $\to (X', \mathcal{S}')$ all maps $f\colon X \to X'$ such that $A \in \mathcal{S}'$ implies $f^{-1}[A] \in \mathcal{S}$.

RESULTS

Proposition 1 [8]. Top *has precisely two realizations by* SSet.

Proposition 2 [3, 7, 2]. Clos *has no realization by* SSet.

Proof: Assume that E: Clos $\to$ SSet is a realization.

Notation: $E(X, \text{cl}) = (X, \mathcal{S}(\text{cl}))$. Then $\bar{E}$: Clos $\to$ SSet, defined by $\bar{E}(X, \text{cl}) =$ $= (X, \mathcal{S}(\text{cl}) \cup \{\emptyset, X\})$, is a realization too. On a 3-element set X there are precisely $4^3 = 64$ closure structures and precisely $2^{(2^3-2)} = 64$ subsets $\mathcal{S}$ of $\mathcal{P}X$ with $\{\emptyset, X\} \subset \mathcal{S}$. Hence $\bar{E}$ induces an order-isomorphism between the ordered sets F_1 of all closure structures on X and F_2 of all subsets $\mathcal{S}$ of $\mathcal{P}X$ with $\{\emptyset, X\} \subset \mathcal{S}$. Since F_1 has precisely 3 atoms and F_2 has 6, this cannot be.

Proposition 3. Top *has a proper class (even an illegitimate collection) of realizations by* Neigh.

Proof: Let C be a strongly rigid proper class of Hausdorff spaces with more than one point. (Such a class exists by [5, 4]; cf. also [11]). For every subclass Γ of C define a realization E_Γ: Top $\to$ Neigh by $E_\Gamma(X, \mathcal{O}) = (X, N_\Gamma(\mathcal{O}))$ where $U \in$ $\in N_\Gamma(\mathcal{O})\,(x)$ provided U is an open neighbourhood of x in $(X, \mathcal{O})$ or there exists $(X', \mathcal{O}')$ in Γ, a continuous map $f\colon (X, \mathcal{O}) \to (X', \mathcal{O}')$, and a neighbourhood V of $f(x)$ in $(X', \mathcal{O}')$ with $U = f^{-1}[V]$.

The realizations E_Γ are pairwise different, since, if $(X, \mathcal{O})$ belongs to $\Gamma \setminus \Gamma'$, then for any $x \in X$, $N_\Gamma(\mathcal{O})\,(x)$ consists of all neighbourhoods of x in $(X, \mathcal{O})$ and $N_{\Gamma'}(\mathcal{O})\,(x)$ consists of all open neighbourhoods of x in $(X, \mathcal{O})$.

Proposition 4. Clos *has precisely one realization by* Neigh.

Proof: For every closure space (X, cl) define a map $N_{\text{cl}}\colon X \to \mathcal{P}\mathcal{P}X$ by $N_{\text{cl}}(x) = \{U \subset X \mid x \notin \text{cl}\,(X \setminus U)\}$. Then E: Clos $\to$ Neigh, defined by $E(X, \text{cl}) =$ $= (X, N_{\text{cl}})$ is a realization.

For uniqueness, consider an arbitrary realization $\tilde{E}$: Clos $\to$ Neigh.

Notation: $\tilde{E}(X, \text{cl}) = (X, \tilde{N}_{\text{cl}})$. Let (X, cl) be a closure space. Then the following hold:

(a) $\tilde{N}_{\text{cl}}(x) \neq \emptyset$ *for every* $x \in X$.

Proof: Assume $\tilde{N}_{\text{cl}}(x_0) = \emptyset$ for some $x_0 \in X$. Let (X', cl') be an arbitrary closure space, let x be an arbitrary element of X', and let $f\colon X \to X'$ be the constant map with value x. Then continuity of f: $(X, \mathrm{cl}) \to (X', \mathrm{cl}')$ implies $\tilde{N}_{\mathrm{cl}'}(x) = \emptyset$. This in turn impliest that every map between closure spaces is a morphism. Contradiction.

(b) $X \in \tilde{N}(x)$ *for every* $x \in X$.

Proof: This follows from (a), since every constant map between closure spaces is continuous

(c) $X = \{1, 2\}$:

(c1) *if* $\mathrm{cl}\{1\} = \mathrm{cl}\{2\} = X$, *then* $\tilde{N}_{\mathrm{cl}}(1) = \tilde{N}_{\mathrm{cl}}(2) = \{X\}$,

(c2) *if* $\mathrm{cl}\{1\} = \{1\}$ *and* $\mathrm{cl}\{2\} = \{2\}$, *then* $\tilde{N}_{\mathrm{cl}}(1) = \{\{1\}, X\}$ *and* $\tilde{N}_{\mathrm{cl}}(2) = \{\{2\}, X\}$,

(c3) *if* $\mathrm{cl}\{1\} = X$ *and* $\mathrm{cl}\{2\} = \{2\}$, *then one of the following two cases holds:*

Case A: $\tilde{N}_{\mathrm{cl}}(1) = \{\{1\}, X\}$ *and* $\tilde{N}_{\mathrm{cl}}(2) = \{X\}$,

Case B: $\tilde{N}_{\mathrm{cl}}(1) = \{X\}$ *and* $\tilde{N}_{\mathrm{cl}}(2) = \{\{2\}, X)$.

Proof: follows immediately from the fact that, there are only 4 neighbourhood structures on $\{1, 2\}$, which satisfy (b).

(d) $X = \{1, 2, 3\}$: *if* $\mathrm{cl}\{1\} = \mathrm{cl}\{2\} = X$ *and* $\mathrm{cl}\{3\} = \{2, 3\}$, *then one of the following two cases holds:*

Case A: $\tilde{N}_{\mathrm{cl}}(1) = \{\{1, 2\}, X\}$ *and* $\tilde{N}_{\mathrm{cl}}(2) = \tilde{N}_{\mathrm{cl}}(3) = \{X\}$,

Case B: $\tilde{N}_{\mathrm{cl}}(1) = \tilde{N}_{\mathrm{cl}}(2) = \{X\}$ *and* $\tilde{N}_{\mathrm{cl}}(3) = \{X, \{2, 3\}\}$.

Proof: Let (X', cl') be the indiscrete closure space with underlying set $X' = \{1, 2\}$. Then the maps f: $(X', \mathrm{cl}') \to (X, \mathrm{cl})$, defined by $f(x) = x$, and g: $(X', \mathrm{cl}') \to (X, \mathrm{cl})$, defined by $g(x) = x + 1$, are continuous. Hence, by (c1), we obtain:

$$\begin{aligned} &\text{if } U \in \tilde{N}_{\mathrm{cl}}(1), && \text{then } 2 \in U, \\ &\text{if } U \in \tilde{N}_{\mathrm{cl}}(2), && \text{then } 1 \in U, \\ &\text{if } U \in \tilde{N}_{\mathrm{cl}}(2), && \text{then } 3 \in U, \\ &\text{if } U \in \tilde{N}_{\mathrm{cl}}(3), && \text{then } 2 \in U. \end{aligned}$$

Next, let $(\bar{X}, \overline{\mathrm{cl}})$ be the closure space, defined by $\bar{X} = \{1, 3\}$, $\overline{\mathrm{cl}}\{1\} = \bar{X}$ and $\overline{\mathrm{cl}}\{3\} = \{3\}$. Then the map h: $(\bar{X}, \overline{\mathrm{cl}}) \to (X, \mathrm{cl})$, defined by $h(x) = x$, is continuous. Hence, by (c3), one of the following cases must hold:

Case A: $U \in \tilde{N}_{\mathrm{cl}}(3) \to 1 \in U$,

Case B: $U \in \tilde{N}_{\mathrm{cl}}(1) \to 3 \in U$.

Since (X, cl) is not indiscrete, $\tilde{N}_{\mathrm{cl}}(1) = \tilde{N}_{\mathrm{cl}}(2) = \tilde{N}_{\mathrm{cl}}(3) = \{X\}$ cannot hold. This implies (d).

(e) Case B *cannot hold.*

Proof: Assume that case B holds. Let (X, cl) be as in (d), let (X', cl') be an arbitrary closure space, let x be an element of X', let U be a subset of X' with $x \in U$, and let f: $X' \to X$ be defined by

$$f(y) = \begin{cases} 3, & \text{if } y = x, \\ 2, & \text{if } y \in U \setminus \{x\}, \\ 1, & \text{if } y \in X' \setminus U. \end{cases}$$

Then the following conditions are equivalent:

(1) $U \in \tilde{N}_{\mathrm{cl}'}(x)$,

(2) $f\colon (X', \tilde{N}_{\mathrm{cl}'}) \to (X, \tilde{N}_{\mathrm{cl}})$ is a morphism in Neigh,

(3) $f\colon (X', \mathrm{cl}') \to (X, \mathrm{cl})$ is continuous,

(4) $\mathrm{cl}'\{x\} \subset U$.

Hence in particular, if (X', cl') is a topological T_1-space, then $\tilde{N}_{\mathrm{cl}'}(x) = \{U \subset X \mid x \in U\}$ for every $x \in X'$. Since there exist different T_1-topologies on an infinite set, $\tilde{E}$ is not injective on objects. Contradiction.

(f) $\tilde{E} = E$.

Proof: In view of (e), Case A must hold. Again, let (X, cl) be as in (d) let (X', cl') be an arbitrary closure space, let x be an element of X', let U be a subset of X' with $x \in U$, and let $f\colon X' \to Y$ be defined by

$$f(y) = \begin{cases} 1, & \text{if } y = x, \\ 2, & \text{if } y \in U \setminus \{x\}, \\ 3, & \text{if } y \in X \setminus U. \end{cases}$$

Then the following conditions are equivalent:

(1) $U \in \tilde{N}_{\mathrm{cl}'}(x)$,

(2) $f\colon (X', \tilde{N}_{\mathrm{cl}'}) \to (X, \tilde{N}_{\mathrm{cl}})$ is a morphism in Neigh,

(3) $f\colon (X', \mathrm{cl}') \to (X, \mathrm{cl})$ is continuous,

(4) $x \notin \mathrm{cl}'(X \setminus U)$.

Thus $\tilde{N}_{\mathrm{cl}} = N_{\mathrm{cl}}$, i.e., $\tilde{E} = E$.

Proposition 5. Clos *has precisely two realizations by* UNeigh.

Proof. As in the proof of Proposition 4, two cases arise. Case A leads to the realization E_1: Clos $\to$ UNeigh, defined by $E_1(X, \mathrm{cl}) = (X, <_1(\mathrm{cl}))$, where $A <_1 (\mathrm{cl})\, B$ iff $A \cap \mathrm{cl}\,(X \setminus B) = \emptyset$, i.e., iff B is a neighbourhood of A in the familiar sense. Case B does not lead to a contradiction but to the realization E_2: Clos $\to$ UNeigh, defined by $E_2(X, \mathrm{cl}) = (X, <_2(\mathrm{cl})'$ where $A <_2 (\mathrm{cl})\, B$ iff $(X \setminus B) \cap \mathrm{cl}\, A = \emptyset$, i.e., iff $X \setminus A$ is a neighbourhood of $X \setminus B$ in the familiar sense.

Remark. Since the construct Rere of reflexive relations has a realization E: Rere $\to$ Clos, given by

$$x \in \mathrm{cl}\, A \leftrightarrow \exists\, a \in A\ a\varrho x,$$

since the restriction of E to objects with finite underlying sets is an isomorphism, and since the proof of Proposition 5 depends only on finite closure space, Rere has precisely two realizatioos in UNeigh (resp. in Neigh).

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REFLECTIONS IN LOCALLY PRESENTABLE CATEGORIES

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Dedicated to the memory of Milan Sekanina

Abstract. For each locally presentable category it is proved that all full subcategories closed under limits and α-filtered colimits are reflective.

Key words. Locally presentable category, reflective subcategory.

MS Classification. 18 A 40

M. Makkai and A. M. Pitts have recently proved that each locally finitely presentable category $\mathscr{K}$ has the following property: all full subcategories closed under limits and filtered colimits are reflective in $\mathscr{K}$, see [8]. One is impelled to ask: 1. does this hold for locally presentable categories of infinite rank?, and 2. can filtered colimits be substituted by α-filtered colimits? The proof presented in [8] does not seem to give an answer. We were particularly interested in the latter question since the affirmative answer is the best result possible absolutely (i.e., independently of set theory). We have namely proved in [9] that the well-known Vopěnka's principle (which is a large cardinal principle much stronger than the existence of measurable cardinals, see [7]) is logically equivalent to the following statement: if $\mathscr{K}$ is a locally presentable category then each full subcategory of $\mathscr{K}$ closed under limits is closed under α-filtered colimits for some regular cardinal α.

The aim of the present paper is to answer both of the above questions affirmatively:

Theorem. *Let $\mathscr{K}$ be a locally presentable category, and α a regular cardinal. Then each full subcategory of $\mathscr{K}$ closed under limits and α-filtered colimits is reflective in $\mathscr{K}$.*

Proof. We can suppose that $\mathscr{K} = \mathbf{Set}^M$ for a small category M. This will not lose generality since for each locally presentable category $\mathscr{K}$ there exists a small category M such that $\mathscr{K}$ is equivalent to a full, reflective subcategory $\mathscr{K}'$ of $\mathbf{Set}^M$ closed under β-filtered colimits in $\mathbf{Set}^M$ for some β. (See 8.5 in [6]).

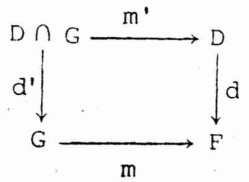
Then $\mathcal{H}'$ has the above property: each full subcategory $\mathcal{L}$ of $\mathcal{H}'$ closed under limits and α-filtered colimits in $\mathcal{H}'$ is closed under $(\alpha + \beta)$-filtered colimits in $\mathbf{Set}^M$, thus is reflective in $\mathbf{Set}^M$, and hence is reflective in $\mathcal{H}'$. It follows immediately that the equivalent category $\mathcal{H}$ has that property too.

Thus, we are to prove that each full subcategory $\mathcal{L}$ of $\mathbf{Set}^M$ closed under limits and α-filtered colimits is reflective in $\mathbf{Set}^M$. Without loss of generality, we may suppose that

$$\alpha > \operatorname{card}(\operatorname{mor} M).$$

It follows that each object F of $\mathbf{Set}^M$ is an α-directed union of all of its α-small subobjects, where an object D is *α-small* provided that $\sum_{X \in \operatorname{obj} M} \operatorname{card} DX < \alpha$. (In fact, for each $X \in \operatorname{obj} M$ and each $A \subset FX$, $\operatorname{card} A < \alpha$, consider the subfunctor $D_A \subset F$ defined on objects by $D_A Y = \bigcup_{f: X \to Y} Ff(A)$.)

A subobject $G \xrightarrow{m} F$ in $\mathbf{Set}^M$ is said to be *α-pure* provided that for each subobject $D \xrightarrow{d} F$ with D α-small, in the pullback



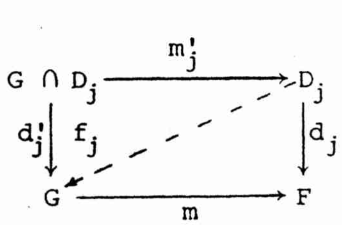
there exists a morphism $f: D \to G$ with $d' = f \,.\, m'$.

A. $\mathcal{L}$ is closed under α-pure subfunctors; i.e., if $G \xrightarrow{m} F$ is α-pure and $F \in \mathcal{L}$, then we will prove that $G \in \mathcal{L}$. Let $(D_j)_{j \in J}$ be the diagram of all α-small subfunctors $D_j \subseteq F$. Observe that J is an α-directed poset and F is the colimit of that diagram (where the colimit morphisms are the inclusion maps $D_j \xrightarrow{d_j} F$). For each j we have, by the α-purity, a morphism $f_j: D_j \to G$ with

(1) $$f_j \,.\, m_j' = d_j'$$

in the following pullback of inclusion maps

(2)



Observe that

(3) $$G \cap G_j \xrightarrow{m_j'} D_j \underset{m.f_j}{\overset{d_j}{\rightrightarrows}} F \qquad \text{is an equalizer}$$

since $f_j(x) = x$ iff x is an element of $G \cap D_j$. (Unfortunately, f_j' s need not be compatible with the diagram (D_j).)

Define a diagram $H : J \to \mathscr{L}$ as follows. For each $j \in J$, H_j is a product of copies of F indexed by $\{k \in J \mid j \leqq k\}$, and if $j \leqq j'$ then $H_{jj'}: H_j \to H_{j'}$ is the canonical projection. Since H is α-filtered, the colimit $colim\ H = (H_j \xrightarrow{h_j} \hat{H})_{j \in J}$ belongs to $\mathscr{L}$. For each $j \in J$ define $\Delta_{j'}\delta_j : D_j \to H_j$ by the following compositions with the projections $\pi_k: H_j \to F, j \leqq k:$

(4) $$\pi_k \,.\, \Delta_j = d_j$$

and

(5) $$\pi_k \,.\, \delta_j = m \,.\, f_k \,.\, d_{jk}$$

where $d_{jk}: D_j \to D_k$ is the inclusion map. It is easy to verify that if $j \leqq j'$, then $\Delta_{j'} \,.\, d_{jj'} = H_{jj'} \,.\, \Delta_j$ and hence, we have a compatible collection $h_j \,.\, \Delta_j: D_j \to \hat{H}$ yielding

$$\Delta = colim\ \Delta_j: F \to \hat{H}, \qquad \Delta \,.\, d_j = h_j \,.\, \Delta_j.$$

Analogously, $\delta_{j'} \,.\, d_{jj'} = H_{jj'} \,.\, d_j$ and thus we have

$$\delta = colim\ \delta_j: F \to \hat{H}, \qquad \delta \,.\, d_j = h_j \,.\, \delta_j.$$

We will prove that

$$G \xrightarrow{m} F \underset{\delta}{\overset{\Delta}{\rightrightarrows}} \hat{H}$$

is an equalizer—since $F, \hat{H} \in \mathscr{L}$ and $\mathscr{L}$ is closed under limits, this implies $G \in \mathscr{L}$. Since in $\mathbf{Set}^{\mathbf{M}}$ finite limits commute with α-filtered colimits, it is sufficient to show that each

$$G \cap D_j \xrightarrow{m_j'} D_j \underset{\delta_j}{\overset{\Delta_j}{\rightrightarrows}} H_j$$

is an equalizer. First $\Delta_j \,.\, m_j' = \delta_j \,.\, m_j'$ because, for each $k \geqq j$,

$$\begin{aligned} \pi_k \,.\, \Delta_j \,.\, m_j' &= d_j \,.\, m_j' && \text{by (4)}, \\ &= m \,.\, d_j' && \text{by (2)}, \\ &= m \,.\, d_k' \,.\, d_{jk}', \end{aligned}$$

where $d_{jk}': G \cap D_j \to G \cap D_k$ is the inclusion map, and

$$\begin{aligned} \pi_k \,.\, \delta_j \,.\, m_j' &= m \,.\, f_k \,.\, d_{jk} \,.\, m_j' && \text{by (5)}, \\ &= m \,.\, f_k \,.\, m_k' \,.\, d_{jk}' \\ &= m \,.\, d_k' \,.\, d_{jk}' && \text{by (1)}. \end{aligned}$$

Further, suppose that $p: P \to D_j$ fulfils $\Delta_j . p = \delta_j . p$. By composing with $\pi_j: H_j \to F$ we obtain, via (4) and (5),

$$d_j . p = m . f_j . p$$

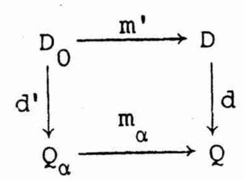
and thus, by (3), p factors through m_j'.

B. A morphism $f: P \to Q$ in $\mathbf{Set}^M$ will be called a γ-epimorphism if $\sum_{X \in obj\, M} card\, [QX - f_X(PX)] < \gamma$. We are going to prove that for each object P in $\mathbf{Set}^M$ there exists a cardinal γ such that every morphism with the domain P factors as a γ-epimorphism followed by an α-pure monomorphism.

We will first show a standard process of enlarging any monomorphism m_0: $Q_0 \to Q$ in $\mathbf{Set}^M$ to an α-pure one. This is done inductively, by defining an α-chain of monomorphism $m_i: Q_i \to Q$ for $i \leqq \alpha$, with m_α α-pure. First, m_0 is the given monomorphism, and $m_i = \bigcup_{j<i} m_j$ for each limit ordinal i. Given m_i: $Q_i \to Q$ consider all pairs of subobjects $(D_0 \xrightarrow{m'} D,\ D_0 \xrightarrow{d'} Q_i)$ where D is α-small. Since $\mathbf{Set}^M$ is wellpowered and has, essentially, only a set of α-small subobjects, all such pairs have a (small) set of representatives. Let us choose a set T_i of representative pairs of monomorphisms $(D_0 \xrightarrow{m'} D,\ D_0 \xrightarrow{d'} Q_i)$ such that D is α-small and that there is a monomorphism $d: D \to Q$ with $d . m' = m_i . d'$ a pullback; choose one d and denote it $d = u_{m',d'}$. Then put

$$m_{i+1} = m_i \cup \bigcup_{(m',d') \in T_i} u_{m',d'} .$$

Let us verify that m_α is α-pure. Given a subobject $d: D \to Q$ with D α-small, we form the pullback of m_α and d:



Since D_0 is α-small, d' factors through some Q_i, $i < \alpha$. That is, if $m_{i\alpha}: Q_i \to Q_\alpha$ denotes the connecting monomorphism, there is d'': $D_0 \to Q_i$ with $d' = m_{i\alpha} . d''$. It is clear that $d . m' = m_i . d''$ is a pullback, and hence, we can suppose that $(m', d'') \in T_i$. Then we have $u_{m',d''} \subset m_{i+1}$, i.e. there is a morphism f_0: $D \to Q_{i+1}$ with $u_{m',d''} = m_{i+1} . f_0$, which composed with m_{i+1} yields f: $D \to Q_\alpha$ with $f . m' = d'$. Thus, m_α is α-pure.

Now, inspecting the construction of m_α we see that $m_{0\alpha}$: $Q_0 \to Q_\alpha$ is a δ-epimorphism for the following cardinal $\delta = \bigvee_{i<\alpha} \delta_i$ (independent of Q): $\delta_0 = 0$ and if i is a limit ordinal, then $\delta_i = \bigvee_{j<i} \delta_j$. Given δ_i, let δ_{i+1} be the cardinality of a set of representative pairs of subobjects $(D_0 \xrightarrow{m'} D, D_0 \xrightarrow{d'} Q')$ where D is any α-small object and Q' is any object for which a δ_i-epimorphism $Q_0 \to Q'$ exists. Thus, for each object Q_0 we have found a cardinal δ such that any monomorphism with domain Q_0 factors as a δ-epimorphism followed by an α-pure monomorphism.

Finally, given an object P, let γ be the join of all δ's associated with quotient objects Q_0 of P. Then each morphism $f: P \to Q$ factors as an epimorphism $e: P \to Q_0$ followed by a monomorphism $m_0: P_0 \to Q$, and factoring the latter as a δ-epimorphism $m_{0,\alpha}: Q_0 \to Q_\alpha$ followed by an α-pure monomorphism $m_\alpha: Q_\alpha \to Q$, we obtain the required factorization: $m_{0,\alpha} . e$ is a γ-epimorphism (since $\gamma > \delta$) and $f = m_\alpha . (m_{0,\alpha} . e)$.

C. $\mathscr{L}$ is reflective in $\mathbf{Set}^M$. In fact, the embedding functor $\mathscr{L} \to \mathbf{Set}^M$ satisfies the solution set condition: for each object P of $\mathbf{Set}^M$ consider a representative set of γ-epimorphisms with the domain P and codomain in $\mathscr{L}$ (for γ as in B.). Then A. and B. show that this set is a solution set of the embedding functor.

Remarks. (1) The above proof shows that the theorem can be slightly strengthened: each full subcategory of $\mathscr{K}$ closed under limits and reduced powers modulo α-complete filters is reflective in $\mathscr{K}$. (If α is compact, the filters can be replaced by ultrafilters. In particular, each full subcategory of $\mathscr{K}$ closed under limits and ultraproducts is reflective.)

(2) We have been partly inspired by [5]. Some ingredients of our proof are not really new; see the characterization of α-algebraically closed (= α-pure) embeddings in [4], (5–7), and the well-known procedure of constructing algebraic closures.

(3) The assumption that $\mathscr{K}$ be locally presentable cannot be omitted in the above theorem. For example, the dual $\mathbf{Ord}^{\mathrm{op}}$ of the usual category of ordinals is not reflective in its extension by an initial object. However, that extension is complete and complete, and $\mathbf{Ord}^{\mathrm{op}}$ is closed in it under (small) limits and non-empty colimits.

(4) We have shown in [2] that, under some set-theoretical assumptions, the collection $Ref(\mathscr{K})$ of all full reflective subcategories can be badly behaved even if $\mathscr{K}$ is locally finitely presentable: for $\mathscr{K}$ = graphs we have exhibited two members of $Ref(\mathscr{K})$ whose intersection is not a member of $Ref(\mathscr{K})$.

The situation is different with the collection $Ref_\alpha(\mathscr{K})$ of all full reflective subcategories of $\mathscr{K}$ closed under α-filtered colimits.

Proposition. *For each locally presentable category $\mathscr{K}$ and each regular cardinal α, $Ref_\alpha(\mathscr{K})$ is a small complete lattice in which meets are intersections.*

Proof. Since $Ref_\alpha(\mathscr{K})$ coincides with the collection of all full subcategories of $\mathscr{H}$ closed under limits and α-filtered colimits, the only fact to be proved is that $Ref_\alpha(\mathscr{H})$ is small. This follows from an easy inspection of the above proof. First, since $\alpha \leqq \beta$ implies $Ref_\alpha(\mathscr{H}), \subseteq Ref_\beta(\mathscr{H})$ we can suppose, without loss of generality, that $\mathscr{H}$ is locally α-presentable. Then for each $\mathscr{L} \in Ref_\alpha(\mathscr{H})$ the reflector of $\mathscr{L}$ preserves α-filtered colimits and hence, $\mathscr{L}$ is determined by the reflections of α-presentable $\mathscr{H}$-objects in $\mathscr{L}$. There is, essentially, a set only of α-presentable objects P, and for each of them we have provided in our proof a solution set of morphisms with domain P and codomains in $\mathscr{L}$ the size of which was independent of $\mathscr{L}$. Thus $Ref_\alpha(\mathscr{H})$ is small.

The fact that $Ref_\alpha(\mathscr{H})$ is closed under intersections also follows by [3] (a remark before 5.3.).

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COMPLETE PERMUTABILITY OF PARTITIONS IN A SET

Part I

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(Received May 16, 1988)

Dedicated to the memory of Milan Sekanina

Abstract. J. Hashimoto introduced in [7] a notion of a complete permutability of partitions on a set as a generalization of commutativity of two partitions. By H. Draškovičová [5] this is transferred unchanged to partitions in a set. Unfortunately, her notion fails to generalize the commutativity. In the present paper, this disadvantage is removed by a little modification of the definition. This definition reproduces that of Hashimoto in the case of partitions "on". The relations of the modified notion to that of the associability, which represents another generalization of the commutativity, are found. Properties of the introduced complete permutability are discussed in Theorems 1.9, 1.10 and 1.11. An analogous definition is given for congruence relations in an algebra. Some sufficient conditions are found that guarantee the possibility of the preceding results to be applied for congruence relations. In particular, a characterization of complete permutability for Ω-groups is derived.

Key words. Partition "on" and "in", ST-relation, congruence relation, complete permutability, associability, commutativity.

MS Classification. 06 F 15

A partition *in* a set G is a system A (possibly empty) of nonempty mutually disjoint subsets of G [2, 3, 4]. The elements of A are called *blocks* of the partition A. The union $\bigcup A$ of all blocks of A is said to be a *domain* of A. If the domain of A is equal to G, $\bigcup A = G$, the partition A is called a partition *on* G.

It is a well-known fact, that there exists an one-one correspondence between all partitions on a set and all equivalence relations in the same set, and analogously, an one-one correspondence between all partitions in the set G and all symmetric and transitive relations (ST-relations) in G. We shall find it useful to hold, if need be, the partitions in G (on G) for ST-relations (equivalence relations) in G and vice versa. If A is an ST-relation in G, the corresponding partition in G is $\bigcup A/A$, where in this case $\bigcup A = \{x \in G: xAx\}$.

Let (G, Ω) be a universal algebra with the system of operations Ω and let A be an ST-relation (a partition) in the set G. We say, that A is a *congruence relation* *in* the algebra (G, Ω) if A preserves the operations of Ω (of arity ≥ 1). Congruence relations A in (G, Ω) with $\bigcup A = G$ are said to be *congruence relations on* the algebra (G, Ω). We denote by

$P(G)$ the system of all partitions (ST-relations) in the set G,
$\pi(G)$ the system of all partitions (equivalence relations) on the set G,
$\mathscr{K}(G)$ the system of all congruence relations in the algebra (G, Ω)
(the system of the corresponding ST-relations in the set G),
$\mathscr{C}(G)$ the system of all congruence relations on the algebra (G, Ω),
(the system of the corresponding equivalence relations in the set G).

Some known and relevant facts related to these notions are summarized in the following

Theorem. *The set $\pi(G)$ is a complete, semimodular and relatively complemented lattice* [13], [11] Th. 67. *The lattice $P(G)$ is complete, semimodular and Brouwerian, it is not relatively complemented* [4] Ths. 4.5, 4.1 and 5.3, *$\pi(G)$ is a closed sublattice of $P(G)$. If (G, Ω) is an algebra, then the set $\mathscr{C}(G)$ is a closed sublattice of $\pi(G)$* (see e.g. [11] Th. 84; [1] VI § 4 Th. 8). *The lattice $\mathscr{C}(G)$, where (G, Ω) is an Ω-group, is modular* (see e.g. [8] IV, 2.2). *If G is a lattice or an l-group, then $\mathscr{C}(G)$ is a distributive lattice* (see e.g. [11] Th. 90, [1] XIII § 9 Th 16). *All the lattices $P(G)$, $\pi(G)$, $\mathscr{K}(G)$, $\mathscr{C}(G)$ are algebraic* [12] 1.6; [1] XII § 9 Th. 16, VI § 4 Th. 9; [10] § 5.

The domain $\bigcup A$ of a congruence relation $A \in \mathscr{K}(G)$ is a subalgebra of the algebra (G, Ω). The nullblock $A(0) = \{x \in G: xA0\}$ of a congruence relation A in an Ω-group (G, Ω) is an ideal in $\bigcup A$ and there holds $A = \bigcup A / A(0)$ [9] I 1.3 and 1.4. □

1. The notion of the complete permutability of partitions was introduced in [7], p. 90 for partitions *on* a set and in [5], Def. 1.3, unchanged transferred to partitions *in* a set. The mentioned definition in [7] reads

(*) *A system $\{A_\iota : \iota \in \Gamma\}$ of partitions on a set G is called* completely permutable *if for arbitrary subsets $\emptyset \neq \Lambda \subseteq \Gamma$ and $\{x^\iota : \iota \in \Lambda\} \subseteq G$ it holds*

whenever $x^\mu (C_\mu \vee C_\nu)\, x^\nu$, $\mu, \nu \in \Lambda$, where $C_\alpha = \bigwedge\limits_{\substack{\iota \in \Lambda \\ \iota \neq \alpha}} A_\iota$, $\alpha \in \Lambda$, then there exists $x' \in G$ such that $x^\iota A_\iota x$, $\iota \in \Lambda$.

Note that the definition (*) was introduced in [7] for congruence relations on a universal algebra (G, Ω). Since the lattice $\mathscr{C}(G)$ of all congruence relations on (G, Ω) is a closed sublattice of the lattice of all partitions $\pi(G)$ on the set G, the replacement of congruence relations by partitions is unessential (viz. $\vee_{\mathscr{C}} \equiv \vee_P$)

For a reason, which will be explained in the sequel, it is suitable to modify the definition (*) for partitions in a set as follows

1.1. Definition. *A system of partitions in a set G $\{A_\iota : \iota \in \Gamma\}$ with* card $\Gamma \geq 2$ *is called* completely permutable [finitely permutable] *if for an arbitrary [finite] subset $\Lambda \subseteq \Gamma$ with* card $\Lambda \geq 2$ *and for an arbitrary system of elements $\mathfrak{A} =$*

$= \{x^\iota : \iota \in \Lambda\} \subseteq G$ *fulfilling* $x^\mu(C_\mu \vee C_\nu)\, x^\nu$, μ, $\nu \in \Lambda$, $\mu \neq \nu$, *where* $\alpha \in \bigwedge C_\alpha = \bigwedge_{\substack{\iota \in \Lambda \\ \iota \neq \alpha}} A_\iota$, *the condition* (1.1Z) *or* (1.2Z) *is valid*

(1.1Z) $x \in G$ *exists such that* $x^\iota A_\iota x$, $\iota \in \Lambda$, *or*

(1.2Z) *there exist* $\alpha \in \Lambda$ *and* $A^1_\alpha \in \bigcup A_\alpha / A_\alpha$ *which satisfy* a) $\mathfrak{A} \subseteq A^1_\alpha$ and b) $A^1_\alpha \cap \cap \bigcup A_\iota \neq \emptyset$ *for some* $\iota \in \Lambda$ *implies* $A^1_\alpha \in \bigcup A_\iota / A_\iota$. (By $\vee$ there is meant the supremum $\vee_P$ in $P(G)$.)

Note. If we should admit card $\Lambda = 1$ in the Definition 1.1, then it would hold $\bigcup_{\iota \in \Gamma} \cup A^\iota = G$.

Indeed $\iota \in \Gamma$, choose $y \in G$ and $\Lambda = \{\iota_0\}$ for some $\iota_0 \in \Gamma$. Then the requirement for the singleton $\mathfrak{A} = \{y\} \subseteq G\,(y = x^{\iota_0})$ is satified trivially and thus (1.1Z) or (1.2Z) do not be fulfilled unless it holds $y \in \bigcup A_{\iota_0}$, i.e. $\bigcup_{\iota \in \Gamma} \bigcup A_\iota = G$. □

1.2. Definition ([9] IV Def. 4.1; [5] Def. 1.2) *A system* $\{A_\iota : \iota \in \Gamma\}$ *of partitions in a set G is called* associable *if it satisfies; For any system* $\mathfrak{A} = \{x^\iota : \iota \in \Gamma\}$ *of elements of the set G fulfilling* $x^\mu(\bigvee_{\iota \in \Gamma} A_\iota)\, x^\nu$, μ, $\nu \in \Gamma$, *one of the following conditions holds*

(1.1A) $x \in G$ *exists such that* $x^\iota A_\iota x$, $\iota \in \Gamma$, *or*

(1.2A) $\alpha \in \Gamma$ *and* $A^1_\alpha \in \bigcup A_\alpha / A_\alpha$ *exist such that* a) $\mathfrak{A} \subseteq A^1_\alpha$ and b) if $A^1_\alpha \cap \bigcup A_\beta \neq \emptyset$ *for some* $\beta \in \Gamma$, $A^1_\alpha \in \bigcup A_\beta / A_\beta$.

Any nonempty subset of an associable system is associable [5] and [9] IV 4.5. Then the following two Propositions are evidently true.

1.3. Proposition. *Any associable system of (at least two) partitions in a set is completely permutable.* □

1.4. Proposition. *If* $\{A_\iota : \iota \in \Gamma\}$ *with* card $\Gamma \geq 2$ *is a completely permutable [finitely permutable] system of partitions in a set G,* $K \subseteq \Gamma$ *with* card $K \geq 2$, *then the system* $\{A_\iota : \iota \in K\}$ *is completely permutable [finitely permutable] as well.* □

1.5. Proposition. *Let* $\{A_1, A_2\}$ *be a system of two partitions in a set G. Then the following are equivalent.*

a) *The system* $\{A_1, A_2\}$ *is completely permutable*;

b) *The system* $\{A_1, A_2\}$ *is associable*;

c) *The partitions* A_1 *and* A_2 *commute.*

Proof. c implies b. Let the partitions A_1 and A_2 commute and let $x^1(A_1 \vee A_2)\, x^2$. By [9] III 3.1.1(1), $A_1 \vee A_2 = A_1 A_2 \cup A_1 \cup A_2$, thus $x^1 A_1 A_2 x^2$ or $x^1 A_1 x^2$ or $x^1 A_2 x^2$. In the first case $x \in G$ exists such that $x^1 A_1 x A_2 x^2$, consequently (1.1A) holds. In the second case $x^1, x^2 \in A^1_1$ ($\alpha = 1$) for some block $A^1_1 \in \bigcup A_1 / A_1$. If for $i = 2$ there exists an element $x \in A^1_1 \cap \bigcup A_2$, then $x^2 A_1 A_2 y$ for some $y \in G$. From the commutativity of A_1 and A_2 it follows $x^2 A_2 A_1 y$, thus $x^2 A_2 x^2$. Hence

$x^1A_1x^2A_2x^2$ and therefore the condition (1.1A) is satisfied. The last case $x^1A_2x^2$ is symmetric to the preceding one.

b implies a is evident.

a implies c. Let the system $\{A_1, A_2\}$ be completely permutable, $\mathfrak{A} = \{x^1, x^2\} \subseteq G$ and $x^1A_2A_1x^2$. Then $x^1(A_1 \vee A_2)\, x^2$. If the condition (1.1Z) is satisfied, then $x \in G$ exists such that $x^1A_1xA_2x^2$, hence $x^1A_1A_2x^2$. Let (1.2Z) hold, let $\alpha \in \{1, 2\}$ and a block $A^1_\alpha \in \bigcup A_\alpha/A_\alpha$ exist such that $\mathfrak{A} \subseteq A^1_\alpha$. For any $\iota \in \{1, 2\}$ there holds x^1 or $x^2 \in A^1_\alpha \cap \bigcup A_\iota$. Thus $A^1_\alpha \in \bigcup A_\iota/A_\iota$ $(\iota = 1, 2)$, i.e. $(x^1, x^2) \in A_1 \cap A_2 \subseteq A_1A_2$. The reverse inclusion can be proved symmetrically. Hence $A_1A_2 = A_2A_1$, the desired commutativity. □

1.6. Corollary. *Any two partitions belonging to a completely permutable [finitely permutable] system of partitions in a set commute.*

Proof follows from 1.4 and 1.5. □

1.7. Remark. For partitions on a set the definition 1.1 of the complete permutability is identical with the definition (*) (appart from the requirement card $\Gamma \geq 2$ and card $\Lambda \geq 2$).

Proof. For partitions $\{A_\iota\colon \iota \in \Gamma\}$ on a set it holds $x^\mu(C_\mu \vee C_\nu)\, x^\nu$, $\mu \neq \nu$, $\mu, \nu \in \Lambda \Leftrightarrow x^\mu(C_\mu \vee C_\nu)\, x^\nu$, $\mu, \nu \in \Lambda$. Further the condition (1.2Z) for partitions "on" mplies the condition (1.1Z) (with an arbitrary $x \in A^1_\alpha$). □

1.8. The principal disadvantage of the definition 1.3 [5] for the complete permutability of partitions in a set is that it is not equivalent to the commutativity in case of two partitions. The 1.3 [5] version of the complete permutability of two partitions A_1, A_2 implies the relation $A_1 \vee A_2 = A_1A_2$, while their commutativity implies $A_1 \vee A_2 = A_1A_2 \cup A_1 \cup A_2$ by [9] 3.1.1(1). Thus the complete permutability by [5] implies the commutativity ([9] III 3.1.1(3)), the converse does not hold in general ([9] III 3.1.1(1)).

As we see, the notions of associability and complete permutability generalize the notion of commutativity. It is usefull when two generalizations are comparable. Our definition 1.1 admits such a comparison. This is given by Proposition 1.3. Such a comparison is not true for the complete permutability version [5]. Namely, all the partitions of a system $\{A_\iota\colon \iota \in \Gamma\}$, which is completely permutable in the sense of [5] have the same domain. Indeed, in this case $A_\mu \vee A_\nu = A_\mu A_\nu$, $\mu, \nu \in \Gamma$, so that $\bigcup A_\mu = \bigcup A_\nu$ by [9] III 3.1.1(6).

Now, we give some properties of complete permutability. To this end let us recall two notions..

A subset H of a set G is said to *respect* a partition A in G if H contains each block of A, which it intersects ([9] IV Def. 4.8).

Let $\mathbf{A} = \{A_\iota\colon \iota \in \Gamma\}$ be a system of partitions in a set G and $\emptyset \neq H \subseteq G$. Under $\mathbf{A} \sqcap H$ we understand the system $\{A_\iota \sqcap H\colon \iota \in \Gamma\}$ ([9] IV Def. 4.8.2). As for the symbol $A_\iota \sqcap H$, see [3] I 2.3: $A_\iota \sqcap H = \{A^1 \cap H\colon A^1 \in \bigcup A/A,\ A^1 \cap H \neq \emptyset\}$.

1.9. Theorem. *If a system* $\mathbf{A} = \{A_\iota : \iota \in \Gamma\}$ *with* card $\Gamma \geq 2$ *of partitions in a set G is completely permutable [finitely permutable], then the following conditions are true.*

(A.1Z) *For any* $K \subseteq \Gamma$ *with* card $K \geq 2$ *and* $G_K = \bigcap_{\iota \in K} \bigcup A_\iota \neq \emptyset$ *it holds that* $\{A_\iota \sqcap G_K : \iota \in K\}$ *is a completely permutable [finitely permutable] system of partitions on* G_K;

(A.2Z) $\bigcup A_\alpha$ *respects the partitions* A_β, $\alpha, \beta \in \Gamma$.

Note. Let $\bar{\bigvee}$ be the symbol for supremum in the lattice $P(G_K)$. Then for $\emptyset \neq \Lambda \subseteq K$ there holds $G_\Lambda \sqcap \bigvee_{\iota \in \Lambda} A_\iota = \bar{\bigvee}_{\iota \in \Lambda} (G_\Lambda \sqcap A_\iota)$. Analogously for infimum.

The result follows after some easy manipulation.

Proof of Theorem will be carried out for the complete permutability. The case of finite permutability is analogous.

First, from 1.6 it follows that A_α and A_β commute ($\alpha, \beta \in \Gamma$). Suppose that $\bigcup A_\beta$ does not respect A_α, i.e. that for some $x, y \in G$ it holds $x, y \in A^1_\alpha \in \bigcup A_\alpha / A_\alpha$, $x \in \bigcup A_\beta$, $y \bar{\in} \bigcup A_\beta$. Then $aA_\beta x A_\alpha y$ for some $a \in G$ and thus $aA_\alpha A_\beta y$, which means that $y \in \cup A_\beta$ — a contradiction. Thus (A.2Z) holds.

To prove (A.1Z), let us choose $\Lambda \subseteq K$ with card $\Lambda \geq 2$ and denote $A_\iota \sqcap G_K = \bar{A}_\iota (\iota \in K)$, $\bar{C}_\alpha = \bigwedge_{\substack{\iota \in \Lambda \\ \iota \neq \alpha}} \bar{A}_\iota (\alpha \in \Lambda)$. Consider $\mathfrak{A} = \{x^\iota : \iota \in \Lambda\} \subseteq G_K$ with $x^\mu (\bar{C}_\mu \bar{\vee} \bar{C}_\nu) x^\nu$, $\mu \neq \nu$, $\mu, \nu \in \Lambda$. Then $x^\mu (C_\mu \vee C_\nu) x^\nu$, $\mu \neq \nu$, $\mu, \nu \in \Lambda$, where $C_\alpha = \bigwedge_{\substack{\iota \in \Lambda \\ \iota \neq \alpha}} A_\iota$. Consequently (1.1Z) or (1.2Z) hold for the completely permutable system $\{A_\iota : \iota \in \Gamma\}$ of partitions in G. Now, (1.1Z) reads that $x \in G$ exists with $x^\iota A_\iota x$, $\iota \in \Lambda$. We shall prove that $x \in G_K$ and so it will be showed $x^\iota \bar{A}_\iota x$, $\iota \in \Lambda$. On the one hand it holds $x, x^\iota \in A^1_\iota$ for some $A^1_\iota \in \bigcup A_\iota / A_\iota$ $(\iota \in \Lambda)$. On the other hand $\Lambda \subseteq G_K$ implies $x^\iota \in \bigcup A_\alpha$, $\alpha \in K$. Since $\bigcup A_\alpha$ respects A^ι and $x^\iota \in \bigcup A_\alpha$ we have $x \in A^1_\iota \subseteq \bigcup A_\alpha$, $\alpha \in K$, so that $x \in G_K$, whence $x^\iota \bar{A} x_\iota$, $\iota \in \Lambda$.

From the condition (1.2Z) we get an anlogous condition for the system $\{\bar{A}_\iota : \iota \in \Lambda\}$ of partitions on G_K, which by 1.7 implies that (1.1Z) for the system $\{\bar{A}_\iota : \iota \in \Lambda\}$ is true, completing the proof. □

1.10. Theorem. *A system* $\mathbf{A} = \{A_\iota : \iota \in \Gamma\}$ *with* card $\Gamma \geq 2$ *of partitions in a set G is completely permutable [finitely permutable] iff the following conditions* (B1) *and* (B2) *are true.*

(B1) *Any two partitions of the system* $\mathbf{A}$ *commute*;

(B2) *If* $\Lambda \subseteq \Gamma$, card $\Lambda \geq 3$ [$\aleph_0 >$ card $\Lambda \geq 3$], $\mathfrak{A} = \{x^\iota : \iota \in \Lambda\} \subseteq G$, $x^\mu (C_\mu \vee C_\nu) x^\nu$, $\mu, \nu \in \Lambda$, $\mu \neq \nu$, *then there exists* $x \in G$ *such that* $x^\iota A_\iota x$, $\iota \in \Lambda$.

Proof will be carried out for the complete permutability. The case of finite permutability is analogous.

(B1) and (B2) imply evidently the complete permutability of $\mathbf{A}$.

Conversely, let $\boldsymbol{A}$ be completely permutable. By 1.6, (B1) holds. To prove (B2), let $\Lambda \subseteq \Gamma$ with card $\Lambda \geq 3$ and $\mathfrak{A} = \{x^\iota: \iota \in \Lambda\} \subseteq G$ with $x^\mu(C_\mu \vee C_\nu)\, x^\nu$, $\mu, \nu \in \Lambda$, $\mu \neq \nu$. If (1.1Z) is true, then $x \in G$ exists with $x^\iota A_\iota x$, $\iota \in \Lambda$. Let (1.2Z) be satisfied. We shall prove that $A_\alpha^1 \cap \bigcup A_\iota \neq \emptyset$ for all $\iota \in \Lambda$ and thus that A_α^1 is a block of every A_ι ($\iota \in \Lambda$). From this one derives that an arbitrary $x \in A_\alpha^1$ fulfils $x^\iota A_\iota x$, $\iota \in \Lambda$. Choose $\iota \in \Lambda$. There are $\mu, \nu \in \Lambda$ such that μ, ν and ι are different (recall that card $\Lambda \geq 3$). From $x^\mu(C_\mu \vee C_\nu)\, x^\nu$ we get $x^\mu A_\iota x^\nu$. It follows that e.g. $x^\mu \in \bigcup A_\iota$. Since $x^\mu \in A_\alpha^1$ we get then $x^\mu \in A_\alpha^1 \cap \bigcup A_\iota$ which was to be proved. □

1.11. Theorem. *A system $\{A_\iota: \iota \in \Gamma\}$ with* card $\Gamma \geq 2$ *of partitions in a set G is completely permutable [finitely permutable] iff the following condition is satisfied.*

If $\Lambda \subseteq \Gamma$ with card $\Lambda \geq 2[\aleph_0 >$ card $\Lambda \geq 2]$ *and $\mathfrak{A} = \{x^\iota: \iota \in \Lambda\} \subseteq G$ with $x^\mu(C_\mu \vee C_\nu)\, x^\nu$, $\mu \neq \nu$, $\mu, \nu \in \Lambda$ are given, where $C_\alpha = \bigwedge_{\iota \in \Lambda, \iota \neq \alpha} A_\iota$, then one of the following conditions* is *fulfilled*

(C1) *$x \in G$ exists such that $x^\iota A_\iota x$, $\iota \in \Lambda$, or*

(C2) card $\Lambda = 2$ *and there exist $\alpha \in \Lambda$ and $A_\alpha^1 \in \bigcup A_\alpha / A_\alpha$ such that $\mathfrak{A} \subseteq A_\alpha^1$ and $A_\alpha^1 \cap \bigcup A_\beta = \emptyset$ for $\beta \in \Lambda$, $\beta \neq \alpha$.*

Proof. Obviously the condition of the Theorem implies that (1.1Z) and (1.2Z) are fulfilled. It remains to prove that the complete permutability [finite permutability] implies that the condition of the Theorem is fulfilled.

Suppose that $\Lambda \subseteq \Gamma$ with card $\Lambda \geq 2$ and $\mathfrak{A} = \{x^\iota: \iota \in \Lambda\} \subseteq G$ with $x^\mu(C_\mu \vee C_\nu)\, x^\nu$, $\mu \neq \nu$, $\mu, \nu \in \Lambda$ are given. Provided card $\Lambda \geq 3$ holds, (C1) is true by Theorem 1.10. Suppose card $\Lambda = 2$ and denote $\Lambda = \{1, 2\}$. Then $x^1(C_1 \vee C_2)\, x^2$ means that $x^1(A_2 \vee A_1)\, x^2$. By 1.10, A_1 and A_2 commute and by [9] III 3.1.1(1) $A_1 \vee A_2 = A_1 A_2 \cup A_1 \cup A_2$. Then $x^1 A_1 A_2 x^2$ or $x^1 A_1 x^2$ or $x^1 A_2 x^2$. In the first case $x^1 A_1 x A_2 x^2$ for some $x \in G$, thus we have (C1). The second case leads to the relation $x^1, x^2 \in A_1^1$ for some block $A_1^1 \in \bigcup A_1 / A_1$. If $A_1^1 \cap \bigcup A_2 \neq \emptyset$, then since $\bigcup A_1$ respects A_1 we have $A_1^1 \subseteq \bigcup A_2$ and the condition (C1) is fulfilled for an arbitrary $x \in \{x^1, x^2\}$. Analogously for the third case. □

1.11. Example of an associable (and therefore completely permutable) system $\{A_\iota: \iota \in \Gamma\}$ of partitions in the set $G = \{1, 2, 3, \ldots, 11\}$, $\Gamma = \{1, 2, 3, 4\}$ (for which, in addition, $G_\Lambda \neq G_K$ whenever $\Lambda \neq K$, $\Lambda, K \subseteq \Gamma$, is satisfied).

All blocks of any partition A_1 to A_4 are singletons:

A_1:	1	2	3	4		6	7	8			
A_2:	1	2	3		5	6			9	10	
A_3:	1	2		4	5		7		9		11
A_4:	1		3	4	5			8		10	11

$G_{\{1,2,3,4\}} = \{1\}$, $G_{\{1,2,3\}} = \{1, 2\}$, $G_{\{1,2,4\}} = \{1, 3\}$, $G_{\{1,3,4\}} = \{1, 4\}$,
$G_{\{2,3,4\}} = \{1, 5\}$, $G_{\{1,2\}} = \{1, 2, 3, 6\}$, $G_{\{1,3\}} = \{1, 2, 4, 7\}$, $G_{\{1,4\}} = \{1, 3, 4, 8\}$,
$G_{\{2,3\}} = \{1, 2, 5, 9\}$, $G_{\{2,4\}} = \{1, 3, 5, 10\}$, $G_{\{3,4\}} = \{1, 4, 5, 11\}$.

Evidently for card $\Lambda = 1$ the sets G_Λ differ from one another and from the preceding ones, as well.

2. The definition of a completely permutable system of congruence relations in an algebra (G, Ω) can be formulated analogously as in Definition 1.1 with the distinction that in the join $C_\mu \vee C_\nu$ under $\vee$ there is to understand the supremum $\vee_{\mathscr{K}}$ in the lattice $\mathscr{K}(G)$. Unfortunately, suprema in $\mathscr{K}(G)$ do not coincide with those in $P(G)$ hence the lattice $\mathscr{K}(G)$ is not a sublattice of the lattice $P(G)$. Consequently, the theory concerning partitions cannot be applied directly to congruence relations. Thus it is useful to study conditions which assure "the closedness" of some subsets of $\mathscr{K}(G)$ in $P(G)$. In the following we point out some cases.

2.1. Definition. *Let* $\{A_\iota : \iota \in \Gamma\}$ *with* card $\Gamma \geq 2$ *be a system of congruence relations in a universal algebra* (G, Ω). *This system is called* completely permutable [finitely permutable] *if the system of corresponding partitions in the set G is completely permutable [finitely permutable] according to the lattice* $\mathscr{K}(G)$ *of congruence relations in the algebra* (G, Ω). (Now, in the Definition 1.1 under $\vee$ supremum $\vee_{\mathscr{K}}$ in $\mathscr{K}(G)$ is meant.)

Under certain conditions the permutability of congruence relations can be related to the lattice $P(G)$. Later, one of these will be given.

2.2. Proposition ([9] I 1.2) *Let* (G, Ω) *be an algebra and* $\{A_\iota : \iota \in \Gamma\} \subseteq \mathscr{K}(G)$. *Then* $\bigvee_{\mathscr{K}\, \iota \in \Lambda} A_\iota = \bigvee_{P\, \gamma} B_\gamma$, *where by* B_γ *there is meant the congruence relation* $A_{\iota_1} \vee_{\mathscr{K}} \dots \vee_{\mathscr{K}} A_{\iota_n}$ *for an arbitrary finite choice* $A_{\iota_1}, \dots, A_{\iota_n}$ *in* $\{A_\iota : \iota \in \Lambda\}$. □

2.3. Theorem ([9] I 1.2.0) *Let* (G, Ω) *be an algebra and* $\{A_\iota : \iota \in \Lambda\}$ *an up-directed subset of the lattice* $\mathscr{K}(G)$. *Then* $\bigvee_{\mathscr{K}\, \iota \in \Lambda} A_\iota = \bigvee_{P\, \iota \in \Lambda} A_\iota$. □

2.4. Corollary. *Let* (G, Ω) *be an algebra and* $\mathbf{A} = \{A_\iota : \iota \in \Gamma\} \subseteq \mathscr{K}(G)$. *Let every finite subset* Λ *of* $\mathbf{A}$ *be up-directed. Then* $\bigvee_{\mathscr{K}\, \iota \in \Lambda} A_\iota = \bigvee_{P\, \iota \in \Lambda} A_\iota$.

Proof follows from 2.3 and 2.2. □

2.5. Proposition ([9] IV 4.8.1 (b)) *If A is a congruence relation in an Ω-group G and H a subgroup of the additive group G, then H respects the partition A iff* $A(0) \subseteq H$ *(where* $A(0)$ *is the block of the partition A containing the neutral element* **0** *of the group G).* □

2.6. Theorem. *Let a system* $\{A_\iota : \iota \in \Gamma\}$ *with* card $\Gamma \geq 2$ *of congruence relations in an Ω-group* (G, Ω) *be completely permutable [finitely permutable]. Then the following conditions* a) *and* b) *are true*

a) $A_\alpha(0) \subseteq \bigcup A_\beta$, $\alpha, \beta \in \Gamma$;

b) *For every* $K \subseteq \Gamma$ *with* card $K \geq 2$ [$\aleph_0 >$ card $K \geq 2$] *the system* $\{G_K/(A_\iota(0)) : \iota \in K\}$ *of congruence relations on* $G_K = \bigcap_{\iota \in K} \bigcup A_\iota$ *is completely permutable [finitely permutable].*

Proof. The condition a) expresses the requirement (A.2Z), Theorem 1.9 (see also 2.5). As for the condition b), there holds $G_K/(A_\iota(0)) = A_\iota \sqcap G_K$. Reference to a) shows that $A_\iota(0) \subseteq G_K$. Thus, our conditions a) and b) express the conditions (1.1Z) and (1.2Z) of Theorem 1.9 provided G is an Ω-group. Finally, for the congruence relations "on" (on G_K) the equality $\vee_P = \vee_{\mathscr{E}}$ is valid. □

Part II of the present paper will contain applications of the results discussed in this Part I.

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О РЕГУЛЯРНОСТИ ЭЛЕМЕНТОВ РЕЛЯТИВА

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(Поступило в редакцию 17 го мая 1988 г.)

Посвящается памяти Милана Секанины

Резюме. Критерий регулярности элемента полугруппы бинарных отношений обобщается на случай релятивов (алгебр отношений). Из этого извлекаются некоторые следствия для полугруппы B-отношений, где B — булева алгебра.

Ключевые слова: релятив, алгебра отношений, полугруппа отношений над булевой алгеброй.

Классификация АМО: 04 A 05, 20 M 20

Релятивой или *алгеброй отношений* называется универзальная алгебра R сигнатуры $\{+, \cdot, -, 0, I, \circ, *, E\}$, где $+$, $\cdot$ и $\circ$ — бинарные операции, $-$ и $*$ — унарные, а 0, I и E нульарные, причем $\{_R \mid +, \cdot, -, 0, I\}$ — булева алгебра, $\{R \mid \circ, *, E\}$ — моноид инволюцией, $(x + y) \circ z = x \circ z + y \circ z$, $(x + y)^* = x^* + y^*$ и $x^* \circ \overline{x \circ y} \leq \bar{y}$ для любых $x, y, z \in R$. В настоящей заметке приняты обозначения, использовавшиеся в [2], где можно найти и дальнейщие ссылки. Важным примером релятива служит множество всех B-отноцений, где B — булева алгебра (см. [1]). Напомним, что если $B = 2$, то B-отношения — это обычные бинарные отношения. В связи с этим возникает задача обобщения свойств релятива бинарных отнощений на произвольные релятива. Одна из таких задач и решается в настоящей заметке.

Элемент ϱ релятива R называется *регулярным*, если $\varrho = \varrho \circ \delta \circ \varrho$ для некоторого $\sigma \in R$.

Если ϱ — элемент релятива R, то положим

$$\tilde{\varrho} = \overline{\varrho^* \circ \bar{\varrho} \circ \varrho^*}.$$

Теорема 1. *Следующие свойства элемента ϱ релятива R равносильны:* (1) *ϱ регулярен;* (2) $\varrho \leq \varrho \circ \tilde{\varrho} \circ \varrho$; (3) $\varrho = \varrho \circ \tilde{\varrho} \circ \varrho$ (ср. [7], теорема 2).

Доказательство. (1) $\Rightarrow$ (2). Если $\varrho = \varrho \circ \sigma \circ \varrho$, то, применяя соотношение (S 6):

$$(a \circ b)(c \circ d) \leq a \circ ((a^* \circ c)(b \circ d^*)) \text{ od} \quad \text{(см. [2], с. 130, свойство (S 6))},$$

где $a = E$, получаем

$$(\varrho \circ \sigma)(\bar{\varrho} \circ \varrho^*) \leq (\bar{\varrho}(\varrho \circ \sigma \circ \varrho)) \circ \varrho^* = \bar{\varrho}\varrho \circ \varrho^* = 0.$$

Вторичное применение той же формулы при $d = E$ дает

$$(\varrho^* \circ \bar{\varrho} \circ \varrho^*)\,\sigma = \varrho^* \circ ((\varrho \circ \sigma)\,(\bar{\varrho} \circ \varrho^*)) = 0.$$

Ввиду [3], с. 155, упр. 1, отсюда вытекает

$$\sigma \leq \overline{\varrho^* \circ \bar{\varrho} \circ \varrho^*} = \tilde{\varrho}$$

и, ввиду [2], лемма 1 (2),

$$\varrho = \varrho \circ \sigma \circ \varrho \leq \varrho \circ \tilde{\varrho} \circ \varrho.$$

(2) ⇒ (3). Используя неравенство $x^* \circ \overline{x \circ y} \leq \bar{y}$ и двойственное соотношение, получим

$$\varrho \leq \varrho \circ \tilde{\varrho} \circ \varrho = \varrho \circ \overline{\varrho^* \circ \bar{\varrho} \circ \varrho^*} \circ \varrho \leq \overline{\bar{\varrho} \circ \varrho^*} \circ \varrho \leq \bar{\bar{\varrho}} = \varrho.$$

(3) ⇒ (1). Тривиально.

Теорема 2. *Если ϱ — регулярный элемент релятива R, то $\tilde{\varrho} \circ \varrho \circ \tilde{\varrho}$ — наибольший инверсный элемент элемента ϱ* (ср. [7], с. 97, следствие).

Доказательство. Пусть $\sigma = \tilde{\varrho} \circ \varrho \circ \tilde{\varrho}$. Ввиду теоремы 1,

$$\varrho \circ \sigma \circ \varrho = \varrho \circ \tilde{\varrho} \circ \varrho \circ \tilde{\varrho} \circ \varrho = \varrho$$

и

$$\sigma \circ \varrho \circ \sigma = \tilde{\varrho} \circ \varrho \circ \tilde{\varrho} \circ \varrho \circ \tilde{\varrho} \circ \varrho \circ \tilde{\varrho} = \tilde{\varrho} \circ \varrho \circ \tilde{\varrho} = \sigma,$$

т. е. σ — инверсный для ϱ. Если $\tau \in R$, $\varrho \circ \tau \circ \varrho = \varrho$ и $\tau \circ \varrho \circ \tau = \tau$, то, используя [2], леммы 1 (2) и 1 (4), а также (S 6) из [2], получим

$$\begin{aligned}(\varrho^* \circ \bar{\varrho} \circ \varrho^*)\,\tau &= (\varrho^* \circ \bar{\varrho} \circ \varrho^*)\,(\tau \circ \varrho \circ \tau) \leq \\ &\leq \varrho^* \circ (\varrho \circ \tau \circ \varrho)\,(\bar{\varrho} \circ \varrho^* \circ \tau^*) \circ \tau = \\ &= \varrho^* \circ (E \circ \varrho)\,(\bar{\varrho} \circ \varrho^* \circ \tau^*) \circ \tau \leq \\ &\leq \varrho^* \circ (E \circ (E^* \circ \bar{\varrho})\,(\varrho \circ \tau \circ \varrho) \circ (\varrho^* \circ \tau^*)) \circ \tau = \\ &= \varrho^* \circ \bar{\varrho}\varrho \circ \varrho^* \circ \tau^* \circ \tau = 0.\end{aligned}$$

В силу [3], с. 155, упр. 1, отсюда вытекает, что

$$\tau \leq \overline{\varrho^* \circ \bar{\varrho} \circ \varrho^*} = \tilde{\varrho}.$$

Учитывая [2], лемма 1 (2), получаем $\tau = \tau \circ \varrho \circ \tau \leq \tilde{\varrho} \circ \varrho \circ \tilde{\varrho}$.

Пусть M — непустое множество. Булева алгебра B называется *M-полной*, если она содержит точные верхние и нижние грани любых своих подмножеств, мощность которых не превосходит мощности множества M. Отображение ϱ: $M \times M \to B$ называется *B-отношением*. Произведение $\varrho \circ \sigma$ B-отношений ϱ и σ определяется равенством

$$\varrho \circ \sigma(a, b) = \sum_{x \in M} \varrho(a, x)\,\sigma(x, b)$$

для любых a, $b \in M$ (см. [1]). Если $B = 2$, то B-отношения — это обычные отношения с обычным произвелением. Совокупность всех таких отношений на M оказывается релятивом, если в качестве решеточных операций рассмотреть теоретико-множественные объедиение, пересечение и дополнение, для любых a, $b \in M$ определить $\varrho^*(a, b) = \varrho(b, a)$ и

$$E(a, b) = \begin{cases} 1, & \text{если } a = b, \\ 0, & \text{если } a \neq b, \end{cases}$$

и положить $0 = \varnothing$ и $I = M \times M$. Назовем B-отношение ϱ *рефлексивным*, если $E \leq \varrho$, и *антисиметричным*, если $\varrho\, \varrho^* \leqq E$.

Теорема 3. *Рефлексивное антисиметричное B-отношение ϱ регулярно тогда и только тогда, когда $\varrho = \varrho \circ \varrho$* (ср. [8], *а также* [5] и [7]).

Доказательство. Если $\varrho = \varrho \circ \varrho$, то $\varrho = \varrho \circ \varrho \circ \varrho$, т. е. ϱ регулярно. Если ϱ регулярно, то, в силу теоремы 1, $\varrho = \varrho \circ \tilde{\varrho} \circ \varrho$. Поскольку ϱ рефлексивно, то $\varrho(x, x) = 1$ для всех $x \in M$ и, следовательно,

$$\begin{aligned} \tilde{\varrho}(x, z) &= \overline{\sum_{s,t \in M} \varrho(s, x)\, \overline{\varrho(s, t)}\, \varrho(z, t)} = \\ &= \prod_{s,t \in M} (\overline{\varrho(s, x)} + \varrho(s, t) + \overline{\varrho(z, t)}) \leqq \\ &\leqq \overline{\varrho(x, x)} + \varrho(x, z) + \overline{\varrho(z, z)} = \varrho(x, z), \end{aligned}$$

для дюбых $x, z \in M$. Поскольку ϱ антисимметрично, то

$$\varrho(x, z)\, \varrho(z, x) = \varrho(x, z)\, \varrho^*(x, z) = 0,$$

если $x \neq z$. Следовательно, если $x \neq z$, то

$$\tilde{\varrho}(x, z)\, \varrho(z, x) \leq \varrho(x, z)\, \varrho(z, x) = 0.$$

Кроме того, при любых x, y и $з$ получаем

$$\begin{aligned} \tilde{\varrho}(y, z)\, \varrho(z, x) &= \varrho(y, y)\, \tilde{\varrho}(y, z)\, \varrho(з, x) \leqq \\ &\leqq \sum_{s,t \in M} \varrho(y, s)\, \tilde{\varrho}(s, t)\, \varrho(t, x) = \varrho\tilde{\varrho}\varrho(y, x) = \varrho(y, x), \end{aligned}$$

откуда при $x \neq y$ вытекает

$$\varrho(x, y)\, \tilde{\varrho}(y, z)\, \varrho(z, x) \leq \varrho(x, y)\, \varrho(y, x) = 0.$$

Следовательно, для любого $x \in M$ имеет место

$$\begin{aligned} 1 = \varrho(x, x) &= \sum_{y,z \in M} \varrho(x, y)\, \tilde{\varrho}(y, z)\, \varrho(z, x) = \\ &= \sum_{z \in M} \varrho(x, x)\, \tilde{\varrho}(x, z)\, \varrho(z, x) = \\ &= \varrho(x, x)\, \tilde{\varrho}(x, x)\, \varrho(x, x) = \tilde{\varrho}(x, x). \end{aligned}$$

Таким образом, $E \leq \tilde{\varrho}$, что, ввиду [2], лемма 1 (2), влечет

$$\varrho \circ \varrho = \varrho \circ E \circ \varrho \leq \varrho \circ \tilde{\varrho} \circ \varrho = \varrho = \varrho \circ E \leq \varrho \circ \varrho.$$

Определение рефлексивности и антисимметричности B-отношения дословно переносится на элемент произвольного релятива. При этом аналог теоремы 3 тривиальным образом верен для любого релятива, где $x = x^*$ для любого элемента x, ибо из ее посылок вытекает, что $E \leq x = xx^* \leq E$. То же самое можно сказать и о релятиве

$$(2^G \mid \cup, \cap, |, \setminus, \varnothing, G, ., {}^{-1}, \{e\}),$$

где G — группа с единицей e (см. [5], [6]). В самом деле, допустим, что $A, B \in 2^G$, $e \in A$, $A \cap A^{-1} = \{e\}$ и $ABA = A$. Тогда $e = a_1 b a_2$, где a_1, $a_2 \in A$ и $b \in B$. Отсюда

$$a_1^{-1} = ba_2 = eba_2 \in ABA = A$$

и, следовательно, $a_1 = e$. Аналогично получаем, что $a_2 = e$. Таким образом, $e = b \in B$, откуда

$$a'a'' = a' e a'' \in ABA = A.$$

для любых $a', a'' \in A$. Таким образом,

$$AA \subseteq A \subseteq A\{e\} \subseteq AA,$$

т. е. $AA = A$. Однако, вопрос о справедливости аналога теоремы 3 для произвольного релятива остается открытым.

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MULTIPLICATIVE STRUCTURES OVER SUP-LATTICES

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Abstract. Modules over a not necessarily commutative multiplicative sup-lattice A are described as the Eilenberg–Moore algebras of a fairly elementary monad (T, η, μ) over **Set** with $TX = A^X$ which was considered before for commutative A, in particular when A is a frame. These modules are shown to carry a generalized metric structure, inducing another monadic functor.

Key words: sup-lattice, multiplicative sup-lattice, frame, locale, quantale, module, monadic functor.

MS Classification: 06 D 99; 06 A 23, 18 C 15, 18 C 20

INTRODUCTION

For a *frame* A (= complete lattice with $x \wedge \bigvee y_i = \bigvee x \wedge y_i$) Machner [4] gave a rather technical description of the algebras of the following monad $\tau_A = (T, \eta, \mu)$ on **Set**:

$$TX = A^X, (Tf)(\varphi)(y) = \bigvee \{\varphi(x) \mid x \in f^{-1}y\} \; (f: X \to Y, \varphi \in A^X, y \in Y),$$

$$\eta_X : X \to A^X \quad \text{with} \quad \eta_X(x)(x') = \delta_{xx'} \quad \text{(Kronecker's delta)},$$

$$\mu_X : A^{A^X} \to A^X \quad \text{with} \quad \mu_X(\Phi)(x) = \bigvee \{\Phi(\varphi) \wedge \varphi(x) \mid \varphi \in A^X\} \; (\Phi \in A^{A^X}, x \in X).$$

However, from Joyal's and Tierney's work [3] one now has a nice characterization of these algebras: interpreting A as a commutative monoid (with $\wedge$ as multiplication) over the *sup-lattice* (= complete lattice in which one considers $\bigvee$ the only structural element) A, Eilenberg–Moore algebras with respect to τ_A are nothing but modules over the monoid A, i.e. sup-lattices M which come equipped with an associative and unary action $A \otimes M \to M$ of sup-lattices.

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In this short note we present this observation in the non-commutative case. More precisely, we show that the above monad exists for every sup-lattice A which comes equipped with an associative, but not necessarily commutative multiplication and a one-sided unit (so in particular for every quantale in the sense of [1], and that the algebras are the same as in the localic case described above. We also observe that they carry a generalized metric structure which we discuss in terms of adjoint functors.

1. SUP-LATTICES

The category **SupLat** has as its objects partially ordered sets X which admit arbitrary suprema (in particular, one has $0 = \bigvee\emptyset$ and $1 = \bigvee X$), and as its morphisms $f: X \to Y$ mappings which preserve suprema. Every such morphism has a right adjoint $f_* : Y \to X$, given by the formula

$$\frac{f(x) \leq y}{x \leq f_*(y)},$$

(or $f_*(y) = \bigvee\{x \mid f(x) \leq y\}$); f_* preserves all infima, so it can be interpreted as a morphism $f^0 : Y^0 \to X^0$ in **SupLat** with X^0 the sup-lattice provided with the partial order opposite to that one of X. (Recall that the existence of arbitrary suprema implies the existence of arbitrary infima.) Obviously,

$$(-)^0 : \mathbf{SupLat} \to \mathbf{SupLat}$$

is a contravariant isomorphism of categories, yielding a strong self-duality of the category **SupLat**.

A *bimorphism* $f: X \times Y \to Z$ of sup-lattices satisfies the laws

$$f(\bigvee x_i, y) = \bigvee f(x_i, y), \qquad f(x, \bigvee y_i) = \bigvee f(x, y_i).$$

The *tensor product* of two sup-lattices X, Y is given by a universal bimorphism

$$X \times Y \to X \otimes Y, \qquad (x, y) \mapsto x \otimes y,$$

so that $\mathrm{Bihom}(X \times Y, Z) \cong \mathrm{Hom}(X \otimes Y, Z)$. Therefore, bimorphisms can be always written as **SupLat**-morphisms on the tensor product.

2. MODULES OVER MULTIPLICATIVE SUP-LATTICES

A sup-lattice A is called *multiplicative* when it comes equipped with a nullary operation $\varepsilon : 1 \to A$ (i.e. an element $\varepsilon \in A$) and a binary operation

$$A \otimes A \to A, \qquad \alpha \otimes \beta \mapsto \alpha\beta,$$

in **SupLat**. A *left A-module* M is a sup-lattice together with an action

$$A \otimes M \to M, \qquad \alpha \otimes x \mapsto \alpha x,$$

in **SupLat** such that

$$(\alpha\beta)\, x = \alpha(\beta x) \qquad \text{and} \qquad \varepsilon x = x \;\; (\alpha, \beta \in A, x \in M)$$

hold. The morphisms of the category A-**Mod** of left A-modules are morphisms $f: M \to N$ in **SupLat** such that $f(\alpha x) = \alpha f(x)$. A *right A-module M* is a left A^*-module where A^* has the multiplicative structure given by ε and $\alpha * \beta = \beta\alpha$. We write **Mod**-$A$ for A^*-**Mod**.

If A with its multiplicative structure is itself a left (right resp.) A-module, then A is called a *left* (*right* resp.) *monoid* over **SupLat**; it is a *monoid* if it is both a left and right A-module.

Every frame (= *locale*) is a monoid when putting $\alpha\beta = \alpha \wedge \beta$ and $\varepsilon = 1$; in fact, frames are those monoids over **SupLat** with $\varepsilon = 1$ and $\alpha^2 = \alpha$. (The Joyal – Tierney [3] proof survives dropping commutativity.) Prime examples of locales are the lattices of open sets of a topological space.

More generally, *quantales* in the sense of Borceux and van den Bossche [1] are, by definition, right monoids over **SupLat** with $\varepsilon = 1$ and $\alpha^2 = \alpha$. Those were introduced to describe, inter alia, the lattice of closed right ideals in a C^*-algebra.

For a multiplicative A, a left A-module M, and every $\alpha \in M$, the **SupLat**-morphism $\alpha(-) : M \to M$ has a right adjoint, denoted by $(-)^\alpha$, so

$$\frac{\alpha x \leq y}{x \leq y^\alpha}.$$

One has a **SupLat**-morphism

$$M^0 \otimes A \to M^0, \qquad y \otimes \alpha \mapsto y^\alpha,$$

which provides M^0 with a right A-module structure:

$$\frac{\dfrac{x \leq y^\varepsilon}{\varepsilon x \leq y}}{x \leq y} \qquad \frac{\dfrac{\dfrac{\dfrac{x \leq y^{\alpha\beta}}{(\alpha\beta)\, x \leq y}}{\alpha(\beta x) \leq y}}{\beta x \leq y^\alpha}}{x \leq (y^\alpha)^\beta}$$

This way one obtains a strong duality

$$(-)^0 : A\text{-}\mathbf{Mod} \to \mathbf{Mod}\text{-}A.$$

For A *commutative* this gives a strong self-duality of A-**Mod** (which is the self-duality of **SupLat** mentioned before when taking A to be the 2-element chain).

3. MONADICITY OF LEFT A-MODULES

Theorem 1. *For a left monoid A over* **SupLat**, *A-**Mod** is monadic over* **Set**.

Proof: For every set X, $A^X = \mathbf{Set}(X, A)$ carries the structure of a left A-module, with $(\alpha\varphi)(x) = \alpha\varphi(x)$ $(\alpha \in A,\ \varphi \in A^X,\ x \in X)$, which is simply a direct product of X copies of the left A-module A. It is indeed the free left A-module over X, since every **Set**-map $f: X \to M$ into a left A-module M factors through

$$\eta_X : X \to A^X, \qquad \text{with} \qquad \eta_X(x)(x) = \varepsilon \qquad \text{and} \qquad \eta_X(x)(x') = 0 \qquad \text{for} \qquad x \neq x',$$

by a unique morphism in **SupLat**, namely

$$g : A^X \to M \qquad \text{with} \qquad g(\varphi) = \bigvee\{\varphi(x)\, f(x) \mid x \in X\}$$

for all $\varphi \in A^X$.

It is elementary to show that the forgetful A-**Mod** $\to$ **Set** creates coequalizers of absolute pairs, so it is monadic (cf. [5]). But it is not difficult either to see directly how τ_A-algebras (M, m) correspond to left A-modules M (here τ_A is the monad induced by A-**Mod** $\to$ **Set** which may be described as in the Introduction, replacing $\wedge$ by the multiplication of A): for a left A-module M, the Eilenberg–Moore structure m is a morphism $A^M \to M$ in A-**Mod** with $m\eta_M = 1_M$, so

$$m(\varphi) = \bigvee\{\varphi(x)\, x \mid x \in M\};$$

on the other hand, given an Eilenberg–Moore structure m on a set M, A acts on M by

$$\alpha x = m(\alpha\eta_M(x)). \qquad \square$$

Analogously one can show that **Mod**-A is monadic over **Set** when A is a right monoid. So one has:

Corollary 1. *For a commutative monoid A over* **SupLat**, *both A-**Mod** and $(A$-**Mod**$)^{op}$ are monadic over* **Set**. □

4. THE INDUCED HEYTING STRUCTURE

For a left monoid A and a left A-module M and every $x \in M$, the **SupLat**-Morphism $(-)\,x$: $A \to M$ has a right adjoint, denoted by $x \to (-)$, so

$$\frac{\alpha x \leq y}{\alpha \leq x \to y}.$$

One has a **SupLat**-morphism

$$M \otimes M^0 \to A^0, \qquad x \otimes y \mapsto (x \to y),$$

satisfying the following laws for all $x, y \in M$:

Proposition 1.

(1) $$x \leq y \Leftrightarrow \varepsilon \leq x \to y,$$

(2) $$\bigvee_{z \in X} (z \to y)(x \to z) = x \to y.$$

Proof: (1) is trivial, and it implies

$$x \to y = \varepsilon(x \to y) \leq (y \to y)(x \to y) \leq \text{l.h.s. of (2)}.$$

For the other inequality needed in (2), first observe that trivially

$$(x \to z)\, x \leq z \tag{*}$$

for all $x, z \in M$; therefore,

$$((z \to y)(x \to z))\, x = (z \to y)((x \to z)\, x) \leq (z \to y)\, z \leq y,$$

hence $(z \to y)(x \to z) \leq x \to y$ for all $x, y, z \in M$. □

Passing to the induced Heyting structure causes no problems when forming direct products:

Proposition 2. *For families $(x_i)_i$, $(y_i)_i$ in the direct product $\prod_i M_i$ in A-**Mod** one has*

$$(x_i)_i \to (y_i)_i = \bigwedge_i (x_i \to y_i).$$

Proof: Since the partial order in $\prod_i M_i$ is componentwise, we have

$$\frac{\dfrac{\dfrac{\dfrac{\alpha \leq (x_i) \to (y_i)}{\alpha(x_i) \leq (y_i)}}{\forall\, i : \alpha x_i \leq y_i}}{\forall\, i : \alpha \leq x_i \to y_i}}{\alpha \leq \bigwedge_i (x_i \to y_i)}$$ □

However, morphisms require more detailed considerations:

Proposition 3. *For left A-modules M, N and a* **Set**-*map $f : M \to N$ one has·*

(1) *$x \to y \leq f(x) \to f(y)$ $(x, y \in M)$ holds if and only if f is monotone* (i.e. $x \leq y \Rightarrow f(x) \leq f(y)$) *and satisfies $\alpha f(x) \leq f(\alpha x)$ $(\alpha \in A,\ x \in M)$.*

(2) *For f monotone and onto, $f(x) \to f(y) \leq x \to y$ $(x, y \in M)$ implies $f(\alpha x) \leq \alpha f(x)$ $(\alpha \in A,\ x \in M)$.*

(3) *$f(\alpha x) \leq \alpha f(x)$ $(\alpha \in A,\ x \in M)$ implies $f(x) \to f(y) \leq x \to y$ $(x, y \in M)$ if and only if f reflects the order* (i.e. $f(x) \leq f(y) \Rightarrow x \leq y$).

Proof: (1) "⇒" f is monotone by Prop. 1 (1). From $\alpha \leq x \to \alpha x \leq f(x) \to f(\alpha x)$ one obtains $\alpha f(x) \leq f(\alpha x)$. "⇐" In $\alpha \leq f(x) \to f(\alpha x)$ we may substitute $\alpha = x \to y$ to obtain with (*)

$$x \to y \leq f(x) \to f((x \to y)\, x) \leq f(x) \to f(y)$$

since f is monotone.

(2) We may write, for $\alpha \in A$ and $x \in M$ given, $\alpha f(x) = f(y)$ and have $\alpha \leq f(x) \to f(y) \leq x \to y$, hence $\alpha x \leq y$, so $f(\alpha x) \leq f(y) = \alpha f(x)$.

(3) "$\Rightarrow$" Reflection of the order follows from Prop. 1 (1) again. "$\Leftarrow$" With $\alpha = f(x) \to f(y)$ one obtains from (*)

$$f((f(x) \to f(y))\, x) \leq (f(x) \to f(y))\, f(x) \leq f(y),$$

hence $(f(x) \to f(y))\, x \leq y$, so $f(x) \to f(y) \leq x \to y$. □

5. THE METRIC POINT OF VIEW

If, for a left monoid A over **SupLat** with $\varepsilon = 1$ and for a left A-module M, we write

$$d(x, y) = x \to y, \quad \alpha + \beta = \beta\alpha, \quad \alpha \prec \beta \Leftrightarrow \beta \leq \alpha, \quad \Theta = \varepsilon;$$

then Prop. 1 gives

(1) $$d(x, y) = \Theta = d(y, x) \Leftrightarrow x = y,$$

(2) $$d(x, y) \prec d(x, z) + d(z, y)$$

for all $x, y, z \in M$.

For a partially ordered (**Set**-based) semigroup $(S, +, \prec)$ (so $(S, +)$ is a not necessarily commutative semigroup and $(S, \prec)$ is a poset with the binary $+$ monotone in each variable) such that there is a bottom element Θ with $\Theta + \Theta = \Theta$, we consider the category

$$S\text{-}\mathbf{Met}$$

whose objects are pairs (M, d) with a set M and a function $d : M \times M \to S$ that satisfies (1) and (2), and whose morphisms $f : (M, d) \to (M', d')$ are non-expanding maps, i.e.

$$d'(f(x), f(y)) \prec d(x, y).$$

Putting $(x \leq y \Leftrightarrow d(x, y) = \Theta)$ defines a functor S-**Met** → **PoSet** (the category of partially ordered sets and monotone maps).

If we denote by A^+ the partially ordered semigroup as described above (so A^+ is, as a semigroup, A^* and, as a poset, A^0) then Propositions 2 and 3 give immediately:

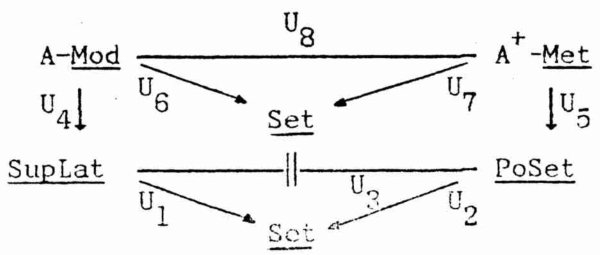
Corollary 2. *There is a faithful functor A-**Mod** → A^+-**Met** that preserves products and reflects isomorphisms.*

Next we shall point out that the functor is actually monadic.

6. SUMMARY IN TERMS OF ADJOINTS

For a left monoid A over **SupLat** with $\varepsilon = 1 \neq 0$ one has:

Theorem 2. *In the diagram*



of forgetful functors, each one has a left adjoint; U_1, U_3, U_4, U_6, U_8 *are monadic whereas* U_2, U_5 *and* U_7 *induce trivial monads.*

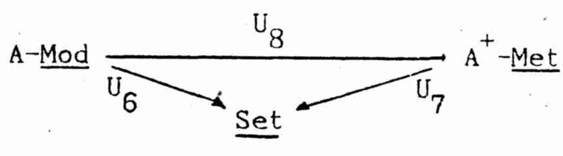
Proof: Denoting the left adjoint of U_i by F_i, one has F_1X the power set PX of the set X, $F_2X = X$ with the discrete order, and F_3X the system of down-sets in the poset X (cf. [2]). F_4 is tensoring with A, so F_4F_1 gives an alternative way of constructing the left adjoint F_6 as in Theorem 1, i.e.

$$A \otimes PX \cong A^X.$$

For a poset X, the metric structure of $F_5X = X$ is given by

$$d(x, y) = \begin{cases} \varepsilon & \text{if } x \leqq y \\ 0 & \text{otherwise} \end{cases}$$

(recall that 0 is the bottom element in A, i.e. the top element in A^+). Since $U_7 = U_2U_5$ trivially has a left adjoint, we just need to show existence of F_8: this can be derived from Corollary 2 above and Theorem 3 of [6], applied to the triangle



(we do not have an explicit construction of F_8).

Monadicity of U_1, U_3, U_4, U_6, U_8 is easily checked with the Beck–Paré criterion (cf. [5]); U_2, U_5 and U_7 obviously induce identical monads (to have $U_5F_5 = \mathrm{Id}$, one needs $1 \neq 0$ in A). □

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REPRESENTABILITY OF CONCRETE CATEGORIES BY NON-CONSTANT MORPHISMS

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Dedicated to the memory of Milan Sekanina

Abstract. We prove that the category of all compact Hausdorff spaces (or all metrizable spaces) admits a representation of every concrete category iff there does not exist a proper class of measurable cardinals.

Key words: almost universal category, compact Hausdorff space, metrizable space.

MS Classification. 18 B 15, 18 B 30, 54 C 05

In 1974, V. Koubek [4] proved that the category **Par** of paracompact Hausdorff spaces (and continuous maps) is *almost universal.* It means that any concrete category $\mathscr{K}$ has an embedding F (= one-to-one functor) into **Par** such that $g : FA \to FB$ is of the form $F(f)$ iff g is non-constant. Such embeddings F are called *almost full.* Due to constant maps, this is the strongest universality which topological spaces may offer. The second author proved that the categories **Metr** of metrizable spaces ([7]) and **Comp** of compact Hausdorff spaces ([8]) are almost universal (in both cases, morphisms are continuous maps) provided that the following statement is true

(M) It does not exist a proper class of measurable cardinals.

It remained open whether one really needs (M) for these results. We show that the answer is yes (for **Comp**, it solves Research Problem 12 in [6]).

Str(Δ) denotes the concrete category of structures of type Δ (= a set of possibly infinitary relation and operation symbols) and homomorphisms (maps preserving relations and operations). A full embedding of concrete categories is called a *realization* if it commutes with underlying set functors ([6]). A category $\mathscr{A}$ is called *universal* if any category can be fully embedded into $\mathscr{A}$. A basic (and deep) result is that the category **Graph** (= **Str**(Δ) where Δ consists of one binary relation) is universal iff (M) is fulfilled (see [6]). The mentioned results of [7] and [8] are proved by constructing almost full embeddings **Graph** – **Metr** and **Graph**^{op} – **Comp**.

Proposition 1. *The existence of an almost universal concrete category admitting a realization into* **Str**(Δ) *implies the universality of* **Graph**.

Proof: Let $(\mathscr{K}, U)$ be an almost universal concrete category and $F : \mathscr{K} \to \to \mathbf{Str}(\Delta)$ a realization. We will show that any concrete category $\mathscr{H}$ can be fully embedded into **Graph**.

Let $\mathscr{H}^+$ be the category obtained from $\mathscr{H}$ by adding an initial object I and a terminal object T; i.e. $obj\mathscr{H}^+ = obj\mathscr{H} \cup \{I, T\}$, $I \neq T$ and $obj\mathscr{H} \cap \{I, T\} = \emptyset$, $\mathscr{H}$ is a full subcategory of $\mathscr{H}^+$, $\mathscr{H}^+(I, H)$ and $\mathscr{H}^+(H, T)$ are one-element for any $H \in obj\mathscr{H}^+$, $\mathscr{H}^+(H, I) = \mathscr{H}^+(T, H) = \emptyset$ for any $H \in obj\mathscr{H}$. The underlying set functor of $\mathscr{H}$ can be easily extended to $\mathscr{H}^+$. Hence $\mathscr{H}^+$ is concrete and there is an almost full embedding $G : \mathscr{H}^+ \to \mathscr{K}$. Since F is a realization, the composition $E = F . G : \mathscr{H}^+ \to \mathbf{Str}(\Delta)$ is an almost full embedding. Therefore $E(m_T) : : E(I) \to E(T)$ is non-constant (m_H is a unique morphism $I \to H$) and we can find $x, y \in E(I)$ such that their images in $E(m_T)$ are distinct. Then $x_H = E(m_H)(x)$, $y_H = E(m_H)(y)$ are distinct for any $H \in obj\,\mathscr{H}$ and $E(f)(x_H) = x_{\bar{H}}$, $E(f)(y_H) = y_{\bar{H}}$ for any $f : H \to \bar{H}$. Consequently, $g : E(H) \to E(\bar{H})$ is non-constant iff $g(x_H) = x_{\bar{H}}$ and $g(y_H) = y_{\bar{H}}$. Hence E gives a full embedding of $\mathscr{H}$ into $\mathbf{Str}(\Delta')$ where Δ' is obtained from Δ by adding two new constants interpreted as x_H and y_H. But $\mathbf{Str}(\Delta')$ has a full embedding into **Graph** (see [6]).

Theorem 1. **Metr** *is almost universal iff* (M) *holds.*

Proof: As already mentioned in the introduction, (M) implies the almost universality of **Metr**. For the converse, we realize **Metr** into structures with one ω-are relation; $(x_0, x_2, \ldots, x_n, \ldots)$ belongs to the relation iff the sequence $x_1, \ldots, x_n, \ldots$ converges to x_0. Proposition 1, **Graph** is universal. As stated in the introduction, it implies (M) (see [5]).

Remark 1: The same result is true for metrizable spaces with morphisms taken as

(a) uniformly continuous maps,

(b) non-expanding maps.

In case (a), we represent metrizable spaces by structures with an ω-ary relation again; but $(x_0, x_1, \ldots, x_n, \ldots)$ belongs to the relation iff $\lim\limits_{n \to \infty} d(x_{2n}, x_{2n+1}) = 0$ where d is the distance. In case (b) we use ω binary relations R_n, $n > 0$ an integer; xR_ny iff $d(x, y) < 1/n$. The opposite implications are proved in [7].

Proposition 1 is not applicable to **Comp** because **Comp** cannot be fully embedded into **Str**(Δ) without (M) (see [5]).

Proposition 2. *Let there exist an almost universal concrete category* $\mathscr{K}$ *admitting a full embedding* $F : \mathscr{K}^{op} \to \mathbf{Str}(\Delta)$ *with the property:*

For every $K \in obj\mathscr{K}$ *there is a subset* Y_K *of (the underlying set of)* $F(K)$ *such that for any* $f : \bar{K} \to K$ *in* $\mathscr{K}$

(i) *$F(f)$ maps Y_K into $Y_{\bar{K}}$.*

(ii) *$F(f)$ maps the whole $F(K)$ into $Y_{\bar{K}}$ iff f is constant.*

Then **Graph** *is universal.*

Proof: We will follow the proof of Proposition 1. Let $\mathscr{L}$ be a concrete category. Since $\mathscr{H} = \mathscr{L}^{op}$ is concrete ([6], p. 33), we may take $\mathscr{H}^+$, an almost full embedding $G : \mathscr{H}^+ \to \mathscr{K}$ and the composition $E = F . G : (\mathscr{H}^+)^{op} \to \mathbf{Str}(\Delta)$. Then E is an embedding and $E(h)$ maps $Y_{G(H)}$ into $Y_{G(\bar{H})}$ for any $h : \bar{H} \to H$ in $\mathscr{H}^+$. Moreover $g : E(H) \to E(\bar{H})$ does not map the whole $E(H)$ into $Y_{G(\bar{H})}$ iff $g = E(h)$ for rome $h : \bar{H} \to H$.

Choose $x \in E(T)$ such that $y = E(m_T)(x) \notin Y_{G(I)}$. Then $x_H = E(n_H)(x)$ (n_H is a unique morphism $H \to T$) does not belong to $Y_{G(H)}$. We have $E(h)(x_H) = x_{\bar{H}}$ for any $h : \bar{H} \to H$. Hence E gives a full embedding of $\mathscr{L} = \mathscr{H}^{op}$ into $\mathbf{Str}(\Delta')$ where Δ' is obtained from Δ adding a new constant interpreted as x_H.

Theorem 2. Comp *is almost universal iff* (M) *holds.*

Proof: As already mentioned in the introduction, (M) implies the almost universality of **Comp**. Let F : $\mathbf{Comp}^{op} \to \mathbf{Ring}$ send a compact Hausdorff space X to its ring of continuous real-valued functions (**Ring** is the category of rings with unit and with unit preserving homomorphisms). It is well known that F is a full embedding (cf. e.g. [3], p. 152). Taking for Y_X the set of all constant real-valued functions on X, it is easy to check that (i) and (ii) of Proposition 2 are fulfilled. Hence the almost universality of **Comp** implies (M).

Remark 2. The almost universality of **Comp** implies the existence of a *stiff* proper class $\mathscr{S}$ of compact Hausdorff spaces (i.e. if $S, \bar{S} \in \mathscr{S}$ and $f : S \to \bar{S}$ is a morphism then either f is constant or $S = \bar{S}$ and f is the identity). One does not need the full force of (M) for it, the existence of a *rigid* proper class $\mathscr{R}$ of graphs is sufficient (cf. [8]), $\mathscr{R}$ is rigid if $X, Y \in \mathscr{R}$ and $f : X \to Y$ is a morphism then $X = Y$ and f is the identity).

Our method yields that, conversely, the existence of a stiff proper class of compact Hausdorff spaces implies the existence of a rigid propei class of graphs (not to enlarge $\mathscr{S}$ to $\mathscr{S}^+$ but kill constant maps by choosing $x_S \notin Y_S$, $S \in \mathscr{S}$).

For metrizable spaces, the following statements are equivalent:

(a) **Metr** contains a stiff proper class of objects,

(b) The category of metrizable spaces and uniformly continuous maps contains a stiff proper class of objects.

(c) The category of metrizable spaces and non-expanding maps contains a stiff proper class of objects.

(d) **Graph** contains a rigid proper class of objects.

Remark 3. The existence of a rigid proper class $\mathscr{R}$ of graphs is really weaker than (M). Indeed, it is easy to show (cf. [1]) that the existence of $\mathscr{R}$ is exactly the negation of the *Vopěnka's Principle* which is well known in set theory (see [2], *VP* [= Vopěnka's Principle] says that, for each first-order language, every class of models such that none of them has an elementary embedding into another is a set). Hence

$$VP \Rightarrow non(M).$$

It is known in set theory that *VP* is stronger than $non(M)$ (even, it cannot be shown that *VP* is consistent with $ZFC + non(M)$). It follows by Gödel's second incompleteness theorem and by the fact that *VP* yields a model of $ZFC + non(M)$. Indeed, *VP* implies the existence of a supercompact cardinal $\varkappa$ ([2], 33.15, 33.14 (a)) and the set V_K of all sets of rank less than $\varkappa$ is a model of $ZFC + non(M)$ (by [2], the Corollary to 33.10).

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BASES RATIONNELLES DES SYSTÉMES DE QUADRIQUES

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Abstract. On démontre que la variété base d'un système linéaire de quadriques de S_r à Jacobienne de rang $r - k$ est: un S_k double, ou une variété réductible, qui possède deux sub-variétés rationnelles, ou une variété rationnelle irréductible. Après on donne des exemples significatifs.

Key words. Quadric, linear system of quadrics.

MS Classification. 51 A 05

1°

Dans l'espace linéaire S_r de coordonnées projectives homogènes x_i ($i = 0, 1, \dots, r$), choisissons $d + 1$ quadriques linéairement indépendantes:

$$f_0 = 0, f_1 = 0, \dots, f_d = 0$$

avec:

$$f_q = \sum_{i,k=0}^{r} a_q^{ik} x_i x_k \qquad (a_q^{ik} = a_q^{ki}).$$

Le système linéaire L_d de dimension d, qui en résulte, est exprimé par l'équation:

$$\sum_{q=0}^{d} \varrho_q f_q = 0.$$

Supposons que la matrice Jacobienne à $r + 1$ lignes et $d + 1$ colonnes:

$$J = \left\| \frac{\partial f_q}{\partial x_i} \right\| \qquad \begin{pmatrix} q = 0, 1, \dots, d \\ i = 0, 1, \dots, r \end{pmatrix}$$

soit à rang $m = r - k$.

La matrice Jacobienne égalisée à zéro est le lieu géometrique des points de S_r conjugués entre eux-mêmes par rapport à toutes les quadriques du système. Si la matrice Jacobienne est identiquement nulle, cela signifie que tout l'espace est le lieu des points conjugués. Si le rang est $r - h$ ($h \geqq 0$) un point générique de S_r est conjugué avec un S_h.

Donnés deux systèmes linéaires L_a et L_b, qui ont en commun un système linéaire L_c, leur système-union résulte de dimension $a + b - c$.

Nous dirons que le système $L_{d/m}(m \leq d)$ de dimension d et à Jacobienne de rang m est *réductible*, lorsqu'il est l'union de systèmes subordonnés, parmi les quels au moins un $L_{g/c}(c \leq g)$ n'a pas des quadriques en commun avec les autres; autrement dît, il sera nommé: *irréductible*.

Il existe le:

Théoreme. *Un système linéaire irréductible de quadriques* $L_{d/r-k}(r - k \leq d, k \geqq 0)$ *a pour variété base seulement une des variétés suivantes:*

I. *Un* S_k *double et, dans ce cas, les quadriques sont des* S_k*-cônes avec* S_k*-sommet en commun:*

II. *Une variété réductible, qui possède deux sub-variétés rationnelles de dimensions* h *et* $r - h + k - 1 (1 \leq h \leq r - 2)$:

III. *Une variété rationnelle irréductible de dimension:* $\dfrac{r + k - 1}{2}$.

Démonstration. Considérons avant tout le cas particulier: $k = 0$. Soit le système linéaire irréductible de quadriques $L_{d/r}(r \leq d)$ de S_r.

D'après un théorème connu (voir: [2] et [5]) les quadriques de L_d, qui passent par un point générique P de S_r, ont en commun une droite, qui résulte corde de la variété base V du système.

Soit R le complexe des droites constitué par toutes les cordes de V, qui à cause du théorème cité, remplissent tout S_r.

Entrecoupons ce complexe par un hyperplan S_{r-1}. Chaque droite du complexe sera entrecoupée dans un point. Il est ainsi possible établir une correspondance biunivoque entre les points de l'hyperplan et les droites du complexe R.

Pour cela le complexe R est rationnel.

Supposons que les quadriques aient en commun un point double A. Il en résulte que toutes les droites, qui sortent de A, sont cordes de la variété constituée par le point A. Elles remplissent tout S_r et il est évident qu'elles forment un complexe rationnel.

Les quadriques ne peuvent pas avoir un autre point double B en commun, sinon par un point générique P il passerait plus qu'une corde: la droite PA et la droite PB.

Par conséquent le rang de la Jacobienne serait $< r$, contre l'hypothése.

Il s'ensuit que les quadriques sont des cônes avec S_0-sommet en commun.

'Hors de ce cas, puisque R est rationnel, il en résulte que les coordonnées de droite de ses droites, les p_{ik}, c'est à dire les mineurs extraits de la matrice:

$$\begin{Vmatrix} x_0 & x_1 & x_2 & \cdots & x_r \\ y_0 & y_1 & y_2 & \cdots & y_r \end{Vmatrix}$$

sont fonctions rationnelles de $r - 1$ paramètres indépendantes.

Nous pouvons indiquer avec $M(x_0, x_1, \ldots, x_r)$ et $N(y_0, y_1, \ldots, y_r)$ deux points quelconques de la variété base et avec MN la corde qui les joint.

Si la variété base est réductible, il aura dans elle deux variétés subordonnées W et Z de dimensions respectives h et $r - h - 1$ $(1 \leq h \leq r - 2)$. On rejette le cas $h = 0$ et $h = r - 1$ parce que dans ce cas les quadriques contiennent au moins un hyperplan. Donc elles résultent couples d'hyperplans ayant un hyperplan en commun.

Les autre hyperplans de la couple devraient possèder un S_0: il en résulte que leur nombre est ∞^{r-1}. Pour cela le système de quadriques aurait dimension $d = r - 1$, contre l'hypothèse qu'il soit $r \leq d$.

Les variétés W et Z ont les dimensions citées parce que, en projetant W par un point générique P on obtient une variété T de dimension $h + 1$, qui entrecoupe Z seulement dans un point Q externe à la variété intersection de W avec Z.

En effet, en résultant la droite PQ corde de V, seulement dans ce cas P résulte conjugué avec un seul point par rapport à toutes les quadriques du système.

Notons:

$x_0 = 1, x_1 = \Phi_1(\sigma_1, \sigma_2, \ldots, \sigma_h), x_2 = \Phi_2(\sigma_1, \sigma_2, \ldots, \sigma_h), \ldots, x_r = \Phi_r(\sigma_1, \sigma_2, \ldots, \sigma_h)$ les équations paramétriques de W.

On peut indiquer celles de Z avec:

$$y_0 = 1, y_1 = \Psi_1(\tau_1, \tau_2, \ldots, \tau_{r-h-1}), y_2 = \Psi_2(\tau_1, \tau_2, \ldots, \tau_{r-h-1}), \ldots, y_r = \\ = \Psi_r(\tau_1, \tau_2, \ldots, \tau_{r-h-1}).$$

Les premiers rp_{ik}, extraits de la matrice (1) résultent:

$$\begin{aligned} p_{01} &= \Psi_1 - \Phi_1 \\ p_{02} &= \Psi_2 - \Phi_2 \\ &\ldots\ldots\ldots\ldots \\ p_{0r} &= \Psi_r - \Phi_r. \end{aligned}$$

Il s'ensuit que les différences:

$$\Psi_i - \Phi_i \qquad (i = 1, 2, \ldots, r),$$

sont rationnelless par rapport aux paramètres σ_m, τ_n. Pour cela l'éventuelle partie irrationnelle de Ψ_i et Φ_i doit s'éclipser par différence.

On en déduit que:

$$\begin{aligned} \Phi_i &= C_i + E_i \qquad (i = 1, 2, \ldots, r), \\ \Psi_i &= D_i + E_i \end{aligned}$$

où C_i et D_i sont fonctions rationnelles et E_i est l'éventuelle partie irrationnelle de Φ_i et Ψ_i.

Fixons un point M sur W: ce comporte: $C_i = C$ et $E_i = E$ (C et E = constants).

En variant le point N sur Z, il s'ensuit que, pour que la difference $\Psi_i - \Phi_i$ soit toujours rationnelle, il faut que E_i résulte toujours égal à E, c'est à dire: $E_i =$

= constant. Analoguement, si nous tenons fixé Ψ_t et nous faisons varier Φ_t sur W, on obtient le même résultat.

Mais si E_t est constant dans les deux variétés, il n'est plus irrationnel. Pour cela Φ_t e Ψ_t sont rationnelles, comme il fallait démontrer.

Supposons maintenant que V soit irréductible. Si h est sa dimension, puisque ses cordes sont ∞^{2h} chaque corde possède ∞^1 points.

On obtient:

$$2h - 1 = r$$

c'est à dire:

$$h = \frac{r-1}{2},$$

il s'ensuit que r est nécessairement impair.

En répétant le raisonnement, que nous avons fait précédemment, choisissons deux points $M(x_0, x_1, \dots, x_r)$ et $N(y_0, y_1, \dots, y_r)$ sur cette variété.

Écrivons les équations de V en forme paramétrique:

$$x_0 = 1, x_1 = \Theta_1(\sigma_1, \sigma_2, \dots, \sigma_{\frac{r-1}{2}}), x_2 = \Theta_2(\sigma_1, \sigma_2, \dots, \sigma_{\frac{r-1}{2}}), \dots, x_r =$$
$$= \Theta_r(\sigma_1, \sigma_2, \dots, \sigma_{\frac{r-1}{2}}), y_0 = 1, y_1 = \Theta_1(\tau_1, \tau_2, \dots, \tau_{\frac{r-1}{2}}), y_2 = \Theta_2(\tau_1, \tau_2, \dots,$$
$$\dots, \tau_{\frac{r-1}{2}}), \dots, y_r = \Theta_r(\tau_1, \tau_2, \dots, \tau_{\frac{r-1}{2}}.$$

Les premiers p_{ik} résultent:

$$p_{01} = \Theta_1(\tau_1, \tau_2, \dots, \tau_{\frac{r-1}{2}}) - \Theta_1(\sigma_1, \sigma_2, \dots, \sigma_{\frac{r-1}{2}})$$
$$p_{02} = \Theta_2(\tau_1, \tau_2, \dots, \tau_{\frac{r-1}{2}}) - \Theta_2(\sigma_1, \sigma_2, \dots, \sigma_{\frac{r-1}{2}})$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$p_{0r} = \Theta_r(\tau_1, \tau_2, \dots, \tau_{\frac{r-1}{2}}) - \Theta_r(\sigma_1, \sigma_2, \dots, \sigma_{\frac{r-1}{2}})$$

En répétant les considérations précédentes, nous pouvons fixer le point M et par conséquent les $\Theta_i(\sigma_1, \sigma_2, \dots, \sigma_{\frac{r-1}{2}})$ et faire varier le point N sur toute la variété.

Par le raisonnement précédent, puisque les p_{0k} $(k = 1, 2, \dots, r)$ résultent rationnelles, les Θ_i aussi seront rationnelles, comme il fallait démontrer.

Soit maintenant le système $L_{d/r-1}(r - 1 \leq d)$ irréductible. Entrecoupons ce système par un hyperplan. D'après un théorème bien connu de Terracini (voir: [7]), on obtient un système de quadriques irréductible de S_{r-1} ayant la même dimension et le même rang.

Indiquons ce système par $L'_{d/r-1}$. Puisque l'espace qui le contient a dimension $r - 1$, ce système satisfait à la démonstration précédente et sa variété est une des suivantes:

I. un point double de S_{r-1}:

II. une variété réductible de S_{r-1}, qui possède deux sub-variétés rationnelles de dimensions h_1 et $r - h_1 - 2 (1 \leq h_1 \leq r - 3)$:

III. une variété irréductible de S_{r-1} de dimension $\frac{r-2}{2}$ (r = nombre pair).

Telles variétés sont évidemment les sections hyperplanes de la variété base du système $L_{d/r-1}$.

Puisque on obtient un tel résultat dans un S_{r-1} quelconque, il s'ensuit que la variété base de $L_{d/r-1}$ est une des suivantes:

I. une droite double de S_r. (En effet un hyperplan générique de S_r coupe dans un seul point double seulement une droite double).

II. Une variété réductible, qui possède deux sub-variétés radionnelles de dimensions: $h = h_1 + 1$ et $r - h = r - h_1 - 1$. (Les variétés sont nécessairement rationnelles parce que leur sections avec un hyperplan générique sont rationnelles. Si, en effet, une seule coordonnée, par exemple $x_m = \Phi_m(\sigma_1, \sigma_2, \ldots, \sigma_{h+1})$ était fonction irrationnelle des paramètres, la section hyperplane $x_s = 0 (s \neq m, 0 \leq s \leq \leq r)$ serait irrationnelle, contre la démonstration précédente. Les dimensions de cette variété, sont évidemment $h_1 + 1$ et $r - h_1 - 1$, pour que les sections hyperplanes résultent de dimensions h_1 et $r - h_1 - 2$.)

III. Une variété rationnelle irréductible de dimension $\frac{r}{2}$ $\Big($cette variété doit être rationnelle pour les mêmes motifs susdits et elle doit avoir dimension $\frac{r}{2}$, pour que la section hyperplane ait dimension $\frac{r-2}{2}\Big)$.

Soit maintenant le système $L_{d/r-2} (r - 2 \leq d)$. En sectionnant ce système avec un hyperplan, on obtient un système $L'_{d/r-2} (r - 2 \leq d)$ de S_{r-1}, qui par rapport à S_{r-1}, se trouve dans les mêmes conditions du système $L_{d/r-1}$ par rapport à S_r. Pour cela les conclusions précédentes sont valides et la variété base de $L_{d/r-2}$ est une des suivantes:

I. Un plan double;

II. une variété réductible, qui possède deux sub-variétés rationnelles de dimensions h et $r - h + 1 = r - h + 2 - 1$;

III. une variété rationnelle irréductible de dimension $\frac{r+1}{2}$ (r = nombre impair).

Soit maintenant le système $L_{d/r-3} (r - 3 \leq d)$. En sectionnant ce système par un hyperplan et en répétant le raisonnement précédent on trouvera que le variété base sera une des suivantes:

I. Un S_3 double;

II. une variété réductible, qui possède deux sub-variétés rationnelles de dimensions h et $r - h + 2 = r - h + 3 - 1$;

III. une variété rationnelle irréductible de dimension $\frac{r+2}{2}$ (r = nombre pair).

Ainsi poursuivant il est évident qu'on parvient a démontrer le théoreme.

2°

Donnons maintenant des exemples significatifs.

1°) Dans S_5 il existe un système linéaire de quadriques $L_{5/5}$, dont la variété base est constituée par une V_2^3 rationnelle de S_4 et par un plan, ayant en commun avec la V_2^3 une génératrice.

Les équations canoniques de V_2^3 sont:

$$\frac{x_0}{x_1} + \frac{x_1}{x_2} = \frac{x_3}{x_4}, \qquad x_5 = 0$$

celles du plan:

$$x_0 - x_1 = 0, \qquad x_1 - x_2 = 0, \qquad x_3 - x_4 = 0, \qquad x_5 = 0.$$

Et le système devient:

$$\varrho_0(x_0x_2 - x_1^2) + \varrho_1(x_0x_4 - x_1x_3) + \varrho_2(x_1x_4 - x_2x_3) + \\ + \varrho_3(x_0 - x_1)\, x_5 + \varrho_4(x_1 - x_2)\, x_5 + \varrho_5(x_3 - x_4)\, x_5 = 0.$$

2°) Si la V_2^3 est constituée d'une quadrique de S_3 et d'un plan de S_4 d'équations:

$$(1) \qquad x_0 - x_1 = 0, \qquad x_2 - x_3 = 0, \qquad x_5 = 0,$$

qui possède en commun une génératrice avec la quadrique:

$$(2) \qquad \frac{x_0}{x_1} = \frac{x_2}{x_3}, \qquad x_4 = x_5 = 0$$

on obtient trois quadriques de S_4:

$$x_0x_3 - x_1x_2 = 0, \qquad x_4(x_0 - x_1) = 0, \qquad x_4(x_2 - x_3) = 0.$$

Nous pouvons considérer un plan de S_5, qui a en commun avec la V_2^3 réductible, c'est à dire avec la quadrique (2), une autre génératrice:

$$x_0 - x_2 = 0, \qquad x_3 - x_1 = 0, \qquad x_4 = 0$$

et de cette manière on obtient le système linéaire de S_5:

$$\varrho_0(x_0x_3 - x_1x_2) = \varrho_1(x_0 - x_1)\, x_4 = \varrho_2(x_2 - x_3)\, x_4 + \\ + \varrho_3(x_0 - x_3)\, x_5 = \varrho_4(x_3 - x_4)\, x_5 + \varrho_5 x_4 x_5 = 0$$

qui est un $L_{5/5}$ de S_5.

3°

Dans S_6 considérons le système linéaire $L_{5 \cdot 5}$ de S_6:

$$\varrho_0(x_0x_3 - x_1x_2) + \varrho_1(x_0x_5 - x_1x_4) + \varrho_2(x_2x_5 - x_3x_4) + \\ + \varrho_3x_1x_6 + \varrho_4x_3x_6 + \varrho_5x_5x_6 = 0$$

qui a pour base la V_3^4 rationnelle de S_5:

$$\frac{x_0}{x_1} = \frac{x_2}{x_3} = \frac{x_4}{x_5}, \qquad x_6 = 0$$

et l' S_3 d'équations:

$$x_1 = x_3 = x_5 = 0,$$

ayant en commun avec la V_3^4 le plan:

$$x_0 = x_1 = x_3 = x_5 = 0.$$

Si d'un point P nous projetons S_3 nous obtenons un S_4, qui coupe la V_3^4 réductible, constituée d'une V_1^3, qui se trouve sur le plan intersection et d'une droite s. Le point P et la droite s individualisent un plan de cordes de la variété base, qui sorte de P. Pour cela le rang de la Jacobienne du système est:

$$6 - 1 = 5.$$

4°

Considérons la variété de Segre individualisée par les couples des points de deux plans de S_5:

$$y_{0p} = x_0x_p, \\ y_{1p} = x_1x_p, \qquad (p = 3, 4, 5), \\ y_{2p} = x_2x_p.$$

En éliminant x_0, x_1, x_2, x_0, on obtient las équations:

$$y_{0k}y_{1h} - y_{0h}y_{1k} - 0, \\ y_{0k}y_{2h} - y_{0h}y_{2k} - 0, \qquad (k \neq h = 3, 4, 5), \\ y_{1k}y_{2h} - y_{1h}y_{2k} = 0.$$

Il s'agit de neuf quadriques de S_8, qui forment un système $L_{8/7}$, dont la variété base est donnée par:

$$(3) \quad \frac{y_{03}}{y_{04}} = \frac{y_{13}}{y_{14}} = \frac{y_{23}}{y_{24}}; \qquad \frac{y_{03}}{y_{05}} = \frac{y_{13}}{y_{15}} = \frac{y_{23}}{y_{25}}; \qquad \frac{y_{04}}{y_{05}} = \frac{y_{14}}{y_{15}} = \frac{y_{24}}{y_{25}}.$$

Les équations paramétriques de cette variété sont:

$$y_{03} = 1,\ y_{13} = \nu,\ y_{23} = \sigma,\ y_{04} = \tau,\ y_{14} = \nu\tau,\ y_{24} = \sigma\tau,$$
$$y_{05} = \omega,\ y_{15} = \nu\omega,\ y_{26} = \sigma\omega.$$

Il s'agit d'une V_4: ∞^1 cordes de cette variété contenues dans un plan passent par chaque point de S_8. Donc le rang de la Jacobinne du système est 7.

5°

Le précédent système en génère un autre. Si aux trois groups de fractions (3) on égalise respectivement les rapports suivants:

$$\frac{y_{33}}{y_{34}},\quad \frac{y_{33}}{y_{35}},\quad \frac{y_{34}}{y_{35}},$$

on obtient un L_{17} de S_{11}, qui a pour base une V_5^{10} de S_5, qui résulte une variété de Segre, individualisée par les couples de points d'un S_2 et d'un S_3 de S_6.

Le rang du système est 11 et par un point P de S_{11} il passe une seule corde de la variété base.

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ANTIMORPHISMS OF PARTIALLY ORDERED SETS

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Abstract. In this paper we prove some fixed point theorems for local antimorphisms which need not be either isotone or antitone mappings. We give, in a way necessary and sufficient conditions for the existence of fixed points on partially ordered sets. We also introduce the concepts: inf, sup-antimorphisms, and, in connection with that we also have some additional results. With such an extension, a general fixed point theorem is obtained which includes a recent result of the author, and also contains, as special cases, some results of Abian, Shmuely, Kurepa, Metcalf and Payne, and some others.

Key words. Lattices, order-reversing (antitone) mapping, fixed point posets.

MS Classification. 05 A 15.

1. INTRODUCTION AND COMMENTARY

Let P be a partially ordered set. A function f from P to P is order-preserving (or *isotone* or increasing) if for all $x, y \in P$, $x \leq y$ implies $f(x) \leq f(y)$. If f satisfies the condition that for $x, y \in P$, $x \leq y$ implies $f(y) \leq f(x)$, then f is said to be *antitone* (or decreasing). P has a fixed point under f if $f(x) = x$ for some $x \in P$. P has the fixed point property if it has a fixed point under all order-preserving functions.

An ordered set P is said to be *complete* provided any non-void subset X of P determines its own infimum $\inf X \in P$ and supremum $\sup X \in P$.

Several authors have treated the problem of characterizing posets with the fixed point property: Abian A., Abian and Brown, Davis A., Edelman, Höft H. and Höft M., Kurepa D., Rival, Smithson, Tarski, Tasković, Ward and Wong, among others.

Tarski [12], Abian and Brown [2], and others have studied fixed points of isotone mappings on partially ordered sets. In [1] and [11] fixed points of certain antitone mappings are studied.

In a poset P functions f are considered such that, for any nonempty $A \subset P$

$$(1)\qquad f(\sup A) = \inf f(A), \qquad \text{where } f(A) = \{f(a) \mid a \in A\}.$$

A function f in a poset P satisfying (1) is referred to as a *join antimorphism.* One considers also meet antimorphism satisfying, for any nonempty $A \subset P$,

$$(2) \qquad f(\inf A) = \sup f(A), \qquad \text{where } f(A) = \{f(a) \mid a \in A\}.$$

It is easily seen that every function f, defined on a complete poset (lattice) L, satisfying (1) or (2) is also antitone, that is, join or meet antimorphisms are antitone mappings. On the other hand, it is easy to construct an antitone mapping on a complete poset (lattice) which is neither a join antomorphism nor a meet antimorphism. Namely, let L be the lattice on the Figure 1 and $f: L \to L$ defined by

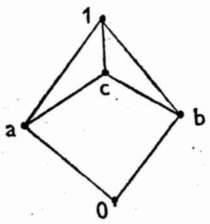

fig. 1

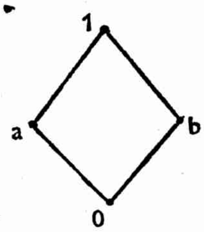

fig. 2

$f(0) = f(b) = f(a) = 1, f(c) = a, f(1) = 0$. Evidently, f is antitone, but $f(\sup\{a, b\}) = = f(c) = a \neq \inf\{f(a), f(b)\} = \inf\{1, 1\} = 1$. The mapping f is not a join antimorphism. If we define $g: \{0, a, b, 1\} \to \{0, a, b, 1\}$, $g(\{a, b, 1\}) = \{0\}$ and $g(0) = 1$, then g is antitone, but not a meet antimorphism (Fig. 2.).

Sufficiency for antimorphisms. Let P be a complete partially ordered set(poset) and $f: P \to P$ an antitone mapping satisfying the conditions: $f(x) \leq x$ or $f^2(x) \leq x$ for all $x \in P$. Then f is a meet antimorphism.

The analogous statement for join antimorphisms is also valid, when $x \leq f(x)$ or $x \leq f^2(x)$ for all $x \in P$ (see [11]).

Proof. Let $A \subset P$ be a nonempty set, $f(x) \leq x (x \in P)$ and $i = \inf A$. Then $f(x) \leq f(i)$ for every $x \in A$. Thus, $f(i)$ is an upper bound for $f(A)$. Let $s = \sup f(A)$, and then $s \leq f(i)$. Assume $s < f(i)$. From $f(x) \leq s$ $(x \in A)$, it follows that $s < < f(i) \leq i$ and hence $s < f(i) \leq f(s)$, i.e., $s < f(s)$-contradiction. That is $f(i) = s$, i.e., $f(\inf A) = \sup f(A)$.

When $f^2(x) \leq x$, $x \in P$ we have $s \leq f(i)$. From $f(x) \leq s$, $x \in A$, it follows that $f(s) \leq f^2(x)$, i.e., $f(s) \leq x$, $x \in A$. We conclude that $f(s)$ is a lower bound for A. Then $f(i) \leq s$, which implies $f(i) = s$, i.e., $f(\inf A) = \sup f(A)$. This completes the proof of sufficiency for antimorphisms.

In this paper we examine fixed points of mappings $f: P \to P$ which are comparable to the identity mapping $i_P: P \to P$, in the sense that for any $x \in P$, $f(x) \leq x$ or $x \leq f(x)$. For any $f: P \to P$ it is natural to consider the following sets

$$P^f := \{x \mid x \in P \wedge x \leq f(x)\}, \qquad P_f := \{x \mid x \in P \wedge f(x) \leq x\}.$$

If $f: P \to P$ is any mapping of P into P, let $I(P, f)$ be the set of all invariant points of P relative to f; i.e., $I(P,f) := \{x \mid x \in P \wedge f(x) = x\}$.

In this paper we prove some fixed point theorems for local antimorphisms which need not be either isotone or antitone mappings. We give, in a way, necessary and sufficient conditions for the existence of fixed points on partialy ordered sets.

Inf, Sup-antimorphisms. We also introduce the concepts: inf, sup-antimorphisms, and, in connection with that we also prove a result.

Let P be a poset. The mapping $f: P \to P$ satisfying, for nonempty sets P^f, $P_f \subset P$, the condition

$$f(\inf P^f) = \inf f(P_f), \qquad \text{where } f(P_f) = \{f(x) \mid x \in P_f\},$$

is called an *inf-antimorphism*. Similarly, if f satisfies the condition

$$f(\sup P^f) = \sup f(P_f), \qquad \text{for } \emptyset \neq P^f, P_f \subset P,$$

then such an f is said to be a *sup-antimorphism*.

2. FIXED POINTS OF LOCALLY MEET AND LOCALLY JOIN ANTIMORPHISMS

We start with a statement on join or meet antimorphisms.

Theorem 1. *Let $(P, \leq)$ be a partially ordered set and f a mapping from P into P such that:*

(A) *The set P^f is nonempty, the point $s := \sup P^f$ exists and satisfies $f(s) \leq s$,*

(B) *s is a lower bound (minorant) for the set $f(P^f)$,*

(C) (Locally join antimorphism) $f(\sup P^f) = \inf f(P^f)$.

Then:

(1.1.) *The set $I(P,f) := I$ is nonempty,*

(1.2.) *Neither of the conditions* (A), (B), (C) *can be deleted if* (1.1) *is to be valid.*

Dually, if

(A′) *The set P_f is nonempty, the point $I_m := \inf P_f$ exists and satisfies $I_m \leqq f(I_m)$,*

(B′) *I_m is an upper bound (majorant) for the set $f(P_f)$,*

(C′) (Locally meet antimorphism) $f(\inf P_f) = \sup f(P_f)$;

then the set $I(P,f)$ is nonempty and neither of the conditions (A′), (B′), (C′) *can be deleted if* (1.1.) *is to be valid.*

Proof. By the assumption (A), the set P^f is nonempty, the point $s = \sup P^f$ exists and $f(s) \leq s$. From (B), we have, for all $x \in P^f$ is $s \leq f(x)$, which using (C) implies $s \leq \inf f(P^f) = f(\sup P^f) = f(s)$. Our conclusion follows from (A) and $s \leq f(s)$, that is $f(s) = s$ and thus $s \in I(P, f)$, i.e., the set $I(P,f)$ is nonempty. This completes the proof of (1.1).

(1.2). Now we prove that the conditions (A), (B) and (C) cannot be removed. We show that by the following examples (1, 2, 3).

Example 1. (Fig. 3.) Let P be the set (interval) $[0, 2]$ and define $f: P \to P$ by $f(x) = 2$ for $x \in [0, 1]$ and $f(x) = 1$ for $x \in (1, 2]$, where P is totally ordered by the ordinary ordering $\leq$. Then conditions (B) and (C) are satisfied. Condition (A) is not satisfied ($f(s) = f(1) = 2 \geq 1$). The set I is empty.

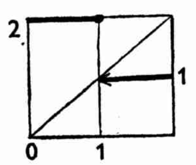

fig 3

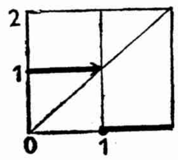

fig.4

Example 2. (Fig. 4.) Let $P = [0, 2]$ and define $f: [0, 2] \to [0, 2]$ by $f(x) = 1$ for $x \in [0, 1)$ and $f(x) = 0$ for $x \in [1, 2]$, where P is totally ordered by the ordinary ordering $\leq$. Condition (A) is satisfied ($f(\sup P^f) = f(s) = f(1) = 0 \leq 1 = s$), condition (B) is satisfied ($s = 1$ is a minorant for the set $f(P^f) = \{1\}$), but condition (C) is not satisfied ($f(\sup P^f) = f(1) = 0 \neq 1 = \inf f(P^f)$). Again $I = \emptyset$.

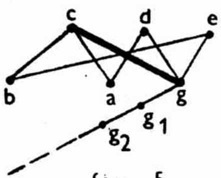

fig 5

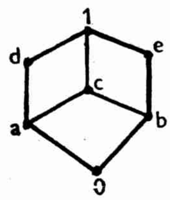

fig.6

Example 3. (Fig. 5.) Let the poset $P = \{a, b, c, d, e, g, g_n \ (n = 1, 2, 3, \ldots)\}$ be ordered by the order relation $\leq$ so that $a \leq c$, $a \leq d$, $b \leq e$, $b \leq c$, $g \leq e$, $g \leq d$, $g \leq c$, $g_1 \leq g$, $g_{n+1} \leq g_n$ $(n \in \mathrm{N})$, and if the elements a, g, b are incomparable, then the elements c, d, e are also incomparable. Define $f: P \to P$ by $f(a) = d$, $f(b) = e$, $f(d) = f(e) = f(c) = g$, $f(g) = g_1$ and $f(g_n) = g_{n+1}$ $(n = 1, 2, \ldots)$. Condition (A) is satisfied ($P^f = \{a, b\}$, $f(\sup P^f) = f(c) = g \leq c = \sup P^f = \sup \{a, b\} = s$), condition (C) is satisfied ($f(\sup P^f) = f(c) = g = \inf f(P^f) = \inf \{d, e\} = g$), but condition (B) is not satisfied ($s = \sup P^f = c$ is not minorant for the set $f(P^f) = \{d, e\}$). Furthermore, f does not have a fixed point.

By dual considerations one proves the part of the Theorem which concerns the point $I = \inf P_f$. It sufficies to make the following substitutions: $\sup \to \inf$, $P^f \to P_f$, $\leq \to \geq$. This completes the proof of the Theorem.

Some corollaries. Now we shall apply the results above by considering the following consequence. They bring into connection the results (sufficient conditions) which were obtained in the case when the set $I(P,f)$ is nonempty.

Corollary 1. (Kurepa [6]) *Let P be a nonempty right conditionally complete set and f a decreasing selfmapping of P such that for at least one member $x \in P$ we have*

$$x \leq f(x) \quad \text{or} \quad f(x) \leq x, \quad \text{i.e.,} \quad \neg\, (\forall\, x \in P,\ x \,\|\, f(x)).$$

Let us assume that

1. $f(\sup A) = \inf f(A)$,
2. *Each point of P_f is comparable with each point of P^f,*
3. *If $s := \sup P^f \in P$ exists then $f(s) \leq s$.*

Then the set $I(P,f)$ is nonempty and $f(s) = s = \inf P_f$.

Proof. Let us prove that the sufficient conditions of this statement implies the validity of the sufficient conditions (A), (B) and (C). First, direct condition 1) implies (C), because 1) is valid for each $A \subset P$, also $A = P^f$. Otherwise, as the set P^f is bounded (from condition 2)) then according to conditional completeness; has the supremum denoted by $s := \sup P^f$, and from 3) we have $f(s) \leqq s$, i.e., our condition (A) is valid.

We prove that the condition (B) is valid, i.e. that s is a minorant for the set $f(P^f)$. From 2) also each point of P^f is comparable to each point of P_f. So, the sets P_f and P^f have minorants and majorants respectively. But, the set P is conditionally complete and so these sets of minorants and majorants have a supremum and infimum denoted by s and i. Also, from the conditions of the Corollary $f: P \to P$ is an antitone mapping, so $f(P^f) \subset P_f$, and as s is a minorant for P_f, s will be a minorant for $f(P^f)$, i.e., the condition (B) is valid. It means that Corollary 1 is the consequence of our Theorem 1.

Corollary 2. (Tasković [14]) *Let $(P, \leq)$ be a partially ordered set and f a mapping from P into P such that* (A), (C) *and*

(a) $x, y \in P^f \wedge x \leq y \Rightarrow f(y) \leq f(x)$,
(b) P^f *is a totally ordered set.*

Then the set $I(P,f)$ is nonempty.

Proof. Since $f: P \to P$ is a decreasing mapping on P^f (from (a)) and the condition (b) is valid, condition (B) is satisfied. This, with (A) and (C) proves Corollary 2.

In the following $(P, \leq)$ will denote a nonempty partially ordered set P with partial order $\leq$. A subset A of P is a *toset* (chain) just in case A is totally ordered. For $x \in P$ and $A \subset P$, define $L(x) = \{y : y \in P, y \leq x\}$, $M(x) = \{y : y \in P, x \leq y\}$, and $M(A) = \cup\{M(x) : x \in A\}$.

A partially ordered set $(P, \leq)$ is a *mod* if and only if the following hold:

(1) For all $x, y \in P$ sup $\{x, y\}$ exists,

(2) For all $x \in P$, $L(x)$ is a toset,

(3) Each nonempty subset of P which is bounded above (below) has a supremum (infimum) in P,

(4) If $x < y$, then there is a $z \in P$ such that $x < z < y$.

A function $f: P \to P$ is nonoscillatory from above if and only if for each nonmaximal x and maximal toset $A \subset M(x)\backslash\{x\}$, $\cap\{f[x, u]) : u \in A\} = \{f(x)\}$. The function f is nonoscillatory from below if and only if for each nonminimal x, $\cap\{f([u, x]) : u < x\} = \{f(x)\}$.

Corollary 3. (Metcalf and Payne [7]) *Let P be a totally ordered mod. Suppose that $f: P \to P$ is a function satisfying:*

(5) *If $x \leq y$ and $f(y) \leq f(x)$, then $[f(y), f(x)] \subset f([x, y])$.*

(6) *The function f is either nonoscillatory from above or from below.*

(7) *There exists a, $b \in P$ such that $a \leq b$, $a \leq f(a)$, and $f(b) \leq b$.*

Then f has a fixed point.

Proof. First, let us prove that our condition (A) of Theorem 1 is satisfied, i.e., that $f(s) \leq s$. The whole situation we observe on the interval $[a, b]$, according to condition (3) of Corollary 3. So, let us adapt pur signs for P^f and P_f for this situation, and let the corresponding set P^f be denoted by

$$A^f := \{x : x \in [a, b] \quad \text{and} \quad t \leq f(t), \quad \text{for all } t \in [a, x]\}.$$

By the assumptions of the Corollary, the set A^f is nonempty, and the point $s := \sup A^f$ exists. It will first be shown that $f(s) \leq s$. Suppose, to the contrary, that $s < f(s)$, and let

$$A_f := \{x : s \leq x \leq f(s) \text{ and } f(x) < x\},$$

so that $s = \sup A^f = \inf A_f$. Then, for $x \in A_f$, $f(x) < x \leq f(s)$, so that condition (5) of Corollary 3 yields

$$[x, f(s)] \subset [f(x), f(s)] \subset f([s, x]), \qquad \text{for all } x \in A_f.$$

For $x \in A_f$ the sets $[x, f(s)]$ are increasing as x is decreasing, while the sets $f([s, x])$ are decreasing as x is decreasing. Thus,

$$(s, f(s)] = \bigcup_{x \in A_f} [x, f(s)] \subset \bigcap_{x \in A_f} f([s, x]);$$

however, the intersection on the right hand side has at most one element, since f is nonoscillatory from the right, which contradicts $s < f(s)$. Thus, $f(s) \leq s$, i.e., the condition (A) is satisfied. On the other hand, in an analogous proof of Corollary 1 we prove that the conditions (B) and (C) are satisfied. This proves Corollary 3.

We next demonstrate that the following condition introduced by Abian [1] is a form of continuity.

Abian's conditions. Let $f: P \to P$ where P is a mod. If $A \subset P$ is a toset, then $f(\inf A) = \sup f(A)$ and $f(\sup A) = \inf f(A)$ whenever both sides of the equalities exist.

Corollary 4. (Abian [1]) *Let $f: P \to P$ where P is a totally ordered mod. If f is decreasing and satisfied Abian's condition, then f has a fixed point.*

Proof. Since Abian's conditions imply our condition (C), because $f(\sup A) = \inf f(A)$ for all $A \subset P$, we also have $A := P^f$. In the other hand, the set P is a totally ordered set and mod, thus the point $s := \sup P^f$ exists; and as s is a minorant for $f(P^f)$, because s is a minorant also for the set P_f. Also, from Abian's conditions we have $f(s) = f(\inf P_f) = \sup f(P_f) \leq \sup P^f =: s$, i.e., $f(s) \leq s$. It means that Abian's statement is the consequence of our Theorem 1.

Corollary 5. (Tasković [13]) *Let P be a totally ordered conditionally complete set and $f: P \to P$ antitone mapping satisfiing the conditions* (4) *and* (5). *Then f has a fixed point.*

Proof. Evidently, the proof of this statement is analogous to the proof of the preceding statement of Abian.

Corollary 6. (Shmuely [11]) *Let L be a complete atomic lattice and let $f: L \to L$ be an antitone mapping satisfying the conditions:*

(a) $x \leq f^2(x)$ *for every* $x \in L$,

(b) $a \leq f(a)$ *for each atom* $a \in L$.

Then f has a fixed point.

Proof. Since $f: P \to P$ is an antitone mapping and $x \leq f(x)$ for every $x \in L$, we have from sufficiency for antimorphism, that our condition (C) is satisfied. In this paper $P^f(A)$ denotes the family of all subsets A of L satisfying $\sup A \leq f(\sup A)$. Notice that $\{0\} \in P^f(A)$ and $P^f(A)$ is ordered by set inclusion. Here we use the following statement of Shmuely [11]:

Lemma (Shmuely [11]) *Under the assumption of Corollary* 6, *$P^f(A)$ ordered by inclusion, has a maximal element.*

Now, let $A_0 \subset L$ be a maximal element of $P^f(A)$ and put $s := \sup A_0 (= \sup P^f(A))$. Obviously, $s \leq f(s)$. Assuming $s < f(s)$ we can find an atom $r \in L$ and $r \notin A_0$, such that $r \leq f(s)$. Also, $s \leq f(r)$, because f is antitone and (a) is valid. This together with $r \leq f(r)$ yields, from (C),

$$f(\sup\{r, s\}) = \inf\{f(r), f(s)\} \geqslant \sup\{r, s\},$$

contradicting the maximality of A_0. Thus $f(s) = s$, i.e., obviously $f(s) \leq s$, and s is minorant for the set $f(P^f)$, because $s = f(s) = f(\sup P^f(A)) = \inf f(P^f(A))$. It means, the conditions (A) and (B) are satisfied and thus Shmuely's statement is the consequence of Theorem 1.

3. FIXED POINTS OF INF, SUP-ANTIMORPHISMS

Theorem 2. *Let* $(P, \leq)$ *be a partially ordered set and* f *a mapping from* P *into* P *such that:*

D) *The set* P^f *is nonempty, the point* $s := \sup P^f$ *exists and satisfies* $s \leq f(s)$,

(E) s *is an upper bound (majorant) for the set* $f(P_f)$,

(F) (Sup-antimorphism) $f(\sup P^f) = \sup f(P_f)$.

Then: (2.1) *The set* $I(P,f)$ *is nonempty,* (2.2) *Neither of the conditions* (D), (E), (F) *can be deleted if* (2.1) *is to be valid,*

Dually, if

(D′) *The infimum of the set* P_f *defined by* $I_m = \inf P_f$ exists and $f(I_m) \leqq I_m$,

(E′) I_m *is a lower bound (minorant) for the set* $f(P^f)$,

(F′) (Inf-antimorphism) $f(\inf P_f) = \inf f(P^f)$,

then set $I(P,f)$ *is nonempty and neither of the conditions* (D′), (E′), (F′) *can be deleted if* (2.1) *is to be valid.*

Proof. The set P^f being, by assumption, nonempty, the point $s = \sup P^f$ exists, and from (D), $s \leq f(s)$. From (E), it follows that s is an upper bound for the set $f(P_f)$ and thus we have $x \in P_f, f(x) \leq s$, i.e., $\sup f(P_f) \leq s$, which implies $f(s) = f(\sup P^f) = \sup f(P_f) \leq s$. Our conclusion follows from (D) and $f(s) \leq s$, that is $f(s) = s$ and thus $s \in I(P,f)$, i.e., the set $I(P,f)$ is nonempty. This completes the proof of (2.1).

(2.2). Now we prove that the conditions (D), (E) and (F) not be removed. We show that by the following examples.

Example 4. (Figure 6) Let P be the lattice (poset) on the Figure 6 and let $f: P \to P$ be defined by $f(a) = f(b) = f(c) = 1, f(d) = f(1) = a, f(0) = a, f(e) = 0$. Conditions (D) and (E) are satisfied ($P^f = \{0, a, b, c\}$, $s = \sup P^f = c \leq f(s) = f(c) = 1$, $s = c$ is majorant for the set $f(P_f) = f(\{d, e, 1\}) = \{f(d), f(e), f(1)\} = \{0, a\}$) but condition (F) is not satisfied ($f(\sup P^f) = f(c) = 1 \neq a = \sup f(P_f)$). Furthemore, f does not have a fixed point.

Example 5. Also, neither of the conditions (D) and (E) can be deleted if (2.1) is to be valid, which is illustrated by the following examples for $P = [0, 2]$ and $f: P \to P$, defined geometrically by

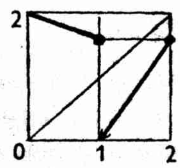

fig. 7

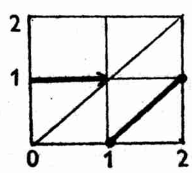

fig. 8

In Figure 7., conditions (D) and (F) are satisfied but condition (E) is not. Further, in Figure 8., conditions (E) and (F) are fulfilled, but condition (D) is not, and f (Fig. 7. and 8.) has not fixed point.

(2′) By dual considerations one proves the part of Theorem 2. ((2.1), (2.2)) which concerns the point $I_m = \inf P_f$; it sufficies to make the following substitutions: $s \to I_m$, $P^f \to P_f$, $\sup \to \inf$, $\leq \to \geqslant$.

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NECESSARY AND SUFFICIENT CONDITIONS FOR FINALLY VANISHING OSCILLATORY SOLUTIONS IN SECOND ORDER DELAY EQUATIONS

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Abstract. Necessary and sufficient conditions have been found to ensure that all oscillatory solutios of the equation

(1) $$(r(t)\,y'(t))' + a(t)\,y(g(t)) = f(t), \qquad g(t) \leqq t$$

approach zero. By way of several theorems it is shown that this behavior of equation (1) is associated with the presence of nonoscillatory solutions with certain properties.

Key words. Oscillatory, nonoscillatory, asymptotic, delay, forced.

MS Classification. 34 K 25.

1. INTRODUCTION

In [9], this author found conditions on $a(t)$, $r(t)$, $f(t)$ and $g(t)$ to ensure that all nontrivial oscillatory solutions of the equation

(1) $$(r(t)\,y'(t))' + a(t)\,y(g(t)) = f(t)$$

approach zero asymptotically. It was shown that an oscillatory solution $y(t)$ of (1) satisfies $\lim_{t\to\infty} y(t) = 0$ subject to:

$$\int^{\infty} 1/r(t)\,\mathrm{d}t < \infty, \qquad \int^{\infty} |\,a(t)\,|\,\mathrm{d}t < \infty \qquad \text{and} \qquad \int^{\infty} |\,f(t)\,|\,\mathrm{d}t < \infty.$$

In section 3 of this work, we would present necessary and sufficient conditions to achieve asymptotic approach to zero of all oscillatory solutions of (1). This behavior of the oscillatory solutions of (1) is closely associated with (1) having a nonoscillatory solution with certain properties. The connection between oscillation and nonoscillation becomes very interesting under rather restrictive constraint $a(t) > 0$, in which case the ratio $|\,f(t)\,|/a(t)$ (Wallgren [14]) plays a significant role. This connection is examined in several theorems in this section without the restriction that $a(t) > 0$.

The proof of [9, Theorem 2] is lengthy and requires a stringent condition that the retardation $g(t)$ be slight by requiring $t - g(t) \leq B$, $B > 0$ a constant. We remedy this situation by giving an alternative proof based on $t > g(t)$ and $g(t) \to \infty$ entirely.

It turns out that we can deduce restrictions on the growth of oscillatory solutions from growth condition on $r(t)$. We examine this in section 4 and come up with alternative theorem to ensure asymptotic decay to zero of the oscillatory trajectories of (1).

Even though a voluminous literature exists about many oscillatory and non-oscillatory criteria for homogeneous and nonhomogeneous equations such as (1), the asymptotic nature of nonoscillatory or oscillatory solutions of these equations has not been so extensively studied, and for that matter the literature is very scanty with regard to oscillatory solutions. For asymptoticity on nonoscillation, the reader will find a good account in the works of Hammett [5], Londen [6] and this author [8, 10, 11, 12]. An excellent reference list is included by Graef [3] and Graef and Spikes [4] for any interested reader.

Throughout this study, all theorems proven are supported by examples to show that they are not vacuous. Although the results found apply well to ordinary differential equations, the presence of retarded term makes application of common techniques which work for ordinary differential equations a nontrivial matter. Travis [13] shows how a theorem of Bhatia [1] fails in such passage to retarded equations (cf, [9]). In what follows all results are easily extendable to the nonlinear equation

(2) $$(r(t)\, y'(t))' + a(t)\, h(y(g(t))) = f(t).$$

2. DEFINITION AND ASSUMPTIONS

It will be assumed for the rest of this paper that

(i) $r(t)$, $a(t)$, $g(t)$, $f(t) : R \to R$ and continuous; R is real line,

(ii) $r(t) > 0$, $r'(t) \geqq 0$, $g(t) > 0$ on some positive half real line R^+.

(iii) $g(t) \to \infty$ as $t \to \infty$, $g(t) \leq t$ and $g'(t) > 0$ for $t \geqq t_0$ where $t_0 > 0$.

We call a function $Q(t) \in C[t_0, \infty)$ *oscillatory* if it has arbitrarily large zeros in $[t_0, \infty)$. Otherwise $Q(t)$ is called *nonoscillatory*. In this work, the term "solution" applies to those solutions (of equations under consideration) which can be extended to the right of some positive point on R, say t_0.

3. MAIN RESULTS

Theorem (1). *Suppose*

(3) $$\int^{\infty} |a(t)|\, dt < \infty,$$

(4) $$\int^{\infty} |f(t)|\,dt < \infty$$

and

(5) $$\int^{\infty} 1/r(t)\,dt < \infty,$$

than all oscillatory solutions of (1) *approach zero as* $t \to \infty$.

Proof. Let $y(t)$ be an oscillatory solution of equation 1. Let $T > t_0$ be large enough so that

(6) $$\int_T^{\infty} |a(t)|\,dt < 1/4$$

and

(7) $$\int_T^{\infty} |f(t)|\,dt < 1/4.$$

Suppose to the contrary that $\limsup_{t\to\infty} |y(t)| > d > 0$. Let $T_1 > T$ be so large that $g(T_1) \geqq T$ and $y(T_1) = 0$. There is $T' > T_2 > T_1$ such that $y(T_2) = 0$ and

(8) $$\text{Max}\,\{|y(t) : T \leq t \leq T'\} = |y(T')| > d > 0.$$

Let $[x_1, x_2]$ designate the smallest closed interval containing T' such that $y(x_1) = y(x_2) = 0$. Designate $M = \text{Max}\,\{|y(t)| : x_1 \leq t \leq x_2\}$. Note that $T_2 \leq x_1$. It is clear that $M \geqq d$ and

(9) $$|y(t)| \leq M$$

and

(10) $$|y(g(t))| \leq M$$

for $t \in [x_1, x_2]$. Also let $T_M \in [x_1, x_2]$ be such that $M = |y(T_M)|$. Now

$$M = \int_{x_1}^{T_M} y'(t)\,dt,$$

which gives

(11) $$M \leqq \int_{x_1}^{T_M} |y'(t)|\,dt.$$

also

$$-M = \int_{T_M}^{x_2} y'(t)\,dt,$$

which gives

(12) $$M \leqq \int_{T_M}^{x_2} |y'(t)|\,dt.$$

From (11) and (12) we get

$$2M \leqq \int_{x_1}^{x_2} |y'(t)|\,dt = \int_{x_1}^{x_2} |y'|^{1/2} (r(t)\,|y'(t)|)^{1/2} (r(t))^{-1/2}\,dt.$$

By Schwarz's inequality

$$4M^2 \leqq \Big(\int_{x_1}^{x_2} 1/r(t)\,dt\Big)\Big(\int_{x_1}^{x_2} (r(t)\,y'(t))\,y'(t)\,dt\Big). \tag{13}$$

Integration by parts yield

$$4M^2 \leqq \Big(\int_{x_1}^{x_2} 1/r(t)\,dt\Big)\Big(-\int_{x_1}^{x_2} (r(t)\,y'(t))'y(t)\,dt\Big). \tag{14}$$

From (14) by using equation 1 we have

$$4M^2 \leqq \int_{x_1}^{x_2} 1/r(t)\,dt\Big(\int_{x_1}^{x_2} a(t)\,y(g(t))\,y(t)\,dt - \int_{x_1}^{x_2} y(t)\,f(t)\,dt\Big). \tag{15}$$

From (9), (10) and (15)

$$4 \leqq \Big(\int_{x_1}^{x_2} 1/r(t)\,dt\Big)\Big(\int_{x_1}^{x_2} |\,a(t)\,|\,dt + \frac{1}{M}\int_{x_1}^{x_2} |\,f(t)\,|\,dt\Big),$$

i.e.

$$\frac{4}{\int_{x_1}^{x_2} 1/r(t)\,dt} \leqq \frac{1}{4} + \frac{1}{4d}. \tag{16}$$

Unless $d = 0$, (16) yields a contradiction since $\int_{x_1}^{x_2} 1/r(t)\,dt$ can be made arbitrarily small by choice of large T. The proof is complete.

Remark (1). The above theorem improves our theorem 2 in [9] by eliminating the requirement that $g(t) = t - \tau(t)$ with $\tau(t)$ bounded. If conditions (3) and (4) hold then condition (5) is necessary as the following example shows.

Example (1). The equation

$$y''(t) + \frac{1}{t^2}\,y(t) = -\frac{\cos(\log t)}{t^2}, \qquad t > 0, \tag{17}$$

has $y(t) = \sin(\log t)$ as a solution.

The decomposition of $a(t)$ as $a(t) = a_1(t) + a_2(t)$ can be effectively used by assuming conditions on $a_1(t)$ and $a_2(t)$. Our next theorem uses such a decomposition toward obtaining necessary and sufficient condition for all oscillatory solutions of (1) to approach zero asymptotically.

Theorem. (2). *Suppose* $a(t) = a_1(t) + a_2(t)$, $a_1(t) > 0$, $|\,a_2/a_1\,| \leq k_1$ *for some* $k_1 > 0$ *and large* t. *Further suppose that* $|\,f(t)\,|/a_1(t)$ *approaches a limit as* $t \to \infty$. Let

$$\int^{\infty} 1/r(t)\,dt < \infty, \quad \text{and} \quad \int^{\infty} a_1(t)\,dt < \infty.$$

Then

$$\lim_{t\to\infty} (|f(t)|/a_1(t)) = 0$$

is a necessary and sufficient further condition for all oscillatory solutions of (1) *to approach zero as* $t \to \infty$.

Proof. The sufficiency is obvious. To prove necessity we rewrite (1) as

$$\frac{(r(t)\,y'(t))'}{a_1(t)} + y(g(t)) + \frac{a_2(t)}{a_1(t)}\,y(g(t)) = \frac{f(t)}{a_1(t)}. \tag{18}$$

This yields

$$\frac{|(r(t)\,y'(t))'|}{a_1(t)} \geqq \frac{|f(t)|}{a_1(t)} - (1 + k_1)\,|y(g(t))|, \tag{19}$$

where $y(t)$ is an oscillatory solution such that $y(t) \to 0$ as $t \to \infty$. Now if

$$\liminf_{t\to\infty} \frac{|f(t)|}{a_1(t)} > 0,$$

then $\dfrac{(r(t)\,y'(t))'}{a_1(t)}$ is bounded away from zero. Thus $(r(t)\,y'(t))'$ assumes a constant sign making $y(t)$ nonoscillatory. This contradiction completes the proof of the theorem.

Example (2). The equation

$$(t^2y'(t))' + \frac{1 - 2\sin(\log t)}{t^2}\,y(t) = \frac{6}{t^3} + \frac{10}{t^3}(\sin(\log t) - \cos(\log t)) +$$

$$+ \frac{1 - 4\sin^2(\log t)}{t^5}, \qquad t > 0, \tag{20}$$

has $y = \dfrac{1 + 2\sin(\log t)}{t^3}$ as an eventually vanishing solution. Here all conditions of Theorem 2 are easily verified. Hence all oscillatory solutions of (20) approach zero as $t \to \infty$.

Corollary (1). *Suppose* $a(t) > 0$, $\int^{\infty} 1/r\,\mathrm{d}t < \infty$, *and* $\int^{\infty} a(t)\,\mathrm{d}t < \infty$. *Further suppose* $\lim_{t\to\infty} \dfrac{|f(t)|}{a(t)}$ *exists. Then a necessary and sufficient condition for* (1) *to have all oscillatory solutions approaching zero is* $\lim_{t\to\infty} \dfrac{|f(t)|}{a(t)} = 0$.

Proof. Follows from Theorem 2.

Sufficiency part of the proof of Theorem 2 leads us to the following theorem.

Theorem (3). *Suppose* $a(t) = a_1(t) + a_2(t)$, $a_1(t) > 0$, $a_2(t)/a_1(t)$ *bounded for large* t, $\int^{\infty} a_1(t)\,\mathrm{d}t < \infty$, *and* $\int^{\infty} 1/r\,\mathrm{d}t < \infty$. *Further suppose that* $f(t)/a_1(t)$ *is bounded. Then all oscillatory solutions of* (1) *approach zero as* $t \to \infty$.

Proof. It is clear that $\int^{\infty} |a(t)|\,dt < \infty$. Since $\int^{\infty} a_1(t)\,dt < \infty$, and $f(t)/a_1(t)$ is bounded as $t \to \infty$, we have $\int^{\infty} |f(t)|\,dt < \infty$. The conclusion now follows by Theorem 1.

Example (3). Consider the equation

$$(21) \qquad (t^2 y'(t))' + \frac{1 - 2\sin(\log(t))}{t^3}\, y(t) = \frac{6}{t^3} + $$
$$+ \frac{10}{t^3}(\sin(\log t) - \cos(\log t)) + \frac{1 - 4\sin^2(\log t)}{t^6}, \qquad t > 0,$$

which has $y(t) = \dfrac{1 + 2\sin(\log t)}{t^3}$ as a vanishing oscillatory solution. In fact, since all conditions of Theorem 3 are satisfied all oscillatory solutions of (21) tend to 0 as $t \to \infty$.

Remark (2). Theorem 3 and Example 3 show that the existence of the limit

$$\lim_{t\to\infty} \frac{|f(t)|}{a_1(t)}$$

is essential in Theorem 2. In fact, if all oscillatory solutions of (1) approach 0, then (19) in the proof of Theorem 2 shows that $\liminf (|f(t)|/a_1(t)) = 0$. In example 3 we see that $\dfrac{f(t)}{a_1(t)}$ is bounded, $\liminf\limits_{t\to\infty} \dfrac{|f(t)|}{a_1(t)} = 0$ but $\lim\limits_{t\to\infty} \dfrac{|f(t)|}{a_1(t)}$ does not exist.

Our next theorem gives sufficient conditions when oscillatory solutions do not have limits.

Theorem (4). *Suppose* $a(t) = a_1(t) + a_2(t)$, $a_1(t) > 0$ *and* $a_2(t)/a_1(t)$ *is bounded for large t. Further suppose that* $\liminf\limits_{t\to\infty} |f(t)|/a_1(t) > 0$. *Let* $y(t)$ *be an oscillatory solution of* (1). *Then* $\limsup\limits_{t\to\infty} |y(t)| > 0$.

Proof. Suppose to the contrary that $y(t) \to 0$ as $t \to \infty$. From (1), we get inequality (19)

$$\frac{|(r(t)\,y'(t))'|}{a_1(t)} \geqq \frac{f(t)}{a_1(t)} - (1 + k_1)\,|y(g(t))| \qquad \text{where } \frac{|a_2(t)|}{a_1(t)} \leqq k_1,$$

$k_1 > 0$. A contradiction is immediately reached, since $(r(t)\,y'(t))'$ assumes a constant sign. The proof is complete. The following example satisfies the conditions and conclusion of Theorem 4.

Example (4). Consider the equation

$$(22) \qquad y''(t) + y(t - 2\pi) = 1.$$

All oscillatory solutions of (22) satisfy $\limsup\limits_{t\to\infty} |y(t)| > 0$ since all conditions of theorem 4 are satisfied. In fact $y(t) = 1 + \cos t$ is one such solution.

Remark (3). Note that Theorem 4 does not require $\int^{\infty} 1/r(t) < \infty$.

Theorem (5). *Suppose* $a(t) = a_1(t) + a_2(t)$, $a_1(t) > 0$, $a_2(t)/a_1(t)$ *bounded for large* t, $\int^{\infty} a_1(t)\, dt < \infty$, $\int^{\infty} 1/r\, dt < \infty$, *and* $f(t)/a_1(t)$ *is bounded for large* t. *Then all solutions of* (1) *are nonoscillatory if* $\liminf_{t\to\infty} \frac{|f(t)|}{a_1(t)} > 0$.

Proof. It is easily seen that $\int^{\infty} |f(t)|\, dt < \infty$ and $\int^{\infty} |a(t)|\, dt < \infty$. Since $\int^{\infty} 1/r(t)\, dt < \infty$, by Theorem 1, all oscillatory solutions approach zero. Let now $y(t)$ be a solution of (1), If $y(t)$ is oscillatory then $y(t) \to 0$ as $t \to \infty$. From (1), we obtain (in a manner of inequality (19))

$$\frac{|(r(t)\, y'(t))'|}{a_1(t)} \geqq \frac{|f(t)|}{a_1(t)} - (1 + k_1)\,|y(g(t))|,$$

which clearly gives a contradiction by making $y(t)$ nonoscillatory.

Example (5). The equation

$$\left(\frac{1}{2} t^2 y'(t)\right)' + \frac{1 + \sin t}{t^2} y(t) = \frac{1}{t^2} + \frac{1 + \sin t}{t^4}, \tag{23}$$

has $y(t) = \frac{1}{t^2}$ a nonoscillatory solution. In fact, taking $a_1(t) = 1/t^2$, $a_2(t) = \sin t/t^2$, $r(t) = 1/2t^2$ and $f(t) = (t^2 + \sin t + 1)/t^4$ we find that all conditions of Theorem 5 hold. Hence all solutions of (23) are nonoscillatory.

Our next theorem generalizes Theorem 2.6 of Wallgren [14].

Theorem (6). *Suppose* $r(t)$ *is bounded,* $a(t) = a_1(t) + a_2(t)$, $a_1(t) > \varrho > 0$, $|a_2(t)|/a_1(t) \leqq k_1$, *for large* t *and* $\lim_{t\to\infty} |f(t)|/a_1(t) = \infty$. *Then all solutions of* (1) *are unbounded.*

Proof. From equation 1

$$\frac{(r(t)\, y'(t))'}{a_1(t)} + \left(1 + \frac{a_2}{a_1}\right) y(g(t)) = f(t)/a_1(t).$$

Thus

$$\frac{|(r(t)\, y'(t))'|}{a_1(t)} \geqq \frac{|f(t)|}{a_1(t)} - (1 + k_1)\,|y(g(t))|.$$

If $y(t)$ is bounded, then above inequality shows that $\frac{|(r(t)\, y'(t))'|}{a_1(t)} \to \infty$ as $t \to \infty$. Since $a_1(t) \geqq \varrho > 0$ we get $|(r(t)\, y'(t))'| \to \infty$ as $t \to \infty$. Since $r(t)$ is bounded $y'(t) \to \pm\infty$ as $t \to \infty$. The conclusion follows by contradiction.

Example (6). The equation

$$y''(t) + 2y(t - \pi/2)\, e^{\pi/2} = e^{\pi/2}\, 2(t - \pi/2), \tag{24}$$

has $y = e^t \sin t + t$ and $y = t$ as solutions. All conditions of Theorem 6 are satisfied.

Theorem (7). *Suppose* $\int^{\infty} | a(t) | \, dt < \infty$ and $\int^{\infty} f(t) \, dt = \infty$ *then all oscillatory solutions of* (1) *are unbounded.*

Proof. From (1) we get

$$r(t) y'(t) - r(T) y'(T) + \int_T^t a(s) y(g(s)) \, ds = \int_T^t f(s) \, ds. \tag{25}$$

If $y(t)$ is oscillatory and bounded then (25) yields

$$r(t) y'(t) - r(T) y'(T) + m \int_T^t | a(s) | \, ds \geqq \int_T^t f(s) \, ds, \tag{26}$$

where $| y(t) | \geqq m$ for $t \leqq T$. (26) readily leads to a contradiction which proves this theorem.

Example (7). The following equation shows that under the conditions of Theorem 7, bounded nonoscillatory solutions can exist.

$$(t^{5/2} y'(t))' + \frac{1}{t^2} y(t) = \frac{1}{2\sqrt{t}} - \frac{1}{t^3}, \tag{27}$$

has $y(t) = -1/t$ as a solution.

Theorem (8). *If under the hypothesis of theorem 7 we require* $r(t)$ *to be bounded, all other conditions being the same then all solutions of* (1) *are unbounded.*

Proof. We only need to prove it when $y(t)$ is nonoscillatory. Following the proof of Theorem 7, if $| y(t) | \leq m$ then (26) yields $r(t) y'(t) \to \infty$ as $t \to \infty$ and since $r(t)$ is bounded, we have $y'(t) \to \infty$ as $t \to \infty$ which forces $y(t)$ to be unbounded. The proof is now complete by contradiction.

4. EFFECT OF LARGE $r(t)$ AND NONOSCILLATION

Example (8). The equation

$$(e^t y'(t))' + e^{-2\pi} y(t - 2\pi) = e^{-t} \sin t - 3e^{-t} \cos t + e^{-2t} \sin t, \tag{29}$$

has $y = e^{-2t} \sin t$ as an oscillatory solution approaching zero. But this equation is not covered by Theorem 1 since $\int^{\infty} | a(t) | \, dt = \infty$. However, it will be shown by our next theorem, that all oscillatory solutions of (29) approach zero as $t \to \infty$. In fact, Theorem 9 measures the growth of solutions of (1) in terms of $r(t)$. By taking $r(t)$ large enough, the sizes of $a(t)$ and $f(t)$ can be compensated for. As an outcome of this approach, we observe that oscillatory trajectories of (1) eventually vanish if (1) has a nonoscillatory solution satisfying certain properties.

Theorem (9). *Suppose* $\int^{\infty} 1/r(t)\,\mathrm{d}t < \infty$,

(30) $$\int^{\infty} |a(x)| \left(\int_x^{\infty} 1/r(s)\,\mathrm{d}s\right) \mathrm{d}x < \infty$$

and

(31) $$\int^{\infty} |f(x)| \left(\int_x^{\infty} 1/r(s)\,\mathrm{d}s\right) \mathrm{d}x < \infty;$$

then all oscillatory solutions of (1) tend to zero asymptotically.

Proof. We proceed as in Theorem 1 with $y(t)$ as an oscillatory nonvanishing solution of (1), and arrive at conclusions (9) and (10) namely $|y(t)| \le M$ and $|y(g(t))| \le M$ for $t \in [x_1, x_2]$, $y(x_1) = y(x_2) = 0$. Let $x_0 \in [x_1, x_2]$ such that $M = |y(x_0)|$. Integrating (1) for $t \in [x_0, x_2]$, we have

(32) $$r(t)\,y'(t) + \int_{x_0}^{t} a(x)\,y(g(x))\,\mathrm{d}x = \int_{x_0}^{t} f(x)\,\mathrm{d}x,$$

since $y'(x_0) = 0$. Dividing (32) by $r(t)$ and integrating between $[x_0, x_2]$ we have

$$\pm M = -\int_{x_0}^{x_2} 1/r(t) \int_{x_0}^{t} a(x)\,y(g(x))\,\mathrm{d}x\,\mathrm{d}t + \int_{x_0}^{x_2} 1/r(t) \int_{x_0}^{t} f(x)\,\mathrm{d}x\,\mathrm{d}t,$$

which gives

$$M \leqq \int_{x_0}^{x_2} 1/r(t) \int_{x_0}^{t} |a(x)|\,|y(g(x))|\,\mathrm{d}x\,\mathrm{d}t + \int_{x_0}^{x_2} 1/r(t) \int_{x_0}^{t} |f(x)|\,\mathrm{d}x\,\mathrm{d}t.$$

Since $|y(g(t))| \le M$ for $t \in [x_0, x_2] \subset [x_1, x_2]$ we have

(33) $$1 \leqq \int_{x_0}^{x_2} \left(\int_x^{x_2} 1/r(s)\,\mathrm{d}s\right) |a(x)|\,\mathrm{d}x + \frac{1}{M} \int_{x_0}^{x_2} |f(x)| \left(\int_x^{x_2} 1/r(s)\,\mathrm{d}s\right) \mathrm{d}x,$$

where, in (33), the integrals have been rearranged by change of order of integration. Unless M becomes arbitrarily small, (33) leads to a contradiction.

Our next theorem highlights nonoscillation in obtaining some results about oscillatory solutions. We will need the equation

(34) $$(r(t)\,y'(t))' + a(t)\,y(g(t)) = 0.$$

Theorem (10). *Suppose equation* (34) *has a nonoscillatory solution* $y(t)$ *such that* $\operatorname{sgn}(y(t)) = \operatorname{sgn}(y'(t))$. *Further suppose that* $a(t) > 0$, $\int^{\infty} 1/r(t)\,\mathrm{d}t < \infty$ *and* $\int^{\infty} |f(x)| \left(\int_x^{\infty} 1/r(t)\,\mathrm{d}t\right) \mathrm{d}x < \infty$. *Then all oscillatory solutions of* (1) *approach zero as* $t \to \infty$.

Proof. Let T be large enough so that for $t \geqq T$, $y(t)$ and $y(g(t))$ are of the same sign. Without any loss of generality, we can assume that for $t \geqq T$ we have

(35) $$y(t) > 0,\ y(g(t)) > 0,\ y'(t) > 0 \quad \text{and} \quad y'(g(t)) > 0.$$

Dividing by $y(g(t))$ and integrating between T and t we have

$$(36)\qquad \frac{r(t)\,y'(t)}{y(g(t))} - \frac{r(T)\,y'(T)}{y(g(T))} + \int_T^t \frac{r(x)\,y'(x)\,y'(g(x))\,g'(x)\,\mathrm{d}x}{y^2(g(x))} + $$
$$+ \int_T^t a(x)\,\mathrm{d}x = 0.$$

(36) yields on further manipulation

$$\int_T^t \frac{y'(s)}{y(g(s))}\,\mathrm{d}s - \frac{r(T)\,y'(T)}{y(g(T))} \int_T^t 1/r(s)\,\mathrm{d}s + \int_T^t \frac{1}{r(t)} \int_T^s \frac{r(x)\,y'(x)\,y'(g(x)\,g'(x)\,\mathrm{d}x\,\mathrm{d}s}{y^2(g(x))} =$$
$$(37)\qquad = -\int_T^t 1/r(s) \int_T^s a(x)\,\mathrm{d}x\,\mathrm{d}s.$$

From (35), the first and third term on the left are positive; the second term is finite. Since the first term on the right hand side is negative, we arrive at the conclusion

$$\lim_{t\to\infty} \int_T^t 1/r(s) \int_T^s a(x)\,\mathrm{d}x\,\mathrm{d}s = \lim_{t\to\infty} \int_T^t \left(\int_x^t 1/r(s)\,\mathrm{d}s\right) a(x)\,\mathrm{d}x < \infty.$$

The proof is now complete by the application of Theorem 9.

The following example gives an application of this theorem.

Example (9). The equation

$$(38)\qquad (e^{t/2}y'(t))' + \frac{e^{t/2}}{2(e^t - 1)}\,y(t) = 0,$$

has $y(t) = 1 - e^{-t}$ as a nonoscillatory solution. Hence all oscillatory solutions of

$$(39)\qquad (e^{t/2}y'(t))' + \frac{e^{t/2}}{2(e^t-1)}\,y(t) = 4e^{-2t}\sin t - \frac{9}{2}e^{-2t}\cos t + \frac{e^{-2t}}{2(e^t-1)}\sin t$$

approach zero. In fact $y = (e^{-5/2\,t}\sin t)$ is one such solution of (39). It is easily verified that all conditions of Theorem 10 are satisfied. We also note that all conditions of Theorem 9 are satisfied. Indeed, Theorem 10 is a recapitulation of Theorem 9 in terms of the nonoscillatory solutions of the homogeneous part of (1).

Example 9 suggests the following theorem.

Theorem (11). *Suppose* (1) *has a nonoscillatory solution* $y(t)$ *such that* $\operatorname{sgn}(y(t)) = \operatorname{sgn}(y'(t))$. *Further suppose that* $a(t) > 0$, $\int^\infty 1/r(t)\,\mathrm{d}t < \infty$ *and* $\int^\infty f(x)\,|\,(\int^\infty 1/r(t)\,\mathrm{d}t)\mathrm{d}x < 0$. *Then all oscillatory solutions of* (1) *approach zero as* $t \to \infty$.

Proof. We proceed as in Theorem 10 and arrive at conclusion (35). Dividing (1) by $y(g(t))$ and integrating we get

$$\frac{r(t)\,y'(t)}{y(g(t))} - \frac{r(T)\,y'(T)}{y(g(T))} + \int_T^t \frac{r(x)\,y'(x)\,y'(g(x)\,g'(x)\,dx}{y^2(g(x))} +$$
$$+ \int_T^t a(x)\,dx = \int_T^t \frac{f(x)}{y(g(x))}\,dx.$$

Since $y(g(x)) > y(g(T)) > 0$, further integration yields

$$\int_T^t \frac{y'(s)}{y(g(s)}\,ds - \frac{r(T)\,y'(T)}{y(g(T))}\int_T^t 1/r(s)\,ds + \int_T^t 1/r(s)\int_T^s \frac{r(x)\,y'(x)\,y'(g(x))\,g'(x)\,dx\,ds}{y^2(g(x))} \leq$$

(40) $$\leq \frac{1}{y(g(T))}\int_T^t 1/r(s)\int_T^s |\,f(x)\,|\,dx\,ds.$$

In view of (35) and condition on $f(x)$, (39) yields

$$\lim_{t\to\infty}\int_T^t \frac{1}{r(t)}\int_T^s a(x)\,dx\,ds = \int_T^\infty a(x)\,[\int_x^\infty 1/r(s)\,ds]\,dx < \infty.$$

The conclusion follows by Theorem 9.

Example (10). In equation (39), the nonoscillatory solution $y(t) = 1 - e^{-t} + + e^{-5/2\,t}\sin t$ for sufficiently large t satisfies the requirements of this theorem.

Theorem (12). *Suppose* $\int^\infty 1/r(t)\,dt < \infty$ *and there exist nonnegative functions* $H_1(t)$, $H_2(t)$ *such that* $\operatorname{sgn}(H_i(t)) = \operatorname{sgn}(H_i'(t))$, $i = 1, 2$. *Further suppose that* H_1 *and* H_2 *satisfy*

(41) $$(r(t)\,H_1'(t))' + |\,a(t)\,|\,H_1(g(t)) \leq 0,$$

(42) $$(r(t)\,H_2'(t))' + |\,f(t)\,|\,H_2(g(t)) \leq 0.$$

Then all oscillatory solutions of (1) *approach zero as* $t \to \infty$.

Proof. Following identically the proof of Theorem 10 we obtain (cf. this author [7, Theorem 2])

$$\int^\infty |\,a(x)\,|\int_x^\infty 1/r(t)\,dt\,dt < \infty$$

and

$$\int^\infty |\,f(x)\,|\int_x^\infty 1/r(t)\,dt\,dx < \infty,$$

which are the conditions of Theorem 9.

Our next theorem gives an alternative version of Theorem 4.

Theorem (13). *Suppose* $\liminf_{t\to\infty}\int^t (f(t) - |\,a(t)\,|)\,dt > 0$. *Then any oscillatory solution* $y(t)$ *of* (1) *satisfies* $\limsup_{t\to\infty} |\,y(t)\,| > 0$.

Proof. Let $y(t)$ be an oscillatory solution. Then $y'(t)$ is oscillatory. Let T be large enough so that $y'(T) = 0$. From (1)

$$r(t)\,y'(t) + \int_T^t |a(x)|\,|y(g(x))|\,dx \geqq \int_T^t f(x)\,dx. \tag{43}$$

Suppose to the contrary that $y(t) \to 0$ as $t \to \infty$. Without any loss we can assume that T is large enough so that for $t \geqq T$, $|y(g(t))| \leq 1$. From (43) and this fact

$$(r(t)\,y'(t)) \geqq (\liminf_{t\to\infty} \int_T^t (f(x) - |a(x)|)\,dx) > 0. \tag{44}$$

But (44) implies that $y'(t)$ is eventually positive and $y(t)$ is nonoscillatory. This contradiction completes the proof.

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ALGEBRAIC THEORY OF FAST MIXED-RADIX TRANSFORMS: I. GENERALIZED KRONECKER PRODUCT OF MATRICES

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Abstract. A new operation over matrices is introduced which is a generalization of the Kronecker (direct) product and its basic properties are derived. It is shown that matrices formed in this way define a class of the so called fast mixed-radix transforms as a natural generalization of the mixed-radix fast Fourier transforms. The new operation allows a straightforward and simple derivation of the appropriate factorization associated with the fast algorithm. The paper will be continued.

Key words. Generalized Kronecker product of matrices, fast mixed-radix transform, fast Fourier transform, factorization of matrices.

MS Classification: 15 A 23, 15 A 04, 68 Q 25, 65 F 30, 65 T 05.

INTRODUCTION

Linear transforms $\mathbf{x} \to \mathbf{y} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ denotes a fixed matrix and $\mathbf{x}$ and $\mathbf{y}$ are data vectors of appropriate sizes, are widely used in various applications. Multiplication of a vector $\mathbf{x}$ by the matrix $\mathbf{A}$ may become a crucial operation on a computer if many such transforms are to be accomplished and/or $\mathbf{A}$ is a large matrix with many non-zero elements. In such a case it is desirable to find for the given matrix $\mathbf{A}$ a "fast" algorithm that reduces the amount of scalar multiplications and additions accomplishing $\mathbf{A}\mathbf{x}$. One is usually profiting from the knowledge of the concrete structure of $\mathbf{A}$ to find such a factorization $\mathbf{A} = \mathbf{A}^{(m)}\mathbf{A}^{(m-1)} \dots \mathbf{A}^{(1)}$ into sparse matrices $\mathbf{A}^{(i)}$ that $\mathbf{A}^{(i)}\mathbf{x}^{(i-1)}$ may be viewed with $\mathbf{x} = \mathbf{x}^{(0)}$ and $\mathbf{y} = \mathbf{x}^{(m)}$ as the i-th step ($i = 1, 2, \dots m$) of a fast algorithm. Product of such matrices is said to be a fast (linear) transform.

The above approach is typical in the field of digital signal processing [1–5, 7, 8], where the mostly used transforms are orthogonal [3]. Chief among them is the *discrete Fourier transform* (DFT). A fast algorithm computing DFT is called *fast Fourier transform* (FFT). Discussion of various commonly used FFTs may be found e.g. in [1–4, 7].

I. J. Good [5] shows that the structure of the multidimensional FFT is characteristic for a class of linear transforms, the matrices of which may be expressed as Kronecker (direct) product [6], i.e. $\mathbf{A} = \mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \ldots \otimes \mathbf{A}_m$. Then it is easy to see that $\mathbf{A}^{(i)} = \mathbf{I}_1 \otimes \ldots \otimes \mathbf{I}_{i-1} \otimes \mathbf{A}_i \otimes \mathbf{I}_{i+1} \otimes \ldots \otimes \mathbf{I}_m$ defines the i-th step of the corresponding fast algorithm ($\mathbf{I}_j$ denotes identity matrices of appropriate sizes) and thus Kronecker product is a typical operation forming matrices of this class of (fast) transforms. Similarly another class of linear transforms may be based on the structure of another FFT, the so called mixed-radix FFT. I. J. Good develops in [5] the appropriate factors $\mathbf{A}^{(i)}$ and illustrates a close relationship between both classes of fast transforms. Hereafter we shall call transforms of the latter class *mixed-radix transforms* (MRTs) and the corresponding fast algorithms *fast mixed-radix transforms* (FMRTs).

There arises a natural question whether one can find a simple algebraic operation over matrices typical for MRTs and having properties admitting the derivation of factors $\mathbf{A}^{(i)}$ of FMRT by simple and easy algebraic manipulations so as this is in the case of the Kronecker product.

This paper gives a positive answer to this question. In Sect. 2 we define in two ways a new operation over matrices which may be viewed as a generalization of the Kronecker product. Several basic algebraic properties of this generalized Kronecker product are proved which allow the desired easy derivation of the FMRTs.

1. NOTATION AND INTRODUCTORY REMARKS

1.1 Notation

- $\mathbf{N}$ ... the set of natural numbers.
- $\mathbf{Z}$ ... the set of integers.
- $\mathbf{Z}_N = \{0, 1, \ldots, N-1\}$, $N \in \mathbf{N}$.
- $\mathbf{C}$ ... the field of complex numbers.
- $\mathbf{R}$ ... an arbitrary associative and commutative ring with unity, all matrices and vectors mentioned later on are over $\mathbf{R}$ if not stated otherwise.
- If $\mathbf{A}$ is a matrix of size $N \times K$ ($N, K \in \mathbf{N}$), then we shall denote $A(n, k)$ its entry in $(n+1)$-th row and $(k+1)$-th column, $n \in \mathbf{Z}_N$, $k \in \mathbf{Z}_K$. The set of all matrices of size $N \times K$ will be denoted as $\mathcal{M}(N \times K)$. We write $\mathbf{A} = (\mathbf{A}^{n_1,k_1})$, $\mathbf{A}^{n_1,k_1} \in \mathcal{M}(N_2 \times K_2)$, $n_1 \in \mathbf{Z}_{N_1}$, $k_1 \in \mathbf{Z}_{K_1}$ for a matrix $\mathbf{A}$ which is structured into $N_1 \times K_1$ blocks $\mathbf{A}^{n_1,k_1}$ of size $N_2 \times K_2$ ($N = N_1N_2$, $K = K_1K_2$), $n_1 + 1$ is the row position and $k_1 + 1$ the column position of the block $\mathbf{A}^{n_1,k_1}$.
- $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1})^T$, $N \in \mathbf{N}$ denotes a column vector of length N, (^{T} is transposition).

– $|\mathbf{A}|$ ... determinant of a square matrix $\mathbf{A}$.
– $\mathbf{I}_N$ ... identity matrix of order N.
– $[i:j] = \{k \mid i \leq k \leq j,\ k \in \mathbf{Z}\}$, $i, j \in \mathbf{Z}$, $i \leq j$.
– Let $N_k \in \mathbf{N}$ for $k \in [i:j]$, then $N_{i,j} = N_i N_{i+1} \dots N_j$ if $i \leq j$ and $N_{i,j} = 1$ otherwise.
– $\delta_{i,j}$, $\delta(i,j)$ ... Kronecker's symbol.
– $n \mid m$ ... integer n is a divisor of integer m.
– $\mathscr{P}(M)$ ... permutation group of the set M.
We shall not distinguish between a permutation $P \in \mathscr{P}(\mathbf{Z}_N)$ and the corresponding matrix $\mathbf{P} \in \mathscr{M}(N \times N)$, $P(n,k) = \delta_{n,P(k)}$.

1.2 Definition. A mapping $\mathscr{N}\colon [i:j] \to \mathbf{N}$ is said to be a (finite) *number system* (NS). We shall write also $\mathscr{N} = (N_i, N_{i+1}, \dots, N_j)$ to visualize the function values $\mathscr{N}(k) = N_k$ for $k \in [i:j]$. Alternatively the notation $\mathscr{N}_{i,j}$ will be used instead of $\mathscr{N}$ to emphasize the index domain $[i:j]$.

1.3 Remark. Combining a NS $\mathscr{N}_{i,j}$ with a permutation $p \in \mathscr{P}([i:j])$, we arrive at a permuted NS $\mathscr{N}_{i,j}p = (N_{p(i)}, N_{p(i+1)}, \dots, N_{p(j)})$.

1.4 Lemma. *Let $\mathscr{N} = (N_1, N_2, \dots, N_m)$ be a number system associated with $N = N_{1,m}$. Then the mapping $[.]_{\mathscr{N}}\colon \mathbf{Z}_{N_1} \times \mathbf{Z}_{N_2} \times \dots \times \mathbf{Z}_{N_m} \to \mathbf{Z}_N$ defined as $[n_1, n_2, \dots, n_m]_{\mathscr{N}} = n_1 N_{2,m} + n_2 N_{3,m} + \dots + n_{m-1} N_m + n_m = n$ is a bijection.*

Proof. We proceed by induction on m. For $m = 1$ $[.]_{\mathscr{N}}$ is an identical mapping. Let $m > 1$. Clearly $n = kN_m + n_m$ with $k = [n_1, \dots, n_{m-1}]_{\mathscr{N}'}$, and $\mathscr{N}' = (N_1, N_2, \dots, N_{m-1})$. By induction hypothesis $0 \leq k \leq N_{1,m-1} - 1 \Rightarrow 0 \leq kN_m + n_m \leq N - N_m + n_m \leq N - 1 \Rightarrow n \in \mathbf{Z}_N$. $[.]_{\mathscr{N}}$ is injective: $n = n' = [n'_1, n'_2, \dots, n'_{m-1}]_{\mathscr{N}'} N_m + n'_m \Rightarrow N_m \mid (n_m - n'_m) \Rightarrow n_m = n'_m$ in view of $0 \leq |n_m - n'_m| \leq N_m - 1$. Hence $[n_1, n_2, \dots, n_{m-1}]_{\mathscr{N}'} = [n'_1, n'_2, \dots, n'_{m-1}]_{\mathscr{N}'}$ and by induction hypothesis $n_i = n'_i$ for each $i \in [1:m-1]$. ∎

1.5 Definition. The ordered m-tuple $(n_1, n_2, \dots, n_m)$ is called a *mixed-radix integer representation* of $n = [n_1, n_2, \dots, n_m]_{\mathscr{N}}$ with respect to the number system $\mathscr{N}$.

Hereafter we shall omit the subscript $\mathscr{N}$ and write simply $[n_1, n_2, \dots, n_m]$ whenever the NS is implicitely determined from the context. In particular the NS $\mathscr{N} = (N_1, N_2, \dots, N_m)$ associated with the factorization $N = N_{1,m}$ is assumed if not stated otherwise.

1.6 Lemma. *Let $N = N_{1,m}$, $m \geqslant 2$. Then for each $i \in [1:m-1]$ it holds $[[n_1, n_2, \dots, n_i], [n_{i+1}, n_{i+2}, \dots, n_m]] = [n_1, n_2, \dots, n_m]$.*

Proof. $[n_1, \ldots, n_i] \in \mathbf{Z}_{N_{1,i}}, [n_{i+1}, \ldots, n_m] \in \mathbf{Z}_{N_{i+1,m}}, N = N_{1,i}N_{i+1,m} \Rightarrow$ $\Rightarrow [[n_1, \ldots, n_i], [n_{i+1}, \ldots, n_m]] = [n_1, \ldots, n_i]\, N_{i+1,m} + [n_{i+1}, \ldots, n_m] =$ $= (n_1N_{2,i} + n_2N_{3,i} + \ldots + n_i)\, N_{i+1,m} + n_{i+1}N_{i+2,m} + \ldots + n_m = [n_1, n_2, \ldots, \ldots, n_m]$. ∎

1.7 Definition. Let us have a NS $\mathcal{N} = (N_i, \ldots, N_j)$ and $N = N_{i,j}$. We define a mapping $\varphi_{\mathcal{N}} : \mathscr{P}([i:j]) \to \mathscr{P}(\mathbf{Z}_N)$ as follows:
$\varphi_{\mathcal{N}}(p) = P$, where $P([n_i, \ldots, n_j]_{\mathcal{N}}) = [n_{p(i)}, \ldots, n_{p(j)}]_{\mathcal{N}p}$.
It holds $\varphi_{\mathcal{N}}(1) = \mathbf{I}_N$ (here 1 is the identical permutation in $\mathscr{P}([i:j])$). But in general $\varphi_{\mathcal{N}}$ is not a homomorphism of permutation groups, e.g. $N_1 = 2$, $N_2 = 3$, $p(1) = 2$, $p(2) = 1$ is a counter-example.

1.8 Lemma. *Let* $\mathbf{A}_i \in \mathscr{M}(N_i \times K_i)$ *for* $i \in [1:m]$, $m \geqq 2$, $N = N_{1,m}$, $K = K_{1,m}$, $\mathcal{N} = (N_1, \ldots, N_m)$ *and* $\mathcal{K} = (K_1, \ldots, K_m)$. *If we put* $\mathbf{A} = \mathbf{A}_1 \otimes \ldots \otimes \mathbf{A}_m$, $\mathbf{A}_p = \mathbf{A}_{p(1)} \otimes \ldots \otimes \mathbf{A}_{p(m)}$, $P_{\mathcal{N}} = \varphi_{\mathcal{N}}(p)$ *and* $P_{\mathcal{K}} = \varphi_{\mathcal{K}}(p)$ *for an arbitrary permutation* $p \in \mathscr{P}([1:m])$, *then it holds* $\mathbf{A}_p = \mathbf{P}_{\mathcal{N}}\mathbf{A}\mathbf{P}_{\mathcal{K}}^T$, *or equivalently* $A_p(P_{\mathcal{N}}(n), P_{\mathcal{K}}(k)) = A(n, k)$ *for each* $n \in \mathbf{Z}_N$ *and* $k \in \mathbf{Z}_K$.

Proof. $A_p(P_{\mathcal{N}}([n_1, \ldots, n_m]), P_{\mathcal{K}}([k_1, \ldots, k_m])) = A_p([n_{p(1)}, \ldots, n_{p(m)}]_{\mathcal{N}p}, [k_{p(1)}, \ldots, k_{p(m)}]_{\mathcal{K}p}) = A_{p(1)}(n_{p(1)}, k_{p(1)}) \ldots A_{p(m)}(n_{p(m)}, k_{p(m)}) = A_1(n_1, k_1) \ldots \ldots A_m(n_m, k_m) = A([n_1, \ldots, n_m], [k_1, \ldots, k_m])$ in view of commutativity of multiplication in the ring $\mathbf{R}$. ∎

1.9 Convention. Later on we shall agree on the following notation: $p_{i,j}$ and $1_{i,j}$ stand for an arbitrary and identical permutation, respectively belonging to $\mathscr{P}([i:j])$; $s_{i,j} \in \mathscr{P}([i:j])$ denotes a permutation defined by $s_{i,j}(i + k) = j - k$, $k \in [0:j-i]$. Similarly $P_{i,j} = \varphi_{\mathcal{N}_{i,j}}(p_{i,j})$, $\mathbf{I}_{N_{i,j}} = \varphi_{\mathcal{N}_{i,j}}(1_{i,j})$ and $S_{i,j} = \varphi_{\mathcal{N}_{i,j}}(s_{i,j})$ are the associated permutations belonging to $\mathscr{P}(\mathbf{Z}_{N_{i,j}})$. $S_{i,j}$ is called the *digit reversal* with respect to the NS $\mathcal{N}_{i,j}$. Subscripts i, j may be omitted whenever $i = 1$ and $j = m$. We shall write also $S_{\mathcal{N}}$ to emphasize that $S_{\mathcal{N}}$ is the digit reversal with respect to $\mathcal{N}$.

1.10 Theorem. *Let* $\mathcal{N} = (N_1, \ldots, N_m)$, $m \geqq 2$ *and* $p = p_{1,i} \cup p_{i+1,m} \in \mathscr{P}([1:m])$ *for some* $i \in [1:m-1]$. *Then* $\varphi_{\mathcal{N}}(p) = \mathbf{P} = \mathbf{P}_{1,i} \otimes \mathbf{P}_{i+1,m}$.

Proof. We are going to verify $\mathbf{P} = \tilde{\mathbf{P}}$ where $\tilde{\mathbf{P}} = \mathbf{P}_{1,i} \otimes \mathbf{P}_{i+1,m}$. Let $n = [n_1, \ldots, n_m]$, $k = [k_1, \ldots, k_m] \in \mathbf{Z}_{N_{1,m}}$ be arbitrary. Using 1.6 we get $\tilde{P}(n, k) = \tilde{P}([[n_1, \ldots, n_i], [n_{i+1}, \ldots, n_m]], [[k_1, \ldots, k_i], [k_{i+1}, \ldots, k_m]]) = P_{1,i}([n_1, \ldots, \ldots, n_i], [k_1, \ldots, k_i])\, P_{i+1,m}([n_{i+1}, \ldots, n_m], [k_{i+1}, \ldots, k_m]) = \delta([n_1, \ldots, n_i], [k_{p_{1,i}(1)}, \ldots, k_{p_{1,i}(i)}])\, \delta([n_{i+1}, \ldots, n_m], [k_{p_{i+1,m}(i+1)}, \ldots, k_{p_{i+1,m}(m)}]) = \delta([n_1, \ldots, \ldots, n_m], [k_{p(1)}, \ldots, k_{p(m)}]) = \delta_{n,P(k)} = P(n, k)$. ∎

1.11 Corollary. *Let* $p_1 = p_{1,i} \cup 1_{i+1,m}$ *and* $p_2 = 1_{1,i} \cup p_{i+1,m}$ *then* $p = p_{1,i} \cup \cup p_{i+1,m} = p_1 p_2 = p_2 p_1$ *and* $P = \varphi_{\mathcal{N}}(p) = \varphi_{\mathcal{N}}(p_1)\, \varphi_{\mathcal{N}}(p_2) = \varphi_{\mathcal{N}}(p_2)\, \varphi_{\mathcal{N}}(p_1)$ *where* $\varphi_{\mathcal{N}}(p_1) = \mathbf{P}_{1,i} \otimes \mathbf{I}_{N_{i+1,m}}$, $\varphi_{\mathcal{N}}(p_2) = \mathbf{I}_{N_{1,i}} \otimes \mathbf{P}_{i+1,m}$.

Proof. $\mathbf{P} = (\mathbf{P}_{1,i} \otimes \mathbf{I}_{N_{i+1,m}})\, (\mathbf{I}_{N_{1,i}} \otimes \mathbf{P}_{i+1,m}) = (\mathbf{I}_{N_{1,i}} \otimes \mathbf{P}_{i+1,m})\, (\mathbf{P}_{1,i} \otimes \otimes \mathbf{I}_{N_{i+1,m}})$ is a well-known property of $\otimes$. The factors are equal to $\varphi_{\mathcal{N}}(p_1)$ and $\varphi_{\mathcal{N}}(p_2)$ due to 1.10 and by $\varphi_{\mathcal{N}_{1,i}}(1_{1,i}) = \mathbf{I}_{N_{1,i}}$ and $\varphi_{\mathcal{N}_{i+1,m}}(1_{i+1,m}) = \mathbf{I}_{N_{i+1,m}}$. ∎

1.12 Corollary. *Let* $i \in [1 : m-1]$, $m \geqq 2$ *be arbitrary and* $\mathbf{S}_i = \varphi_{(N_{1,i},\, N_{i+1,m})}(s)$. *Then it holds* $\varphi_{\mathcal{N}}(s_{1,m}) = \mathbf{S} = \mathbf{S}_i(\mathbf{S}_{1,i} \otimes \mathbf{S}_{i+1,m}) = (\mathbf{S}_{i+1,m} \otimes \mathbf{S}_{1,i})\, \mathbf{S}_i$.

Proof. It is sufficient to show $\mathbf{S} = \mathbf{S}_i\mathbf{P}$ with $\mathbf{P} = \varphi_{\mathcal{N}}(p)$, $p = s_{1,i} \cup s_{i+1,m}$. For each $n = [n_1, \ldots, n_m] \in \mathbf{Z}_{N_{1,m}}$ we can write in view of 1.6 $S_iP(n) = S_iP([n_1, \ldots, n_m]) = S_i([n_{p(1)}, \ldots, n_{p(m)}]) = S_i([n_i, n_{i-1}, \ldots, n_1, n_m, n_{m-1}, \ldots, \ldots, n_{i+1}]) = S_i([[n_i, \ldots, n_1], [n_m, \ldots, n_{i+1}]]) = [[n_m, \ldots, n_{i+1}], [n_i, \ldots, n_1]] = [n_m, \ldots, n_1] = S(n)$. Then $\mathbf{P} = \mathbf{S}_{1,i} \otimes \mathbf{S}_{i+1,m}$ by 1.10 and also $\mathbf{S} = \mathbf{S}_i\mathbf{P}\mathbf{S}_i^T\mathbf{S}_i$ where $\mathbf{S}_i\mathbf{P}\mathbf{S}_i^T = \mathbf{S}_{i+1,m} \otimes \mathbf{S}_{1,i}$ by 1.8. ∎

2. GENERALIZED KRONECKER PRODUCT OF MATRICES

By definition, the Kronecker product $\mathbf{A} = \mathbf{A}_1 \otimes \mathbf{A}_2$, $\mathbf{A}_1 \in \mathcal{M}(N_1 \times K_1)$, $\mathbf{A}_2 \in \mathcal{M}(N_2 \times K_2)$ is a matrix having block form $\mathbf{A} = (\mathbf{A}^{n_1,k_1}) \in \mathcal{M}(N \times K)$, $N = N_1 N_2$, $K = K_1 K_2$ where for each $n_1 \in \mathbf{Z}_{N_1}$ and $k_1 \in \mathbf{Z}_{K_1}$

$$\mathbf{A}^{n_1,k_1} = A_1(n_1, k_1)\, \mathbf{A}_2 . \tag{2.1}$$

Clearly, either of the following two equations is equivalent to (2.1):

$$\mathbf{A}^{n_1,k_1} = \mathbf{A}_2 \overrightarrow{\mathbf{A}}_1^{n_1,k_1},\ \overrightarrow{\mathbf{A}}_1^{n_1,k_1} = = \operatorname{diag}(A_1(n_1, k_1), \ldots, A_1(n_1, k_1)) \in \mathcal{M}(K_2 \times K_2), \tag{2.2}$$

$$\mathbf{A}^{n_1,k_1} = \overleftarrow{\mathbf{A}}_1^{n_1,k_1} \mathbf{A}_2,\ \overleftarrow{\mathbf{A}}_1^{n_1,k_1} = = \operatorname{diag}(A_1(n_1, k_1), \ldots, A_1(n_1, k_1)) \in \mathcal{M}(N_2 \times N_2). \tag{2.3}$$

Allowing different elements to enter into the diagonal of $\overrightarrow{\mathbf{A}}_1^{n_1,k_1}$ or $\overleftarrow{\mathbf{A}}_1^{n_1,k_1}$, a Kronecker product generalized in two ways may be obtained according to the following definition.

2.1 Definition. *Generalized Kronecker product of matrices.*

Let $N = N_1 N_2$, $K = K_1 K_2$, $\mathbf{A}_1 \in \mathcal{M}(N_1 \times K_1 K_2)$, $\mathbf{A}_2 \in \mathcal{M}(N_2 \times K_2)$, $\mathbf{B}_1 \in \mathcal{M}(N_1 N_2 \times K_1)$ and $\mathbf{B}_2 \in \mathcal{M}(N_2 \times K_2)$. Then the matrix $\mathbf{A} = \mathbf{A}_1 \otimes_R \mathbf{A}_2 \in \mathcal{M}(N \times K)$ ($\mathbf{B} = \mathbf{B}_1 \otimes_L \mathbf{B}_2 \in \mathcal{M}(N \times K)$) is said to be a right (left) generalized Kronecker product of matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ ($\mathbf{B}_1$ and $\mathbf{B}_2$) if $A([n_1, n_2], [k_1, k_2]) = A_1(n_1, [k_1, k_2])\, A_2(n_2, k_2)$ and $B([n_1, n_2], [k_1, k_2]) = B_1([n_1, n_2], k_1)\, B_2(n_2, k_2)$ holds for each $n_i \in \mathbf{Z}_{N_i}$ and $k_i \in Z_{K_i}$ with $i = 1, 2$.

Clearly, $\mathbf{A} = (\mathbf{A}^{n_1,k_1})$ where

$$\mathbf{A}^{n_1,k_1} = \mathbf{A}_2 \overrightarrow{\mathbf{A}}_1^{n_1,k_1},$$

(2.4) $$\overrightarrow{\mathbf{A}}_1^{n_1,k_1} = \operatorname{diag}(A_1(n_1, [k_1, 0]), A_1(n_1, [k_1, 1]), \ldots, \ldots, A_1(n_1, [k_1, K_2 - 1]))$$

and $\mathbf{B}_2 = (\mathbf{B}^{n_1,k_1})$ where

$$\mathbf{B}^{n_1,k_1} = \overleftarrow{\mathbf{B}}_1^{n_1,k_1}\mathbf{B}_2,$$

(2.5) $$\overleftarrow{\mathbf{B}}^{n_1,k_1} = \operatorname{diag}(B_1([n_1, 0], k_1), B_1([n_1, 1], k_1), \ldots, \ldots, B_1([n_1, N_2 - 1], k_1)).$$

2.2 Remark. Kronecker product $\otimes$ may be considered as a special case of both $\otimes_R$ and $\otimes_L$ writing instead of $\mathbf{A} = \mathbf{A}_1 \otimes \mathbf{A}_2$ either $\mathbf{A} = \mathbf{A}_{1,R} \otimes_R \mathbf{A}_2$ or $\mathbf{A} = \mathbf{A}_{1,L} \otimes_L \mathbf{A}_2$ where $A_{1,R}(n_1, [k_1, k_2]) = A_{1,L}([n_1, n_2], k_1) = A_1(n_1, k_1)$.

2.3 Lemma. *For* $\mathbf{A}_1 \in \mathcal{M}(N_1 \times K_1K_2)$ *and* $\mathbf{B}_1 \in \mathcal{M}(N_1N_2 \times K_1)$ *it holds* $\mathbf{A}_1 \otimes_R \mathbf{I}_{K_2} = \overrightarrow{\mathbf{A}}_1 = (\overrightarrow{\mathbf{A}}_1^{n_1,k_1})$ *and* $\mathbf{B}_1 \otimes_L \mathbf{I}_{N_2} = \overleftarrow{\mathbf{B}}_1 = (\overleftarrow{\mathbf{B}}_1^{n_1,k_1})$ *where* $\overrightarrow{\mathbf{A}}_1^{n_1,k_1}$ *and* $\overleftarrow{\mathbf{B}}_1^{n_1,k_1}$ *are diagonal matrices* of (2.4) *and* (2.5), *respectively. Moreover* $\mathbf{S}_{(N_1,K_2)}\overrightarrow{\mathbf{A}}_1\mathbf{S}^T_{(K_1,K_2)} = \operatorname{diag}(\mathbf{A}_{1,0}, \mathbf{A}_{1,1}, \ldots, \mathbf{A}_{1,K_2-1})$ *and* $\mathbf{S}_{(N_1,N_2)}\overleftarrow{\mathbf{B}}_1\mathbf{S}^T_{(K_1,N_2)} = \operatorname{diag}(\mathbf{B}_{1,0}, \mathbf{B}_{1,1}, \ldots, \mathbf{B}_{1,N_2-1})$ *where* $\mathbf{A}_{1,k_2}, \mathbf{B}_{1,n_2} \in \mathcal{M}(N_1 \times K_1)$, $A_{1,k_2}(n_1, k_1) = A_1(n_1, [k_1, k_2])$ *and* $B_{1,n_2}(n_1, k_1) = B_1([n_1, n_2], k_1)$ *for each* $n_i \in \mathbf{Z}_{N_i}$ *and* $k_i \in \mathbf{Z}_{K_i}$, $i = 1, 2$.

Proof. By definition 2.1, $\overrightarrow{A}_1([n_1, k_2'], [k_1, k_2]) = A_1(n_1, [k_1, k_2])\, \delta_{k_2',k_2}$ is the element positioned in $(k_2' + 1)$-th row and $(k_2 + 1)$-th column of the block $\overrightarrow{\mathbf{A}}_1^{n_1,k_1}$, which says that $\overrightarrow{\mathbf{A}}_1^{n_1,k_1}$ is exactly the diagonal matrix of (2.4). At the same time it is the element in $([k_2', n_1] + 1)$-th row and $([k_2, k_1] + 1)$-th column of $\mathbf{S}_{(N_1,K_2)}\overrightarrow{\mathbf{A}}_1\mathbf{S}^T_{(K_1,K_2)}$, which means that the only non-zero blocks of size $N_1 \times K_1$ are those with $k_2 = k_2'$, i.e. $A_1(n_1, [k_1, k_2])$ is the element in $(n_1 + 1)$-th row and $(k_1 + 1)$-th column of $(k_2 + 1)$-th diagonal block $\mathbf{A}_{1,k_2}$. For $\mathbf{B}_1$ is the argumentation analogical. ∎

2.4 Theorem. *Duality principle.*

Under assumptions of definition 2.1 *it holds* $(\mathbf{A}_1 \otimes_R \mathbf{A}_2)^T = \mathbf{A}_1^T \otimes_L \mathbf{A}_2^T$ and $(\mathbf{B}_1 \otimes_L \mathbf{B}_2)^T = \mathbf{B}_1^T \otimes_R \mathbf{B}_2^T$.

Proof. $\mathbf{A} = \mathbf{A}_1 \otimes_R \mathbf{A}_2 \Rightarrow A^T([k_1, k_2], [n_1, n_2]) = A([n_1, n_2], [k_1, k_2]) = A_1(n_1, [k_1, k_2])\, A_2(n_2, k_2) = A_1^T([k_1, k_2], n_1)\, A_2^T(k_2, n_2) \Rightarrow \mathbf{A}^T = \mathbf{A}_1^T \otimes_L \mathbf{A}_2^T$. $(\mathbf{B}^T)^T = \mathbf{B} = \mathbf{B}_1 \otimes_L \mathbf{B}_2 = (\mathbf{B}_1^T)^T \otimes_L (\mathbf{B}_2^T)^T = (\mathbf{B}_1^T \otimes_R \mathbf{B}_2^T)^T \Rightarrow \mathbf{B}^T = \mathbf{B}_1^T \otimes_R \mathbf{B}_2^T$. ∎

We shall prove some basic properties of $\otimes_R$ and $\otimes_L$ analogical to those of the ordinary Kronecker product $\otimes$ (cf. [6]). Moreover, these properties of $\otimes$ are obtained by 2.2 as a special case of the corresponding properties of $\otimes_R$ or $\otimes_L$ (see 2.5, 2.6, 2.11 and 2.12).

2.5 Theorem. *Either of the operations* $\otimes_R$ *and* $\otimes_L$ *is associative and distributive*:

1° *If* $\mathbf{A}_i \in \mathcal{M}(N_i \times K_{i,3})$ *and* $\mathbf{B}_i \in \mathcal{M}(N_{i,3} \times K_i)$ for $i = 1, 2, 3$ *then*

$$(\mathbf{A}_1 \otimes_R \mathbf{A}_2) \otimes_R \mathbf{A}_3 = \mathbf{A}_1 \otimes_R (\mathbf{A}_2 \otimes_R \mathbf{A}_3),$$
$$(\mathbf{B}_1 \otimes_L \mathbf{B}_2) \otimes_L \mathbf{B}_3 = \mathbf{B}_1 \otimes_L (\mathbf{B}_2 \otimes_L \mathbf{B}_3).$$

2° *If* $\mathbf{A}_i, \mathbf{A}_i' \in \mathcal{M}(N_i \times K_{i,2})$ *and* $\mathbf{B}_i, \mathbf{B}_i' \in \mathcal{M}(N_{i,2} \times K_i)$ *for* $i = 1, 2$ *then*

$$(\mathbf{A}_1 + \mathbf{A}_1') \otimes_R \mathbf{A}_2 = \mathbf{A}_1 \otimes_R \mathbf{A}_2 + \mathbf{A}_1' \otimes_R \mathbf{A}_2,$$
$$\mathbf{A}_1 \otimes_R (\mathbf{A}_2 + \mathbf{A}_2') = \mathbf{A}_1 \otimes_R \mathbf{A}_2 + \mathbf{A}_1 \otimes_R \mathbf{A}_2',$$
$$(\mathbf{B}_1 + \mathbf{B}_1') \otimes_L \mathbf{B}_2 = \mathbf{B}_1 \otimes_L \mathbf{B}_2 + \mathbf{B}_1' \otimes_L \mathbf{B}_2,$$
$$\mathbf{B}_1 \otimes_L (\mathbf{B}_2 + \mathbf{B}_2') = \mathbf{B}_1 \otimes_L \mathbf{B}_2 + \mathbf{B}_1 \otimes_L \mathbf{B}_2'.$$

Proof. We shall prove the assertion only for $\otimes_R$ because for $\otimes_L$ it follows by the duality principle.

1° $\mathbf{A}_1 \in \mathcal{M}(N_1 \times K_1K_{2,3})$, $\mathbf{A}_2 \in \mathcal{M}(N_2 \times K_{2,3}) \Rightarrow \mathbf{B} = \mathbf{A}_1 \otimes_R \mathbf{A}_2 \in \mathcal{M}(N_{1,2} \times K_1K_{2,3})$. $\mathbf{A}_2 \in \mathcal{M}(N_2 \times K_2K_3)$, $\mathbf{A}_3 \in \mathcal{M}(N_3 \times K_3) \Rightarrow \tilde{\mathbf{B}} = \mathbf{A}_2 \otimes_R \mathbf{A}_3 \in \mathcal{M}(N_{2,3} \times K_{2,3})$. Hence $\mathbf{A} = \mathbf{B} \otimes_R \mathbf{A}_3 \in \mathcal{M}(N_{1,2}N_3 \times K_{1,2}K_3)$ and $\tilde{\mathbf{A}} = \mathbf{A}_1 \otimes_R \tilde{\mathbf{B}} \in \mathcal{M}(N_1N_{2,3} \times K_1K_{2,3})$ are correctly defined matrices of the same size $N_{1,3} \times K_{1,3}$. We are going to prove $\mathbf{A} = \tilde{\mathbf{A}}$. In view of 1.6, $B([n_1, n_2], [[k_1, k_2], k_3]) = B([n_1, n_2], [k_1, [k_2, k_3]]) = A_1(n_1, [k_1, [k_2, k_3]])\, A_2(n_2, [k_2, k_3])$. Thus $A([[n_1, n_2], n_3], [[k_1, k_2], k_3]) = B([n_1, n_2], [[k_1, k_2], k_3])\, A_3(n_3, k_3) = (A_1(n_1, [k_1, [k_2, k_3]])\, A_2(n_2, [k_2, k_3]))\, A_3(n_3, k_3) = A_1(n_1, [k_1, [k_2, k_3]]) \,.\, \tilde{B}([n_2, n_3], [k_2, k_3]) = \tilde{A}([n_1, [n_2, n_3]], [k_1, [k_2, k_3]])$ holds by the associativity of multiplication in the ring $\mathbf{R}$. Using 1.6 once more, we get $A([n_1, n_2, n_3], [k_1, k_2, k_3]) = \tilde{A}([n_1, n_2, n_3], [k_1, k_2, k_3])$.

2° follows immediately by definition 2.1 and by the distributivity of multiplication in the ring $\mathbf{R}$. ∎

2.6 Theorem. *Let* $\mathbf{A}_i' \in \mathcal{M}(M_i \times N_i)$, $\mathbf{A}_i \in \mathcal{M}(N_i \times K_{i,2})$, $\mathbf{B}_i \in \mathcal{M}(N_{i,2} \times K_i)$ *and* $\mathbf{B}_i' \in \mathcal{M}(K_i \times L_i)$ *for* $i = 1, 2$. *Then it holds*

$$(\mathbf{A}_1' \otimes \mathbf{A}_2')(\mathbf{A}_1 \otimes_R \mathbf{A}_2) = \mathbf{A}_1'\mathbf{A}_1 \otimes_R \mathbf{A}_2'\mathbf{A}_2,$$
$$(\mathbf{B}_1 \otimes_L \mathbf{B}_2)(\mathbf{B}_1' \otimes \mathbf{B}_2') = \mathbf{B}_1\mathbf{B}_1' \otimes_L \mathbf{B}_2\mathbf{B}_2'.$$

Proof. Let us denote $\mathbf{A}' = \mathbf{A}_1' \otimes \mathbf{A}_2' \in \mathcal{M}(M_1M_2 \times N_1N_2)$, $\mathbf{A} = \mathbf{A}_1 \otimes_R \mathbf{A}_2 \in \mathcal{M}(N_1N_2 \times K_1K_2)$, $\tilde{\mathbf{A}}_1 = \mathbf{A}_1'\mathbf{A}_1 \in \mathcal{M}(M_1 \times K_1K_2)$ and $\tilde{\mathbf{A}}_2 = \mathbf{A}_2'\mathbf{A}_2 \in \mathcal{M}(M_2 \times K_2)$. We see that $\mathbf{C} = \mathbf{A}'\mathbf{A}$ and $\tilde{\mathbf{C}} = \tilde{\mathbf{A}}_1 \otimes_R \tilde{\mathbf{A}}_2$ are correctly defined matrices of the same size $M_1M_2 \times K_1K_2$. We are going to show $\mathbf{C} = \tilde{\mathbf{C}}$. As $A'([m_1, m_2], [n_1, n_2]) = A_1'(m_1, n_1)\, A_2'(m_2, n_2)$ by 2.2 and $A([n_1, n_2], [k_1, k_2]) = A_1(n_1, [k_1, k_2]) \,.\, A_2(n_2, k_2)$ by 2.1, we have $C([m_1, m_2], [k_1, k_2]) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} (A_1'(m_1, n_1) \,.$

$$\cdot\, \mathbf{A}_2'(m_2, n_2))\,(A_1(n_1, [k_1, k_2])\,A_2(n_2, k_2)) = (\sum_{n_1=0}^{N_1-1} A_1'(m_1, n_1)\,A_1(n_1, [k_1, k_2]))\,.$$
$$.\,(\sum_{n_2=0}^{N_2-1} A_2'(m_2, n_2)\,A_2(n_2, k_2)) = \tilde{A}_1(m_1, [k_1, k_2])\,\tilde{A}_2(m_2, k_2) = \tilde{C}([m_1, m_2],$$
$[k_1, k_2])$ by 2.1 and in view of commutativity, associativity and distributivity of multiplication in the ring $\mathbf{R}$.

The assertion for $\otimes_L$ is easy to prove by the duality principle:

$$(\mathbf{B}_1 \otimes_L \mathbf{B}_2)(\mathbf{B}_1' \otimes \mathbf{B}_2') = ((\mathbf{B}_1' \otimes \mathbf{B}_2')^T (\mathbf{B}_1 \otimes_L \mathbf{B}_2)^T)^T =$$
$$= ((\mathbf{B}_1'^T \otimes \mathbf{B}_2'^T)(\mathbf{B}_1^T \otimes_R \mathbf{B}_2^T))^T = (\mathbf{B}_1'^T\mathbf{B}_1^T \otimes_R \mathbf{B}_2'^T\mathbf{B}_2^T)^T =$$
$$= ((\mathbf{B}_1\mathbf{B}_1')^T \otimes_R (\mathbf{B}_2\mathbf{B}_2')^T)^T = \mathbf{B}_1\mathbf{B}_1' \otimes_L \mathbf{B}_2\mathbf{B}_2'. \blacksquare$$

The associativity of $\otimes_R$ and $\otimes_L$ allows one to extend the notion of the generalized right and left Kronecker product to m factors ($m \geqq 2$):

2.7 Definition. *Mixed-radix transform.*

Let $N = N_{1,m}$, $K = K_{1,m} (m \geqq 2)$, $\mathbf{A}_i \in \mathcal{M}(N_i \times K_{i,m})$ and $\mathbf{B}_i \in \mathcal{M}(N_{i,m} \times K_i)$ for $i \in [1 : m]$. Then the linear transform defined by the matrix $\mathbf{A} = \mathbf{A}_1 \otimes_R \mathbf{A}_2 \otimes_R \dots \otimes_R \mathbf{A}_m \in \mathcal{M}(N \times K)$ or $\mathbf{B} = \mathbf{B}_1 \otimes_L \mathbf{B}_2 \otimes_L \dots \otimes_L \mathbf{B}_m \in \mathcal{M}(N \times K)$ is said to be a mixed-radix transform (MRT).

2.8 Remark. It is easy to see by induction on m and in view of 1.6 that $\mathbf{A} = \mathbf{A}_1 \otimes_R \mathbf{A}_2 \otimes_R \dots \otimes_R \mathbf{A}_m$ iff $A([n_1, \dots, n_m], [k_1, \dots, k_m]) = A_1(n_1, [k_1, \dots, k_m])\,A_2(n_2, [k_2, \dots, k_m]) \dots A_m(n_m, k_m)$ for each $n_i \in \mathbf{Z}_{N_i}$ and $k_i \in \mathbf{Z}_{K_i}$, $i \in [1 : m]$. Similarly $\mathbf{B} = \mathbf{B}_1 \otimes_L \mathbf{B}_2 \otimes_L \dots \otimes_L \mathbf{B}_m$ iff $B([n_1, \dots, n_m], [k_1, \dots, k_m]) = B_1([n_1, \dots, n_m], k_1)\,B_2([n_2, \dots, n_m], k_2) \dots B_m(n_m, k_m)$ for each $n_i \in \mathbf{Z}_{N_i}$ and $k_i \in \mathbf{Z}_{K_i}$, $i \in [1 : m]$.

2.9 Theorem. *Fast mixed-radix transform.*

If $\mathbf{A}$ *and* $\mathbf{B}$ *are* MRT *matrices defined in 2.7 then the following factorizations, called* fast mixed-radix transforms (FMRTs), *take place*:
$\mathbf{A} = \mathbf{A}^{(m)}\mathbf{A}^{(m-1)} \dots \mathbf{A}^{(1)}$ *and* $\mathbf{B} = \mathbf{B}^{(1)}\mathbf{B}^{(2)} \dots \mathbf{B}^{(m)}$ *where for* $i \in [1 : m]$
$\mathbf{A}^{(i)} = \mathbf{I}_{N_{1,i-1}} \otimes (\mathbf{A}_i \otimes_R \mathbf{I}_{K_{i+1,m}}) \in \mathcal{M}(N_{1,i}K_{i+1,m} \times N_{1,i-1}K_{i,m})$ *and*
$\mathbf{B}^{(i)} = \mathbf{I}_{K_{1,i-1}} \otimes (\mathbf{B}_i \otimes_L \mathbf{I}_{N_{i+1,m}}) \in \mathcal{M}(K_{1,i-1}N_{i,m} \times K_{1,i}N_{i+1,m})$.

Proof. First we shall prove the factorization of $\mathbf{A}$ by induction on m.

1. $m = 2$: $\mathbf{A}^{(2)}\mathbf{A}^{(1)} = (\mathbf{I}_{N_1} \otimes \mathbf{A}_2)(\mathbf{A}_1 \otimes_R \mathbf{I}_{K_2}) = \mathbf{I}_{N_1}\mathbf{A}_1 \otimes_R \mathbf{A}_2\mathbf{I}_{K_2} = \mathbf{A}_1 \otimes_R \mathbf{A}_2 = \mathbf{A}$ is an immediate consequence of theorem 2.6.

2. $m > 2$: $\mathbf{A} = \mathbf{A}_1 \otimes_R \mathbf{A}'$ where $\mathbf{A}' = \mathbf{A}_2 \otimes_R \dots \otimes_R \mathbf{A}_m$. By induction hypothesis $\mathbf{A}' = \mathbf{A}'^{(m)}\mathbf{A}'^{(m-1)} \dots \mathbf{A}'^{(2)}$ with $\mathbf{A}'^{(i)} = \mathbf{I}_{N_{2,i-1}} \otimes (\mathbf{A}_i \otimes_R \mathbf{I}_{K_{i+1,m}})$, $\mathbf{A} = (\mathbf{I}_{N_1} \otimes \mathbf{A}')(\mathbf{A}_1 \otimes_R \mathbf{I}_{K_{2,m}}) = (\mathbf{I}_{N_1} \otimes \mathbf{A}')\,\mathbf{A}^{(1)}$ and $\mathbf{I}_{N_1} \otimes \mathbf{A}' = \mathbf{I}_{N_1} \otimes (\mathbf{A}'^{(m)}\mathbf{A}'^{(m-1)} \dots \mathbf{A}'^{(2)}) = (\mathbf{I}_{N_1} \otimes \mathbf{A}'^{(m)})(\mathbf{I}_{N_1} \otimes \mathbf{A}'^{(m-1)}) \dots (\mathbf{I}_{N_1} \otimes \mathbf{A}'^{(2)})$ where $\mathbf{I}_{N_1} \otimes \mathbf{A}'^{(i)} = \mathbf{I}_{N_1} \otimes \mathbf{I}_{N_{2,i-1}} \otimes (\mathbf{A}_i \otimes_R I_{K_{i+1,m}}) = \mathbf{I}_{N_{1,i-1}} \otimes (\mathbf{A}_i \otimes_R \mathbf{I}_{K_{i+1,m}}) = \mathbf{A}^{(i)}$ for $i \in [2 : m]$.

The factorization of $\mathbf{B}$ is an immediate consequence of the factorization of $\mathbf{A}$ when putting $\mathbf{A} = \mathbf{B}^T$, $\mathbf{A}_i = \mathbf{B}_i^T$ and using the duality principle (N_i and K_i interchange their roles): $\mathbf{B} = ((\mathbf{B}_1 \otimes_L \mathbf{B}_2 \otimes_L \dots \otimes_L \mathbf{B}_m)^T)^T = (\mathbf{B}_1^T \otimes_R \mathbf{B}_2^T \otimes_R \dots \otimes_R \otimes_R \mathbf{B}_m^T)^T = (\mathbf{A}_1 \otimes_R \mathbf{A}_2 \otimes_R \dots \otimes_R \mathbf{A}_m)^T = \mathbf{A}^T = (\mathbf{A}^{(m)}\mathbf{A}^{(m-1)} \dots \mathbf{A}^{(1)})^T = \mathbf{A}^{(1)T}\mathbf{A}^{(2)T} \dots \mathbf{A}^{(m)T}$ where $\mathbf{B}^{(i)} = \mathbf{A}^{(i)T} = (\mathbf{I}_{K_{1,i-1}} \otimes (\mathbf{A}_i \otimes_R \mathbf{I}_{N_{i+1,m}}))^T = \mathbf{I}_{K_{1,i-1}} \otimes (\mathbf{A}_i^T \otimes_L \mathbf{I}_{N_{i+1,m}}) = \mathbf{I}_{K_{1,i-1}} \otimes (\mathbf{B}_i \otimes_L \mathbf{I}_{N_{i+1,m}})$. ∎

Similarly as for FFTs (see [4, p. 88]), still more FMRTs may be obtained by inserting a factored identity matrix between two factors of the appropriate matrix product of $\mathbf{A}$ or $\mathbf{B}$. E.g., if $\mathbf{P}_i \in \mathscr{P}(\mathbf{Z}_{N_{1,i-1}}K_{i,m})$ is not an identity permutation for all $i \in [2 : m]$ then $\tilde{\mathbf{A}}^{(m)} = \mathbf{A}^{(m)}\mathbf{P}_m^T$, $\tilde{\mathbf{A}}^{(i)} = \mathbf{P}_{i+1}\mathbf{A}^{(i)}\mathbf{P}_i^T$, $i \in [2 : m-1]$ and $\tilde{\mathbf{A}}^{(1)} = \mathbf{P}_2\mathbf{A}^{(1)}$ define another FMRT. We have $\mathbf{A} = \tilde{\mathbf{A}}^{(m)}\tilde{\mathbf{A}}^{(m-1)} \dots \tilde{\mathbf{A}}^{(1)}$ because $\mathbf{P}_i^T\mathbf{P}_i$ is an identity matrix which, being inserted between factors $\mathbf{A}^{(i)}$ and $\mathbf{A}^{(i-1)}$, leaves the matrix product unchanged.

As in fact the factorization of $\mathbf{B}$ in theorem 2.9 is obtained by matrix transpose of $\mathbf{A} = \mathbf{B}^T$, all FMRTs may be derived from the factorization $\mathbf{A} = \mathbf{A}^{(m)}\mathbf{A}^{(m-1)} \dots \dots \mathbf{A}^{(1)}$ by inserting factored identity matrix and/or by matrix transpose.

Due to 2.3 the structure of the generating factors $\mathbf{A}^{(i)}$ may be presented in a very simple form as a block diagonal matrix with $N_{1,i-1}$ identical blocks $\vec{\mathbf{A}}_i$ along the diagonal, i.e. $\mathbf{A}^{(i)} = \text{diag}\,(\vec{\mathbf{A}}_i, \vec{\mathbf{A}}_i, \dots, \vec{\mathbf{A}}_i)$ where $\vec{\mathbf{A}}_m = \mathbf{A}_m$ and for $i \in [1 : m-1]$ each $\vec{\mathbf{A}}_i = (\vec{\mathbf{A}}_i^{n_i,k_i}) \in \mathscr{M}(N_iK_{i+1,m} \times K_{i,m})$ is a matrix with $N_i \times K_i$ diagonal blocks $\vec{\mathbf{A}}_i^{n_i,k_i} = \text{diag}\,(A_i(n_i, [k_i, 0]), A_i(n_i, [k_i, 1]), \dots, A_i(n_i, [k_i, K_{i+1,m}-1])) \in \mathscr{M}(K_{i+1,m} \times K_{i+1,m})$.

We shall now derive an important FMRT by inserting identity matrices factored by the permutation of the digit reversal (see 1.9). The resulting factorization attains a more compact form if it is applied rather to the modified matrices $\mathbf{A}^- = \mathbf{S}_{\mathscr{N}}\mathbf{A}\mathbf{S}_{\mathscr{K}}^T$ and $\mathbf{B}^- = \mathbf{S}_{\mathscr{N}}\mathbf{B}\mathbf{S}_{\mathscr{K}}^T$ obtained by writing rows and columns of $\mathbf{A}$ and $\mathbf{B}$ in digit-reversed order than for the $\mathbf{A}$ and $\mathbf{B}$ themselves. That is why the linear transform defined by $\mathbf{A}^-$ or $\mathbf{B}^-$ will be termed *digit-reversed* MRT (DRMRT) and the corresponding fast algorithm *fast digit-reversed* MRT (FDRMRT).

2.10 Theorem. *Fast digit-reversed* MRT.

Let $\mathbf{A}^- = \mathbf{S}_{\mathscr{N}}\mathbf{A}\mathbf{S}_{\mathscr{K}}^T$ *and* $\mathbf{B}^- = \mathbf{S}_{\mathscr{N}}\mathbf{B}\mathbf{S}_{\mathscr{K}}^T$ *where* $\mathscr{N} = (N_1, \dots, N_m)$, $\mathscr{K} = (K_1, \dots, K_m)$ *and* $\mathbf{A}$ *and* $\mathbf{B}$ *are* MRT *matrices defined in* 2.7. *Then the following factorizations,* called fast digit-reversed MRTs, *are true*: $\mathbf{A}^- = \mathbf{A}^{-(m)}\mathbf{A}^{-(m-1)} \dots \mathbf{A}^{-(1)}$ *and* $\mathbf{B}^- = \mathbf{B}^{-(1)}\mathbf{B}^{-(2)} \dots \mathbf{B}^{-(m)}$, *where* $\mathbf{A}^{-(i)} = \text{diag}\,(\mathbf{A}_{i,\alpha_i(0)}, \mathbf{A}_{i,\alpha_i(1)}, \dots, \dots, \mathbf{A}_{i,\alpha_i(K_{i+1,m}-1)}) \otimes \mathbf{I}_{N_{1,i-1}}$, $\mathbf{B}^{-(i)} = \text{diag}\,(\mathbf{B}_{i,\beta_i(0)}, \mathbf{B}_{i,\beta_i(1)}, \dots, \mathbf{B}_{i,\beta_i(N_{i+1,m}-1)}) \otimes \otimes \mathbf{I}_{K_{1,i-1}}$ for $i \in [1 : m-1]$, $\mathbf{A}^{-(m)} = \mathbf{A}_m \otimes I_{N_{1,m-1}}$ *and* $\mathbf{B}^{-(m)} = \mathbf{B}_m \otimes \mathbf{I}_{K_{1,m-1}}$. $\mathbf{A}_{i,k}$ $(\mathbf{B}_{i,n})$ *are matrices of size* $N_i \times K_i$ *associated with* $\mathbf{A}_i$ $(\mathbf{B}_i)$ *according to lemma* 2.3, *but arranged along the diagonal in digit-reversed order by* $\alpha_i^T = \varphi_{\mathscr{K}_{i+1,m}}(s_{i+1,m})$ $(\beta_i^T = \varphi_{\mathscr{N}_{i+1,m}}(s_{i+1,m}))$. *For* $i = m - 1$ *this ordering is natural because* α_{m-1} and β_{m-1} *are identical permutations.*

Proof. As the factorization of $\mathbf{B}^-$ is easy to be derived by that of $\mathbf{A}^-$ in view of the duality principle, we shall be concerned with $\mathbf{A}^-$ only. We can write by theorem 2.9 $\mathbf{A}^- = \mathbf{S}_{\mathscr{N}}\mathbf{A}\mathbf{S}_{\mathscr{K}}^T = \mathbf{A}^{-(m)}\mathbf{A}^{-(m-1)} \ldots \mathbf{A}^{-(1)}$ where $\mathbf{A}^{-(i)} = \mathbf{S}^{(i+1)}\mathbf{A}^{(i)}\mathbf{S}^{(i)T}$ and $\mathbf{S}^{(i)} = \varphi_{\mathscr{N}^{(i)}}(s)$ is the digit reversal with respect to $\mathscr{N}^{(i)} = (N_1, \ldots, N_{i-1}, K_i, \ldots, K_m)$ for each $i \in [1 : m + 1]$. $\mathbf{A}^{-(m)} = \mathbf{S}^{(m+1)}(\mathbf{I}_{N_{1,m-1}} \otimes \mathbf{A}_m)\, \mathbf{S}^{(m)T} = \mathbf{A}_m \otimes \mathbf{I}_{N_{1,m-1}}$ by 1.8. Let $i \in [1 : m - 1]$ be arbitrary and let us denote $\mathscr{N}_i = \mathscr{N}_{i,m}^{(i)} = (K_i, \ldots, K_m)$, $\mathscr{N}_i' = \mathscr{N}_{i,m}^{(i+1)} = (N_i, K_{i+1}, \ldots, K_m)$ and $\mathbf{S}_i = \varphi_{\mathscr{N}_i}(s_{i,m})$, $\mathbf{S}_i' = \varphi_{\mathscr{N}_i'}(s_{i,m})$ the associated permutations. First we shall prove that $\mathbf{A}^{-(i)} = \mathbf{S}_i'(\mathbf{A}_i \otimes_R \mathbf{I}_{K_{i+1,m}})\, \mathbf{S}_i^T \otimes \mathbf{I}_{N_{1,i-1}}$. For $i = 1$ this is evident because $\mathbf{A}^{-(1)} = \mathbf{S}^{(2)}(\mathbf{A}_1 \otimes_R \mathbf{I}_{K_{2,m}})\, \mathbf{S}^{(1)T}$ and $\mathbf{S}^{(2)} = \mathbf{S}_1'$ and $\mathbf{S}^{(1)} = \mathbf{S}_1$. For $i > 1$ one can split $\mathbf{S}^{(i+1)}$ and $\mathbf{S}^{(i)T}$ into two parts using 1.12, namely $\mathbf{S}^{(i+1)} = \mathbf{S}_{i-1}^{(i+1)}(\mathbf{S}_{1,i-1} \otimes \mathbf{S}_i')$ and $\mathbf{S}^{(i)T} = (\mathbf{S}_{1,i-1}^T \otimes \mathbf{S}_i^T)\, \mathbf{S}_{i-1}^{(i)T}$ where $\mathbf{S}_{i-1}^{(i)} = \varphi_{(N_{1,i-1}, K_{i,m})}(s)$, $\mathbf{S}_{i-1}^{(i+1)} = \varphi_{(N_{1,i-1}, N_i K_{i+1,m})}(s)$ and $\mathbf{S}_{1,i-1} = \varphi_{\mathscr{N}_{1,i-1}}(s_{1,i-1})$. Hence $\mathbf{A}^{-(i)} = \mathbf{S}_{i-1}^{(i+1)}(\mathbf{S}_{1,i-1} \otimes \mathbf{S}_i')\,(\mathbf{I}_{N_{1,i-1}} \otimes (\mathbf{A}_i \otimes_R \mathbf{I}_{K_{i+1,m}}))\,(\mathbf{S}_{1,i-1}^T \otimes \mathbf{S}_i^T)\, \mathbf{S}_{i-1}^{(i)T} = \mathbf{S}_{i-1}^{(i+1)} \cdot (\mathbf{S}_{1,i-1}\mathbf{S}_{1,i-1}^T \otimes \mathbf{S}_i'(\mathbf{A}_i \otimes_R \mathbf{I}_{K_{i+1,m}})\, \mathbf{S}_i^T)\, \mathbf{S}_{i-1}^{(i)T} = \mathbf{S}_i'(\mathbf{A}_i \otimes_R \mathbf{I}_{K_{i+1,m}})\, \mathbf{S}_i^T \otimes \mathbf{I}_{N_{1,i-1}}$ by 1.8. It remains to verify $\mathbf{S}_i'(\mathbf{A}_i \otimes_R \mathbf{I}_{K_{i+1,m}})\, \mathbf{S}_i^T = \operatorname{diag}(\mathbf{A}_{i,\alpha_i(0)}, \ldots, \mathbf{A}_{i,\alpha_i(K_{i+1,m}-1)})$. $\mathbf{S}_i'$ and $\mathbf{S}_i^T$ may be split using 1.12 once more: $\mathbf{S}_i' = (\boldsymbol{\alpha}_i^T \otimes \mathbf{I}_{N_i})\, \tilde{\mathbf{S}}_i'$ and $\mathbf{S}_i^T = \tilde{\mathbf{S}}_i^T(\boldsymbol{\alpha}_i \otimes \mathbf{I}_{K_i})$ where $\tilde{\mathbf{S}}_i' = \varphi_{(N_i, K_{i+1,m})}(s)$ and $\tilde{\mathbf{S}}_i = \varphi_{(K_i, K_{i+1,m})}(s)$. Hence by 2.3 $(\boldsymbol{\alpha}_i^T \otimes \mathbf{I}_{N_i})\, \tilde{\mathbf{S}}_i'(\mathbf{A}_i \otimes_R \mathbf{I}_{K_{i+1,m}})\, \tilde{\mathbf{S}}_i^T(\boldsymbol{\alpha}_i \otimes \mathbf{I}_{K_i}) = (\boldsymbol{\alpha}_i^T \otimes \mathbf{I}_{N_i}) \operatorname{diag}(\mathbf{A}_{i,0}, \mathbf{A}_{i,1}, \ldots, \mathbf{A}_{i,K_{i+1,m}-1})\,(\boldsymbol{\alpha}_i \otimes \mathbf{I}_{K_i}) = \operatorname{diag}(\mathbf{A}_{i,\alpha_i(0)}, \ldots, \mathbf{A}_{i,\alpha_i(K_{i+1,m}-1)})$. ∎

2.11 Corollary. *If* $\mathscr{N} = \mathscr{K}$ *then*

$$|\mathbf{A}| = |\mathbf{A}^-| = \prod_{i=1}^{m} (|\mathbf{A}_{i,0}|\ |\mathbf{A}_{i,1}| \ldots |\mathbf{A}_{i,N_{i+1,m}-1}|)^{N_{1,i-1}}, \qquad \mathbf{A}_{m,0} = \mathbf{A}_m$$

and

$$|\mathbf{B}| = |\mathbf{B}^-| = \prod_{i=1}^{m} (|\mathbf{B}_{i,0}|\ |\mathbf{B}_{i,1}| \ldots |\mathbf{B}_{i,N_{i+1,m}-1}|)^{N_{1,i-1}}, \qquad \mathbf{B}_{m,0} = \mathbf{B}_m .$$

In particular $\mathbf{A}$ $(\mathbf{B})$ *is invertible iff* $\mathbf{A}_{i,n}$ $(\mathbf{B}_{i,n})$ *are invertible for each* $i \in [1 : m]$ *and* $n \in \mathbf{Z}_{N_{i+1,m}}$.

Proof. $\mathscr{N} = \mathscr{K}$ and $|\mathbf{S}|\ |\mathbf{S}^T| = 1 \Rightarrow |\mathbf{A}| = |\mathbf{S}|\ |\mathbf{A}|\ |\mathbf{S}^T| = |\mathbf{A}^-| = \prod_{i=1}^{m} |\mathbf{A}^{-(i)}|$ where $|\mathbf{A}^{-(i)}| = (|\mathbf{A}_{i,\alpha_i(0)}|\ |\mathbf{A}_{i,\alpha_i(1)}| \ldots |\mathbf{A}_{i,\alpha_i(N_{i+1,m}-1)}|)^{N_{1,i-1}} = (|\mathbf{A}_{i,0}|\ |\mathbf{A}_{i,1}| \ldots |\mathbf{A}_{i,N_{i+1,m}-1}|)^{N_{1,i-1}}$. The same holds for $|\mathbf{B}|$. Finally, a square matrix over a commutative ring $\mathbf{R}$ with unity is invertible iff its determinant is an invertible element in $\mathbf{R}$. ∎

2.12 Corollary. *Let* $\mathcal{N} = \mathcal{K}$ *and* **A** (**B**) *be an invertible* MRT *matrix. Then* $\mathbf{A}^{-1}$ ($\mathbf{B}^{-1}$) *is an* MRT *matrix uniquely determined by* $\mathbf{A}^{-1} = \mathbf{A}_1^* \otimes_L \mathbf{A}_2^* \otimes_L \dots \otimes_L \otimes_L \mathbf{A}_m^*$ ($\mathbf{B}^{-1} = \mathbf{B}_1^* \otimes_R \mathbf{B}_2^* \otimes_R \dots \otimes_R \mathbf{B}_m^*$) *where* $A_i^*([n_i, \dots, n_m], n_i') = A^{-1}_{i,[n_{i+1},\dots,n_m]}(n_i, n_i')$ $(\mathbf{B}_i^*(n_i', [n_i, \dots, n_m]) = \mathbf{B}^{-1}_{i,[n_{i+1},\dots,n_m]}(n_i', n_i))$ for $i \in [1:m-1]$ *and* $\mathbf{A}_m^* = \mathbf{A}_m^{-1}$ ($\mathbf{B}_m^* = \mathbf{B}_m^{-1}$).

Proof. Let $\mathbf{A}^* = \mathbf{A}_1^* \otimes_L \mathbf{A}_2^* \otimes_L \dots \otimes_L \mathbf{A}_m^*$. As $\mathbf{A}_{i,n}^* = \mathbf{A}_{i,n}^{-1}$ for each $i \in [1:m]$ and $n \in \mathbf{Z}_{N_{i+1,m}}$ ($\mathbf{A}_{m,0}^* = \mathbf{A}_m^*$ and $\mathbf{A}_{m,0} = \mathbf{A}_m$), we have $\mathbf{A}^{-(i)}\mathbf{A}^{*-(i)} = \mathbf{I}_N$ for each $i \in [1:m]$, which means that $\mathbf{A}^-\mathbf{A}^{*-} = \mathbf{I}_N$. Consequently $\mathbf{A}\mathbf{A}^* = \mathbf{S}^T\mathbf{A}^-\mathbf{S}\mathbf{S}^T\mathbf{A}^{*-}\mathbf{S} = \mathbf{S}^T\mathbf{A}^-\mathbf{A}^{*-}\mathbf{S} = \mathbf{S}^T\mathbf{S} = \mathbf{I}_N$. $\mathbf{A}^*\mathbf{A} = \mathbf{I}_N$ follows analogically. The same argumentation may be applied to **B**. ∎

2.13 Remark. As $\otimes$ is a special case of both $\otimes_R$ and $\otimes_L$ in the sense of 2.2, lemma 1.8 suggests with $\mathbf{P}_{\mathcal{N}} = \mathbf{S}_{\mathcal{N}}$ and $\mathbf{P}_{\mathcal{K}} = \mathbf{S}_{\mathcal{K}}$ another definition of the so called *digit-reversed generalized Kronecker product* $\otimes_{R^-}$ or $\otimes_{L^-}$, namely by $\mathbf{S}_{\mathcal{N}}\mathbf{A}\mathbf{S}_{\mathcal{K}}^T = \mathbf{A}^-$ where $\mathbf{A} = \mathbf{A}_1 \otimes_R \dots \otimes_R \mathbf{A}_m$ and $\mathbf{A}^- = \mathbf{A}_m^- \otimes_{R^-} \dots \otimes_{R^-} \mathbf{A}_1^-$ or by $\mathbf{S}_{\mathcal{N}}\mathbf{B}\mathbf{S}_{\mathcal{K}}^T = \mathbf{B}^-$ where $\mathbf{B} = \mathbf{B}_1 \otimes_L \dots \otimes_L \mathbf{B}_m$ and $\mathbf{B}^- = \mathbf{B}_m^- \otimes_{L^-} \dots \otimes_{L^-} \mathbf{B}_1^-$. Accepting the symmetrically reversed number systems $\mathcal{N}s$ and $\mathcal{K}s$ as the basic ones, we can adopt $A^-([n_m, \dots, n_1], [k_m, \dots, k_1]) = A_m^-(n_m, k_m)\, A_{m-1}^-(n_{m-1}, [k_m, k_{m-1}]) \dots A_1^-(n_1, [k_m, \dots, k_1])$ and $B^-([n_m, \dots, n_1], [k_m, \dots, k_1]) = B_m^-(n_m, k_m)\, B_{m-1}^-([n_m, n_{m-1}], k_{m-1}) \dots B_1^-([n_m, \dots, n_1], k_1)$ as the defining relations for $\otimes_{R^-}$ and $\otimes_{L^-}$, respectively (cf. 2.8).

The following relations between $\otimes_R$ and $\otimes_{R^-}$ ($\otimes_L$ and $\otimes_{L^-}$), or more precisely between **A** and $\mathbf{A}^-$ (**B** and $\mathbf{B}^-$), are easy to establish:

(1) $\mathbf{A}_i^-$ ($\mathbf{B}_i^-$) is obtained by writing columns (rows) of $\mathbf{A}_i$ ($\mathbf{B}_i$) in digit-reversed order, i.e. $\mathbf{A}_i^- = \mathbf{A}_i\mathbf{S}_{\mathcal{K}_{i,m}}^T$ ($\mathbf{B}_i^- = \mathbf{S}_{\mathcal{N}_{i,m}}\mathbf{B}_i$); specifically for $i = m$ we get $\mathbf{A}_m^- = \mathbf{A}_m$ ($\mathbf{B}_m^- = \mathbf{B}_m$).

(2) Let $i \in [1:m-1]$. Then $\mathbf{A}_{i,k}^- = \mathbf{A}_{i,\alpha_i(k)}$, $k \in \mathbf{Z}_{K_{i+1,m}}$ and $\mathbf{B}_{i,n}^- = \mathbf{B}_{i,\beta_i(n)}$, $n \in \mathbf{Z}_{N_{i+1,m}}$ where α_i and β_i have been defined in 2.10, and $A^-_{i,[k_m,\dots,k_{i+1}]}(n_i, k_i) = A_i^-(n_i, [k_m, \dots, k_i])$ and $B^-_{i,[n_m,\dots,n_{i+1}]}(n_i, k_i) = B_i^-([n_m, \dots, n_i], k_i)$.

(3) Let $i \in [1:m-1]$. Then the matrices $\mathbf{A}_i$ ($\mathbf{B}_i$) arise from the family of matrices $\{\mathbf{A}_{i,k}\}_{k \in \mathbf{Z}_{K_{i+1,m}}}$ ($\{\mathbf{B}_{i,n}\}_{n \in \mathbf{Z}_{N_{i+1,m}}}$) by grouping all columns (rows) with the same position in each $\mathbf{A}_{i,k}$ ($\mathbf{B}_{i,n}$) into blocks, more precisely $\mathbf{A}_i = (\mathbf{A}_{i,0}, \mathbf{A}_{i,1}, \dots \mathbf{A}_{i,K_{i+1,m}-1})\, \mathbf{S}_{(K_i, K_{i+1,m})}$ ($\mathbf{B}_i = \mathbf{S}^T_{(N_i, N_{i+1,m})}(\mathbf{B}_{i,0}, \mathbf{B}_{i,1}, \dots, \mathbf{B}_{i,N_{i+1,m}-1})^{BT}$ where ^{BT} stands for transposition of whole blocks).

On the other hand, the matrices $\mathbf{A}_i^-$ ($\mathbf{B}_i^-$) are obtained from $\{\mathbf{A}_{i,k}^-\}_{k \in \mathbf{Z}_{K_{i+1,m}}}$ ($\{\mathbf{B}_{i,n}^-\}_{n \in \mathbf{Z}_{N_{i+1,m}}}$) by placing all $\mathbf{A}_{i,k}^-$ ($\mathbf{B}_{i,n}^-$) side by side into one row (column), more precisely $\mathbf{A}_i^- = (\mathbf{A}_{i,0}^-, \dots, \mathbf{A}_{i,K_{i+1,m}-1}^-)$ ($\mathbf{B}_i^- = (\mathbf{B}_{i,0}^-, \dots, \mathbf{B}_{i,N_{i+1,m}-1}^-)^{BT}$).

(4) Following the analogy of (2.4) and (2.5), we have for $m = 2$: $\mathbf{A}^- = (\mathbf{A}^{-n_2,k_2})$,

and $\mathbf{B}^- = (\mathbf{B}^{-n_2,k_2})$ where $\mathbf{A}^{-n_2,k_2} = A_2(n_2, k_2)\,\mathbf{A}^-_{1,k_2}$ and $\mathbf{B}^{-n_2,k_2} = B_2(n_2, k_2)\,.$ $.\,\mathbf{B}^-_{1,n_2}$, which may serve as the starting-point motivation for the definition of $\otimes_{R^-}$ and $\otimes_{L^-}$, similarly as (2.4) and (2.5) did for $\otimes_R$ and $\otimes_L$.

From (4) we get immediately $\mathbf{I}_{K_2} \otimes_{R^-} \mathbf{A}^-_1 = \operatorname{diag}(\mathbf{A}^-_{1,0}, \ldots, \mathbf{A}^-_{1,K_2-1})$ and $\mathbf{I}_{N_2} \otimes$ $\otimes_{L^-} \mathbf{B}^-_1 = \operatorname{diag}(\mathbf{B}^-_{1,0}, \ldots, \mathbf{B}^-_{1,N_2-1})$ as an analogy of 2.3. Thus $\otimes_{R^-}$ and $\otimes_{L^-}$ provide an algebraic method of forming block diagonal matrices with generally different blocks of equal sizes along the diagonal, which is a natural extension of $\mathbf{I}_{K_2} \otimes \mathbf{A}_1(\mathbf{I}_{N_2} \otimes \mathbf{B}_1)$ where all blocks $\mathbf{A}^-_{1,k_2}(\mathbf{B}^-_{1,n_2})$ are equal to $\mathbf{A}_1(\mathbf{B}_1)$. Using this and (2) it is easy to rewrite $\mathbf{A}^{-(i)}$ and $\mathbf{B}^{-(i)}$ of the FDRMRT from 2.10 in terms of $\otimes_{R^-}$ and $\otimes_{L^-}$ as follows: $\mathbf{A}^{-(i)} = (\mathbf{I}_{K_{i+1,m}} \otimes_{R^-} \mathbf{A}^-_i) \otimes \mathbf{I}_{N_{1,i-1}}$, $\mathbf{B}^{-(i)} =$ $= (\mathbf{I}_{N_{i+1,m}} \otimes_{L^-} \mathbf{B}^-_i) \otimes \mathbf{I}_{K_{1,i-1}}$ for $i \in [1 : m - 1]$ and $\mathbf{A}^{-(m)} = \mathbf{A}^-_m \otimes \mathbf{I}_{N_{1,m-1}}$, $\mathbf{B}^{-(m)} = \mathbf{B}^-_m \otimes \mathbf{I}_{K_{1,m-1}}$ in view of (1).

It is easy to establish properties of $\otimes_{R^-}$ and $\otimes_{L^-}$ analogical to those stated by 2.4–2.6, 2.11, 2.12 for $\otimes_R$ and $\otimes_L$, either applying the relations (1)–(2) directly or paraphrasing the appropriate proofs.

In the sense of lemma 1.8 $\otimes_R$, $\otimes_L$ and $\otimes_{R^-}$, $\otimes_{L^-}$ may be viewed as operations associated with $1 \in \mathscr{P}([1 : m])$ and $s \in \mathscr{P}([1 : m])$, respectively. In general of course one can associate an operation $\otimes_{R_p}$ or $\otimes_{L_p}$ with any permutation $p \in \mathscr{P}([1 : m])$ by the formula $\mathbf{P}_{\mathscr{N}}\mathbf{A}\mathbf{P}^T_{\mathscr{K}} = \mathbf{A}^p = \mathbf{A}^p_{p(1)} \otimes_{R_p} \cdots \otimes_{R_p} \mathbf{A}^p_{p(m)}$ or $\mathbf{P}_{\mathscr{N}}\mathbf{B}\mathbf{P}^T_{\mathscr{K}} = \mathbf{B}^p =$ $= \mathbf{B}^p_{p(1)} \otimes_{L_p} \cdots \otimes_{L_p} \mathbf{B}^p_{p(m)}$ and derive a fast algorithm by inserting identity matrices factored by means of $\mathbf{P}^{(i)} = \varphi_{\mathscr{N}^{(i)}}(p)$ so as this was done in the proof of 2.10 with $\mathbf{P}^{(i)} = \mathbf{S}^{(i)}$. But for most permutations p a complex structure of the resulting factors $\mathbf{A}^{p(m)}$ or $\mathbf{B}^{p(m)}$ is to be expected, which makes the appropriate $\otimes_{R_p}$ and $\otimes_{L_p}$ less attractive for practical applications. Let us observe that it was exactly the property 1.12 of the digit reversal that has brought about the neat form of the factors.

2.14 Remark. *Multidimensional* MRT.

$\mathbf{A}' = \mathbf{A}'_1 \otimes \mathbf{A}'_2 \otimes \ldots \otimes \mathbf{A}'_r$ is said to be a matrix of an *r-dimensional* MRT ($r \geqq 2$) if each $\mathbf{A}'_j \in \mathscr{M}(N'_j \times K'_j)$ is an MRT matrix. Clearly $\mathbf{A}' = \mathbf{A}'^{(r)}\mathbf{A}'^{(r-1)} \ldots$ $\ldots \mathbf{A}'^{(1)}$ where $\mathbf{A}'^{(j)} = \mathbf{I}_{N'_{1,j-1}} \otimes \mathbf{A}'_j \otimes \mathbf{I}_{K'_{j+1,r}}$, $j \in [1 : r]$. Each $\mathbf{A}'^{(j)}$ may be again decomposed according to 2.9: Assume $N'_j = N_1 \ldots N_m$, $K'_j = K_1 \ldots K_m$ and $\mathbf{A}'_j =$ $= \mathbf{A}_1 \otimes_R \cdots \otimes_R \mathbf{A}_m$, $\mathbf{A}_i \in \mathscr{M}(N_i \times K_{i,m})$ for a fixed j. Then $\mathbf{A}'^{(j)} = \mathbf{I}_{N'_{1,j-1}} \otimes$ $\otimes \mathbf{A}^{(m)} \ldots \mathbf{A}^{(1)} \otimes \mathbf{I}_{K'_{j+1,r}} = \mathbf{A}_j^{(m)} \ldots \mathbf{A}_j^{(1)}$ where $\mathbf{A}_j^{(i)} = \mathbf{I}_{N'_{1,j-1}N_{1,i-1}} \otimes (\mathbf{A}_i \otimes_R$ $\otimes_R \mathbf{I}_{K_{i+1,m}}) \otimes \mathbf{I}_{K'_{j+1,r}}$ is one step of the final fast r-dimensional MRT. In view of 2.3 we can write also $\mathbf{A}_j^{(i)} = \mathbf{I}_{N'_{1,j-1}N_{1,i-1}} \otimes (\tilde{\mathbf{A}}_i \otimes_R \mathbf{I}_{K_{i+1,m}K'_{j+1,r}})$ where $\tilde{\mathbf{A}}_i \in$ $\in \mathscr{M}(N_i \times K_{i,m}K'_{j+1,r})$ is obtained from $\mathbf{A}_i$ repeating $K'_{j+1,r}$-times the entry of each column in $\mathbf{A}_i$. In this way steps of fast multidimensional MRT have the same structure as those of fast one-dimensional MRT. We can proceed similarly if $\mathbf{A}'_j = \mathbf{B}_1 \otimes_L \cdots \otimes_L \mathbf{B}_m$.

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A GRAMMATICAL INFERENCE FOR C-FINITE LANGUAGES

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Abstract. For any language L, any finite set of contexts C, and any positive integer i we construct a linear grammar $FG(L, C, i)$ generating a language, whose ith fragment coincides with the ith fragment of the given language. If there exists some positive integer k such that for any $i \geq k$ the grammars $FG(L, C, i)$ and $FG(L, C, k)$ coincide, then the grammar $FG(L, C, k)$ generates the given language. A necessary and sufficient condition for this coincidence is given.

Key words. Grammatical inference, linear grammar, context, derivative, C-finite language complete set of contexts.

MS Classification. 68 Q 50.

1. INTRODUCTION

In special cases of grammars (e.g. regular, linear or context-free ones) nonterminal symbols can be considered the sets of all words generated by them. M. Novotný and his collaborators investigate possibilities of constructing grammars, where the role of nonterminals is played by special sets of words, so called derivatives and syntactic categories. The noneffective constructions based on this idea can be seen in [1], [7], [8], [10], the effective ones in [6], [11]. Similar ideas are used in algorithm inferring a linear harmonic grammar, which has been proposed by K. Tanatsugu [12].

This paper presents an effective algorithm inferring a linear grammar from a sample called fragment of the language (the set of all words of the language that are not larger than a given positive integer). The idea of using derivatives as nonterminals in effective constructions is due to M. Novotný ([9]).

2. PRELIMINARY DEFINITIONS AND NOTATION

By N we denote the set of all positive integers. An *alphabet* V is a finite set, whose elements are called *symbols*. The set of all words over an alphabet V –

including the empty word λ – is denoted by V^*. For any $x, y \in V^*$ we denote by xy their concatenation and for any $P, Q \subseteq V^*$ we put $PQ = \{xy;\ x \in P,\ y \in Q\}$. For any $a \in V$ a^k denotes the word of k concatenated a's. The lenght of the word x denoted by $|x|$ is the number of symbols used in its formation. An element $(u, v) \in V^* \times V^*$ is called a *context over V* or simply a *context*. We put $|(u, v)| = |u| + |v|$. For two arbitrary contexts $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ we define the operation $w_1 \circ w_2 = (u_1u_2, v_2v_1)$ and it is easy to see that $(V^* \times V^*, \circ, (\lambda, \lambda))$ is a monoid. Any set of contexts C generates the submonoid in the above mentioned monoid. By $[C]$ we denote its carrier (i.e. any $w \in [C]$ is of the form $w = w_1 \circ \ldots \circ w_k$, where $w_1, \ldots, w_k \in C$). A *language L* over an alphabet V is an arbitrary subset of V^*. For any $Q \subseteq V^*$ we put $\|Q\| = \max \{|t|;\ t \in Q\}$ if Q is finite $\|Q\| = \infty$ otherwise. A *grammar* is an ordered quadruple $G =$ fragment of the set Q. Let $w = (u, v)$ be a context and $Q \subseteq V^*$. Then the set Q_w $= (S, V, R, s_0)$, where V and S are disjoint alphabets called *terminal* and *nonterminal* ones respectively, $R \subseteq (V \cup S)^* \times (V \cup S)^*$ finite set of *rules* and $s_0 \in S$ starting symbol. The relation of *direct derivation* denoted by $\rightarrow$ and its transitive-reflexive closure denoted by $\rightarrow^*$ are defined in the usual manner. Grammars are said to be *regular* and *linear*, if their sets of rules are of the form $R \subseteq S \times V^* \cup S \times V^*S$ and $R \subseteq S \times V^* \cup S \times V^*SV^*$ respectively. We put $L(G) = \{t;\ t \in V^*, s_0 \rightarrow^* t\}$ and $L(G)$ is said to be the *language generated by grammar G*. For any positive integer i and any $Q \subseteq V^*$ the set $iQ = \{t;\ t \in Q,\ |t| \leq i\}$ is called *ith fragment of the set Q*. Let $w = (u, v)$ be a context and $Q \subseteq V^*$. Then the set $Q_w = \{t;\ utv \in Q\}$ is said to be the derivative of the set Q by the context w. Clearly $(Q_x)_y = Q_{x \circ y}$ for any contexts $x, y \in V^* \times V^*$ and any set $Q \subseteq V^*$. For any sets P, $Q \subseteq V^*$ we set $P \Subset Q$ if and only if there exists some positive integer i such that P is the ith fragment of Q. Obviously for any system of sets $T \subseteq 2^{V^*}$ the pair $(T, \Subset)$ is a partially ordered set.

3. CONSTRUCTION OF *FG*-GRAMMARS

Let L be an arbitrary language over an alphabet V, C finite set of nontrivial contexts (i.e. contexts different from (λ, λ)). We set

$$P(i) = \{(iL)_w;\ w \in [C],\ (iL)_w = \emptyset\} \cup \{iL\}.$$

(Many constructions in this paper depend on fixed sets L and C. For the sake of notation convenience we shall omit them as parameters.)

Clearly $(iL)_w = \emptyset$ for any $w \in [C]$ with the property $|w| > i$, thus the set $P(i)$ is finite. By $M(i)$ we denote the set of all maximal elements in the ordered set $(P(i), \subset)$. Note that $iL \in M(i)$. Let us have a mapping$^-$ of $\bigcup_{i \in N} P(i)$ into $\bigcup_{i \in N} M(i)$ with the following properties:

(i) $Q \in P(i)$ implies $\bar{Q} \in M(i)$,
(ii) $Q \subseteq \bar{Q}$.

Any mapping with those properties will be called a *C-mapping of the language L.* Any pair $(Q, u\bar{Q}_w v)$, where $Q \in M(i)$, $w \in C$ and $Q_w \in P(i)$ is said to be an *FG-rule of the ith fragment*. Now, let us define the mapping c of $\bigcup_{i \in N} \{\{i\} \times P(i)\}$ into N in the following way:

$$c(i, Q) = \max \{i - | w |; w \in [C], Q = (iL)_w\}.$$

An arbitrary *FG*-rule of the ith fragment (Q, uPv) is said to be *suitable* if for any $t \in \{u\} P\{v\} - Q$ the condition $| t | > c(i, Q)$ holds. Now we can construct the grammar $FG(L, C, i)$ belonging to the ith fragment of the language L. We put

$R_1(i)$ – the set of all suitable *FG*-rules of the ith fragment,

$R_2(i) = \{(Q, t); Q \in M(i), t \in Q - \{urv; (Q, uPv) \in R_1(i), r \in P\}\}$.

The ordered quadruple $FG(L, C, i) = (V, M(i), R_1(i) \cup R_2(i), iL)$ is a linear grammar, where we suppose without loss of generality that the sets V and $M(i)$ are disjoint. In the next section we show that the construction of a grammar $FG(L, C, i)$ is relatively independent on mapping $\bar{\ }$, the only importance is that it has the properties of a C-mapping.

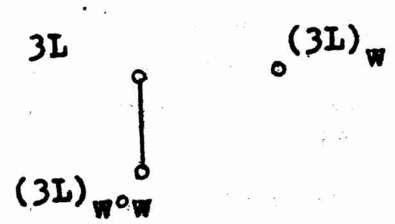

fig. 1

3.1 Example. (a) Let $V = \{a\}$, $C = \{w = (a, \lambda)\}$ and $3L = \{\lambda, a^2\}$. The ordered set $(P(3), \subset)$ is shown in fig. 1. We have two *FG*-rules $3L \to a(3L)_w$ and $(3L)_w \to a3L$ and it is easy to see that both ones are suitable. Thus the grammar $FG(L, C, 3)$ contains the following rules:

$3L \to a(3L)_w \mid \lambda$,
$(3L)_w \to a3L$.

This grammar generates all even powers of the symbol a. (b) Let V and C be the same ones as in (a) but assume that the sample $\{\lambda, a^2\}$ is the fourth fragment of some language. The ordered set $(P(4), \subset)$ is of the same structure as in (a) but the rule $(4L)_w \to a4L$ is not suitable since $a^3 \in \{a\}\, 4L - (4L)_w$ and $c(4, (4L)_w) = 3$. Thus we obtain the grammar $FG(L, C, 4)$ with the rules:

$$4L \to a(4L)_w \mid \lambda,$$
$$(4L)_w \to a.$$

This grammar generates exactly the given sample. (c) If the fourth fragment of some language $4L = \{\lambda, a^2, a^4\}$ and $C = \{(a, \lambda)\}$, the construction of the grammar $FG(L, C, 4)$ leads to the same one as in (a) (up to renaming nonterminals). □

The example 3.1. shows that the grammar $FG(L, C, i)$ generates all words of iL. Moreover the suitability of FG-rules guarantees that if the grammar $FG(C, L, i)$ generates some words that are not contained in iL, then they must be larger than i. Let us prove this fact exactly. In what follows we suppose that we are given fixed sets V, L and C. □

3.2. Lemma *Let $i \in N$, $Q \in P(i)$ and $w \in C$ such that $Q_w \in P(i)$. Then:*

(i) *$t \in Q$ implies $|t| \leq c(i, Q)$;*

(ii) *$c(i, iL) = i$,*

(iii) *$c(i, Q) - |w| \leq c(i, Q_w)$.*

Proof. The statements (i) and (ii) are trivial, we prove (iii). Let $x \in [C]$ be a context such that $Q = (iL)_x$ and $c(i, Q) = i - |x|$. We have $c(i, Q_w) = c(i, (iL)_{x \circ w}) = \max\{i - |y|;\, y \in [C],\, (iL)_{x \circ w} = (iL)_y\} > i - |x \circ w| = i - |x| - |w| = c(i, Q) - |w|$. □

3.3. Lemma *For any $i \in N$ the following assertions hold.*

(i) *$Q \in M(i)$ and $t \in Q$ imply $Q \to^* t$ in the grammar $FG(L, C, i)$,*

(ii) *$L(FG(L, C, i)) \supseteq iL$.*

Proof. (i) By induction on lenght of the word t.

(a) If $|t| = 0$ (i.e. $t = \lambda$), then there exists the rule $Q \to \lambda$ in $R_2(i)$ since $\lambda \notin \{u\} P\{v\}$ for any rule $Q \to uPv$ in $R_1(i)$.

(b) Let $|t| > 0$ and suppose that the assertion holds for any word r such that $|r| < |t|$. If $R_1(i)$ does not contain any rule $Q \to uPv$ with the property $t = urv$, then $R_2(i)$ contains the rule $Q \to t$. If $R_1(i)$ contains some rule $Q \to uPv$ such that $t = urv$, then $P = Q_w$ where $w = (u, v)$, $r \in Q_w$ and $r \in P$ since Q_w is a fragment of P. Furthermore $|r| < |t|$ implies $P \to^* r$ and $Q \to uPv \to^* urv = t$ completes the proof of the assertion (i).

(ii) is a consequence of (i). □

3.4. Lemma *For any suitable FG-rule of the ith fragment $Q \to uPv$ holds*:

$c(i, Q) - |(u, v)| \leq c(i, P)$.

Proof. Let $(u, v) = w$. If $P = Q_w$, then by 3.2. (iii) the assertion holds. Assume that $P = \bar{Q}_w \neq Q_w$. Then there exists some word t with the property $t \in P$ and $t \notin Q_w$ since Q_w is a fragment of P. Consequently $utv \in \{u\} P\{v\} - Q$ and this implies $|utv| > c(i, Q)$ since the rule $Q \to uPv$ is suitable. By 3.2. (i) we have $|t| \leq c(i, P)$ and $c(i, Q) - |w| < |t| \leq c(i, P)$ completes the proof. □

3.5. Lemma *For any $i \in N$ the following assertions hold.*

(i) *$Q \to^* t$ in the grammar $FG(L, C, i)$ and $|t| \leq c(i, Q)$ imply $t \in Q$.*

(ii) *$iL \supseteq iL(FG(L, C, i))$.*

Proof. (i) By induction on lenght of derivation.

(a) If t can be derived in one step from Q, then there exists a rule $Q \to t$ in $R_2(i)$ and $t \in Q$ trivially.

(b) Suppose that t can be derived in n steps ($n > 1$) and that the assertion holds for any $k < n$. Consequently there exists a rule $Q \to uPv$ such that $t = urv$ and r can be derived from P in $n - 1$ steps. We have $|t| = |urv| \leq c(i, Q)$, i.e. $|r| \leq \leq c(i, Q) - |(u, v)|$ and by 3.4. $c(i, Q) - |(u, v)| \leq c(i, P)$. Thus $|r| \leq c(i, P)$ and $r \in P$. Finally $t = urv \in \{u\} P\{v\}$ and $|t| \leq c(i, Q)$ implies $t \in Q$ since otherwise we would have a contradiction with the suitability of the FG-rule $Q \to uPv$.

(ii) is a consequence of (i) and 3.2. (ii). □

3.3. (ii) and 3.5. (ii) yield the following result.

3.6. Theorem $iL = iL(FG(L, C, i))$. □

A language L is said to be *FG-grammatizable*, if there exists a finite set of non-trivial contexts C, C-mapping$^-$ and a positive integer k such that for any $i \geqq k$ the grammars $FG(L, C, i)$ and $FG(L, C, k)$ coincide up to renaming nonterminals. □

4. C-FINITE LANGUAGES, COMPLETE SETS OF CONTEXTS

Let L be an arbitrary language over an alphabet V, C a finite set of nontrivial contexts. We define the equivalence relation R on $[C]$ in the following way:

For any $x, y \in [C]$ xRy if and only if $L_x = L_y$. A language L is said to be C-finite if the set $[C]/R$ is finite (c.f. [10]).

4.1. Lemma *$(iL)_x \subseteq (iL)_y$ holds for any $i \in N$ and any $x, y \in [C]$ such that xRy and $|y| \leq |x|$.*

Proof. If $|y| > i$ or $|x| > i \geqq |y|$, then the assertion is trivial. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and assume that $i \geqq |x| \geqq |y|$. First we prove $(iL)_x \subseteq$

$\subseteq (iL)_y$. For any $t \in (iL)_x$ we have $x_1tx_2 \in iL$, consequently $x_1tx_2 \in L$ and xRy implies $y_1ty_2 \in L$. Furthermore $|y_1ty_2| \leq |x_1tx_2| \leq i$, hence $y_1ty_2 \in iL$ and $t \in (iL)_y$. Now we prove that for any $t \in (iL)_y$ with the property $t \leq \max\{|r|; r \in (iL)_x\} = m$ the condition $t \in (iL)_x$ holds. Let $t \in (iL)_y$ and $|t| \leq m$. Similarly we have $x_1tx_2 \in L$ and clearly $m \leq i - |x|$. Thus $|x_1tx_2| = |t + |x| \leq m + i - - m = i$ and consequently $t \in (iL)_x$ which completes the proof. □

4.2. Lemma *Let $x, y \in [C]$ be two contexts such that L_x is infinite and $x\bar{R}y$. Then there exists $k \in N$ such that for any $i \geqq k$ $(iL)_x$ is not a fragment of $(iL)_y$.*

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. $x\bar{R}y$ implies that there exists a word $t \in (L_x - L_y) \cup (L_y - L_x)$. If there exists $t \in L_x - L_y$, then $x_1tx_2 \in L$ and $y_1ty_2 \notin L$. We put $k = |x_1tx_2|$. Obviously $t \in (iL)_x - (iL)_y$ for any $i \geqq k$, hence $(iL)_x$ is not a subset of $(lL)_y$. The second subcase $t \in L_y - L_x$ implies that $y_1ty_2 \in L$ and $x_1tx_2 \notin L$. We put $k \geqq |t|$ sufficiently large such that there exists $u \in (kL)_x$ with the property $|u| \geqq |t|$ (this is possible since L_x is an infinite set of words). For any $i \geqq k$ we have $t \in (iL)_y - (iL)_x$ and $|t| \leq \max\{|u|; u \in (iL)_x\}$. Thus $(iL)_x$ is not a fragment of $(iL)_y$. □

4.3. Lemma *Let $x, y \in [C]$ be two contexts such that L_x is a finite set. Then there exists $k \in N$ such that for any $i \geqq k$ $(iL)_x \Subset (iL)_y$ if and only if $(kL)_x \Subset (kL)_y$.*

Proof. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and let us set $k = \max\{|u|; u \in L_x\} + + \max\{|x|, |y|\}$. Clearly $(iL)_x = L_x$ for any $i \geqq k$.

(a) We prove „if" part of the assertion. Let $(kL)_x \Subset (kL)_y$ and $i \geqq k$. Obviously $(iL)_x = (kL)_x \subseteq (kL)_y \subseteq (iL)_y$. Assume that there exists a word $t \in (iL)_y - (iL)_x$ (otherwise $(iL)_x = (iL)_y$ and „if" part of the proof is trivial). If $t \in (kL)_y$ then $|t| > > \max\{|u|; u \in (iL)_x\}$ since $(iL)_x = (kL)_x \Subset (kL)_y$. If $t \notin (kL)_y$, then $|y_1ty_2| > k$, i.e. $|t| > k - |y| = \max\{|u|; u \in (iL)_x\} + \max\{|x|, |y|\} - |y| \geqq \max\{|u|; u \in (iL)_x\}$.

(b) To prove „only if" part of the assertion let us suppose that $(kL)_x$ is not a fragment of $(kL)_y$. If there exists $t \in (kL)_x - (kL)_y$, then $t \in (iL)_x - (iL)_y$ for any $i \geqq k$ since otherwise $t \in (iL)_y$ implies $|y_1ty_2| > k$, i.e. $|t| > k - |y| \geqq \geqq \max\{|u|; u \in (iL)_x\}$ which would be a contradiction. If there exists $t \in (kL)_y - - (kL)_x$ with the property $|t| \leq \max\{|u|; u \in L_x\}$, then clearly $t \in (iL)_x = (kL)_x$ and consequently $t \in (iL)_y - (iL)_x$ for any $i \geqq k$. □

4.4. Lemma *Let the set $\{L_w; w \in [C], L_w$ is infite$\}$ be finite. Then the set $\{L_w; w \in [C], L_w$ is finite$\}$ is finite too.*

Proof. If L is finite the assertion is trivial, suppose that L is infinite. Let $n \geqq 1$ be an integer such that for any infinite derivative Q of L by the context from $[C] - - \{(\lambda, \lambda)\}$ there exist contexts $w_1, \ldots, w_k \in C$ such that $k < n$ and $Q = L_w$ where $w = w_1 \circ \ldots \circ w_k$. Setting $m = \max\{\{0\} \cup \{\|L_w\|; L_w$ is finite, $w = w_1 \circ \ldots \circ w_k$,

$w_i \in C$ for $1 \leq i \leq k \leq n\}\}$ we prove that for any finite derivative Q the condition $\| Q \| \leq m$ holds. Let L_w be an arbitrary finite derivative where $w = w_1 \circ \ldots \ldots \circ w_s$ and $w_i \in C$ for $1 \leq i \leq s$. If L_{w_1} is finite, then clearly $\| L_w \| \leq \| L_{w_1} \| \leq m$. Assume that $\| L_{w_1} \|$ is infinite and let j be an integer with the following property; setting $x = w_1 \circ \ldots \circ w_j$ L_x is infinite and $L_{x \circ w_{j+1}}$ is finite. There exists a context $y = y_1 \circ \ldots \circ y_k$ where $y_i \in C$ for $1 \leq i \leq k$, $k < n$ and $L_x = L_y$. We have $\| L_{x \circ w_{j+1}} \| = \| L_{y \circ w_{j+1}} \| \leq m$ and clearly $\| L_w \| \leq \| L_{x \circ w_{j+1}} \|$. □

4.5. Corollary *For any language L and any finite set of nontrivial contexts C the following statements are equivalent*:

(i) *L is C-finite.*

(ii) *There exists* $m \in N$ *such that* card $(M(i)) \leq m$ *for any* $i \in N$.

Proof. By 4.1. (i) implies (ii) since it sufices to put $m = \text{card}([C]/R)$. Conversely suppose that L is not C-finite. We set $D = \{L_w;\ w \in [C],\ L_w \textit{ is infinite}\}$ and by 4.4. D is an infinite set. Furthermore by 4.1. and 4.2. for any two different derivatives P, $Q \in D$ there exist contexts $x, y \in [C]$ and an integer k such that $P = L_x$, $Q = L_y$ and for any $i \geqq k (iL)_x \neq (iL)_y$ and $(iL)_x$, $(iL)_y \in M(i)$. This completes the proof. □

In what follows we show that for any language L and any finite set of nontrivial contexts C there exists an integer k and *a* C-mapping $^-$ such that for any $i \geqq k$ the sets of FG-rules $R_1(i)$ and $R_1(k)$ coincide if and only if L is C-finite. The necessity of this condition follows by 4.5., we show sufficiency. Let L be a C-finite language and $D = \{Q_1, \ldots, Q_n\}$ be the set of all derivatives of L by the contexts from $[C]$. We choose the set of contexts $Y = \{y_1, \ldots, y_n\} \subseteq [C]$ in the following way:

(i) $Q_i = L_{y_i}$ for $1 \leq i \leq n$,

(ii) $x \in [C]$ and xRy_i imply $| y_i | \leq | x |$.

4.1. guarantees $M(i) \subseteq \{(iL)_y;\ y \in Y\}$. Let us put $C_0 = C \cup \{(\lambda, \lambda)\}$. By 4.1., 4.2., 4.3. and construction of Y it follows that for any contexts $x, y \in Y$ and $w \in C_0$ there exists an integer k_{pwy} such that for any $i \geq k_{xwy}$ $(iL)_{x \circ w} \Subset (iL)_y$ if and only if $(k_{xwy}L)_{x \circ w} \Subset (k_{xwy}L)_y$. We put $k = \max\{k_{xwy};\ x, y \in Y, w \in C_0\}$. We have $(iL)_{x \circ w} \Subset (iL)_y$ if and only if $(kL)_{x \circ w} \Subset (kL)_y$ for any x, $y \in Y$, $w \in C_0$. Furthermore if $(iL)_x = (iL)_y$ for some $x, y \in Y$ and $i \geqq k$, then $x = y$ since by construction of the index k $(iL)_x = (iL)_y$ holds for any $i \geqq k$, i.e. $L_x = L_y$. Denoting by X the subset of Y such that $M(k) = \{(kL)_x;\ x \in X\}$ we can estabilish the following assertion.

4.6. Lemma *Let L be a C-finite language. Then there exists* $k \in N$ *and a finite set of contexts* $X \subseteq [C]$ *such that for any* $i \geqq k$ *hold*:

(i) $M(i) = \{(iL)_x;\ x \in X\}$, $x, y \in X$ *and* $x \neq y$ *imply* $(iL)_x \neq (iL)_y$.

(ii) $(iL)_{xow} \in (iL)_y$ *if and only if* $(kL)_{xow} \in (kL)_y$ *for any* $x, y \in X$ *and any* $w \in [C]$.

(iii) $c(i, (iL)_x) = i - |x|$.

Proof. Let Y, $X \subseteq Y$ and k be the above constructed sets and index. (i) and (ii) has been already proved, we prove (iii). Assume that $c(i, (iL)_x) > i - |x|$ for some $x \in X$ and $i \geqq k$. Consequently there exists a context $w \in [C]$ such that $(iL)_x = (iL)_w$ and $|w| < |x|$, by construction of the set X we have $x\bar{R}w$. Let $z \in Y$ and $y \in X$ be the contexts such that wRz and $(iL)_z \in (iL)_y$. We have $(iL)_x \in (iL)_W \in (iL)_z \in (iL)_y$ and this implies $(iL)_x = (iL)_y$ since $(iL)_x, (iL)_y \in M(i)$. Thus $(iL)_x = (iL)_z$, consequently $x = z$ and we have $x\bar{R}wRz = x$ which is a contradiction. □

Let L be a C-finite language, X a set of contexts and let k be the least integer such for any $i \geqq k$ the conditions 4.6. (i), (ii) and (iii) hold. Then X is said to be the *principal set* of *contexts of the language* L and $k = d_1(L, C)$ is said to be the first degree of the language L.

4.7. Lemma *Let L be a C-finite language, X its principal set of contexts and let* $\tilde{}$: $P(d_1(L, C)) \to M(d_1(L, C))$ *be an arbitrary mapping with the property* $Q \in \tilde{Q}$. *Then there exists a C-mapping* $\bar{}$ *such that hold*:

(i) $\bar{Q} = \tilde{Q}$ *for any* $Q \in P(d_1(L, C))$.

(ii) *The sets of FG-rules of the* $d_1(L, C)$th *and ith fragment coincide for any* $i \geqq d_1(L, C)$.

Proof. By 4.6. (ii) it suffices to put $\overline{(iL)_{xow}} = (iL)_y$ if and only if $\widetilde{(kL)_{xow}} = (kL)_y$ for any $i \geqq k = d_1(L, C)$, any $x, y \in X$ and any $w \in C$. □

4.7. guarantees not only the existence of the C-mapping $\bar{}$ but also the independence of choice of restriction $\bar{}$ on $P(i)$ for any $i \in N$. In other words we can construct the restriction $\bar{}$ on $P(i)$ arbitrarily, i.e. effectively. Any mapping with the property 4.7. (ii) will be called *a principal mapping* of the language L. Now we can estabilish the assertion guaranteeing coincidence of the sets $R_1(i)$ and $R_1(k)$ for some $k \in N$ and any $i \geqq k$.

4.8. Lemma *Let L be a C-finite language, X its principal set of contexts and* $\bar{}$ *its principal mapping. Let* $w = (u, v) \in C$ *and* $x, y \in X$ *be the contexts such that* $(iL)_x \to u(iL)_y v$ *is an FG-rule for any* $i \geqq d_1(L, C)$. *Then there exists* $k \geqq d_1(L, C)$ *such that the following statements are equivalent*:

(i) $x \circ wRy$.

(ii) *FG-rule* $(iL)_x \to u(iL)_y v$ *is suitable for any* $i \geqq k$.

Proof. (a) We prove that (i) implies (ii) for any $i \geqq d_1(L, C)$. Let $x \circ wRy$, $i \geqq d_1(L, C)$ and $t \in \{u\} (iL)_y\{v\} - (iL)_x$. Obviously the word t is of the form $t = urv$, where $r \in (iL)_y$ and $r \notin (iL)_{x\circ w}$. However $x \circ wRy$ implies $r \in L_{x\circ w}$ and consequently $|r| > i - |x \circ w|$. Thus $|t| = |urv| > i - |x \circ w| + |w| = i - |x| = c(i, (iL)_x)$ (by 4.6. (ii)).

(b) To prove that (ii) implies (i) suppose that $x \circ w\bar{R}y$. $(iL)_{x\circ w} \subset (iL)_y$ holds for any $i \geqq d_1(L, C)$ thus by 4.2. $L_{x\circ w}$ is finite. Let $m \geqq d_1(L, C)$ be an integer such that $L_{x\circ w} \in P(i)$ for any $i \geqq m$. Furthermore there exists $j \geqq m$ and a word r with the property $r \in (jL)_y - (jL)_{x\circ w}$ since otherwise we would have a contradiction with $x \circ w\bar{R}y$. We put $k = \max\{j, |x \circ w| + |r|\}$. For any $i \geqq k$ we have $urv \in \{u\} (iL)_y \{v\} - (iL)_x$ and $|urv| \leq i - |x| = c(i, (iL)_x)$ (by 4.6. (iii)), i.e. the FG-rule is not suitable. □

By 4.8. there exists an integer k such that for any $i \geqq k$ the sets of rules $R_1(i)$ and $R_1(k)$ coincide. The least one of these integers denoted by $d_2(L, C)$ will be caled the *second degree* of the language L.

It remains to estabilish the necessary and sufficient condition guaranteeing the coincidence of the sets $R_2(i)$ and $R_2(k)$ for some fixed $k \in N$ and any $i \geqq k$. Let L be an arbitrary language, C a finite set of nontrivial contexts. The set C is said to be complete with respect to L if there exists a nonnegative integer m such that for any context $x \in [C]$ and any word $t \in L_x$ with the property $|t| > m$ there exists a context $(u, v) \in C$ and a word $r \in V^*$ such that $t = urv$ (c.f. [10]).

4.9. Lemma *Let L be a language, C a finite set of nontrivial contexts. Let $x \in [C]$ and $t \in L_x$ be a word such that there does not exist any context $(u, v) \in C$ and a word $r \in V^*$ with the property $t = urv$. Then there exists positive integer k such that the grammar $FG(L, C, i)$ contains the rule $\overline{(iL)_x} \to t$ for any $i \geqq k$.*

Proof. We put $k = |x| + |t|$. Clearly $t \in (iL)_x$ and $t \in \overline{(iL)_x}$ for any $i \geqq k$. However $t \notin \{urv;\ (\overline{(iL_x)}, uQv) \in R_1(i), r \in Q\}$, thus $R_2(i)$ contains the rule $\overline{(iL)_x} \to t$ for any $i \geqq k$. □

Finally we estabilish the main theorem.

4.10. Theorem *Let L be a language, C a finite set of nontrivial contexts. Then the following statements are equivalent:*

(i) *L is FG-grammatizable,*

(ii) *L is C-finite and C is complete with respect to L.*

Proof. By 4.5. and 4.9. (i) implies (ii). Furthermore by 4.7. and 4.8. it follows that C-finiteness of the language L guarantees coincidence of the sets $R_1(i)$ and $R_1(k)$ for some fixed k and any $i \geqq k$. It remains to prove that C-finiteness of L and completeness of the set C with the respect to L guarantee coicidence of the sets $R_2(i)$ and $R_2(k)$ for some fixed $k \in N$ and any $i \geqq k$. Let X be a principal set of contexts, $^-$ a principal mapping and $k = d_2(L, C)$ the second degree of L. Completness of the set C guarantees that there exists at most finite number of the words $t \in L_x$ $(x \in [C])$ which can't be expressed in the form $t = urv$ for some $(u, v) \in C$ and by 4.9. for any word t with this property there exists $k \in N$ such that the grammar $FG(C, L, i)$ contains the rule $\overline{(iL)_x} \to t$ for any $i \geqq k$. If the grammar $FG(L, C, j)$ contains some rule $(jL)_x \to t = urv$ where $(u, v) \in C$ and $j \geqq k$, then this grammar does not contain the rule $(jL)_x \to u(jL)_y v$. By construction of the second degree of L the rule $(iL)_x \to u(iL)_y v$ is not contained in the grammar $FG(L, C, i)$ for any $i \geqq k$. By 4.8. we have $x \circ (u, v) \bar{R} y$ and consequently by 4.2 $L_{x \circ (u, v)}$ is finite since by 4.6. (ii) $(iL)_{x \circ (u, v)} \subset (iL)_y$ holds for any $i \geq k$. Thus there exists at most finite number of the words $t = urv \in L_x$ where $x \in X$ and $(u, v) \in C$ such that the rule $(jL)_x \to t$ is contained in the grammar $FG(L, C, j)$ for some $j \geqq k$. Moreover the nonexistence of the rule $(iL)_x \to u(iL)_y u$ implies that the rule $(iL)_x \to t$ is contained in the grammar $FG(L, C, i)$ for any $i \geqq j$ and this completes the proof. □

The conditions "to be C-finite" and "to be complete" are mutualy independent (c.f. [10]).

4.11. Examples (a) Any finite language is FG-grammatizable since any set of contexts is complete with respect to any finite language and any finite language is C-finite for any set of contexts C.

(b) Any regular language is FG-grammatizable. It suffices to put $C = \{(a, \lambda); a \in V\}$. Clearly the set C is complete with respect to any language over the alphabet V and any regular language is C-finite ([3]). Moreover this construction leads to a regular grammar.

(c) Any even linear language is FG grammatizable (i.e. language generated by a grammar whose rules are either of the form $P \to vQu$ where $| u | = | v |$, or $P \to t$). We put $C = \{(a, b); a, b \in V\}$. The set C is complete with the respect to any language and any even linear language is C-finite ([10]). □

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A FOUR-POINT PROBLEM FOR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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Abstract. The paper deals with the four-point problem $u'' = f(t, u, u')$, $u(c) - u(a) = A$, $u(b) - u(d) = B$, where $a, b, c, d, A, B \in R$, $a < c < d < b$. The sufficient conditions for the existence of solutions of this problem are established.

Key words. Four-point boundary value problem, a priori estimate, Carathéodory conditions.

MS Classification 34 B 10.

The questions of existence and uniqueness of solutions of the two-point boundary value problem for differential equations of the second order have a long history, going back to Picard (1893).

The boundary problems

$$u'' = f(t, u, u'), \tag{0.1}$$

$$\sum_{j=1}^{2} (a_{ij}u^{(j-1)}(a) + b_{ij}u^{(j-1)}(b)) = c_i, \qquad i = 1, 2, \tag{0.2}$$

where $a, b, a_{ij}, b_{ij}, c_i \in (-\infty, +\infty)$, $a < b$, and f is a continuous function or satisfies the local Carathéodory conditions, are solved for example in [3], [5], [7], [8], [12],. In [10], [12] the linear conditions (0.2) are generalized for the case of nonlinear ones.

The three-point problems for differential equations of the second order were studied in [1], [2], [9], and [11]. The problem of existence of solutions of the equation

$$u'' = f(t, u),$$

satisfying the conditions

$$u(0) = u(a) = u(2a), \qquad a \in (-\infty, +\infty)$$

is solved in [1], [2].

The theorems of existence and uniqueness of solutions of the equation (0.1) satisfying the conditons

$$u(a) = c_1, \quad u(b) = u(t_0) + c_2, \quad a, b, t_0, c_1, c_2 \in (-\infty, +\infty), a < t_0 < b,$$

are proved in [11] and for the linear differential equation in [9].

I

Our paper deals with the problem of existence of solutions of the equation

$$(1.1) \qquad u'' = f(t, u, u'),$$

defined on the interval $[a, b]$ and satisfying the conditions

$$(1.2) \qquad u(c) - u(a) = A, \quad u(b) - u(d) = B,$$

where $A, B \in (-\infty, \infty)$, $-\infty < a < c < d < b < +\infty$.

We shall use the following notations:

$R = (-\infty, +\infty)$, $R_+ = [0, +\infty)$, $D = [a, b] \times R^2$, $D_+ = [a, b] \times R_+^2$,

$$\tau = \begin{cases} \max\{c - a, b - c\} & \text{for } d - a > b - c, \\ \max\{d - a, b - d\} & \text{for } d - a \leqq b - c, \end{cases} \qquad g_0(t) = \alpha t^2 + \beta t + \gamma, \text{ where}$$

$$\alpha = (B/(b - d) - A/(c - a))(b - c + d - a)^{-1},$$

$$\beta = (A(b + d)/(c - a) - B(c + a)/(b - d))(b - c + d - a)^{-1},$$

$\gamma \in R$, $r_0 = \max\{|g_0(t)| : a \leqq t \leqq b\}$, $r_1 = \max\{|g_0'(t)| : a \leqq t \leqq b\}$.

$AC^1(a, b)$ is the set of all real functions which are absolutely continuous with their first derivatives on $[a, b]$.

$\mathrm{Car}_{\mathrm{loc}}(D)$ is the set of all real functions satisfying the local Carathéodory conditions on D, i.e. $f \in \mathrm{Car}_{\mathrm{loc}}(D)$ iff

$f(., x, y) : [a, b] \to R$ is measurable for every $(x, y) \in R^2$,

$f(t, ., .) : R^2 \to R$ is continuous for almost every $t \in [a, b]$,

$\sup\{|f(., x, y)| : |x| + |y| \leqq \varrho\} \in L(a, b)$ for any $\varrho \in (0, +\infty)$.

Definition. A function $u \in AC^1(a, b)$ which fulfils (1.1) for almost every $t \in [a, b]$ will be called a solution of the equation (1.1). Each solution of (1.1) which satisfies the conditions (1.2) will be called a solution of the problem (1.1), (1.2).

In the whole paper we suppose that $f \in \mathrm{Car}_{\mathrm{loc}}(D)$ and $\lambda \in \{-1, 1\}$.

Theorem 1. *Let there exist $r \in (0, +\infty)$ such that on the set D the inequalities*

$$(1.3) \qquad \lambda[f(t, x, y) - 2\alpha] \operatorname{sgn} x \geqq 0 \quad \text{for } |x| > r,$$

(1.4) $$| f(t, x, y) | \leqq \omega(t, | x |, | y |)$$

are fulfilled, where $\omega \in \mathrm{Car}_{\mathrm{loc}}(D_+)$ *is a non-negative function, non-decreasing with respect to its second and third variables and satisfying the conditions*

(1.5) $$\limsup_{\varrho \to +\infty} \frac{1}{\varrho} \int_a^b \omega(t, \varrho(b-a), \varrho)\, \mathrm{d}t < 1.$$

Then the problem (1.1), (1.2) has at least one solution.

Corollary. *Let there exist* $r \in (0, +\infty)$ *such that on the set D the inequalities* (1.3) *and*

(1.6) $$| f(t, x, y) | \leqq h_1(t) | x | + h_2(t) | y | + \omega(t, | x | + | y |)$$

are fulfilled, where $h_1, h_2 \in L(a, b)$ *are non-negative functions satisfying*

(1.7) $$(b - a) \int_a^b h_1(t)\, \mathrm{d}t + \int_a^b h_2(t)\, \mathrm{d}t < 1$$

and $\omega \in \mathrm{Car}_{\mathrm{loc}}([a, b] \times R_+)$ *is a non-negative function, non-decreasing with respect to its second variable and satisfying the condition*

(1.8) $$\lim_{\varrho \to +\infty} \frac{1}{\varrho} + \int_a^b \omega(t, \varrho)\, \mathrm{d}t = 0.$$

Then the problem (1.1), (1.2) has at least one solution.

Theorem 2. *Let there exist* $r \in (0, +\infty)$ *such that on the set D the inequalities* (1.3) *and*

(1.9) $$| f(t, x, y) | \leqq a_1 | x | + a_2 | y | + \omega(t, | x | + | y |)$$

are fulfilled, where $a_1, a_2 \in (0, +\infty)$ *satisfy*

(1.10) $$a_1(2(b - a)/\pi)^2 + a_2(2(b - a)/\pi) < 1$$

and ω *is the function from Corollary.*

Then the problem (1.1), (1.2) has at least one solution.

Theorem 3. *Let there exist* $r \in (0, +\infty)$ *such that on the set D the inequalities* (1.3) *and* (1.9) *are fulfilled, where* $a_1, a_2 \in (0, +\infty)$ *satisfy*

(1.11) $$a_1 \tau (b - a) (2/\pi)^2 + a_2 \tau\, 2/\pi < 1$$

and $\omega : [a, b] \times R_+ \to R_+$ *is a function such that*

$$(1.12)\qquad \begin{cases} \omega(.,\varrho) \in L^2(a,b) & \textit{for any } \varrho \in R_+, \\ \omega(t,.) \in C(R_+) & \textit{is non-decreasing}, \\ \lim\limits_{\varrho \to +\infty} \dfrac{1}{\varrho} (\int\limits_a^b \omega^2(t,\varrho)\,\mathrm{d}t)^{1/2} = 0. \end{cases}$$

Then the problem (1.1), (1.2) has at least one solution.

II

Lemma 1. ([6], Theorem 256, p. 219). *If* $f \in AC(t_1, t_2), f' \in L^2(t_1, t_2)$ *and* $f(t_0) = 0$, *where* $-\infty < t_1 < t_2 < +\infty$, $t_0 \in [t_1, t_2]$, *then*

$$\int\limits_{t_1}^{t_2} f^2(t)\,\mathrm{d}t = (2(t_2 - t_1)/\pi)^2 \int\limits_{t_1}^{t_2} f'^2(t)\,\mathrm{d}t.$$

Lemma 2. *Let* $\varepsilon \in (0, +\infty)$ *satisfy the inequality*

$$(2.1)\qquad \varepsilon\tau(b - a)(2/\pi)^2 < 1.$$

Then the problem

$$(2.2)\qquad v'' = \lambda\varepsilon v,$$

$$(2.3)\qquad v(c) - v(a) = 0, \qquad v(b) - v(d) = 0$$

has only the trivial solution.

Proof. Let v be a solution of the problem (2.2), (2.3). By (2.3), there exist $t_1 \in (a, c)$, $t_2 \in (d, b)$ such that $v'(t_1) = v'(t_2) = 0$. Therefore, in view of (2.2), we have $t_0 \in (t_1, t_2)$ such that $v''(t_0) = v(t_0) = 0$. It follows from Lemma 1, that

$$\int\limits_a^b v'^2(t)\,\mathrm{d}t \leqq (2\tau/\pi)^2 \int\limits_a^b v''^2(t)\,\mathrm{d}t$$

and

$$(2.4)\qquad \int\limits_a^b v^2(t)\,\mathrm{d}t \leqq (2/\pi)^4 (\tau(b - a))^2 \int\limits_a^b v''^2(t)\,\mathrm{d}t.$$

Hence, by (2.2), (2.4), we find, that

$$\int\limits_a^b v''^2(t)\,\mathrm{d}t \leqq (\varepsilon(2/\pi)^2 \tau(b - a))^2 \int\limits_a^b v''^2(t)\,\mathrm{d}t$$

and by (2.1) (2.4), we find, that $v(t) = 0$ for $t \in [a, b]$.

Lemma 3. *Let* $a_1, a_2 \in (0, +\infty)$ *satisfy* (1.10) *and* $h_1, h_2 \in L(a, b)$ *be such that*

$$(2.5)\qquad |h_i(t)| \leqq a_i, \qquad i = 1, 2, \qquad a \leqq t \leqq b.$$

Then the problem

$$v'' = h_1(t)\, v + h_2(t)\, v', \tag{2.6}$$

$$v(t_0) = v'(t_1) = 0, \qquad t_0, t_1 \in (a, b), \tag{2.7}$$

has only the trivial solution.

Proof. Let v be a solution of the problem (2.6), (2.7). Then, by Lemma 1, we have

$$\int_a^b v'^2(t)\,\mathrm{d}t \leqq (2(b-a)/\pi)^2 \int_a^b v''^2(t)\,\mathrm{d}t \tag{2.8}$$

and

$$\int_a^b v^2(t)\,\mathrm{d}t \leqq (2(b-a)/\pi)^4 \int_a^b v''^2(t)\,\mathrm{d}t. \tag{2.9}$$

Therefore, by (2.5), (2.6), (2.8), (2.9), we obtain

$$\Big(\int_a^b v''^2(t)\,\mathrm{d}t\Big)^{1/2} \leqq \big((a_1\, 2(b-a)/\pi)^2 + a_2\, 2(b-a)/\pi\big) \Big(\int_a^b v''^2(t)\,\mathrm{d}t\Big)^{1/2}.$$

From the last inequality, according to (1.10) and (2.9), it follows $v(t) = 0$ for $t \in [a, b]$.

Lemma 4. *Let* $g \in \mathrm{Car}_{\mathrm{loc}}(D)$ *and* $\varepsilon \in (0, +\infty)$ *satisfy* (2.1). *If there exists* $g^* \in L(a, b)$ *such that*

$$|\, g(t, x, y)\,| \leqq g^*(t) \qquad \text{on } D,$$

then the problem

$$v'' = \lambda\varepsilon v + g(t, v, v'), \qquad (2.3)$$

is solvable.

Proof. See [4] or [8], Theorem 2.4, p. 25.

Lemma 5. *Let* $a_1, a_2 \in (0, +\infty)$ *and let for any* $h_1, h_2 \in L(a, b)$ *satisfying* (2.5) *the problem* (2.6), (2.7) *have only the trivial solution. Then there exists such* $\gamma \in (0, +\infty)$, *that for any* $h_1, h_2 \in L(a, b)$ *satisfying* (2.5), *the inequality*

$$\left|\frac{\partial G(t, s)}{\partial t}\right| + |\, G(t, s)\,| \leqq \gamma, \qquad a \leqq t, s \leqq b \tag{2.10}$$

is fulfilled, where G *is the Green function of the problem* (2.6), (2.7).

Proof. See [8], Lemma 2.2, p. 12.

III

Lemmas for a priori estimates

Lemma 6. *Let* $r \in (0, +\infty)$ *and* $\omega \in \mathrm{Car}_{\mathrm{loc}}(D_+)$ *be a non-negative function, non-decreasing with respect to its second and third variables and satisfying* (1.5).

Then there exists $r^* \in (r, +\infty)$ *such that for any function* $v \in AC^1(a, b)$ *the conditions*

$$v(a) = v(c), \qquad v(d) = v(b), \tag{3.1}$$

$$\lambda v''(t) \operatorname{sgn} v(t) > 0 \qquad \text{for } | v(t) | > r,\ t \in [a, b], \tag{3.2}$$

$$| v''(t) | \leqq \omega(t, | v |, | v' |), \qquad \text{for } a < t < b \tag{3.3}$$

imply the estimate

$$| v(t) | + | v'(t) | \leqq r^* \qquad \text{for } a \leqq t \leqq b. \tag{3.4}$$

Proof. The condition (3.1) implies the existence of $t_1, t_2 \in (a, b)$ such that $v'(t_1) = v'(t_2) = 0$. If $| v(t) | > r$ on (a, b) then, by (3.2), v' has to be strictly monotonous on (a, b) and we get the contradiction. Therefore there exists $t_0 \in (a, b)$ such that $| v(t_0) | \leqq r$.

Put $\varrho_0 = \max\{| v'(t) | : a \leqq t \leqq b\}$. Integrating the inequality $| v'(t) | \leqq \varrho_0$ from t_0 to t, we have $| v(t) | \leqq r + (b - a)\,\varrho_0$. Let $t^* \in [a, b]$ be such that $| v'(t^*) | = \varrho_0$. Integrating (3.3) from t_1 to t^*, we get

$$\varrho_0 \leqq \int_a^b \omega(t, r + (b - a)\,\varrho_0, \varrho_0)\,\mathrm{d}t. \tag{3.5}$$

Hence, by (1.5), there exists $\delta > 0$ such that

$$(1 + \delta) \limsup_{\varrho \to +\infty} \frac{1}{\varrho} \int_a^b \omega(t, \varrho(b - a), \varrho)\,\mathrm{d}t < 1.$$

Consequently there exists $\varrho^* > 0$ such that for any $\varrho > \varrho^*$ the inequalities

$$r + \varrho(b - a) \leqq (1 + \delta)\,\varrho(b - a)$$

and

$$\frac{1}{\varrho} \int_a^b \omega(t, (1 + \delta)(b - a)\,\varrho, \qquad (1 + \delta)\,\varrho)\,\mathrm{d}t < 1 \tag{3.6}$$

are satisfied. By (3.5) and (3.6), we obtain $\varrho_0 \leqq \varrho^*$. Putting

$$r^* = r + (b - a + 1)\,\varrho^*,$$

we get the estimate (3.4).

Lemma 7. *Let* $r \in (0, +\infty)$, $a_1, a_2 \in (0, +\infty)$ *satisfy* (1.10) and $\omega \in \mathrm{Car}_{\mathrm{loc}}([a, b] \times$

$\times R_+)$ *is a non-negative function, non-decreasing with respect to its second variable and satisfying* (1.8).

Then there exists $r^* \in (r, +\infty)$ *such that for any function* $v \in AC^1(a, b)$ *the conditions* (3.1), (3.2) *and*

$$(3.7) \qquad |v''(t)| \leqq a_1 |v(t)| + a_2 |v'(t)| + \omega(t, |v| + |v'|), \qquad t \in (a, b)$$

imply the estimate (3.4).

Proof. From (3.1), (3.2) it follows that there exist $t_0, t_1, t_2 \in (a, b)$ such that $v'(t_1) = v'(t_2) = 0$ and $v(t_0) = c_0$, where $|c_0| \leqq r$. Put $y(t) = v(t) - c_0$ for $a \leqq t \leqq b$ and consider the equation

$$(3.8) \qquad y'' = h_1(t)\, y + h_2(t)\, y' + h_0(t),$$

where $h_i(t) = a_i \,.\, k(t)\, v''(t) \operatorname{sgn} v^{(i-1)}(t), i = 1, 2, h_0(t) = \omega(t, |v| + |v'|)\, k(t)\, v''(t) + h_1(t)\, c_0$, $k(t) = (a_1 |v| + a_2 |v'| + \omega(t, |v| + |v'|))^{-1}$. Since $|h_i(t)| \leqq a_i$, $i = 1, 2$, it follows from Lemma 3 that the problem

$$(3.9) \qquad y'' = h_1(t) \cdot y + h_2(t)\, y',$$

$$(3.10) \qquad y(t_0) = y'(t_1) = 0$$

has only the trivial solution. Consequently, by Lemma 5, the solution

$$y(t) = \int_a^b G(t, s)\, h_0(s)\, ds$$

of the problem (3.8), (3.10) satisfies

$$|y(t)| + |y'(t)| \leqq \gamma \int_a^b |h_0(s)|\, ds \leqq \gamma(1 + r) \int_a^b (|h_1(s)| + \omega(s, r + |y| + |y'|))\, ds.$$

Let $\varrho_0 = \max\{|y(t)| + |y'(t)| : a \leqq t \leqq b\}$. Then

$$(3.11) \qquad \varrho_0 \leqq \gamma(r + 1) \int_a^b (|h_1(s)| + \omega(s, r + \varrho_0))\, ds.$$

In view of (1.8) there exists $\varrho^* > 0$ such that for any $\varrho > \varrho^*$ the inequality

$$(3.12) \qquad \gamma(1 + r) \int_a^b (|h_1(t)| + \omega(t, r + \varrho))\, dt < \varrho$$

is satisfied. From (3.11), (3.12) it is clear that $\varrho_0 \leqq \varrho^*$. Putting $r^* = \varrho^* + r$, we get the estimate (3.4).

Lemma 8. *Let* $r \in (0, +\infty)$, $a_1, a_2 \in (0, +\infty)$ *satisfy* (1.11) *and* $\omega : [a, b] \times + R_+ \to R_+$ *satisfy* (1.12).

Then there exists $r^* \in (r, +\infty)$ *such that for any function* $v \in AC^1(a, b)$ *the conditions* (3.1), (3.2) *and* (3.7) *imply the estimate* (3.4).

Proof. In the same way as in the proof of Lemma 6 we can find zeros of v' and the point t_0 such that $|v(t_0)| \leqq r$. By Lemma 1 we obtain

$$(3.13)\qquad (\int_a^b v'^2(t)\,dt)^{1/2} \leqq 2\tau/\pi(\int_a^b v''^2(t)\,dt)^{1/2}$$

and

$$(3.14)\qquad (\int_a^b (v(t) - v(t_0))^2\,dt)^{1/2} \leqq \tau(b-a)(2/\pi)^2(\int_a^b v''^2(t)\,dt)^{1/2}.$$

Let us put $\varrho_0 = (\int_a^b v''^2(t)\,dt)^{1/2}$. Then, by the Hölder inequality, we get

$$(3.15)\qquad |v'(t)| = |\int_{t_1}^t v''(s)\,ds| \leqq \varrho_0(b-a)^{1/2}$$

and

$$(3.16)\qquad |v(t)| \leqq |\int_{t_0}^t v'(s)\,ds| + r \leqq \varrho_0(b-a)^{3/2} + r.$$

From (3.7) it follows, by virtue of (3.13), 3.14), (3.15) and (3.16)

$$\varrho_0 \leqq (a_1\tau(b-a)(2/\pi)^2 + a_2 2\tau/\pi)\varrho_0 + a_1 r\sqrt{b-a} + \\ + (\int_a^b \omega^2(t, r + \varrho_0(b-a+1)^2)\,dt)^{1/2}.$$

In view of (1.11) and (1.12), there exists $\varrho^* > 0$ such that for any $\varrho > \varrho^*$ the inequality

$$(a_1\tau(b-a)(2/\pi)^2 + a_2 2\tau/\pi)\varrho + a_1 r\sqrt{b-a} + \\ + (\int_a^b \omega^2(t, r + \varrho(b-a+1)^2\,dt)^{1/2} < \varrho$$

is valid and consequently $\varrho_0 \leqq \varrho^*$. Putting

$$r^* = r + \varrho^*((b-a)^{1/2} + (b-a)^{3/2}),$$

in accordance to (3.15), (3.16), we get the estimate (3.4).

IV

Proofs of Theorems

Proof of Theorem 1. Let $\varepsilon_0 \in (0, +\infty)$ satisfy

$$(4.1)\qquad \varepsilon_0(b-a)^2 + \limsup_{\varrho \to +\infty} \frac{1}{\varrho} \int_a^b \omega(t, \varrho(b-a), \varrho)\,dt < 1$$

and r^* be the constant constructed by means of Lemma 6 for the function

$\tilde{\omega}(t, |x|, |y|) = \omega(t, |x| + r_0, |y| + r_1) + \varepsilon_0 |x| + 2|\alpha|$ and for the constant $\tilde{r} = r + r_0$. Put

$$\chi(r^*, s) = \begin{cases} 1 & \text{for } 0 \leqq s \leqq r^*, \\ 2 - s/r^* & \text{for } r^* < s < 2r^*, \\ 0 & \text{for } s \geqq 2r^*, \end{cases}$$

$$g(t, x, y) = f(t, x + g_0(t), y + g_0'(t)) - 2\alpha,$$
$$\tilde{g}(t, x, y) = \chi(r^*, |x| + |y|)\, g(t, x, y)$$

and consider the equation

$$v'' = \lambda \varepsilon v + \tilde{g}(t, v, v'), \qquad \varepsilon \in (0, \varepsilon_0]. \tag{4.2}$$

Since ε and $\tilde{g}$ satisfy the assumptions of Lemma 4, the problem (4.2), (2.3) has a solution v. Clearly v satisfies (3.1). Let $v(t) > \tilde{r}$ for some $t \in [a, b]$. Then $v(t) + g_0(t) > r$ and

$$\lambda v''(t) = \lambda \chi(r^*, |v| + |v'|)\,(f(t, v + g_0(t), v' + g_0'(t)) - 2\alpha) + \varepsilon v(t) > 0.$$

Analogously, if $v(t) < -\tilde{r}$, then $v(t) + g_0(t) < -r$ and $\lambda v''t) < 0$. Consequently v satisfies (3.2) with the constant $\tilde{r}$. Further

$$|v''(t)| \leqq |f(t, v + g_0(t), v' + g_0'(t)) - 2\alpha| + \varepsilon |v(t)| \leqq$$
$$\leqq \omega(t, |v| + r_0, |v'| + r_1) + 2|\alpha| + \varepsilon_0 |v(t)| = \tilde{\omega}(t, ||v|, |v'|).$$

According to (4.1) there exists $\delta > 0$ such that

$$\varepsilon_0 (b - a)^2 + (1 + \delta) \limsup_{\varrho \to +\infty} \frac{1}{\varrho} \int_a^b \omega(t, \varrho(b - a), \varrho)\, dt < 1. \tag{4.3}$$

It follows from (4.3) that there exists $\varrho^* > 0$ such that for any $\varrho > \varrho^*$ the inequalities

$$r_0 + \varrho(b - a) \leqq (1 + \delta)\, \varrho(b - a), \qquad r_1 + \varrho \leqq (1 + \delta)\, \varrho,$$

$$\varepsilon_0 (b - a)^2 + \frac{1}{\varrho} \int_a^b (\omega(t, (1 + \delta)\, \varrho(b - a), (1 + \delta)\, \varrho) + 2|\alpha|)\, dt < 1.$$

The latter inequality implies that $\tilde{\omega}$ satisfies (1.5). Hence, by Lemma 6, the estimate (3.4) is valid and v is a solution of the equation $v'' = \lambda \varepsilon v + g(t, v, v')$. Thus $u = v + g_0$ is a solution of the equation

$$u'' = \lambda \varepsilon (u - g_0(t)) + f(t, u, u') \tag{4.4}$$

and satisfies the conditions (1.2). Therefore for any $\varepsilon \in (0, \varepsilon_0]$ there exists a solution u_ε of the problem (4.4), (1.2) satisfying the estimate $|u_\varepsilon| + |u_\varepsilon'| \leqq r^* + r_0 + r_1$ for $a \leqq t \leqq b$. From this it follows that all functions of the set $\{u_\varepsilon : \varepsilon \in (0, \varepsilon_0]\}$ are uniformly bounded with their derivatives and so also equi-continuous on

$[a, b]$. Therefore, by the Arzelà–Ascoli lemma, there exists a sequence $(\varepsilon_k)_{k=1}^{\infty}$, $\varepsilon_k \to 0$ for $k \to \infty$, and a sequence $(u_{\varepsilon_k})_{k=1}^{\infty}$ uniformly converging together with $(u'_{\varepsilon_k})_{k=1}^{\infty}$ on $[a, b]$ such that $u_0(t) = \lim_{k\to\infty} u_{\varepsilon_k}(t)$ is a solution of the problem (1.1), (1.2).

Proof of Theorem 2. Let $\varepsilon_0 \in (0, +\infty)$ satisfy the inequality $\varepsilon_0(2/\pi)^2 (b - a)^2 + a_1(2/\pi)^2 (b - a)^2 + a_2(2/\pi)(b - a) < 1$ and r^* be the constant constructed by means of Lemma 7 for the function $\tilde{\omega}(t, |x| + |y|) = \omega(t, |x| + |y| + r_0 + r_1) + a_1 r_0 + a_2 r_1 + 2|\alpha|$ and for the constants $a_1 + \varepsilon_0, a_2, \tilde{r} = r + r_0$. Then, using Lemma 7, we can prove Theorem 2 in a similar way as Theorem 1.

Proof of Theorem 3. Theorem 3 can be proved in the same way as Theorem 2, only by means of Lemma 8.

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ON ITERATION GROUPS OF CERTAIN FUNCTIONS

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Abstract. This paper contains a characterization of iteration groups formed, up to conjugacy, by certain functions of the form

$$\operatorname{Arctan} \frac{a \tan x + b}{c \tan x + d}, \qquad | ad - bc | = 1,$$

under the condition that graphs of different elements of such a group do not intersect each other.

Key words. Iteration groups, Linear differential equations.

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I. INTRODUCTION

For description of global transformations of linear differential equations, it is important to characterize all groups of those transformations that keep a given equation unchanged, see [5] and [6]. This characterization requires the following result concerning iteration groups of certain functions.

II. NOTATION, DEFINITIONS AND SOME BASIC FACTS

In accordance with O. Borůvka [2], the fundamental groups $\mathscr{F}_1$ is defined as the group of all functions $f: \mathbf{R} \to \mathbf{R}$ given by the formula

$$f(t) = \operatorname{Arctan} \frac{a \tan t + b}{c \tan t + d},$$

$a, b, c, d \in \mathbf{R}$, $| ad - bc | = 1$, where Arctan denotes this branch of $\arctan x + k\pi$ that makes function f continuous on $\mathbf{R}$. Then the elements of the fundamental group $\mathscr{F}_1$ are real analytic bijections of $\mathbf{R}$ onto $\mathbf{R}$, they are increasing exactly when $ad - bc = 1$. The group operation "$\circ$" is the composition of functions; for brevity the symbol $\circ$ is sometimes omitted.

Consider the following groups, whose elements are some functions of the fundamental group $\mathscr{F}_1$, restricted to an open interval $I \subset \mathbf{R}$.

$\mathscr{F}_2$: f: $(0, \infty) \to (0, \infty)$,

$$f(t) = \operatorname{Arctan} \frac{a \tan t}{b \tan t + 1/a}, \qquad a \in (0, \infty),\ b \in \mathbf{R}.$$

$\mathscr{F}_{3m}$: for each positive integer m
f: $(0, m\pi) \to (0, m\pi)$,

$$f(t) = \operatorname{Arctan} \frac{a \tan t}{b \tan t + 1/a}, \qquad a \in (0, \infty),\ b \in \mathbf{R}.$$

$\mathscr{F}_{4m}$: for each positive integer m
f: $(0, m\pi - \pi/2) \to (0, m\pi - \pi/2)$,
$f(t) = \operatorname{Arctan}(a \tan t)$, $a \in (0, \infty)$.

Let the topology on $\mathscr{F}_1$ be the relative topology on

$$\{(a, b, c, d) \in \mathbf{R}^4;\ |ad - bc| = 1\},$$

where $\mathbf{R}^4$ is considered with the usual topology.

Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two groups whose elements are (some) bijections of intervals I_1 and I_2 onto themselves, respectively. We say that the groups $\mathscr{G}_1$ and $\mathscr{G}_2$ are C^k-conjugate (with respect to φ) for some positive integer k if there is a C^k-diffeomorphism φ of interval I_1 onto interval I_2, i.e. $\varphi(I_1) = I_2$, $\varphi \in C^k(I_1)$, $\mathrm{d}\varphi(x)/\mathrm{d}x \neq 0$ on I_1,
such that

$$\mathscr{G}_2 = \varphi \circ \mathscr{G}_1 \circ \varphi^{-1} := \{\varphi \circ f \circ \varphi^{-1};\ f \in \mathscr{G}_1\}.$$

If $\mathscr{G}_1$ is a topological group the topology on $\mathscr{G}_2$ is induced by the conjugacy.

Let α be an element of a group. For any integer k define the element $\alpha^{[k]}$ as follows:

$\alpha^{[0]}$ is the unit element of the group,
$\alpha^{[k]} = \alpha^{[k-1]} \circ \alpha$ for positive k,
$\alpha^{[k]} = (\alpha^{-1})^{[-k]}$ for negative k,
α^{-1} being the inverse to α. Element $\alpha^{[k]}$ is called the k th iterate of α.

A group is said to be partially (linearly) ordered if the set of its elements is partially (linearly) ordered and, for each its elements α, β and γ, the relation $\alpha \leqq \beta$ implies both $\alpha \circ \gamma \leqq \beta \circ \gamma$ and $\gamma \circ \alpha \leqq \gamma \circ \beta$.

A partially ordered group is called archimedean if the following implication holds:

if $\alpha^{[n]} \leqq \beta$ is satisfied for some elements α and β and for all integers n, then α is the unit element of the group.

The following theorem is due to O. Hölder [3]: *There exists an order preserving isomorphism of any linearly ordered archimedean group into a subgroup of the additive group of real numbers* $\mathbf{R}$.

For proof see also for example A. I. Kokorin and V. M. Kopytov [4].

A group is said to be a cyclic group if there exists an element α of it such that all elements are iterates of α. Element α of this property is called a generator of the cyclic group. If, in addition,

$$\alpha^{[m]} \neq \alpha^{[n]}$$

for generator α and different integers m and n, then the group is an infinite cyclic group.

Now, consider an open interval $I \subset \mathbf{R}$. Let $n \geqq 1$ be an integer and $\mathscr{G}$ denote a group of some C^n-diffeomorphisms of I into I. Moreover, suppose that graphs of different elements of $\mathscr{G}$ do not intersect each other (on I).

III. THEOREM

If $\mathscr{G}$ is C^n-conjugate to a closed subgroup of increasing elements of the group $\mathscr{F}_1$, or $\mathscr{F}_2$, or $\mathscr{F}_{3m}$, or $\mathscr{F}_{4m}$,

then either $\mathscr{G}$ is trivial,

or $\mathscr{G}$ is an infinite cyclic group with a generator $h_e \in C^n(I)$, $\mathrm{d}h_e(x)/\mathrm{d}x > 0$ and $h_e(x) \neq x$ on I,

or $\mathscr{G}$ is C^n-conjugate to the group of all translations $\{h_c;\ c \in \mathbf{R}\}$,

$$h_c\colon \mathbf{R} \to \mathbf{R}, \qquad h_c(x) = x + c.$$

Proof

Since different elements of the group $\mathscr{G}$ do not intersect each other on I, $\mathscr{G}$ can be linearly ordered in the following manner:

for $h_1, h_2 \in \mathscr{G}$ we write $h_1 \leqq h_2$,

if either $h_1(x_0) < h_2(x_0)$ for some (then any) number $x_0 \in I$, or $h_1 = h_2$.

Moreover, $\mathscr{G}$ is archimedean, because for $h \neq \mathrm{id}_I$ there holds $h(x) \neq x$ on I an the sequences

$$\{h^{[i]}(x_0)\}_{i=1}^{\infty} \qquad \text{and} \qquad \{h^{[i]}(x_0)\}_{i=-1}^{-\infty}$$

converge to both ends of interval I for any $x_0 \in I$. Due to the Hölder Theorem there exists an order preserving isomorphism of $\mathscr{G}$ onto a subgroup $\tilde{\mathscr{G}}$ of the additive group $\mathbf{R}$.

If $\mathscr{G}$ is trivial then $\mathscr{G} = \{\mathrm{id}_I\}$ and $\tilde{\mathscr{G}} = \{0\}$.

Let $\mathscr{G}$ be not trivial and $\tilde{\mathscr{G}} = \{ie;\ i \in \mathbf{Z},\ 0 \neq e \in \mathbf{R}\}$ be an infinite cyclic group generated by a nonzero number e. Denote by h_e this element of group $\mathscr{G}$ that corresponds to the number e. Evidently $h_e \in C^n(I)$, $\mathrm{d}h_e(x)/\mathrm{d}x > 0$ and $h_e(x) \neq x$ on I. Moreover,

$$\mathscr{G} = \{h_e^{[i]};\ i \in \mathbf{Z}\},$$

h_e being a generator of the infinite cyclic group $\mathscr{G}$.

From now, let $\mathscr{G}$ be not trivial, neither it be an infinite cyclic group.

1. Consider first the case when $\mathscr{G}$ is C^n-conjugate to a closed subgroup of the fundamental group $\mathscr{F}_1$ with respect to a C^n-diffeomorphism φ of $\mathbf{R}$ onto I. Let $h \in \mathscr{G}$, $h \neq \mathrm{id}_I$. Then

$$\varphi^{-1}h\varphi(t) = \mathrm{Arctan}\,\frac{a_{11}\tan x + a_{12}}{a_{21}\tan x + a_{22}} \in \mathscr{F}_1$$

and $a_{11}a_{22} - a_{12}a_{21} = 1$ because $\mathrm{d}h(x)/\mathrm{d}x > 0$ on I.
Case 1a. Let

$$C^{-1}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} C = \begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix},$$

$b \in \mathbf{R}$, for a non-singular 2 by 2 matrix $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$. Without loss of generality, let $\det C = 1$. Denote by ψ one of the continuous functions, element of the group $\mathscr{F}_1$, given by the formula

$$\psi(t) = \mathrm{Arctan}\,\frac{c_{11}\tan t + c_{12}}{c_{21}\tan t + c_{22}}.$$

It can be verified that

$$\psi^{-1}\varphi^{-1}h\varphi\psi(t) = \mathrm{Arctan}\,(b^2 \tan t) \in \mathscr{F}_1.$$

Since $h(x) \neq x$ on I, we have

$$\psi^{-1}\varphi^{-1}h\varphi\psi(0) = k\pi$$

for some integer $k \neq 0$.
Case 1b. Let

$$C^{-1}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} C = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix},$$

$\det C = 1$ and $\psi \in F_1$ be defined as in case 1a. Then

$$\psi^{-1}\varphi^{-1}h\varphi\psi(t) = \mathrm{Arctan}\,(\tan t \pm 1) \in \mathscr{F}_1,$$

$$\psi^{-1}\varphi^{-1}h\varphi\psi(\pi/2) = \pi/2 + k\pi$$

for some $k \in \mathbf{Z} \setminus \{0\}$, otherwize h intersects id_I.
Case 1c. Let

$$C^{-1}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} C = \begin{pmatrix} \cos\omega\pi & \sin\omega\pi \\ -\sin\omega\pi & \cos\omega\pi \end{pmatrix},$$

$\omega \in \mathbf{R} \setminus \mathbf{Z}$, $\det C = 1$ and ψ be defined as above. Then

$$\psi^{-1}\varphi^{-1}h\varphi\psi(\mathrm{t}) = t + \omega\pi \in \mathscr{F}_1.$$

Now, let h and g be two different elements of the group $\mathscr{G}$ that do not belong to the same infinite cyclic group. Denote

$$h_1 := \varphi^{-1}h\varphi \in \mathscr{F}_1 \qquad \text{and} \qquad g_1 := \varphi^{-1}g\varphi \in \mathscr{F}_1.$$

Suppose first that

$$\psi_1^{-1}h_1\psi_1(t) = \mathrm{Arctan}\,(b_1^2 \tan t), \qquad \text{case 1a for } h,$$

and

$$\psi_2^{-1}g_1\psi_2(t) = \mathrm{Arctan}\,(b_2^2 \tan t), \qquad \text{case 1a for } g,$$

hold for suitable elements ψ_1 and ψ_2 of the fundamental group $\mathscr{F}_1$. Due to the initial values of $\psi_1^{-1}h_1\psi_1$ and $\psi_2^{-1}g_1\psi_2$ at 0, and with respect to the fact that the relation

$$\psi(t + n\pi) = \psi(t) + n\pi,$$

holds for every increasing element ψ of $\mathscr{F}_1$, there exist integers n_1 and n_2 such that either $h_1^{[n_1]}$ and $g_1^{[n_2]}$ coincide and then h and g belong to the some infinite cyclic group, or $h_1^{[n_1]}$ and $g_1^{[n_2]}$ intersect each other, the same being true for $h^{[n_1]}$ and $g^{[n_2]}$. However both cases were excluded from our considerations.

The same argument shows that neither the situation when

$$\psi_1^{-1}h_1\psi_1(t) = \mathrm{Arctan}\,(\tan t + 1), \qquad \text{case 1b} \qquad \text{for } h,$$

and

$$\psi_2^{-1}g_1\psi_2(t) = \mathrm{Arctan}\,(\tan t + 1), \qquad \text{case 1b} \qquad \text{for } g,$$

nor the case when

$$\psi_1^{-1}h_1\psi_1(t) = \mathrm{Arctan}\,(b^2 \tan t), \qquad \text{case 1a} \qquad \text{for } h,$$

and

$$\psi_2^{-1}g_1\psi_2(t) = \mathrm{Arctan}\,(\tan t + 1), \qquad \text{case 1b} \qquad \text{for } g,$$

can occur.

If one of the functions, say h, is of the form described in case 1c, i.e.

$$\psi_1^{-1}h_1\psi_1(t) = t + \omega\pi, \qquad \omega \in \mathbf{R} \setminus \mathbf{Z},$$

t
hen g cannot be of the form in case 1a

$$\psi_2^{-1}g_1\psi_2(t) = \text{Arctan}\,(b^2 \tan t) \qquad \text{for } k \neq 1,$$

or of the form of the case 1b

$$\psi_2^{-1}g_1\psi_2(t) = \text{Arctan}\,(\tan t + 1),$$

because then there again exist integers n_1 and n_2 such that $h^{[n_1]}$ and $g^{[n_2]}$ intersect each other.

Hence in this case 1 when the group $\mathscr{G}$ is C^n-conjugate to a closed subgroup of the whole fundamental group $\mathscr{F}$, it remains to consider only the situation when

$$\psi_1^{-1}h_1\psi_1(t) = t + \omega_1\pi, \qquad \omega_1 \in \mathbf{R}\setminus\mathbf{Z}$$

and either

$$\psi_2^{-1}g_1\psi_2(t) = \text{Arctan}\,(\tan t),$$

or

$$\psi_2^{-1}g_1\psi_2(t) = t + \omega_2\pi, \qquad \omega_2 \in \mathbf{R}\setminus\mathbf{Z}.$$

In the first of these cases

$$\psi_2^{-1}g_1\psi_2(t) = t + k_1\pi \qquad \text{for some } k_1 \in \mathbf{Z}\setminus\{0\}$$

due to the initial value of this function at 0. Since

$$\psi_1^{-1}g_1\psi_1(t) = (\psi_2\psi_1)^{-1}\,\psi_2 g_1\psi_2^{-1}(\psi_2\psi_1)\,(t)$$

and $\psi_2\psi_1$ is again an increasing element of the fundamental group $\mathscr{F}_1$, i.e. $\psi_2\psi_1(t + k\pi) = \psi_2\psi_1(t) + k\pi$, we have

$$\psi_1^{-1}g_1\psi_1(t) = (\psi_2\psi_1)^{-1}\,(\psi_2\psi_1(t) + k\pi) = t + k\pi, \qquad k \in \mathbf{Z}.$$

Hence ω_1 is an irrational number, otherwise h_1 and g_1 belong to the same infinite cyclic group and the same is true for the functions h and g, that was already excluded. However, when ω_1 is irrational, then the union of graphs of functions $h_1^{[n_1]}$ and $g_1^{[n_2]}$ for all n_1 and n_2 from $\mathbf{Z}$ is a dense set in $\mathbf{R}^2$. Now we have

$$h = \psi_1\varphi(\text{id} + \omega_1\pi)\,\varphi^{-1}\psi_1^{-1} \qquad \text{and} \qquad g = \psi_1\varphi(\text{id} + k\pi)\,\varphi^{-1}\psi_1^{-1},$$

where $\psi_1\varphi$ is a C^n-diffeomorphism of $\mathbf{R}$ onto I. Since the group $\mathscr{G}$ is closed, we conclude that it is C^n-conjugate to the group of all translations

$$t \mapsto t + c, \qquad \text{for all } c \in \mathbf{R}.$$

Now, let

$$\psi_1^{-1}h_1\psi_1(t) = t + \omega_1\pi, \qquad \omega_1 \in \mathbf{R}\setminus\mathbf{Z}, \qquad \text{case 1c for } h,$$

and

$$\psi_2^{-1}g_1\psi_2(t) = t + \omega_2\pi, \qquad \omega_2 \in \mathbf{R}\setminus\mathbf{Z}, \qquad \text{case 1c for } g.$$

Then

$$h_1^{[n_1]}(t) = \psi_1(\psi_1^{-1}(t) + n_1\omega_1\pi),$$
$$g_1^{[n_2]}(t) = \psi_2(\psi_2^{-1}(t) + n_2\omega_2\pi)$$

and the condition $h_1^{[n_1]}(t) \neq g_1^{[n_2]}(t)$ on $\mathbf{R}$ implies

$$\psi_3(t + n_1\omega_1\pi) \neq \psi_3(t) + n_2\omega_2\pi$$

for $\psi_3 := \psi_2^{-1}\psi_1 \in \mathcal{F}$, otherwise $h_1^{[n_1]}$ coincides with $g_1^{[n_2]}$ that shows that h_1 and g_1 belong to the same infinite cyclic group, the case already excluded from our considerations. Since

$$\psi_3(t + \pi) = \psi_3(t) + \pi,$$

we have

$$\psi_3(t) = t + p(t),$$

where p is a π-periodic function: $p(t + \pi) = p(t) \in C^3(\mathbf{R})$. Hence

$$t + n_1\omega_1\pi + p(t + n_1\omega_1\pi) \neq t + p(t) + n_2\omega_2\pi,$$

or

$$p(t + n_1\omega_1\pi) - p(t) \neq (n_2\omega_2 - n_1\omega_1),$$

for all $t \in \mathbf{R}$ and all $n_1, n_2 \in \mathbf{Z}$, $n_1^2 + n_2^2 \neq 0$.

If $n_2\omega_2 - n_1\omega_1 = 0$ for some n_1 and n_2 then either

$$p(t + n_1\omega_1\pi) > p(t) \qquad \text{on } \mathbf{R},$$

or

$$p(t + n_1\omega_1\pi) < p(t) \qquad \text{on } \mathbf{R}.$$

Neither of these cases is possible for any continuous periodic function p.

Hence $n_2\omega_2 - n_1\omega_1 \neq 0$ for all integers n_1 and n_2, $n_1^2 + n_2^2 \neq 0$, that means that ω_1 and ω_2 are rationally independent. Then for each number $t_0 \in \mathbf{R}$ the set

$$\{g_1^{[n_2]} \circ h_1^{[n_1]}(t_0);\ n_1, n_2 \in \mathbf{Z}\}$$

is dense in $\mathbf{R}$, because for different couples (n_1, n_2) and (n_1^*, n_2^*) the values, $g_1^{[n_2]} \circ h_1^{[n_1]}(t_0)$ and $g_1^{[n_2^*]} \circ h_1^{[n_1^*]}(t_0)$ are different, there are infinite number of coumples (n_1, n_2) satisfying $|n_1\omega_1 + n_2\omega_2| < \varepsilon$ for any given $\varepsilon > 0$ and, moreover, ψ_1 and ψ_2 are C^n-diffeomorphisms of $\mathbf{R}$ onto $\mathbf{R}$ for any $n \in \mathrm{N}$ satisfying

$$\psi_1(t) = t + p_1(t), \qquad \psi_2(t) = t + p_2(t),$$

with π-periodic functions p_1 and p_2.

Since φ is a C^n-diffeomorphism of $\mathbf{R}$ onto I, and the group $\mathcal{G}$ is archimedean and closed, the union of graphs of all its elements is the whole square I^2. In such a situation we may apply Theorem 1 of G. Blanton and J. A. Baker [1] which states: "Each group whose elements are C^n-diffeomorphisms of an interval I onto I and such that to each point $(x_0, y_0) \in I \times I$ there exists just one element h of the group satisfying $h(x_0) = y_0$, is formed by functions

$$\chi(\chi^{-1}(x) + c),$$

where χ is a C^n-diffeomorphism of $\mathbf{R}$ onto I and c ranges through the real numbers". In our case we may write

$$G = \chi \circ h_c \circ \chi^{-1},$$

where $h_c: \mathbf{R} \to \mathbf{Z}$, $h_c(t) = t + c$, $c \in \mathbf{R}$.

2. Now, suppose that

$$\varphi^{-1} h \varphi(t) = \mathrm{Arctan} \frac{a \tan t}{b \tan t + 1/a}, \qquad t \in R_+,$$

$a \in \mathbf{R}_+$, $b \in \mathbf{R}$, is an element of the two-parametric group $\mathscr{F}_2$ of increasing functions. Since $\lim_{t \to 0_+} \varphi^{-1} h \varphi(t) = 0$, we have

$$\varphi^{-1} h \varphi(\pi) = \pi,$$

hence $\varphi^{-1} h \varphi = \mathrm{id}_{\mathbf{R}_+}$ that is excluded from our considerations.

3*m*. If

$$\varphi^{-1} h \varphi(t) = \mathrm{Arctan} \frac{a \tan t}{b \tan t + 1/a}, \qquad \varphi^{-1} h \varphi; (0, m\pi) \to (0, m\pi),$$

$$a \in \mathbf{R}_+, b \in \mathbf{R}, \qquad \text{then} \qquad \lim_{t \to 0_+} \varphi^{-1} h \varphi(t) = 0 \qquad \text{and} \qquad \lim_{t \to \pi_-} \varphi^{-1} h \varphi(t) = \pi,$$

because h as well as $\varphi^{-1} h \varphi$ are increasing functions. Hence $m = 1$, otherwise $h = \mathrm{id}_I$ that contradicts to our assumptions. However, if $a \neq 1$ and $b \neq 0$ then the equation

$$\arctan \frac{a \tan t}{b \tan t + 1/a} = t,$$

i.e.

$$a \tan t = (b \tan t + 1/a) \tan t$$

is satisfied for $t_1 \in (0, \pi)$ where

$$\tan t_1 = \frac{a^2 - 1}{ab}.$$

This case is excluded from our considerations. Even the case $b = 0$ impossible since then

$$\varphi^{-1}h\varphi(t) = \arctan(a^2 \tan t)$$

intersects $\mathrm{id}_{(0,\pi)}$ at $\pi/2$.

If $a = 1$ then

$$\begin{aligned}\varphi^{-1}h\varphi(t) &= \arctan\frac{\tan t}{b\tan t + 1}\\ &= \operatorname{arccot}\frac{1 + b\tan t}{\tan t}\\ &= \operatorname{arccot}(\cot t + b), \qquad t \in (0, \pi),\end{aligned}$$

hence h is conjugate to $x \mapsto x + b$, $x \in \mathbf{R}$ for a fixed $b \in \mathbf{R}$ by means of the function $\varphi \circ \operatorname{arccot} : \mathbf{R} \to I$.

Now, let h and g be two different elements of the stationary group $\mathcal{G}$ that do not belong to the same infinite cyclic group. Then

$$\psi^{-1}h\psi(x) + x + b_1 \qquad \text{and} \qquad \psi^{-1}g\psi(x) = x + b_2$$

on $\mathbf{R}$ where $\psi = \varphi \circ \operatorname{arccot} \in C^n(\mathbf{R})$, and b_1/b_2 is irrational. Since the union of the graphs of functions

$$x \mapsto x + n_1b_1 + n_2b_2 \qquad \text{for all } n_1, n_2 \in \mathbf{Z}$$

is dense in $\mathbf{R}^2$, and the group $\mathcal{G}$ is closed, it is C^n-conjugate to the group of all translations:

$$\{x \mapsto x + c,\ c \in \mathbf{R}\}.$$

4*m*. Finally, if

$$\varphi^{-1}h\varphi(t) = \operatorname{Arctan}(a\tan t), \qquad a > 0,$$

$$\varphi^{-1}h\varphi\colon (0, m\pi - \pi/2) \to (0, m\pi - \pi(2),$$

$$\text{then} \qquad \lim_{t\to 0+}\varphi^{-1}h\varphi(t) = 0 \qquad \text{and} \qquad \lim_{t\to\pi/2-}\varphi^{-1}h\varphi(t) = \pi/2,$$

and hence $m = 1$. In this case h is conjugate to the function $x \to x + \ln a$, $x \in \mathbf{R}$ by means of the C^n-diffeomorphism $\varphi \circ \arctan \circ \exp\colon \mathbf{R} \to I$.

Now, analogously to case 3*m*, if h and g are two different elements of $\mathcal{G}$ that do not belong to the same infinite cyclic group, they are C^n-conjugate to $x + b_1$ and $x + b_2$, respectively, with respect to the some C^n-diffeomorphism, the quotient b_1/b_2 being irrational. Hence the group $\mathcal{G}$ is C^n-conjugate to the group

$$\{x \mapsto x + c;\ c \in \mathbf{R}\},$$

that finishes the proof of the theorem.

IV. REMARK

The present paper gives technical details of the proof of Theorem 6.3.5 in the monograph [6], where main steps of the proof were outlined.

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ASYMPTOTIC BEHAVIOUR OF THE EQUATION $\dot{z} = G(t, z)[h(z) + g(t, z)]$

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In honour of the 60th birthday anniversary of Prof. M. Ráb

Abstract. Asymptotic properties of the solutions of an equation $\dot{z} = G(t, z)[h(z) + g(t, z)]$ with real-valued function G and complex-valued functions h, g are studied. The technique of the proofs of results is based on the modified Liapunov function method. The results are applied to the generalized Riccati equation $\dot{z} = q(t, z) - p(t)z^2$.

Key words. Asymptotic behaviour, Liapunov function, Riccati equation.

MS Classification. 34 D 05, 34 D 20.

1. INTRODUCTION

Consider the equation

$$\dot{z} = h(z),$$

where h is a holomorphic function in a simply connected region Ω containing zero which satisfies the conditions $h(z) = 0 \Leftrightarrow z = 0$, $h^{(j)}(0) = 0$ $(j = 1, \ldots, n - 1)$, $h^{(n)}(0) \neq 0$, where $n \geqq 2$ is an integer. The paper is concerned with the asymptotic behaviour of the solutions of the perturbed equation

$$\dot{z} = G(t, z)[h(z) + g(t, z)], \tag{1.1}$$

where G is a real-valued function and h, g are complex-valued functions, t or z being a real or complex variable, respectively. The general results for the equation (1.1) are formulated in Section 2. The last section is devoted to the application of these results to the equation

(1.2) $$\dot{z} = q(t, z) - p(t)\, z^2.$$

This application gives the generalization of some results of M. Ráb [6]. The technique of the proofs is based on the Liapunov function method with "Liapunov-like" function $W(z)$ defined in [1].

The case $n = 1$, which is qualitatively different from the case $n \geqq 2$, was investigated in several papers; for the list of these papers see [1] or [2]. The asymptotic behaviour of the solutions of the Riccati equation

(1.3) $$\dot{z} = q(t) - p(t)\, z^2,$$

which is a special case of (1.2) was studied by M. Ráb and Z. Tesařová. Some results dealing with the asymptotic properties of the solutions of (1.1) under the assumption $n \geqq 2$ were published in [2] or [3]. Unfortunately, the assumptions of these results make necessary the existence of the trivial solution of (1.1). Moreover, the inequalities of the type (2.3) were assumed to be satisfied at some points arbitrarily close to the point $z = 0$. This fact is very restrictive and the results are not applicable to the equations (1.2), (1.3). In the present paper and in [4] we attempt to remove this limitation.

In contradistinction to the present paper the paper [4] deals with the sufficient conditions assuring the existence of the solutions $z(t)$ of (1.1) for $t \to \infty$ and

(1.4) $$\liminf_{t \to \infty} | z(t) | \leqq \delta,$$

where $\delta \geqq 0$ is a given nonnegative number. Then the conditions which guarantee

(1.5) $$\limsup_{t \to \infty} | z(t) | \leqq \delta$$

for any solution $z(t)$ of (1.1) satisfying (1.4) are obtained. Even though these results generalize several results of [8], they do not allow to get the results of the type of Theorem 3 and 4 of the present paper.

In the whole paper we use the following notation:

C	– set of all complex numbers
N	– set of all positive integers
R	– set of all real numbers
I	– interval $[t_0, \infty)$
Ω	– simply connected region in C such that $0 \in \Omega$
$C(\Gamma)$	– class of all continuous real-valued functions defined on the set Γ
$\tilde{C}(\Gamma)$	– class of all continuous complex-valued functions defined on the set Γ

$\mathscr{H}(\Omega)$	– class of all complex-valued functions holomorphic in the region Ω
Int Γ	– interior of a Jordan curve with the geometric image Γ
Cl Γ	– closure of a set $\Gamma \subset C$
Bd Γ	– boundary of a set $\Gamma \subset C$
$k, W(z)$	– see [1, pp. 66–67]
$\lambda_+, \lambda_- \mathscr{T}^+, \mathscr{T}^-, \varphi$	– see [1, pp. 73–74]
$B(0, \delta)$	– the set $\{z \in C : \lvert z \rvert \leqq \delta\}$.

Let $\mathscr{S}^+ \in \mathscr{T}^+/\varphi$ and $\mathscr{S}^- \in \mathscr{T}^-/\varphi$ be fixed. Then $\mathscr{S}^+ = \{\hat{K}(\lambda) : 0 < \lambda < \lambda_+\}$, $\mathscr{S}^- = \{\hat{K}(\lambda) : \lambda_- < \lambda < \infty\}$, where $\hat{K}(\lambda)$ are the geometric images of Jordan curves such that: $0 \in \hat{K}(\lambda)$, the equality $W(z) = \lambda$ holds for $z \in \hat{K}(\lambda) - \{0\}$ and $\hat{K}(\lambda_1) - \{0\} \subset \operatorname{Int} \hat{K}(\lambda_2)$ for $0 < \lambda_1 < \lambda_2 < \lambda_+$ or $\hat{K}(\lambda_2) - \{0\} \subset \operatorname{Int} \hat{K}(\lambda)$ for $\lambda_- < \lambda_1 < \lambda_2 < \infty$. Define

$$K(\lambda_1, \lambda_2) = \bigcup_{\lambda_1 < \mu < \lambda_2} \hat{K}(\mu) - \{0\} \qquad \text{for } 0 \leqq \lambda_1 < \lambda_2 \leqq \lambda_+$$

and

$$K(\lambda_1, \lambda_2) = \bigcup_{\lambda_2 < \mu < \lambda_1} \hat{K}(\mu) - \{0\} \qquad \text{for } \lambda_- \leqq \lambda_2 < \lambda_1 \leqq \infty.$$

2. MAIN RESULTS

Suppose $G(t, z)\,[h(z) + g(t, z)] \in \tilde{C}(I \times \Omega)$, $G \in C(I \times (\Omega - \{0\}))$, $g \in \tilde{C}(I \times (\Omega - \{0\}))$, $h \in \mathscr{H}(\Omega)$. Assume that $h(z) = 0 \Leftrightarrow z = 0$ and $h^{(j)}(0) = 0$ $(j = 1, 2, \ldots, n - 1)$, $h^{(n)}_{(0)} \neq 0$, where $n \geqq 2$ is an integer.

Theorem 1. *Let $\delta \geqq 0$, $\vartheta_1 > 0$, $\vartheta \leqq \lambda_+$. Suppose there is a function $E(t) \in C(I)$ such that*

$$\sup_{t_0 \leqq s \leqq t < \infty} \int_s^t E(\xi)\,d\xi = \varkappa < \infty, \tag{2.1}$$

$$\vartheta_1 e^{\varkappa} < \vartheta \tag{2.2}$$

and

$$G(t, z) \operatorname{Re}\left\{k h^{(n)}_{(0)} \left(1 + \frac{g(t, z)}{h(z)}\right)\right\} \leqq E(t) \tag{2.3}$$

holds for $t \geqq t_0$, $z \in K(\vartheta_1, \vartheta)$, $\lvert z \rvert > \delta$.

If a solution $z(t)$ of (1.1) satisfies

$$z(t_1) \in \operatorname{Cl} K(0, \gamma),$$

where $t_1 \geqq t_0$, $0 < \gamma e^{\varkappa} < \vartheta$ *and*

$$z(t) \notin B(0, \delta) - K(0, \vartheta_1)$$

for all $t \geqq t_1$ *for which* $z(t)$ *exists, then*

$$z(t) \in \mathrm{Cl}\, K(0, \beta) \qquad \textit{for } t \geqq t_1,$$

where $\beta = e^{\varkappa} \max [\gamma, \vartheta_1]$.

Proof. Let $\mathcal{M} = \{t \geqq t_1 : | z(t) | > \delta, z(t) \in K(\vartheta_1, \vartheta)\}$. For $t \in \mathcal{M}$ we have

$$\dot{W}(z) = G(t, z)\, W(z) \operatorname{Re}\left\{k h_{(0)}^{(n)} \left[1 + \frac{g(t, z)}{h(z)}\right]\right\},$$

where $z = z(t)$. Using (2.3) we obtain

$$\dot{W}(z(t)) \leqq E(t)\, W(z(t)) \tag{2.4}$$

for $t \in \mathcal{M}$. Suppose there is a $t^* > t_1$ such that $z(t^*) \in K(\beta, \vartheta)$. Without loss of generality it may be assumed that $z(t) \in K(0, \vartheta)$ for $t \in [t_1, t^*]$. There exists a γ_1 such that $\beta < \gamma_1 e^{\varkappa} < W(z(t^*))$. Obviously $\vartheta_1 < \gamma_1 < W(z(t^*))$, $\gamma_1 > \gamma$. Put $t_2 = \sup \{t \in [t_1, t^*] : z(t) \in \mathrm{Cl}\, K(0, \gamma_1)\}$. From (2.4) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \{W(z(t)) \exp [- \int_{t_2}^{t} E(s)\, \mathrm{d}s]\} \leqq 0, \qquad t \in [t_2, t^*].$$

Integration over $[t_2, t^*]$ yields

$$W(z(t^*)) \exp [- \int_{t_2}^{t^*} E(s)\, \mathrm{d}s] - W(z(t_2)) \leqq 0.$$

Using (2.1) and $W(z(t_2)) = \gamma_1$, we get

$$W(z(t^*)) \leqq \gamma_1 \exp [\int_{t_2}^{t^*} E(s)\, \mathrm{d}s] \leqq \gamma_1 e^{\varkappa} < W(z(t^*))$$

and we have a contradiction. Therefore

$$z(t) \in \mathrm{Cl}\, K(0, \beta) \qquad \text{for } t \geqq t_1.$$

Theorem 2. *Let* $\vartheta_j > 0$, $\vartheta \leqq \lambda_+$, $s_j \in I$, $\delta_j \geqq 0$ *for* $j \in N$. *Suppose there are functions* $E_j(t) \in C(I)$ *such that*

$$\int_{t_0}^{\infty} E_j(s)\, \mathrm{d}s = -\infty \qquad (j = 2, 3, \ldots),$$

$$\sup_{s_j \leqq s \leqq t < \infty} \int_{s}^{t} E_j(\xi)\, \mathrm{d}\xi = \varkappa_j < \infty \qquad (j = 1, 2, \ldots),$$

$$\vartheta_j e^{\varkappa_j} < \vartheta \qquad (j = 1, 2, \ldots), \tag{2.5}$$

and,

$$G(t, z) \operatorname{Re}\left\{kh^{(n)}_{(0)}\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_j(t) \tag{2.6}$$

holds for $t \geqq s_j$, $z \in K(\vartheta_j, \vartheta)$, $|z| > \delta_j$, $j \in N$. *Denote*

$$\vartheta^* = \inf_{j \in N} [\vartheta_j e^{\varkappa_j}].$$

If a solution $z(t)$ *of* (1.1) *satisfies*

$$z(t_1) \in K(0, \vartheta e^{-\varkappa_1}),$$

where $t_1 \geqq s_1$, *and*

$$z(t) \notin B(0, \delta_j) - K(0, \vartheta_j) \tag{2.7}$$

for all $t \geqq t_1$ *for which* $z(t)$ *exists and all* $j \in N$, *then to any* ε, $\vartheta^* < \varepsilon < \lambda_+$, *there is a* $T > 0$ *such that*

$$z(t) \in K(0, \varepsilon)$$

for $t \geqq t_1 + T$.

Proof. Put $\mathcal{M}_j = \{t \geqq s_j : |z(t)| > \delta_j,\ z(t) \in K(\vartheta_j, \vartheta)\}$. For $t \in \mathcal{M}_j$ we obtain

$$\dot{W}(z) = G(t, z)\, W(z) \operatorname{Re}\left\{kh^{(n)}_{(0)}\left[1 + \frac{g(t, z)}{h(z)}\right]\right\}.$$

Using (2.6) we get

$$\dot{W}(z(t)) \leqq E_j(t)\, W(z(t)). \tag{2.8}$$

By Theorem 1 we have $z(t) \in K(\vartheta)$ for $t \geqq t_1$. Let ε, $\vartheta^* < \varepsilon < \lambda_+$ be given. Without loss of generality it may be supposed that $\varepsilon < \vartheta$. Choose a fixed positive integer j such that

$$\vartheta_j e^{\varkappa_j} < \varepsilon.$$

Put $\sigma = \max [s_j, t_1]$. Let $T > |s_j - s_1|$ be such that

$$\int_\sigma^t E_j(s)\, ds < \ln \frac{\varepsilon}{2\vartheta}$$

for $t \geqq t_1 + T$. Clearly $t_1 + T > \sigma$.

We claim that $z(t) \in K(\varepsilon)$ for $t \geqq t_1 + T$. If it is not the case, there exists a $t^* \geqq$ $\geqq t_1 + T$ for which

$$z(t^*) \notin K(\varepsilon). \tag{2.9}$$

Using Theorem 1 we have

$$z(t) \in K(\varepsilon e^{-\varkappa_j}, \vartheta) \cup [\mathring{K}(\varepsilon e^{-\varkappa_j}) - \{0\}] = K(\vartheta_j, \vartheta)$$

for $t \in [\sigma, t^*]$. In view of (2.7), $|z(t)| > \delta_j$. The inequality (2.8) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\{W(z(t))\exp[-\int_{\sigma}^{t}E_j(s)\,\mathrm{d}s]\}\leqq 0, \qquad t\in\mathcal{M}_j.$$

Integration over $[\sigma, t^*]$ yields

$$W(z(t^*))\exp[-\int_{\sigma}^{t^*}E_j(s)\,\mathrm{d}s]-W(z(\sigma))\leqq 0.$$

Therefore

$$W(z(t^*))\leqq W(z(\sigma))\exp[\int_{\sigma}^{t^*}E_j(s)\,\mathrm{d}s]\leqq \vartheta\frac{\varepsilon}{2\vartheta}=\frac{\varepsilon}{2}<\varepsilon,$$

which contradicts (2.9) and proves $z(t)\in K(\varepsilon)$ for $t\geqq t_1+T$.

Analogously we can prove the following two theorems corresponding to the case $\vartheta\geqq\lambda_-$:

Theorem 1′. *Let* $\delta\geqq 0$, $\vartheta\geqq\lambda_-$. *Suppose there is a function* $E(t)\in C(I)$ *such that*

$$\sup_{t_0\leqq s\leqq t<\infty}\int_{s}^{t}E(\xi)\,\mathrm{d}\xi=\varkappa<\infty,$$

$$\vartheta e^{\varkappa}<\vartheta_1<\infty$$

and

$$-G(t,z)\operatorname{Re}\left\{kh_{(0)}^{(n)}\left[1+\frac{g(t,z)}{h(z)}\right]\right\}\leqq E(t)$$

holds for $t\geqq t_0$, $z\in K(\vartheta_1,\vartheta)$, $|z|>\delta$.

If a solution $z(t)$ *of* (1.1) *satisfies*

$$z(t_1)\in \mathrm{Cl}\,K(\infty,\gamma),$$

where $t_1\geqq t_0$, $\vartheta<\gamma e^{-\varkappa}<\infty$ *and*

$$z(t)\notin B(0,\delta)-K(\infty,\vartheta_1)$$

for all $t\geqq t_1$ *for which* $z(t)$ *exists, then*

$$z(t)\in\mathrm{Cl}\,K(\infty,\beta)\qquad \textit{for } t\geqq t_1,$$

where $\beta=e^{-\varkappa}\min[\gamma,\vartheta_1]$.

Theorem 2′. *Let* $\vartheta\geqq\lambda_-$, $\vartheta_j<0$, $s_j\in I$, $\delta_j\geq 0$ *for* $j\in N$. *Suppose there are functions* $E_j(t)\in C[t_0,\infty)$ *such that*

$$\int_{t_0}^{\infty}E_j(s)\,\mathrm{d}s=-\infty\qquad (j=2,3,\ldots),$$

$$\sup_{s_j \leqq s \leqq t < \infty} \int_s^t E_j(\xi)\, d\xi = \varkappa_j < \infty \qquad (j = 1, 2, \ldots),$$

$$\vartheta e^{\varkappa_j} < \vartheta_j \qquad (j = 1, 2, \ldots)$$

and,

$$-G(t, z) \operatorname{Re}\left\{k h_{(0)}^{(n)}\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_j(t)$$

holds for $t \geqq s_j$, $z \in K(\vartheta_j, \vartheta)$, $|z| > \delta_j$, $j \in N$. *Denote*

$$\vartheta^* = \sup_{j \in N} [\vartheta_j e^{-\varkappa_j}].$$

If a solution $z(t)$ *of* (1.1) *satisfies*

$$z(t_1) \in K(\infty, \vartheta e^{\varkappa_1}),$$

where $t_1 \geqq s_1$, *and*

$$z(t) \notin B(0, \delta_j) - K(\infty, \vartheta_j)$$

for all $t \geqq t_1$ *for which* $z(t)$ *exists and all* $j \in N$, *then to any* ε, $\lambda_- < \varepsilon < \vartheta^*$, *there is a* $T > 0$ *such that*

$$z(t) \in K(\infty, \varepsilon)$$

for $t \geqq t_1 + T$.

3. APPLICATION TO THE EQUATION $\dot{z} = q(t, z) - p(t) z^2$

Supposing that $q \in \tilde{C}(I \times C)$, $p \in \tilde{C}(I)$ and $a \in C$, $a \neq 0$, the equation

$$\dot{z} = q(t, z) - p(t) z^2 \tag{3.1}$$

can be written in the form

$$\dot{z} = G(t, z)[h(z) + g(t, z)], \tag{3.2}$$

where $h(z) = -az^2$, $G(t, z) = 1$ and $g(t, z) = q(t, z) + az^2 - p(t) z^2$. In view of [1, Example 1], where $\Omega = C$, $b = -a$, we get $h'(z) = -2az$, $h''(z) = -2a$, $n = 2$, $W(z) = \exp[\operatorname{Re}(2\bar{a}z^{-1})]$, $\lambda_+ = \lambda_- = 1$, $k = -\bar{a}$. The sets $\bar{K}(\lambda)$, where $0 < \lambda < \lambda_+ = 1$ or $1 = \lambda_- < \lambda < \infty$, are circles with centres $\dfrac{\bar{a}}{\ln \lambda}$ and radii $\dfrac{|a|}{|\ln \lambda|}$, $K(0, 1) = \{z \in C : \operatorname{Re}(az) < 0\}$, $K(\infty, 1) = \{z \in C : \operatorname{Re}(az) > 0\}$.

For $a \in C$, $a \neq 0$, $A > 0$, $B > 0$, $\delta \in \left(0, \dfrac{\pi}{4}\right]$ denote

$$\Omega_{A,B}(a) = \{z \in C : -A \operatorname{Re}[a^2 z^2] - B |\operatorname{Im}[a^2 z^2]| > 0\},$$

$$\Omega_\delta(a) = \left\{ z = \mu e^{i\varphi} : \mu \in R - \{0\}, \operatorname{Arg} \bar{a} + \frac{\pi}{2} - \delta < \varphi < \operatorname{Arg} \bar{a} + \frac{\pi}{2} + \delta \right\}.$$

Obviously,

$$\Omega_{A,B}(a) \subset \Omega_{\pi/4}(a) = \{z \in C : \operatorname{Re}(a^2z^2) < 0\}$$

for any $A, B > 0$, and, to any $A, B > 0$ there exists a $\delta_0 \in \left(0, \frac{\pi}{4}\right)$ such that

$$\Omega_\delta(a) = \Omega_{A,B}(a) \qquad \text{for } \delta \in (0, \delta_0].$$

First we shall prove the following lemma:

Lemma 1. *Assume there are $a \in C$ and $C \geqq 0$ such that*

$$\operatorname{Re}[\bar{a}p(t)] > 0 \qquad \text{for } t \in I, \tag{3.3}$$

$$\liminf_{t\to\infty} \operatorname{Re}[\bar{a}p(t)] > 0, \qquad \limsup_{t\to\infty} |\operatorname{Im}[\bar{a}p(t)]| < \infty, \tag{3.4}$$

$$\operatorname{Re}[aq(t, z)] \geqq -C\,|\operatorname{Im}[a^2z^2]| \qquad \text{for } t \in I,\ z \in \Omega_{\pi/4}(a) \tag{3.5}$$

and

$$q(t, 0) \neq 0 \qquad \text{for } t \in I. \tag{3.6}$$

Then every solution $z(t)$ of (3.1) *satisfying at $t_1 \geqq t_0$ the condition* $\operatorname{Re}[az(t_1)] \geqq 0$ *fulfils* $\operatorname{Re}[az(t)] \geqq 0$ *for all $t > t_1$ for which $z(t)$ exists.*

Moreover, $\operatorname{Re}[az(t)] > 0$ *provided $z(t) \neq 0$.*

Proof. Choose $A, B > 0$ so that

$$\operatorname{Re}[\bar{a}p(t)] \geqq |a|^2 A, \qquad |\operatorname{Im}[\bar{a}p(t)]| \leqq |a|^2 (B - C)$$

for $t \geqq t_1$. There exists $\delta_0 \in \left(0, \frac{\pi}{4}\right)$ with the property $\Omega_{\delta_0}(a) \subset \Omega_{A,B}(a)$. For $t \geqq t_1$ such that $z = z(t) \in \Omega_{\delta_0}(a)$ we obtain

$$\begin{aligned}
\frac{d}{dt} \operatorname{Re}[az(t)] &= \operatorname{Re}[a\dot{z}(t)] = \operatorname{Re}[aq(t, z)] - \operatorname{Re}[ap(t)\,z^2] = \\
&= \operatorname{Re}[aq(t, z)] - |a|^{-2} \operatorname{Re}[\bar{a}p(t)\,a^2z^2] = \\
&= \operatorname{Re}[aq(t, z)] - |a|^{-2} \{\operatorname{Re}[\bar{a}p(t)] \operatorname{Re}[a^2z^2] - \\
&- \operatorname{Im}[\bar{a}p(t)] \operatorname{Im}[a^2z^2]\} \geqq -C\,|\operatorname{Im}[a^2z^2]| - A \operatorname{Re}[a^2z^2] - \\
&- (B - C)\,|\operatorname{Im}[a^2z^2]| \geqq -A \operatorname{Re}[a^2z^2] - B\,|\operatorname{Im}[a^2z^2]| > 0.
\end{aligned}$$

If $z(t) = 0$ we have

$$\frac{d}{dt} \operatorname{Re}[az(t)] = \operatorname{Re}[aq(t, 0)] > 0$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}[az(t)] = \operatorname{Re}[aq(t, 0)] = 0. \tag{3.7}$$

Because of (3.6) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}[az(t)] = \operatorname{Im}[aq(t, 0)] \neq 0$$

in the case (3.7). In view of the fact that Re $[az] = 0$ implies $z \in \Omega_{\delta_0}(a) \cup \{0\}$, we get Re $[az(t)] \geqq 0$ for all $t \geqq t_1$ for which $z(t)$ is defined. Clearly, Re $[az(t)] > 0$ if $z(t) \neq 0$.

Remark. If the condition (3.6) of Lemma 2 is replaced by Re $[aq(t, 0)] > 0$, we get the assertion Re $[az(t)] > 0$ for all $t > t_1$ for which $z(t)$ exists.

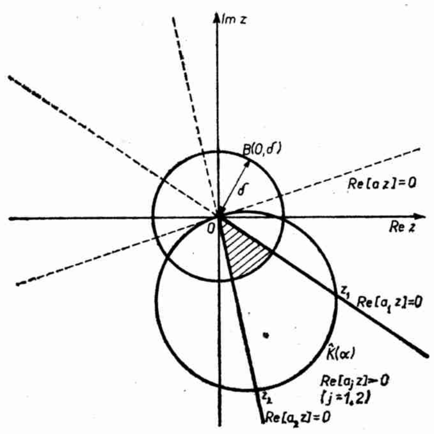
The next lemma will be useful in our further considerations.

Lemma 2. *Let* $\delta > 0$, $a_1, a_2 \in C$ *and let* a_1, a_2 *be linearly independent. If* $a = (a_1 + a_2)/2$,

$$1 < \alpha \leqq \exp\left[\delta^{-1} \min_{m=1,2}\left(|a_m| \left|\operatorname{Im}\frac{a_{3-m}}{a_m}\right|\right)\right] \tag{3.8}$$

and

$$\operatorname{Re}[a_m z] > 0 \qquad (m = 1, 2), \qquad \textit{then} \qquad z \notin B(0, \delta) - K(\infty, \alpha).$$



Proof. Since Re $[a_m z] > 0$ $(m = 1, 2)$ implies Re $[az] > 0$, it is sufficient to prove that $\delta \leqq \min[|z_1|, |z_2|]$, where $z_m \neq 0$ is the intersection of $\hat{K}(\alpha)$ with the line Re $[a_m z] = 0$.

Supposing

$$W(z_m) = \exp\{\mathrm{Re}\,[2\bar{a}z_m^{-1}]\} = \alpha \qquad \text{and} \qquad \mathrm{Re}\,[a_m z_m] = 0,$$

there exists a $\tau_m \in \boldsymbol{R}$ such that

$$z_m = i\bar{a}_m\tau_m \qquad \text{and} \qquad \mathrm{Re}\,\frac{2\bar{a}}{i\bar{a}_m\tau_m} = \ln\alpha.$$

Hence

$$\tau_m = [\ln\alpha]^{-1}\,\mathrm{Re}\,\frac{\bar{a}_{3-m}}{i\bar{a}_m}$$

and

$$z_m = \frac{i\bar{a}_m}{\ln\alpha}\,\mathrm{Re}\,\frac{\bar{a}_{3-m}}{i\bar{a}_m} = -\frac{i\bar{a}_m}{\ln\alpha}\,\mathrm{Im}\,\frac{a_{3-m}}{a_m}.$$

Therefore

$$|z_m| = \frac{|a_m|}{|\ln\alpha|}\left|\mathrm{Im}\,\frac{a_{3-m}}{a_m}\right|.$$

In view of (3.8) we obtain $\delta \leqq \min\,[|z_1|, |z_2|]$.

Applying Theorem 2′ and using Lemma 1 and Lemma 2 we obtain

Theorem 3. *Suppose there are $a_1, a_2 \in C$ linearly independent such that the following inequalities are fulfilled for $m = 1, 2$:*

(3.9) $$\mathrm{Re}\,[\bar{a}_m p(t)] > 0 \qquad \textit{for } t \in I,$$

(3.10) $$\liminf_{t\to\infty} \mathrm{Re}\,[\bar{a}_m p(t)] > 0, \qquad \limsup_{t\to\infty} |\mathrm{Im}\,[\bar{a}_m p(t)]| < \infty,$$

(3.11) $$\mathrm{Re}\,[a_m q(t, z)] \geqq 0 \qquad \textit{for } t \in I, z \in \boldsymbol{C},$$

(3.12) $$\mathrm{Re}\,[a_m q(t, 0)] > 0 \qquad \textit{for } t \in I.$$

Assume there exists $D(t) \in C(I)$ such that

$$|q(t, z)| \leqq D(t) \qquad \textit{for } t \geqq t_0, z \in \boldsymbol{C}$$

and

(3.13) $$\lim_{t\to\infty} D(t) = 0.$$

Then any solution $z(t)$ of (3.1) *satisfying* $\mathrm{Re}\,[a_m z(t_1)] > 0$ $(m = 1, 2)$, *where $t_1 \geqq t_0$ satisfies the condition*

$$\lim_{t\to\infty} z(t) = 0.$$

Moreover, $\mathrm{Re}\,[a_m z(t)] > 0$ $(m = 1, 2)$ *for $t \geqq t_1$.*

Proof. Put $a = (a_1 + a_2)/2$. Choose $\vartheta = \lambda_- = 1$, $s_1 = t_1$,

$$\delta_1 = 2\sqrt{|a|\frac{\max_{t\in I} D(t)}{\min_{t\in I} \operatorname{Re}[\bar{a}p(t)]}}, \qquad \delta_j = j^{-1} \qquad (j = 2, 3, \dots),$$

$\vartheta_j = \exp\{\delta_j^{-1} \min[|a_m|\,|\operatorname{Im}(a_{3-m}a^{-1})|]\}$, $\varkappa_j = 0$, $E_j(t) = 2\,|a|\,\delta_j^{-2}D(t) - 2\operatorname{Re}[\bar{a}p(t)]$. For $j \geqq 2$ let $s_j \geqq t_0$ be such that $E_j(t) < 0$ for $t \geqq s_j$. Then $-G(t, z)\operatorname{Re}\left\{kh''(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} = 2\operatorname{Re}[\bar{a}z^{-2}q(t, z)] - 2\operatorname{Re}[\bar{a}p(t)] \leqq E_j(t)$ foi $t \geqq s_j$, $z \in K(\vartheta_j, \vartheta)$, $|z| > \delta_j$. Further it holds that $\vartheta < \vartheta_j$ and $\vartheta^* = \sup_{j\in N} \vartheta_j = \infty$. In view of Lemma 1 and following Remark we have $\operatorname{Re}[a_m z(t)] > 0$ ($m = 1, 2$) for all $t \geqq t_1$ for which $z(t)$ exists. By use of Lemma 2 we infer that

$$z(t) \notin B(0, \delta_j) - K(\infty, \vartheta_j)$$

for all $t \geqq t_1$ for which $z(t)$ exists and all $j \in N$. Applying Theorem 2' we find out that to any ε, $1 < \varepsilon < \infty$ there is a $T > 0$ such that $z(t) \in K(\infty, \varepsilon)$ for $t \geqq t_1 + T$, which implies

$$\lim_{t\to\infty} z(t) = 0.$$

The replacement of the condition (3.13) by

$$\int_{t_0}^{\infty} D(t)\,\mathrm{d}t < \infty \tag{3.14}$$

leads to the following theorem:

Theorem 4. *Let the assumptions of Theorem 3 be fulfilled with the exception that* (3.13) *is replaced by* (3.14). *Then the conclusion of Theorem 3 remains true.*

Proof. Set $a = (a_1 + a_2)/2$, $\vartheta = \lambda_- = 1$, $s_1 = t_1$,

$$\delta_1 = \sigma\sqrt{2\,|a|\int_{t_0}^{\infty} D(t)\,\mathrm{d}t},$$

$$\delta_j = \min_{m=1,2}[|a_m|\,|\operatorname{Im}(a_{3-m}a_m^{-1})|]/j \qquad (j = 2, 3, \dots),$$

$$\vartheta_j = \exp\{\delta_j^{-1}\min[|a_m|\,|\operatorname{Im}(a_{3-m}a_m^{-1})]\} \qquad (j = 1, 2, \dots),$$

where

$$\sigma = 2\max\left\{\ln^{-1/2} W(z(t_1)), \frac{\sqrt{2\,|a|\int_{t_0}^{\infty} D(t)\,\mathrm{d}t}}{\min_{m=1,2}[|a_m|\,|\operatorname{Im}(a_{3-m}a_m^{-1})|]}\right\}. \tag{3.15}$$

For $j \geqq 2$ let $s_j \geqq t_0$ be such that

$$2\,|a| \sup_{s_j \leqq s \leqq t < \infty} \int_s^t D(\xi)\,d\xi \leqq \delta_j^2.$$

Put

$$E_j(t) = 2\,|\,a\,|\,\delta_j^{-2}D(t) - 2\,\mathrm{Re}\,[\bar{a}p(t)],$$

$$\varkappa_j = \sup_{s_j \leqq s \leqq t < \infty} \int_s^t E_j(\xi)\,\mathrm{d}\xi.$$

Then $\vartheta e^{\varkappa_1} \leqq e^{\sigma^{-2}} < \vartheta_1$, $\vartheta e^{\varkappa_j} = e^{\varkappa_j} \leqq e < \vartheta_j$ $(j = 2, 3, \ldots)$,

$$\vartheta^* = \sup_{j \in N} [\vartheta_j\,\mathrm{e}^{-\varkappa_j}] \geqq \sup_{j \in N} [\vartheta_j\,\mathrm{e}^{-1}] = \sup_{j \in N} \mathrm{e}^{j-1} = \infty$$

and

$$-G(t, z)\,\mathrm{Re}\left\{kh''(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} = 2\,\mathrm{Re}\,[\bar{a}z^{-2}q(t, z)] - 2\,\mathrm{Re}\,[\bar{a}p(t)] \leqq E_j(t).$$

In view of (3.15) we have $W(z(t_1)) > e^{\sigma^{-2}}$, whence $z(t_1) \in K(\infty, \vartheta e^{\varkappa_1})$. Analogously as in the proof of Theorem 3 we infer that $\mathrm{Re}\,[a_m z(t)] > 0$ and $z(t) \notin B(0, \delta_j) - K(\infty, \vartheta_j)$ for all $t \geqq t_1$ for which $z(t)$ exists. The application of Theorem 2' yields the desired result.

Remark. In a special case $p(t) = 1$, $q(t, z) \equiv q(t)$ the conditions (3.9)–(3.12) are reduced to $\mathrm{Re}\, a_m > 0$, $\mathrm{Re}\,[a_m q(t)] > 0$ $(m = 1, 2)$ and we can put $D(t) = |\,q(t)\,|$. Thus we get some results of M. Ráb [6].

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ON SOME NON-LINEAR BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Existence and uniqueness of the solution to some boundary value problems for the second-order differential equation in a critical case is proved by using the method of upper and lower solutions. Further boundary value problems with a parameter are investigated.

Key words. Neumann's conditions, periodic conditions, three and four point conditions. Peano's phenomen, Bernstein–Nagumo condition, boundary value problem with a parameter, isotone and antitone operator.

MS Classification. 34 B 15, 34 B 27.

The method of upper and lower solutions has been firstly used to solve the nonlinear boundary value problems (for short BVP-s) in a noncritical case (see e.g. [7]). In the last time some papers have appeared they use this method, sometimes with other arguments, in a critical case (e.g. [5], [2], [6]).

Here on the basis of this method combined with apriori estimates the solution of the differential equation

$$x'' = f(t, x, x') \tag{1}$$

is searched for which satisfies one of the following boundary conditions

(2_1) $\quad x'(a) = 0, \quad x'(b) = 0, \quad a < b \quad$ (Neumann's conditions)

(2_2) $\quad x(a) - x(b) = 0, \quad x'(a) - x'(b) = 0, \quad a < b,$
(periodic conditions),

(2_3) $\quad x'(a) = 0, \quad x(b) - x(c) = 0, \quad a < c < b,$
(three point conditions),

$$(2_4) \qquad x(c) - x(a) = 0, \qquad x(b) - x(d) = 0, \qquad a < c < \leqq d < b,$$
(three or four points conditions).

We shall assume that $f \in C([a, b] \times R^2, R)$ and we shall show that all BVP-s (1), (2_j) $j = 1, 2, 3, 4$, have similar properties. Besides the existence, the problem of uniqueness of a solution to the BVP (1), (2_j) is studied together with the case that the set of all solutions to that problem is connected in the space $C([a, b], R)$ provided with the sup-norm (Peano's phenomenon). Further a BVP with a parameter is investigated and finally the theory of isotone and antitone operators (see [1], [8]) is applied to the investigation of a special case of the BVP (1), (2_j), $j = 1, 2, 3, 4$.

In what follows j will be an arbitrary, but fixed number, from the set $\{1, 2, 3, 4\}$.

LINEAR PROBLEM

The eigenvalue problem $x'' = \lambda x$, (2_j), has an eigenvalue $\lambda = 0$ and the corresponding eigenfunction $x_0(t) = c \neq 0$. This problem has no positive eigenvalue as the following lemma indicates.

Lemma 1. *Let $K < 0$. Then the problem (2_j),*

$$(3) \qquad x'' + Kx = 0,$$

has only the trivial solution.

Proof. Here and in the sequel only the case (3), (2_4) will be proved. In the other cases the proof is similar. By (3), each nontrivial solution $x(t)$ of (3) has neither a positive local maximum nor a negative local minimum.

Let $x(a) > 0$. Then $x(t)$ possesses a nonnegative local minimum in $[a, c]$ and hence $x'(c) \geqq 0$, $x''(c) > 0$. This implies that $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$ in $(c, b]$ and hence the second of conditions (2_4) is not fulfilled. In case $x(a) < 0$ we come to contradiction, too. If $x(a) = 0$, then $x(t) = 0$ in $[a, c]$ and by the considerations as above we get that $x(t) = 0$ in $[c, b]$, too.

Lemma 2. *Let $K < 0$. Then there exists the Green function $G(t, s)$ of the problem (3), (2_j). This function is continuous in $[a, b] \times [a, b]$ and $\frac{\partial G}{\partial t}$ is continuous in the triangles $a \leqq t \leqq s \leqq b$, $a \leqq s \leqq t \leqq b$.*

Proof. Let $g(t) \in C([a, b], R)$ and let $C(t, s) = [e^{\sqrt{-K}(t-s)} - e^{-\sqrt{-K}(t-s)}]/(2\sqrt{-K})$ be the Cauchy function for (3). Then the general solution of the equation $x'' + Kx = g(t)$ is of the form

(4) $$x(t) = c_1 e^{\sqrt{-K}t} + c_2 e^{-\sqrt{-K}t} + \int_a^t C(t, s) g(s) \,ds$$

and

$$x'(t) = \sqrt{-K}(c_1 e^{\sqrt{-K}t} - c_2 e^{-\sqrt{-K}t}) + \int_a^t \frac{\partial C(t, s)}{\partial t} g(s) \,ds, \qquad a \leqq t \leqq b.$$

By substituting $x(t)$ into (2_4) for c_1, c_2 we get the system of two conditions

$$c_1(e^{\sqrt{-K}c} - e^{\sqrt{-K}a}) + c_2(e^{-\sqrt{-K}c} - e^{-\sqrt{-K}a}) = -\int_a^c C(c, s) g(s) \,ds,$$

$$c_1(e^{\sqrt{-K}b} - e^{\sqrt{-K}d}) + c_2(e^{-\sqrt{-K}b} - e^{-\sqrt{-K}d}) = -\int_a^b C(b, s) g(s) \,ds + \int_a^d C(d, s) g(s) \,ds.$$

With respect to Lemma 1 this system has a unique solution (c_1, c_2). Putting this solution into (4) we get that

$$x(t) = \int_a^b G(t, s) g(s) \,ds, \qquad a \leqq t \leqq b,$$

with a uniquely determined function $G(t, s)$ and this function has all required properties.

Lemma 3. *Let $K < 0$. Then the Green function $G(t, s)$ for the problem* (3), (2_j), *satisfies the inequality*

(5) $$G(t, s) \leqq 0, \qquad a \leqq t, s \leqq b.$$

Proof. If suffices to show that for each function $x(t) \in C^2([a, b], R)$ satisfying the boundary conditions (2_j) the following implication holds:

If

(6) $$x''(t) + Kx(t) \geqq 0 \qquad \text{in } [a, b],$$

then

(7) $$x(t) \leqq 0 \qquad \text{for each } t \in [a, b].$$

Again only the problem (3), (2_4) will be considered. The solution $x(t)$ of (6) has the following property: If $x(t_0) > 0$, $x'(t_0) \geqq 0$ for a $t_0 \in (a, b)$, then $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$ in $[t_0, b]$, while in the case $x(t_0) > 0$, $x'(t_0) < 0$ we have that $x(t) > 0$, $x'(t) < 0$, $x''(t) > 0$ in $[a, t_0]$.

If $x(a) > 0$, then $x'(a) \geqq 0$ leads to the inequalities $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$ in $(a, b]$ which contradicts the second condition in (2_4). If $x(a) > 0$, $x'(a) < 0$, then $x(c) > 0$, $x'(c) \geqq 0$ and again we get contradiction with the second condition in (2_4). Hence $x(a) \leqq 0$, $x(c) \leqq 0$ and clearly $x(t) \leqq 0$ in $[a, c]$. This implies that there is no point $t_0 \in (a, b)$ with the property $x(t_0) > 0$, $x'(t_0) < 0$. Hence if $x(t_0) > 0$ in (c, b), then $x'(t) > 0$ in $(t_0, b]$ and again we come to contradiction with the second condition in (2_4). Therefore (7) is true.

PEANO'S PHENOMENON

Lemma 4. *Assume that*

(i) $f(t, ., y)$ *is nondecreasing in* R *for each* $(t, y) \in [a, b] \times R$,

(ii) *for each* $r > 0$ *there is an* $L_r > 0$ *such that*

$$| f(t, x, y) - f(t, x, z) | \leqq L_r | y - z |,$$

for each pair of points $(t, x, y), (t, x, z) \in [a, b] \times [-r, r] \times [-r, r]$.

If $x(t)$, $y(t)$ *are two solutions of* (1) *on* $[a, b]$ *and* $x(t) - y(t) \geqq 0$ in $[t_1, t_2] \subset [a, b]$, $x'(t_1) - y'(t_1) > 0 (x'(t_1) - y'(t_1) = 0)$, *then*

$$x(t) - y(t) > 0,\ x'(t) - y'(t) > 0 \text{ in } (t_1, b]\ (x'(t) - y'(t) \geqq 0 \text{ in } [t_1, t_2]).$$

Proof. Denote $v(t) = x(t) - y(t)$ in $[a, b]$. Then

$$v''(t) = [f(t, x(t), x'(t)) - f(t, y(t), x'(t))] + \\ + [f(t, y(t), x'(t)) - f(t, y(t), y'(t))] \text{ in } [a, b]. \tag{8}$$

Consider the case $v'(t_1) > 0$ and $v(t) \geqq 0$ in $[t_1, t_2]$. Then there is a maximal t_3, $t_1 < t_3 \leqq b$ such that $v'(t) > 0$, $v(t) > 0$ in (t_1, t_3). If $v'(t_3) = 0$, then from (8) we would have

$$v''(t) \geqq - | f(t, y(t), x'(t)) - f(t, y(t), y'(t)) | \geqq - L_r v'(t) \tag{9}$$

in $[t_1, t_3]$ with a suitable $r > 0$ and hence,

$$v'(t_3) \geqq v'(t_1) \exp [-L_r(t_3 - t_1)] > 0,$$

which gives that $v'(t) > 0$ must hold in $[t_1, b]$ and thus, $v(t) > 0$ in $(t_1, b]$. If $v'(t_1) = 0$, then from (9) we only get that $v'(t) \geqq 0$ in $[t_1, t_2]$.

Remark 1. By this lemma, there are no two solutions $x(t)$, $y(t)$ of (1) on $[a, b]$ such that $x(t_i) = y(t_i)$, $i = 1, 2$, and $x(t) > y(t)$ in (t_1, t_2). Hence, if $x(t_1) = y(t_1)$, $x'(t_1) = y'(t_1)$ and there are points $t_n \to t_1 +$ as $n \to \infty$ such that $x(t_n) > y(t_n)$, then $x(t) > y(t)$, $x'(t) > y'(t)$ in $(t_1, b]$.

Theorem 1 (Peano's phenomenon). *If the conditions of Lemma 4 are satisfied, and* $x(t)$, $y(t)$ *are two solutions of* (1), (2_j), *then*

(a) $x(t) - y(t) = c = \text{const}$ *in* $[a, b]$;

(b) *if* $c > 0 (c < 0)$, *then for each* c_1, $0 \leqq c_1 \leqq c$ $(0 \geqq c_1 \geqq c)$ *the function* $y(t) + c_1$ *is a solution of the problem* (1), (2_j).

Proof. Only the case (1), (2_4) will be considered. Denote $v(t) = x(t) - y(t)$ in $[a, b]$. By properly denoting the solutions $x(t)$, $y(t)$ we may assume that $v(a) \geqq 0$. By Lemma 4 the case $v(a) \geqq 0$, $v'(a) > 0$ would lead to contradiction with (2_4). If $v(a) > 0$, $v'(a) = 0$, then by this lemma $v(t)$ is a nondecreasing function in a maximal interval where $v(t) \geqq 0$, hence in $[a, b]$. If $v'(t_0) > 0$ for a $t_0 \in (a, b)$, then $v(t)$ would be increasing in $[t_0, b]$ which contradicts the second condition in (2_4). Thus $v(t) \equiv v(a) > 0$. Since $v(c) = v(a)$, the case $v(a) > 0$, $v'(a) < 0$ would imply that there is a point t_0, $a < t_0 < c$, such that $v(t_0) > 0$, $v'(t_0) > 0$ and, in view of Lemma 4, we again come to contradiction with (2_4). The case $v(a) = 0$, $v'(a) < 0$ can be inverted to the case $v(a) = 0$, $v'(a) > 0$ by relabelling the solutions $x(t)$, $y(t)$. If $v(a) = v'(a) = 0$, then either $v(t) \equiv 0$ in $[a, b]$, or by Remark 1, there is a point t_0, $a \leqq t_0 < b$ such that $v(t) \equiv 0$ in $[a, t_0]$ and either $v(t) > 0$, $v'(t) > 0$ in $(t_0, b]$ or $v(t) < 0$, $v'(t) < 0$ in $(t_0, b]$. In the last two cases we come to contradiction with (2_4). The statement (a) is completely proved.

To prove (b), suppose that $c > 0$ and $0 \leqq c_1 \leqq c$. Then $(y(t) + c_1)'' = y''(t) = f(t, y(t), y'(t)) = f(t, y(t) + c_1, (y(t) + c_1)')$ for each $t \in [a, b]$, since $x''(t) = f(t, y(t) + c, y'(t)) = f(t, y(t), y'(t)) = y''(t)$ in $[a, b]$ and $f(t, ., y)$ is non-decreasing in R.

Theorem 2. *If f satisfies the strengthened condition* (i)

(i′) $f(t, ., y)$ *is increasing in R for each* $(t, y) \in [a, b] \times R$, *then there exists at most one solution of* (1), (2_j).

Proof. Only the case (1), (2_4) is proved. Suppose that there are two solutions $x(t)$, $y(t)$ of (1), (2_4) and that the function $v(t) = x(t) - y(t)$ has a positive local maximum at t_0. If $a < t_0 < b$, then $v(t_0) > 0$, $v'(t_0) = 0$, $v''(t_0) \leqq 0$. On the other hand, by (i′) $v''(t_0) = x''(t_0) - y''(t_0) = f(t_0, x(t_0), x'(t_0)) - f(t_0, y(t_0), x'(t_0)) > 0$ which gives a contradiction. If $t_0 = a$ or $t_0 = b$, then v attains a positive local maximum at c or at d, and hence the same conclusion follows.

METHOD OF LOWER AND UPPER SOLUTIONS

The notion of a lower and upper solution can be defined for the problem (1), (2_j).

Definition 1. We say that $\alpha(t) \in C^2([a, b], R)$ $(\beta(t) \in C^2([a, b], R))$ is a lower solution for (1), (2_j) (an upper solution for (1), (2_j)) if

$$\alpha''(t) \geqq f(t, \alpha(t), \alpha'(t)), \qquad (\beta''(t) \leqq f(t, \beta(t), \beta'(t)) \qquad \text{for every } t \in [a, b]$$

and in case (2_1)

(11) $$\alpha'(a) \geqq 0, \qquad \alpha'(b) \leqq 0 \qquad (\beta'(a) \leqq 0, \beta'(b) \geqq 0);$$

in case (2_2)

(12) $$\alpha(a) - \alpha(b) = 0, \qquad \alpha'(a) - \alpha'(b) \geqq 0 \quad (\beta(a) - \beta(b) = 0, \beta'(a) - \beta'(b) \leqq 0);$$

in case (2_3)

(13) $$\alpha'(a) = 0, \qquad \alpha(b) - \alpha(c) \leqq 0 \qquad (\beta'(a) = 0, \beta(b) - \beta(c) \geqq 0);$$

in case (2_4)

(14) $$\alpha(c) - \alpha(a) = 0, \qquad \alpha(b) - \alpha(d) \leqq 0 \qquad (\beta(c) - \beta(a) = 0, \beta(b) - \beta(d) \geqq 0).$$

Remark 2. If we denote

$$g(t) = \alpha''(t) - f(t, \alpha(t), \alpha'(t)), \qquad h(t) = \beta''(t) - f(t, \beta(t), \beta'(t)), \qquad t \in [a, b],$$

and $v(t)$ $(w(t))$ is the solution of (3) for $K < 0$ which satisfies the same boundary conditions as $\alpha(t)$ $(\beta(t))$, e.g. in case (2_4)

$$v(c) - v(a) = \alpha(c) - \alpha(a), \qquad v(b) - v(d) = \alpha(b) - \alpha(d),$$
$$(w(c) - w(a) = \beta(c) - \beta(a), \qquad w(b) - w(d) = \beta(b) - \beta(d)),$$

then

(15) $$g(t) \geqq 0, \qquad h(t) \leqq 0 \text{ in } [a, b]$$

and by using the identita $(x(t)\,x'(t))' = -Kx^2(t) + x'^2(t)$ which is true in $[a, b]$ for each solution $x(t)$ of (3) we get that

(16) $$v(t) \leqq 0, \qquad (w(t) \geqq 0) \text{ in } [a, b].$$

Hence if $G(t, s)$ is the Green function for the problem (3), (2_j), then the lower solution $\alpha(t)$ and the upper solution $\beta(t)$ for that problem satisfy the relations

$$\alpha(t) = v(t) + \int_a^b G(t, s)\,[f(s, \alpha(s), \alpha'(s)) + K\alpha(s) + g(s)]\,ds,$$

$$\beta(t) = w(t) + \int_a^b G(t, s)\,[f(s, \beta(s), \beta'(s)) + K\beta(s) + h(s)]\,ds,$$

and in view of Lemma 3, (15), (16), we have

(17) $$\alpha(t) \leqq T\alpha(t), \qquad \beta(t) \geqq T\beta(t), \qquad t \in [a, b],$$

where $T\colon C^1([a, b], R) \to C^2([a, b], R)$ is the operator defined by

(18) $$Tx(t) = \int_a^b G(t, s)\,[f(s, x(s), x'(s)) + Kx(s)]\,\mathrm{d}s, \qquad a \leqq t \leqq b.$$

The meaning of T is based on the equivalence of the problem (1), (2_j) to the integro-differential equation

(19) $$x(t) = \int_a^b G(t, s)\,[f(s, x(s), x'(s)) + Kx(s)]\,\mathrm{d}s, \qquad a \leqq t \leqq b.$$

The existence of the BVP (1), (2_j) will be proved by using the method developed by K. Schmitt in [7]. First we shall deal with a modified problem (2_j),

(20) $$x'' + Kx = F(t, x, x'),$$

where $K < 0$ and F is continuous on $[a, b] \times R^2$.

Lemma 5. *Let there exist a constant* $L > 0$ *such that*

$$| F(t, x, y) | \leqq L$$

for all $(t, x, y) \in [a, b] \times R^2$. *Then the* BVP (20), (2_j) *has a solution.*

Proof. Let $C^1 = C^1([a, b], R)$ be endowed with the norm $\| x \|_1 = \sup_{a \leqq t \leqq b} | x(t) | + \sup_{a \leqq t \leqq b} | x'(t) |$. Then $(C^1, \| . \|_1)$ is a Banach space. Define the mapping T_1: $C^1 \to C^1$ by setting for each $x \in C^1$

$$T_1 x(t) = \int_a^b G(t, s)\, F(s, x(s), x'(s))\,\mathrm{d}s, \qquad a \leqq t \leqq b,$$

where G is the Green function for (3), (2_j). If

$$N = \sup_{[a,b]\times[a,b]} | G(t, s) | (b - a), \qquad N_1 = \sup_{[a,b]\times[a,b]} \left| \frac{\partial G(t, s)}{\partial t} \right| (b - a),$$

then we have that $| T_1 x(t) | \leqq \mathrm{NL}$, $| (T_1 x)'(t) | \leqq N_1 L$. Therefore T_1 maps the closed, bounded and convex set

$$B_1 = \{x \in C^1 : | x(t) | \leqq NL, | x'(t) | \leqq N_1 L, a \leqq t \leqq b\}$$

into itself. Furthermore $T_1 B_1$ is compact. Hence, by the Schauder fixed point theorem T_1 has a fixed point in B_1. This is a solution of (20), (2_j).

Lemma 6. *Assume that the assumption of Lemma 5 is fulfilled and that there exist a lower solution* $\alpha(t)$ *and an upper solution* $\beta(t)$ *of the problem* (20), (2_j) *such that* $\alpha(t) \leqq \beta(t)$, $a \leqq t \leqq b$. *Then there exists a solution* $x(t)$ *of* (20), (2_j) *with the property*

(21) $$\alpha(t) \leqq x(t) \leqq \beta(t), \qquad \textit{for every } t \in [a, b].$$

Proof. Define the function $H(t, x, y)$ on $[a, b] \times R^2$ by setting

$$H(t, x, y) = \begin{cases} F(t, \beta(t), y) + \dfrac{K}{2} \dfrac{x - \beta(t)}{1 + x^2} & \text{if } x > \beta(t), \\ F(t, x, y) & \text{if } \alpha(t) \leqq x \leqq \beta(t), \\ F(t, \alpha(t), y) + \dfrac{K}{2} \dfrac{x - \alpha(t)}{1 + x^2} & \text{if } x < \alpha(t). \end{cases}$$

Since F is bounded, H is also bounded. H is, together with F, continuous on $[a, b] \times R^2$. Hence, by Lemma 5, there exists a solution $x(t)$ of $x'' + Kx = H(t, x, x')$, (2_j). We now show that (21) is true. Denote $v(t) = x(t) - \beta(t)$, $t \in [a, b]$. If $v(t) \leqq 0$ on $[a, b]$ were not true, then there would exist a point $t_0 \in [a, b]$ at which $v(t)$ attains its positive absolute maximum in $[a, b]$.

If $t_0 \in (a, b)$, then $v(t_0) > 0$, $v'(t_0) = 0$, $v''(t_0) \leqq 0$. On the other hand, $v''(t_0) = x''(t_0) - \beta''(t_0) = -K(x(t_0) - \beta(t_0)) + \dfrac{K}{2} \dfrac{x(t_0) - \beta(t_0)}{1 + x^2(t_0)} > 0$ which is a contradiction. The case $t_0 = a$ or $t_0 = b$ also leads to contradiction, since the conditions (2_j), (11)–(14) imply that there is an inner point $t_1 \in (a, b)$ at which $v(t)$ attains its positive absolute maximum.

Similarly $x(t) \geqq \alpha(t)$, $a \leqq t \leqq b$, can be proved. This completes the proof of Lemma 6.

Definition 2 ([2], p. 174). We say that the function f satisfies a Bernstein–Nagumo condition if for each $M > 0$ there exists a continuous function h_M: $[0, \infty) \to [a_M, \infty)$ with $a_M > 0$ and $\int^{\infty} \dfrac{s\,ds}{h_M(s)} = +\infty$ such that for all x, $|x| \leqq M$, all $t \in [a, b]$ and all $y \in R$

$$|f(t, x, y)| \leqq h_M(|y|).$$

Lemma 7 ([3], p. 503, [2], p. 174). *Let f satisfy a Bernstein–Nagumo condition. Let $x(t)$ be any solution of (1) on $[a, b]$ satisfying the condition $|x(t)| \leqq M$, $a \leqq t \leqq b$. Then there exists a number $N > 0$ depending only on M, h_M such that $|x'(t)| \leqq N$ on $[a, b]$. More exactly, N can be taken as the root of the equation*

$$\int_{2M/(b-a)}^{N} \frac{s\,ds}{h_M(s)} = 2M.$$

Theorem 3 (Compare with [5], pp. 20–30). *If $\alpha(t)$, $\beta(t)$ are lower and upper solutions for the* BVP (1), (2_j) *such that $\alpha(t) \leqq \beta(t)$ on $[a, b]$ and f satisfies a Bernstein–Nagumo condition, then there exists a solution $x(t)$ of* (1), (2_j) *with $\alpha(t) \leqq x(t) \leqq \beta(t)$, $a \leqq t \leqq b$.*

Proof. Let $M = \max [\sup_{t\in[a,b]} |\alpha(t)|, \sup_{t\in[a,b]} |\beta(t)|]$. By Lemma 7, there exists an $N > 0$ such that for each solution $x(t)$ of (1) the implication holds: If $|x(t)| \leqq M$ on $[a, b]$, then $|x'(t)| \leqq N$ on the same interval. Let N be such that $N > |\alpha'(t)|$, $N > |\beta'(t)|$ for every $t \in [a, b]$.

Define $F(t, x, y)$ on the set $w \times R$ where $w = \{(t, x) \in R^2 : \alpha(t) \leqq x \leqq \beta(t), t \in [a, b]\}$ by setting

$$F(t, x, y) = \begin{cases} f(t, x, N) + Kx, & \text{if } y > N, \\ f(t, x, y) + Kx, & \text{if } |y| \leqq N, \\ f(t, x, -N) + Kx, & \text{if } y < -N \end{cases}$$

and extend to $[a, b] \times R^2$ by the relation

$$F(t, x, y) = \begin{cases} F(t, \beta(t), y), & \text{if } x > \beta(t), \\ F(t, \alpha(t), y), & \text{if } x < \alpha(t). \end{cases}$$

Then F is bounded and $F(t, \alpha(t), \alpha'(t)) = f(t, \alpha(t), \alpha'(t)) + K\alpha(t)$, $F(t, \beta(t), \beta'(t)) = f(t, \beta(t), \beta'(t)) + K\beta(t)$, hence $\alpha(t)$ is a lower solution and $\beta(t)$ is an upper solution of (20), (2_j). By Lemma 6 there exists a solution $x(t)$ of that problem such that $\alpha(t) \leqq x(t) \leqq \beta(t)$, $t \in [a, b]$. In view of the definition of the function F, $x(t)$ is the solution of the equation $x'' = f_1(t, x, x')$ where

$$f_1(t, x, y) = \begin{cases} f(t, x, N), & \text{if } y > N, \\ f(t, x, y), & \text{if } -N \leqq y \leqq N, \\ f(t, x, -N), & \text{if } y < -N \end{cases}$$

and

$$|f_1(t, x, y)| \leqq h_M(|y|) \quad \text{for all } t \in [a, b], |x| \leqq M, \quad \text{and} \quad |y| \leqq N.$$

By Lemma 7, each solution $z(t)$ of the equation $x'' = f_1(t, x, x')$ satisfying $|z(t)| \leqq M$ fulfils $|z'(t)| \leqq N$ and thus $x(t)$ satisfies the inequality $|x'(t)| \leqq N$ in $[a, b]$ which implies that $x(t)$ is a solution of (1), (2_j). The theorem is proved.

Denote

$$(22) \quad \varphi(c) = \min_{a\leqq t\leqq b} f(t, c, 0), \qquad \psi(c) = \max_{a\leqq t\leqq b} f(t, c, 0) \qquad \text{for each } c \in R.$$

The functions φ, ψ are continuous and $\varphi(c) \leqq \psi(c)$ for every $c \in R$.

A necessary condition for the existence of a solution to (1), (2_j) is given by the lemma.

Lemma 8. *The following statements are true:*

1. $x(t) \equiv c$, $a \leqq t \leqq b$, *is a solution of* (1), (2_j) *if and only if* $\varphi(c) = \psi(c) = 0$.
2. *If there exists a solution* $x(t)$ *of* (1), (2_j), *then*

(23) $$\psi(c_3) \geqq 0, \qquad \varphi(c_4) \leqq 0,$$

where $c_3 = \min_{a \leqq t \leqq b} x(t)$, $c_4 = \max_{a \leqq t \leqq b} x(t)$.

3. *If* $\psi(c) < 0$ *in an interval* $[c_1, c_2]$ *or* $\varphi(c) > 0$ *in that interval, then there is no solution* $x(t)$ *of* (1), (2_j) *such that*

(24) $$c_1 \leqq x(t) \leqq c_2 \quad \text{for all } t \in [a, b].$$

The proof of the statement 1 is trivial. The second statement follows from the fact that for each solution $x(t)$ of (1), (2_j) there exists a point $t_0 \in [a, b]$ such that $x(t) \geqq x(t_0) = c_3$ $(x(t) \leqq x(t_0) = c_4)$ for every $t \in [a, b]$ and $x'(t_0) = 0$, $x''(t_0) \geqq 0$ $(x'(t_0) = 0$, $x''(t_0) \leqq 0)$. The third statement follows from the second one.

A sufficient condition for the existence of a solution to (1), (2_j) is established in the following corollary to Theorem 3.

Corollary 1. *If f satisfies a Bernstein–Nagumo condition and there exists a pair* $c_1 \leqq c_2$ *such that*

(25) $$\psi(c_1) \leqq 0 \leqq \varphi(c_2),$$

then there exists a solution $x(t)$ *of* (1), (2_j) *satisfying* (24).

Proof. By (10)–14), $\beta(t) \equiv c_2$, $a \leqq t \leqq b$, is an upper solution of (1), (2_j) iff $f(t, c_2, 0) \geqq 0$ in $[a, b]$ and $\alpha(t) \equiv c_1$, $t \in [a, b]$, is a lower solution of (1), (2_j) iff $f(t, c_1, 0) \leqq 0$ in the same interval. Both inequalities are satisfied in $[a, b]$ when (25) is true.

Corollary 2. *If f satisfies a Bernstein–Nagumo condition and there exists a sequence of pairs* $\{c_{1k}\}$, $\{c_{2k}\}$, $k = 1, 2, \ldots$, *such that*

$$c_{1k} \leqq c_{2k}, \qquad c_{2k} < c_{1,k+1}, \qquad \psi(c_{1k}) \leqq 0 \leqq \varphi(c_{2k}), \qquad k = 1, 2, \ldots,$$

then there exist infinitely many solutions of (1), (2_j).

BOUNDARY VALUE PROBLEM WITH A PARAMETER

Consider the problem (2_j),

(1_s) $$x'' = f(t, x, x') + s,$$

with a real parameter s.

Then the following statements are true:

1. If $\beta(t)$ is an upper solution of the problem (1_{s_1}), (2_j), then $\beta(t)$ is also an upper solution for (1_s), (2_j) for each $s \geqq s_1$.

2. If $\alpha(t)$ is a lower solution for the problem (1_{s_1}), (2_j), then $\alpha(t)$ is also a lower solution for (1_s), (2_j) for each $s \leqq s_2$.

3. Let $f(t, ., y)$ be nondecreasing in R for each $(t, y) \in [a, b] \times R$. Then the following statements holds: If $\beta(t)$ is an upper solution and $\alpha(t)$ a lower solution of (1_s), (2_j), then for each $c > 0$ the function $\beta(t) + c$ is also an upper solution and $\alpha(t) - c$ is a lower solution for the same problem.

4. Let $f(t, ., y)$ be nondecreasing in R for each $(t, y) \in [a, b] \times R$. If $s_1 \leqq s_2$ and there exists an upper solution $\beta_1(t)$ for the problem (1_{s_1}), (2_j) and a lower solution $\alpha_1(t)$ for the problem (1_{s_2}), (2_j), then for each s, $s_1 \leqq s \leqq s_2$, there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of $(1\ )$, (2_j) such that $\alpha(t) \leqq \beta(t)$ on $[a, b]$.

Proof. By the statements 1 and 2, β_1 is an upper solution and α_1 is a lower solution of (1_s), (2_j) for each s, $s_1 \leqq s \leqq s_2$. Then by taking sufficiently great $c > 0$, on the basis of the statement 3, we get that $\alpha(t) = \alpha_1(t) - c$ and $\beta(t) = \beta_1(t) + c$, $a \leqq t \leqq b$, are a lower and an upper solution for (1_s), (2_j) with the desired property.

Let $\varphi(c)$ and $\psi(c)$ be defined by (22). Then the following statements hold:

5. $\beta(t) = c$, $a \leqq t \leqq b$, is an upper solution for (1_s), (2_j) for each $s \geqq -\varphi(c)$. $\alpha(t) \equiv c$, $a \leqq t \leqq b$, is a lower solution for (1_s), (2_j) for each $s \leqq -\psi(c)$.

6. If $c_1 < c_2$ and $\psi(c_1) \leqq \varphi(c_2)$, then for each s such that

$$-\varphi(c_2) \leqq s \leqq -\psi(c_1),$$

c_1 is a lower solution, c_2 is an upper solution for (1_s), (2_j).

On the basis of the last statement we prove the theorem.

Theorem 4. *If f satisfies a Bernstein–Nagumo condition and is such that there exist two sequences*

$$c_1 > c_2 > \ldots > c_n > \ldots \to -\infty, \qquad d_1 < d_2 < d_3 < \ldots < d_n < \ldots \to \infty$$

as $n \to \infty$ where $c_1 < d_1$ and there exists a number s_1 with the property

$$-\varphi(d_n) < s_1 < -\psi(c_n), \qquad n = 1, 2, \ldots, \tag{26}$$

then the set of all s for which there exists a solution for (1_s), (2_j) is an interval containing s_1 as an inner point.

Proof. Since c_1 is a lower solution and d_1 is an upper solution for (1_{s_1}), (2_j), there exists a solution $x_{s_1}(t)$ to (1_{s_1}). (2_j). Clearly s_1 can vary in the open interval $(-\varphi(d_1), -\psi(c_1))$. Suppose that $\tilde{s} < s_1$ and that there exists a solution $x_{\tilde{s}}(t)$ to $(1_{\tilde{s}})$, (2_j). Then for s, $\tilde{s} < s < s_1$, $x_{\tilde{s}}(t)$ is an upper solution to (1_s), (2_j) and, in view of the statement 6 and (26) c_n with sufficiently great n, is a lower solution whereby $c_n < x_{\tilde{s}}(t)$ for each $t \in [a, b]$. Hence by Theorem 3 there exists a solution $x_s(t)$ of the problem (1_s), (2_j). Similar considerations for $\tilde{s} > s > s_1$ can be carried out.

Corollary 3. *If f satisfies a Bernstein – Nagumo condition, $f(t, ., 0)$ is nondecreasing in R for each $t \in [a, b]$ and there are numbers $c_1 < d_1$, s_1 such that*

$$-\varphi(d_1) < s_1 < -\psi(c_1), \tag{27}$$

then the conclusion of Theorem 4 is true.

Proof. Since both functions $\varphi(c)$, $\psi(c)$ are nondecreasing, the inequalities (27) imply the inequalities (26) and the result follows.

Remark 3. In the proof of Theorem 4 we have shown the following implications: If $\tilde{s} \leqq s \leqq s_1$, then for each solution $x_{\tilde{s}}(t)$ of $(1_{\tilde{s}})$, (2_j) and each constant $c_n \leqq x_{\tilde{s}}(t)$, $a \leqq t \leqq b$, satisfying (26), there exists a solution $x_s(t)$ of (1_s), (2_j) such that

$$c_n \leqq x_s(t) \leqq x_{\tilde{s}}(t), \qquad a \leqq t \leqq b.$$

If $s_1 \leqq s \leqq \tilde{s}$, then for each solution $x_{\tilde{s}}(t)$ of $(1_{\tilde{s}})$, (2_j) and each constant $d_n \geqq x_{\tilde{s}}(t)$, $a \leqq t \leqq b$, for which (26) is true there exists a solution $x_s(t)$ of (1_s), (2_j) with the property

$$x_{\tilde{s}}(t) \leqq x_s(t) \leqq d_n, \qquad a \leqq t \leqq b.$$

By this remark and by Corollary 3 we get the following theorem. In this theorem the Banach space $C^1 = C^1([a, b], R)$ is provided with the same norm as above.

Theorem 5 (Comparison theorem). *If f satisfies a Bernstein – Nagumo condition, $f(t, ., y)$ is increasing in R for each $(t, y) \in [a, b] \times R$ and the condition (27) is fulfilled, then there exists an interval I such that for each $s \in I$ there exists a unique solution $x_s(t)$ for (1_s), (2_j) whereby*

$$s_1 < s_2 \text{ implies that } x_{s_1}(t) \geqq x_{s_2}(t) \text{ in } [a, b] \text{ for any two } s_1, s_2 \in I \tag{28}$$

and the solution $x_s(t)$ continuously depends in C^1 on $s \in I$.

Proof. The existence and uniqueness of the solution to (1_s), (2_j) for each s from an interval I follows from Corollary 3 and Theorem 2. The last remark gives the implication (28).

Fix a constant $K < 0$ and denote $G(t, u)$ the Green function for (3), (2_j). Then for each $s \in I$ the solution $x_s(t)$ of (1_s), (2_j) satisfies the integral equation

$$x_s(t) = \int_a^b G(t, u)\,[f(u, x_s(u), x_s'(u)) + Kx_s(u) + s]\,du = \tag{29}$$

$$= \frac{s}{K} + \int_a^b G(t, u)\,[f(u, x_s(u), x_s'(u)) + Kx_s(u)]\,du, \qquad a \leqq t \leqq b.$$

Then

$$x_s'(t) = \int_a^b \frac{\partial G(t, u)}{\partial t}\,[f(u, x_s(u), x_s'(u)) + Kx_s(u)]\,du, \qquad a \leqq t \leqq b. \tag{30}$$

Let $\{s_n\}$ be a nonincreasing sequence in I converging to $s \in I$. Then $x_{s_n}(t)$ is a nondecreasing sequence converging to a function $x(t) \leqq x_s(t)$ pointwise in $[a, b]$. Further both sequences $\{x_{s_n}\}$, $\{x'_{s_n}\}$ are uniformly bounded on $[a, b]$. The uniform boundedness of $\{x_{s_n}(t)\}$ follows from the inequalities $x_{s_1}(t) \leqq x_{s_n}(t) \leqq \ldots \leqq x_s(t)$ for each $n = 1, 2, \ldots$, and each $t \in [a, b]$. The uniform boundedness of $\{x'_{s_n}(t)\}$ follows on the basis of the Bernstein–Nagumo condition from that of $\{x_{s_n}(t)\}$. As $x''_{s_n}(t) = f(t, x_{s_n}(t), x'_{s_n}(t)) + s_n$, the sequence $\{x''_{s_n}(t)\}$ is uniformly bounded on $[a, b]$, too and hence, by the Ascoli theorem, there is a subsequence $\{x_{s_{n(k)}}(t)\}$ such that $\{x_{s_{n(k)}}(t)\}$ converges uniformly to $x(t)$ and $\{x'_{s_{n(k)}}(t)\}$ to $x'(t)$ on $[a, b]$. From (29), (30), by the limit process for $s = s_{n(k)}$ we get that

$$x(t) = \frac{s}{K} + \int_a^b G(t, u)\left[f(u, x(u), x'(u)) + Kx(u) + s\right] du, \qquad a \leqq t \leqq b.$$

This implies that $x(t)$ is a solution of (1_s), (2_j) which, on the basis of the uniqueness result, gives that $x(t) \equiv x_s(t)$, $a \leqq t \leqq b$, and the proof in this case is complete. Similarly we can proceed when $\{s_n\}$ is a nondecreasing sequence. In both cases the whole sequences $\{x_{s_n}(t)\}$, $\{x'_{s_n}(t)\}$ converge uniformly (to the function $x_s(t)$ and $x'_s(t)$, respectively). Since any convergent sequence $\{s_n\} \subset I$ contains a monotonic convergent subsequence, the proof by contradiction gives that also in the general case $\{x_{s_n}(t)\}$ converges uniformly on $[a, b]$ to $x_s(t)$ and $\{x'_{s_n}(t)\}$ to $x'_s(t)$ what we had to prove.

Theorem 6. *If f satisfies a Bernstein–Nagumo condition and is such that there exist two sequences*

$$s_1 < s_2 < \ldots < s_n < \ldots \to \infty, \qquad s_{-1} > s_{-2} > \ldots > s_{-n} > \ldots \to -\infty,$$

as $n \to \infty$ with $s_{-1} \leqq s_1$ and the sequences

$$d_1 < d_2 < \ldots < d_n < \ldots \to \infty, \qquad c_1 > c_2 > \ldots > c_n > \ldots \to -\infty,$$

as $n \to \infty$ where $c_1 < d_1$, with the property

(31) $$s_n \leqq -\psi(c_n), \qquad s_{-n} \geqq -\varphi(d_n), \qquad n = 1, 2, \ldots,$$

then the problem (1_s), (2_j) has a solution for each $s \in R$.

Proof. By (31), and the statement 6, for each $s \in [s_{-n}, s_n]$ c_n is a lower solution and d_n is an upper solution of (1_s), (2_j). Hence by Theorem 3, there exists a solution $x_s(t)$ for (1_s), (2_j) such that $c_n \leqq x_s(t) \leqq d_n$, $a \leqq t \leqq b$.

A SPECIAL CASE OF f

When $f = f(t, x)$, then this function satisfies a Bernstein – Nagumo condition. Now the functions $\varphi(c)$, $\psi(c)$ will mean

$$\varphi(c) = \min_{a \leqq t \leqq b} f(t, c), \qquad \psi(c) = \max_{a \leqq t \leqq b} f(t, c). \tag{32}$$

Consider the case

$$f(t, .) \text{ is nondecreasing in } R \text{ for each } t \in [a, b].$$

Then $\varphi(c)$ and $\psi(c)$ are nondecreasing, too. Since the conditions of Lemma 4 are fulfilled, Peano's phenomenon can occur for the problem (2_j),

$$x'' = f(t, x). \tag{33}$$

Further, by the statement 4, if there exist a lower and an upper solution for (33), (2_j), then there exist a lower solution $\alpha(t)$ and an upper rolution $\beta(t)$ for that problem such that $\alpha(t) \leqq \beta(t)$ on $[a, b]$ and by Theorem 3 we get the following theorem.

Theorem 7. *If $f(t, .)$ is nondecreasing in R for each $t \in [a, b]$ and there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ for the problem* (33), (2_j), *then there exists a solution $x(t)$ of that problem satisfying*

$$\alpha(t) - c \leqq x(t) \leqq \beta(t) + c, \qquad a \leqq t \leqq b,$$

for a $c \geqq 0$ such that $\alpha(t) - c \leqq \beta(t) + c$ for all $t \in [a, b]$.

Now we shall apply the theory of antitone operators (see [8]). Consider the vector space $C = C([a, b], R)$ with the sup-norm. Then C is a Banach space which can be ordered by the rule $x \leqq y$ iff $x(t) \leqq y(t)$ for every $t \in [a, b]$ for two functions $x, y \in C$. C with this ordering is an ordered Banach space. The positive cone in this space is made of all nonnegative continuous functions on $[a, b]$. P is normal. If $\alpha \leqq \beta$ are two points of C, then the subset $[\alpha, \beta] = \{z \in C: \alpha \leqq z \leqq \beta\}$ is called an ordered interval.

Suppose that $K < 0$ is a constant and consider the operator T defined by (18). Since

$$Tx(t) = \int_a^b G(t, s)\,[f(s, x(s)) + Kx(s)]\,\mathrm{d}s, \qquad a \leqq t \leqq b, \tag{34}$$

$T: C \to C$. We can easily show that T is a completely continuous operator. If the function $f(t, x) + Kx$ is nondecreasing in $x \in R$ for each fixed $t \in [a, b]$, then T is antitone, which means that for any two elements $x, y \in C$, $x \leqq y$ implies that $Tx \geqq Ty$. By Theorem 1 in [8], p. 533, we get the following theorem (compare with Theorem 10 in [8], p. 552).

Theorem 8. *Let there exist two numbers $K < 0$ and $c_1 \in R$ such that the function*

$$f(t, x) + Kx \leqq c_1 \qquad \text{for each } (t, x) \in [a, b] \times R, \tag{35}$$

or

$$f(t, x) + Kx \geqq c_1 \qquad \text{for each } (t, x) \in [a, b] \times R$$

and let the function $f(t, x) + Kx$ be nondecreasing in $x \in R$ for each $t \in [a, b]$. Then there exists a unique solution of (33), (2_j).

Proof. Since $G(t, s) \leqq 0$ for all $(t, s) \in [a, b] \times [a, b]$, the inequality $f(t, x) + Kx \leqq c_1$ implies that

$$Tx(t) \geqq \int_a^b G(t, s)\, c_1 \, ds = \frac{c_1}{K} \qquad \text{for all } x(t) \in C.$$

Similarly in the second case of (35) T is bounded from above. Then the existence of a solution to (33), (2_j) follows from Theorem 1 cited above. As $f(t, .)$ is increasing for each $t \in [a, b]$, the uniqueness of that solution is implied by Theorem 2.

In case

the function $f(t, x) + Kx$ is nonincreasing in $x \in R$ for each $t \in [a, b]$,

the operator T given by (34) is isotone, i.e. if $x, y \in C$ and $x \leqq y$, then $Tx \leqq Ty$. By Corollary 2.2 ([1], p. 369) we get the following theorem.

Theorem 9. *Let there exist a number $K < 0$ such that the function $f(t, x) + Kx$ is nonincreasing in $x \in R$ for each fixed $t \in [a, b]$ and let there exist a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of the problem* (33), (2_j) *whereby $\alpha(t) \leqq \beta(t)$, $a \leqq t \leqq b$. Then there exist a minimal solution $u(t)$ and a maximal solution $v(t)$ of* (33), (2_j) *in the order interval $[\alpha, \beta]$. Moreover, the sequences $\{\alpha_p\}_{p=0}^{\infty}$, $\{\beta_p\}_{p=0}^{\infty}$ defined by*

$$\alpha_0(t) = \alpha(t), \qquad \alpha_{p+1}(t) = T\alpha_p(t), \qquad \beta_0(t) = \beta(t), \qquad \beta_{p+1}(t) = T\beta_p(t),$$
$$a \leqq t \leqq b, \qquad p = 0, 1, 2, \ldots,$$

are such that

$$\alpha_0(t) \leqq \alpha_1(t) \leqq \ldots \leqq \alpha_p(t) \leqq \ldots \leqq u(t) \leqq v(t) \leqq \ldots \leqq \beta_p(t) \leqq \ldots \leqq$$
$$\leqq \beta_1(t) \leqq \beta_0(t), \qquad a \leqq t \leqq b,$$

and $\lim_{p \to \infty} \alpha_p(t) = u(t)$, $\lim_{p \to \infty} \beta_p(t) = v(t)$ *uniformly on* $[a, b]$.

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SINGULAR QUADRATIC FUNCTIONALS AND TRANSFORMATION OF LINEAR HAMILTONIAN SYSTEMS

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Abstract. Singular quadratic functionals with a single singular end-point are investigated using the transformation theory of linear Hamiltonian systems. In particular, there are established results for self-adjoint 2n-order functionals.

Key words. Transformation of linear Hamiltonian systems and functionals, singularity condi-on, self-adjoint functionals of higher order.

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1. INTRODUCTION

The theory of singular quadratic functionals as introduced by Morse and Leighton [11] and followed by [12, 13] involves the study of functional

$$J[y; s_1, s_2] = \int_{s_1}^{s_2} [p(t)\, y'^2(t) - q(t)\, y^2(t)]\, dt, \tag{1}$$

$a < s_1 < s_2 < b$ as $s_1 \to a$, $s_2 \to b$ and y belongs to the prescribed class of "admissible arcs" defined on (a, b). Morse and Leighton [11] discovered a condition termed the „singularity condition" which with the classical condition (disconjugacy of the corresponding Euler equation) yields necessary and sufficient condition for singular functional to be nonnegative, i.e. $\liminf_{s_1 \to a,\, s_2 \to b} J[y; s_1, s_2] \geqq 0$. Comprehensive bibliography concerning the problem may be found in [16].

In this paper we solve the problem of minimizing of the singular quadratic functionals corresponding to linear Hamiltonian systems. The principal idea we use is the application of the transformation theory of linear Hamiltonian systems and corresponding quadratic functionals. In particular, we establish results for singular functionals in terms of a singular condition similar to that of [11, 17] and for regular functionals in terms of a phase matrix. In the case of the second order linear differential equation this approach was originally proposed by J. Krbiľa [10] for associated regular functionals (on the compact interval) and later in [8, 9] for singular functionals (1). However, the result in [8, 9] is incorrect as provides the counter example in [5].

Statement of the problem. We suppose the second order variational problem corresponding with the linear Hamiltonian system

$$y' = B(t)\,y + C(t)\,z, \qquad z' = -A(t)\,y - B^T(t)\,z, \tag{2}$$

where $A(t)$, $B(t)$, $C(t)$ are $n \times n$ matrices of real-valued functions continuous on the interval $I = [a, \infty)$, the matrices $A(t)$, $C(t)$ are symmetric.

We suppose (1) to be *identically normal* on I, i.e. the trivial solution $(y, z) \equiv (0, 0)$ is the only one solution of (2) for which $y(t) = 0$ on a nondegenerate subinterval of I.

We consider the functional

$$J[y, z; a, b] = \int_a^b [z^T(t)\,C(t)\,z(t) - y^T(t)\,A(t)\,y(t)]\,dt, \tag{3}$$

$a < b < \infty$. Integrals employed throughout are Lebesgue integrals and their extensions.

We say that vector functions $y(t)$, $z(t)$ are *admissible* curves on $I = [a, \infty)$ with respect to (2) if

i) $z(t)$ is (Lebesgue) measurable on I and $y(t)$ is a solution of $y' = B(t)\,y + C(t)\,z(t)$ a.e., satisfying boundary conditions $y(a) = 0$, $\lim_{t\to\infty} y(t) = 0$;

ii) $\int_a^b z^T(t)\,z(t)\,dt < \infty$ for every b, $a < b < \infty$.

We seek conditions under which

$$\liminf_{t\to\infty} J[y, z; a, t] \geqq 0 \tag{4}$$

for all admissible functions $y(t)$, $z(t)$ on $[a, \infty)$ with respect to (2). Whenever (4) holds for the admissible class of curves we say that $[a, \infty)$ affords a *minimum limit* to J.

Remark 1. Some special cases of the problem have been investigated in the past. If $n = 1$, $C(t) \neq 0$ then (2) and (3) corresponds to the second order equation $(p(t)\, y')' + q(t)\, y = 0$ and to (1), respectively (Case I). If $B(t) = 0$, $C(t)$ being invertible then we have quadratic functionals of n dependent variables corresponding to the second order linear system

$$\int_a^b (y^{T\prime} C^{-1} y' - y^T A y)\, dt \to (C^{-1} Y')' + AY = 0,$$

investigeted by Tcmastik [17, 18] (Case II).

The condition of the identical normality of (2) eliminates pathologies in the investigation of conjugate points present in an abnormal differential system (2), see [16]; in the terminology of [4] this condition is called "controllability condition".

The introduced definition of admissible functions agrees with that of [4, 16] for the compact interval and with that of [17, 18] for Case II.

2. PRELIMINERIES

Corresponding to (2), we have the matrix equation

$$(2)^* \qquad \begin{aligned} Y' &= B(t)\, Y + C(t)\, Z, \\ Z' &= -A(t)\, Y - B^T(t)\, Z. \end{aligned}$$

In accordance with [4, 16] we use the following notation. We say that $(Y(t), Z(t))$ is a solution of (2)* if $Y(t), Z(t) \in \mathscr{AC}(I)$ (absolutely continuous) and (2)* satisfy a.e. on I. If $(Y(t), Z(t))$ is a solution of (2)* then $Y^T(t)\, Z(t) - Z^T(t)\, Y(t) = K$, where K is a constant $n \times n$ matrix. If $K = 0$ then $(Y(t), Z(t))$ is called *conjoined* (an alternate terminology for this concept is isotropic; see [4]). Two points $a, b \in \mathbf{R}$ are *conjugate* with respect to (2) if there exists a non-trivial solution $(y(t), z(t))$ of (2) such that $y(a) = 0$, $y(b) = 0$. (2) is disconjugate on I if there exist no two distinct points from I that are conjugate with respect to (2).

Let be (2) disconjugate on $[a, \infty)$. Then there exists a conjoined solution $(Y_0(t), Z_0(t))$ of (2)* such that the matrix $Y_0(t)$ is nonsingular on (a, ∞) and

$$\lim_{t \to \infty} [\int^t Y_0^{-1}(t)\, C(t)\, (Y_0^T)^{-1}(t)]^{-1} = 0.$$

The solution $(Y_0(t), Z_0(t))$ with these properties is called *principal* at infinity. A principal solution (Y_a, Z_a) at a is defined similarly; one can verify that this solution satisfies the initial condition $Y_a(a) = 0$, $Z_a(a) = N$ where N is a non-singular matrix. A solution $(Y(t), Z(t))$ of (2)* is called *antiprincipal* at infinity if it is conjoined, $Y(t)$ is non-singular for large t and

$$\lim_{t\to\infty} [\int^{t} Y^{-1}(t)\, C(t)\, Y^{T-1}(t)]^{-1} = M,$$

where M is a non-singular matrix.

If $(Y(t), Z(t))$ is a solution of (2)* such that $Y(t)$ is invertible for all t then $W(t) = Z(t)\, Y^{-1}(t)$ is a solution of the Riccati equation

$$W' + A(t) + WB(t) + B^T(t)\, W + WC(t)\, W = 0. \tag{5}$$

The solution $(Y(t), Z(t))$ is conjoined if and only if the corresponding solution $W(t)$ of (5) is symmetric. If $(Y_a(t), Z_a(t))$ is the principal solution at a then the solution $W_a(t) = Z_a(t)\, Y_a^{-1}(t)$ of (5) is called the *distinguished* solution at a.

Our method will be based on the transformation of linear Hamiltonian system given in the following two theorems.

Theorem A. [1, Theorem 6.3]. *Let $D(t)$, $E(t) \in \mathcal{AC}(I)$ be $n\times n$ matrices $D(t)$ being non-singular, for which $D^T(t)\, E(t) = E^T(t)\, D(t)$.*

Then the transformation

$$\begin{aligned} y &= D(t)\, u, \\ z &= E(t)\, u + D^{T-1}(t)\, v \end{aligned} \tag{6}$$

transforms (2) into the system

$$\begin{aligned} u' &= B_0(t)\, u + C_0(t)\, v, \\ v' &= -A_0(t)\, u - B_0^T(t)\, v, \end{aligned} \tag{7}$$

where $B_0(t) = D^{-1}(-D' + BD + CE)$, $C_0(t) = D^{-1}CDT^{-1}$, $A_0(t) = D^T(E' + AD + B^TE) + (-D' + BD + CE)$.

Remark 2. The transformation (6) keeps the identical normality, disconjugacy on the given interval I, and a principal (antiprincipal, conjoined) solution is transformed into that of the same type.

Obviously, the transformation (6) with $E(t) = 0$, $D' = B(t)\, D$ transforms (2)* into the "off-diagonal" system

$$\begin{aligned} U' &= \bar{C}(t)\, V, \\ V' &= -\bar{A}(t)\, U, \end{aligned} \tag{8}$$

where $\bar{C}(t) = D^{-1}CD^{T-1}$, $\bar{A}(t) = D^TAD$.

Theorem B. [5, Theorem 1]. *There exist $n\times n$ matrices $D(t)$, $E(t) \in \mathcal{AC}$, $D(t)$ being nonsingular, such that the transformation $U = D(t)\, Y, V = E(t)\, Y + D^{T-1}(t)\, Z$ transforms the system (8) into the system*

(9)
$$Y' = Q(t)\,Z,$$
$$Z' = -Q(t)\,Y,$$

where $Q(t) = D^{-1}CD^{T^{-1}}$. *The matrix* $A(t) = \int_a^t Q(s)\,ds$ *is called a phase matrix of the system* (2)*.

3. TRANSFORMATION OF FUNCTIONALS AND SINGULARITY CONDITION

The symbol $y \in \mathcal{D}[a, b] : z$ will denote those functions $y \in \mathcal{AC}[a, b]$ for which there exists a $z(t)$ measurable, satisfying condition ii) from the definition of admissible functions and such that $y' = B(t)\,y + C(t)\,z(t)$ a.e. on $[a, b]$.

Theorem 1. Let $y \in \mathcal{D}[a, b] : z$. Then functions u, v given by the transformation (6) satisfy

$$\int_a^b (z^TCz - y^TAy)\,dt = \int_a^b (v^TC_0v - u^TA_0u)\,dt + [y^TED^{-1}y]_a^b.$$

Proof. According to Theorem A it holds $u' = B_0(t)\,u + C_0(t)\,v$ and $u = D^{-1}y$. Using the transformation (6) we get

$$(10)\quad \int_a^b (z^TCz - y^TAy)\,dt = \int_a^b [(u^TE^T + v^TD^{-1})\,C(Eu + D^{T^{-1}}v) - u^TD^TADu]\,dt =$$
$$= \int_a^b (v^TD^{-1}CD^{T^{-1}}v + u^TE^TCEu + v^TD^{-1}CEu + u^TE^TCD^{T^{-1}}v - u^TD^TADu)\,dt.$$

Further it holds $(u^T)' = u^TB_0^T + v^TC_0 = u^T(-D^{T'} + D^TB^T + E^TC)\,D^{T^{-1}} + v^TD^{-1}CD^{T^{-1}}$, thus

$$(u^TD^TEu)' = u^T(-D^{T'} + D^TB^T + E^TC)\,D^{T^{-1}}D^TEu + v^TD^{-1}CD^{T^{-1}}D^TEu +$$
$$+ u^TD^{T'}Eu + u^TD^TE'u + u^TD^TED^{-1}(-D' + BD + CE)\,u + u^TD^TED^{-1}CD^{T^{-1}}v =$$
$$= v^TD^{-1}CEu + u^TE^TCD^{T^{-1}}v +$$
$$+ u^T(-D^{T'}E + D^TB^TE + E^TCE + D^{T'}E + D^TE' - E^TD' + E^TBD + E^TCE)\,u =$$
$$= v^TD^{-1}CEu + u^TE^TCD^{T^{-1}}v +$$
$$+ u^T(D^TE' - E^TD' + D^TB^TE + 2E^TCE + E^TBD)\,u.$$

Integrating the last equality we get

$$(11)\quad \int_a^b (u^TE^TCEu + v^TD^{-1}CEu + u^TE^TCD^{T^{-1}}v)\,dt = [u^TD^TEu]_a^b +$$
$$+ \int_a^b [-u^T(D^TE' - E^TD' + D^TB^TE + E^TCE + E^TBD)\,u]\,dt.$$

Finally, by substitution (11) into (10) we have

$$\int_a^b (z^T Cz - y^T Ay)\,\mathrm{d}t = \int_a^b [v^T C_0 v - u^T(D^T AD + D^T E' - E^T D' + D^T B^T E + E^T CE + E^T BD)\,u]\,\mathrm{d}t = \int_a^b (v^T C_0 v - u^T A_0 u)\,\mathrm{d}t + [u^T D^T Eu]_a^b = \int_a^b (v^T C_0 v - u^T A_0 u)\,\mathrm{d}t + [y^T ED^{-1} y]_a^b. \blacksquare$$

We can use Theorem 1 to have a non-negativity of functionals. In the following if C is symmetric $n \times n$ matrix (i.e. $C^T = C$), $C \geqq 0$ means that C is non-negative definite.

Theorem 2. *Let $C(t) \geqq 0$ on $[a, \infty)$. In order that* (4) *holds for all admissible functions $y(t)$, $z(t)$ on $[a, \infty)$ with respect to* (2) *it is necessary and sufficient*

i) (2) *is disconjugate on $[a, \infty)$,*

ii) *singularity condition is satisfied,* i.e. *for all $y(t)$, $z(t)$ admissible on $[a, \infty)$ with respect to* (2) *such that*

$$\liminf_{t \to \infty} \int_a^t (z^T Cz - y^T Ay)\,\mathrm{d}t < \infty.$$

it holds

$$\liminf_{t \to \infty} y^T(t)\, W_a(t)\, y(t) \geqq 0,$$

where $W_a(t)$ is the distinguished solution of (5).

Proof. I. Note that if $y(t)$, $z(t)$ are admissible functions with respect to (2) then $y \in \mathscr{D}[a, b] : z$ and by virtue of the boundary condition at a it holds $[y^T ED^{-1} y]_{t=a} = 0$.

Let (2)* be disconjugate on $[a, \infty)$ and (Y, Z) be a principal solution of (2)* at a. Then $W_a(t) = Z(t)\, Y^{-1}(t)$ is the distinguish solution of (5) at a and the transformation (6) with

$$D(t) = Y(t), \qquad E(t) = Z(t)$$

yields

$$B_0 = Y^{-1}(-Y' + BY + CZ) = Y^{-1}(-BY - CZ + BY + CZ) = 0,$$
$$C_0 = Y^{-1} C Y^{T-1},$$
$$A_0 = Y^T(Z' + AY + B^T Z) = Y^T(-AY - B^T Z + AY + B^T Z) = 0.$$

By Theorem 1

$$\int_a^b (z^T Cz - y^T Ay)\,\mathrm{d}t = \int_a^b (v^T C_0 v)\,\mathrm{d}t + [y^T W_a y]_{t=b}$$

holds for all corresponding couples of functions $y(t)$, $z(t)$ and $u(t)$, $v(t)$. From the inequality

$$\liminf_{t\to\infty} \int_a^b (z^TCz - y^TAy)\,ds \geqq \liminf_{t\to\infty} \int_a^t (v^TC_0v)\,dt + \liminf_{t\to\infty} y^TW_ay,$$

it follows the sufficiency of the singular condition.

II. We now follow a method which was used in the scalar case by Morse and Leighton [11]. Suppose there exists a couple of admissible functions y, z such that $\liminf_{t\to\infty} J[y, z; a, t] < \infty$ and the singularity condition is not satisfied for this couple i.e., $\liminf_{t\to\infty} y^T(t)\,W_a(t)\,y(t) = -k^2$, where $W_a(t)$ is the distinguish solution of (5) at a and k is a real constant. Let $e \in (a, \infty)$. We construct a couple of vector functions

$$(y_e(t), z_e(t)) = \begin{cases} (y(t), z(t)) & \text{for } t \in (e, \infty), \\ (Y_a(t)\,c, Z_a(t)\,c) & \text{for } t \in (a, e], \end{cases}$$

where (Y_a, Z_a) is the principal solution of (2)* at a, c is a constant vector such that $(y(e), z(e)) = (Y_a(e)\,c, Z_a(e)\,c)$. It holds

$$\int_a^t (z_e^TCz_e - y_e^TAy_e)\,ds = \int_a^e c^T(Z_aCZ_a - Y_a^TAY_a)\,c\,dt + \int_e^t (z^TCz - y^TAy)\,ds =$$

$$= -c^TY_a^T(e)\,W_a(e)\,Y_a(e)\,c + \int_e^t (z^TCz - y^TAy)\,ds =$$

$$= -y^T(e)\,W_a(e)\,y(e) + \int_e^t (z^TCz - y^TAy)\,ds.$$

Since $\liminf_{t\to\infty} (-y^T(t)\,W_a(t)\,y(t)) = -k^2$ and $\liminf_{t\to\infty} \int_a^t (z^TCz - y^TAy)\,ds < \infty$ choosing e sufficiently large, we have $-y^T(e)\,W_a(e)\,y(e) < -2k^2/3$ and $\liminf_{t\to\infty} \int_e^t . (z^TCz - y^TAy)\,ds < k^2/3$.

Consequently, we have $\liminf_{t\to\infty} \int_a^t (z_e^TCz_e - y_eAy_e)\,dt < -k^2/3$ which is a contradiction. ■

Remark that in special Cases I and II (see Remark 1) the singularity condition complies with that one introduced in [11] and [17], respectively.

The following theorem gives sufficient condi ions for singularity condition to be satisfied. Since every system can be transformed to "off-diagonal" form (see Remark 2) we suppose $B(t) = 0$ in (2)* without loss of generality.

Theorem 3. *Let* $B(t) = 0$, $C(t) \geqq 0$ on $[a, \infty)$. *If the system* (2)* *is disconjugate on* $(a - \varepsilon, \infty)$ *for some* $\varepsilon > 0$, $\int_a^\infty C(s)\,ds < \infty$ *and* $\int_a^\infty \max | a_{ij}(s) |\,ds < \infty$ *then* $[a, \infty)$ *affords a minimum limit to* J.

Proof. Let (Y_a, Z_a) be a principal solution of (2)* at a. In the light of the fact that $W = W_a = Z_a Y_a^{-1}$ is a solution of the Riccati equation

$$W' + A(t) + WC(t)W = 0,$$

it holds

$$W(t) = W(b) - \int_b^t W(s)\,C(s)\,W(s)\,ds - \int_b^t A(s)\,ds, \qquad a < b < t$$

and using the symmetry of $W(t)$ we get

$$W(t) = W(b) - \int_b^t Z_a Y_a^{-1} C Y_a^{T-1} Z_a^T\,ds - \int_b^t A(s)\,ds. \tag{12}$$

The fact that (Y_a, Z_a) is a principal solution and disconjugacy of (2) on $(a - \varepsilon, \infty)$ for some $\varepsilon > 0$ imply that (Y_a, Z_a) is a antiprincipal solution of (2)* at infinity. Thus $\int_b^t Y_a^{-1} C Y_a^{T-1}\,ds$ is bounded as well as $Z_a(t) = -\int_b^t A Y_a\,ds$.

Now, we use the following lemma [17, Lemma 6.3].

Lemma. *If* $Q(t)$ *is a positive definite matrix on* $[a, \infty)$, $\int_a^t Q(s)\,ds$ *is bounded and* $A(t)$ *is bounded matrix then* $\int_a^t A^T(s)\,Q(s)\,A(s)\,ds$ *is bounded.*

According to this Lemma the first integral in (12) is bounded and thus $W(t)$ is bounded. Hence $\lim_{t\to\infty} y^T(t)\,W(t)\,y(t) = 0$ i.e., the singularity condition is satisfied. ■

In the following, we denote $l_n(Q)$ the maximal eigenvalue of the matrix $Q(t)$. If $\int_a^t l_n(Q) < \pi$ then (9) is disconjugate on $[a, t]$ (see e.g. [16, p. 366]). This fact together with Theorem 3 is used in the following example.

Example 1. Let $Q(t) \geqq 0$ on $[a, \infty)$ and $\int_a^\infty l_n(Q) < \pi$. Then it holds

$$\liminf_{t\to\infty} \int_a^t (z^T(s)\,Q(s)\,z(s) - y^T(s)\,Q(s)\,y(s))\,ds \geqq 0,$$

for all $y(t)$, $z(t)$ admissible on $[a, \infty)$ with respect to (9) i.e. $y, z \in \mathscr{AC}$ such that $y' = Q(t)\,z$, $y(a) = 0 = \lim_{t\to\infty} y(t)$.

This example corresponds in the scalar case to the well-known fact that $\int_a^b q(t)(y'^2 - y^2)\,dt > 0$, $y(a) = 0 = y(b)$, $y \not\equiv 0$, whenever $\int_a^b q(t)\,dt < \pi$.

Till now we have used transformation of the functional (3) into the functional $\int_a^b (v^T C_0 v)\,dt$ which is allways non-negative (if $C \geqq 0$). Now we use another method consisting in the fact that every system (2) can be transformed into the system (9) whose solutions are the so called trigonometric matrices (see [2]). This method follows the idea of [8, 9, 10] consisting in the fact that the equation $(p(t)\,y')' + q(t)\,y = 0$ can be (globally) transformed into the equation $u'' + u = 0$ whose solutions are the sine and cosine functions.

The following statement sketches the application of this idea.

Corollary 1. *Let* $A(t)$, $C(t) \in \mathscr{C}[a, b]$, $B(t) = 0$ *and* $Q(t)$ *be a derivative of the phase matrix of* (2)* *satisfying* $\int_a^b l_n(Q) < \pi$. *Then*

$$\int_a^b (z^T(s)\,C(s)\,z(s) - y^T(s)\,A(s)\,y(s))\,ds \geqq 0,$$

for all $y(t)$, $z(t)$ *admissible on* $[a, \infty)$ *with respect to* (2).

Especially, if $n = 1$ *we get results of* [10].

Proof. It follows immediately from Remark 2 and Theorem B.

4. SELF-ADJOINT FUNCTIONALS OF HIGHER ORDER

Consider a self-adjoint linear differential equation of the 2n order

$$\sum_{k=0}^{n} (-1)^k \left[p_k(t)\,u^{(k)}\right]^{(k)} = 0, \tag{13}$$

where $p_k(t) \in \mathscr{C}^k[a, \infty)$, $k = 0, \ldots, n$ and $p_n(t) > 0$ for $t \in I = [a, \infty)$.

Putting

$$y = (u, u', \ldots, u^{(n-1)})^T, \qquad z = (z_1, \ldots, z_n)^T, \qquad z_k = \sum_{j=k}^{n} (-1)^{j-k} \left[p_j u^{(j)}\right]^{(j-k)}, \tag{14}$$

we can write the equation (13) as a linear Hamiltonian system (2) where

$$A = -\operatorname{diag}[p_0, p_1, \ldots, p_{n-1}], \qquad C = p_n^{-1} \operatorname{diag}[0, \ldots, 0, 1]$$

$$B = (b_{ij}), \qquad b_{ij} = \begin{cases} 1 & i = j + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

It is easy to verify that $C(t) \geqq 0$ and the system (2) with A, B, C given by (15) is identically normal. In accordance with [4] we call points a, b conjugate with respect to (13) if there exists a nontrivial solution of (13) having zeros of multiplicity n at a and b. We say that (13) is disconjugate on I if there exists no couple points from I conjugate with respect to (13).

The equation (13) is Euler—Lagrange equation for quadratic functional

$$J_s(u) = \int_a^b [p_n(u^{(n)})^2 + p_{n-1}(u^{(n-1)})^2 + \dots + p_0 u^2]\, dt.$$

The functional $J_s(u)$ will be investigated on the class of admissible functions $u(t)$ on $[a, \infty)$ i.e. $u \in \mathscr{C}^{n-1}$, $u^{(n)} \in \mathscr{AC}$, $u^{(i)}(a) = 0 = \lim u^{(i)}(t)$, $i = 0, \dots, n-1$, $(p_n u^{(n)})^{(k)}$ are measurable and $\int_a^b (p_n u^{(n)})^{(k)} (p_n u^{(n)})^{(j)} < \infty$ for every $b > a$; $k, j = 0, \dots, n-1$.

Note that admissible functions defined in such a way are admissible functions of the corresponding system (2) with matrices (15) as well as the definition of conjugate points with respect to (13) corresponds to that one of (2). Hence, we can apply Theorem 3.

Corollary 2. *If the equation* (13) *is disconjugate on* $(a - \varepsilon, \infty)$ *for some* $\varepsilon > 0$ *and*

$$\int_a^\infty p_n^{-1}(t)\, t^{2n-2}\, dt < \infty, \qquad \int_a^\infty |p_i(t)|\, t^{2n-2(i+1)}\, dt < \infty, \qquad i = 0, \dots, n-1,$$

then

$$\liminf_{t \to \infty} \int_a^t \sum_{k=0}^n p_k(s)[u^{(k)}(s)]^2\, ds \geqq 0$$

for all admissible functions $u(t)$ *on* $[a, \infty)$.

Proof. Using Theorem A and Remark 2 we transform (2) with matrices (15) into "off-diagonal" system (8). The equation $D' = BD$ yields

$$D = (d_{ij}), \qquad d_{ij} = \begin{cases} 0 & i > j, \\ t^{j-i}/(j-i)! & i \leqq j. \end{cases}$$

Then

$$D^{-1} = (\bar{d}_{ij}), \qquad \bar{d}_{ij} = \begin{cases} 0 & i > j \\ (-1)^{i+j} t^{j-i}/(j-i)! & i \leqq j \end{cases}$$

and by a straightforward computation we get $\bar{C}(t) = D^{-1} C D^{T-1} = (\bar{c}_{ij})$ and $\bar{A}(t) = D^T A D = (\bar{a}_{ij})$

$$\bar{c}_{ij} = (-1)^{i+j} \frac{t^{2n-i-j}}{(n-i)!\,(n-j)!} \qquad \bar{a}_{ij} = \sum_{k=1}^{\min\{i,j\}} \frac{t^{2k-i-j}}{(i-k)!\,(j-k)!}\, p_{k-1}(t).$$

Now, Theorem 3 can be used to obtain the desired result.

Example 2. Consider the self-adjoint equation of the fourth order

$$(p(t)\,y'')'' + q(t)\,y = 0 \qquad t \in (0, \infty), \tag{16}$$

where $p(t) > 0$, $p \in \mathscr{C}^2$ and (i) $q(t) < 0$ for $t \in (0, \infty)$,

(ii) $$\int_1^\infty t^2 q > -\infty,$$

(iii) $$\int_1^\infty t^2 p^{-1} < \infty.$$

Assumptions (i), (ii) and $\int^\infty p^{-1} < \infty$ ensure disconjugacy of (16) on $[a, \infty)$ where a is sufficiently large (see [7]). Hence, according to Corollary 2 it holds

$$\liminf_{t\to\infty} \int_a^t (p(s)\,u''^2 + q(s)\,u^2)\,\mathrm{d}s \geqq 0,$$

for all admissible functions $u(t)$ i.e. $u \in \mathscr{C}^3[a, \infty)$, $u^{(i)}(a) = 0$, $\lim_{t\to\infty} u^{(i)}(t) = 0$, $i = 0, 1$.

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A SIMPLE PROOF OF A SEMI-FREDHOLM PRINCIPLE FOR PERIODICALLY FORCED SYSTEMS WITH HOMOGENEOUS NONLINEARITIES

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Abstract. We prove that a generalized version of a semi-Fredholm principle for the existence of periodic solutions for forced systems with homogeneous nonlinearities recently obtained by Lazer and McKenna can be proved by a simple homotopy argument, which answers a question raised by those authors.

Key words. Periodic solution, periodically forced system, semi-Fredholm principle.

MS Classification. 34 C 25.

1. INTRODUCTION

In a recent paper, Lazer and McKenna [1] have proved the existence of T-periodic solutions for systems of the form

$$u''(t) + V'(u(t)) = p(t), \tag{1}$$

when $V \in C^1(\mathbf{R}^n, \mathbf{R})$ is positively homogeneous of degree two, positive semidefinite and $p \in C^1(\mathbf{R}, \mathbf{R}^n)$ is T-periodic. They use Leray – Schauder degree theory together with two perturbations arguments through systems of the form

$$u''(t) + \varepsilon u'(t) + V'(u(t)) = p(t), \tag{2}$$

with $\varepsilon > 0$ and V positive definite and

$$u''(t) + \delta u(t) + V'(u(t)) = p(t), \tag{3}$$

with $\delta > 0$ and V positive semidefinite. They remark that it does not seem possible to prove the theorem more directly by connecting (1) rather (2) to a linear equation by a homotopy.

We show in this paper that it is indeed possible and, without further complication, we can deal with a more general system which may also depend nonlinearly of u'.

II. A SEMI-FREDHOLM PRINCIPLE FOR PERIODIC SOLUTIONS OF FORCED SYSTEMS WITH HOMOGENEOUS NONLINEARITIES

Recall that a function $W : \mathbf{R}^n \to \mathbf{R}$ is said to be positive (resp. negative) semidefinite if $W(x) \geqq 0$ (resp. $W(x) \leqq 0$) for all $x \in \mathbf{R}^n$, and is said to be positively homogeneous of degree $k \geqq 0$ if $W(tx) = t^k W(x)$ for all $t \geqq 0$ and $x \in \mathbf{R}^n$. We shall call W semidefinite if it is either positive or negative semidefinite. Recall also that if $W \in C^1(\mathbf{R}^n, \mathbf{R})$ and positive homogeneous of degree $k \geqq 1$, then Euler's identity implies that

$$(x, W'(x)) = kW(x)$$

for all $x \in \mathbf{R}^n$. Of course, W' denotes the gradient of W and (x, y) the inner product of x and y in $\mathbf{R}^n$.

We may now state and prove in a direct way a semi-Fredholm principle in the sense of Lazer–McKenna for a larger class of systems.

Theorem 1. *If U and V are in $C^1(\mathbf{R}^n, \mathbf{R})$, positive homogeneous of degree two, semidefinite and such that the system*

$$u''(t) + U'(u'(t)) + V'(u(t)) = 0, \tag{4}$$

has no T-periodic solution other than 0, then for each $p \in L^1(0, T; \mathbf{R}^n)$ the problem

$$\begin{aligned} u''(t) + U'(u'(t)) + V'(u(t)) &= p(t), \\ u(0) - u(T) = u'(0) - u'(T) &= 0 \end{aligned} \tag{5}$$

has at least one solution.

Proof. Let $a = \pm 1$ and $b = \pm 1$ be such that aU and bV are positive semidefinite. Observe that the linear system

$$u''(t) + au'(t) + bu(t) = 0, \tag{6}$$

has no T-periodic solution other than 0, because if u is any T-periodic solution of (6), then, taking the inner product of (6) with $u'(t)$, integrating over $[0, T]$ and using the periodicity, we get

$$a \int_0^T |u'(t)|^2 \, dt = 0,$$

so that u is constant, and this constant must be zero as shown by integrating (6) over $[0, T]$. Consequently, it follows from one version of the Leray–Schauder's continuation theorem (see e.g. [2], Theorem IV.5) that (5) will have at least one solution if we can find $r > 0$ such that for each $\lambda \in [0, 1]$ and each possible solution u of the problem

$$\begin{aligned} u''(t) + (1 - \lambda)(au'(t) + bu(t)) + \lambda[U'(u'(t)) + V'(u(t))] &= \lambda p(t), \\ u(0) - u(T) = u'(0) - u'(T) &= 0, \end{aligned} \tag{7}$$

one has $\| u \|_1 < r$, where

$$\| u \|_1 = \max_{t \in [0,T]} | u(t) | + \max_{t \in [0,T]} | u'(t) |.$$

If it is not the case, we can find sequences (λ_k) in $[0, 1]$ and (u_k) in $C^1([0, T], \mathbf{R}^n)$ such that $\| u_k \|_1 > k$ and u_k is a solution of (7) with $\lambda = \lambda_k$ $(k \in \mathbf{N}^*)$. Letting $w_k = u_k / \| u_k \|_1$, so that $\| w_k \|_1 = 1$, for all $k \in \mathbf{N}$, and using the positive homogenity of degree one of U' and V', we get

$$\begin{aligned} w_k''(t) + (1 - \lambda_k)(aw_k'(t) + bw_k(t)) + \lambda_k[U'(w_k'(t)) + V'(u_k(t))] &= \\ = \lambda_k(p(t)/\| u_k \|_1), \\ w_k(0) - w_k(T) = w_k'(0) - w_k'(T) = 0, \end{aligned} \tag{8}$$

for all $k \in \mathbf{N}^*$, which immediately implies that the sequence $(\| w_k'' \|_{L^1})$ is bounded independently of k. Hence, the sequences (w_k) and (w_k') are equibounded and equiuniformly continuous on $[0, T]$, and Ascoli–Arzela's theorem implies the existence of subsequences (λ_{j_k}) of (λ_k), (w_{j_k}) of (w_k) and of $w \in C^1([0, T], \mathbf{R}^n)$ verifying

$$w(0) - w(T) = w'(0) - w'(T) = 0 \tag{9}$$

and such that $w_{j_k} \to w$ and $w_{j_k}' \to w'$ uniformly on $[0, T]$ and $\lambda_{j_k} \to \lambda^*$ for some $\lambda^* \in [0, 1]$. Therefore, if we take the integrated form, from 0 to t, of the differential system in (8) for $k = j_k$ and let $k \to \infty$, we see that

$$w'(t) - w'(0) + \int_0^t \{(1 - \lambda^*)(aw'(s) + bw(s)) + \lambda^*[U'(w'(s)) + V'(w(s))]\} \, ds = 0$$

for all $t \in [0, T]$, and hence w' is absolutely continuous on $[0, T]$ and satisfies the differential equation

$$w''(t) + (1 - \lambda^*)(aw'(t) + bw(t)) + \lambda^*[U'(w'(t)) + V'(w(t))] = 0. \tag{10}$$

If $\lambda^* = 1$, it follows from the assumption on (4) that $w = 0$, a contradiction with $\| w \|_1 = 1$. If $0 \leqq \lambda^* < 1$, then, taking the inner product of (10) with $w'(t)$, integrating over $[0, T]$ and using the conditions (9), we get

$$(1-\lambda^*)\,a\int_0^T |w'(t)|^2\,dt + \lambda^*\int_0^T (U'(w'(t)), w'(t))\,dt = 0,$$

i.e.

$$(1-\lambda^*)\int_0^T |w'(t)|^2\,dt + 2a\lambda^*\int_0^T U(w'(t))\,dt = 0,$$

which implies, by the positive semidefiniteness of aU that

$$\int_0^T |w'(t)|^2\,dt = 0,$$

and hence that w is constant on $[0, T]$, say $w(t) = \overline{w}$ for all $t \in [0, T]$. But then (10) implies, after an inner product with w,

$$(1-\lambda^*)\,b|\overline{w}|^2 + \lambda^*(V'(\overline{w}), \overline{w}) = 0,$$

i.e.

$$(1-\lambda^*)\,|\overline{w}|^2 + 2\lambda^* aV(\overline{w}) = 0,$$

so that $\overline{w} = 0$, as aV is positive semidefinite, and hence $w = 0$, a contradiction with $\|w\|_1 = 1$. Hence, the proof is complete.

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