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## ON SOME SHEAVES OVER A DIFFERENTIAL SPACE\*

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### INTRODUCTION

Let  $C$  be a non empty set of real functions defined on a set  $M$ . The set  $M$  will be interpreted as a topological space with weakest topology  $\tau_C$  in which all functions from  $C$  are continuous.

It is known ([7]) that the set  $C$  is called the differential structure on  $M$  iff the set  $C$  is closed with respect to the localization ( $C = C_M$ ) and  $C$  is closed with respect to the superpositions with the smooth functions on  $R^n$ .

It is easy to show that if  $C$  is the set of real functions on  $M$  closed with respect to the superposition with the smooth functions on  $R^n$  then  $C$  is a linear ring over  $R$  containing all constant functions and that topological space  $(M, \tau_C)$  is a  $C$ -regular ([7]).

The pair  $(M, C)$ , where  $C$  is a differential structure on  $M$  is called the differential space.

Similarly as in theory of differential manifolds we define a tangent vector to the differential space  $(M, C)$  at the point  $p \in M$  as well as the smooth tangent vector field on  $(M, C)$  ([7]).

The set  $M_p$  of all tangent vectors to differential space  $(M, C)$  at the point  $p \in M$  has a natural structure of linear space over  $R$  and the set  $\mathfrak{X}(M)$  of all smooth tangent vector fields on  $(M, C)$  has a natural structure of  $C$ -module.

In this paper by  $\mathfrak{C}$  we shall denote the sheaf of all smooth real functions on  $(M, C)$  and by  $\mathfrak{X}$  we shall denote the sheaf of all smooth tangent vector fields on  $(M, C)$ .

A sheaf  $\mathfrak{N}$  over differential space  $(M, C)$  is called the sheaf of  $\mathfrak{C}$ -modules ([2]) if

- (i)  $\mathfrak{N}(U)$  is  $\mathfrak{C}(U)$ -module for every open  $U \in \tau_C$ ,
- (ii)  $\varrho_V^U(\alpha \cdot \xi) = \alpha|_V \cdot \varrho_V^U(\xi)$  for  $\alpha \in \mathfrak{C}(U)$  and  $\xi \in \mathfrak{N}(U)$ ,

where  $V \subset U$  and  $\varrho_V^U: \mathfrak{N}(U) \rightarrow \mathfrak{N}(V)$  is restricting homomorphism in the sheaf  $\mathfrak{N}$ .

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## 2. THE SHEAVES OF $\mathbb{C}$ -MODULES OVER A DIFFERENTIAL SPACE

Let  $\mathfrak{N}$  be an arbitrary sheaf of  $\mathbb{C}$ -modules over a differential space  $(M, C)$ .  
It is not difficult to prove.

**Lemma 1.** *If  $U \in \tau_C$  and  $\eta \in \mathfrak{N}(U)$  then for any point  $p \in U$  there exists an open neighbourhood  $B \in p$  and  $\bar{\eta} \in \mathfrak{N}(M)$  such*

$$\varrho_B^U(\eta) = \varrho_B^M(\bar{\eta}),$$

or equivalently, as we will write usually

$$\eta|_B = \bar{\eta}|_B.$$

Now let  $\mathfrak{N}_1, \dots, \mathfrak{N}_k, \mathfrak{N}_{k+1}$ ,  $k \in \mathbb{N}$  be any sheaves of  $\mathbb{C}$ -modules over a differential space  $(M, C)$ .

We introduce the following definition

**Definition 1.** Any map

$$f: \mathfrak{N}_1(U) \times \dots \times \mathfrak{N}_k(U) \rightarrow \mathfrak{N}_{k+1}(U)$$

satisfying the condition:

$$(LF) \quad \text{if } \eta_i|_V = \eta'_i|_V, \quad \eta_i, \eta'_i \in \mathfrak{N}(U), \quad i = 1, 2, \dots, k, \quad V \subset U$$

and  $V \in \tau_C$  then

$$f(\eta_1, \dots, \eta_k)|_V = f(\eta'_1, \dots, \eta'_k)|_V,$$

will be called the *LF-mapping* of  $\mathbb{C}(U)$ -modules  $\mathfrak{N}_1(U), \mathfrak{N}_2(U), \dots, \mathfrak{N}_k(U)$  into  $\mathbb{C}(U)$ -module  $\mathfrak{N}_{k+1}(U)$ .

The set of all LF-mappings of  $\mathbb{C}(U)$ -modules  $\mathfrak{N}_1(U), \dots, \mathfrak{N}_k(U)$  into  $\mathbb{C}(U)$ -module  $\mathfrak{N}_{k+1}(U)$  will be denoted by  $LF(\mathfrak{N}_1(U), \dots, \mathfrak{N}_k(U); \mathfrak{N}_{k+1}(U))$ .

Evidently this set can be equipped with the structure of  $\mathbb{C}(U)$ -module.

Now we shall give some examples of LF-mappings important in the theory of differential space.

1. A smooth tangent vector field on differential space defined as a map  $X: C \rightarrow C$  satisfying well known condition is of course LF-mapping.

2. For any smooth tangent vector fields  $X, Y$ ,  $X \circ Y$  is an LF-mapping, too.

3. One can easily show that the operator of exterior derivative is also an LF-mapping.

4. Likely a linear connection  $D$  in a module  $\mathfrak{N}$ , treated as a map  $D: \mathfrak{N}(U) \rightarrow \Lambda^1(\mathfrak{X}(U), \mathfrak{N}(U))$  satisfying the condition

$$D(\alpha\xi) = d\alpha \cdot \xi + \alpha D\xi,$$

for any  $\alpha \in \mathbb{C}(U)$  and  $\xi \in \mathfrak{N}(U)$ , is an LF-mapping, too.

We shall prove.

**Lemma 2.** *If  $f_i : \mathfrak{N}_1(U) \times \dots \times \mathfrak{N}_k(U) \rightarrow \mathfrak{N}_{k+1}(U)$ ,  $i = 1, 2$  are the LF-mappings satisfying condition*

$$(1) \quad f_1(\eta_1 | U, \dots, \eta_k | U) = f_2(\eta_1 | U, \dots, \eta_k | U),$$

for all  $(\eta_1, \dots, \eta_k) \in \mathfrak{N}_1(V) \times \dots \times \mathfrak{N}_k(V)$ , where  $U \subset V$ ,  $U, V \in \tau_C$  then  $f_1 = f_2$ .

*Proof.* Let  $(\eta_1, \dots, \eta_k) \in \mathfrak{N}_1(U) \times \dots \times \mathfrak{N}_k(U)$  and let there be an open covering of  $U$  such that for any  $B \in \mathfrak{B}$  there exists  $(\xi_1^B, \dots, \xi_k^B) \in \mathfrak{N}_1(V) \times \dots \times \mathfrak{N}_k(V)$  such that

$$\eta_i | B = \xi_i^B | B,$$

for any  $i = 1, 2, \dots, k$ . Hence if  $f_i$ ,  $i = 1, 2$  are LF-mappings then

$$(2) \quad f_1(\eta_1, \dots, \eta_k) | B = f_2(\xi_1^B | U, \dots, \xi_k^B | U) | B,$$

and

$$(3) \quad f_2(\eta_1, \dots, \eta_k) | B = f_2(\xi_1^B | U, \dots, \xi_k^B | U) | B.$$

From (1), (2) and (3) we get

$$(4) \quad f_1(\eta_1, \dots, \eta_k) | B = f_2(\eta_1, \dots, \eta_k) | B,$$

for all  $B \in \mathfrak{B}$ . From (4) and definition of sheaf we obtain

$$f_1(\eta_1, \dots, \eta_k) = f_2(\eta_1, \dots, \eta_k),$$

for any  $(\eta_1, \dots, \eta_k) \in \mathfrak{N}_1(U) \times \dots \times \mathfrak{N}_k(U)$  or equivalently

$$f_1 = f_2.$$

**Lemma 3.** *For any LF-mapping  $f : \mathfrak{N}_1(U) \times \dots \times \mathfrak{N}_k(U) \rightarrow \mathfrak{N}_{k+1}(U)$  and for any open set  $V \subset U$  there exists one and only one LF-mapping*

$$f_V : \mathfrak{N}_1(V) \times \dots \times \mathfrak{N}_k(V) \rightarrow \mathfrak{N}_{k+1}(V),$$

such that

$$f_V(\eta_1 | V, \dots, \eta_k | V) = f(\eta_1, \dots, \eta_k) | V,$$

for all  $(\eta_1, \dots, \eta_k) \in \mathfrak{N}_1(U) \times \dots \times \mathfrak{N}_k(U)$ .

*Proof:* Let  $(\xi_1, \dots, \xi_k) \in \mathfrak{N}_1(V) \times \dots \times \mathfrak{N}_k(V)$  and  $\mathfrak{B}$  be an open covering of  $V$  such that for any  $B \in \mathfrak{B}$  there exists  $(\eta_1^B, \dots, \eta_k^B) \in \mathfrak{N}_1(U) \times \dots \times \mathfrak{N}_k(U)$  such that

$$\xi_i | B = \eta_i^B | B,$$

for  $i = 1, 2, \dots, k$ .

Now, let  $f : \mathfrak{N}_1(U) \times \dots \times \mathfrak{N}_k(U) \rightarrow \mathfrak{N}_{k+1}(U)$  be an LF-mapping. Let us consider a family

$$(5) \quad (\varrho_B^U(f(\eta_1^B, \dots, \eta_k^B)))_{B \in \mathfrak{B}},$$

of an elements of  $\mathbb{C}(B)$ -modules  $\mathfrak{N}_{k+1}(B)$  for  $B \in \mathfrak{B}$ .

Of course the elements of family (5) depend upon the choice of  $(\xi_1, \dots, \xi_k) \in \mathfrak{N}_1(V) \times \dots \times \mathfrak{N}_k(V)$ .

Now we shall show that the elements of the family (5), are agreeable on the intersections of sets of the covering. Indeed, let  $B, B' \in \mathfrak{B}$  and  $B \cap B' \neq \emptyset$ . Then evidently

$$\xi_i|_{B \cap B'} = \bar{\eta}_i^B|_{B \cap B'} = \bar{\eta}_i^{B'}|_{B \cap B'},$$

for any  $i = 1, 2, \dots, k$ .

As the map  $f$  is the LF-mapping then

$$\begin{aligned} (f(\bar{\eta}_1^B, \dots, \bar{\eta}_k^B))|_{B \cap B'} &= (f(\bar{\eta}_1^{B'}, \dots, \bar{\eta}_k^{B'}))|_{B \cap B'} = \\ &= (f(\bar{\eta}_1^B, \dots, \bar{\eta}_k^B)|_B)|_{B \cap B'} = (f(\bar{\eta}_1^{B'}, \dots, \bar{\eta}_k^{B'})|_{B'})|_{B \cap B'}. \end{aligned}$$

From here and from definition of the sheaf follows that there exists one and only one element  $f_{\mathfrak{B}}(\xi_1, \dots, \xi_k) \in \mathfrak{N}_{k+1}(V)$  such that

$$f_{\mathfrak{B}}(\xi_1, \dots, \xi_k)|_B = f(\bar{\eta}_1^B, \dots, \bar{\eta}_k^B)|_B,$$

for any  $B \in \mathfrak{B}$ .

Now, let us put

$$(6) \quad f_V(\xi_1, \dots, \xi_k) := f_{\mathfrak{B}}(\xi_1, \dots, \xi_k),$$

for an arbitrary  $(\xi_1, \dots, \xi_k) \in \mathfrak{N}_1(V) \times \dots \times \mathfrak{N}_k(V)$ .

We shall show next that the definition (6) does not depend on the choice of the covering  $\mathfrak{B}$  of the set  $V$ .

To this end let us take other open covering  $\mathfrak{A}$  of  $V$  such that for any  $A \in \mathfrak{A}$  there exists point  $(\bar{\eta}_1^A, \dots, \bar{\eta}_k^A) \in \mathfrak{N}_1(U) \times \dots \times \mathfrak{N}_k(U)$  such that

$$\xi_i|_A = \bar{\eta}_i^A|_A,$$

for  $i = 1, 2, \dots, k$ .

By definition (6) we have

$$(7) \quad f_{\mathfrak{A}}(\xi_1, \dots, \xi_k)|_A = f(\bar{\eta}_1^A, \dots, \bar{\eta}_k^A)|_A,$$

for any  $A \in \mathfrak{A}$ .

Now, let  $\mathfrak{A} \vee \mathfrak{B} = \{A \cap B : A \in \mathfrak{A} \wedge B \in \mathfrak{B}\}$ . Of course  $\mathfrak{A} \vee \mathfrak{B}$  is an open covering of  $V$ , refinement of a covering  $\mathfrak{A}$  and  $\mathfrak{B}$ . From (6) and (7) as well as definition of LF-mapping it follows

$$\begin{aligned} f_{\mathfrak{A}}(\xi_1, \dots, \xi_k)|_{A \cap B} &= f(\bar{\eta}_1^A, \dots, \bar{\eta}_k^A)|_{A \cap B} = \\ &= f(\bar{\eta}_1^B, \dots, \bar{\eta}_k^B)|_{A \cap B} = f_{\mathfrak{B}}(\xi_1, \dots, \xi_k)|_{A \cap B}, \end{aligned}$$

for all  $A \in \mathfrak{A}$  and  $B \in \mathfrak{B}$  such that  $A \cap B \neq \emptyset$ .

From here and definition of the sheaf we obtain

$$f_{\mathfrak{A}}(\xi_1, \dots, \xi_k) = f_{\mathfrak{B}}(\xi_1, \dots, \xi_k),$$

for any  $(\xi_1, \dots, \xi_k) \in \mathfrak{N}_1(V) \times \dots \times \mathfrak{N}_k(V)$ .

The verification that  $f_V$  is LF-mapping satisfying the condition

$$f_V(\eta_1 | V, \dots, \eta_k | V) = f(\eta_1, \dots, \eta_k) | V,$$

for all  $(\eta_1, \dots, \eta_k) \in \mathfrak{N}_1(U) \times \dots \times \mathfrak{N}_k(U)$  is not difficult.

**Lemma 4.** Let  $\mathfrak{B}$  be an open covering of  $U$  and

$$\{f^B : \mathfrak{N}_1(B) \times \dots \times \mathfrak{N}_k(B) \rightarrow \mathfrak{N}_{k+1}(B)\}_{B \in \mathfrak{B}},$$

family of LF-mappings such that

$$f^B | B \cap B' = f^{B'} | B \cap B',$$

for all  $B, B' \in \mathfrak{B}$  such that  $B \cap B' \neq \emptyset$ . Then there exists one and only one LF-mapping

$$f : \mathfrak{N}_1(U) \times \dots \times \mathfrak{N}_k(U) \rightarrow \mathfrak{N}_{k+1}(U),$$

such that

$$f | B = f^B,$$

for any  $B \in \mathfrak{B}$ .

Proof: Let  $(f^B)_{B \in \mathfrak{B}}$  be a family of LF-mappings of the form

$$f^B : \mathfrak{N}_1(B) \times \dots \times \mathfrak{N}_k(B) \rightarrow \mathfrak{N}_{k+1}(B),$$

satisfying the condition

$$(8) \quad f^B | B \cap B' = f^{B'} | B \cap B',$$

for all  $B, B' \in \mathfrak{B}$ ,  $B \cap B' \neq \emptyset$ . Let  $(\eta_1, \dots, \eta_k) \in \mathfrak{N}_1(U) \times \dots \times \mathfrak{N}_k(U)$  and let us take under consideration the family

$$\{f^B(\eta_1 | B, \dots, \eta_k | B)\}_{B \in \mathfrak{B}},$$

of the elements of  $\mathfrak{C}(B)$ -module  $\mathfrak{N}_{k+1}(B)$ .

From our assumption (8) it follows that

$$f^B(\eta_1 | B, \dots, \eta_k | B) | B \cap B' = f^{B'}(\eta_1 | B', \dots, \eta_k | B') | B \cap B',$$

for any  $B, B' \in \mathfrak{B}$ ,  $B \cap B' \neq \emptyset$ . Now, from here and from the fact that  $\mathfrak{N}_{k+1}$  is a sheaf it follows that there exists an element  $f(\eta_1, \dots, \eta_k) \in \mathfrak{N}_{k+1}(U)$  such that

$$f(\eta_1, \dots, \eta_k) | B = f^B(\eta_1 | B, \dots, \eta_k | B),$$

for any  $B \in \mathfrak{B}$  and  $(\eta_1, \dots, \eta_k) \in \mathfrak{N}_1(U) \times \dots \times \mathfrak{N}_k(U)$ .

Hence there exists one and only one LF-mapping

$$f \in LF(\mathfrak{N}_1(U), \dots, \mathfrak{N}_k(U); \mathfrak{N}_{k+1}(U)),$$

such that

$$f | B = f^B,$$

for any  $B \in \mathfrak{B}$ .

Let  $\mathfrak{N}_1, \dots, \mathfrak{N}_k, \mathfrak{N}_{k+1}$ ,  $k \in N$  be an arbitrary sheaves of  $\mathbb{C}(U)$ -modules over a differential space  $(M, C)$ . Let us denote by  $LF(\mathfrak{N}_1, \dots, \mathfrak{N}_k; \mathfrak{N}_{k+1})$  the category whose objects are  $\mathbb{C}(U)$ -modules  $LF(\mathfrak{N}_1(U), \dots, \mathfrak{N}_k(U); \mathfrak{N}_{k+1}(U))$ ,  $U \in \tau_C$ , of the LF-mappings.

The above proved lemmas imply the theorem

**Theorem 1.** *The three-triple  $(LF(\mathfrak{N}_1, \dots, \mathfrak{N}_k; \mathfrak{N}_{k+1}), F, \tau_C)$  is a sheaf over a differential space, where  $F$  is a contravariant functor from the category  $\tau_C$  into category  $LF(\mathfrak{N}_1, \dots, \mathfrak{N}_k; \mathfrak{N}_{k+1})$ .*

For the arbitrary sheaves  $\mathfrak{N}_1, \dots, \mathfrak{N}_k, \mathfrak{N}_{k+1}$  over a differential space we shall denote by  $LF(\mathfrak{N}_1, \dots, \mathfrak{N}_k; \mathfrak{N}_{k+1})$  the sheaf  $(LF(\mathfrak{N}_1, \dots, \mathfrak{N}_k; \mathfrak{N}_{k+1}), F, \tau_C)$ . This sheaf will be called the sheaf of LF-mappings.

Now we shall give some examples of the most important sheaves of LF-mappings over a differential spaces.

Of course, one of the fundamental sheaf of LF-mappings over a differential space is a sheaf of the tangent vector fields on a differential space which we denote by  $\mathfrak{X}$ .

Now, let  $\mathfrak{N}_1, \dots, \mathfrak{N}_k, \mathfrak{N}_{k+1}$ ,  $k \in N$  be the sheaves of  $\mathbb{C}(U)$ -modules over a differential space  $(M, C)$  and

$$\omega : \mathfrak{N}_1(U) \times \dots \times \mathfrak{N}_k(U) \rightarrow \mathfrak{N}_{k+1}(U),$$

$U \in \tau_C$ ,  $\mathbb{C}(U)$ - $k$ -linear map. It is not difficult to show that  $\omega$  is an LF-mapping. Consequently the triple

$$(L_{\mathbb{C}}(\mathfrak{N}_1, \dots, \mathfrak{N}_k; \mathfrak{N}_{k+1}), F, \tau_C),$$

is a sheaf of LF-mappings on a differential space, where  $F$  is a contravariant functor from the category  $L_{\mathbb{C}}(\mathfrak{N}_1, \dots, \mathfrak{N}_k; \mathfrak{N}_{k+1})$  of  $\mathbb{C}(U)$ -modules  $L_{\mathbb{C}(U)}(\mathfrak{N}_1(U), \dots, \mathfrak{N}_k(U); \mathfrak{N}_{k+1}(U))$  of  $\mathbb{C}(U)$ - $k$ -linear mappings into the category  $\tau_C$ . This sheaf is also denoted by

$$L_{\mathbb{C}}(\mathfrak{N}_1, \dots, \mathfrak{N}_k; \mathfrak{N}_{k+1}),$$

and called the sheaf of smooth tensor fields over a differential space.

Evidently in the particular case when  $\mathfrak{N}_1 = \mathfrak{N}_2 = \dots = \mathfrak{N}_k = \mathfrak{X}$  we have a sheaf  $L_{\mathbb{C}}^k(\mathfrak{X}, \mathfrak{N}_{k+1})$  of  $\mathbb{C}$ - $k$ -forms on the differential space with a value in the  $\mathbb{C}$ -module  $\mathfrak{N}_{k+1}$ . The sheaf of all exterior form on differential space is denoted usually by  $\Lambda^k(\mathfrak{X}, \mathbb{C})$ .

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