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SOME REMARKS ON TRIANGULATING A d-CUBE

Johannes Böhm

To Professor Dr. Otto Krötenheerdt on his 60^{th} birthday

Abstract. For simplicial approximation of fixed points of continous mappings more efficient algorithms are important for optimal work. This requires to find geometrical decompositions into simplices with a minimal number of tiles. This note is concerned with discussing triangulations of the d-cube. We give three methods to obtain vertex preserving facet-to-facet triangulations. The first method gives the standard decomposition of the d-cube into the maximal number d! simplices. The second method is due to J.F.Sallee [6]. The third method is new. The last two methods give the same numbers of simplices for $d \leq 5$ being the least possible values. For $d \geq 6$ the third method gives lower values. For example, for $d = 6$ we get 324 while the second method gives 364 simplices. Finally for arbitrary triangulations of the d-cube we establish formulas for the numbers of simplices as a function of special simplex numbers for d=2,3,4, 5 and 6.

1. Let a d-cube C_d ($d \geq 2$) in the Euclidean d-dimensional space be given. We consider triangulations of C_d, that means total decompositions of C_d into d-simplices S_j, $j=1,\ldots,s$. We assume that these triangulations are vertex preserving and facet-to-facet ones. Topological results for C_d remain true for triangulations of a parallelepipedon, since we can map the d-cube into that parallelepipedon by a homeomorphism (affine transformation). Without loss of generality C_d can be assumed to be a unit

cube so that the vertices of the cube have the coordinates $(x_1,\dots,x_d)$ with $x_j \in \{0,1\}$. A simplex of a triangulation is given by its coordinate matrix the lines of which are the coordinates of the vertices of that simplex. A triangulation $T = \{S_1,S_2,\dots,S_s\}$ of an arbitrary polyhedron is called "optimal", iff its number s of simplices is minimal. Optimal triangulations of the d-cube were given for d = 3 and d = 4 by P.S. Mara [5] and R.W. Cottle [4]. These triangulations are essentially unique. For $d > 4$ J.F.Sallee [6] and later W.D.Smith [7] have given important contributions to this subject. In case d = 5 the author could show that there are essentially three different types of optimal triangulations of the 5-cube into 67 simplices (cf. [3]).

An important instrument in decomposing a convex d-dimensional polytope is the method of coning off this body P to one of its vertices. Let $p \in \text{vert } P$ and let $\mathcal{F}_p = \{F_1,\dots,F_n\}$ be the set of all (d-1)-dimensional facets of P with $p \notin F_j$ for $j = 1,\dots,n$. We call F_j a facet opposite to p. Then the cones $P_j = \text{conv}(\{p\}, F_j)$ give always a decomposition of P by $\{P_1,\dots,P_n\}$. We say this decomposition is generated by coning off P to the vertex p. If we first triangulate the facets F_j and then cone off to p, we get a triangulation of P.

2. Now we first describe a method of (vertex preserving) triangulating a d-cube into as many simplices as possible. We work inductively.
For d = 2 the triangulation of a quadrangle is essentially unique. By drawing a diagonal of the quadrangle we dissect it into two triangles. Since there exist two diagonals of a quadrangle, we have exactly two different triangulations of it. But these ones are not essentially different, because we can find a mapping of the quadrangle onto itself which also maps the one triangulation onto the other one.

For d = 3 we choose the vertex p = (0,0,0). There are three opposite facets of the 3-cube to p. We triangulate these ones. Then we cone off the 3-cube to p. Thus we get a triangulation into 6 tetrahedra. For obtaining a unique solution we triangulate the three facets in such a manner that all the dissecting diagonals of the three quadrangles contain the vertex (1,1,1).

Now assume that in this manner we have triangulated the (d-1)-cube into (d-1)! simplices such that the intersection of all simplices of this triangulation is a body diagonal of the (d-1)-cube of length $\sqrt{d-1}$. Then in the d-cube we triangulate the d facets opposite to (0,...,0) such that for each facet the common body diagonal of the (d-1)! (d-1)-simplices contains the vertex (1,...,1) of C_d. Thus this triangulation of the d-cube consists of exactly d! simplices. The intersection of all these simplices is the body diagonal $[\![(0,\dots,0),(1,\dots,1)]\!]$, the simplices are mutually congruent and every simplex is an orthoscheme. The volume of each simplex is 1/d!. Since the least volume of a d-simplex with all d+1 vertices being vertices of the d-cube is 1/d! and the volume of the unit d-cube is 1, the maximal number s of simplices of a triangulation of the d-cube with the properties described above is d!. This triangulation is called standard triangulation of the d-cube. Therefore we have

Theorem 1. The standard triangulation of a d-cube consists of d! pairwise congruent simplices. This is a vertex preserving and facet-to-facet triangulation of a d-cube with the most simplices.

For $d \geqq 3$ there are other vertex preserving triangulations of a d-cube into d! simplices which are essentially different from the standard triangulation. We get these triangulations, if we choose the common body diagonal of the simplices of the standard triangulation of at least one r-facet, $r < d$, in such a way that it does not contain the vertex (1,...,1) which lies opposite to the vertex (0,...,0) to which we cone off. For example in case d = 3 we have 4 essentially different triangulations of this kind into 6 tetrahedra where the three triangulated facets (quadrangles opposite to the vertex (0,0,0)) are coned off to (0,0,0). This follows from the fact that if the diagonals of exactly r of these quadrangles contain the vertex (1,1,1), then the other (3-r) quadrangles do not. Since

$r = 0,1,2,3$, we have these 4 types (cf. [1, p.24]).

3. The next method for getting a vertex preserving triangulation of the d-cube has been given by J.F.Sallee [6], very clearly described in [7]. The conjecture that this triangulation is an optimal one of the d-cube could be rejected by Sallee himself who claimed to have an example for $d = 6$ with less simplices (344).

A vertex v_j of a polytope Q is called neighbouring to a vertex v_k of Q, iff v_j and v_k are joined by an edge of Q. We say we have cut off the vertex v of C_d, iff we can decompose C_d into the two polytopes $P_1 = \text{conv}(\text{vert } C_d \setminus \{v\})$ and $S = \text{conv}\{v' : v' = v \vee v' \text{ is neighbouring to } v\}$ and cancel S such that P_1 remains. Now in C_d we cut off all the $\frac{1}{2} \cdot 2^d = 2^{d-1}$ vertices $v = (x_1, x_2, \ldots, x_d)$ of C_d with $\sum_{j=1}^{d} x_j \equiv 1 \pmod 2$. The remaining polytope is the truncated d-cube Z_d. After triangulating the facets of Z_d opposite to the vertex $v_o = (0,0,\ldots,0)$ we cone off Z_d to the vertex v_o.

Z_d has $2^d - 2^{d-1} = 2^{d-1}$ vertices, 2d facets of type Z_{d-1} and 2^{d-1} simplicial facets (for $d \geq 3$). In particular, Z_3 is a regular simplex. Constructing Z_2 analogously we get an edge.

Let P(d) be the number of simplices which arise by the triangulation now being described. Then Z_d can be triangulated into $P(d)-2^{d-1}$ simplices. Each vertex of Z_d does not touch d facets (of type Z_{d-1}) of Z_d and $(2^{d-1}-d)$ simplicial facets of Z_d. Therefore, coning off Z_d to v_o we obtain $d(P(d-1)-2^{d-2}) + (2^{d-1}-d)$ simplices. Thus we get the equation

$P(d)-2^{d-1} = d(P(d-1)-2^{d-2})+(2^{d-1}-d)$ or

$$P(d) = dP(d-1) - d2^{d-2} + 2^d - d \quad . \tag{1}$$

This recursion formula is valid for $d \geq 3$. For $d = 1$ we can fix $P(1) = 1$ so that for $d = 2$ we get the only and right value $P(2) = 2 \cdot 1 - 2 \cdot 2^0 + 2^2 - 2 = 2$. This is Sallee's recursion formula for $d > 1$ with the initial value $P(1) = 1$. From (1) we get an explicit representation of P(d) in the form

$$P(d) = -\frac{1}{2}d! + 2^d + d!\cdot\left(\frac{2^{-1}-1}{0!} + \frac{2^0-1}{1!} + \frac{2^1-1}{2!} + \dots + \frac{2^{d-2}-1}{(d-1)!}\right) \quad (d \geq 1) \qquad (2)$$

(cf. [1])

and the asymptotic formula

$$P(d) \sim d!\cdot\left(\frac{1}{2}e^2 - e - \frac{1}{2}\right) \qquad \text{(cf. [7]).}$$

Equation (2) can be obtained by giving the general solution $P_h(d) = c_0 d!$ of the homogeneous part of (1), adding a special solution of (1), for example, $P_i(d) = 2^d + d!\cdot\left(\frac{2^{-1}-1}{0!} + \frac{2^0-1}{1!} + \frac{2^1-1}{2!} + \dots + \frac{2^{d-2}-1}{(d-1)!}\right)$ and applying the initial value $P(1)=1$ which gives $c_0 = -\frac{1}{2}$.

Therefore we have

Theorem 2. By cutting off all vertices of the d-cube being mutually not neighbouring ones and then coning off the remaining truncated d-cube Z_d to a vertex, we get a triangulation of the d-cube into P(d) simplices.

For the first values of d we have

$P(1) = 1$, $P(2) = 2$, $P(3) = 5$, $P(4) = 16$, $P(5) = 67$,

$P(6) = 364$.

Thus for $d \leq 5$ this triangulation is an optimal one (cf. [4] and [3]).

4. Seeking for further general methods of triangulating a d-cube in order to get lower simplex numbers and to approach to an optimal triangulation, we present a third method for $d \geq 3$. The vertex $v = (y_1,\dots,y_d)$ of the d-cube with

$$\sum_{j=1}^{d} y_j = r$$

lies in the hyperplane H_r

$$\sum_{j=1}^{d} x_j - r = 0 .$$

We are especially interested in the hyperplanes H_{r_o}, H_{r_o+1} and H_{r_o-1} with $r_o := \left\lceil \frac{d+1}{2} \right\rceil$. For even d we have $r_o = \frac{d}{2}$ and for odd d we have $r_o = \frac{d+1}{2}$. In the hyperplane H_{r_o} there are exactly $\binom{d}{r_o}$ vertices of C_d. Now we dissect C^d by the hyperplane H_{r_o} into two polytopes P_o^d and P_1^d with

$$P_o^d \cup P_1^d = C_d, \; f_o^{d-1} := P_o^d \cap P_1^d \in H_{r_o} \quad \text{and}$$

$$v_o := (0,\dots,0) \in P_o^d, \; v_1 := (1,\dots,1) \in P_1^d .$$

Further we cut off all vertices which lie in H_{r_o-1} (belonging to P_o^d) and in H_{r_o+1} (belonging to P_1^d). Let Q_o^d and Q_1^d be the remaining polytopes. For even d the two polytopes P_o^d and P_1^d (and also Q_o^d and Q_1^d) are always congruent ones. – For coning off Q_o^d to v_o we triangulate the facets of Q_o^d opposite to v_o. Some of these facets of Q_o^d are simplices. We get those ones by cutting off the above vertices. The further facets f_j $(j=1,\dots,d)$ of Q_o^d lie in that facet of C_d belonging to the hyperplane $x_j - 1 = 0$. They can be triangulated by an inductive method because f_j is congruent with a $(d-1)$-polytope Q_1^{d-1} . Therefore we only have to consider triangulations of f_o^{d-1}.

The vertices of f_o^{d-1} are those ones of the d-cube whose sum of its coordinates is r_o. For coning off we choose the vertex $\bar{v}_o$ as apex with the coordinates $x_1 = \ldots = x_{d-r_o} = 0$, $x_{d-r_o+1} = \ldots = x_d = 1$, i.e. $\bar{v}_o = (\overbrace{0,\ldots,0}^{d-r_o},\overbrace{1\ldots,1}^{r_o})$. In f_o^{d-1} the opposite facets of $\bar{v}_o$ are of two types. The first one consists of $d-r_o$ facets $\bar{f}_1,\ldots,\bar{f}_{d-r_o}$ which are the intersections $\bar{f}_j := H_{r_o} \cap f_j$. The second type consists of r_o facets $\bar{f}'_{d-r_o+1},\ldots,\bar{f}'_d$ which are the intersections $\bar{f}'_j := H_{r_o} \cap f'_j$ where f'_j is a facet of Q_1^d which is contained in the hyperplane $x_j = 0$. The triangulation of Q_1^d gives topologically identical results for even d. For odd d the results can be obtained in a similar manner: The facets of Q_1^d opposite to v_1 firstly are the simplices arising after cutting off the vertices in the hyperplane H_{r_o+1}. Secondly they are facets f_e^{d-1} and f'_j. For odd d (> 5) the facets f'_j are also congruent with that ones which arise in the following way: In C_d we cut off those vertices which lie in H_{r_o-2}. Thus we get C'_d. After the decomposition of C'_d into $\hat{Q}_o^d$ and $\hat{Q}_1^d$ by the hyper plane H_{r_o-1} with $v_o \in \hat{Q}_o^d$ we consider the facets $\hat{f}_j$ of $\hat{Q}_o^d$ lying in the hyperplane $x_j=1$. The facet f'_j is congruent with $\hat{f}_j$. Therefore in case of odd d we have to work with $\hat{Q}_o^{d-1}$ being congruent with $\hat{f}_j$.

For our calculation we need triangulations of the intersection $C_d \cap H_r$ with $r = r_o+1, r_o+2,\ldots,d-1$ or $r = r_o-1, r_o-2,\ldots,1$. These intersections are always congruent for the pairs of r (1,d-1), (2,d-2),... . Let $\hat{f}_o^{d-1} := H_{r_o+1} \cap C_d$. Our triangulations of f_o^{d-1} and $\hat{f}_o^{d-1}$, inductively obtained, may consists of $p(d)$ and $\hat{p}(d)$ simplices, respectively. Then we can establish

<u>Lemma 1.</u> For $d \geqq 4$ we have

$$p(d) = \begin{cases} d \cdot p(d-1) & \text{if d is even,} \\ \frac{d-1}{2} \cdot p(d-1) + \frac{d+1}{2} \cdot \hat{p}(d-1) & \text{if d is odd,} \end{cases}$$

$$p(2) = p(3) = \hat{p}(4) = \hat{p}(5) = 1 .$$

P r o o f. In f_o^{d-1} we cone off to the vertex $\bar{v}_o := (0,\dots,0,\overbrace{1,\dots,1}^{r_o})$. Let d be even. Then there exist r_o facets of C_d containing the vertex v_o whose intersections with f_o^{d-1} are congruent with f_o^{d-2}, and r_o facets of C_d containing the vertex v_1 whose intersections with f_o^{d-1} are also congruent with f_o^{d-2}. We see this, for example, if we cut f_o^{d-1} with the hyperplane $x_1=1$. Then in this hyperplane there remains the convex hull of the cube vertices $(1,x_2,x_3,\dots,x_d)$ with $\sum_{j=2}^{d} x_j = r_o-1$. This is an intersection of a (d-1)-cube with the hyperplane H_{r_o}. Therefore, there exist exactly $2r_o = d$ facets of f_o^{d-1} being congruent with f_o^{d-2}. Thus we have $p(d) = d\cdot p(d-1)$ for even d. – If d is odd then there exist r_o facets of C_d containing the vertex v_o whose intersections with f_o^{d-1} are congruent with $\hat{f}_o^{d-2}$, and (r_o-1) facets of C_d containing the vertex v_1 whose intersections with f_o^{d-1} are congruent with f_o^{d-2}. We have $p(d) = r_o\cdot\hat{p}(d-1) + (r_o-1)\cdot p(d-1)$. The initial values for $p(2)$, $p(3)$, $\hat{p}(4)$ and $\hat{p}(5)$ are 1, because f_o^1 is a segment, f_o^2 is a triangle $\hat{f}_o^3$ is a tetrahedron and $\hat{f}_o^4$ is a 4-simplex. (All these simplices are regular ones.) This proves the lemma.

In this manner we can calculate $\hat{p}(d)$. More generally we consider the intersection

$$\hat{\dot{\dot{f}}}{}^{q\,d-1}_{o} := C_d \cap H_{r_o+q}\,, \quad r_o+q < d \quad .$$

If $r_o+q = d-1$ then $\hat{\dot{\dot{f}}}{}^{q\,d-1}_{o}$ is a regular simplex. An inductive triangulation of $\hat{\dot{\dot{f}}}{}^{q\,d-1}_{o}$ by coning off to a vertex of it gives

$\overset{\hat{\vdots}\,q}{p}(d)$ simplices. Therefore initial values are

$$\overset{\hat{\vdots}\,q}{p}(2q+2) = \overset{\hat{\vdots}\,q}{p}(2q+3) = 1 .$$

Then we have

<u>Lemma 2.</u> Let $q > 0$. For $d \geqq 2q+4$ we have

$$\overset{\hat{\vdots}\,q}{p}(d) = \begin{cases} \frac{d-2q}{2} \cdot \overset{\hat{\vdots}\,q}{p}(d-1) + \frac{d+2q}{2} \cdot \overset{\hat{\vdots}\,q}{p}(d-1) & \text{if } d \text{ is even,} \\ \frac{d-2q-1}{2} \cdot \overset{\hat{\vdots}\,q}{p}(d-1) + \frac{d+2q+1}{2} \cdot \overset{\hat{\vdots}\,q+1}{p}(d-1) & \text{if } d \text{ is odd,} \end{cases}$$

$$\overset{\hat{\vdots}\,q}{p}(2q+2) = \overset{\hat{\vdots}\,q}{p}\,(2q+3) = 1 .$$

The proof of lemma 2 runs analogously to that of lemma 1. Now we can recursively calculate the values for $p(d)$. The first values for $2 \leqq d \leqq 11$ are

$p(2) = p(3) = 1$,

$p(4) = 4 \cdot 1 = 4$, $\hat{p}(4) = 1$,

$p(5) = 2 \cdot 4 + 3 \cdot 1 = 11$, $\hat{p}(5) = 1$,

$p(6) = 6 \cdot 11 = 66$, $\hat{p}(6) = 2 \cdot 11 + 4 \cdot 1 = 26$, $\hat{\hat{p}}(6) = 1$,

$p(7) = 3 \cdot 66 + 4 \cdot 26 = 302$, $\hat{p}(7) = 2 \cdot 26 + 5 \cdot 1 = 57$, $\hat{\hat{p}}(7) = 1$,

$p(8) = 8 \cdot 302 = 2416$, $\hat{p}(8) = 3 \cdot 302 + 5 \cdot 57 = 1191$,

$\hat{\hat{p}}(8) = 2 \cdot 57 + 6 \cdot 1 = 120$,

$p(9) = 4 \cdot 2416 + 5 \cdot 1191 = 15\,619$, $\hat{p}(9) = 3 \cdot 1191 + 6 \cdot 120 = 4293$,

$p(10) = 10 \cdot 15619 = 156\,190$, $\hat{p}(10) = 4 \cdot 15619 + 6 \cdot 4293 = 88\,234$,

$p(11) = 5 \cdot 156190 + 6 \cdot 88234 = 1\,310\,354$.

After these triangulations of the facets of Q_0^d and Q_1^d being opposite to v_0 and v_1, respectively, we can cone off to these two vertices. For low dimensions we get the following results.

<u>d=3</u> ($r_0=2$): f_0^2 is a (regular) triangle ($p(3)=1$), Q_0^3 is a (regular) tetrahedron, Q_1^3 is the empty set. We have 3 vertices in H_{r_0-1} and 1 vertex in H_{r_0+1}. Therefore the triangulation consists of $s = 1+0+3+1 = 5$ tiles.

<u>d=4</u> ($r_0=2$): f_0^3 is an octahedron ($p(4) = 4$), Q_0^4 and Q_1^4 are congruent. Coning off Q_0^4 to v_0 we get 4 tiles and coning off Q_1^4 to v_1 we also get 4 tiles. Together with the 8 vertices lying in the two hyperplanes H_1 and H_3 and being cut off we obtain $2\cdot 4 + 8 = 16$ tiles.

<u>d=5</u> ($r_0=3$): f_0^4 can be triangulated with $p(5) = 11$ tiles. The simplex number for Q_0^5 is $p(5) + 5\cdot p(4) + \binom{5}{2} = 41$ and for Q_1^5 it is $p(5) = 11$. The number of the vertices in H_2 and H_4 being cut off is $\binom{5}{2} + \binom{5}{1} = 15$. Thus for this triangulation the number of simplices is $s = 41 + 11 + 15 = 67$.

<u>d=6</u> ($r_0=3$): f_0^5 can be triangulated with $p(6) = 66$ tiles. The simplex number for Q_0^6 (and also for Q_1^6) is $p(6) + 6p(5) + \binom{6}{2} = 147$. The number of the vertices being cut off in Q_0^6 is $\binom{6}{2} = 15$. Thus for this triangulation the number of simplices is $2(147+15) = 324$.

In this manner we get for

<u>d=7:</u> Q_0^7: $p(7) + 7\cdot p(6) + 7\cdot 6\cdot p(5) + 7\binom{6}{2} + \binom{7}{3} = 1366$

Q_1^7: $p(7) + 7\cdot \hat{p}(6) + \binom{7}{2} = 505$

number of vertices being cut off: $\binom{7}{3} + \binom{7}{2} = 56$; $s = 1927$.

<u>d=8:</u> Q_0^8 (and Q_1^8) : $p(8) + 8\cdot p(7) + 8\cdot 7\cdot \hat{p}(6) + 8\binom{7}{2} + \binom{8}{3} = 6512$

number of vertices being cut off: $2\cdot\binom{8}{3} = 112$

$s = 13\,136$.

<u>d=9:</u> Q_0^9: $p(9) + 9\cdot p(8) + 9\cdot 8\cdot p(7) + 9\cdot 8\cdot 7\cdot \hat{p}(6) + 9\cdot 8\cdot\binom{7}{2} + {}$

$+ 9\cdot\binom{8}{3} + \binom{9}{4} = 74\,353$

Q_1^9: $p(9) + 9\cdot \hat{p}(8) + 9\cdot 8\cdot \hat{p}(7) + 9\cdot\binom{8}{2} + \binom{9}{3} = 30\,778$

number of vertices being cut off: $\binom{9}{4} + \binom{9}{3} = 210$

$s = 105\ 341.$

d=10: Q_0^{10}(and Q_1^{10}): $p(10) + 10\cdot p(9) + 10\cdot 9\cdot \hat{p}(8) + 10\cdot 9\cdot 8\cdot \hat{p}(7) + 10\cdot 9\cdot \binom{8}{2} + 10\cdot \binom{9}{3} + \binom{10}{4} = 464\ 180$

number of vertices being cut off: $2\cdot\binom{10}{4} = 420$

$s = 928\ 780.$

To give another proof of these simplex numbers we add the volumes of the simplices of these triangulations. If the numbers given above are right, we obtain volume 1 of the cube. Multiplying the volume of each simplex of that triangulation of the d-cube by d! we obtain the weight of that simplex. This is a value w which is a natural number. In our case we have $1 \leq w \leq r_0$. We can show this by induction. But we remark that there are simplices triangulating a cube with $w > r_0$ (cf. section 5; for example, in case d = 6 we have $w \leq 9$ and there exist simplices with w = 9; cf. also [7], appendix 1). Here the sum of the weighted simplices must be d!. We show this for half of the cube P_0^{10}. In the first line of table 1 we have the numbers of special simplices and in the second line we find the weight of these simplices.

simplex number:	$p(10)$	$10\cdot p(9)$	$10\cdot 9\cdot \hat{p}(8)$	$10\cdot 9\cdot 8\cdot \hat{p}(7)$	$\binom{10}{4}$
weight:	5	4	3	2	1

simplex number:	$\binom{10}{4}$	$10\cdot\binom{9}{3}$	$10\cdot 9\cdot\binom{8}{2}$
weight:	3	2	1

Table 1

The sum of these weighted values is 1 814 400. Indeed, it is $2\cdot 1814400 = 3628800 = 10!$. Also, the proofs for $d < 10$ give the right numbers d! . For $d \leq 5$ these values agree with those ones of Sallee and Smith (cf. [6], [7]). They are less than those ones for $d > 5$. Because of the recursion formula (1), for a lower value $P(d-1)$ we also obtain a lower value for $P(d)$ (cf. table 2). Especially for $d = 6$ Sallee's formula (2) gives $P(6) = 364$ and the method demonstrated in this section gives $s = 324$. This number is even less than Sallee's number 344 mentioned above. Therefore we have -

Theorem 3. For a 6-cube there is 324 an upper bound for the simplex number of an optimal triangulation.

dimension d	simplex number via formula (2)	simplex number obtained in this section	optimal simplex number
2	2	2	2
3	5	5	5
4	16	16	$16 = 4^2$
5	67	67	67
6	364	$324 = 18^2$	
7	2445	1927	
8	19296	13136	
9	173015	105341	
10	1728604	928780	

Table 2

5. In this last section we will establish formulas for the number of simplices in an arbitrary vertex preserving triangulation T of the d-cube ($d \leq 6$, also cf. [1], [2]). Let $s^{(d)} = s^{(d)}(T)$ be the number of the simplices in the triangulation T, let $s_0^{(d)}$ be the minimal number of $s^{(d)}$ (for an optimal triangulation) and let k be the number of simplices in T with special properties described in the corresponding tables. We call a facet f of a simplex of T an exterior facet, if f lies in a facet of the d-cube. For simplices of T having d exterior

facets we use the following assertion established in [1] (also cf. [5]), here formulated as

<u>Lemma 3.</u> In a vertex preserving triangulation of the d-cube there exist at most 2^{d-1} simplices each possessing d exterior facets.

Now we classify the simplices of T by the numbers of their exterior facets and by their weight. Let $k := k_j$ $(0 \leq j \leq d)$ be the number of simplices in T with j exterior facets. Then lemma 3 means

$$0 \leq k_d \leq 2^{d-1} . \tag{3}$$

To obtain a formula for $s^{(d)}$ as a function of simplex numbers k we establish some equations.

(i) We use the fact that $s^{(d)}$ is the sum of all k_j, i. e.

$$s^{(d)} = \sum_{j=0}^{d} k_j . \tag{4}$$

k simplices of T may have the weight w. Then the balance of the volumes of the simplices of T and of the volume of the unit cube gives

$$\frac{1}{d!}\left(\sum w \cdot k\right) = 1 . \tag{5}$$

In section 2 we saw that $s^{(d)} \leq d!$. Together with (4) and (5) we get an equation of the kind

$$s^{(d)} = d! - \sum (w-1) \cdot k \leq d! . \tag{6}$$

(ii) Using the knowledge that the d-cube possesses 2d facets being (d-1)-cubes and assuming that the minimal number $s_o^{(d)}$ is known, then the balance of the numbers k_j gives the equation

$$\sum_{j=1}^{d} j \cdot k_j = 2d \cdot s_o^{(d-1)} + a_d . \tag{7}$$

$a_d (\geqq 0)$ is the excess of the (d-1)-simplices over the minimal number in the facets with

$$a_d = \sum_{j=1}^{2d} a_{d-1}^{(j)}$$

and $a_{d-1}^{(j)}$ ($\geqq 0$) is the excess of simplices over the minimal number in each facet j (j=1,...,2d).

(iii) Comparing with the minimal number $s_o^{(d)}$ equation (6) gives

$$s_o^{(d)} \leqq d! - \sum (w-1)\cdot k \quad . \tag{8}$$

We apply inequation (8) for the facets of the d-cube, i.e. we use (8) for each of the 2d facets ((d-1)-cubes) in the form

$$\sum ((w-1)\cdot k)^{(d-1)} = (d-1)! - s_o^{(d-1)} - a_{d-1}^{\prime(j)} \quad .$$

Therefore we have for the sum of all the 2d facets

$$\sum\sum ((w-1)\cdot k)^{(d)} = 2d((d-1)! - s_o^{(d-1)}) - a_d \quad . \tag{9}$$

We can establish analogous equations with respect to facets of the cube with lower dimensions than d-1.

In this way we now discuss the dimensions d = 2,3,...,6 .

d = 2

This case is trivial. We only have simplices (triangles) with weight 1 (cf. table 3). Because of (3) we have $k_2 \leqq 2 \cdot \frac{1}{2} 2$.

k:=	k_o	k_1	k_2
number of exterior facets	0	1	2
weight	1	1	1

Table 3

Because of (4) and (5) we can write $s^{(2)} = k_o + k_1 + k_2 = 2! = 2$. Because of $a_2 = 0$ and $s_o^{(1)} = 1$ equation (7) gives $k_1 + 2k_2 = 4$. Therefore we have to consider the two equations

$$k_0 + k_1 + k_2 = 2 \qquad (10)$$
$$k_1 + 2k_2 = 4 \quad . \qquad (11)$$

Eliminating k_2 we obtain

$$2k_0 + k_1 = 0$$

and therefore we have $k_0 = k_1 = 0$ and $k_2 = 2$. That also means that there are no triangles with zero or with one exterior facet. Thus our desired formula is

$$\boxed{s^{(2)} = k_2 = 2}$$

In addition to this we have $s_0^{(2)} = 2$ and $a_2' = 0$.

d = 3

Let k be the number of simplices with the properties given in table 4. Only the simplices with no exterior facets have weight 2. These ones are always of the same type (they are even congruent). We call this type U_0. Because of (3) we have $k_3 \leq 2^2 = 4$.

k:=	k_0	k_1	k_2	k_3
number of exterior facets	0	1	2	3
weight	2	1	1	1
type	U_0			

Table 4

Because of (4) and (6) we can write $s^{(3)} = k_0+k_1+k_2+k_3 = 3! - k_0 \leq 6$. Because of $s_0^2 = 2$ and $a_3 = 0$ equation (7) gives $k_1+2k_2+3k_3 = 12$. For this dimension equation (9) gives the identity $0 = 6\cdot(2! - 2) - 0 = 0$. Therefore we have to consider the two equations

$$2k_0 + k_1 + k_2 + k_3 = 6 \qquad (12)$$
$$k_1 + 2k_2 + 3k_3 = 12 \qquad (13)$$

Eliminating k_3 we obtain

$$2k_o + \frac{1}{3}(2k_1+k_2) = 2 \tag{14}$$

and therefore we have $0 \leqq k_o \leqq 1$. Thus our desired formula is

$$\boxed{s^{(3)} = 6 - k_o} \quad .$$

Since $k_o \leqq 1$ the last equation implies $5 \leqq s^{(3)} \leqq 6$. The value $s_o^{(3)} = 5$ can only be realized by $k_o = 1$. Because of (14) then we have $k_1 = k_2 = 0$, (13) gives $k_3 = 4$. Such a triangulation of the 3-cube exists and is unique (cf. [1], [2], [5]). Of course, our methods in section 3 and 4 also give this triangulation. Therefore we can write $s^{(3)} = s_o^{(3)} + a_3' = 5 + a_3'$ $(0 \leqq a_3' \leqq 1)$; or because of the above formula for $s^{(3)}$

we obtain

$$k_o = 1 - a_3' \quad . \tag{15}$$

d = 4

Let k be the number of simplices of T with the properties given in table 5.

k:=	k_o'	$\bar{k}_o'$	k_o^*	k_1'	k_1^*	k_2	k_3	k_4
number of exterior facets	0	0	0	1	1	2	3	4
weight	3	2	1	2	1	1	1	1
type	W_o	W_1	W_2	U_o				

Table 5

It is $k_o = k_o' + \bar{k}_o' + k_o^*$, $k_1 = k_1' + k_1^*$. Because of (3) we have $k_4 \leqq 8$. The weight of a simplex with no exterior facets is at most 3. Simplices of type W_o are congruent with $W_o^{(4)}$, those of type W_1 are congruent with $W_1^{(4)}$. Simplices of type U_o contain a facet which is congruent with U_o^3. The coordinate matrices of these simplices are

$$W_o^{(4)} = \begin{pmatrix} 0000 \\ 0111 \\ 1011 \\ 1101 \\ 1110 \end{pmatrix}, \quad W_1^{(4)} = \begin{pmatrix} 0000 \\ 0011 \\ 0111 \\ 1101 \\ 1110 \end{pmatrix}, \quad U_o^{(3)} = \begin{pmatrix} 000 \\ 011 \\ 101 \\ 110 \end{pmatrix} .$$

(cf. [1; (13), (19)] ; there the combinatorical method for getting these results is given). Since (4) and (6) we can write $s^{(4)} = k_o+k_1+k_2+k_3+k_4 = 41 - 2k_o' - \bar{k}_o' - k_1' \leq 24$. Because of $s_o^{(3)} = 5$ equation (7) gives $k_1'+k_1^*+2k_2+3k_3+4k_4 = 40 + a_4 (0 \leq a_4 \leq 8)$. Equation (9) gives $k_1' = 8 - a_4$ because of (15). In the last two equations we eliminate a_4 and get $2k_1'+k_1^*+2k_2+3k_3+4k_4 = 48$. Thus we have a system of linear equations for k_1' and k_4:

$$2k_1' + k_4 = 24 - 3k_o' - 2\bar{k}_o' - k_o^* - k_1^* - k_2 - k_3 \tag{16}$$

$$2k_1' + 4k_4 = 48 - \bar{k}_1' - 2k_2 - 3k_3 . \tag{17}$$

The determinant of the coefficients is $6 \neq 0$. So we get

$$k_1' = 8 - \tfrac{2}{3}(3k_o' + 2\bar{k}_o' + k_o^*) - \tfrac{1}{6}(3k_1^* + 2k_2 + k_3)$$

$$k_4 = 8 + \tfrac{1}{3}(3k_o' + 2\bar{k}_o' + k_o^*) - \tfrac{1}{3}(k_2 + 2k_3) .$$

Substituting k_1' into the equation

$$s^{(4)} = 24 - 2k_o' - \bar{k}_o' - k_1' \tag{18}$$

we obtain

$$\boxed{s^{(4)} = 16 + \tfrac{1}{3}(\bar{k}_o' + 2k_o^*) + \tfrac{1}{6}(3k_1^* + 2k_2 + k_3)} \ .$$

At once this formula gives $s^{(4)} \geqq 16$. For $s^{(4)} = 16$ there is only one arithmetical realization: First we get $\bar{k}_o' = k_o^* = k_1^* = k_2 = k_3 = 0$. Then with these values we have $k_1' = 8 - 2k_o'$ and $k_4 = 8 + k_o'$. Because of $k_4 \leqq 8$ we have $k_o' = 0$ and therefore $k_1' = k_4 = 8$. A geometrical realization is that one described in sections 3 and 4 (also cf. [4] and [5]). Since 16 is the least value of $s^{(4)}$ from (18) we can imply

$$2k_o' + \bar{k}_o' + k_1' \leqq 8 = 2k_o' + \bar{k}_o' + k_1' + a_4' \quad . \tag{19}$$

a_4' $(0 \leqq a_4' \leqq 8)$ is the excess of 4-simplices over an optimal triangulation of the 4-cube.

<u>d = 5</u>

Let k be the number of simplices of T with the properties given in table 6.

k:=	$\mathring{k}_o'$	k_o'	$\bar{k}_o'$	$\bar{\bar{k}}_o'$	k_o^*	k_1'	$\bar{k}_1'$	k_1''	k_1^*	k_2''
number of exterior facets	0	0	0	0	0	1	1	1	1	2
weight	5	4	3	2	1	3	2	2	1	2
type	$\mathring{X}_o$	X_o	X_1	X_2		W_o	W_1	U_o		$2xU_o$

k:=	k_2^*	k_3	k_4	k_5
number of exterior	2	3	4	5
weight	1	1	1	1

Table 6

It is $k_o = \overset{o}{k}{}'_o + k'_o + \bar{k}'_o + \bar{\bar{k}}'_o + k^*_o$, $k_1 = k'_1 + \bar{k}'_1 + k''_1 + k^*_1$, $k_2 = k''_2 + k^*_2$. Because of (3) we have $k_5 \leq 16$. The weight of a simplex with no exterior facets is at most 5. Simplices of type $\overset{o}{X}_o$ are congruent with $\overset{o}{X}{}^{(5)}_o$ having the coordinate matrix

$$\overset{o}{X}{}^{(5)}_o = \begin{pmatrix} 00000 \\ 00111 \\ 01011 \\ 10011 \\ 11101 \\ 11110 \end{pmatrix} .$$

For simplices of type X_o there exist 5 different sorts being not mutually congruent. For simplices of type X_1 there exist 9 such different sorts (cf. [3]).

Since (4) and (6) we can write $s^{(5)} = k_o + k_1 + k_2 + k_3 + k_4 + k_5 = 51 - 4\overset{o}{k}{}'_o - 3k'_o - 2\bar{k}'_o - \bar{\bar{k}}'_o - 2k'_1 - \bar{k}'_1 - k''_1 - k''_2$. Because of $s^{(4)}_o = 16$ equation (7) gives $k'_1 + \bar{k}'_1 + k''_1 + k^*_1 + 2k''_2 + 2k^*_2 + 3k_3 + 4k_4 + 5k_5 = 160 + a_5$ $(0 \leq a_5 \leq 80)$. (9) gives $2k'_1 + \bar{k}'_1 + k''_1 + 2k''_2 + a_5 = 80$ because of (19). Thus we have a system of linear equations for k''_1 , k''_2 and k_5:

$$2k''_1 + 2k''_2 + k_5 = 120 - 5\overset{o}{k}{}'_o - 4k'_o - 3\bar{k}'_o - 2\bar{\bar{k}}'_o - k^*_o - 3k'_1 - 2\bar{k}'_1 - k^*_1 - k^*_2 - k_3 - k_4 \qquad (20)$$

$$k''_1 + 2k''_2 + 5k_5 = 160 + a_5 - k'_1 - \bar{k}'_1 - k^*_1 - 2k^*_2 - 3k_3 - 4k_4 \qquad (21)$$

$$k''_1 + 2k''_2 \quad = 80 - a_5 - 2k'_1 - \bar{k}'_1 \quad . \qquad (22)$$

The determinant of the system is $-10 \neq 0$. Therefore we have a unique solution

$$k''_1 = 24 - (5\overset{o}{k}{}'_o + 4k'_o + 3\bar{k}'_o + 2\bar{\bar{k}}'_o + k^*_o) - \frac{1}{5}(4k^*_1 + 3k^*_2 + 2k_3 + k_4)$$
$$- \frac{1}{5}(6k'_1 + 5\bar{k}'_1) + \frac{3}{5}a_5$$

$$k''_2 = 28 + \frac{1}{2}(5\overset{o}{k}{}'_o + 4k'_o + 3\bar{k}'_o + 2\bar{\bar{k}}'_o + k^*_o) + \frac{1}{10}(4k^*_1 + 3k^*_2 + 2k_3 + k_4) +$$

$$- \frac{2}{5}k_1' - \frac{4}{5}a_5$$

$$k_5 = 16 + \frac{2}{5}a_5 + \frac{1}{5}k_1' - \frac{1}{5}(k_1^* + 2k_2^* + 3k_3 + 4k_4) \qquad (\leq 16) \quad .$$

Substituting k_1'' and k_2'' into

$$s^{(5)} = 120 - 4\overset{o}{k}_0' - 3k_0' - 2\bar{k}_0' - \bar{\bar{k}}_0' - 2k_1' - \bar{k}_1' - k_1'' - k_2'' \tag{23}$$

we obtain

$$\boxed{s^{(5)} = 68 - \frac{1}{2}(3\overset{o}{k}_0' + 2k_0' + \bar{k}_0') + \frac{1}{10}(5k_0^* + 4k_1^* + 3k_2^* + 2k_3 + k_4) + \frac{1}{5}(a_5 - 2k_1')}$$

For $s^{(5)}$ of this formula the first therm 68 is obtained from $68 = 36 + 2\cdot s_0^{(4)}$.

In [3] we have shown that $s_0^{(5)} = 67$. Therefore from (23) we can imply

$$4\overset{o}{k}_0' + 3k_0' + 2\bar{k}_0' + \bar{\bar{k}}_0' + 2k_1' + \bar{k}_1' + k_1'' + k_2'' \leq$$

$$53 = 4\overset{o}{k}_0' + 3k_0' + 2\bar{k}_0' + \bar{\bar{k}}_0' + 2k_1' + \bar{k}_1' + k_1'' + k_2'' + a_5' \quad . \tag{24}$$

a_5' $(0 \leq a_5' \leq 53)$ is the excess of 5-simplices over an optimal triangulation of the 5-cube.

The two methods presented in the sections 3 and 4 giving an optimal triangulation with 67 simplices yield the following simplex numbers:

via section 3: $k_0 = k_0' = 1$, $k_1 = k_1'' = 20$, $k_2 = k_2'' = 30$, $k_3 = k_4 = 0$, $k_5 = 16$

via section 4: $k_0 = \bar{k}_0' = 6$, $k_1 = 7$ $(k_1' = 5,\ k_1'' = 2)$, $k_2 = k_2'' = 29$, $k_3 = 10$, $k_4 = 0$, $k_5 = 15$.

These two triangulations are essentially different. There exist further triangulations of the 5-cube into 67 simplices which are also essentially different from these two ones (cf. [3]).

<u>d = 6</u>

Let k be the number of simplices of T with the properties given in table 7.

k:=	k'_0	$\overline{k}'_0$	$\overline{\overline{k}}'_0$	$\overline{\overline{\overline{k}}}'_0$	$\overline{\overline{\overline{\overline{k}}}}'_0$	$\overline{\overline{\overline{\overline{\overline{k}}}}}'_0$	$\overline{\overline{\overline{\overline{\overline{\overline{k}}}}}}'_0$	$\overline{\overline{\overline{\overline{\overline{\overline{\overline{k}}}}}}}'_0$	k^*_0	$\overline{\overline{k}}'_1$	$\overline{\overline{k}}''_1$	$\overline{\overline{k}}'''_1$
number of exterior facets	0	0	0	0	0	0	0	0	0	1	1	1
weight	9	8	7	6	5	4	3	2	1	5	4	3
type	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7		X_0	X_1	X_2

k:=	$\overline{\overline{k}}''''_1$	$\overline{k}'_1$	$\overline{k}''_1$	k'_1	k^*_1	$\overline{k}'_2$	$\overline{k}''_2$	k'_2	k^*_2
number of exterior facets	1	1	1	1	1	2	2	2	2
weight	2	3	2	2	1	3	2	2	1
type	X_3	W_0	W_1	U_0		$2xW_0$	$2xW_1$	$3xU_0$	

k:=	k'_3	k^*_3	k_4	k_5	k_6
number of exterior facets	3	3	4	5	6
weight	2	1	1	1	1
type	$6xU_0$				

Table 7

It is $k_o=k'_o+\bar{k}'_o+\bar{\bar{k}}'_o+\bar{\bar{\bar{k}}}'_o+\bar{\bar{\bar{\bar{k}}}}'_o+\bar{\bar{\bar{\bar{\bar{k}}}}}'_o+\bar{\bar{\bar{\bar{\bar{\bar{k}}}}}}'_o+\bar{\bar{\bar{\bar{\bar{\bar{\bar{k}}}}}}}'_o+k^*_o$, $k_1=\bar{\bar{k}}'_1+\bar{\bar{k}}''_1+\bar{\bar{k}}'''_1+\bar{\bar{k}}''''_1+\bar{k}'_1+\bar{k}''_1+k'_1+k^*_1$, $k_2 = \bar{k}'_2+\bar{k}''_2+k'_2+k^*_2$, $k_3=k'_3+k^*_3$.

Because of (3) we have $k_6 \leqq 32$. The weight of a simplex with no exterior facets is at most 9 (cf. [7] , appendix 1). A representant of type Y_o is the simplex $Y_o^{(6)}$ having the coordinate matrix

$$Y_o^{(6)} = \begin{pmatrix} 000000 \\ 000111 \\ 001110 \\ 011001 \\ 101001 \\ 110010 \\ 110100 \end{pmatrix} .$$

Because of (4) and (6) we can write

$$s^{(6)}=k_o+k_1+k_2+k_3+k_4+k_5+k_6= 61 -8k'_o-7\bar{k}'_o-6\bar{\bar{k}}'_o-5\bar{\bar{\bar{k}}}'_o-4\bar{\bar{\bar{\bar{k}}}}'_o-3\bar{\bar{\bar{\bar{\bar{k}}}}}'_o-2\bar{\bar{\bar{\bar{\bar{\bar{k}}}}}}'_o-\bar{\bar{\bar{\bar{\bar{\bar{\bar{k}}}}}}}'_o - 4\bar{\bar{k}}'_1-3\bar{\bar{k}}''_1-2\bar{\bar{k}}'''_1 -\bar{\bar{k}}''''_1-2\bar{k}'_1-\bar{k}''_1-k'_1-2\bar{k}'_2-\bar{k}''_2-k'_2-k'_3 . \qquad (25)$$

Because of $s_o^{(5)}= 67$ equation (7) gives $\bar{\bar{k}}'_1+\bar{\bar{k}}''_1+\bar{\bar{k}}'''_1+\bar{\bar{k}}''''_1+\bar{k}'_1+\bar{k}''_1+k'_1+k^*_1+2\bar{k}'_2+2\bar{k}''_2+2k'_2+2k^*_2+3k'_3+3k^*_3+4k_4+5k_5+6k_6=12\cdot s_o^{(5)}+a_6=804+ a_6$ $(0 \leqq a_6 \leqq 12\cdot 53 = 636)$. Equation (9) gives $4\bar{\bar{k}}'_1+3\bar{\bar{k}}''_1+2\bar{\bar{k}}'''_1+\bar{\bar{k}}''''_1+2(\bar{k}'_1+2\bar{k}'_2)+(\bar{k}''_1+2\bar{k}''_2)+k'_1+2k'_2+3k'_3 = 12\cdot(120-s_o^{(5)})-a_6 = 636 - a_6$ because of (24). With respect to the 4-facets of the cube equation (9) gives $2(\bar{k}'_1+2\bar{k}'_2)+(\bar{k}''_1+2\bar{k}''_2)+(k'_1+3k'_2+6k'_3) = 12\cdot 80-a''_6 = 960 - a''_6$ $(0 \leqq a''_6 \leqq 12\cdot 80 = 960)$ because of (22) (and (19)). a''_6 is the excess of the 4-simplices over an optimal triangulation of the 4-cubes. Thus we have a system of linear equations for k'_1, k'_2, k'_3 and k_6:

$$
\begin{aligned}
2k'_1+2k'_2+2k'_3+k_6 &= 720 - A\\
k'_1+2k'_2+3k'_3+6k_6 &= 804 + a_6 - B\\
k'_1+2k'_2+3k'_3 &= 636 - a_6 - C\\
k'_1+3k'_2+6k'_3 &= 960 - a''_6 - D \quad .
\end{aligned}
$$

We obtain A,B,C,D from the above equations. Therefore it is

$$
\begin{aligned}
A = {} & 9k'_0+8\bar{k}'_0+7\bar{\bar{k}}'_0+6\bar{\bar{\bar{k}}}'_0+5\bar{\bar{\bar{\bar{k}}}}'_0+4\bar{\bar{\bar{\bar{\bar{k}}}}}'_0+3\bar{\bar{\bar{\bar{\bar{\bar{k}}}}}}+2\bar{\bar{\bar{\bar{\bar{\bar{\bar{k}}}}}}}'_0+k^{*}_0+5\bar{\bar{k}}'_1+4\bar{\bar{k}}''_1+3\bar{\bar{k}}'''_1+2\bar{\bar{k}}''''_1 +\\
& +3\bar{k}'_1+2\bar{k}''_1+k^{*}_1+3\bar{k}'_2+2\bar{k}''_2+k^{*}_2+k^{*}_3+k_4+k_5\\
B = {} & \bar{\bar{k}}'_1+\bar{\bar{k}}''_1+\bar{\bar{k}}'''_1+\bar{\bar{k}}''''_1+\bar{k}'_1+\bar{k}''_1+k^{*}_1+2\bar{k}'_2+2\bar{k}''_2+2k^{*}_2+3k^{*}_3+4k_4+5k_5\\
C = {} & 4\bar{\bar{k}}'_1+3\bar{\bar{k}}''_1+2\bar{\bar{k}}'''_1+\bar{\bar{k}}''''_1+2\bar{k}'_1+\bar{k}''_1+4\bar{k}'_2+2\bar{k}''_2\\
D = {} & 2\bar{k}'_1+\bar{k}''_1+4\bar{k}'_2+2\bar{k}''_2
\end{aligned}
$$

The determinant of the system is $12 \neq 0$. Thus we have a unique solution for k'_1, k'_2, k'_3, k_6. Especially we get

$$
\begin{aligned}
k'_1+k'_2+k'_3 = 346 - \frac{a_6}{6} - \frac{1}{12}\big(& 54k'_0+48\bar{k}'_0+42\bar{\bar{k}}'_0+36\bar{\bar{\bar{k}}}'_0+30\bar{\bar{\bar{\bar{k}}}}'_0+24\bar{\bar{\bar{\bar{\bar{k}}}}}'_0+18\bar{\bar{\bar{\bar{\bar{\bar{k}}}}}}'_0\\
& +12\bar{\bar{\bar{\bar{\bar{\bar{\bar{k}}}}}}}'_0+6k^{*}_0+33\bar{\bar{k}}'_1+26\bar{\bar{k}}''_1+19\bar{\bar{k}}'''_1+12\bar{\bar{k}}''''_1+19\bar{k}'_1+12\bar{k}''_1+5k^{*}_1\\
& +20\bar{k}'_2+12\bar{k}''_2+4k^{*}_2+3k^{*}_3+2k_4+k_5\big) \quad .
\end{aligned}
$$

Substituting this sum into the representation (25) of $s^{(6)}$ we obtain

$$
\boxed{
\begin{aligned}
s^{(6)} = 374 & - \tfrac{1}{2}\big(7k'_0+6\bar{k}'_0+5\bar{\bar{k}}'_0+4\bar{\bar{\bar{k}}}'_0+3\bar{\bar{\bar{\bar{k}}}}'_0+2\bar{\bar{\bar{\bar{\bar{k}}}}}'_0+\bar{\bar{\bar{\bar{\bar{\bar{k}}}}}}'_0\big) - \tfrac{5}{12}\big(3\bar{\bar{k}}'_1+2\bar{\bar{k}}''_1+\bar{\bar{k}}'''_1\big)\\
& - \tfrac{1}{12}\big(5\bar{k}'_1+4\bar{k}'_2\big) + \tfrac{1}{12}\big(6k^{*}_0+5k^{*}_1+4k^{*}_2+3k^{*}_3+2k_4+k_5\big) + \tfrac{1}{6}\cdot a_6 .
\end{aligned}
}
$$

For $s^{(6)}$ of this formula the first term 374 is obtained from

$$374 = 240 + 2s_0^{(5)} \text{ with } s_0^{(5)} = 67.$$

The two methods presented in the sections 3 and 4 yield triangulations into 364 and 324 simplices, respectively, with the following simplex numbers:

(i) $k_0=0$, $k_1 = 132$ $(\bar{\bar{k}}_1''=12, k_1' = 120)$, $k_2=k_2' = 120$, $k_3=k_3' = 80$, $k_4=k_5=0$, $k_6=32$, $a_6=a_6'' =0$, $s^{(6)}= 364$;

(ii) $k_0=0$, $k_1=132$ $(\bar{\bar{k}}_1''=72, \bar{k}_1'=60)$, $k_2=k_2'=24$, $k_3=k_3'=108$, $k_4=30$, $k_5=0$, $k_6=30$, $a_6=0$, $a_6''=120$, $s^{(6)}= 324$.

We obtain these simplex numbers recursively. The coordinate matrices of the simplices of the corresponding triangulation are given in [2] and [3].

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Verfasser: Johannes Böhm
Sektion Mathematik
Friedrich-Schiller-Universität Jena