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APPROXIMATION OF SPHERICAL CAPS BY POLYGONS

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To Professor Dr. Otto Krötenheerdt on his 60th birthday

1. Introduction

In Euclidean 3-space let S^2 denote the two-dimensional unit sphere. We shall only consider closed subsets of S^2. A closed set on S^2 is said to be <u>convex</u> if it contains, whith each pair of its points, the small arc or a semicircular arc of a great circle determined by them. Throughout we shall assume that the non-empty convex set C is a proper subset of S^2 and different from a great circle. Such a set C is contained in a closed hemisphere. The intersection of the closed hemispheres of S^2, the centres of which are the points of C, is also a (closed) convex set C^*, <u>the polar set</u> of C. As is well known, the polar of C^* is C. The perimeter of C and the area of C^* satisfy the relation

$$p(C) + a(C^*) = 2\pi. \tag{1}$$

A convex proper subset of S^2 with interior points will be called a <u>convex cap</u>. If a and p denote the area and the perimeter of a convex cap, we have the isoperimetric inequality

$$(2\pi-a)^2 + p^2 \geq 4\pi^2 \tag{2}$$

with equality only for the circular caps. A <u>convex k-gon</u> (k≥1) is a convex cap, which is the intersection of at most k closed hemispheres. The boundary of a convex k-gon P consists of at most k arcs of great circles, the sides of P. By a <u>regular k-gon</u> (k≥2) we mean a regular polygon with exactly k sides. If a and p denote the area and the perimeter of a convex k-gon (k≥2) then

$$\cos\frac{P}{2k} \le \frac{\cos\frac{\pi}{k}}{\cos\frac{2\pi-a}{2k}} \qquad (3)$$

with equality only for the regular k-gons.

Let P be a regular k-gon, where $k \ge 2$. We round the corners of P off by k congruent circular arcs, each touching two sides of P. The convex cap C which is bounded by these k arcs and k segments on the sides of P will be called a <u>smooth regular k-gon with case P</u>. The smooth regular k-gons with given case P form a pencil joining P with its in-circle. It is convenient to consider the in-circle of P as a degenerate smooth regular k-gon.

Let P^* be the polar k-gon of P. C^*, the polar cap of C, arises from P^* by joining each two consecutive vertices of P^* by congruent circular arcs. We shall call C^* a <u>regular arc-sided k-gon with kernel P^*</u>. The regular arc-sided k-gons with given kernel P^* form a pencil joining P^* with its circum-circle.

We shall deal with the approximation of convex caps by convex polygons. There is a wide range of literature concerning the approximation of convex bodies by polytopes in Euclidean spaces (see [7]). But little is known about the analogous problems in non-Euclidean spaces ([2],[4],[5],[8]). In this paper some results obtained for plane convex discs ([1],[3],[6]) are extended to the 2-sphere. We shall see that the situation in S^2 is in a way more satisfactory than that in the plane.

We restrict the mutual position of a cap C and an approximating polygon P by supposing that either $P \supset C$ or $P \subset C$. That means that P is approximating C either from the exterior or from the interior. There are several methods of measuring the deviation between two convex caps X and Y, where $X \subset Y$. Two of the most usual methods are given by the <u>area deviation</u> between X and Y

$$\delta^A(X,Y) = a(Y) - a(X) \qquad (4)$$

and by the <u>perimeter deviation</u>

$$\delta^P(X,Y) = p(Y) - p(X), \qquad (5)$$

where a(M) and p(M) denote the area and the perimeter of the set

M respectively.

Throughout this paper we use a and p to denote positive constants less than 2π and satisfying the isoperimetric inequality (2). Let $\mathcal{C}(a,p)$ be the class of all convex caps with area not less than a and perimeter not greater than p, and let $\mathcal{P}_k$ be the class of all convex k-gons. Two measures for the "nearness" of the caps from $\mathcal{C}(a,p)$ to convex k-gons are given by the functions

$$\Delta_e^A(a,p,k) = \min \delta^A(C,P), \qquad [C \subset P] \qquad (6)$$

$$\Delta_e^P(a,p,k) = \min \delta^P(C,P), \qquad (7)$$

where the minimum extends over all caps C from $\mathcal{C}(a,p)$ and all convex k-gons P containing C. (The subscript "e" refers to approximation from the exterior.) The existence of the minima follows from the Blaschke selection theorem. Both functions are interesting only if

$$\cos\frac{p}{2k} > \frac{\cos\frac{\pi}{k}}{\cos\frac{2\pi-a}{2k}}, \qquad (8)$$

which means that p is less than the perimeter of a regular k-gon of area a. Otherwise $\mathcal{C}(a,p)$ would contain a k-gon, so that $\Delta_e^A = \Delta_e^P = 0$. In addition to Δ_e^A and Δ_e^P the functions

$$m_e^A(a,p,k) = \min a(P), \qquad [C \subset P] \qquad (9)$$

$$m_e^P(a,p,k) = \min p(P), \qquad (10)$$

will be considered, where the minimum is taken over all caps C from $\mathcal{C}(a,p)$ and all convex k-gons P containing C. In a similar way we define four further functions Δ_i^A, Δ_i^P, M_i^A, M_i^P by replacing in (9) and (10) the minimum by the maximum and extending the extrema over all caps C from $\mathcal{C}(a,p)$ and all k-gons P contained in C.

2. Theorems

We are now in a position to state the following theorems.

THEOREM 1. Let $k \geq 2$ be an integer and let a and p be positive constants less than 2π and satisfying the inequalities (2) and (8). If $C \in \mathcal{C}(a,p)$ and $P \in \mathcal{P}_k$ with $C \subset P$ satisfy one of the conditions

(i) $\delta^A(C,P) = \Delta_e^A(a,p,k)$, or

(ii) $a(P) = m_e^A(a,p,k)$, or

(iii) $p(P) = m_e^P(a,p,k)$, or

(iv) $\delta^P(C,P) = \Delta_e^P(a,p,k)$,

then C is a smooth regular k-gon of area a and perimeter p, and P is the case of C.

THEOREM 2. Let $k \geq 3$ be an integer and let a and p be positive constants less than 2π and satisfying the inequalities (2) and (8). If $C \in \mathcal{C}(a,p)$ and $P \in \mathcal{P}_k$ with $P \subset C$ satisfy one of the conditions

(i) $\delta^P(P,C) = \Delta_i^P(a,p,k)$, or

(ii) $p(P) = M_i^P(a,p,k)$, or

(iii) $a(P) = M_i^A(a,p,k)$, or

(iv) $\delta^A(P,C) = \Delta_i^A(a,p,k)$,

then C is a regular arc-sided k-gon of area a and perimeter p, and P is the kernel of C.

Note that Theorem 2 follows from Theorem 1 by spherical polarity, so that it suffices to prove Theorem 1. Two analogous theorems for plane convex discs are contained in the papers [3] and [1].

3. Proofs

Proof of Theorem 1. Let us assume that $C \in \mathcal{C}(a,p)$ and $P \in \mathcal{P}_k$ with $C \subset P$ satisfy condition (i). In the following 10 lemmas the properties of C and P are developed, the last showing that C and

P correspond with the assertion of Theorem 1. First we observe that C lies in some open hemisphere and that at least one vertex of P is outside C. Otherwise C would be a k-gon in contradiction to (8).

LEMMA 1. The midpoint of every side of P lies on the boundary of C.

Proof. By a segment on S^2 we mean always an arc of a great circle of length less than π. Let M be the midpoint of the side A_1A_2 of P, and suppose that C and the segment MA_2 are disjoint. A small rotation of A_1A_2 about a point of A_1A_2 near to M reduces a(P) without changing a(C). This, however, contradicts condition (i).

LEMMA 2. P has exactly k sides.

Proof. Let A_1 be some vertex of P outside C. If P has fewer than k sides, we can reduce a(P) by cutting off from P a small triangle with vertex A_1.

From the lemmas 1 and 2 it follows that the boundary of C consists of k segments (possibly points) on the sides of P, and j arcs $b_1,\dots,b_j$ $(1 \le j \le k)$, each having its endpoints on two adjacent sides of P.

LEMMA 3. b_i is a circular arc (possibly of spherical radius $\pi/2$), for $1 \le i \le j$.

Proof. Let V_1 and V_2 be any two distinct points in the relative interior of b_i. The arc $\widehat{V_1V_2}$ lies entirely in the interior of P and has a positive distance ρ from the boundary of P. We cover $\widehat{V_1V_2}$ by a finite number of its subarcs in such a way that each has a length less than ρ and overlaps the following. It suffices to show that each of these subarcs, say $\widehat{W_1W_2} = s$, is a circular arc. If this is not so, we replace s by the circular arc $\hat{s}$ of the same length, say λ, which joins W_1 with W_2 and lies on the same side of the great circle through W_1 and W_2. Because $\lambda < \rho$, the set $D = (C \setminus \operatorname{conv} s) \cup \operatorname{conv} \hat{s}$ is contained in P and in some open hemisphere.

We proceed to show that

$$a(\operatorname{conv} s) < a(\operatorname{conv} \hat{s}). \tag{11}$$

$\hat{s}$ is a subarc of the boundary of a circular cap $\hat{C}$ of radius $r < \pi/2$. Let s' denote the complementary subarc of bd $\hat{C}$. The curve $s \cup s'$ is the boundary of a closed set T with $p(T) = p(\hat{C})$. We observe that T is a subset of some open hemisphere. Indeed: if s' is at most a semicircle, then T is contained in the interior of P. If s' is larger than a semicircle, then consider the hemisphere H concentric with $\hat{C}$. It is easy to show that, for any point X on the boundary of H,

$$\widehat{XW}_1 + \widehat{XW}_2 \geq 2 \arccos(\sin r \cos \frac{\alpha}{2}) > \alpha \sin r = \lambda,$$

where $\alpha < \pi$ is the central angle of $\hat{s}$, whereas

$$\widehat{YW}_1 + \widehat{YW}_2 \leq \lambda$$

for any point $Y \in s$. Thus T is contained in the interior of H. Since $s \neq \hat{s}$ and $p(T) = p(\hat{C})$, it follows from the isoperimetric inequality that $a(T) < a(\hat{C})$, which implies (11). From (11) we deduce that $a(C) < a(D) \leq a(\operatorname{conv} D)$. Because $\operatorname{conv} D \subset P$ and $p(\operatorname{conv} D) \leq p(D) = p(C)$, we have a contradiction to assumption (i). Thus s is, in fact, a circular arc, and Lemma 3 is proved.

<u>LEMMA 4.</u> The arcs $b_1, \ldots, b_j$ have the same radius, say r.

<u>Proof.</u> Suppose that b_1 and b_2 have different radii. Let s_1 and s_2 be two subarcs of b_1 and b_2 contained in the interior of P and corresponding to two chords c_1 and c_2 of the same length. We interchange the positions of the circular segments $\operatorname{conv} s_1$ and $\operatorname{conv} s_2$. That means we cut them off from C and join them to c_2 and c_1 respectively, By this process we obtain a non-convex set D which is contained in P and in some open hemisphere when c_1 is sufficiently small. Since $p(D) = p(C)$ and $a(D) = a(C)$ we have $a(\operatorname{conv} D) > a(C)$ and $p(\operatorname{conv} D) < p(C)$ contradicting assumption (i).

<u>LEMMA 5.</u> Every vertex of P is exterior to C.

<u>Proof.</u> Let A_1 be a vertex of P on the boundary of C. We may clearly assume that A_2 is outside C. By Lemma 1 there is an isosceles triangle $\Delta = A_1V_1V_2 \subset C$ with apex A_1, and V_1 and V_2 on

the segments A_1A_2 and A_1A_k respectively. Let $s = \widehat{W_1W_2}$ be a subarc of the circular arc rounding off the corner of P at A_2, where the segment V_1V_2 and the chord W_1W_2 have the same length. Similarly as in the proof of Lemma 4 we interchange the positions of Δ and conv s obtaining a disc D with $p(D) = p(C)$ and $a(D) = a(C)$. If Δ is sufficiently small, then D is non-convex and contained in P and in some open hemisphere, which again leads to a contradiction to (i).

By the lemmas 3, 4 and 5, the boundary of C contains k circular arcs $b_1,\dots,b_k$ of the same radius r. We next prove that C has a smooth boundary. That means in particular that $r < \pi/2$.

<u>LEMMA 6.</u> Through every boundary point of C there passes just one supporting great circle.

<u>Proof.</u> The boundary of C intersects the side A_iA_{i+1} of $P = A_1\dots A_k$ in a segment $U_{i1}U_{i2}$ for $i = 1,\dots,k$. Let b_i be the circular subarc of the boundary of C joining $U_{i-1,2}$ with U_{i1}. Let t be the great circle tangent to b_2 at U_{12}. To prove Lemma 6 we have to show that t coincides with the great circle determined by the side A_1A_2 of P. Suppose that this is not true. We prolongate the arc b_2 beyond U_{12} and choose two points X and Z on b_2 and on the prolongation respectively, such that $U_{12}X = U_{12}Z$. The great circle XZ intersects the boundary of C at a point Y between X and Z. We denote the convex hull of $\widehat{XU_{12}Y}$ by S_1, the angle between XY and the arc $\widehat{XU_{12}}$ by $\sphericalangle X$, and the angle between the segment YX and the arc $\widehat{YU_{12}}$ by $\sphericalangle Y$. Let $\widehat{VW}$ be a subarc of b_2 contained in the interior of P and such that $VW = XY$. Write S_2 for the convex hull of $\widehat{VW}$. By exchanging the positions of S_1 and S_2 we obtain a non-convex set D. If ZX is sufficiently small, the following conditions are satisfied:

$\sphericalangle X < \sphericalangle Y$;

S_1 and S_2 are disjoint, both in the original and in the exchanged positions;

S_1 is in the new position contained in the interior of P.

Then S_2 in the new position is contained in C. Obviously, we have $D \subset P$, $p(D) = p(C)$, $a(D) = a(C)$. Hence $a(\text{conv} D) > a(C)$, $p(\text{conv} D) < p(C)$. By the assumption on C and P, this is impossible.

LEMMA 7. $A_iU_{i1} = U_{i2}A_{i+1}$, for $i = 1,\ldots,k$.

Proof. Let M and M' be the midpoints of A_1A_2 and $U_{11}U_{12}$ respectively, and let us assume that $A_1U_{11} < U_{12}A_2$. Then M' lies between A_1 and M. By Lemma 1, we have $M \in U_{11}U_{12}$ and thus $U_{11} \neq U_{12}$. A rotation through a small angle ϕ about a point X of the segment MM' transforms A_1A_2 into a new position $A_1'A_2'$, where $A_2' \in A_2A_3$. $A_1'A_2'$ intersects b_2 at a point Y. When X = M then $YX < XU_{11}$, and when X = M' then $YX > XU_{11}$ (use Lemma 6). Thus, for a sufficiently small angle ϕ, there exists a point X between M and M' such that $YX = XU_{11}$. Moreover, X approaches M' as ϕ tends to 0. Let S be the set bounded by the segments YX, XU_{12} and the arc $\widehat{U_{12}Y}$. We cut S off from C and rejoin it to C such that XY coincides with XU_{11}. The non-convex disc D obtained by this process has the properties $a(D) = a(C)$, $p(D) = p(C)$, $D \subset P'$, where P' is the convex k-gon $A_1'A_2'A_3\ldots A_kA_1'$. Since X is near to M' we have $a(P') < a(P)$. Again, this contradicts the assumption on C and P.

LEMMA 8. $p(C) = p$.

Proof. Supposing $p(C) < p$, we choose a point $X \in P\setminus C$ near the boundary of C. For $C' = \text{conv}(C \cup \{X\})$ we have $a(C') > a(C)$, $p(C') < p$ and $C' \subset P$, which is impossible.

LEMMA 9. $a(C) = a$.

Proof. Suppose that $a(C) > a$. When $k \geq 3$, choose $A_1' \in A_1A_2$ near A_1 and let P' be the convex k-gon $A_1'A_2 \ldots A_kA_1'$. When k = 2, let $P' \subset P$ be a digon with vertices A_1, A_2 and sides near those of P. In any case, we have for $C' = C \cap P'$

$$a(C') > a,\ p(C') < p,\ \delta^A(C',P') < \delta^A(C,P),$$

which contradicts the assumption on C and P.

The following lemma completes the proof of Theorem 1 under the assumption (i).

LEMMA 10. P is regular.

Proof. From Lemma 4, 6 and 7 it follows that P is an equiangular k-gon and that the circular arcs $b_1, \dots, b_k$ on the boundary of C are congruent. Thus we can assume $k \geq 4$. We denote the interior angle of P by 2ϕ, and the central angle of b_i by 2ψ. Comparing the two representations of a(P)

$$\begin{aligned} a(P) &= 2k\phi - (k-2)\pi, \\ a(P) &= a + 2k\phi + 2k\psi\cos r - k\pi \end{aligned} \tag{12}$$

we obtain

$$a + 2k\psi\cos r = 2\pi. \tag{13}$$

(13), together with

$$\sin\psi = \frac{\cos\phi}{\cos r}, \tag{14}$$

implies

$$\frac{\psi}{\sin\psi} = \frac{2\pi - a}{2k\cos\phi}. \tag{15}$$

By use of (14) we find for the perimeter of P

$$p(P) = p - 2k\sin r \arcsin\frac{\cos\phi}{\cos r} + 2k\arcsin(\tan r\cot\phi). \tag{16}$$

Suppose that P is not regular. Let $\overline{P}$ be the regular k-gon with

$$a(\overline{P}) = a(P). \tag{17}$$

If the area of the in-circle of $\overline{P}$ is at most a, then there is a smooth regular k-gon $\overline{C}$ with case $\overline{P}$ and $a(\overline{C}) = a$. By (12) and (17), the interior angle of $\overline{P}$ is equal to 2ϕ. (15) and (14) show that the circular arcs on the boundary of $\overline{C}$ have central angle 2ψ and radius r. Thus, applying (16) to $\overline{P}$ and $\overline{C}$ we find

$$p(\overline{P}) = p(\overline{C}) - 2k\sin r \arcsin\frac{\cos\phi}{\cos r} + 2k\arcsin(\tan r\cot\phi). \tag{18}$$

By (17), we have $p(\overline{P}) < p(P)$. Hence, by (16) and (18)

$$p(\overline{C}) < p. \tag{19}$$

Since $a(\overline{C}) = a$ and $a(\overline{P}) = a(P)$, the convex cap $\overline{C} \in \mathcal{C}(a,p)$ satisfies the condition $\delta^A(\overline{C},\overline{P}) = \Delta_e^A(a,p,k)$. But (19) contradicts Lemma 8.

If the area of the in-circle of $\overline{P}$ is greater than a, then there is a circle $\overline{C}$ of area a in the interior of $\overline{P}$. Because $p(\overline{C}) \leqq p$, $\overline{C}$ is from $\mathscr{C}(a,p)$. Since $\delta^A(\overline{C},\overline{P}) = \Delta_e^A(a,p,k)$, we have a contradiction to Lemma 1. Thus the supposition that P is not regular was wrong and Lemma 10 is proved.

Let now (ii) be satisfied. From $a(P) = a(C) + \delta^A(C,P)$ it follows that $a(P)$ attains its minimum if $a(C)$ and $\delta^A(C,P)$ are minimal, that is in the indicated case. Moreover, we see that

$$m_e^A(a,p,k) = a + \Delta_e^A(a,p,k).$$

(iii) By the isoperimetric inequality (3), $p(P)$ attains its minimum if P is regular and $a(P)$ minimal, as required. Moreover, we have

$$\cos\frac{m_e^P(a,p,k)}{2k} \quad \cos\frac{2\pi - m_e^A(a,p,k)}{2k} = \cos\frac{\pi}{k}.$$

(iv) From $\delta^P(C,P) = p(P) - p(C)$ it follows that $\delta^P(C,P)$ is minimal if $p(P)$ is minimal and $p(C)$ maximal, in accordance with Theorem 1. Furthermore, we have

$$\Delta_e^P(a,p,k) = m_e^P(a,p,k) - p.$$

This completes the proof of Theorem 1.

Let $k \geq 2$ and the constants a,p with $0 < a,p < 2\pi$ be given such that the inequalities (2) and (8) are satisfied. We conclude the paper by showing that a smooth regular k-gon is uniquely determined by the parameters $a(C) = a$ and $p(C) = p$.

A regular k-gon P is the case of a smooth regular k-gon of perimeter p if and only if

(a) $p < p(P)$, and

(b) $p \geq$ perimeter of the in-circle of P.

Let 2ϕ be the interior angle of P. A straight forward calculation shows that the conditions (a) and (b) are satisfied exactly if

$$\arcsin \frac{\cos \frac{\pi}{k}}{\cos \frac{p}{2k}} < \phi \leq \arccos \left(\sqrt{1-\left(\frac{p}{2\pi}\right)^2} \sin \frac{\pi}{k} \right). \tag{20}$$

Each of these values ϕ determines uniquely a smooth regular k-gon of perimeter p and case P. The corresponding value of r follows from

$$\cos \frac{p(P)}{2k} = \frac{\cos \frac{\pi}{k}}{\sin \phi} .$$

or, by (16), from

$$f(r,\phi) \equiv \frac{p}{2k} - \sin r \arcsin \frac{\cos \phi}{\cos r} + \arcsin (\tan r \cot \phi)$$
$$- \arccos \frac{\cos \frac{\pi}{k}}{\sin \phi} = 0. \tag{21}$$

It is easy to show that the equation $f(r,\phi) = o$ has exactly one solution $r = r(\phi)$ contained in the interval

$$0 < r \leq r_i = \arccos \frac{\cos \phi}{\sin \frac{\pi}{k}} \quad (= \text{in-radius of } P) \tag{22}$$

with equality if and only if $\phi = \arccos \left(\sqrt{1-(p/2\pi)^2} \sin \pi/k \right)$.
Note that by (13) and (14)

$$a(C) = 2\pi - 2k \cos r(\phi) \arcsin \frac{\cos \phi}{\cos r(\phi)} .$$

Making use of (21) and (22) we finally obtain

$$\frac{da(C)}{d\phi} = \frac{2k}{\sin \phi} \left(\sqrt{1-\frac{\cos^2\phi}{\cos^2 r}} - \frac{\cos \frac{\pi}{k} \cos \phi \tan r}{\sqrt{\sin^2\phi - \cos^2 \frac{\pi}{k}}} \right) \geq 0$$

with equality if and only if $\phi = \arccos \left(\sqrt{1-(p/2\pi)^2} \sin \pi/k \right)$.
Observe that

$$a(C) \to 2\pi - 2k \arccos \frac{\cos \frac{\pi}{k}}{\cos \frac{p}{2k}} < a$$

as $\phi \to \arcsin \frac{\cos(\pi/k)}{\cos(p/2k)}$, and

$$a(C) = 2\pi\,(1 - \sqrt{1 - (\frac{p}{2\pi})^2}\,) \geq a$$

for $\phi = \arccos(\sqrt{1-(p/2\pi)^2}\,\sin \pi/k)$. Hence, there is exactly one ϕ in the interval given by (20) such that $a(C) = a$. The corresponding value of r is the (unique) solution of the equation $f(r,\phi) = 0$. This proves the statement of uniqueness.

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